Groupoids: C^*-algebras, Rapid Decay and Amenability

University of Hawai‘i at Mānoa

Doctor of Philosophy in Mathematics, May 2018

John Clark Robertson

Dissertation Committee:
Rufus Willett, Chairperson
Erik Guentner
Robert Harron
Sarah Post
Jason Kumar

Abstract

In this paper we study properties of groupoids by looking at their C^*-algebras. We introduce a notion of rapid decay for transformation groupoids and we show that this is equivalent to the underlying group having the property of rapid decay. We show that our definition is equivalent to a number of other properties which are in direct correspondence to the group case. Additionally, given two bilipschitz equivalent discrete groups we construct an isomorphism of the corresponding transformation groupoids and are able to reformulate the open problem of showing invariance of rapid decay under quasi-isometry.

We then begin to examine various notions of amenability when abstracted to measured étale groupoids. In the group case, the following properties are equivalent:

1) G is amenable

2) $C_r^*(G) = C^*(G)$

3) The trivial representation decends from $C^*(G)$ to $C_r^*(G)$.

In the group*oid* case we have 1) $\Rightarrow$ 2) $\Rightarrow$ 3), but it is shown in [19] that $C_r^*(\mathcal{G}) = C^*(\mathcal{G})$ is not enough in general to give amenability of $\mathcal{G}$. In this paper we study property 3) for groupoids, formulate some equivalent statements and show that 3) $\Rightarrow$ 2) is also false in general.

Acknowledgments

I would like to express my deepest gratitude to Professor Rufus Willett. For without his unwavering patience and undeniable genius I would have no dissertation to present. Thank you to my friends and family for your support. A special thanks to Ryan Felix and Dr. Efren Ruiz for everything, I couldn't have done this without you. Thank you to the entire Mathematics Department at UH Mānoa, you all all amazing, to my graduate committee for your flexibility, to Clément Dell'Aiera for your friendship and mathematical insight and to Dr. Guentner for putting up with me.

Contents

Chapter 1

Basics

1.1 Groups, Groupoids and their C^*-Algebras

The study of C^*-algebras has had a long and rich history, and researchers have used discrete groups to generate new examples and understand old ones. The goal of this first section is to introduce the reader to this kind of construction and motivate the use of groupoids in the study of C^*-algebras.

The quick way to define what a C^*-algebra is, is to declare it to be any closed $*$-subalgebra of bounded operators on some Hilbert space. This works fine for our purposes, as our constructions usually involve creating a $*$-algebra, representing it as bounded operators on some Hilbert space and completing. If G is a discrete group, we can use its information to move around basis vectors in the Hilbert space $\ell^2(G)$ and complete the appropriate $*$-algebra to get a C^*-algebra. More precisely, for a discrete group, G, we define the regular representation, $\lambda : G \to \mathbb{B}(\ell^2(G))$ to be the group homomorphism (the range of λ is a subset of the unitary group) defined by

$$\lambda_g(\delta_h) = \delta_{gh}$$

where $\delta_h \in \ell^2(G)$ is an element of the canonical orthonormal basis. The notation can be confusing. Some people like to write $\lambda(g) \in \mathbb{B}(\ell^2(G))$ as the image of g under the map λ, but then describing how it operates on basis vectors becomes $\lambda(g)(\delta_h) = \delta_{gh}$ and the order of operations is now less clear, so we'll stick to the λ_g notation. The $*$-algebra we'll need is called the group ring,

$$\mathbb{C}(G) = \left\{ \sum_{g \in G} a_g g \ : \ a_g \in \mathbb{C} \text{ and } a_g = 0 \text{ for all but finitely many g} \right\}$$

and we declare the multiplication to be

$$\begin{aligned}(\sum_{g\in G} a_g g)(\sum_{h \in G} b_h h) &= \sum_{g,h\in G} a_g b_h gh \\ &= \sum_{t\in G}(\sum_{g\in G} a_g b_{g^{-1}t})t.\end{aligned}$$

and the involution to be

$$(\sum_{g\in G} a_g g)^* = \sum_{g\in G} \overline{a_g} g^{-1}$$

The left regular representation (or any representation) extends from the group to the group ring naturally,

$$\lambda(\sum_{g\in G} a_g g) = \sum_{g\in G} a_g \lambda_g.$$

It's sometimes convenient to view elements of the group ring as functions on the group under the identification $f \leftrightarrow \sum_{g\in G} a_g g$ where $f(g) = a_g$. The usual formula for convolution of functions

$$(f * g)(s) = \sum_{ab=s} f(a)g(b) = \sum_{a\in G} f(a)g(a^{-1}s)$$

and involution $f^*(s) = \overline{f(s^{-1})}$ corresponds to the multiplication and involution in the group ring that we have defined above.

Now that we have a representation of $\mathbb{C}(G)$, each element gives rise to a bounded linear operator on $\ell^2(G)$ and therefore has an operator norm. The reduced C^*-algebra of G is given by $\overline{\mathbb{C}(G)}^{||\cdot||_{\mathbb{B}(\ell^2(G))}}$ and denoted $C^*_r(G)$. We sometimes write $||f||_r = ||\lambda(f)||_{\mathbb{B}(\ell^2(G))}$.

For a Hilbert space, $\mathcal{H}$ we denote the group of unitary operators on $\mathcal{H}$ by $U(\mathcal{H})$. Another C^*-algebra associated to G is called the full (or max) C^*-algebra and is defined by completing the group ring under the norm given by

$$||f||_{max} = \sup_{\pi} \{||\pi(f)|| \ : \ \pi : G \to U(\mathcal{H}) \text{ is a homomorphism } \}$$

and is denoted $C^*(G)$.

One should stop here and investigate what kind of C^*-algebras these constructions are giving rise to, but in the spirit of casting math spells we proceed ahead.

Definition 1.1.1. A groupoid is a small category such that every morphism is invertible.

The above definition is usually a bit cryptic for much of any understanding, unless category theory is your thing. As the name suggests, a groupoid is group*ish* in the sense that is has a multiplication and every element has an inverse. The following list of facts will will be handy when working with a groupoid, $\mathcal{G}$: The multiplication is a map $m : \mathcal{G}^2 \to \mathcal{G}$ is defined on a subset $\mathcal{G}^2 \subset \mathcal{G} \times \mathcal{G}$, and we write the image of (g, h) as gh. Multiplication is also associative. For each $g \in \mathcal{G}$, g^{-1} always exists and one can always multiply $g^{-1}g$ and gg^{-1}, however these need not be equal. Further, it is always the case that $(g^{-1})^{-1} = g$. The special subset of $\mathcal{G}$ where $g = g^{-1}g$ is denoted $\mathcal{G}^{(0)}$ and is called the unit space or base space. The range map $r : \mathcal{G} \to \mathcal{G}^{(0)}$ and source map $s : \mathcal{G} \to \mathcal{G}^{(0)}$ determine what elements can be multiplied: $(g, h) \in \mathcal{G}^2 \iff s(g) = r(h)$. Additionally, $g^{-1}g = s(g)$, $gg^{-1} = r(g)$, and more generally $r(gh) = r(g)$ and $s(gh) = s(h)$. We define $\mathcal{G}^x := r^{-1}(x)$ and $\mathcal{G}_x := s^{-1}(x)$.

Remark 1.1.2. A groupoid is a group if and only if the base space is a point. In this way, a groupoid is a natural generalization of a group.

It should not be surprising that mathematicians have declared various topologies on groupoids and studied the results.

Definition 1.1.3. A topological groupoid is a groupoid $\mathcal{G}$ equipped with a locally compact topology such that $\mathcal{G}^{(0)} \subset \mathcal{G}$ is Hausdorff in the relative topology, the range, source, and inverse maps are all continuous and the multiplication map is continuous with respect to the relative topology on $\mathcal{G}^2$ as a subset of $\mathcal{G} \times \mathcal{G}$.

The following is a nice property to have when working with a topological groupoid:

Definition 1.1.4. A topological groupoid is called étale if the range and source maps are local homeomorphisms.

Definition 1.1.5. We call the subset of $\mathcal{G}$ such that $r(\gamma) = s(\gamma)$ the isotropy groupoid. One checks that this is indeed a groupoid, and further that for each $x \in \mathcal{G}^{(0)}$, $\mathcal{G}_x$ is a group.

The following example is the motivating example for étale groupoids, and probably the best way for someone to begin thinking about groupoids in general.

Example 1.1.6. Let G be a discrete group with identity e, and a homeomorphic action on a compact topological space X. For an $x \in X$, denote gx the image of x by $g \in G$. The transformation groupoid, $X \rtimes G$, is constructed as follows:

$$X \rtimes G := \{(gx, g, x) \in X \times G \times X \ : \ x \in X, g \in G\}$$

it has unit space $\mathcal{G}^{(0)} = (X \times G)^{(0)} := \{(gx, g, x) \in X \rtimes G \ : \ g = e\}$. One often identifies $\mathcal{G}^{(0)}$ with X (and vice-versa) under the map $(x, e, x) \mapsto x$. The range and source maps are given by

$$r((gx, g, x)) = gx, \quad s((gx, g, x)) = x,$$

multiplication as $(hgx, h, gx)(gx, g, x) = (hgx, hg, x)$, and $(gx, g, x)^{-1} = (x, g^{-1}, gx)$. With the subspace topology inherited from $X \times G \times X$, it's not too hard to see that $X \rtimes G$ is a (locally compact) étale groupoid.

In this paper, unless otherwise stated, all groupoids are étale, and Hausdorff. This implies that each of the sets $\mathcal{G}_x$ and $\mathcal{G}^x$ are discrete for all $x \in \mathcal{G}^{(0)}$. The compactly supported continuous functions, $C_c(\mathcal{G})$, is a $*$-algebra with convolution defined by $(f * g)(\gamma) = \sum_{\gamma_1\gamma_2=\gamma\in\mathcal{G}} f(\gamma_1)g(\gamma_2)$ and star operation $f^*(\alpha) = \overline{f(\alpha^{-1})}$. Each $f \in C_c(\mathcal{G})$ can be seen as a (bounded) convolution operator on $\ell^2(\mathcal{G}_x)$ via the formula for a $\xi \in \ell^2(\mathcal{G}_x)$:

$$f * \xi(\alpha) = \sum_{\gamma_1\gamma_2=\alpha} f(\gamma_1)\xi(\gamma_2) = \sum_{\beta\in\mathcal{G}_x} f(\alpha\beta^{-1})\xi(\beta).$$

Definition 1.1.7. The reduced C^*-algebra of $\mathcal{G}$, $C^*_r(\mathcal{G})$, is defined to be the completion of

$C_c(\mathcal{G})$ under the norm

$$||f||_{C_r^*(\mathcal{G})} := \sup_{x \in \mathcal{G}^{(0)}} ||f||_{\mathbb{B}(\ell^2(\mathcal{G}_x))}.$$

1.2 An Explicit Construction

The isomorphism $C(X) \rtimes_r G \cong C_r^*(X \rtimes G)$ is well known but proofs, as in [17], commonly over simplify the situation. In this section we provide a straightforward, albeit long, simple minded proof with most details worked out. We also adopt a non-standard construction of $C(X) \rtimes_r G$ as it becomes easier to work with later on, especially when defining a notion of rapid decay for groupoids.

Our construction of the C^*-algebra $C(X) \rtimes_r G$ begins with the $*$-algebra

$$C(X) \rtimes_{alg} G := \left\{ \sum_{F \subset G} f_g g \ : \ f_g \in C(X), \quad g \in G, \quad F \text{ a finite subset of } G \right\},$$

where elements are subject to the relation $gf(\cdot) = f(g^{-1}\cdot)g$, which allows closure of multiplication, and we define the $*$-operation to be $(f_g g)^* = g^{-1}\overline{f}_g$. Throughout, F will always be a finite subset of the appropriate group.

We summon a faithful family of representations of $C(X) \rtimes_{alg} G$ as follows: Fix an $x \in X$ and define

$$\pi_x : C(X) \rtimes_{alg} G \to B(\ell^2(G \cdot x \times G))$$

$$g \mapsto U_g, \qquad U_g(\delta_{y,h}) = \delta_{gy,gh}$$

$$f \mapsto \text{Multiplication Operator on the first coordinate}$$

$$||\sum_{g \in G} f_g g||_{C(X) \rtimes_r G} := \sup_x ||\pi_x(\sum_{g \in G} f_g g)||_{B(\ell^2(G \cdot x \times G))}$$

and complete $C(X) \rtimes_{alg} G$ with respect to it. Note that we may simply write g for U_g where there can be no confusion.

As you may have noticed, this is not the usual definition of the reduced crossed product as in ([3] 4.1.2). However, they show that such a construction does not depend on the (faithful) representation of $C(X)$. As $\bigoplus_{x\in X} \ell^2(G\cdot x)$ is a faithful representation of $C(X)$, the canonical isomorphism

$$\bigoplus_{x\in X} \ell^2(G\cdot x\times G) \cong \bigoplus_{x\in X} \left(\ell^2(G\cdot x)\otimes \ell^2(G)\right) \cong \left(\bigoplus_{x\in X} \ell^2(G\cdot x)\right)\otimes \ell^2(G)$$

shows that our construction does indeed give the same C^*-algebra.

