

# *P*-ADIC ANALYSIS AND MOCK MODULAR FORMS

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# Abstract

A *mock modular form*  $f^+$  is the holomorphic part of a harmonic Maass form  $f$ . The non-holomorphic part of  $f$  is a period integral of a cusp form  $g$ , which we call the *shadow* of  $f^+$ . The study of mock modular forms and mock theta functions is one of the most active areas in number theory with important works by Bringmann, Ono, Zagier, Zwegers, among many others. The theory has many wide-ranging applications: additive number theory, elliptic curves, mathematical physics, representation theory, and many others.

We consider arithmetic properties of mock modular forms in three different settings: zeros of a certain family of modular forms, coupling the Fourier coefficients of mock modular forms and their shadows, and critical values of modular  $L$ -functions.

For a prime  $p > 3$ , we consider  $j$ -zeros of a certain family of modular forms called Eisenstein series. When the weight of the Eisenstein series is  $p - 1$ , the  $j$ -zeros are  $j$ -invariants of elliptic curves with supersingular reduction modulo  $p$ . We lift these  $j$ -zeros to a  $p$ -adic field, and show that when the weights of two Eisenstein series are  $p$ -adically close, then there are  $j$ -zeros of both series that are  $p$ -adically close.

A direct method for relating the coefficients of shadows and mock modular forms is not known. This is considered to be among the first of Ono's Fundamental Problems for mock modular forms. The fact that a shadow can be cast by infinitely many mock

modular forms, and the expected transcendence of generic mock modular forms pose serious obstructions to this problem. We solve these problems when the shadow is an integer weight cusp form. Our solution is  $p$ -adic, and it relies on our definition of an algebraic *regularized mock modular form*.

We use *mock modular forms* to compute generating functions for the critical values of modular  $L$ -functions. To obtain this result we derive an Eichler-Shimura theory for weakly holomorphic modular forms and mock modular forms. This includes an “Eichler-Shimura isomorphism”, a “multiplicity two” Hecke theory, a correspondence between *mock modular periods* and classical periods, and a “Haberland-type” formula which expresses Petersson’s inner product and a related antisymmetric inner product on  $M_k^!$  in terms of periods.

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# Chapter 1

## Introduction

Throughout this text, we will explore the arithmetic properties of certain complex-valued meromorphic functions which live on the complex upper half plane  $\mathbb{H}$ . We will consider three different avenues: roots of a certain family of modular forms called Eisenstein series (Chapter 2), coupling the Fourier coefficients of mock modular forms and their shadows (Chapter 3), and critical values of certain modular  $L$ -functions (Chapter 4).

We begin by considering the fundamental elliptic function, i.e. the Weierstrass  $\wp$ -function. We describe its relation to elliptic curves and certain complex-valued functions called Eisenstein series which live on  $\mathbb{H}$ . The Eisenstein series actually give us the very first examples of mock modular forms. We provide basic definitions and results in this introductory chapter.

### 1.1 Elliptic functions and curves

Throughout this section, fix  $z \in \mathbb{H}$ , a point in the complex upper half plane. Then we may consider the lattice  $\Lambda_z = \mathbb{Z} + z\mathbb{Z}$ . Factoring the complex plane  $\mathbb{C}$  by this lattice and identifying opposite boundaries of the resulting fundamental parallelogram gives us a genus 1 Riemann surface, i.e. a torus. In this section we consider meromorphic functions on the quotient space  $\mathbb{C}/\Lambda_z$  and then briefly describe elliptic curves over fields other than  $\mathbb{C}$ .

### 1.1.1 Elliptic functions

**Definition 1.1.1.** A meromorphic function  $f(w) : \mathbb{C} \rightarrow \mathbb{C}$  is called an *elliptic function* with respect to the lattice  $\Lambda_z$  if  $f(w + \lambda) = f(w)$  for every  $\lambda \in \Lambda_z$ .

The simplest examples are holomorphic elliptic functions as the following proposition concludes.

**Proposition 1.1.2.** *A holomorphic elliptic function is everywhere constant.*

*Proof.* Apply Liouville's Theorem. □

The most important example of an elliptic function is the *Weierstrass  $\wp$ -function* which is defined as follows:

$$\wp(w) = \wp(w, z) := \frac{1}{w^2} + \sum_{\substack{\lambda \in \Lambda_z \\ \lambda \neq 0}} \left( \frac{1}{(w + \lambda)^2} - \frac{1}{\lambda^2} \right). \quad (1.1.1)$$

*Remark.* The sum in (1.1.1) converges absolutely and uniformly on compact subsets of  $\mathbb{C} - \Lambda_z$  (see [32]), and the only poles of  $\wp(w)$  are double poles at each lattice point. It is easy to see that  $\wp(w)$  is  $\Lambda_z$ -periodic.

The following proposition explains exactly why this is the most important elliptic function. It essentially gives birth to all others.

**Proposition 1.1.3** (see [32]). *The field of elliptic functions (with respect to a lattice  $\Lambda_z$ ) over  $\mathbb{C}$  is generated by  $\wp(w)$  and  $\wp'(w)$  (i.e. the derivative with respect to  $w$ ).*

### 1.1.2 A certain differential equation

We now quickly (see [32] for more details) show that  $\wp(w)$  satisfies the differential equation

$$\wp'(w)^2 = f(\wp(w)) \quad \text{where} \quad f(x) = 4x^3 - g_2x - g_3 \in \mathbb{C}[x]. \quad (1.1.2)$$

*Remark.* The well known quantities,  $g_2$  and  $g_3$ , which are associated with the lattice  $\Lambda_z$  are defined below.



We begin by considering the Laurent expansion of  $\wp(w)$  about the point  $w = 0$ :

$$\wp(w) = \frac{1}{w^2} + \sum_{\substack{k=2 \\ k \text{ even}}}^{\infty} (k+1)G_{k+2}w^k \quad (1.1.3)$$

where the coefficients in this expansion are defined as

$$G_k = G_k(z) := \sum_{\substack{\lambda \in \Lambda_z \\ \lambda \neq 0}} \lambda^{-k} = \sum_{\substack{m, n \in \mathbb{Z} \\ (m, n) \neq (0, 0)}} \frac{1}{(mz + n)^k}. \quad (1.1.4)$$

*Remark.* The series  $G_k$  are called *Eisenstein series* and are a two-dimensional generalization of the classical Riemann  $\zeta$ -function. They are defined in this way for integers  $k > 2$  and vanish for odd integers  $k$ . We will show in the next section that Eisenstein series are among the first examples of modular forms.

