

Investors Want Both More Risk and Better Information

D.J. Johnstone
University of Sydney Business School
University of Sydney
NSW 2006
Australia

david.johnstone@sydney.edu.au

Draft Only (June 15, 2019)

Abstract

Market folklore says that investors abhor uncertainty. But if there were no uncertainty, investors could earn only the risk-free interest rate. So how much risk do investors want resolved? Using a payoffs form of CAPM, I find that the mean-variance investor buying the risky asset is attracted to invest by higher payoff risk. Higher payoff risk, accounted for in the rational asset price, translates unconditionally to higher ex ante expected utility (the payoff mean makes no difference). Also contrary to intuition, resolution of parameter risk can heighten payoff risk - and hence increase expected utility. By distinguishing payoff risk from parameter risk, more is revealed about why decision makers always want more information. Ultimately, both potential buyers and existing owners of a risky asset want more information about it before they trade, but they want it for different reasons.

Key words: information, parameter risk, payoff risk, CAPM, Bayesian predictive distribution

JEL categories: G10, G11, D80, D81

1 Introduction

Investment lore says that "markets abhor uncertainty". Yet suppose that there were no uncertainty, or that information resolved all uncertainty. All assets would be risk-free and would return only the risk-free interest rate.

Consider the stock market as a single asset paying a random cash amount V at period end, with mean $E[V]$ and variance $\text{var}(V)$. The issue is how large would prospective investors like $\text{var}(V)$ to be, either in relation to $E[V]$ or unconditionally?

Typically, $\text{var}(V)$ will be higher if firms do risky things with positively correlated payoffs. The "physical" randomness in business (e.g. weather, technological changes, market mood) affects the market's assessment of $\text{var}(V)$, but so does the available information. In asking whether investors have an ex ante preferred (possibly high) $\text{var}(V)$, we are implicitly asking how much information they want.

It might seem that higher $\text{var}(V)$ should of itself reduce the investor's expected utility when buying the risky asset. But a rationally priced market takes account of $\text{var}(V)$, so the question of how risk affects an investor's overall expected utility can only be solved in the context of an asset pricing model.

Under the capital asset pricing model (CAPM), higher $\text{var}(V)$ drives the price of the risky market lower, which means that the representative investor holds less initial wealth in the market (and more in the risk-free asset). It must be asked therefore whether the lower fraction of wealth invested at a higher expected market rate of return ("cost of capital"), when combined with a bigger residual fraction left in the risk-free asset, has higher or lower ex ante expected utility?

In light of this introduction, my paper addresses the following research questions:

(1) *Since an asset with higher $\text{var}(V)$ is priced down by CAPM, and the investor can buy the same share of that asset while leaving more of her initial wealth earning the risk-free interest rate, how risky would she like the risky asset to be?*

(2) *If there exists a theoretically definable "optimal risk" $\text{var}(V)$, does a CAPM investor "abhor information" that is expected ex ante to resolve some of that risk?*

Having chosen to allocate a portion of cash wealth to the risky asset, the stock holder's expectation (implied by holding onto the risky asset) is that resolution of uncertainty will show "she is right" and will bring a price increase (capital gain). As an owner, resolution of uncertainty is now welcome, and cannot come soon enough, which raises a further research question:

(3) *How do we reconcile a prospective investor's want for risk with an existing owner's*

want for resolution? Do potential investors (buyers) and existing asset owners have different information ideals?

The very idea underpinning these research questions - i.e. that an investor might be attracted by an asset having more risk - seems odd, but becomes interesting once we remember that the asset's investment appeal depends on its cost, and a more risky asset might offer "better value" depending on how its CAPM price P responds to its risk.

Changes in $E[V]$ and $\text{var}(V)$ lead to a different P , which is reason to look for the parity between $E[V]$ and $\text{var}(V)$ that makes the accordant CAPM price $P \equiv f(E[V], \text{var}(V))$ most attractive to an expected utility maximizing investor. A priori, we cannot know which of the available combinations $(E[V], \text{var}(V), P)$ is most preferred ex ante - so further analysis is needed.

The final question in the paper, consistent with any attempt to understand the investment benefits of better information, comes from the mathematical theory of gambling and economic Darwinism (e.g. Sandroni 2000, Blume and Easley 2006, MacLean, Sanegre, Zao and Ziemba 2004). That theory overlaps decision theory and finance but goes one step further by considering conditions under which maximum ex ante expected utility converts to maximum (or at least "satisfactory") ex post average realized utility (i.e. the investor "makes money").

(4) How can we reconcile the higher ex ante expected utility brought by some level of ex ante uncertainty with the need to avoid investment mistakes by resolving as much uncertainty as possible?

1.1 Contribution to the Literature

I extend the theory of financial information by treating the risky asset or market as just one portion of the investor's chosen portfolio. The investor is interested in the expected return from the market, but the full measure of investor welfare is the expected utility from her weighted portfolio of risky and risk-free assets.

Like Coles and Loewenstein (1988) and Coles et al. (1995), my analysis calls on the payoffs or "certainty equivalent" version of CAPM (see the original and thorough exposition of the payoffs CAPM in Fama and Miller 1972). The market produces an exogenous risky payoff V , as if "from nature", and the investor infers its distribution $f(V|\cdot)$, based on available information.

That distribution has relevant parameters (mean, variance,...) and information is used to assess probability distributions for those parameters. A distinction is made between uncertainty about V ("payoff uncertainty") and uncertainty about the parameters of $f(V)$ ("parameter uncertainty"), because these uncertainties, whilst fungible, are not the same and do not necessarily, both or individually, decrease with better information.

To view the full picture of investor welfare (expected utility) terminal wealth is defined as the sum of the terminal cash proceeds from the risky asset plus the residual cash held earning interest in the bank. Based on this whole-of-portfolio approach, some unexpected and interesting implications appear:

(i) A mean-variance investor buying into a CAPM-priced asset prefers *greater* payoff variance $\text{var}(V)$, *unconditionally*. The risky asset's mean payoff $E[V]$ does not affect the ex ante expected utility of an investor who pays the theoretical CAPM price to buy the asset.

(ii) Although investors who perceive less risk are ready to pay a higher asset price and earn a lower expected rate of return (cet. par.), their overall portfolio of risky and riskless assets is found to carry lower ex ante expected utility. More or better information, intended to bring lower perceived risk, and lower cost of capital, cannot therefore - of itself - be the information objective.

(iii) Although CAPM investors prefer greater payoff risk ex ante, they maximize ex ante expected utility by acquiring any information with positive marginal expected utility. Paradoxically, therefore, prospective investors abhor neither greater risk nor the information that might resolve it.

(iv) Either higher or lower parameter uncertainty can bring higher payoff uncertainty and therefore higher ex ante expected utility. Information that reduces parameter uncertainty can increase payoff uncertainty, and hence increase the investor's ex ante expected utility from CAPM investment.

(v) Common claims that investors want better information so as to "resolve uncertainty" are justified, provided that the risk or uncertainty described is understood as parameter risk, not overall payoff risk. By distinguishing explicitly between payoff risk and parameter estimation risk, the typically correct¹ claim (e.g. Botosan 1997, Core, Hail and Verdi 2015) that new or better information resolves parameter risk is reconciled with superficially contrary but also correct claims (e.g. Rogers, Skinner and Van Buskirk 2019, pp.92-94) that better information can sometimes serve only to heighten investors' payoff uncertainty.

1.2 Risk or Uncertainty?

I adopt the nowadays widely accepted Bayesian position by which the statistical variation represented in a probability distribution over future asset payoffs can be equally called risk or uncertainty (Kadane 2011).

¹Pathological exceptions can be constructed (see footnote 19).

Risk is historically associated with "physical" or objective probabilities and uncertainty with subjective probability, but in asset pricing there is no operational distinction between risk and uncertainty. It is not possible to identify "which is one and which is the other". Investors can't know whether uncertainty about an asset's payoff is due to irreducible physical variation in nature or to inadequate information and human information processing.

A perception of greater uncertainty or higher risk is of course invited by there being "truly" or "physically" more risk or variability in the market payoff (in the sense for example of the farmer's crop being made more variable by more variable weather, or a manufacturer running an efficient but temperamental machine).