Proposition 1.2.1. Let a countable group G act on a compact space X. Choose an $x\in X$, and for a $y\in G\cdot x$ define $H_y := \ell^2(\{(gy,g)\ :\ g\in G\})$, then

$$\ell^2(G\cdot x\times G) \cong \bigoplus_{y\in G\cdot x} H_y.$$

Proof. This amounts to showing that

$$G\cdot x\times G = \bigsqcup_{y\in G\cdot x} \{(gy,g)\ :\ g\in G\}.$$

Clearly,

$$G\cdot x\times G = \bigcup_{y\in G\cdot x} \{(gy,g)\ :\ g\in G\}$$

and one can show disjointness as follows: for $y_1, y_2\in G\cdot x$, if $(gy_1, g) = (hy_2, h)$, then $g = h$, so $gy_1 = hy_2 = gy_2$, and therefore $y_1 = y_2$. ☐

We make a note that $\pi(f_g g)H_y \subset H_y$, for all y and g and so we may view $\pi_x(f_g g)$ as a diagonal operator on $\bigoplus_{y\in G\cdot x} \ell^2(H_y)$. This clearly extends to any $f\in C(X)\rtimes_{alg} G$. For the operator $\pi_x(\cdot)|_{\ell^2(H_y)}$, we write π_x^y to simplify the notation.

Proposition 1.2.2. Fix an $x\in X$. The operator π_x^y is unitarily equivalent to π_x^z.

Proof. Write $y = a\cdot x$ and $z = b\cdot x$. Define $U : H_y \to H_z$ by $\delta_{gy,g} \mapsto \delta_{gba^{-1}y,g}$. An easy computation shows that U is well-defined, unitary and that $U\pi_x^y U^* = \pi_x^z$. ☐

Proposition 1.2.3. With the notation of the above propositions, we have that

$$||\pi_x(\sum f_g g)||_{B(\ell^2(G\cdot x\times G))} = ||\pi_x(\sum f_g g)|_{\mathcal{H}_x}||_{B(\mathcal{H}_x)}.$$

Proof. Denote $H = \bigoplus_{y\in G\cdot x} H_y$, and let $\xi \in \ell^2(G\cdot x\times G)$ and write $\xi = \sum \xi_y$, with $\xi_y \in H_y$. Now consider

$$\begin{aligned}
||\pi_x(\sum f_g g)(\xi)||^2_{\ell^2(G\cdot x\times G)} &= ||\pi_x(\sum f_g g)(\xi)||^2_H \\
&= ||\pi_x(\sum f_g g)(\sum_y(\xi_y))||^2_H \\
&= \sum_y ||\pi_x(\sum f_g g)(\xi_y)||^2_{H_y} \\
&= \sum_y ||\pi_x(\sum f_g g)|_{H_y}(\xi_y)||^2_{H_y} \\
&\leq \sum_y ||\pi_x(\sum f_g g)|_{H_y}||^2_{B(H_y)}||(\xi_y)||^2_{H_y} \\
&= \sum_y ||\pi_x(\sum f_g g)|_{H_x}||^2_{B(H_x)}||(\xi_y)||^2_{H_y} \\
&= ||\pi_x(\sum f_g g)|_{H_x}||^2_{B(H_x)}||\xi||^2_H.
\end{aligned}$$

Now, we can take a supremum over unit vectors to obtain the inequality

$$||\pi_x(\sum f_g g)||_{B(\ell^2(G\cdot\times G))} \leq ||\pi_x(\sum f_g g)|_{\mathcal{H}_x}||_{B(\mathcal{H}_x)}.$$

Extending H_x by zero gives $H_x \subset \ell^2(G\cdot x\times G)$, and the other inequality is now trivial. The result follows. □

Theorem 1.2.4. Let X be a compact topological space and G a discrete group that acts on X via homeomorphisms. The function

$$\phi : C_c(X \rtimes G) \to C(X) \rtimes_{alg} G$$
$$f \mapsto \sum f_g g$$
$$f_g(gx) := f(gx, g, x)$$

is a $*$-algebra isomorphism that extends to a C^*-algebra isomorphism to give $C(X) \rtimes_r G \cong C_r^*(X \rtimes G)$.

Proof. It is clear that ϕ is bijective. To show ϕ respects the $*$ operation, fix an $f \in C_c(X \rtimes G)$ and compute

$$\begin{aligned} \phi(f^*) &= \sum_{g \in G} (f^*)_g g, \quad \text{where} \\ (f^*)_g(gx) &= f^*(gx, g, x) \\ &= \overline{f(x, g^{-1}, gx)} \end{aligned}$$

And similarly,

$$
\begin{aligned}
(\phi(f))^* &= \big(\sum_{g\in G} f_g g\big)^* \\
&= \sum_{g\in G} g^{-1}\overline{f}_g \\
&= \sum_{g\in G} g\overline{f}_{g^{-1}} \\
&= \sum_{g\in G} \overline{f}_{g^{-1}}(g^{-1}\cdot\ \ )g, \quad \text{where} \\
\overline{f}_{g^{-1}}(g^{-1}(gx)) &= \overline{f}_{g^{-1}}(x) \\
&= \overline{f(x, g^{-1}, gx)}.
\end{aligned}
$$

The above shows that $(f^*)_g(gx) = \overline{f}_{g^{-1}}(g^{-1}(gx))$, and therefore $\phi(f^*) = (\phi(f))^*$. It's not hard to check that ϕ preserves multiplication.

Our next claim is that $||f||_{\mathbb{B}(\ell^2(\mathcal{G}_x))} = ||\pi_x(\phi(f))||_{\mathbb{B}(\ell^2(G\cdot x\times G)}$. When $f \in C_c(\mathcal{G})$ is seen as an operator on $\ell^2(\mathcal{G}_x)$ we will write $\overline{\pi_x}(f)$ for clarity. For an arbitrary $x \in \mathcal{G}^{(0)}$, $\ell^2(\mathcal{G}_x)$ and H_x are unitarily equivalent through $U(\delta_{(gx,g,x)}) = \delta_{(gx,g)}$. Take an $f = \sum_{F\subset G} f_g g \in C(X) \rtimes_{alg} G$, denote $F := \phi^{-1}(f)$, and see that we can directly compute

$$
\begin{aligned}
(U\overline{\pi_x}(\phi^{-1}(f))U^{-1})(\delta_{ax,a}) &= U\bigg(\sum_{y\in G_{ax}} F(y)\delta_{y(ax,a,x)}\bigg) \\
&= U\bigg(\sum_{g\in F\subset G} F(gax, g, ax)\delta_{(gax,ga,x)}\bigg) \\
&= \sum_{g\in F\subset G} F(gax, g, ax)\delta_{(gax,ga)} \\
&= \sum_{g\in F\subset G} f_g(gax)\delta_{(gax,ga)} \\
&= \pi_x(f)(\delta_{ax,a})
\end{aligned}
$$

this gives that $||\overline{\pi}_x(F)||_{\mathbb{B}(\ell^2(G_x))} = ||\pi_x(\sum_{g\in F\subset G} f_g g)||_{\mathbb{B}(H_x)}$, and so

$$\begin{aligned}||\overline{\pi}_x(F)||_{\mathbb{B}(\ell^2(G_x))} &= ||\pi_x(\sum_{g\in F\subset G} f_g g)||_{\mathbb{B}(H_x)} \\ &= ||\pi_x(\sum_{g\in F\subset G} f_g g)||_{\mathbb{B}(\ell^2(G\cdot x\times G))}.\end{aligned}$$

Clearly, we can take a supremum over x on both sides and (finally) obtain that

$$C(X) \rtimes_r G \cong C_r^*(X \rtimes G).$$

□

Chapter 2

Rapid Decay Groupoids

2.1 Motivations and Definitions

It could be said that the study of rapid decay groups began with a technical inequality of U. Haagerup found in [6] which came about in the process of showing that $C_r^*(\mathbb{F}_2)$ has the metric approximation property. However, the first defintion of rapid deacy wan't given until almost 20 years later by A. Connes and was not phrased in terms of an explicit inequality. This inequality of Haagerup's was generalized, shown to be equivalent to Connes' defintion, has been studied extensively and has seen several nice results. Most notably that if G is a "good" group with the property of rapid decay then $C_r^*(G)$ satisfies the Baum-Connes conjecture (which is aparently due to the work of Lafforgue in [11]). Without saying anything about K-theory, in this section we recall some basic definitions and results about rapid decay groups that motivate our definition of rapid decay groupoids. One should see this section as a bare bones introduction to rapid decay for discrete groups which highlights the similarities of our later definition. For more on rapid decay for discrete groups we point the reader to [4] and [5].

Definition 2.1.1. A length function on a discrete group G is a function $l : G \to \mathbb{R}_+$ such that $l(g) = l(g^{-1})$ for any $g \in G$, $l(gh) \leq l(g) + l(h)$ for any $g, h \in G$ and $l(e) = 0$.

Definition 2.1.2. Let $l : G \to \mathbb{N}$ be a length function and for $f = \alpha_g g \in \mathbb{C}(G)$ and a polynomial, P call $P_l(f) := \max_{g \in \text{supp}(f)} P(l(g))$. We say that a discrete group has the property of rapid decay with respect to l if there exists a polynomial P such that

$$||f||_r \leq P_l(f)||f||_2$$

for all $f \in \mathbb{C}(G)$.

When G is finitely generated, the property of rapid decay is independent of the choice of a word length function defined using a generating set i.e. if S_1 and S_2 are two (finite) generating sets for the same group and l_1 and l_2 are the associated algebraic word length functions, rapid decay with respect to l_1 implies rapid decay with respect to l_2 and vice versa, [4] . In this section we assume this to be the case and therefore need not worry about what generating set we use.

Definition 2.1.3. For an $s \in \mathbb{R}^{\geq 0}$, one obtains the s-Sobolev space $H^s(G)$ by completing the group ring via the norm given by

$$||f||_s = \sqrt{\sum_{g \in G} |f(g)|^2(1 + l(g))^{2s}}.$$

The functions of rapid decay are defined to be

$$H^\infty(G) := \bigcap_s H^s(G).$$

The following is a well known result. For a proof we refer the reader to the next section where we have proven this statement in more generality.

Proposition 2.1.4. A discrete group G has the property of rapid decay if and only if there exists a $c > 0$ and s such that for all $f \in \mathbb{C}(G)$,

$$||f||_r \leq c||f||_s.$$

The introduction of these spaces was originally motivated for the purposes of computing

K-theory, and unless G has rapid decay $H^s(G)$ may not even be an algebra. When G does have this property however, there is an s such that $H^s(G)$ is a $*$-subalgebra of $C_r^*(G)$ [10] and has the same K-theory (which is apparently easier to compute, see [8] for a proof). For now, they serve as a tool to rephrase the property of rapid decay for groups which subsequently becomes a desired property when generalizing to the transformation groupoid case.

Definition 2.1.5. For a finitely generated group G with word length function l and compact space X, we say that the transformation groupoid $G \rtimes X$ (or sometimes $C(X) \rtimes_r G$) has rapid decay if there exists a polynomial P such that

$$\text{for all} \sum_{g \in F \subset G} f_g g \in C(X) \rtimes_{alg} G,$$

$$||\sum_g f_g g||_{C(X) \rtimes_r G} \leq \max_{g \in F} P(l(g)) \sqrt{\sum_{F \subset G} ||f_g||^2_{C(X)}}. \quad (\dagger)$$

Sometimes we write $P(f)$ or $P(F)$ instead of $\max_{g \in F} P(l(g))$. We restrict to the finitely generated groups throughout the section. When the context is clear, we sometimes write

$$||f||_{2,C(X)} := \sqrt{\sum_{F \subset G} ||f_g||^2_{C(X)}}.$$

Remark 2.1.6. Recall that in the case of group rapid decay it suffices to consider $f \in \mathbb{R}_+(G)$ (non-negative, real valued functions on the group) while establishing the property of Rapid Decay for groups. The analogue for groupoids is that it suffices to consider only non-negative and real valued f_g's $\in C(X)$. To see this, suppose that all such functions which satisfy ($\dagger$) in Definition 2.1.4 and consider an arbitrary $f = \sum_{g \in F \subset G} f_g g \in C(X) \rtimes_{alg} G$. We may write each $f_g = f_g^1 - f_g^2 + i(f_g^3 - f_g^4)$ such that the support of f^1 is disjoint from f^2, similarly for f^3 and f^4, which puts all $f^i \in f \in \mathbb{R}_+(G)$. Doing this ensures that $|f_g^i|_{C(X)} \leq |f_g|_{C(X)}$ for all i and g.

Now,

$$||f||_{C(X)\rtimes_r G} \leq \sum_{i=1}^{4} ||\sum_{g\in F} f_g^i g||_{C(X)\rtimes_r G} \leq P(F)\sum_{i=1}^{4}\sqrt{\sum_{F\subset G}||f_g^i||^2_{C(X)}} \leq 4P(F)\sqrt{\sum_{F\subset G}||f_g||^2_{C(X)}}.$$

A similar computation shows that it suffices to consider only positive vectors. More precisely, if $||\pi_x(f)\xi|| \leq \max_{g\in F} P(l(g))\sqrt{\sum_{F\subset G}||f_g||^2_{C(X)}}$ for all $\xi \in \ell^2(G\cdot x, G)^+$ and $x \in X$, then $X \rtimes G$ has rapid decay.

2.2 Results Similar to the Group Case

We now provide some stability type results for moving to a more general (transformation) groupoid from a group. The proofs we provide have various degrees of similarity to the group case, and some have no difference at all. This section serves as a collection of evidence to suggest that our generalized definition is a reasonable one. A notion of rapid decay for groupoids has been defined before, as in [7], however our definition is not equivalent and as far as we know not found anywhere else in the literature.