With the Laurent expansion (1.1.3) in mind, we find that

$$\begin{aligned} \wp'(w)^2 &= \frac{4}{w^6} - 24G_4\frac{1}{w^2} - 80G_6 + (36G_4^2 - 168G_8)w^2 + \dots; \\ \wp(w)^3 &= \frac{1}{w^6} + 9G_4\frac{1}{w^2} + 15G_6 + (21G_8 + 27G_4^2)w^2 + \dots; \\ \wp(w)^2 &= \frac{1}{w^4} + 6G_4 + 10G_6w^2 + \dots \end{aligned}$$

Define

$$g_2 := 60G_4 \quad \text{and} \quad g_3 := 140G_6. \quad (1.1.5)$$

Then the following linear combination

$$\wp'(w)^2 - 4\wp(w)^3 + g_2\wp(w) + g_3 \quad (1.1.6)$$

is a holomorphic elliptic function and is therefore constant. When evaluated at the point  $w = 0$ , the linear combination (1.1.6) is zero and therefore vanishes everywhere. This proves identity (1.1.2).

### 1.1.3 Elliptic curves

The differential equation (1.1.2) has a very nice geometric interpretation. We first introduce the following geometric object.

**Definition 1.1.4.** Let  $K$  be a field and  $\bar{K}$  an algebraic closure of  $K$ . Suppose that  $f(x) \in K[x]$  is a cubic polynomial with distinct roots. Then the locus of the equation

$$y^2 = f(x) \tag{1.1.7}$$

in  $\bar{K}$  together with a basepoint  $O$  at infinity is called an *elliptic curve* over  $K$ . We denote the elliptic curve by  $E(\bar{K}/K)$ .

*Remark.* It turns out that we actually consider an elliptic curve  $E(\bar{K}/K)$  to be a subset of  $\mathbb{P}_{\bar{K}}^2$  (two dimensional projective space over  $\bar{K}$ ), i.e. a solution  $x, y \in \bar{K}$  to equation (1.1.7) is written as  $(x, y, 1)$ , and the basepoint  $O$  at infinity is written as  $(0, 1, 0)$ .

Suppose that  $L \subset \bar{K}$  is a field extension of  $K$ . We may restrict an elliptic curve  $E(\bar{K}/K)$  to those points whose coordinates are in  $L$ . We call these the *L-points* of the elliptic curve, and denote this set as  $E(L/K)$ , i.e.

$$E(L/K) := E(\bar{K}/K) \cap \mathbb{P}_L^2.$$

When the field under consideration is the set of complex numbers, we have a geometric interpretation.

**Proposition 1.1.5** (see Silverman [47]). *Let  $f(x) = 4x^3 - g_2x - g_3 \in \mathbb{C}[x]$  where  $g_2$  and  $g_3$  are the quantities associated with the lattice  $\Lambda_z$ . Then  $f(x)$  has distinct roots and its discriminant*

$$\Delta(\Lambda_z) := g_2^3 - 27g_3^2$$

*is nonzero. In particular, the equation  $E : y^2 = f(x)$  is an elliptic curve over  $\mathbb{C}$ . Furthermore, the map  $\pi : \mathbb{C}/\Lambda_z \rightarrow E \subset \mathbb{P}_{\mathbb{C}}^2$  (two dimensional projective space over  $\mathbb{C}$ ) given by*

$$\pi(w) := (\wp(w), \wp'(w), 1)$$

*is an analytic isomorphism of Riemann surfaces.*

When the field  $K$  under consideration has characteristic neither 2 nor 3, we may use a variable substitution to rewrite equation (1.1.7) as

$$y^2 = 4x^3 - g_2x - g_3 \tag{1.1.8}$$

where the quantities  $g_2$  and  $g_3$  are defined as in (1.1.5). For the rest of this section, we consider this to be the case.

One of the most remarkable properties of an elliptic curve is the following result.

**Proposition 1.1.6** (see Silverman [47]). *The points of an elliptic curve  $E(L/K)$  form a group with identity  $O$ .*

*Remark.* When  $K = \mathbb{C}$ , this is a simple corollary of Proposition 1.1.5.

Two very important quantities associated with an elliptic curve  $E$  with equation (1.1.8) are the discriminant

$$\Delta_E := g_2^3 - 27g_3^2 \tag{1.1.9}$$

and the  $j$ -invariant

$$j_E := 1728 \frac{g_2^3}{\Delta_E}. \tag{1.1.10}$$

When we start with an arbitrary choice of  $g_2, g_3 \in K$ , the discriminant of the cubic equation (1.1.8) determines whether or not the equation determines an elliptic curve.

**Proposition 1.1.7** (see Proposition 1.4 in Silverman [47]). *Let  $E(L/K)$  be the  $L$ -points of the equation (1.1.8) with quantities  $g_2, g_3 \in K$ . Then  $E(L/K)$  is an elliptic curve if and only if  $\Delta_E \neq 0$ .*

We also have the following proposition explaining why we call  $j_E$  an invariant of an elliptic curve.

**Proposition 1.1.8** (see Proposition 1.4 in Silverman [47]). *Two elliptic curves are isomorphic over  $\overline{K}$  as groups if and only if they have the same  $j$ -invariant. Also, for any  $j_o \in \overline{K}$ , there exists an elliptic curve defined over  $K(j_o)$  with  $j$ -invariant equal to  $j_o$ .*

**Example 1.1.9.** Consider the curve over  $\mathbb{Q}$  given by the equation

$$E_1 : y^2 = 4x^3 + x.$$

Then  $g_2 = -1$  and  $g_3 = 0$ . Because  $\Delta_{E_1} = -1$ ,  $E_1(\mathbb{Q}/\mathbb{Q})$  is an elliptic curve. Also, we have  $j_{E_1} = 1728$ . Now consider the curve over  $\mathbb{Q}$  given by the equation

$$E_2 : y^2 = 4x^3 + 2x.$$

This is also an elliptic curve because  $\Delta_{E_2} = -8$ . Furthermore, both curves are isomorphic since  $j_{E_2} = 1728$ .

Finally, let  $j_o = 0$ , and choose  $g_2 = 0$  and  $g_3 = -1$ . Then  $\Delta = -27$  and  $j = j_o$ , so the equation

$$E_3 : y^2 = 4x^3 + 1$$

defines an elliptic curve over any field  $K$  not of characteristic 2 or 3.