Ultimately, however, assets are priced subjectively based on perceived rather than unobservable "true" risk. In Bayesian theory, the term "true probability" is for this reason suppressed, or even outlawed.² Probability distributions are understood as all being subjective in some degree, some being merely more inter-subjective or mutually agreeable than others. Even a roulette wheel can be misprogrammed or physically defective in its fairness, so its subjective probability of red is not an exact constant. Probabilities and their resulting asset prices are merely subjective investor perceptions, and react rationally, by way of Bayes theorem and CAPM, to any new information.

1.3 Theoretical Framework

The following analysis combines conventional Bayesian statistical decision theory (subjective expected utility maximization) with conventional asset pricing according to CAPM. That nexus first appeared in the statistical literature on parameter "estimation risk" in portfolio optimization (e.g. Winkler and Barry, 1975; Barry, 1978; and Bawa, Brown and Klein, 1979). Risky asset payoffs are treated as exogenous (from "nature") and the payoff distribution on which assets are priced is understood as a subjective Bayesian distribution, which in principle incorporates all of the investor's information (and information about the quality of that information).

To incorporate parameter uncertainty, the payoff's "predictive distribution" is found by "integrating out" the unknown parameters (i.e. by averaging over the parameter space, or all feasible parameter values, weighting them by their posterior probabilities). This Bayesian method is explained and illustrated throughout my paper.

²See Nau (2001) on the modern Bayesian view of why "probability does not exist" as a physical or objective quantity.

2 Payoff Risk Versus Parameter Risk

In a one-period CAPM, the market offers a random cash payoff V , no different in statistical eyes to the cash amount put out by a fruit machine. Like the proceeds from the fruit machine, the market payoff V is described by a probability distribution $f(V)$ with uncertain parameter values.

It is essential to make explicit mathematically rigorous distinction between payoff uncertainty (uncertainty about the future random V) and parameter uncertainty (uncertainty about the values of the parameters of the payoff distribution, $f(V)$). Confusion easily sets in when we think of parameter risk and payoff risk as one and the same. If there is any doubt about the value of this distinction, think again of the unknown variance of the payoff distribution. If information reveals this parameter to be high, then high parameter certainty translates to low payoff certainty.

In a well specified Bayesian model, payoff risk and parameter risk are not the same, but are intimately related by relevant probability laws. Information allows us to form posterior distributions for the unknown parameters, from which we can infer the posterior or "predictive" distribution of the payoff (see below).

Definitions:

Payoff Risk. A risky or random future cash payoff of amount V exhibits "payoff risk" in the sense that investors don't know what V will be. All they can infer is a probability distribution $f(V|\Omega)$ over V conditioned on current information Ω .

Parameter Risk. Parameter risk, also known as information risk or "estimation risk", is the risk that investors' assessments of the parameters of $f(V|\Omega)$ are incorrect. Note that $f(V|\Omega)$ changes with current information Ω and is therefore Bayesian. That is why the early finance literature (Kalymon 1971; Winkler and Barry 1975; Barry 1978; Klein and Bawa 1976, 1977; Brown 1979; Bawa, Brown and Klein 1979) on portfolio choice under parameter uncertainty relied on Bayesian inference models rather than frequentist "plug in point parameter estimate" methods.

2.1 Why a "Predictive Distribution"?

To display the distinction between payoff risk and parameter risk, consider a simple "urn market" in which the cash payoff V depends on whether the next chip drawn from the urn is *Red*. Suppose that the urn pays $V = 1$ if *Red* and otherwise zero. The urn has just one parameter, i.e. its index $\theta = \Pr(\text{Red})$.

To find the probability distribution of V , when θ is an unknown (i.e. uncertain) parameter, investors write a conditional distribution $f(\theta|\Omega)$ over θ .³ Parameter risk

³The distribution $f(V|\Omega)$ incorporates uncertainty about "which urn is being drawn from". Each different possible urn (regime) has its own different θ , and θ values are distributed as $f(\theta|\Omega)$.

under information Ω is reflected in the dispersion of $f(\theta|\Omega)$, a flatter or more disperse distribution suggesting wider uncertainty about the parameter's true value.

The alternative to recognizing parameter risk is to "plug in" an assumed value for parameter θ , as if it were already known. That approach was set aside in the Bayesian finance literature, essentially because it failed to hedge portfolios against the risk of getting the assumed parameter values badly wrong (see the literature summary in Avramov and Zhou 2010).⁴

To embed parameter estimation risk into the assessment of payoff risk, the Bayesian method invented by Zellner and Chetty (1965), and adopted in finance literature, is to find the "predictive distribution" of V by "integrating out" the unknown parameter or parameters. In the urn model, that requires calculation of the integral

$$\Pr(V = 1|\Omega) = \int_{\theta} f(\theta|\Omega)f(V = 1|\theta)d\theta = E[\theta|\Omega].$$

In this simple model, the predictive probability of $V = 1$ is just the weighted average value of θ , weighted by posterior density $f(\theta|\Omega)$.

The perceived risk or variance of payoff V , after accounting for the parameter uncertainty reflected in $f(\theta|\Omega)$, is therefore, $E[\theta|\Omega](1 - E[\theta|\Omega])$, which is maximized when $E[\theta|\Omega] = 0.5$.

Typically, but not always, more uncertainty about parameter θ translates to more uncertainty about payoff V . In the urn model, a more uniform distribution $f(\theta|\Omega)$, reflecting more ignorance or greater uncertainty about parameter θ , tends to leave $E[\theta|\Omega]$ nearer to 0.5, and hence leaves greater payoff uncertainty.

With better information (like a higher sample size), the posterior distribution of θ tends to shrink, meaning less parameter risk. That reduced parameter uncertainty will bring an increase (decrease) in payoff uncertainty if the revised mean $E[\theta|\cdot]$ is closer to (further from) 0.5, since $\Pr(V = 1) = E[\theta|\cdot] = 0.5$ implies maximum payoff uncertainty. Hence, surprisingly, more certainty about the unknown parameter θ can sometimes bring less certainty about the risky payoff V .

3 Mean-Variance Pricing Model

The model is simplified but conventional. Its assumptions are:

- (i) There is a single investor (buyer) with utility function $U(W)$ for wealth W , initial cash wealth W_0 and personal probability distribution $f(V)$.

⁴The finance literature on parameter estimation risk shows how more information (larger n) can reduce both parameter and payoff uncertainty. The negative relationship in some Bayesian models between parameter uncertainty and larger n is raised in accounting theory as a way of understanding how better accounting information can affect market uncertainty and the market required return on capital (e.g. Botosan 1997, Riedl and Serafeim 2011, Leuz and Schrand 2018).

(ii) At equilibrium (market clearing) price P , the buyer's portfolio is optimized by buying exactly one unit of the risky asset (i.e. the market is cleared). At that price, the last $\$ \delta \rightarrow 0$ spent buying the risky asset (for total price P) produces zero marginal expected utility. All of the earlier outlay, up to aggregate amount P , produces positive expected utility (the "consumer surplus").

(iii) The buyer has utility $U(W)$ for wealth W .

If V is normally distributed and the investor has exponential utility $U(W) = 1 - \exp[-cW]$, the risky asset has price

$$P = \frac{1}{R_f} [E[V] - c \text{var}(V)], \quad R_f = (1 + r_f) \quad (1)$$

where r_f . The price P in (1) is derived in the Appendix (using a convenient shortcut).

3.1 Period-End Wealth Parameters (μ, σ^2)

The potential investor endowed with wealth W_0 anticipates random cash wealth W_1 at period-end ($t = 1$) from her portfolio of risky and risk-free assets, given by

$$W_1 = V + (W_0 - P)R_f,$$

where $R_f = (1 + r_f)$ is the risk-free return factor.⁵

The ex ante mean and variance of terminal wealth W_1 are then, respectively

$$\mu \equiv \mu(W_1) = E[V] + (W_0 - P)R_f, \quad (2)$$

and

$$\begin{aligned} \sigma^2 &\equiv \sigma^2(W_1) = \text{var}[V + (W_0 - P)R_f] \\ &= \text{var}(V), \end{aligned}$$

since the only random variable is V .

3.2 Effects of Higher Variance

If the risky asset has higher perceived $\text{var}(V)$, its price P from (1) is lower for any given expected payoff $E[V]$.