Definition 2.2.1. For a $\sum_{g\in F\subset G} f_g g \in C(X) \rtimes_{alg} G$ and $p \geq 0$, define

$$||\sum_g f_g g||_p := \sqrt{\sum_{F\subset G}(1+l(g))^{2p}||f_g||^2_{C(X)}}$$

Note that this is indeed a norm and let

$$L_{2,p}(X \rtimes G) := \overline{C(X) \rtimes_{alg} G}^{||\cdot||_p}.$$

By $S_2(X \rtimes G)$, we mean the completion of $C(X) \rtimes_{alg} G$ under the local convex topology generated by the sequence of p norms defined above along with the ∞ norm. Recall that, by definition, in this topology $f_n \to f$ in $S_2(X \rtimes G)$ if and only if $f_n \to f$ in $L_{2,p}(X \rtimes G)$ for all p (including ∞). This means that

$$S_2(X \rtimes G) = \bigcap_{p}^{\infty} L_{2,p}(X \rtimes G)$$

Lemma 2.2.2. A transformation groupoid has rapid decay if and only if there is a C and a p such that

$$||\sum_g f_g g||_{C(X)\rtimes_r G} \leq C||\sum_g f_g g||_p$$

$$\text{for every} \quad \sum_g f_g g \in C(X) \rtimes_{alg} G.$$

Proof. Suppose that $||\sum_g f_g g||_{C(X)\rtimes G} \leq C||\sum_g f_g g||_p$ for every $\sum_g f_g g \in C(X) \rtimes_{alg} G$. Since

$$\sqrt{\sum_{F\subset G}(1+l(g))^{2p}||f_g||^2_{C(X)}} \leq \max_{g\in F}(1+l(g))^p \sqrt{\sum_{F\subset G}||f_g||^2_{C(X)}}$$

and thus the transformation groupoid has rapid decay.

Suppose that the transformation groupoid has rapid decay with polynomial P. We shall now choose p and a K large enough so that $(1+n)P(n) \leq K(1+n)^p$ for all n.

Let

$$A_n = \left\{\sum_g f_g g \in C(X) \rtimes_{alg} G \ : \ l(g) = n, \text{ when } f_g \neq 0\right\}$$

and

$$f_n = \sum_g f_g g\bigg|_{A_n}.$$

Now, for an $\sum_g f_g g \in C(X) \rtimes_{alg} G$ we have

$$
\begin{aligned}
||\sum_g f_g g||_{C(X)\rtimes G} &\leq \sum_n ||f_n||_{C(X)\rtimes G} \\
&\leq \sum_n P(n)\sqrt{\sum_{g\in A_n} ||f_g||^2_{C(X)}} \\
&= \sum_n \frac{n+1}{n+1}P(n)\sqrt{\sum_{g\in A_n} ||f_g||^2_{C(X)}}.
\end{aligned}
$$

The Cauchy-Schwarz inequality gives

$$
\sum_n \frac{n+1}{n+1}P(n)\sqrt{\sum_{g\in A_n} ||f_g||^2_{C(X)}} \leq \left(\sum_n \frac{1}{(1+n)^2}\right)^{1/2}\left(\sum_n (1+n)^2(P(n))^2\sum_{g\in A_n} ||f_g||^2_{C(X)}\right)^{1/2}
$$

and one may use our choice of p and K to obtain

$$
\begin{aligned}
\frac{\pi}{\sqrt{6}}\left(\sum_n (1+n)^2(P(n))^2\sum_{g\in A_n} ||f_g||^2_{C(X)}\right)^{1/2} &\leq \frac{\pi}{\sqrt{6}}\left(\sum_n (1+n)^{2p}K^2\sum_{|g|=n} ||f_g||^2_{C(X)}\right)^{1/2} \\
&= \frac{\pi}{\sqrt{6}}K\sqrt{\sum_{F\subset G}(1+l(g))^{2p}||f_g||^2_{C(X)}} \\
&= C||\sum_g f_g g||_p
\end{aligned}
$$

Combining the above gives a C and p, independent of the arbitrarily chosen $\sum_g f_g g \in C(X)\rtimes_{alg} G$, such that $||\sum_g f_g g||_{C(X)\rtimes G} \leq C||\sum_g f_g g||_p$. □

Even though our definition of a rapid decay groupoid is not equivalent, parts of the proofs of the following two statements are similar to ones found in [7] where they can further prove that $S_2(X\rtimes G)$ is a spectral invariant $*$-subalgebra.

Lemma 2.2.3. The groupoid has rapid decay if and only if $S_2(X\rtimes G)$ is contained in

$C(X) \rtimes_r G$.

Proof. Suppose that $X \rtimes G$ has rapid decay i.e. there is a p and C such that

$$||\sum_g f_g g||_{C(X)\rtimes_r G} \leq C||\sum_g f_g g||_p \quad \text{for each} \quad \sum_{F\subset G} f_g g \in C(X) \rtimes_{alg} G.$$

The above inequality dictates that $L_{2,p}(X \rtimes G) \subset C(X) \rtimes_r G$, but since

$$S_2(X \rtimes G) = \bigcap_p^\infty L_{2,p}(X \rtimes G)$$

we have $S_2(X \rtimes G) \subset L_{2,p}(X \rtimes G)$ and therefore $S_2(X \rtimes G)$ is contained in $C(X) \rtimes_r G$.

Now, suppose that $S_2(X \rtimes G)$ is contained in $C(X) \rtimes_r G$. We claim that the inclusion map i is a closed map and whence continuous by the closed graph theorem. The injection $j_0 : C_c(X \rtimes G) \to C_0(X \rtimes G)$ extends to a $1-1$ and norm decreasing inclusion map from $C(X) \rtimes_r G$ to the Banach space $C_0(X \rtimes G)$ ([14], Lemma 2.1.17) and we call this map J_0. Since $||f||_\infty \leq ||f||_r$, J_0 is continuous. Further, since $S_2(X \rtimes G) \subset C_0(X \rtimes G)$, the inclusion map, J_1, is also continuous since $||J_1(f)||_\infty \leq ||f||_s$ for all s. Note that, by definition, $J_1 = J_0 \circ i$ on $S_2(X \rtimes G)$. Let $\phi_n \to \phi \in S_2(X \rtimes G) \subset C_0(X \rtimes G)$ and set $f = \lim i(\phi_n)$. We have $J_0(i(\phi_n)) \to J_0(f)$ because J_0 is continuous, and also that $J_0(i(\cdot)) = J_1(\cdot)$, so $J_0(i(\cdot))$ is continuous. Hence,

$$J_0(i(\phi_n)) \to J_0(i(\phi)) = J_0(\phi)$$

which gives $J_0(f) = J_0(\phi)$ and whence $f = \phi$. □

Theorem 2.2.4. If the groupoid $\mathcal{G} = X \rtimes G$ has the property of rapid decay then $S_2(X \rtimes G)$ is a dense $*$-subalgebra of $C_r^*(X \rtimes G)$.

Proof. For $f = \sum f_g g$ and $\varphi = \sum h_k k$ both in $S_2(X \rtimes G)$, we will show that $f * \varphi \in S_2(X \rtimes G)$ by showing that $||f * \varphi||_p$ is bounded for all p. There is a bit of setup:

Let $||f_g||_{C(X)} = a_g$ and $||h_k||_{C(X)} = b_g$, let $F = \sum a_g g$ and $H = \sum b_k k$. We've proved above that the inclusion map from $S_2(X \rtimes G)$ to $C(X) \rtimes_r G$ is continuous and therefore there is a q and $c > 0$ such that

$$||\sum_g f_g g||_{C(X)\rtimes_r G} \leq C||\sum_g f_g g||_q \quad \text{for every} \quad \sum_g f_g g \in S_2(X \rtimes G).$$

Recall the common notation $\alpha_g(h) = h(g^{-1}\cdot)$ and let $F = \sum a_g g$ and $H = \sum b_k k$. We can now compute

$$\begin{aligned}
||f * \varphi||_0 = ||(\sum f_g g)(\sum h_k k)||_0 &= ||\sum_\beta (\sum_{gk=\beta} f_g \alpha_g h_k)\beta||_0 \\
&= ||\sum_\beta (\sum_g f_g \alpha_g h_{g^{-1}\beta})\beta||_0 \\
&= \sqrt{\sum_\beta |\sum_g f_g \alpha_g h_{g^{-1}\beta}|^2_{C(X)}} \\
&\leq \sqrt{\sum_\beta |\sum_g a_g b_{g^{-1}\beta}|^2} \\
&= ||F * H||_{\ell^2(G)} \\
&\leq ||F||_{C_r^*(G)}||H||_{\ell^2(G)} \\
&\leq ||\sum a_g g||_{C_r^*(G)}||H||_{\ell^2(G)} \\
&= ||\sum a_g g||_{C(X)\rtimes_r G}||\sum h_k k||_0 \\
&\leq C||\sum f_g g||_q||\sum h_k k||_0 \\
&= C||f||_q||\varphi||_0.
\end{aligned}$$

Next, we show that

$$||f * \varphi||_p \leq \left\| \left(\sum (1 + l(g))^p f_g g \right) \left(\sum (1 + l(k))^p h_k k \right) \right\|_0 .$$

Firstly,

$$\begin{aligned}1 + l(g) = 1 + l(ghh^{-1}) &\leq 1 + l(gh) + l(h^{-1}) \\ &\leq 1 + l(gh) + l(h^{-1}) + l(gh)l(h^{-1}) \\ &= \big(1 + l(gh)\big)\big(1 + l(h^{-1})\big)\end{aligned}$$

and therefore

$$(1 + l(g))^p \leq \big(1 + l(gh)\big)^p \big(1 + l(h^{-1})\big)^p.$$

We can now see that

$$\begin{aligned}||f * \varphi||_p^2 &= ||\sum_{\beta}(\sum_{g} f_g \alpha_g h_{g^{-1}\beta})\beta||_p^2 \\ &= \sum_{\beta}(1 + l(\beta))^p || \sum_{g} f_g \alpha_g h_{g^{-1}\beta}||_{C(X)}^2 \\ &\leq \sum_{\beta} || \sum_{g}(1 + l(g^{-1}\beta))^p (1 + l(g))^p f_g \alpha_g h_{g^{-1}\beta})||_{C(X)}^2 \\ &= \sum_{\beta} || \sum_{g}(1 + l(g))^p f_g (1 + l(g^{-1}\beta))^p h_{g^{-1}\beta})||_{C(X)}^2 \\ &= \left\| \Big(\sum (1 + l(g))^p f_g g \Big) \Big(\sum (1 + l(k))^p h_k k \Big) \right\|_0 .\end{aligned}$$

Now we're ready to combine the above:

$$\begin{aligned}||f * \varphi||_p &\leq \left\| \Big(\sum (1 + l(g))^p f_g g \Big) \Big(\sum (1 + l(k))^p h_k k \Big) \right\|_0 \\ &\leq C|| \sum (1 + l(g))^p f_g g||_q \;\; || \sum (1 + l(k))^p h_k k||_0 \\ &= C|| \sum f_g g||_{p+q} \;\; || \sum h_k k||_p \\ &= C||f||_{p+q}||\varphi||_p.\end{aligned}$$

Showing closure under addition and the $*$-operation is trivial. The above computations show that $S_2(X \rtimes G)$ is a $*$-subalgebra of $C_r^*(X \rtimes G)$, whose density follows from the density of $C_c(X \rtimes G)$ in $C_r^*(X \rtimes G)$.

□

2.3 Invariance Under Quasi-Isometry: An Open Problem

In this section we show that G has rapid decay if and only if $\beta G \rtimes G$ has rapid decay where βG is the Stone-Čech compactifiaction of G. This highlights the main difference of our definition to [7], where rapid decay of the transformation groupoid $\beta G \rtimes G$ is shown to be equivalent to polynomial growth of G. Additionally, we use our definition to rephrase a current open problem: Given two quasi-isometric groups, if one has the property of rapid decay, must the other? This question was posed in [5] and [4] and was our main motivation for developing our definition.

Lemma 2.3.1. If X is a compact G-space and the groupoid $X \rtimes G$ has rapid decay then G has rapid decay.

Proof. Suppose that $X \rtimes G$ has rapid decay and let $P(x)$ be the given polynomial.

We must show that there exists a polynomial with the property that for any $f \in \mathbb{C}(G)$,

$$||f||_r \leq P(l(f))||f||_2.$$

Let $f = \sum_{g \in F} \alpha_g g \in \mathbb{C}(G)$ and $x \in X$. We may view f as an element of $C(X) \rtimes_{alg} G$ because $C(X)$ contains the constant function 1 and all of its complex multiples. Given a unit vector $\xi \in \ell^2(G)$ and a finite subset $F \subset G$, we may induce a unit vector in $\ell^2(G \cdot x \times G)$ by defining $\hat{\xi}(g \cdot x, b) = \xi(b)$ when $g = b$ and zero otherwise. We'll consider

$$
\begin{aligned}
||\pi_x(\sum_{g\in F}\alpha_g g)(\hat{\xi})||^2_{\ell^2(G\cdot x\times G)} &= \sum_{(a,b)\in G\cdot x\times G}\left|\sum_{g\in F}\alpha_g\hat{\xi}(g^{-1}a,g^{-1}b)\right|^2\\
&= \sum_{(b,b)\in G\cdot x\times G}\left|\sum_{g\in F}\alpha_g\hat{\xi}(g^{-1}b,g^{-1}b)\right|^2\\
&= \sum_{b\in G}\left|\sum_{g\in F}\alpha_g\xi(g^{-1}b)\right|^2\\
&= ||f*\xi||^2_{\ell^2(G)}
\end{aligned}
$$

Since

$$
||\pi_x(\sum_{g\in F}\alpha_g g)(\hat{\xi})||_{\ell^2(G\cdot x\times G)} \leq ||f||_{C(X)\rtimes G}
$$

and

$$
||f||_{C(X)\rtimes G} \leq \max_{g\in F} P(l(g))\sqrt{\sum_{F\subset G}||f_g||^2_{C(X)}} \quad \text{by groupoid rapid decay}
$$

we have that

$$
||f*\xi||_2 \leq P(f)||f||_2
$$

for all unit vectors and the result follows. □

The following is a version of Fell's trick in terms of the technology we are using.