In Chapter 2, we will want to reduce elliptic curves defined over  $\mathbb{Q}$  by a rational prime number  $p \neq 2, 3$ . To be precise, let  $E(\overline{\mathbb{Q}}/\mathbb{Q})$  be an elliptic curve with equation as in (1.1.8). Suppose further that  $g_2, g_3 \in \mathbb{Z}$ . We will denote the reduction of the equation modulo  $p$  as  $\tilde{E}$ . Now we end up with an equation for  $\tilde{E}$  with coefficients in  $\mathbb{F}_p$  (the finite field with  $p$  elements), and one of three things can happen:

1. The equation no longer defines an elliptic curve (i.e.  $\Delta_{\tilde{E}} = 0$ ).  
We call this *bad reduction*.
2. The equation still defines an elliptic curve (i.e.  $\Delta_{\tilde{E}} \neq 0$ ) which has no  $p$ -torsion elements. We call this type of reduction *supersingular*.
3. The equation still defines an elliptic curve (i.e.  $\Delta_{\tilde{E}} \neq 0$ ) which has  $p$ -torsion elements. We call this type of reduction *ordinary*.

When the elliptic curve has ordinary or supersingular reduction, we say that the reduction is good because we still have an elliptic curve modulo  $p$ . The case of ordinary reduction has been very well studied (see [27, 29]) and gives rise to the associated Kubota-Leopoldt  $p$ -adic  $L$ -functions.

We will focus on the case of elliptic curves with supersingular reduction and we will heavily make use of the beautiful theorem of Deuring which asserts that there are only finitely many supersingular  $j$ -invariants for a fixed prime  $p \neq 2, 3$ .

**Theorem 1.1.10.** *Let  $E$  be an elliptic curve over  $\mathbb{Q}$  with equation (1.1.8). If  $E$  has supersingular reduction modulo  $p$ , then  $j_{\tilde{E}} \in \mathbb{F}_{p^2}$ .*

## 1.2 Modular forms

This section presents many of the basic facts and properties of modular forms (see [39, 46] for more details). After some initial definitions, we discuss operators between spaces of forms that will be essential tools for analyzing the coefficients of modular forms.

### 1.2.1 Modular transformations

We let  $z = x + iy \in \mathbb{H}$ , the complex upper-half plane, with  $x, y \in \mathbb{R}$ , and suppose throughout that  $k$  is a nonnegative even integer.

Let  $\Gamma := \mathrm{SL}_2(\mathbb{Z})$  denote the *full modular group* of 2-by-2 matrices with determinant 1. This matrix group acts on the complex upper half plane  $\mathbb{H}$  by linear fractional transformation, i.e. if  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ , then

$$\gamma z = \frac{az + b}{cz + d}.$$

A *congruence subgroup* of level  $N$  is a subgroup  $\Gamma' \subset \Gamma$  that contains the kernel of the projection map:

$$\mathrm{SL}_2(\mathbb{Z}) \rightarrow \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z}).$$

For a given positive integer  $N$ , the congruence subgroups under our consideration will be

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : c \equiv 0 \pmod{N} \right\}.$$

*Remark.* For  $N = 1$ , we have

$$\mathrm{SL}_2(\mathbb{Z}) = \Gamma_0(1).$$

*Remark.* Throughout this text, we will identify  $-I$  with  $I$  where  $I$  is the 2-by-2 identity matrix. The effect is that we are really considering subgroups of  $\mathrm{PSL}_2(\mathbb{Z})$ . We adhere to the standard convention, in the world of modular forms, by always referring to subgroups of  $\mathrm{SL}_2(\mathbb{Z})$  while always remembering we are factoring out by the subgroup  $\{\pm I\}$ .

**Proposition 1.2.1.** *Let  $S, T, U$  be the matrices given by*

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \text{and} \quad U = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}.$$

*Then  $U = ST$ ,  $S^2 = 1$ ,  $U^3 = 1$ , and  $\mathrm{SL}_2(\mathbb{Z})$  is generated by  $S$  and  $T$ .*

**Definition 1.2.2.** The *cusps* of a congruence subgroup  $\Gamma'$  are the equivalence classes of  $\mathbb{Q} \cup \{i\infty\}$  under the action of  $\Gamma'$ .

We now define the action of a congruence subgroup  $\Gamma'$  on meromorphic functions.

**Definition 1.2.3.** Let  $\gamma \in \Gamma$  and suppose that  $f(z) : \mathbb{H} \rightarrow \mathbb{C}$  is a meromorphic function. We define the *weight  $k$  slash operator* by

$$(f|_k\gamma)(z) := (cz + d)^{-k} f(\gamma z). \tag{1.2.1}$$

**Definition 1.2.4.** Let  $f(z) : \mathbb{H} \rightarrow \mathbb{C}$  be a meromorphic function and  $\Gamma'$  a congruence subgroup of level  $N$ . Then  $f(z)$  is a *weakly holomorphic modular form of weight  $k$  on  $\Gamma'$*  if the following conditions are satisfied:

1. We have

$$(f|_k\gamma)(z) = f(z)$$

for all  $\gamma \in \Gamma'$  and all  $z \in \mathbb{H}$ .

2. If  $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ , then  $(f|_k\gamma)(z)$  has a Fourier expansion in  $q_N := q^{1/N} = e^{2\pi iz/N}$  of the form

$$(f|_k\gamma)(z) = \sum_{n \gg -\infty} a_\gamma(n) q_N^n$$

where the notation “ $n \gg -\infty$ ” means that for finitely many negative  $n$ , we have  $a_\gamma(n) \neq 0$ .

3. The function  $f(z)$  is holomorphic on  $\mathbb{H}$  and meromorphic at the cusps of  $\Gamma'$ .

If  $f(z)$  is holomorphic at all cusps (i.e.  $a_\gamma(n) = 0$  for all  $\gamma$  and  $n < 0$ ), then it is called a *modular form*. If  $f(z)$  vanishes at all cusps (i.e.  $a_\gamma(n) > 0$  for all  $\gamma$  and  $n \leq 0$ ), then it is called a *cusp form*. If  $f(z)$  has vanishing constant term for each  $\gamma$ , then it is called a *weakly holomorphic cusp form*.