Lower P implies that less of the investor's fixed initial wealth W_0 is allocated to the risky asset, leaving a greater residual $(W_0 - P)$ in cash. Since P is lower but $E[V]$

⁵This measure of terminal wealth is applied similarly by Berk (1997).

remains constant, the smaller holding in the risky asset is expected to earn a higher expected rate of return

$$E[R] = E[V]/P.$$

The simultaneous effects of higher $\text{var}(V)$ are thus:

- (i) the investor pays a smaller dollar amount P for the risky asset, and
- (ii) the risky asset pays out a higher expected rate of return $E[R]$.

These individual effects are easily demonstrated, but their combined effect on the investor's ex ante expected utility is not immediately apparent.

Superficially, lower P might imply lower expected utility, and a higher expected return seems to suggest higher expected utility. However, without formal analysis neither intuition has any clear justification, so the question of how expected utility reacts to a change in the asset variance remains to be answered.

Proposition 1: *The ex ante expected utility from the CAPM investor's rational portfolio of risky and risk-free assets is:*

(1.1) increasing in the payoff variance, $\text{var}(V)$,

(1.2) unaffected by the amount of the payoff mean, $E[V]$.

In other words, the investor, who must pay the market clearing price P in (1), is attracted to invest *only* by the risky asset's payoff variance, and is not attracted at all by its mean. The investor "pays for" the mean in P and gets no benefit even when it is large.

3.2.1 Proof Under Exponential Utility

Substituting for P in the equation (2) above for μ gives

$$\begin{aligned} \mu &= E \left[V + \left(W_0 - \frac{1}{R_f} [E[V] - c \text{var}(V)] \right) R_f \right] \\ &= W_0 R_f + c \text{var}(V), \end{aligned} \tag{3}$$

and, as above, $\sigma^2 = \text{var}(V)$.

Note in (3) how the mean payoff $E[V]$ cancels itself out and does not affect the mean terminal ($t = 1$) wealth μ . Instead, $\mu \equiv E[W_1]$ is driven only by the variance of the risky asset, $\text{var}(V)$. Interestingly, higher payoff variance $\text{var}(V)$ implies higher expected terminal wealth, μ .

By a standard result for exponential utility $U(W) = 1 - \exp(-cW)$, the expected utility of terminal wealth $W_1 \sim N(\mu, \sigma^2)$ can be written as

$$EU = 1 - \exp \left[-c \left(\mu - \frac{c}{2} \sigma^2 \right) \right], \quad (4)$$

from which we can see that the certainty equivalent of uncertain W_1 is $\mu - \frac{c}{2} \sigma^2$.

Now, as seen in (3), the rate of increase in μ brought by higher $\sigma^2 = \text{var}(V_M)$ is

$$\frac{d\mu}{d\sigma^2} = c,$$

so the certainty equivalent $\mu - \frac{c}{2} \sigma^2$ increases in $\text{var}(V_M)$ at rate $c - c/2 = c/2$. Thus, a mean-variance investor insists on "milking" payoff variance at a rate of $c/2$ per unit. Put another way, by pricing the risky asset at P , she insists on adding $c/2$ dollars in certainty equivalent for every unit of variance in her terminal wealth.

Similarly, in terms of expected utility, substituting in (4) for μ and σ^2 gives

$$\begin{aligned} EU &= 1 - \exp \left[-c \left(W_0 + c \text{var}(V) - \frac{c}{2} \text{var}(V) \right) \right] \\ &= 1 - \exp \left[-c \left(W_0 R_f + \frac{c}{2} \text{var}(V) \right) \right]. \end{aligned} \quad (5)$$

This result shows that the ex ante expected utility from the investor's optimally weighted portfolio of risky and risk-free assets depends only on the variance of the risky asset's payoff, and increases with that variance, hence proving both Propositions (1.1) and (1.2).⁶

It occurs therefore that a risky asset with higher ex ante payoff variance, $\text{var}(V)$, offers the prospective investor a higher ex ante expected utility.

The investor spends a lower amount P when buying the risky asset, but that dollar amount earns a higher ex ante expected percentage return. The combination of those two effects, along with the higher residual $(W_0 - P)$ left in the risk-free asset earning interest, leaves a random terminal cash payoff $W_1 \equiv V + (W_0 - P)R_f$ with higher ex ante expected utility. Again this result is regardless of the risky asset's mean payoff $E[V]$.

This proof tells us that a more risky asset is priced down by CAPM so much that it offers the investor a preferable terminal wealth pairing (μ, σ^2) compared to the mean-variance pair available from a lower σ^2 asset.

At the other extreme, if there was no risky asset the investor's ex ante expected utility would be only what is obtained by investing all of the initial wealth W_0 in the risk-free asset, that is

$$\begin{aligned} EU &= 1 - \exp[-cW_0R_f] \\ &< 1 - \exp \left[-c \left(W_0R_f + \frac{c}{2} \text{var}(V) \right) \right]. \end{aligned}$$

⁶The two other, far more obvious, contributors to expected utility are starting wealth and the risk-free interest rate.

Proposition 2: *Paradoxically, a more risk averse investor gains higher certainty equivalent from the existence of the risky asset than a less risk averse investor, and increased risk adds more to the certainty equivalent of a more risk averse investor.*

Proof requires only inspection of the certainty equivalent of the combined investment portfolio of risky and risk-free assets, which as seen from (5) is

$$CE = W_0 R_f + \frac{c}{2} \text{var}(V).$$

Note how the certainty equivalent increases linearly with the risky asset's payoff variance, and also with the investor's risk aversion c .

Thus, a more risk averse buyer of random payoff V perceives greater ex ante dollar "value added" obtainable from any given level of risk, $\text{var}(V)$.

That result reconciles clearly with the fact that a risk-neutral ($c = 0$) investor is unassisted by there being either high or low payoff variance. Risk can be added but the risk-neutral certainty equivalent of the portfolio of risky and risk-free assets is always equal to the same amount $W_0 R_f$, the same as if the investor puts all initial wealth in the risk-free asset.

Paradoxically, it is the more risk averse investor who foresees most to gain from the existence of a more risky asset (or stock market). That seems remarkable given that more risk seems to be anathema to more highly risk averse investors.

Numerical Example

The following numerical illustration calculates expected utility directly by integrating over a normal probability density, rather than by using the mean-variance expression of expected exponential utility.

The normally distributed risky asset payoff is V and its ex ante parameters are $E[V]$ and $\text{var}(V)$. Its probability density is then

$$f(V) = \frac{1}{\sqrt{2\pi \text{var}(V)}} \exp\left(-\frac{(V - E[V])^2}{2 \text{var}(V)}\right).$$

The utility realized ex post from the overall optimally balanced portfolio of the risky and risk-free assets is

$$U(V) = 1 - \exp\left[-c(V + (W_0 - P)R_f)\right],$$

where as above $P = \frac{1}{R_f} (E[V] - c \text{var}[V])$.

Hence, the expected utility from the overall investment is

$$\begin{aligned} EU &= \int_{-\infty}^{\infty} U(V) f(V) dV \\ &= 1 - \exp \left[-c \left(W_0 R_f + \frac{c}{2} \text{var}(V) \right) \right], \end{aligned}$$

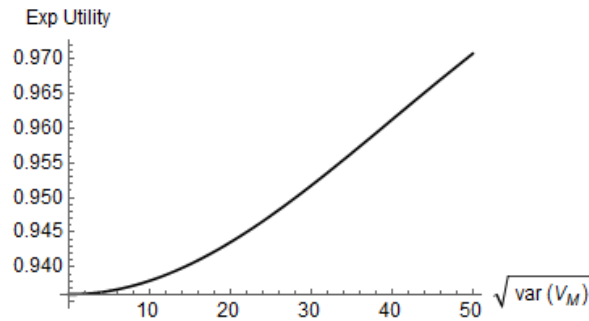
which is the same as (5) above. Again, the risky asset adds to expected utility as long as $\text{var}(V) > 0$.⁷

Expected utility is plotted in Figure 1 against the standard deviation of V , shown in the plot as $\sqrt{\text{var}(V)}$.

The calculations assume $W_0 = 100$, $R_f = 1.1$, and $c = 0.025$. There is no need to assume any particular value for the mean of payoff V , as $E[V]$ makes no difference. As shown by (5), the mean of V is already netted out, or compensated for, within P .