Remark 2.3.2. It's not hard to show that

$$
U:\ell^2(G\cdot x\times G)\to\ell^2(G\cdot x\times G), \quad U(\delta_{a,b})=\delta_{ba,b}
$$

defines a unitary operator. The following computation will be used later:

$$
(U^*\pi(f_g g)U)(\delta_{a,b}) = (U^*\pi(f_g g))(\delta_{ba,b}) = f(g^{-1}ba)U^*((\delta_{g^{-1}ba,g^{-1}b}) = f(g^{-1}ba)\delta_{a,g^{-1}b}.
$$

Proposition 2.3.3. If G has the property of rapid decay and X is a compact G-space, then the groupoid $X\rtimes G$ has rapid decay.

Proof. Suppose G has the property of rapid decay with polynomial P, and let $x \in X$. We will consider a unit vector $\xi \in \ell^2(G \cdot x \times G)$ and $\sum_{F \subset G} f_g g \in C(X) \rtimes_{alg} G$. We may restrict to positive vectors and positive f_g's $\in C(X)$. For a fixed $a \in G \cdot x$, define

$$\xi_a(b) := \xi(a, b)$$

and note that $\xi_a(b) \in \ell^2(G)$. Also note that

$$||\xi||^2_{\ell^2(G \cdot x \times G)} = \sum_{a \in G \cdot x} ||\xi_a||^2_{\ell^2(G)}$$

which implies that each ξ_a is square summable over $G \cdot x$.

Let $\beta_g := ||f_g||_{C(X)}$, and U be the unitary operator in the above remark. Let $f = \sum \beta_g g$. With this notation

$$||f||_{\ell^2(G)} = ||f||_{2,C(X)}.$$

We are now ready to compute

$$\begin{aligned}
||\pi_x(\sum_{F \subset G} f_g g)(\xi)||^2_{\ell^2(G \cdot x \times G)} &= ||U^* \pi_x(\sum_{F \subset G} f_g g) U(\xi)||^2_{\ell^2(G \cdot x \times G)} \\
&= \sum_{(a,b) \in G \cdot x \times G} \Big| \sum_{F \subset G} f(g^{-1}ba) \xi(a, g^{-1}b) \Big|^2 \\
&\leq \sum_{a \in G \cdot x} \sum_{b \in G} \Big| \sum_{F \subset G} \beta_g \xi(a, g^{-1}b) \Big|^2 \\
&= \sum_{a \in G \cdot x} \left(\sum_{b \in G} \Big| \sum_{F \subset G} \beta_g \xi(a, g^{-1}b) \Big|^2 \right) \\
&= \sum_{a \in G \cdot x} ||f * \xi_a||^2_{\ell^2(G)} \\
&\leq \sum_{a \in G \cdot x} ||\xi_a||^2_{\ell^2(G)} P^2(f) ||f||^2_{\ell^2(G)} \text{ by RD of G} \\
&= P^2(f) ||f||^2_{\ell^2(G)} \\
&= P^2(f) ||f||_{2,C(X)}.
\end{aligned}$$

The result now follows from taking a supremum over $x \in X$.

□

Definition 2.3.4. We call two groups G and H *bilipschitz equivalent* if there exists a bijective $\phi : G \to H$ and $C > 1$ such that

$$(*) \quad C^{-1} d_G(g_1, g_2) \leq d_H(\phi(g_1), \phi(g_2)) \leq C d_G(g_1, g_2) \quad \text{for every } g \in G, \text{ and}$$

The result of the next lemma is very similar to one that can be deduced from [12] and [13]. The simple nature of the proof can be attributed to the wild, wild universe of the Stone Čech compactification.

Lemma 2.3.5. If G and H are bilipschitz equivalent discrete groups, then

$$C_r^*(\beta X \rtimes G) \cong C_r^*(\beta H \rtimes H).$$

Proof. Let βG and βH represent the usual Stone Čech compactification of G and H respectively. Let ϕ denote a bilipshitz map, with constant C. Here, G and H both have the discrete topology, and so the map

$$G \xrightarrow{\phi} H \hookrightarrow \beta H$$

is continuous. By the universal property, there is a unique extension to a continuous map

$$\hat{\phi} : \beta G \to \beta H.$$

Note that we can do the same thing with ϕ^{-1}, and because of uniqueness this extends to $\hat{\phi}^{-1}$ making $\hat{\phi}$ a homeomorphism.

This allows us to construct an isomorphism between the transformation groupoids:

$$\varphi : \beta G \rtimes G \to \beta H \rtimes H$$

$$(g\omega, g, \omega) \mapsto (\hat{\phi}(g\omega), \hat{\phi}(g\omega)(\hat{\phi}(\omega))^{-1}, \hat{\phi}(\omega)).$$

Of course, there needs to be some justification here: We show that $\hat{\phi}(g\omega)(\hat{\phi}(\omega))^{-1}$ is an element of H. Consider

$$\hat{\phi}(g\omega)(\hat{\phi}(\omega))^{-1} = \lim_{k\to\omega} \hat{\phi}(gk)(\hat{\phi}(k))^{-1}$$

and the computation

$$\begin{aligned} d(\hat{\phi}(gk)(\hat{\phi}(k))^{-1}, e_H) =& d(\hat{\phi}(gk), \hat{\phi}(k))) \quad \text{by right invariance of the metric} \\ &\leq C\ d(gk, k) \\ &= C\ d(g, e_G) \\ &= |g|. \end{aligned}$$

The above shows that the element $\hat{\phi}(gk)(\hat{\phi}(k))^{-1}$ is in a compact subset of H for all k and by properties of the ultafilter, $\hat{\phi}(g\omega)(\hat{\phi}(\omega))^{-1} = \lim_{k\to\omega} \hat{\phi}(gk)(\hat{\phi}(k))^{-1}$ is in H.

This gives

$$C_r^*(\beta G \rtimes G) \cong C_r^*(\beta H \rtimes H).$$

given by the map

$$\Phi : C_r^*(\beta H \rtimes H) \to C_r^*(\beta G \rtimes G)$$

$$f \mapsto f \circ \varphi.$$

□

Theorem 2.3.6. The property of rapid decay for discrete groups G and H is invariant under quasi-isometry if and only if the property of rapid decay for transformation groupoids $\beta G \rtimes G$ and $\beta H \rtimes H$ is invariant under isomorphism.

Proof. First, the case where G is amenable: if G and H are quasi-isometric then we have that H is amenable too [18]. If $\beta G \rtimes G \cong \beta H \rtimes H$ and $\beta G \rtimes G$ has the property of rapid decay, then by Lemma 2.3.1 G has rapid decay. Since G has rapid decay and is amenable, then G has polynomial growth [4] and therefore H has polynomial growth and therefore has the property of rapid decay. By Proposition 2.3.3, $\beta H \rtimes H$ has rapid decay. Notice that, in the amenable case, we need no assumption on the transformation groupoids to show that rapid decay is invariant under quasi-isometry. So we may proceed to the non-amenable case.

The ($\Leftarrow$) direction is almost a direct consequence of the previous lemma and proposition of this section: suppose the property of rapid decay for transformation groupoids $\beta G \rtimes G$ and $\beta H \rtimes H$ is invariant under isomorphism, then suppose that G and H are quasi-isometric and G has the property of rapid decay. From [18], we may assume that there is a bilipshitz equivalence between G and H and we can therefore construct an isomorphism as in Lemma 2.3.5. Since G has rapid decay, so does $\beta G \rtimes G$, and by assumption, $\beta H \rtimes H$ does too. Again, by Lemma 2.3.1, H has rapid decay.

For the other direction, suppose that the property of rapid decay for discrete groups G and H is invariant under quasi-isometry. Further suppose that $\beta G \rtimes G \cong \beta H \rtimes H$, and that $\beta G \rtimes G$ has the property of rapid decay. It follows from ([16], Remark 10.27) that the action of G on βG is free and therefore the groupoid $G \rtimes \beta G$ is principal. Now, from ([13] Theorem 2.20) we obtain a bilipschitz equivalence of G and H (which is a quasi-isometry). By our assumption and the use of Lemma 2.3.1 and Proposition 2.3.3 $\beta H \rtimes H$ has the property of rapid decay. □

It should be noted that most of the usual groups have already been classified as to whether or not they have the property of rapid decay, and the class of groups one could apply this result to escapes the imagination of the author. However, we hope that this section serves as evidence towards the robust nature of groupids and their usefulness in research.

Remark 2.3.7. The notion of groupoid rapid decay as found in [7] has the benefit of a spectral invariance property, and is phrased in terms of a general r-discrete groupoid (not just a transformation groupoid). However, it does not have the benefit of satisfying proposition 2.3.3. Additionally, it could still be the case that our definition gives rise to the spectral invariance property and can be generalized to étale groupoids.

Chapter 3

Measured étale Groupoids

In this chapter we explore some basic facts about measured étale groupoids and it will ultimately serve as a reference for the following chapter. At this point, we see life as a bit too short for the non-separable case and assume $\mathcal{G}$ to be second countable.

3.1 Representations

In the group case we have a basic fact that any $*$-representation of $\mathbb{C}(G)$ can be restricted to a unitary representation of G (there is a 1-1 correspondence). In the groupoid case things are a bit more complicated but we can still obtain a similar result that characterizes all $*$-representations of $C_c(\mathcal{G})$ but it takes quite a bit of setup.

Definition 3.1.1. A bisection, B, is a subset of $\mathcal{G}$ such that the range and source maps are one-to-one. There is a canonical map $\alpha : s(B) \to r(B)$ (for any bisection) and trying to define it makes it seem more complicated, but I assure you it's not: Given a $\gamma \in s(B)$, let $B\gamma$ denote the unique element in B with $s(B\gamma) = \gamma$ and define $\alpha(\gamma) := r(B\gamma)$.

Definition 3.1.2. A Radon measure, μ ,on $\mathcal{G}^{(0)}$ is said to be invariant if $\mu(r(B)) = \mu(s(B))$ for all open bisections, and quasi-invariant if $\mu(\alpha(s(B))) = 0 \iff \mu(r(B)) = 0$ for all open bisections. We will always assume our measures to be probability measures.

The proof of the following lemma follows from a partition of unity argument and can be found in [14].

Lemma 3.1.3. Let μ be a quasi-invariant measure. Then for any $f \in C_c(\mathcal{G})$, the measures on $\mathcal{G}$ determined by

$$\int f \; dr^*\mu = \int_{\mathcal{G}^{(0)}} \sum_{\gamma \in \mathcal{G}^x} f(\alpha) \; d\mu(x)$$

and

$$\int f \; ds^*\mu = \int_{\mathcal{G}^{(0)}} \sum_{\gamma \in \mathcal{G}_y} f(\alpha) \; d\mu(y)$$

have the same null sets.

Definition 3.1.4. For a given quasi invariant measure let $D_\mu : \mathcal{G} \to (0, \infty)$ be the associated Raydon-Nikodym derivative uniquely determined by the relation

$$\int_{r(B)} f(x) \; d\mu(x) = \int_{s(B)} f(\alpha(y)) D(By) \; d\mu(y)$$

for all open bisections B, see ([14], Proposition 2.1.7) for details. We will just write D when the measure is clear, and for an invariant measure we don't write anything because $D = 1$.

Definition 3.1.5. When $\mathcal{G}$ is étale, we can extend any measure on $\mathcal{G}^{(0)}$ to $\mathcal{G}$ because $r^{-1}(x)$ and $s^{-1}(x)$ are both discrete for all $x \in \mathcal{G}^{(0)}$ so the counting measure, which we call λ, is a natural choice. More precisely, we define $\mu \circ \lambda$ to be the measure on $\mathcal{G}$ uniquely determined by

$$\int_{\mathcal{G}} f \; d\mu \circ \lambda = \int_{\mathcal{G}^{(0)}} \sum_{\gamma \in \mathcal{G}^x} f(\gamma) \; d\mu(x)$$

Definition 3.1.6. ([15], 1.3.12) For a locally compact space X with a radon measure μ, we define a measureable field of Hilbert spaces over a measure space (X, μ) to be a collection of Hilbert spaces $\{\mathcal{H}_x\}_{x \in X}$ together with a linear subspace $\mathcal{E} \subset \Pi_x \mathcal{H}_x$ (called a fundamental space of sections) which is closed under multiplication by $C_c(X)$ and with the following properties:

1) The function $x \mapsto \langle \xi(x), \eta(x) \rangle_x$ is measurable for all $\xi, \eta \in \mathcal{E}$
2) For all $\xi \in \mathcal{E}$, $\displaystyle\int_X \langle \xi(x), \xi(x) \rangle_x \; d\mu(x) < \infty$

3) The subspace $\mathcal{E}$ contains a countable subset $\{\xi_n\}$ that generates $\mathcal{E}$ as a $C_c(X)$-module and such that for all $x \in X$, $\{\xi_n\}$ spans a dense linear subspace of $\mathcal{H}_x$

Definition 3.1.7. Given a measure, μ, and $H = \{(\mathcal{H}_x), \mathcal{E}\}$ a measurable field of Hilbert spaces over $(\mathcal{G}^{(0)}, \mu)$, we define a measurable isometric action of $\mathcal{G}$ on H to be a collection of unitary isomorphisms parametrized by $\gamma \in \mathcal{G}$, $L(\gamma) : \mathcal{H}_{s(\gamma)} \to \mathcal{H}_{r(\gamma)}$, to be one such that

$$L(x)\xi_x = \xi_x \quad \text{for all } x \in \mathcal{G}^{(0)} \quad \text{and} \quad \xi_x \in \mathcal{H}_x$$

$$L(\alpha\beta) = L(\alpha)L(\beta) \quad \text{for all composable pairs } (\alpha, \beta)$$

$$\gamma \to \left\langle \eta_{r(\gamma)}, L(\gamma)\xi_{s(\gamma)} \right\rangle_{r(\gamma)} \quad \text{is measureable for all } \eta, \xi \in \mathcal{E}$$

Definition 3.1.8. Let μ be a quasi-invariant measure on $\mathcal{G}^{(0)}$, and $H = \{(\mathcal{H}_x), \mathcal{E}\}$ a measurable field of Hilbert spaces over $(\mathcal{G}^{(0)}, \mu)$. We say π is a $\mu-$representation of $C_c(\mathcal{G})$ on $\mathcal{H} = L^2(\mathcal{G}, \mu, H)$ if we have we have a measurable isometric action, L_π, of $\mathcal{G}$ on H that satisfies

$$\langle \xi, \pi(f)\eta \rangle = \int_{\mathcal{G}^{(0)}} \sum_{\gamma \in \mathcal{G}^x} \left\langle \xi_x, f(\gamma)D^{-1/2}(\gamma)L_\pi(\gamma)\eta_{s(\gamma)} \right\rangle_x \, d\mu(x). \qquad (\dagger)$$

Remark 3.1.9. One shows that the above is indeed a bounded $*$-representation and, amazingly, all $*$-representations of $C_c(\mathcal{G})$ are of this form ([14], Proposition 2.1.7). Further, one may extend π to a bounded representation of compactly supported Borel functions on $\mathcal{G}$ as in ([14], Lemma 2.1.17). For clarity we make a note that if K is a compact subset of $\mathcal{G}$, and μ is an invariant measure then:

$$\int_{\mathcal{G}^{(0)}} \sum_{\gamma \in \mathcal{G}^x} \left\langle \xi_x, \chi_K(\gamma)L(\gamma)\xi_{s(\gamma)} \right\rangle_x \, d\mu = \langle \xi, \pi(\chi_K)\xi \rangle .$$

Definition 3.1.10. Let $C^*_{max}(\mathcal{G}, \mu)$ denote the separated completion of $C_c(\mathcal{G})$ under the seminorm

$$||f||_{max,\mu} := \sup\left\{||\pi(f)||_{\mathbb{B}(\mathcal{H})} : \ \pi \text{ is a } \mu\text{-representation}\right\}.$$

3.2 Invariant Vectors and the Trivial Representation

In this section we define a notion of an invariant vector for representations of $C_c(\mathcal{G})$ and characterize their existence in terms of an invariant probability measure. For simplicity, in this section we stick to the unital case and assume the unit space to be compact.