The various families of modular forms are  $\mathbb{C}$ -vector spaces. We denote these complex vector spaces by

$M_k(N) :=$  modular forms of weight  $k$  on  $\Gamma_0(N)$

$S_k(N) :=$  cusp forms of weight  $k$  on  $\Gamma_0(N)$

$M_k^!(N) :=$  weakly holomorphic modular forms of weight  $k$  on  $\Gamma_0(N)$

$S_k^!(N) :=$  weakly holomorphic cusp forms of weight  $k$  on  $\Gamma_0(N)$ .

For simplicity, when  $N = 1$  is the full modular group, we omit it from our notation, e.g.  $M_k := M_k(1)$ .

## 1.2.2 Examples

Here we consider some examples of modular forms on  $\mathrm{SL}_2(\mathbb{Z})$  which are related to elliptic functions and elliptic curves.

If  $k$  is a positive integer, then the Bernoulli numbers  $B_k$  appear in the Maclaurin expansion of the function

$$\frac{z}{e^z - 1} = \sum_{k=0}^{\infty} B_k \frac{z^k}{k!} = 1 - \frac{1}{2}z + \frac{1}{12}z^2 + \dots \quad (1.2.2)$$

We also define the classical power-divisor function to be

$$\sigma_{k-1}(n) := \sum_{1 \leq d|n} d^{k-1}.$$

Consider the following  $q$ -series where we take  $q := e^{2\pi iz}$  throughout the rest of this text.

**Definition 1.2.5.** For even integers  $k \geq 2$ , the weight  $k$  Eisenstein series is given by

$$E_k(z) := 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n)q^n \quad (1.2.3)$$

These Eisenstein series are exactly (up to a constant multiple) the Eisenstein series that appear as coefficients of the Weierstrass  $\wp$ -function that we saw earlier in (1.1.4). Indeed, for even integers  $k \geq 4$ , we have

$$2\zeta(k)E_k(z) = G_k(z) \quad (1.2.4)$$

where  $\zeta$  is Riemann's function. This also gives us our first family of modular forms.

**Proposition 1.2.6.** *If  $k \geq 4$ , then the Eisenstein series  $E_k(z)$  is a modular form of weight  $k$  on the full modular group.*

*Remark.* The Eisenstein series  $E_2(z)$  is a strange exception. To prove modularity, we need only see how the series behaves under the transformations  $S$  and  $T$  since these generate  $\mathrm{SL}_2(\mathbb{Z})$ . All  $q$ -series are invariant with respect to  $T$ , however it turns out that (see [32])

$$(E_2|_2S)(z) = E_2(z) + \frac{12}{2\pi iy}.$$

Thus,  $E_2(z)$  is not a modular form. It is also interesting that  $E_2(z)$  does not appear in the Laurent expansion of the Weierstrass  $\wp$ -function.

We can also express the discriminant function of an elliptic curve as a modular form of weight 12.

**Proposition 1.2.7.** *Let  $E(\overline{\mathbb{Q}}/\mathbb{Q})$  be an elliptic curve with equation given by (1.1.8). Then discriminant function for the elliptic curve can be expressed as the  $q$ -series*

$$\Delta(z) := \sum_{n=1}^{\infty} \tau(n) = q - 24q^2 + 252q^3 + \cdots \in S_{12}.$$



*Proof.* We combine identities (1.1.5), (1.1.9), and (1.2.4) to obtain the following identity

$$\Delta(z) = \frac{1}{1728} (E_4(z)^3 - E_6(z)^2).$$

From this, it is easy to see that  $\Delta$  is a cusp form of weight 12. □

A major focus of research activity in modular forms over the past several decades is the following:

**Conjecture** (Lehmer).  $\tau(n) \neq 0$  for all  $n \geq 1$ .

We do not prove the conjecture, however we mention it as the motivating force behind the theory developed in Chapters 3 and 4.

Finally, we can express the  $j$ -invariant of an elliptic curve as a weakly holomorphic modular form of weight 0.

**Proposition 1.2.8.** *Let  $E(\overline{\mathbb{Q}}/\mathbb{Q})$  be an elliptic curve with equation given by (1.1.8). Then the  $j$ -invariant function for the elliptic curve can be expressed as the  $q$ -series*

$$j(z) := \frac{1}{q} + 744 + 196884q + 21493760q^2 + \cdots \in M_0^!$$

*Proof.* We combine identities (1.1.5), (1.1.10), and (1.2.4) to obtain the following identity

$$j(z) = \frac{E_4(z)^3}{\Delta(z)}.$$

From this, it is easy to see that  $j(z)$  has the desired  $q$ -expansion and is weight 0. □

### 1.2.3 Operators on modular forms

We now introduce the operators of Hecke and Atkin which act on spaces of modular forms. Throughout this subsection, let

$$f(z) = \sum_n a(n)q^n \in M_k^!(N)$$

for some integer  $N$ .

**Definition 1.2.9.** For a positive integer  $m$ , the action of the  $m$ th Hecke operator  $T(m)$  on  $f(z)$  is defined to be

$$f(z) |_k T(m) := \sum_n \left( \sum_{d | \gcd(m,n)} d^{k-1} a(mn/d^2) \right) q^n$$

where  $a(mn/d^2) = 0$  if  $d^2 \nmid mn$ . In particular for a prime  $p$ , the action of  $T(p)$  on  $f(z)$  is given by

$$f(z) |_k T(p) := \sum_n (a(pn) + p^{k-1} a(n/p)) q^n.$$

These operators are important tools for finding congruences for the Fourier coefficients of modular forms, as we will see in Chapter 3.

An important fact about these operators is that they take modular forms to modular forms. More precisely,

**Proposition 1.2.10.** *If  $f(z) \in M_k^!(N)$ , then  $f(z) |_k T(m) \in M_k^!(N)$ . Moreover, if  $f(z)$  is holomorphic or a cusp form, then the action of the Hecke operator will preserve these properties.*

**Example 1.2.11.** For an even integer  $k \geq 4$ , consider the Eisenstein series  $E_k(z)$  with  $q$ -expansion as in (1.2.3). For any integer  $m \geq 2$ , we have

$$E_k(z) |_k T(m) = -\frac{2k}{B_k} \sigma_{k-1}(m) E_k(z) \in M_k.$$

The last example illustrates more than one may have expected. The Eisenstein series are actually eigenforms for each Hecke operator  $T(m)$ .