Note that for the case of $\text{var}(V) = 0$, the investment of W_0 is entirely risk-free, leaving its expected utility at $1 - \exp[-c W_0 R_f] = 1 - \exp[-0.025(100)(1.10)] = 0.936$, as shown in Figure 1 at $\sqrt{\text{var}(V)} = 0$.

Figure 1
Plot of EU Obtained from Optimal Portfolio of Risky
and Risk-Free Assets as a Function of $\sqrt{\text{var}(V_M)}$



The main point is that higher perceived ex ante risk makes for a higher expected utility investment portfolio, provided that investment is understood as a combined portfolio of the risky asset and the risk-free asset. Expected utility approaches one (satiation) asymptotically as the risk (variance) of the market payoff gets larger.

From the investor's overall welfare perspective, the relevant ex ante measure or objective function is the expected utility obtainable from the combined investment in risky and risk-free assets. The risky payoff's mean and variance affect its price P , and hence determine the amount $(W_0 - P)$ left for investing in the risk-free asset. That is why the two parts of the investment portfolio must be considered as a whole.

⁷So as to be realistic, $\text{var}(V)$ is constrained relative to $E[V]$ only to ensure that the price P of the asset, calculated according to (1), is positive.

3.2.2 Proof Under Quadratic Utility

Assume quadratic utility for wealth, $U(W) = W - \frac{b}{2}W^2$ ($b > 0$), noting that marginal utility $U'(W) = 1 - bW$ is positive for $W < 1/b$ (this is the domain in which utility is increasing in W). The investor with this utility function, and initial wealth W_0 , prices the risky asset by the equilibrium condition:⁸

$$P = \frac{1}{R_f} [E[V] - g \text{var}(V)], \quad (6)$$

where

$$g = \frac{b}{1 - b[E[V] + (W_0 - P)R_f]}. \quad (7)$$

Substituting μ from (2), g becomes

$$g = \frac{b}{1 - b\mu}.$$

Note that under quadratic utility, unlike exponential (CARA) utility, the aggregate investor's initial wealth W_0 affects the rational asset price P , and thus affects how much (as a proportion) of W_0 is held in the risky asset.

Solving (6) and (7) for P gives⁹

$$P = \frac{1}{R_f} \left[E[V] - \frac{(1 - bW_0R_f) - \sqrt{(1 - bW_0R_f)^2 - 4b^2 \text{var}(V)}}{2b} \right]. \quad (8)$$

Hence, from (2), the mean terminal wealth is

$$\begin{aligned} \mu &= E[V] + \left(W_0 - \frac{1}{R_f} \left[E[V] - \frac{(1 - bW_0R_f) - \sqrt{(1 - bW_0R_f)^2 - 4b^2 \text{var}(V)}}{2b} \right] \right) R_f \\ &= \frac{1 + bW_0R_f - \sqrt{(1 - bW_0R_f)^2 - 4b^2 \text{var}(V)}}{2b}. \end{aligned}$$

With quadratic utility $U(W) = W - \frac{b}{2}W^2$, the expected utility $E[U(W_1)]$ of terminal wealth $W_1 \sim N(\mu, \sigma^2)$ can be written as

$$EU = \mu - \frac{b}{2} [\sigma^2 + \mu^2]. \quad (9)$$

Substituting in (9) for μ , and noting $\sigma^2 = \text{var}(V)$ as above, gives

⁸See Johnstone (2016) for derivation of this price, by the criterion of maximizing expected utility.

⁹This is one of two solutions to a quadratic equation, but the other is economically implausible because the asset price P is increasing in the payoff variance.

$$EU = \frac{1 + 2bW_0R_f - (bW_0R_f)^2 - (1 - bW_0R_f) \sqrt{(1 - bW_0R_f)^2 - 4b^2 \text{var}(V)}}{4b}. \quad (10)$$

Again, the expected utility from an optimally weighted portfolio of the risky and risk-free asset depends only on $\text{var}(V)$, and increases with $\text{var}(V)$, since $(1 - bW_0R_f) > 0$.

Numerical Example

Consider a simple binary security that pays $V = 1$ with probability p and $V = 0$ with probability $(1 - p)$. Binaries have the conceptual advantage that their moments are all determined by a single parameter, p . For mean-variance asset pricing we need only mean $E[V] = p$ and variance $\text{var}(V) = p(1 - p)$.

The following calculations show how the investor's (buyer's) expected utility responds to p . The asset price from (8) is

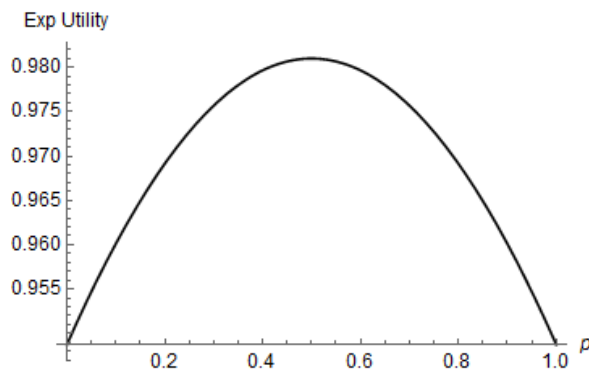
$$P = \frac{1}{R_f} \left[p - \frac{(1 - bW_0R_f) - \sqrt{(1 - bW_0R_f)^2 - 4b^2p(1 - p)}}{2b} \right], \quad (11)$$

and from (10) the expected utility is

$$EU = \frac{1 + 2bW_0R_f - (bW_0R_f)^2 - (1 - bW_0R_f) \sqrt{(1 - bW_0R_f)^2 - 4b^2p(1 - p)}}{4b}. \quad (12)$$

Figure 2 plots expected utility (12) against p , letting $W_0 = 1$, $R_f = 1.1$ and $b = 0.25$.

Figure 2
Plot of EU Obtained from Optimal Portfolio of Risky and Risk-Free Assets as a Function of p



Note how the investor anticipates the same expected utility from the "lottery" V when $p = 0.05$ as when $p = 0.95$, provided as above that V is priced at its CAPM

price (11) according to p . That follows from the mathematical facts that (i) the mean payoff has no effect on overall expected utility, and (ii) the payoff variance, which does effect expected utility, is symmetric around $p = 0.5$.

Figure 2 shows expected utility to be similarly symmetric around $p = 0.5$, which follows from the fact made explicit in (12) that it is driven only by $p(1 - p)$, which is the payoff variance. Note that the term p appears in (12) only in the payoff variance $p(1 - p)$, so the mean payoff p has no effect on expected utility of itself.

Note also that certainty, $p = 0$ or $p = 1$, is no help to the investor. In either case, her ex ante expected utility is equal only to the utility of current wealth invested at the risk-free rate; i.e. $U(W) = 1.1 - \frac{b}{2}(1.1)^2 = 0.94875$. Maximum expected utility occurs at $p = 0.5$, where the risky asset has maximum variance.

Interestingly, therefore, if the investor is faced with a CAPM-priced market that is operated on the random outcome of drawing red or black from an urn, and that Bernoulli variable has fixed probability parameter p , she would prefer a market with $p = 0.5$ (i.e. a coin toss).

That market would offer maximum ex ante payoff risk but also maximum ex ante expected utility. This result follows easily from CAPM but does not seem to be well known in CAPM discussion. Some level of risk is obviously needed, but it is not at all obvious a priori that the optimal risk is maximum risk.

3.3 Intuitive Explanation

That the investor prefers the riskiest possible market goes somewhat against the grain, and seems to clash with some basic precepts in financial economics. In Hadar and Russell (1969), we see a statement that looks superficially at least like the exact opposite:

It is well known that if the utility function is quadratic, then the decision maker will always prefer the prospect with the lower variance (if he is a risk averter) given that the prospects under consideration have identical means, or prefer the prospect with the higher mean so long as the variances are equal.¹⁰ (Hadar and Russell 1969, p.25)

The difference is that my investor must first pay amount P given by (1) to buy the risky asset at its market clearing CAPM price, and the lower price of a higher variance $\text{var}(V)$ asset gives it a "value for money" advantage to a potential investor, manifested in higher expected utility of terminal wealth $E[U(W_1)]$, regardless of its mean $E[V]$.