Definition 3.2.1. With the notation as in Definition 3.1.8, we define the trivial representation by setting $H_x = \mathbb{C}$ for each $x \in \mathcal{G}^{(0)}$, $\mathcal{E} = C_c(\mathcal{G}^{(0)})$ and $L(g) : \mathbb{C} \to \mathbb{C}$ to be the identity map. For a quasi invariant measure μ, the associated representation of $C_c(\mathcal{G})$ (on $\mathcal{H} = L^2(\mathcal{G}, \mu, H) = L^2(\mathcal{G}^{(0)}, \mu)$) is called T_μ.

One can directly compute

$$T_\mu(f)\xi(x) = \sum_{g \in \mathcal{G}^x} f(g) D(g)^{-1/2} \xi(s(g))$$

from definition 3.1.8 and when $\mathcal{G}^{(0)}$ is compact we define the vector state

$$\phi_\mu : C_c(\mathcal{G}) \to \mathbb{C}$$

$$\phi_\mu(f) = \langle 1, T_\mu(f)1 \rangle_{L^2(\mathcal{G}^{(0)},\mu)} .$$

and using the above we obtain the formula

$$\phi_\mu(f) = \int_{\mathcal{G}^{(0)}} \sum_{g \in \mathcal{G}^x} f(g) D(g)^{-1/2} \, d\mu(x)$$

The canonical extension of ϕ_μ to $C^*_{max}(\mathcal{G}, \mu)$ is still denoted ϕ_μ.

Proposition 3.2.2. Let μ be an invariant measure on $\mathcal{G}^{(0)}$, then

$$\phi_\mu(f * g) = \int_{\mathcal{G}^{(0)}} \left(\sum_{\eta \in \mathcal{G}_x} f(\eta) \right) \left(\sum_{\xi \in \mathcal{G}^x} g(\xi) \right) \, d\mu(x).$$

Proof. For an $f, g \in C_c(\mathcal{G})$ we have

$$\begin{aligned}
\phi_\mu(f * g) &= \int_{\mathcal{G}^{(0)}} \sum_{\gamma \in \mathcal{G}^x} \sum_{\gamma_1 \gamma_2 = \gamma} f(\gamma_1) g(\gamma_2) d\mu(x) \\
&= \int_{\mathcal{G}^{(0)}} \sum_{\gamma \in \mathcal{G}^x} \sum_{\gamma_2 \in \mathcal{G}_{s(\gamma)}} f(\gamma\gamma_2^{-1}) g(\gamma_2) d\mu(x) \\
&= \int_{\mathcal{G}^{(0)}} \sum_{\gamma \in \mathcal{G}_x} \sum_{\gamma_2 \in \mathcal{G}_{s(\gamma)}} f(\gamma\gamma_2^{-1}) g(\gamma_2) d\mu(x) \quad \text{by invariance of } \mu \\
&= \int_{\mathcal{G}^{(0)}} \sum_{\gamma \in \mathcal{G}_x} \sum_{\gamma_2 \in \mathcal{G}_x} f(\gamma\gamma_2^{-1}) g(\gamma_2) d\mu(x) \\
&= \int_{\mathcal{G}^{(0)}} \sum_{\gamma_2 \in \mathcal{G}_x} \sum_{\gamma \in \mathcal{G}_x} f(\gamma\gamma_2^{-1}) g(\gamma_2) d\mu(x) \\
&= \int_{\mathcal{G}^{(0)}} \sum_{\gamma_2 \in \mathcal{G}_x} \sum_{\gamma \in \mathcal{G}_{s(\gamma_2)}} f(\gamma\gamma_2^{-1}) g(\gamma_2) d\mu(x) \\
&= \int_{\mathcal{G}^{(0)}} \sum_{\gamma_2 \in \mathcal{G}^x} \sum_{\eta \in \mathcal{G}_x} f(\eta) g(\gamma_2) d\mu(x) \quad \text{where } \eta = \gamma\gamma_2^{-1} \\
&= \int_{\mathcal{G}^{(0)}} \left(\sum_{\eta \in \mathcal{G}_x} f(\eta) \right) \left(\sum_{\xi \in \mathcal{G}^x} g(\xi) \right) d\mu(x)
\end{aligned}$$

□

The following notation will be used to simplify computations

$$\Phi^* : C_c(\mathcal{G}) \to C_c(\mathcal{G}^{(0)}) \quad f \mapsto \Phi(f)$$

$$\Phi^*(f)(x) = \sum_{\gamma \in \mathcal{G}^x} f(\gamma).$$

We also use

$$\Phi_* : C_c(\mathcal{G}) \to C_c(\mathcal{G}^{(0)}) \quad f \mapsto \Phi(f)$$

$$\Phi_*(f)(x) = \sum_{\gamma \in \mathcal{G}_x} f(\gamma).$$

Definition 3.2.3. Suppose that π is a representation of $C_c(\mathcal{G})$ on a Hilbert space $\mathcal{H}$. We call $\xi \in \mathcal{H}$ an invariant vector if

$$\pi(f)\xi = \pi(\Phi^*(f))\xi$$

for all $f \in C_c(\mathcal{G})$.

Proposition 3.2.4. There exists an invariant probability measure on $\mathcal{G}^{(0)}$ if and only if there exists a representation π of $C_c(\mathcal{G})$ on some $\mathcal{H}$ with a non-zero invariant vector.

Proof. ($\Rightarrow$) Suppose there exists an invariant probability measure on $\mathcal{G}^{(0)}$, then the trivial representation T_μ as defined above emits a multitude of invariant vectors, e.g., the constant function 1.

($\Leftarrow$) Suppose that there exists a representation π of $C_c(\mathcal{G})$ on some $\mathcal{H}$ with an invariant vector ξ. The vector state $\phi(f) = \langle \xi, \pi(f)\xi \rangle$ restricts to a linear functional on $C_c(\mathcal{G}^{(0)})$ and admits a representation via integration against an appropriate measure, i.e., there is a measure on $\mathcal{G}^{(0)}$, μ, such that

$$\langle \xi, \pi(f)\xi \rangle = \int_{\mathcal{G}^{(0)}} f \, d\mu \quad \text{for all } f \in C_c(\mathcal{G}^{(0)}).$$

Let B be a compact bisection and note again that every representation of $C_c(\mathcal{G})$ extends to $B(\mathcal{G})$, the set of Borel functions with compact support (via a pointwise limit of $C_c(\mathcal{G})$

functions) ([14], Lemma 1.1.17). We show that $\mu(r(B)) = \mu(s(B))$:

$$\begin{aligned}
\mu(r(B)) &= \int_{\mathcal{G}^{(0)}} \Phi^*(\chi_B)(x)\ d\mu(x) \\
&= \langle \xi, \pi(\Phi^*(\chi_B))\xi \rangle \\
&= \langle \xi, \pi(\chi_B)\xi \rangle \\
&= \langle \pi(\chi_{B^{-1}})\xi, \xi \rangle \\
&= \langle \pi(\Phi^*(\chi_{B^{-1}}))\xi, \xi \rangle \quad \text{by invariance of } \xi \\
&= \langle \pi(\Phi_*(\chi_B))\xi, \xi \rangle \\
&= \langle \xi, \pi(\Phi_*(\chi_B))\xi \rangle \quad \pi(\Phi_*(\chi_B)) \text{ is self adjoint} \\
&= \int_{\mathcal{G}^{(0)}} \Phi_*(\chi_B)(x)\ d\mu(x) \\
&= \mu(s(B)).
\end{aligned}$$

For an arbitrary bisection B one uses the previous argument on a limit of compact bisections.

□

Definition 3.2.5. Recall the isotropy groupoid, $Iso(\mathcal{G})$. We call a groupoid, $\mathcal{G}$, a bundle of groups μ-almost everywhere if for any bisection E we have $\mu(s(E \backslash Iso(\mathcal{G}))) = \mu(r(E \backslash Iso(\mathcal{G}))) = 0$.

Recall the following well known proposition whose proof can be found in [15].

Proposition 3.2.6. Let $\mathcal{G}$ be a locally compact groupoid, and P the canonical conditional expectation $P : C_r^*(\mathcal{G}) \to C_0(\mathcal{G}^{(0)})$ given by restriction. Then

1) The linear functional $\int_{\mathcal{G}^{(0)}} P(f)\ d\mu$ is a tracial state of $C_r^*(\mathcal{G})$ if and only if μ is an invariant measure.

2) If G is principal, each tracial state of $C_r^*(\mathcal{G})$ is of the form $\int_{\mathcal{G}^{(0)}} P(f)\ d\mu$ for some invariant probability measure μ.

In the non-principal case we have the following proposition.

Proposition 3.2.7. Let μ be an invariant measure on $\mathcal{G}^{(0)}$. The state ϕ_μ is a trace if and only if $\mathcal{G}$ is a bundle of groups a.e.

Proof. ($\Rightarrow$) Suppose that ϕ_μ is a trace. Let E be a measurable bisection, and suppose that

$$\mu(s(E \backslash Iso(\mathcal{G})) \neq 0.$$

Define $f = \chi_E$ and $g = \chi_{s(E\backslash(Iso\mathcal{G}))}$, and notice that

$$\begin{aligned}
\phi_\mu(f * g) &= \int_{\mathcal{G}^{(0)}} \Big(\sum_{\eta \in \mathcal{G}_x} f(\eta)\Big)\Big(\sum_{\xi \in \mathcal{G}^x} g(\xi)\Big)\, d\mu(x) \\
&= \int_{\mathcal{G}^{(0)}} \Big(\sum_{\eta \in \mathcal{G}_x} \chi_E(\eta)\Big)\Big(\sum_{\xi \in \mathcal{G}^x} \chi_{s(E\backslash Iso\mathcal{G})}(\xi)\Big)\, d\mu(x) \\
&= \int_{\mathcal{G}^{(0)}} \chi_{s(E\backslash Iso\mathcal{G})}(x) d\mu(x) \\
&= \mu(s(E\backslash Iso\mathcal{G})) \\
&\neq 0.
\end{aligned}$$

However,

$$\begin{aligned}
\phi_\mu(g * f) &= \int_{\mathcal{G}^{(0)}} \Big(\sum_{\eta \in \mathcal{G}_x} g(\eta)\Big)\Big(\sum_{\xi \in \mathcal{G}^x} g(\xi)\Big)\, d\mu(x) \\
&= \int_{\mathcal{G}^{(0)}} \Big(\sum_{\eta \in \mathcal{G}_x} \chi_{s(E\backslash Iso(\mathcal{G}))}(\eta)\Big)\Big(\sum_{\xi \in \mathcal{G}^x} \chi_E(\xi)\Big)\, d\mu(x) \\
&= \int_{\mathcal{G}^{(0)}} \chi_{s(E\backslash Iso(\mathcal{G}))}\chi_{r(E)} d\mu \\
&= 0
\end{aligned}$$

($\Leftarrow$) Suppose that $\mathcal{G}$ is a bundle of groups μ-almost everywhere. AS simple functions are dense in $C_c(\mathcal{G})$, it suffices to show that for two bisections, B and E, $\phi_\mu(\chi_E * \chi_B) = \phi_\mu(\chi_B * \chi_E)$. We have

$$\begin{aligned}
\phi_\mu(\chi_E * \chi_B) &= \int_{\mathcal{G}^{(0)}} \Big(\sum_{\eta \in \mathcal{G}_x} \chi_E(\eta)\Big)\Big(\sum_{\xi \in \mathcal{G}^x} \chi_B(\xi)\Big)\, d\mu(x) \\
&= \mu(s(E) \cap r(B)) \\
&= \mu(\Big(s(E \backslash Iso(\mathcal{G})) \cup s(E \cap Iso(\mathcal{G}))\Big) \cap r(B)) \\
&= \mu(s(E \backslash Iso(\mathcal{G}) \cap r(B)) + \mu(s(E \cap Iso(\mathcal{G})) \cap r(B) \\
&= \mu(s(E \cap Iso(\mathcal{G})) \cap r(B)) \\
&= \mu(s(E \cap Iso(\mathcal{G})) \cap r(B \cap Iso(\mathcal{G}))) \\
&= \mu(r(E \cap Iso(\mathcal{G})) \cap s(B \cap Iso(\mathcal{G}))) \\
&= \int_{\mathcal{G}^{(0)}} \Big(\sum_{\eta \in \mathcal{G}_x} \chi_B(\eta)\Big)\Big(\sum_{\xi \in \mathcal{G}^x} \chi_E(\xi)\Big)\, d\mu(x) \\
&= \phi_\mu(\chi_B * \chi_E)
\end{aligned}$$

□

3.3 The Regular Representation of an étale Groupoid

In this section we define the regular representation of a (measured) étale groupoid. We prove that when the base space is nice enough to have a quasi-invariant Radon measure with full support we can construct the reduced groupoid C^*-algebra in a particularly nice way.