**Definition 1.2.12.** A modular form  $g(z) \in M_k(N)$  is called a *Hecke eigenform* if for every integer  $m \geq 2$ , there exists  $\lambda(m) \in \mathbb{C}$  such that

$$g(z) |_k T(m) = \lambda(m)g(z).$$

We now summarize the important properties of Hecke eigenforms.

**Proposition 1.2.13** (see [39]). *Let  $g(z) \in M_k(N)$  be a Hecke eigenform with  $q$ -expansion*

$$g(z) := \sum_{n=0}^{\infty} b(n)q^n,$$

*and Hecke eigenvalues  $\lambda(m)$  for each  $m \geq 2$ .*

1. *If  $g(z)$  is nonconstant, then  $b(1) \neq 0$ .*
2. *If  $g(z)$  is a cusp form with  $b(1) = 1$ , then  $b(m) = \lambda(m)$ .  
Moreover, if  $\gcd(m, n) = 1$ , then  $b(mn) = b(m)b(n)$ .*
3. *If  $b(0) \neq 0$ , then*

$$\lambda(m) = \sum_{d|m} d^{k-1}.$$

We also consider a couple of operators of Atkin.

**Definition 1.2.14.** For any positive integer  $m$ , Atkin's  $U(m)$  and  $V(m)$  operators act on  $f(z)$  by

$$f(z) |_{k} U(m) := \sum_n a(mn)q^n; \text{ and}$$

$$f(z) |_{k} V(m) := \sum_n a(n)q^{mn}.$$

In particular, for a prime  $p$ , we have

$$T(p) = U(p) + p^{k-1}V(p).$$

We have the following result which describes how modular forms behave with respect to these operators.

**Proposition 1.2.15.** *Let  $f(z) \in M_k^!(N)$  and let  $m$  be a positive integer.*

1. *If  $m \nmid N$ , then*

$$f(z) \mid_k U(m) \in M_k^!(mN).$$

2. *If  $m \mid N$ , then*

$$f(z) \mid_k U(m) \in M_k^!(N).$$

3. *For any  $m$ , we have*

$$f(z) \mid_k V(m) \in M_k^!(mN).$$

*Moreover, if  $f(z)$  is holomorphic or a cusp form, then the action of these operators preserve these properties.*

## 1.2.4 Petersson's scalar product

Now we define an inner product on the space of cusp forms  $S_k(N)$ . A major result of Chapter 4 will be to extend this inner product to forms with poles, i.e. extend it to the space  $M_k^!(N)$ .

**Definition 1.2.16.** The *Petersson inner product* of cusp forms  $f_1, f_2 \in S_k$  is the hermitian (i.e.  $(f_1, f_2) = \overline{(f_2, f_1)}$ ) scalar product defined by

$$(f_1, f_2) := \frac{1}{[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N)]} \int_{\mathbb{H}/\Gamma_0(N)} f_1(z) \overline{f_2(z)} y^k \cdot \frac{dx dy}{y^2} \quad (1.2.5)$$

where the integral is taken over the fundamental domain of the action of the group  $\Gamma_0(N)$  on the complex upper half plane  $\mathbb{H}$ .

The inner product is Hecke invariant as the next proposition states.

**Proposition 1.2.17.** *For any two cusp forms  $f_1, f_2 \in S_k(N)$  and any integer  $m \geq 2$ , we have*

$$(f_1(z) \mid_k T(m), f_2(z)) = (f_1(z), f_2(z) \mid_k T(m)).$$

*Proof.* This follows from a routine calculation. □

## 1.3 Harmonic Maass forms

Here we introduce basic facts about harmonic (weak) Maass forms (see [9, 11, 14, 41] for more details) and we construct Maass-Poincaré series which naturally correspond to the classical Eisenstein series.

### 1.3.1 Basic facts

The *weight  $2 - k$  hyperbolic Laplacian* is defined by

$$\Delta_{2-k} := -y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + i(2-k)y \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right).$$

A *harmonic Maass form of weight  $2 - k$  on  $\Gamma_0(N)$*  is any smooth function  $\mathcal{F} : \mathbb{H} \rightarrow \mathbb{C}$  satisfying:

- (i)  $\mathcal{F}(z) = (\mathcal{F}|_{2-k}\gamma)(z)$  for all  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$ ;
- (ii)  $\Delta_{2-k}\mathcal{F} = 0$ ;
- (iii) The function  $\mathcal{F}(z)$  has at most linear exponential growth at infinity.

We denote the space of such forms by  $H_{2-k}(N)$ . We also require the subspace  $H_{2-k}^*(N)$  of  $H_{2-k}(N)$ , which consists of those  $\mathcal{F} \in H_{2-k}(N)$  with the property that there is a polynomial  $P_{\mathcal{F}}(q)$  for which  $\mathcal{F}(z) - P_{\mathcal{F}}(z) = O(e^{-\epsilon y})$ , as  $y \rightarrow +\infty$ , for some  $\epsilon > 0$ . As in the case of modular forms, when  $N = 1$ , we simply omit the level, e.g.  $H_{2-k} := H_{2-k}(1)$ .

The following proposition describes the Fourier expansions of harmonic Maass forms.

**Proposition 1.3.1** (Bruinier–Funke [11]). *If  $\mathcal{F} \in H_{2-k}$ , then*

$$\mathcal{F}(z) = a_{\mathcal{F}}^-(0)y^{k-1} + \sum_{\substack{n \in \mathbb{Z} \\ n \neq 0}} a_{\mathcal{F}}^-(n)h_{2-k}(2\pi ny)e(nx) + \sum_{n \gg -\infty} a_{\mathcal{F}}^+(n)q^n,$$

where  $e(\alpha) := e^{2\pi i\alpha}$  and  $h_{2-k}(w) := e^{-w} \int_{-2w}^{\infty} e^{-t} t^{k-2} dt$ .