Once it is recognized that under CAPM the prospective investor prefers a more risky asset, the question begging is "why?" - where does the higher mean-variance certainty equivalent under higher $\text{var}(V)$ come from?

¹⁰That is the essential axiom of mean-variance choice; see Liu (2004) and Johnstone and Lindley (2013) for philosophical explanation of primitive risky asset choice under mean-variance.

The answer seems technically clear by the simple mathematics in the proof under exponential utility. First, note from (3) that the mean total cash flow μ from the combined risky and risk-free portfolio increases in $\text{var}(V)$ at rate c . That seems wrong at first, but happens because the risk-free amount $(W_0 - P)$ earning R_f is increasing in $\text{var}(V)$, which in turn is because P is decreasing in $\text{var}(V)$. The CARA utility certainty equivalent of the mean variance pair (μ, σ^2) representing W_1 is $\mu - \frac{c}{2}\sigma^2$. We know that its first term μ is increasing in $\text{var}(V)$ at rate c whereas the second term in the certainty equivalent shows a decrease under higher $\text{var}(V)$ at a rate of only $c/2$. The net change is thus an increase at rate $c/2$. Intuitively, therefore, the extra cash in the bank, when $\text{var}(V)$ is higher, more than makes up for the extra risk (higher σ^2) surrounding V and W_1 .

As already mentioned, this result is merely an economic "fact of life" under the assumed utility function, it is merely what the exponential utility investor implicitly demands by agreeing to pay the asset price P , even if she does not know that consciously. She can obtain her consumer surplus for the reason that the unit market price P payable for the risky asset is fixed, and hence each of the fractions of units up to the very last produce positive (albeit decreasing) marginal utility.

Note of course that investors do not have control over the economic source of their certainty equivalent, namely $\text{var}(V)$. Although $\text{var}(V)$ is a subjective assessment, it reflects the asset's natural cash flow variability, and is therefore often more controlled by nature or by the originator of the asset.¹¹ Moreover, if the asset is the whole market, $\text{var}(V)$ is affected by the cash flow variances and covariances of the different firms' business activities.

4 Parameter Risk Can Add to Expected Utility

The counter-intuitive conclusion in the analysis above is that, for any given W_0 and R_f , higher $\text{var}(V)$ offers a potential investor higher ex ante expected utility from a rationally weighted portfolio of the risky and risk-free assets. Remarkably, therefore, higher *parameter risk* can, by adding to payoff risk, have the curious effect of increasing ex ante expected utility.

Proposition 3: *A risky asset with higher parameter risk will commonly offer higher ex ante expected utility to the rationally weighted investor.*

Proof of the potential "attraction" or more parameter risk is as follows. A standard argument in the Bayesian finance literature on parameter risk uses tractable models to show how parameter risk can add to perceived payoff risk, $\text{var}(V)$. The best known example calls on a Bayesian model of uncertainty about an unknown

¹¹The micro-economic question arises as to whether potential investors in the stock market can benefit by firms adopting business activities and strategies that have innately higher payoff risk.

payoff mean (Kalymon 1971). The risky future payoff $V \sim N(\theta, v^2)$ is taken to have a normal distribution with unknown (i.e. uncertain) mean θ and known v^2 .

To incorporate parameter risk, the investor introduces a probability distribution over the unknown mean, $\theta \sim N(\eta, \varphi^2)$, with specified prior values η and φ^2 . The amount of parameter risk, or uncertainty about parameter θ , is reflected by φ^2 . By incorporating this uncertainty, the parameters of the predictive distribution of payoff V are

$$E[V] = E[E[V|\theta]] = E[\theta] = \eta,$$

and, by the *law of total variance*,

$$\begin{aligned} \text{var}(V) &= E[\text{var}(V|\theta)] + \text{var}([E[V|\theta]]) \\ &= v^2 + \text{var}(\theta) \\ &= v^2 + \varphi^2. \end{aligned} \tag{13}$$

Equation (13) for the variance of cash payoff V recognizes that there are two sources of uncertainty affecting our overall certainty about V . First, even if the investor knew parameter θ for certain, V still has variance v^2 , because the model holds that $V \sim N(\theta, v^2)$ with known v^2 , independent of θ . Second, despite having prior beliefs $\theta \sim N(\eta, \varphi^2)$ about the unknown parameter θ , the investor does not know θ exactly. Rather, the investor's uncertainty about parameter θ is represented by φ^2 .

Conveniently, the joint effect of these two uncertainties is captured mathematically by the sum of the two variances.¹² Thus, the predictive variance of payoff V , meaning the ex ante perceived variance of the coming random payoff V , is $\text{var}(V) = v^2 + \varphi^2$.

It follows directly from (13) that the investor's assessed $\text{var}(V)$, and consequent ex ante expected utility, is higher under higher parameter uncertainty φ^2 . Hence, parameter risk adds to the risk surrounding payoff V , in the same way as if there were some apparent extra natural variation in the economic fundamentals affecting V (uncertainties are fungible). Higher parameter risk leads the market to lower P , but, at the same time, higher expected utility.

4.1 When Low Parameter Risk Implies High Payoff Risk

Common Bayesian models show how lower parameter risk can imply lower payoff risk. The neatest illustration (see above) is the Bayesian model of a normally distributed payoff $V \sim N(\theta, v^2)$, where usually it is assumed (for tractability) that the variance v^2 is known and the mean θ is unknown.

In that model, more information (larger n) always brings a tighter posterior payoff distribution $f(V|\cdot)$. But another form of the same model could equally assume a

¹²For explanation of these two sources of variance see Winkler (2003, p.181), who explains that the predictive variance contains both uncertainty about θ and uncertainty about V given θ .

known mean θ and unknown variance v^2 . When v^2 is taken as the unknown parameter, Bayesian inference (e.g. Winkler 2003) starts with a prior distribution over v^2 , and shows how sample information that exhibits high in-sample variability (a high observed sample variance) can often lead the investor to a posterior distribution over v^2 that is tighter than its prior distribution but which also happens to lie well to the right of the prior distribution. In other words, rational learning can leave the market with an estimate of the unknown payoff variance v^2 that is (i) more precise, and (ii) *higher* than the prior estimate, in which case reduced parameter uncertainty brings greater payoff uncertainty.

The most realistic Bayesian model of a normally distributed payoff $V \sim N(\theta, v^2)$ allows *both* the mean payoff and the payoff variance to be unknown parameters. Unfortunately that model requires numerical methods for solution. The solution appears in a simulated predictive distribution for V , which can reveal higher variance in the future payoff V than was anticipated before receiving even the most highly precise sample data. If the population variance is high, a sufficiently large sample (i.e. sufficiently better information) will reveal that high variability.

In most formal statistical models, the analytical task of assimilating parameter risk into the assessment of payoff risk is straightforward, but requires the Bayesian notion of a predictive distribution. A predictive distribution is really no more than an "average distribution", averaged (or simulated) over all possible parameter values, with parameter values weighted by their posterior probabilities.

In effect, that averaging process is a way of saying that we do not know the parameter values for certain and we want to hedge against that risk rather than ignore it. The conventional alternative is to plug in point estimates of the unknown parameters and act as if they are the known true parameters. The resulting portfolio ignores the possibility that other plausible values of the unknown parameters are quite different to the ones plugged in.¹³

4.2 Payoff Risk Can Be Unaffected By Parameter Risk

To further show how payoff risk does not necessarily, by any probability law, simply increase with higher parameter risk, consider the case of the "urn market" where the risky asset produces $V = 1$ with probability θ and $V = 0$ otherwise.

The investor is uncertain about the value of urn parameter θ and expresses that uncertainty by specifying a subjective probability distribution $f(\theta)$ over θ . Both the

¹³A full Bayesian treatment (e.g. Gelman and Shalizi, 2012) suggests that predictive distribution should be averaged not only over possible parameter values, but over possible models. Just as we cannot simply plug in a known true parameter value, we cannot plug in a known true model. Thus, a fully Bayesian portfolio theory allows the market risk premium to include a premium for "model risk", as well as parameter risk. Note however that these risks are "fungible" and are all assimilated in the predictive distribution $f(V)$, so they are theoretically all wrapped together in beta and the CAPM's market risk premium.

mean and variance of payoff V are affected only by the ex ante expected value of θ , $E[\theta]$, and not by how uncertain the investor is about θ , as indicated by $\text{var}(\theta)$.