Definition 3.3.1. Let $\mathcal{H}_x = \ell^2(\mathcal{G}^x)$ for all $x \in \mathcal{G}^{(0)}$, set $\mathcal{E} = C_c(\mathcal{G})$ and define

$$L_\lambda(g) : \ell^2(\mathcal{G}^{s(g)}) \to \ell^2(\mathcal{G}^{r(g)})$$

$$L_\lambda(g)\xi(\cdot) = \xi(g^{-1}\cdot)$$

Given this data and a quasi-invariant measure μ on $\mathcal{G}^{(0)}$, one uses 3.1.8 to obtain the regular representation, λ, and we make a note that the underlying Hilbert space is $L^2(\mathcal{G}, \mu \circ \lambda)$ and so $\lambda(f) \in \mathbb{B}(L^2(\mathcal{G}, \mu \circ \lambda))$ for all $f \in C_c(\mathcal{G})$.

Definition 3.3.2. We call $C_r^*(\mathcal{G}, \mu)$ the separated completion of $C_c(\mathcal{G})$ under the semi-norm given by

$$||f||_{C_r^*(\mathcal{G},\mu)} = ||\lambda(f)||_{\mathbb{B}(L^2(\mathcal{G},\mu\circ\lambda))}.$$

Remark 3.3.3. If μ is invariant one could define $C_r^*(\mathcal{G}, \mu)$ by completing $C_c(\mathcal{G})$ via the norm

$$||f||^2_{C_r^*(\mathcal{G},\mu\circ\lambda)} := \sup_{||\xi||_{L^2(\mathcal{G},\mu)}=1} \int_{\mathcal{G}} |f * \xi|^2 \, d\mu$$

where

$$\int_{\mathcal{G}} |f * \xi|^2 \, d\mu \circ \lambda = \int_{\mathcal{G}^{(0)}} \sum_{\gamma\in\mathcal{G}^x} | \sum_{\gamma_1\gamma_2=\gamma} f(\gamma_1)\xi(\gamma_2)|^2 \, d\mu(x).$$

Lemma 3.3.4. Let

$$I^r = \{f \in C_c(\mathcal{G}) \, : \, \mu(r(\text{supp}(f))) = 0\}$$

$$I_s = \{f \in C_c(\mathcal{G}) \, : \, \mu(s(\text{supp}(f))) = 0\}$$

then $I^r = I_s$. Recall that we can think of elements of $C_r^*(\mathcal{G})$ as (continuous) functions on $\mathcal{G}$ and therefore can discuss their support. Moreover, if

$$\widetilde{I^r} = \{f \in C_r^*(\mathcal{G}) \, : \, \mu(r(\text{supp}(f))) = 0\}$$

$$\widetilde{I_s} = \{f \in C_r^*(\mathcal{G}) \, : \, \mu(s(\text{supp}(f))) = 0\}$$

then $\widetilde{I^r} = \widetilde{I_s}$.

Proof. For an $f \in C_c(\mathcal{G})$,

$$\mu(r(\text{supp}(f))) = 0$$
$$\Rightarrow \int_{\mathcal{G}^{(0)}} \sum_{\gamma\in\mathcal{G}^x} \chi_{\text{supp}(f)} \, d\mu(x) = 0$$

By the lemma 3.1.3,

$$0 = \int_{\mathcal{G}^{(0)}} \sum_{\gamma \in \mathcal{G}_x} \chi_{\mathrm{supp}(f)} \; d\mu(x)$$

$$\Rightarrow \mu(s(\mathrm{supp}(f))) = 0.$$

Approximation by functions with increasing support (and continuity of measures from below) one can obtain the same result for

$$\widetilde{I^r} = \{ f \in C_r^*(\mathcal{G}) \;:\; \mu(r(\mathrm{supp}(f))) = 0 \} .$$

□

Theorem 3.3.5. For an étale groupoid, $C_r^*(\mathcal{G}, \mu) \cong C_r^*(\mathcal{G}|_{\mathrm{supp}(\mu)})$.

Proof. Let $\pi : C_r^*(\mathcal{G}) \to C_r^*(\mathcal{G}|_{\mathrm{supp}(\mu)})$ denote the canonical extension of the restriction map. We claim that $\ker(\pi) = \widetilde{I^r}$: Let $f \in \ker(\pi)$, then $\pi(f) = 0 \in C_r^*(\mathcal{G}|_{\mathrm{supp}(\mu)})$ and this gives

$$\sup_{x \in \mathcal{G}^{(0)} \,\cap\, \mathrm{supp}(\mu)} ||\pi_x(f)|| = 0.$$

As these are faithful representations, $f = 0$ on $\mathcal{G}^x$ for all $x \in \mathcal{G}^{(0)} \cap \mathrm{supp}(\mu)$ and therefore $\mu(r(\mathrm{supp}(f))) = 0$.

If $f \in \widetilde{I^r}$, then because f is continuous, $\sup_{x \in \mathcal{G}^{(0)} \cap \mathrm{supp}(\mu)} \sum_{\gamma \in \mathcal{G}^x} |f(\gamma)| = 0$. Recall that

$$||f||_{C_r^*(\mathcal{G}|_{\mathrm{supp}(\mu)})} \leq \sup_{x \in \mathcal{G}^{(0)} \cap \mathrm{supp}(\mu)} \sum_{\gamma \in \mathcal{G}^x} |f(\gamma)| = 0 \quad (\dagger)$$

whence $f \in \ker(\pi)$.

Our next clam is that the map $\widehat{\pi} : C_r^*(\mathcal{G}) \to C_r^*(\mathcal{G}, \mu)$ given by the extension of the identity gives rise to the exact sequence

$$0 \to \widetilde{I^r} \hookrightarrow C_r^*(\mathcal{G}) \xrightarrow{\widehat{\pi}} C_r^*(\mathcal{G}, \mu) \to 0.$$

For an $f \in \widetilde{I^r}$, and any ξ, $\eta \in L^2(\mathcal{G}, \mu \circ \lambda)$ we have

$$\langle \xi, \lambda(f)\eta \rangle = \int_{\mathcal{G}^{(0)}} \sum_{\gamma \in \mathcal{G}^x} f(\gamma) \left\langle \xi_x, \lambda(\gamma)\eta_{s(\gamma)} \right\rangle_x \, d\mu(x) = 0$$

and therefore $\widehat{\pi}(f)$ is the zero operator in $L^2(\mathcal{G}, \mu \circ \lambda)$, this means that $f = 0$ in $C_r^*(\mathcal{G}, \mu)$.

Conversely, if $||\widehat{\pi}(f)||_{C_r^*(\mathcal{G}, \mu\circ\lambda)} = 0$, then

$$\int_{\mathcal{G}^{(0)}} \sum_{\gamma \in \mathcal{G}^x} |f(\gamma)|^2 \, d\mu(x) \leq ||\widehat{\pi}(f)||^2_{C_r^*(\mathcal{G}, \mu\circ\lambda)} = 0$$

and thus $f \in \widetilde{I^r}$. Using the above and lemma 3.3.4 we may conclude that

$$C_r^*(\mathcal{G}, \mu) \cong C_r^*(\mathcal{G})/\widetilde{I^r} \cong C_r^*(\mathcal{G}|_{\text{supp}(\mu)}).$$

$\square$

Chapter 4

Amenable Groupoids

In this chapter we study how the various equivalent notions of an amenable group pass to étale groupoids and attempt to understand the situations in which they do not yeild amenability of the groupoid as in [19]. For a detailed study of the subject of amenable groupoids see [1].

4.1 Classic Results

Here we recall some basic characterizations of an amenable group, of which there are many. Our selection provides a point of departure from the group case to look back at the level of groupoids.

Definition 4.1.1. A function $\varphi : G \to \mathbb{C}$ is call positive definite if the $F \times F$ matrix

$$[\varphi(g^{-1}h)]_{g,h\in F}$$

is a positive matrix for every finite $F \subset G$.

Definition 4.1.2. We say a discrete group G is amenable if there exists a net of positive definite functions (φ_i) on G with finite support such that $\varphi_i \to 1$ pointwise.

Definition 4.1.3. The unitary representation $\tau : G \to \mathbb{B}(\mathbb{C})$ $(= \mathbb{C})$ defined by $\tau(g) = 1$ for all $g \in G$ can be canonically extended to a one-dimensional representation of $C^*(G)$ (on $\mathbb{C}$), and therefore viewed as a state on $C^*(G)$. We call τ the trivial representation.

Definition 4.1.4. We say that G has an approximate invariant mean if for any subset $E \subset G$ and $\epsilon > 0$, there exists a $\mu \in \ell^1(G)^+$ with $||\mu||_{\ell^1(G)} = 1$ such that

$$\max_{s \in E} ||s.\mu - \mu||_1 < \epsilon.$$

Theorem 4.1.5. The following statement are equivalent to the definition of amenability, which is a short list of the one found in ([3] Theorem 2.6.8), where proofs can also be found.

(1) G has an approximate invariant mean.

(2) $C^*(G) = C^*_r(G)$.

(3) The trivial representation $\tau : G \to \mathbb{C}$ extends to $C^*_r(G)$.

The following definition of amenability suffices for locally compact étale groupoids.

Definition 4.1.6. We call a groupoid, $\mathcal{G}$, amenable if there exists a net of compactly supported nonnegative functions $\mu_i : \mathcal{G} \to \mathbb{C}$ such that

$$\sum_{g \in \mathcal{G}_{r(\gamma)}} \mu_i(g) \to 1 \quad \text{and} \quad \sum_{g \in \mathcal{G}_{r(\gamma)}} |\mu_i(g) - \mu_i(g\gamma)| \to 0$$

for all $\gamma \in \mathcal{G}$, uniformly on compact subsets of $\mathcal{G}$.

To see that this restricts to the definition of amenability when $\mathcal{G}$ is a group, the statement $\sum_{g \in \mathcal{G}_{r(\gamma)}} \mu_i(g) \to 1$ uniformly on compact subsets of $\mathcal{G}$ says that $||\mu_i||_{\ell^1(\mathcal{G})} \to 1$. For large enough i, one can normalize μ_i (in the $\ell^1(\mathcal{G})$ norm) and obtain an approximate invariant mean. In this way we think of this definition as a generalization of (1) in 4.1.5.

As we will see, (2) and (3) of 4.1.5. are not equivalent to the amenability of $\mathcal{G}$.

Definition 4.1.7. The full or max groupoid C*-algebra $C^*(\mathcal{G})$ is obtained by completing $C_c(\mathcal{G})$ under the norm defined by

$$||f||_{C^*(\mathcal{G})} = \sup_{\pi \in \mathcal{E}} ||\pi(f)||$$

where $\mathcal{E}$ is the set of all (cyclic) $*$-representations of $C_c(\mathcal{G})$ which are bounded on $C_c(\mathcal{G}^{(0)})$.

Definition 4.1.8. A function $h : \mathcal{G} \to \mathbb{C}$ is said to be of positive type if $[h(\alpha\beta^{-1})]_{\alpha,\beta\in F}$ is positive definite for every $x \in \mathcal{G}^{(0)}$ and every finite subset $F \subset \mathcal{G}_x$.

For a $\xi \in C_c(\mathcal{G})$, $\xi^* * \xi$ and $\xi * \xi^*$ are both functions of positive type.

Definition 4.1.9. Call an element in $M_n(C_c(\mathcal{G}))$ *algebraically positive* if it is the finite sum of matrices of the form $[g_p^* * g_q]_{p,q}$. We say m_h is completely algebraically positive if

$$id \otimes h : \mathbb{M}_n(C_c(\mathcal{G})) \to \mathbb{M}_n(C_c(\mathcal{G}))$$

maps algebraically positive elements to algebraically positive elements.

The following argument comes from ([3], Proposition 5.6.16) but we have filled in a few details.

Lemma 4.1.10. Let $h = \xi^* * \xi$ for $\xi \in C_c(\mathcal{G})$. If $\sup |h(\gamma)| \leq 1$, then the multiplier map

$$m_h : C_c(\mathcal{G}) \to C_c(\mathcal{G})$$

$$f \mapsto hf \text{ pointwise multiplication}$$

extends to a c.c.p. map on $C^*(\mathcal{G})$.