Therefore, we have that  $\mathcal{F} = \mathcal{F}^- + \mathcal{F}^+$ , where the *nonholomorphic part*  $\mathcal{F}^-$  (resp. *holomorphic part*  $\mathcal{F}^+$ ) is defined by

$$\begin{aligned}\mathcal{F}^-(z) &:= a_{\mathcal{F}}^-(0)y^{k-1} + \sum_{\substack{n \in \mathbb{Z} \\ n \neq 0}} a_{\mathcal{F}}^-(n)h_{2-k}(2\pi ny)e(nx), \\ \mathcal{F}^+(z) &:= \sum_{n \gg -\infty} a_{\mathcal{F}}^+(n)q^n.\end{aligned}\tag{1.3.1}$$

*Remark.* Recall that, for real  $\beta \geq 0$ , the incomplete Gamma-function is defined by

$$\Gamma(\alpha, \beta) = \int_{\beta}^{\infty} t^{\alpha-1} e^{-t} dt.$$

If  $n < 0$ , then we have that

$$h_{2-k}(2\pi ny)e(nx) = \Gamma(k-1, 4\pi|n|y)q^n.$$

The following proposition gives the main features of the differential operators  $\xi_{2-k} := 2iy^{2-k} \frac{\partial}{\partial \bar{z}}$  and  $D := \frac{1}{2\pi i} \cdot \frac{d}{dz}$ .

**Proposition 1.3.2** (Bruinier–Funke [11], Bruinier–Ono–Rhoades [14]). *The following are true:*

1. *The operator  $\xi_{2-k}$  defines the surjective maps*

$$\begin{aligned}\xi_{2-k} &: H_{2-k}^* \rightarrow S_k, \\ \xi_{2-k} &: H_{2-k} \rightarrow M_k^!.\end{aligned}$$

2. *The operator  $D^{k-1}$  defines maps*

$$\begin{aligned}D^{k-1} &: H_{2-k}^* \rightarrow S_k^!, \\ D^{k-1} &: H_{2-k} \rightarrow M_k^!.\end{aligned}$$

We are now finally in a position to define what a mock modular form is.

**Definition 1.3.3.** A *mock modular form* of weight  $2-k$  is the holomorphic part  $\mathcal{F}^+$  of a harmonic Maass form  $\mathcal{F} \in H_{2-k}$ . The image  $\xi_{2-k}(\mathcal{F})$  is called the *shadow* of the mock modular form  $\mathcal{F}^+$ .

**Example 1.3.4.** All modular forms are trivial examples of mock modular forms. They have shadows which vanish.

**Example 1.3.5.** The first nontrivial example of a mock modular form is the weight 2 Eisenstein series previously introduced in (1.2.3). Recall that

$$E_2(z) = 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n) q^n.$$

Then we can define a function

$$\widehat{E}_2(z) := E_2(z) - \frac{3}{\pi y} \in H_2.$$

A routine calculation shows that  $\widehat{E}_2(z)$  is a harmonic Maass form of weight 2. Clearly,  $E_2(z)$  is a mock modular form and has constant shadow  $\xi_{2-k}(\widehat{E}_2) = 3/\pi$ .

### 1.3.2 Maass-Poincaré series and Eisenstein series

We now construct a weight  $2 - k$  harmonic Maass form whose image under  $\xi_{2-k}$  is not a constant function.

*Remark.* The Maass-Poincaré series  $P_{E_k}(z)$  constructed below should not be confused with the Maass-Poincaré series which have been employed to study  $H_{2-k}^*$  (for example see [9, 14, 41]). Those harmonic Maass forms project to cusp forms under  $\xi_{2-k}$ .

The Eisenstein series  $E_k(z)$  are actually Poincaré series and may be written as

$$E_k(z) := \sum_{\gamma \in \Gamma_{\infty} \backslash \Gamma} (1|_k \gamma)(z),$$

where  $\Gamma = SL_2(\mathbb{Z})$  and  $\Gamma_{\infty} := \{\pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} : n \in \mathbb{Z}\}$  and where  $\Gamma_{\infty} \backslash \Gamma$  is the set of all right cosets of  $\Gamma_{\infty}$  in  $\Gamma$ .

We define  $P_{E_k}(z)$  by

$$P_{E_k}(z) := \sum_{\gamma \in \Gamma_{\infty} \backslash \Gamma} (y^{k-1} |_{2-k} \gamma)(z). \quad (1.3.2)$$

The following theorem provides the main properties of these Poincaré series.

**Theorem 1.3.6.** *If  $k \geq 4$  is even, then the following are true:*

1. *The function  $P_{E_k}(z)$  is a harmonic Maass form of weight  $2 - k$  which satisfies*

$$D^{k-1}(P_{E_k}) = -\frac{(k-1)!}{(4\pi)^{k-1}}E_k(z),$$

$$\xi_{2-k}(P_{E_k}) = (k-1)E_k(z).$$

2. *The Fourier expansion of  $P_{E_k}(z)$  is given by*

$$P_{E_k}(z) = a^+(0) + a^-(0)y^{k-1} + \sum_{n>0} a^+(n)q^n + \sum_{n>0} a^-(n)\Gamma(k-1, 4\pi ny)q^{-n},$$

where

$$a^+(0) = \frac{2 \cdot k!}{B_k} \cdot \frac{\zeta(k-1)}{(4\pi)^{k-1}},$$

$$a^-(0) = 1,$$

$$a^+(n) = \frac{(k-1)!}{(4\pi)^{k-1}} \frac{2k}{B_k} \cdot \sigma_{1-k}(n) \quad \text{for } n > 0, \text{ and}$$

$$a^-(n) = \frac{(k-1)!}{(4\pi)^{k-1}} \frac{2k}{B_k} \cdot \sigma_{1-k}(n) \quad \text{for } n > 0.$$

*Remark.* Theorem 4.1.3 for  $P_{E_k}(z)$  and  $E_k(z)$  follows immediately from Theorem 1.3.6 (1).

*Proof.* That  $P_{E_k}(z)$  is a harmonic Maass form follows by construction. The relations under  $\xi_{2-k}$  and  $D^{k-1}$  are also simple consequences of the fact that  $P_{E_k}$  and  $E_k$  are Poincaré series.

We now establish the claimed Fourier expansion. By Proposition 1.3.1, we have that

$$P_{E_k}(z) = a^+(0) + a^-(0)y^{k-1} + \sum_{\substack{n \gg -\infty \\ n \neq 0}} a^+(n)q^n + \sum_{n>0} a^-(n)\Gamma(k-1, 4\pi ny)q^{-n}.$$



By (1), we then have that

$$\begin{aligned}
-\frac{(k-1)!}{(4\pi)^{k-1}} \cdot E_k(z) &= D^{k-1}(P_{E_k})(z) = -\frac{(k-1)!}{(4\pi)^{k-1}} a^-(0) + \sum_{\substack{n \gg -\infty \\ n \neq 0}} n^{k-1} a^+(n) q^n, \\
(k-1)E_k(z) &= \xi_{2-k}(P_{E_k})(z) = (k-1)\overline{a^-(0)} - \sum_{n>0} (4\pi)^{k-1} n^{k-1} \overline{a^-(n)} q^n.
\end{aligned}$$

Here we have used the fact, for positive  $n$ , that

$$\xi_{2-k}(\Gamma(k-1, 4\pi n y) q^{-n}) = -(4\pi)^{k-1} n^{k-1} q^n.$$

By (1.2.3), we then obtain, apart from the constant term, the desired coefficients of  $P_{E_k}(z)$ .