Specifically,

$$E[V] = E[E[V|\theta]] = E[\theta]$$

and by the law of total variance

$$\begin{aligned} \text{var}(V) &= E[\text{var}(V|\theta)] + \text{var}(E[V|\theta]) \\ &= E[\theta(1-\theta)|\theta] + (E[\theta^2] - E[\theta]^2) \\ &= E[\theta] - E[\theta^2] + E[\theta^2] - E[\theta]^2 \\ &= E[\theta](1 - E[\theta]). \end{aligned}$$

Remarkably, therefore, the perceived risk surrounding payoff V is the same for any given $E[\theta]$ no matter how diffuse (i.e. uncertain) the investor's beliefs over θ . In this market, the CAPM asset price and the investor's expected utility are the same when $f(\theta)$ is (i) a spike at $\theta = 0.5$ (complete parameter certainty) or (ii) uniform over $(0,1)$ (i.e. complete ignorance or indifference about θ).

Thus, in a CAPM market for the urn payoff V , there is no risk premium for the uncertainty $\text{var}(\theta)$ about the urn's parameter θ . There is a risk premium, of course, for payoff uncertainty, $\text{var}(V)$.

5 New Investors Want More Information

If the investor foresees higher ex ante expected utility in a higher $\text{var}(V)$ asset, why would she also desire any more or better information that might reduce $\text{var}(V)$?

The answer is that she is bound by the same economic rationale by which she made that calculation - i.e. the objective is to maximize expected utility. To maximize expected utility, she must draw any information which has non zero probability of changing her beliefs.¹⁴ Any signal or available information with that possibility has positive marginal expected utility, and therefore, if costless, will be drawn. When seen through whatever new beliefs ensue, a different investment portfolio (a revised P) will always have higher expected utility.

That insight is hardly original. The more surprising insight is that the prospective investor does not draw information so as to "resolve" uncertainty, instead she draws information to "assess" uncertainty or expose it more accurately, fully aware that in a general Bayesian model her assessment of $\text{var}(V)$ might increase with more information.¹⁵

¹⁴This rule holds for any formally "proper" utility function (Bernardo and Smith 1994), which means any utility function by which the decision maker's expected utility is maximized only by acting on her "honest" probability beliefs, rather than any others.

¹⁵The meta-statistical interpretation is that by the law of large numbers, a probability distribution based on more information has a higher probability of "better resembling" the corresponding empirical outcome distribution.

That natural possibility whereby investors observe new reasons for uncertainty, or lose confidence in something they earlier believed, is easy to grasp intuitively and is formalized by an important general probability law, see (14) below (Gelman, Carlin, Stern and Rubin, 2004).

Suppose that the information on offer is x (it could be an analyst's forecast say). The completely general, distribution-free, law of uncertainty in probability theory is the law of total variance (14). Without knowing which value of signal x will arise, the prior (unconditional) variance of V is given by the total of (i) the expected conditional variance, $E[\text{var}(V|x)]$, and (ii) the variance of conditional means $\text{var}(E[V|x])$, that is

$$\text{var}(V) = E[\text{var}(V|x)] + \text{var}(E[V|x]). \quad (14)$$

It follows therefore that $E[\text{var}(V|x)] < \text{var}(V)$, which implies that investors "expect" ex ante (before knowing x) that their perceptions of the variance of V will be lower after (or, if) x becomes known. It is not implied, however, that $\text{var}(V|x) < \text{var}(V)$ for all possible observed x , so there is no guarantee of more certainty. To the contrary, it can be that for a subset of possible x , the revised variance $\text{var}(V|x)$ can be much higher than the prior variance $\text{var}(V)$.¹⁶ That happens when the CAPM investor is "lucky", in the sense that the new uncertainty about V is greater and thus the asset is now seen as riskier and the expected utility from buying it - at its now revised CAPM price - is higher.

In summary, the economically rational expected utility seeking investor always wants more information, but in a CAPM framework she is best understood as seeking to "reveal" or "measure" uncertainty rather than always "resolve" it.

That accounting information objective - i.e. facilitating a more informed probability distribution over V - calls for accounting information to show when the firm's cash prospects are "clearly uncertain", in the same way that a stock analyst should say that things about the firm look worrying when in fact they do. As a credo for accounting information, the objective of measuring rather than resolving uncertainty sits well with traditional notions of "value relevance", "decision relevance" and any method of "fundamental analysis" that suggests on good evidence that the firm's future is unclear (or, alternatively, that it is quite certainly looking good, or quite certainly looking bad).

5.1 Ex Post Result of Inadequate Information

If ex ante probability distributions do not match ex post frequencies, the investor's realized utility will *on average* (not always) be lower than expected ex ante. That danger is known in advance, and is one part of why relevant new information always has positive expected utility (the other part is the potential upside that, rather than

¹⁶This law and its equivalent for covariance are essential in any statistical information science. See Johnstone (2015, 2016, 2018) for extensive discussions. See also the recent paper of Dye and Hughes (2018) which is built on the law of total variance.

merely avoiding losses, better information can strengthen beliefs so much that the investor makes a bigger bet and averages a bigger profit (in CAPM, a bigger bet means a higher P and less cash left in the bank).

The following simple illustration shows how a rationally priced market can lose money if priced on deficient information. Consider an urn market on risky event $V \in \{0, 1\}$ priced on the presumption that $p(V = 1) = p = 0.5$. Taking $W_0 = 1$, $R_f = 1.1$ and $b = 0.25$ for example, $E[V] = p = 0.5$ and $\text{var}(V) = p(1 - p) = 0.25$, the market price under risk averse quadratic utility and belief p is from (11)

$$\begin{aligned} P|_{p=0.5} &= \frac{1}{R_f} \left[0.5 - \frac{(1 - bW_0R_f) - \sqrt{(1 - bW_0R_f)^2 - 4b^2 \cdot 0.5(1 - 0.5)}}{2b} \right] \\ &= 0.3737. \end{aligned}$$

It follows therefore that the aggregate investor allocates 37.37% of initial endowment $W_0 = 1$ to the risky market, leaving a residual amount $(W_0 - P|_{p=0.5}) = 0.6263$ of W_0 in the risk-free asset.

But if the "true" exogenous parameter p is $p^* \neq 0.5$,¹⁷ then investors' true expected ex post terminal wealth given the wrong market assessment $p = 0.5$ is

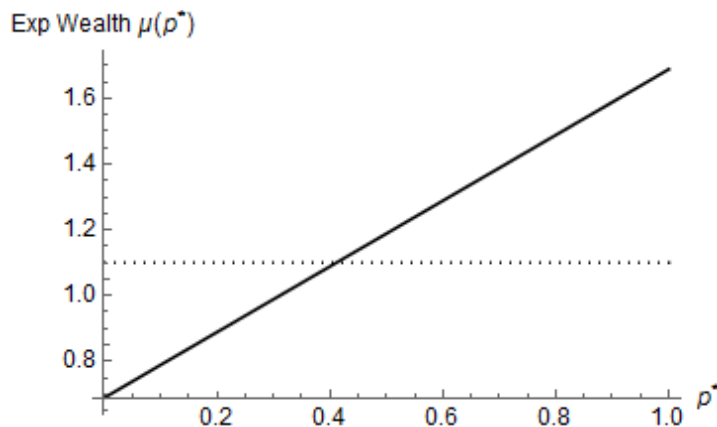
$$\mu(p^*) = p^* + (W_0 - P|_{p=0.5})R_f,$$

and true variance of terminal wealth is

$$\sigma^2(p^*) = \text{var}(V|p^*) = p^*(1 - p^*).$$

Figure 3 shows the market's expected terminal wealth $\mu(p^*)$ for the domain of all possible $p^* \in (0, 1)$.

Figure 3
Plot of Investor's Expected Ex Post Terminal Wealth
 $\mu(p^*) \equiv E[W_1]$ Given "True" Probability p^*



¹⁷This p can be treated as either a "physical" probability or as the relative frequency of $V = 1$ in an empirical reference set of "repeated trials".

The dotted line represents expected terminal wealth when all initial wealth is invested at the risk-free rate. The cutoff true probability is $p^* = 0.41$, which means that the market that judged $p = 0.5$ and priced V accordingly will on average be left with less than the risk-free rate of growth ($r_f = 0.10$) whenever the true probability p^* is less than 0.41.