Proof. Given a suitable partition of unity $\{p_i\} \subset C_c(\mathcal{G})$ such that $\text{supp}(p_i)$ is a bisection for all i, we write

$$\xi(\gamma) = \sum_i \xi_i(s(\gamma)) \quad \text{where} \quad \xi_i(s(\gamma)) = \xi(\gamma)p_i(\gamma)$$

and compute

$$h(\gamma) = \xi^* * \xi(\gamma) = \sum_{\beta\in\mathcal{G}_{r(\gamma)}} \overline{\xi(\beta)}\xi(\beta\gamma) = \sum_{i,j} \overline{\xi_i(r(\gamma))}\xi_j(s(\gamma)).$$

For an $f \in C_c(\mathcal{G})$ we compute

$$
\begin{aligned}
m_h(f^* * f)(\gamma) &= h(\gamma) \sum_{\beta \in \mathcal{G}_{r(\gamma)}} \overline{f(\beta)} f(\beta\gamma) \\
&= \sum_{i,j} \overline{\xi_i(r(\gamma))} \xi_j(s(\gamma)) \sum_{\beta \in \mathcal{G}_{r(\gamma)}} \overline{f(\beta)} f(\beta\gamma) \\
&= \sum_{i,j} \sum_{\beta \in \mathcal{G}_{r(\gamma)}} \overline{\xi_i(r(\gamma))} \xi_j(s(\gamma)) \overline{f(\beta)} f(\beta\gamma) \\
&= \sum_{i,j} \sum_{\beta \in \mathcal{G}_{r(\gamma)}} \overline{f(\beta)\xi_i(r(\gamma))} f(\beta\gamma) \xi_j(s(\gamma)) \\
&= \Big((\sum_i f\xi_i)^* * (\sum_i f\xi_i) \Big)(\gamma) \quad \text{since } (\xi_i)^* = (\xi p_i)^*
\end{aligned}
$$

Similarly, one can show that $m_h(a^* * b) = \Big((\sum_i a\xi_i)^* * (\sum_i b\xi_i) \Big)$. This shows the map $id \otimes m_h$ takes positive elements to positive elements.

Next we decide on a representation $C^*(\mathcal{G}) \subset \mathbb{B}(\mathcal{H})$ (and continue to write f for $C_c(\mathcal{G})$'s image in $\mathbb{B}(\mathcal{H})$) and define a sesquilinear form on the $C_c(\mathcal{G}) \odot \mathcal{H}$ by

$$
\left\langle \sum_i f_i \otimes \eta_i, \sum_j g_j \otimes \xi_j \right\rangle = \sum_{i,j} \left\langle m_h(g_j^* * f_i)\eta_i, \xi_j \right\rangle_{\mathcal{H}}
$$

We have that $[m_h(f_j^* f_i)]_{i,j} \in M_n(C_c(\mathcal{G}))$ is algebraically positive and so $[m_h(f_j^* * f_i)]_{i,j} \in M_n(C^*(\mathcal{G}))$ is a positive matrix, meaning that if $z = (\xi_1, ..., \xi_n) \in \mathcal{H}^n$ then

$$
0 \leq z^*[m_h(f_j^* f_i)]_{i,j} z = \sum_{i,j} \left\langle m_h(f_j^* * f_i)\xi_i, \xi_j \right\rangle_{\mathcal{H}}
$$

Now we have

$$
\left\langle \sum_i f_i \otimes \xi_i, \sum_j f_j \otimes \xi_j \right\rangle = \sum_{i,j} \left\langle m_h(f_j^* * f_i)\xi_i, \xi_j \right\rangle_{\mathcal{H}} \geq 0
$$

i.e., the sesquilinear form we've defined is positive semidefinite. We now declare the zero subspace to be zero elements, complete and the Hilbert space $\overline{\mathcal{H}}$ emerges. Its elements are denoted $(\sum_i g_i \otimes \eta_i)^@$ for the obvious corresponding element in $C_c(\mathcal{G}) \odot \mathcal{H}$.

The goal of the next part is to define a (bounded) representation of $C_c(\mathcal{G}^{(0)})$ on $\overline{\mathcal{H}}$, which is therefore bounded by the full C^*-algebra norm (and whence the map m_h is c.c.p) . Brown and Ozawa do this by first supposing that $f \in C_c(\mathcal{G}^{(0)})$ and $||f||_\infty \leq 1$ and we do not.

For each $f \in C_c(\mathcal{G}^{(0)})$ one defines $\mathbb{M}_h(f) : \overline{\mathcal{H}} \to \overline{\mathcal{H}}$ by

$$\mathbb{M}_h(f)\Big((\sum_i g_i \otimes \eta_i)^{@} \Big) = (\sum_i f * g_i \otimes \eta_i)^{@}.$$

The following computations show that the map makes sense: For an $f \in C_c(\mathcal{G}^{(0)})$ and $g_1, ..., g_n \in C_c(\mathcal{G})$ one can compute

$$\begin{aligned} f * g_i(\gamma) &= \sum_{\gamma_1\gamma_2=\gamma} f(\gamma_1)g_i(\gamma_2) \\ &= f(r(\gamma))g(\gamma) \end{aligned}$$

For simplicity, recall that

$$(a^* * b)(\gamma) = \sum_{\beta \in \mathcal{G}_{r(\gamma)}} \overline{a(\beta)}b(\beta\gamma)$$

using both of the above we can compute

$$\begin{aligned} \Big((f * g_i)^* * (f * g_j) \Big)(\gamma) &= \sum_{\alpha \in \mathcal{G}_{r(\gamma)}} \overline{f(r(\alpha))g_i(\alpha)} f(r(\alpha\gamma))g_j(\alpha\gamma) \\ &= \sum_{\alpha \in \mathcal{G}_{r(\gamma)}} |f(r(\alpha))|^2 g_i(\alpha)g_j(\alpha\gamma) \end{aligned}$$

Next, define $k_p(\gamma) = (||f||_\infty^2 - |f(r(\gamma))|^2)^{1/2} g_p(\gamma)$ (note the difference from Brown-Ozawa) and use the above to obtain

$$
\begin{aligned}
(k_p^* * k_q)(\gamma) &= \sum_{\beta \in \mathcal{G}_{r(\gamma)}} \overline{k_p(\beta)} k_q(\beta\gamma) \\
&= \sum_{\beta \in \mathcal{G}_{r(\gamma)}} \Big((||f||_\infty^2 - |f(r(\beta))|^2)^{1/2} \overline{g_p(\beta)} \Big) \Big((||f||_\infty^2 - |f(r(\beta\gamma))|^2)^{1/2} g_q(\beta\gamma) \Big) \\
&= ||f||_\infty^2 (g_p^* * g_q)(\gamma) - \big((f * g_p)^* * (f * g_q) \big)(\gamma)
\end{aligned}
$$

One now has that the matrix $||f||_\infty^2 [(g_p^* * g_q)]_{p,q} - [(f * g_p)^* * (f * g_q)]_{p,q}$ is a positive matrix, and therefore m_h applied to it is too. This gives

$$
m_h[(f * g_p)^* * (f * g_q)]_{p,q} \leq m_h ||f||_\infty^2 [(g_p^* * g_q)]_{p,q}.
$$

After lightly unraveling some notation, we can obtain

$$
||M_h(f)(\sum_i g_i \otimes \eta_i)^@||_{\overline{\mathcal{H}}} \leq ||f||_\infty ||(\sum_i g_i \otimes \eta_i)^@||_{\overline{\mathcal{H}}}
$$

Next, define $V : \mathcal{H} \to \overline{\mathcal{H}}$ by

$$
V(\eta) = (1 \otimes \eta)^@
$$

We may find the norm of this operator as follows:

$$
\begin{aligned}
||V(\eta)||^2 &= \langle 1 \otimes \eta, 1 \otimes \eta \rangle_{\overline{\mathcal{H}}} \\
&= \langle m_h(1)\eta, \eta \rangle_{\mathcal{H}} \\
&\leq \sup_{\gamma \in \mathcal{G}^{(0)}} |h(\gamma)| ||\eta||_{\mathcal{H}}^2
\end{aligned}
$$

and it follows from the inequality $||h||_\infty \leq ||h||_r \leq ||h||_{full}$ that $||V|| = 1 = \sup_{\gamma \in \mathcal{G}^{(0)}} |h(\gamma)|$.

Next, we find

$$\left\langle \sum_i f_i \otimes \xi_i, V(\eta) \right\rangle_{\overline{\mathcal{H}}} = \left\langle \sum_i f_i \otimes \xi_i, 1 \otimes \eta \right\rangle_{\overline{\mathcal{H}}}$$
$$= \sum_i \langle m_h(f_i)\xi_i, \eta \rangle_{\mathcal{H}}$$

and thus

$$V^*(\sum_i f_i \otimes \xi_i) = \sum_i m_h(f_i)\xi_i$$

It now easily follows that $m_h(f) = V^* M_h(f) V$, as does the result. □

Theorem 4.1.11. If $\mathcal{G}$ is amenable, then $C^*(\mathcal{G}) = C^*_r(\mathcal{G})$.

Proof. If $\mathcal{G}$ is amenable, then let $\{\mu_i\}$ be as in the definition of amenability and set $\xi_i(\gamma) = \sqrt{\mu_i(\gamma) \sum_{g \in \mathcal{G}_{s(\gamma)}} \mu_i(g)}$. It's a straightforward computation to show that $\xi_i \in C_c(\mathcal{G})$ and $h_i = \xi_i^* * \xi \to 1$ uniformly on compact subsets of $\mathcal{G}$. For an $f \in C_c(\mathcal{G})$ supported in a bisection, $f^* * f \in C_c(\mathcal{G}^{(0)})$ (a commutative algebra) and therefore the C^*-identity gives

$$||f||^2_{C^*(\mathcal{G})} = ||f^* * f||_{C^*(\mathcal{G})} = ||f^* * f||_\infty = ||f||^2_\infty$$

With the notation in the lemma, it follows easily from a partition of unity argument that $m_{h_i}(f) \to f$ in $C^*(\mathcal{G})$ for any $f \in C_c(\mathcal{G})$ and so the same is true for an arbitrary element in $C^*(\mathcal{G})$. Let a be in the kernel of the quotient map $\pi : C^*(\mathcal{G}) \to C^*_r(\mathcal{G})$, and see that $\pi \circ m_{h_i}(a) = m_{h_i} \circ \pi(a)$, as this is true for all elements of $C_c(\mathcal{G})$, and therefore $\pi(m_{h_i}(a)) = 0$. Finally, since $m_{h_i}(a) \in C_c(\mathcal{G})$, and π is injective on $C_c(\mathcal{G})$, we have $m_{h_i}(a) = 0$ for all i, therefore $a = 0$. □

As show by Rufus Willett in [19], the converse of theorem 4.1.11 is false.

4.2 A Characterization of Amenable-ish Properties

In this section we present a characterization of certain amenable-ish properties for groupoids, all of which when restricted to groups are equivalent to amenability.

Theorem 4.2.1. Let μ be a quasi-invariant Radon probability measure on $\mathcal{G}^{(0)}$. Consider the following statements:

1. The trivial representation of $C^*_{max}(\mathcal{G},\mu)$ on $L^2(\mathcal{G}^{(0)},\mu)$ descends to $C^*_r(\mathcal{G},\mu)$.

2. The state $\phi_\mu : C^*_{max}(\mathcal{G},\mu) \to \mathbb{C}$ descends to $C^*_r(\mathcal{G},\mu)$.

3. There is a sequence (ξ_n) in $C_c(\mathcal{G})$ which are unit vectors in $L^2(\mathcal{G},\mu\circ\lambda)$ such that for all $f \in C_c(\mathcal{G})$,
$$\langle \xi_n, \lambda(f)\xi_n\rangle \to \phi_\mu$$

4. There is a sequence of compactly supported positive type functions $h_n : \mathcal{G} \to \mathbb{C}$ such that
$$h_n D^{-1/2}\mu\circ\lambda \to D^{-1/2}\mu\circ\lambda$$
in the weak-* topology on $C_c(\mathcal{G})^*$.

Then
$$(4) \Leftrightarrow (3) \Rightarrow (2) \Leftrightarrow (1).$$

Moreover, if we assume $C^*_r(\mathcal{G},\mu)\cap\mathcal{K}(L^2(\mathcal{G},\mu\circ\lambda))) = \emptyset$, then all the statements above are equivalent.

Proof. $(1 \Rightarrow 2)$: It depends on how you define ϕ_μ, but as long as you know that

$$\phi_\mu(f) = \langle 1, T(f)1\rangle_{L^2(\mathcal{G}^{(0)},\mu)}$$

this is trivial.

$(2 \Rightarrow 3)$: If ϕ_μ extends to a state on $C_r^*(\mathcal{G}, \mu \circ \lambda)$, then one may use Glimm's Lemma to obtain a sequence of unit vectors $\xi_n \in L^2(\mathcal{G}, \mu \circ \lambda)$ such that

$$\langle \xi_n, \lambda(f)\xi_n \rangle \to \phi_\mu(f)$$

for all $f \in C_r^*(\mathcal{G}, \mu \circ \lambda)$. As $C_c(\mathcal{G})$ is dense in $L^2(\mathcal{G}, \mu \circ \lambda)$ (completing $C_c(\mathcal{G})$ is how you define this Hilbert space), one may approximate each ξ_n by a function in $C_c(\mathcal{G})$ with the same name. We make a brief note that the use of Glimm's Lemma is the only time we are forced to use our assumption on the compact operators.

$(3 \Rightarrow 4)$ Given such a sequence, one may compute

$$\begin{aligned}
\langle \xi_n, \lambda(f)\xi_n \rangle &= \int_{G^{(0)}} \sum_{g \in \mathcal{G}^x} f(g) D^{-1/2}(g) \sum_{h \in \mathcal{G}^x} \overline{\xi_n(h)} \xi_n(g^{-1}h) \, d\mu(x) \\
&= \int_{G^{(0)}} \sum_{\beta \in \mathcal{G}^x} f(g) D^{-1/2}(g) (\eta_n * \eta_n^*)(g) \, d\mu(x) \quad \text{where } \eta_n = \overline{\xi_n} \\
&= \int_G f h_n D^{-1/2} \, d\mu \circ \lambda \quad (\dagger)
\end{aligned}$$

where $h_n := \eta_n * \eta_n^*$ is positive type and compactly supported. The property of convergence in the weak-* topology on $C_c(\mathcal{G})^*$ is now trivial.