To determine the constant term, we rewrite the terms of  $P_{E_k}(z)$ . Using the notation  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ , and the fact that  $\mathrm{Im}(\gamma(z)) = \frac{y}{|cz+d|^2}$ , we obtain

$$\begin{aligned}
P_{E_k}(z) &= \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} (y^{k-1} |_{2-k} \gamma)(z) \\
&= y^{k-1} \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} (cz+d)^{-1} (c\bar{z}+d)^{1-k} \\
&= y^{k-1} + y^{k-1} \sum_{\substack{c \geq 1 \\ \mathrm{gcd}(c,d)=1}} (cz+d)^{-1} (c\bar{z}+d)^{1-k} \tag{1.3.3} \\
&= y^{k-1} + y^{k-1} \sum_{c \geq 1} c^{-k} \sum_{\substack{d=0 \\ \mathrm{gcd}(c,d)=1}}^{c-1} \sum_{n \in \mathbb{Z}} \left(z + \frac{d}{c} + n\right)^{-1} \left(\bar{z} + \frac{d}{c} + n\right)^{1-k}.
\end{aligned}$$

We now define two functions

$$v(z) := z^{-1} \bar{z}^{1-k} \quad \text{and} \quad V(z) := \sum_{n \in \mathbb{Z}} v(z+n).$$

Since  $V\left(z + \frac{d}{c}\right)$  is 1-periodic, it has a Fourier expansion of the form

$$V\left(z + \frac{d}{c}\right) = \sum_{n \in \mathbb{Z}} a_n(y) e^{2\pi i n \left(x + \frac{d}{c}\right)} dx,$$

where

$$\begin{aligned} a_m(y) &= \int_0^1 V\left(z + \frac{d}{c}\right) e^{-2\pi im(x + \frac{d}{c})} dx = \int_{\frac{d}{c}}^{1 + \frac{d}{c}} V(z) e^{-2\pi imx} dx \\ &= \int_{\mathbb{R}} v(z) e^{-2\pi imx} dx. \end{aligned}$$

Therefore the constant term of  $V\left(z + \frac{d}{c}\right)$  is

$$a_0(y) = \int_{\mathbb{R}} z^{-1} \bar{z}^{1-k} dx$$

Using the properties of the beta-function, and the functional equation for  $\Gamma(s)$ , one can show, for  $\operatorname{Re}(\alpha), \operatorname{Re}(\beta) > 0$  and  $\operatorname{Re}(\alpha + \beta) > 1$ , that

$$\int_{\mathbb{R}} z^{-\alpha} \bar{z}^{-\beta} dx = \frac{\Gamma(\alpha + \beta - 1)}{\Gamma(\alpha)\Gamma(\beta)} 2^{2-\alpha-\beta} i^{\alpha-\beta} \pi y^{1-\alpha-\beta}.$$

Letting  $\alpha = 1$  and  $\beta = k - 1$ , we find that

$$a_0(y) = \frac{\Gamma(k-1)}{\Gamma(1)\Gamma(k-1)} 2^{2-k} i^{k-2} \pi y^{1-k} = \frac{\pi}{(2i)^{k-2}} y^{1-k}.$$

It follows that the constant term in (1.3.3) is given by

$$a^+(0) = y^{k-1} \sum_{c \geq 1} c^{-k} \sum_{\substack{d=0 \\ \gcd(c,d)=1}}^{c-1} a_0(y) = \frac{\pi}{(2i)^{k-2}} \sum_{c \geq 1} \frac{\phi(c)}{c^k},$$

where  $\phi$  is Euler's totient function. Since  $\phi$  is multiplicative, we have an Euler product

$$\sum_{c \geq 1} \frac{\phi(c)}{c^k} = \prod_{p \text{ prime}} \sum_{l \geq 0} \frac{\phi(p^l)}{p^{lk}}.$$

We now simplify the constant term of  $P_{E_k}(z)$  as follows

$$\begin{aligned}
a^+(0) &= \frac{\pi}{(2i)^{k-2}} \prod_{p \text{ prime}} \sum_{l \geq 0} \frac{\phi(p^l)}{p^{lk}} = \frac{\pi}{(2i)^{k-2}} \prod_{p \text{ prime}} \left( 1 + (p-1) \sum_{l \geq 0} \frac{p^l}{p^{(l+1)k}} \right) \\
&= \frac{\pi}{(2i)^{k-2}} \prod_{p \text{ prime}} \left( 1 + \frac{(p-1)}{p^k} \frac{1}{1-p^{1-k}} \right) = \frac{\pi}{(2i)^{k-2}} \prod_{p \text{ prime}} \frac{1-p^{-k}}{1-p^{1-k}} \\
&= \frac{\pi}{(2i)^{k-2}} \frac{\zeta(k-1)}{\zeta(k)}.
\end{aligned}$$

Finally, we may simplify further using the classical evaluation  $\zeta(k) = -\frac{(2\pi i)^k B_k}{2 \cdot k!}$  to obtain the desired form of the constant term.  $\square$

# Chapter 2

## $p$ -adic lifting of $j$ -zeros

Zeros of modular forms is an interesting subject, and there has been a big amount of research connected to this subject during the past several decades (see [1, 2, 12, 18, 43] to name a few). Zeros of Eisenstein series (1.2.3) attract special attention. Following the terminology of [18], we define  $j$ -zeros to be the  $j$ -invariants of zeros of  $E_k$ . The results of this chapter were published with Pavel Guerzhoy in [21].