The other possibility is that the market's assessment of p is wrong but is "lucky" in the sense that it is not "too far wrong" to bring expected loss. When priced on the market judgement $p = 0.5$ the aggregate investor will on average earn a rate of return greater than the risk-free rate for any true $p^* > 0.41$. That is part of the statistical mechanics understood within De Long, Shleifer, Summers, and Waldmann (1991) by which ill-informed investors can be luckily profitable and appear to be sufficiently well informed, astute or even "expert".

The same applies to the whole market. The market consensus belief can be wrong but produce if lucky a high cash payoff to investors. Note in the graph how well the market would have done on average by pricing on the mistaken basis of $p = 0.5$ when p^* is say 0.8 or more. That would be a case of under-betting, brought by under-confidence in $V = 1$, and reflects what could have been gained with better information (i.e. information that prompted an estimate of p nearer p^* and hence a much bigger bet).¹⁸

A rational investor will do best ex post on average, in terms of realized capital growth, when her ex ante probability beliefs best match the actual empirical frequency distribution of the future risky payoff.¹⁹ Investors benefit financially from more accurate probabilities, regardless of whether they imply more certainty or less certainty. That is the finding recurrent throughout the statistical "economic Darwinism" literature (e.g. Alchain 1950, Sandroni 2000, Blume and Easley 2006).

¹⁸Interestingly, the most direct and mathematically sophisticated theory on "making money" in statistical betting contexts is found in management science (see the cited papers by Maclean et al.), and emphasizes the benefits of a conservative approach (because the losses from over-confident over-betting hurt the bettor's growth rate far more than the effects of under-confident underbetting). There is an unexplored parallel here with the motivations for accounting conservatism (in the sense of a reluctance to parade profits and apparent reasons for optimism too early).

¹⁹There is no one measure of the "divergence" between probabilities and empirical frequencies. Different probability "scoring rules" are used in forecasting disciplines to measure the disparity between ex ante probabilities and ex post frequencies. These rules often written as functions capturing the divergence between two sets of probabilities (see Jose, Nau and Winkler 2008; Johnstone 2011; Jose, Johnstone and Winkler 2011). Investors with different utility functions benefit from different "types" of divergence between their ex ante probabilities and the observed frequencies of the same events. Common measures of divergence like mean-squared error (the "Brier probability score") are practical responses to the need to evaluate ex ante probabilities ex post.

6 Existing Investors Want More Information

My analysis so far takes the viewpoint of an investor considering whether to allocate wealth to a risky asset market, and finding under CAPM that her expected utility of terminal total wealth depends only on the market's payoff variance (and on the fixed risk-free rate).

Once having been attracted to the market by the ex ante variance of the risky asset, existing investors have a much more obvious way of thinking - their expected utility is increased by news that adds to either the mean payoff $E[V]$ of the risky asset or reduces its perceived payoff variance $\text{var}(V)$, since either revision brings an increase in the market price of the risky asset that they hold, while at the same time the money amount they hold in the risk-free asset is unchanged.

That "windfall" or addition to the wealth (and thus utility of wealth) of an existing investor is essentially the gain that she anticipated (as enticingly probable) when she bought into the risky asset in the first place. Under a one-period CAPM, she was always going to end up with $\text{var}(V) = 0$ at period-end. Her hope was for increasing $E[V]$ over the period and ultimately a high realized payoff V at expiry.

The message therefore is that prospective or potential new investors prefer a higher variance market, but once they own a high variance asset they want its variance to shrink as quickly as possible - because they believe that more resolution will on average bring them a capital gain.

7 New and Existing Investors Agree

The best summary of what has been found is that current owners and prospective new investors both want more and better information - but they want it for quite different reasons.

Prospective investors want more information because current beliefs tell them that relevant new information would change their probability beliefs and warrant a different investment portfolio (the market would set a different P , and hence would rebalance its allocation wealth W_0 across the risky and risk-free assets).

Existing investors, who have already bought into the asset want validation of their implicitly favorable expectations, and the resulting new P (a price rise). They implicitly hold favorable expectations and look forward to those expectations being realized via new information. If they did not hold favorable expectations, they would not be holding the risky asset.²⁰

The relationship under CAPM between the realization of investment returns and the flow of information (resolution of uncertainty) was clarified in a celebrated early paper by Robichek and Myers (1966). They explained in their "two ships" example

²⁰This desire for quick realization of capital gains is not the only reason for wanting more information. The more usual explanation is that earlier information allows earlier revision (re-optimization) of the investor's portfolio.

that without immediate resolution the return on investment accrues over time at the risk-free rate R_f (the reward for waiting). If there is no resolution before expiry, all return due to risk, rather than merely waiting, whether high or low, positive or negative, arises instantaneously at the moment of expiry. These same information-return patterns are incorporated in Christensen et al. (2010) and Johnstone (2016).

8 Conclusion

My analysis depicts risk as opportunity, and reveals how an exogenously more risky or subjectively less predictable asset payoff, when priced by CAPM, allows investors to take up an ex ante higher expected utility portfolio of the risky and risk-free assets. The riskier market, with its innately higher cost of capital, offers the better ex ante investment opportunity. Specifically, the investor's rational portfolio, which combines the risky asset - obtained at its CAPM price - along with all other wealth held in the risk-free asset, offers higher ex ante expected utility of terminal wealth when the risky asset is more risky (i.e. has higher payoff variance). Prospective investors do not therefore abhor payoff risk, they want more risk.

Given that reduced payoff risk is not an attraction to potential investors in a CAPM market, the economic benefit of obtaining better information cannot simply be "to resolve risk". Rather, the objective of better accounting information in the eyes of potential new investors is not to reduce an asset's risk but to price its risk more accurately, via more accurate ex ante probability assessment (i.e. more accurate assessment of the risky payoff's relevant parameters, typically mean and variance). Thus, while investors in a CAPM market prefer greater *payoff* risk, they do abhor *parameter* risk, or the risk that they have assessed the probability distribution of the risky payoff incorrectly. This understanding of investors' information requirements fits well with the notion of "price discovery" in finance, where information is meant to reveal rather than necessarily always reduce risk.

Investors looking at investing in a CAPM market would ideally like the market to learn via very good information that the asset's payoff variance is very certainly high (statistically akin to when a very large sample from a population of unknown variance shows it to have high variance). That information would bring a lower risky asset price, and a higher expected utility investment opportunity. Paradoxically, current owners of the same asset want the same quality of information, because in their eyes better information will reveal more fully that their implicitly favorable expectations are correct (and they will be rewarded with a price rise). So both potential and existing investors have exactly the same desire for the best information possible, they agree unconditionally on that, but with fundamentally different motivations.

Appendix: Quick Exponential Utility CAPM

The market contains joint normal risky assets $j = 1, 2, 3, \dots, n$ with respective time $t = 1$ payoffs V_j and time $t = 0$ prices P_j (these prices are what we find).

The aggregate investor has time $t = 0$ wealth W_0 and exponential utility function $U(x) = 1 - \exp(-cx)$. By spending fraction ρ_j of wealth on each asset j , and putting the rest $(1 - \sum_j \rho_j)$ of W_0 in risk-free cash, time $t = 1$ payoff is $W_0 (\sum (\rho_j V_j / P_j) + (1 - \sum \rho_j) R_f)$, with R_f representing the risk-free interest factor.