$(4 \Rightarrow 3)$ Given such a sequence of positive definite functions, recall that $\lambda(h_n)$ is positive in $C_r^*(\mathcal{G}, \mu \circ \lambda)$. As such, one may write $\lambda(h_n) = a_n * a_n^*$ for some $a_n \in C_r^*(\mathcal{G}, \mu \circ \lambda)$. Define $\xi_n = \overline{a_n}$, and by using the same computation as in $(\dagger)$ applied to the specific function $f = \chi_{\mathcal{G}^{(0)}}$, one has that

$$\int_{\mathcal{G}} |\xi_n|^2 \, d\mu \circ \lambda = \left\langle \xi_n, \lambda(\chi_{\mathcal{G}^{(0)}}) \xi_n \right\rangle = \int_{\mathcal{G}} \chi_{\mathcal{G}^{(0)}} h_n D^{-1/2} \, d\mu \circ \lambda$$

Now, one uses the assumption that $h_n D^{-1/2} \mu \circ \lambda \to D^{-1/2} \mu \circ \lambda$ in the weak*-topology on

$C_c(\mathcal{G})^*$ to obtain

$$\int_{\mathcal{G}} |\xi_n|^2 \, d\mu \circ \lambda \to \int_{\mathcal{G}^{(0)}} D^{-1/2} \, d\mu = 1.$$

For a sufficiently large N, and $n \geq N$, one may define $\eta_n := \frac{\xi_n}{||\xi_n||_{L^2}}$. Now, one approximates each η_n with an element in $C_c(\mathcal{G})$ (with distance $\frac{1}{n}$ from η_n) and the result follows from a basic computation.

($3 \Rightarrow 2$) This follows from a routine use of the triangle and Cauchy-Schwarz inequalities.

($2 \Rightarrow 1$) Given that

$$\phi_\mu(f) = \langle 1, T(f)1 \rangle$$

and that 1 is a cyclic vector for T, one uses the GNS construction applied to the state ϕ_μ and obtains a representation, π_{ϕ_μ} of $C^*_r(\mathcal{G}, \mu)$. By uniqueness, π_{ϕ_μ} is unitarily equivalent to the trivial representation. □

4.3 Counterexamples with HLS Groupoids

As mentioned before, the converse of Theorem 4.1.11 is false as was shown by Rufus Willett in [19]. Characterizing the situation $C^*_r(\mathcal{G}) = C^*(\mathcal{G})$ was the main motivation for Theorem 4.3.1; Clearly if $C^*_r(\mathcal{G}) = C^*(\mathcal{G})$ then the trivial representation descends to $C^*_r(\mathcal{G})$. Unfortunately, the converse of this statement is also false, and in this section we construct a counterexample using a HLS groupoid similar to the one found in [19]. This example was shown to me by Rufus Willett but is unpublished.

Definition 4.3.1. Let G be a discrete group. We call the sequence (K_n) of subgroups an approximating sequence for G if (K_n) is a decreasing sequence of normal subgroups that approaches the identity. More precisely,

(1) Each K_n is normal and has finite index as a subgroup of G.

(2) $K_{n+1} \subset K_n$ for all n;

(3) and $\cap_n K_n = \{e\}$. The pair $(G, (K_n))$ is called an approximated group.

Definition 4.3.2. Let $(G, (K_n))$ be an approximated group. For each n we define quotient group $G_n := G/K_n$ with π_n the associated quotient map. Write $G_\infty = G$ and π_∞ the identity map. We define the associated HLS groupoid (which is, in fact, a bundle of groups)

$$\mathcal{G} := \cup_{n \in \mathbb{N} \cup \{\infty\}} \{n\} \times \Gamma_n.$$

$$\mathcal{G}^{(0)} := \{(n, g) \in \mathcal{G} \,:\, g = e\}$$

$$r(n, g) = s(n, g) = (n, e)$$

and we declare a basis for the open sets to be $\{(n, g)\}$ for each n and $g \in G_n$ and, for a fixed g and $N \in \mathbb{N}$ $\{(n, \pi_n(g)) \,:\, n \in \mathbb{N} \cup \{\infty\}, n > N\}$.

The topology generated by these open sets gives a locally compact, second countable étale groupoid whose unit space is the one-point compactification of $\mathbb{N}$.

The following lemma can be found in [19]:

Lemma 4.3.3. Let $\mathcal{G}$ be the HLS groupoid associated to an approximated group $(G, (K_n))$. Then G is amenable if and only if $\mathcal{G}$ is amenable.

Theorem 4.3.4. If G is an infinite discrete group and $\mathcal{G}$ any HLS groupoid associated to G. Define μ to be the invariant probability measure on $\mathcal{G}^{(0)}$ be setting $\mu(n) = \frac{1}{2^{n+1}}$ and $\mu(\infty) = \frac{1}{2}$. Then the trivial representation of $C^*_{max}(\mathcal{G}, \mu)$ on $L^2(\mathcal{G}^{(0)}, \mu)$ descends to $C^*_r(\mathcal{G}, \mu)$.

Proof. By Proposition 4.3.1 it is enough to show that there is a sequence (ξ_n) in $C_c(\mathcal{G})$ which are unit vectors in $L^2(\mathcal{G}, \mu \circ \lambda)$ such that for all $f \in C_c(\mathcal{G})$,

$$\langle \xi_n, \lambda(f)\xi_n \rangle \to \phi_\mu(f).$$

Define

$$\xi_n((m,g)) = \begin{cases} \frac{1}{|\sqrt{G_m}|} & m \in \mathbb{N}, m \neq n \\ \frac{\sqrt{1+2^n}}{|G_n|} & m = n \\ 0 & m = \infty \end{cases}$$

Firstly, this is a unit vector:

$$\begin{aligned} ||\xi_n||^2 &= \sum_{m \neq n} \frac{1}{2^{m+1}} (\sum_{g \in G_m} \frac{1}{|G_m|}) + \frac{1}{2^{n+1}} (\sum_{g \in G_n} \frac{1+2^n}{|G_n|}) \\ &= \sum_{m \neq n} \frac{1}{2^{m+1}} + \frac{1}{2^{n+1}} + \frac{1}{2} \\ &= \sum_{m=1} \frac{1}{2^m} \\ &= 1 \end{aligned}$$

Now, we show that $\langle \xi_n, \lambda(f)\xi_n \rangle \to \phi_\mu(f)$ for all $f \in C_c(\mathcal{G})$ by showing that this holds on the dense subset of $C_c(\mathcal{G})$ which consists of functions, f, such that : 1) there exists an N such that for all $n \geq N$, the function $\pi_n : G \to G_n$ is injective on the support of $f|_{G_\infty}$ and 2) for all $g \in G_n$

$$f(n,g) = \begin{cases} f(\infty, h) & \text{there is an h} \in \text{supp}(f|_{G_\infty}) \text{ with } \pi_n(h) = g \\ 0 & \text{otherwise} \end{cases}$$

We may now compute, for such a function when $n \geq N$,

$$\begin{aligned} \langle \xi_n, \lambda(f)\xi_n \rangle &= \sum_{m \in \mathbb{N}, m \neq n} \frac{1}{2^{m+1}} \sum_{g \in G_m} f(m,g) + \sum_{g \in G_n} (\frac{1}{2^{n+1}} + \frac{1}{2}) f(n,g) \\ &= \sum_{m \in \mathbb{N}} \frac{1}{2^{m+1}} f(m,g) + \sum_{g \in G} \frac{1}{2} f(\infty, g) \\ &= \int_{\mathcal{G}^{(0)}} f \; d(\mu \circ \lambda) \\ &= \phi_\mu(f) \end{aligned}$$

which completes the proof. ☐

We now begin to set up a specific counter example, but we need to make sure we don't have any compact operators lying around, and the following makes sure that this is the case.

Lemma 4.3.5. ([3], Proposition 2.5.4) G is an infinite discrete group then $C_r^*(G)$ has no compact operators.

Lemma 4.3.6. ([9], 2.8.3) Suppose that T_n is a bounded operator on $\mathcal{H}_n$ for all n and $T = \oplus_n T_n$. Then T is a compact operator on $\mathcal{H} = \oplus_n \mathcal{H}_n$ if and only if for all n, T_n is compact and $||T_n|| \to 0$.

Lemma 4.3.7. Let $\mathcal{G}$ be a bundle of countably infinite groups and μ an atomic probability measure on $\mathcal{G}^{(0)}$, then $C_r^*(\mathcal{G},\mu) \cap \mathcal{K}(L^2(\mathcal{G},\mu\circ\lambda)) = \emptyset$.

Proof. For an $f \in C_c(\mathcal{G})$, for an $x \in G^{(0)}$ let $f_x = f|_{\mathcal{G}_x}$. It's easy to see that

$$\lambda(f) = \bigoplus_{x\in s(\mathrm{supp}(f)))} \lambda(f_x) \quad \text{and} \quad L^2(\mathcal{G},\mu\circ\lambda) = \bigoplus_{x\in G^{(0)}} \ell^2(G_x,\mu(x)).$$

As $\lambda(f_x)$ is no more than the regular representation of the group G_x, $\lambda(f_x) \in C_r^*(G_x)$ is not a compact operator. Whence, $\lambda(f)$ is not a compact operator. If follows that $C_r^*(\mathcal{G},\mu)$ has no compact operators. ☐

Remark 4.3.8. As follows from [2], if $G = SL(3,\mathbb{Z})$, and $G_n = SL(3,\mathbb{Z}/2^n\mathbb{Z})$ the associated HLS groupoid has the property that $C^*(\mathcal{G}) \neq C_r^*(\mathcal{G})$. Additionally, because $\mathrm{supp}(\mu) = \mathcal{G}^{(0)}$, we have $C^*(\mathcal{G},\mu) = C^*(\mathcal{G})$ and $C_r^*(\mathcal{G},\mu) = C_r^*(\mathcal{G})$, and therefore that $C^*(\mathcal{G},\mu) \neq C_r^*(\mathcal{G},\mu)$. By the above theorem, this provides an example where $C^*(\mathcal{G},\mu) \neq C_r^*(\mathcal{G},\mu)$, but satisfies 3 in proposition 4.2.1. It's easy to see that after crossing each G_n and G with $\mathbb{Z}$, as $\mathbb{Z}$ is an infinite amenable group, there are no compact operators and the trivial representation of $C_{max}^*(\mathcal{G},\mu)$ on $L^2(\mathcal{G}^{(0)},\mu)$ descends to $C_r^*(\mathcal{G},\mu)$. This gives the following string of implications

$$\begin{aligned}\mathcal{G} \text{ is amenable} &\Rightarrow C^*(\mathcal{G}) = C_r^*(\mathcal{G}) \\ &\Rightarrow \text{ the trivial representation of } C_{max}^*(\mathcal{G},\mu) \text{ on } L^2(\mathcal{G}^{(0)},\mu) \text{ descends to } C_r^*(\mathcal{G},\mu)\end{aligned}$$

and the converse of each of these implications is false. Again, if $\mathcal{G}$ is a group then they are all equivalent.

Bibliography

[1] Claire Anantharaman-Delaroche and Jean Renault. *Amenable groupoids*. Vol. 36. L'Enseignement mathématique Geneva, 2000.

[2] MB Bekka. *On the full C^*-algebras of arithmetic groups and the congruence subgroup problem*. Vol. 11. 1999, pp. 705–716.

[3] Nathanial Patrick Brown and Narutaka Ozawa. *C*-algebras and finite-dimensional approximations*. Vol. 88. American Mathematical Soc., 2008.

[4] Indira Chatterji. "Introduction to the Rapid Decay property". In: *arXiv preprint arXiv:1604.06387* (2016).

[5] Łukasz Garncarek. "Mini-course: property of rapid decay". In: *arXiv preprint arXiv: 1603.06730* (2016).

[6] Uffe Haagerup. "An example of a non nuclear C^*-algebra, which has the metric approximation property". In: *Inventiones mathematicae* 50 (1978), pp. 279–293.

[7] Cheng Jun Hou. "Spectral invariant subalgebras of reduced groupoid $C*$-algebras". In: *Acta Mathematica Sinica, English Series* 33 (2017), pp. 526–544.

[8] Paul Jolissaint. "K-theory of reduced $C*$-algebras and rapidly decreasing functions on groups". In: *K-theory* 2 (1989), pp. 723–735.

[9] Richard V Kadison and John R Ringrose. *Fundamentals of the theory of operator algebras, Vol. 2.* American Mathematical Soc., 1997.

[10] Vincent Lafforgue. "A proof of property (RD) for cocompact lattices of SL (3, $\mathbb{R}$) and SL (3, $\mathbb{C}$)". In: *J. Lie Theory* 10 (2000), pp. 255–267.

[11] Vincent Lafforgue. "Banach KK-theory and the Baum-Connes conjecture". In: *European Congress of Mathematics*. Springer. 2001, pp. 31–46.

[12] Xin Li. "Continuous orbit equivalence rigidity". In: *Ergodic Theory and Dynamical Systems* (2016), pp. 1–21.

[13] Xin Li. "Quasi-isometry, Kakutani equivalence, and applications to cohomology". In: *arXiv preprint arXiv:1604.07375* (2016).

[14] Jean Renault. *A groupoid approach to C*-algebras*. Vol. 793. Springer, 2006.

[15] Jean Renault. *C*-algebras and Dynamical Systems*. IMPA, 2009.

[16] John Roe. *Lectures on coarse geometry*. American Mathematical Soc., 2003.

[17] Aidan Sims. "Étale groupoids and their C^*-algebras". In: *arXiv preprint arXiv: 1710. 10897* (2017).

[18] Kevin Whyte. "Amenability, bilipschitz equivalence, and the von Neumann conjecture". In: *Duke Math. J.* 99 (July 1999), pp. 93–112.

[19] Rufus Willett. "A non-amenable groupoid whose maximal and reduced C^*-algebras are the same". In: *Mnster Journal of Mathematics* (2015), pp. 241–252.