Mean wealth W_1 at $t = 1$ is

$$\mu(\boldsymbol{\rho}) = \mu(\rho_1, \rho_2, \dots, \rho_n) = W_0 \left(\sum (\rho_j E[V_j] / P_j) + (1 - \sum \rho_j) R_f \right)$$

and the variance of W_1 is

$$\sigma^2(\boldsymbol{\rho}) = \sigma^2(\rho_1, \rho_2, \dots, \rho_n) = (W_0)^2 \left(\sum_j \sum_k \rho_j \rho_k \text{cov}(V_j, V_k) / P_j P_k \right).$$

With joint normal payoffs, expected utility is

$$EU(\boldsymbol{\rho}) = 1 - \exp \left[\mu(\boldsymbol{\rho}) - \frac{c}{2} \sigma^2(\boldsymbol{\rho}) \right].$$

and hence the certainty equivalent of W_1 is

$$CE(\boldsymbol{\rho}) = \mu(\boldsymbol{\rho}) - \frac{c}{2} \sigma^2(\boldsymbol{\rho}). \tag{15}$$

Substituting for $\mu(\boldsymbol{\rho})$ and $\sigma^2(\boldsymbol{\rho})$ in (15) and differentiating with respect to ρ_j , gives a first order condition for optimal ρ_j

$$\left(\frac{E[V_j]}{P_j} - R_f \right) - cW \sum_{k=1}^n \rho_k \frac{\text{cov}(V_j, V_k)}{P_j P_k} = 0. \tag{16}$$

Condition (16) defines optimal (maximum EU) weights ρ_j ($j = 1, 2, \dots$) given arbitrary asset prices P_1, P_2, \dots . Equilibrium prices require

$$\rho_j = P_j / W \quad \text{for all } j = 1, 2, \dots, n. \tag{17}$$

By solving (16)-(17) to eliminate ρ_1, ρ_2, \dots , the implied price of any asset j is

$$P_j = \frac{E[V_j] - c \text{cov}(V_j, V)}{R_f}$$

where $V = (V_1 + V_2 + \dots + V_n)$ is the aggregate market payoff. Since $\text{var}(V) = \sum \text{cov}[V_j, V]$, it follows that the single asset (aggregate market payoff) has price

$$P = \frac{E[V] - c \text{var}(V)}{R_f}.$$

References

- [1] Alchain A. (1950) Uncertainty, evolution and economic theory. *Journal of Political Economy*. 58: 211–221.
- [2] Avramov, D. and Zhou, G. (2010) Bayesian portfolio analysis. *Annual Review of Financial Economics*. 2: 25-47.
- [3] Barry, C.B. (1978) Effects of uncertain and nonstationary parameters upon capital market equilibrium. *Journal of Financial and Quantitative Analysis*. 13: 419-433.
- [4] Barry, C.B. and Brown, S.J. (1985) Differential information and security market equilibrium. *Journal of Financial and Quantitative Analysis*. 20: 4-7-422.
- [5] Barry, C.B. and Winkler, R.L. (1976) Nonstationarity and portfolio choice. *Journal of Financial and Quantitative Analysis*. 11: 217–235.
- [6] Bawa V.S., Brown S.J. and Klein R.W. (1979) *Estimation Risk and Optimal Portfolio Choice*. Amsterdam: North-Holland.
- [7] Bernardo, J.M. and Smith, A.F.M. (1994) *Bayesian Theory*. New York: Wiley.
- [8] Blume, L. and Easley, D. (2006) If you're so smart, Why aren't you rich? Belief selection in complete and incomplete markets. *Econometrica*. 74: 929-966.
- [9] Botosan, C. (1997) Disclosure level and the cost of equity capital. *The Accounting Review*. 72: 323-349.
- [10] Brown, S.J. (1979) The effect of estimation risk on capital equilibrium. *Journal of Financial and Quantitative Analysis*. 14: 215-220.
- [11] Christensen, P.O., de la Rosa, L.E. and Feltham, G.A. (2010) Information and the cost of capital: An ex ante perspective. *Accounting Review*. 85: 817-848.
- [12] Coles, J.L. and Loewenstein, U. (1988) Equilibrium pricing and portfolio composition in the presence of uncertain parameters. *Journal of Financial Economics*. 22: 279-303.
- [13] Coles, J., Loewenstein, U. and Suay, J. (1995) On equilibrium pricing under parameter uncertainty. *Journal of Financial and Quantitative Analysis*. 30: 347-364.
- [14] Core, J., Hail, L. and Verdi, R.S. (2015) Mandatory disclosure quality, inside ownership, and cost of capital. *European Accounting Review*. 24: 1-29.

- [15] De Long, J.B., Shleifer, A., Summers, L.H. and Waldmann, R.J. (1991) The Survival of noise traders in financial markets. *The Journal of Business*. 64: 1-19.
- [16] Fama, E.F. 1976. *Foundations of Finance*. New York: Basic Books.
- [17] Fama, E.F. (1977) Risk adjusted discount rates and capital budgeting under uncertainty. *Journal of Financial Economics*. 5: 3-24.
- [18] Fama, E.F. and Miller, M. 1972. *The Theory of Finance*. Hinsdale, Illinois: Dryden Press.
- [19] Gelman, A., Carlin, J.B., Stern, H.S. and Rubin, D.B. (2004) *Bayesian Data Analysis*. 2nd ed. Boca Raton, FL.: CRC Press.
- [20] Gelman, A. and Shalizi, C.R. (2012) Philosophy and the practice of Bayesian statistics. *British Journal of Mathematical and Statistical Psychology*. 66: 8–38.
- [21] Hadar, J. and Russell, W. (1969) Rules for Ordering Uncertain Prospects. *American Economic Review*. 59: 25–34.
- [22] Johnstone, D.J. (2011) Economic interpretation of probabilities estimated by maximum likelihood or score. *Management Science*. 57: 308-314.
- [23] ——— (2015) Information and the cost of capital in a mean-variance efficient market. *Journal of Business Finance and Accounting*. 42: 79-100.
- [24] ——— (2016) The effect of information on uncertainty and the cost of capital. *Contemporary Accounting Research*. 33: 752-774.
- [25] Johnstone, D.J. Jose, V.R. and Winkler, R.L. (2011) Tailored scoring rules for probabilities. *Decision Analysis*. 8: 256–268.
- [26] Johnstone, D.J. and Lindley, D.V. (2013) Mean-variance and expected utility: The Borch paradox. *Statistical Science*. 28: 223–237.
- [27] Jose, V.R., Nau, R.F. and Winkler, R.L. (2008) Scoring rules, generalized entropy, and utility maximization. *Operations Research*. 56: 1146-1157.
- [28] Kadane, J.B. (2014) *Principles of Uncertainty*. Boca Raton, FL.: Chapman and Hall.
- [29] Kalymon, B.A. (1971) Estimation risk in the portfolio selection model. *Journal of Financial and Quantitative Analysis*. 6: 559-582.
- [30] Kelly, J.L. (1956) A new interpretation of information rate. *Bell System Technical Journal*. 35: 917–926.

- [31] Klein, R.W. and Bawa, V.S. (1976) The effect of estimation risk on optimal portfolio choice. *Journal of Financial Economics* 3: 215-231.
- [32] ——— (1977) The effect of limited information and estimation risk on optimal portfolio diversification. *Journal of Financial Economics* 5: 89-111.
- [33] Liu, L. (2004) A new foundation for the mean–variance analysis. *European Journal of Operational Research*. 158: 229–242.
- [34] Luenberger, D. (1998) *Investment Science*. New York: Oxford University Press.
- [35] MacLean, L.C., Sanegre, R., Zao, Y. and Ziemba, W.T. (2004) Capital growth with security. *Journal of Economic Dynamics and Control*. 28: 937-954.
- [36] MacLean, L.C., Ziemba, W.T. and Li, Y. (2005) Time to wealth goals in capital accumulation and the optimal trade-off of growth versus security. *Quantitative Finance*. 5: 343-357.
- [37] Nau, R. (2001) De Finetti was right: probability does not exist. *Theory and Decision*. 51: 89–124.
- [38] Robichek, A.A. and Myers, S.C. (1966) Conceptual problems in the use of risk-adjusted discount rates. *Journal of Finance*. 21: 727-30.
- [39] Rogers, J.L., Skinner, D.J. and Van Buskirk, A. (2009) Earnings guidance and market uncertainty. *Journal of Accounting and Economics*. 48: 90-109.
- [40] Sandroni, A. (2000) Do markets favor agents able to make accurate predictions? *Econometrica*. 68: 1303-1342.
- [41] Winkler, R.L. (2003) *An Introduction to Bayesian Inference and Decision*. 2nd ed. Sugar Land, Texas: Probabilistic Publishing.
- [42] Winkler, R. L. and Barry, C. B. (1975), A Bayesian model for portfolio selection and revision. *The Journal of Finance*. 30: 179-192.
- [43] Zellner, A. and Chetty, V. (1965) Prediction and decision problems in regression models from the Bayesian point of view. *Journal of the American Statistical Association*. 60: 608-616.