### 2.1 Lifting supersingular $j$ -invariants

Denote by  $\Psi_k(X)$  the polynomial that encodes the  $j$ -zeros of  $E_k$ :

$$\Psi_k(X) = \prod_{\substack{j=j(\tau), \\ \text{where } E_k(\tau)=0}} (X - j)$$

Let  $p > 3$  be a prime. The coefficients of  $\Psi_{p-1}$  are  $p$ -integral. It is a well-known observation of Deligne (see [35] for a full exposition) that  $\tilde{\Psi}_{p-1}(X)$ , the modulo  $p$  reduction of  $\Psi_{p-1}(X)$ , is the supersingular polynomial at  $p$ . The roots of  $\tilde{\Psi}_{p-1}(X)$  over  $\overline{\mathbb{F}}_p$  are supersingular  $j$ -invariants. This polynomial, considered as a polynomial over  $\mathbb{F}_p$ , splits into a product of factors over  $\mathbb{F}_p$ ,

$$\tilde{\Psi}_{p-1}(X) = \prod_i \tilde{\psi}_i(X), \tag{2.1.1}$$

where the monic polynomials  $\tilde{\psi}_i(x) \in \mathbb{F}_p[X]$  are either linear or irreducible quadratic which follows from Theorem 1.1.10. We will consider  $\Psi_k$  as a polynomial over the field of  $p$ -adic numbers  $\mathbb{Q}_p$ . A standard application of Hensel's lemma allows us

to lift the supersingular  $j$ -invariants to characteristic zero in a canonical way. The possible presence of irreducible (over  $\mathbb{F}_p$ ) quadratic factors in decomposition (2.1.1) makes it necessary to introduce the unique (see [31, Section 3.3]) unramified quadratic extension  $K = \mathbb{Q}_p(\zeta)$  of  $\mathbb{Q}_p$ , where  $\zeta$  is a primitive root of unity of degree  $p^2 - 1$ . The ring of integers of  $K$  will be denoted as  $\mathcal{O}$ . The following proposition is an immediate consequence of Hensel's lemma.

**Proposition 2.1.1.** *For every irreducible factor  $\tilde{\psi}_i(X)$  in decomposition (2.1.1) there are exactly  $\deg(\tilde{\psi}_i(X))$  elements  $u \in \mathcal{O}$  such that*

$$\tilde{\psi}_i(\tilde{u}) = 0 \quad \text{and} \quad \Psi_{p-1}(u) = 0.$$

This proposition motivates the following definition.

**Definition 2.1.2.** We call an element  $u \in \mathcal{O}$  from Proposition 2.1.1 a *lifting* of a supersingular  $j$ -invariant to characteristic zero.

## 2.2 A $p$ -adic connection between $j$ -zeros

Throughout this chapter

$$k = k(a, M) = p - 1 + Mp^a(p^2 - 1)$$

with non-negative integers  $a$  and  $M$ . The subject of our investigation in this chapter is the polynomial  $\Psi_k(X)$  and its zeros.

Define  $\epsilon = \epsilon(p), \gamma = \gamma(p) \in \{0, 1\}$  such that

$$\epsilon \equiv \frac{p-1}{4} \pmod{3} \quad \text{and} \quad \gamma \equiv \frac{p-1}{6} \pmod{2}.$$

Let  $\delta(k) = \lfloor k/12 \rfloor$ . We define the polynomial  $\varphi_k(X)$  (found in [18]) by:

$$\Psi_k(X) = X^\epsilon (X - 1728)^\gamma \varphi_k(X) \tag{2.2.1}$$

A result of Gekeler [18, Corollary 2.6] implies the following factorization over  $\mathbb{F}_p$

$$\tilde{\varphi}_k(X) = \tilde{\varphi}_{p-1}(X)^{d+1} X^{ed/3} (X - 1728)^{\gamma d/2} \quad (2.2.2)$$

where  $d = M(p^{a+1} + p^a)$ . Note that all exponents in this factorization are integers. This factorization implies, in particular, that

$$\Psi_k(u) \equiv 0 \pmod{p}$$

for every lifting  $u$  of a supersingular  $j$ -invariant, and it is natural to ask about a connection between the roots of the polynomials  $\Psi_{p-1}(X)$  and  $\Psi_k(X)$ . Numerical examples show that the roots of  $\Psi_k(X)$  may not belong to  $K$ . We thus consider these roots as elements of the algebraic closure  $\overline{\mathbb{Q}_p}$  of  $\mathbb{Q}_p$ . The principal result provides a partial answer to the above question.

**Theorem 2.2.1.** *Let  $u$  be a lifting of a supersingular  $j$ -invariant. There is  $r \in \overline{\mathbb{Q}_p}$  such that  $\Psi_k(r) = 0$  and*

$$\text{ord}_p(r - u) > a.$$

If  $p > 13$ , the polynomial  $\Psi_k(X)$  is not irreducible. Indeed, since the Galois group preserves distances (see e.g. [31, Chapter 3]), the factorizations (2.1.1), (2.2.1), and (2.2.2) imply the following factorization over  $\mathbb{Q}_p$ :

$$\Psi_k(X) = \prod_u \psi_{u,k}(X). \quad (2.2.3)$$

The product in (2.2.3) is taken over all pairwise non-conjugate by  $\text{Gal}(K/\mathbb{Q}_p)$  liftings  $u$  of supersingular  $j$ -invariants. The polynomials  $\psi_{u,k}(X) \in \mathbb{Z}_p[x]$  are monic of degree

$$\deg \psi_{u,k}(X) = (d/e(u) + 1) \deg \tilde{\psi}_u = (M(p^{a+1} + p^a)/e(u) + 1) \deg \tilde{\psi}_u,$$

where  $e(u)$  is the ramification degree of the relevant  $j$ -zero, i.e.

$$e(u) = \begin{cases} 3 & \text{if } u \equiv 0 \pmod{p} \\ 2 & \text{if } u \equiv 1728 \pmod{p} \\ 1 & \text{otherwise.} \end{cases}$$

In particular, when  $M = 0$ , and  $k(a, 0) = p - 1$ , we drop the index  $k$  by setting

$$\psi_u = \psi_{u,p-1}.$$

There are speculations, based on numerical evidence, on the irreducibility of the polynomials  $\Psi_k$  over  $\mathbb{Q}$ . The above remarks show that over  $\mathbb{Q}_p$  a similar question is meaningful only about the individual polynomials  $\psi_{u,k}$ . As an application to Theorem 2.2.1, we have the reducibility of every factor  $\psi_{u,k}$  of  $\Psi_k$  over  $\mathbb{Q}_p$ .

**Theorem 2.2.2.** *If  $M \geq 1$  and  $a \geq 1$ , then  $\psi_{u,k}(X)$  is reducible over  $\mathbb{Q}_p$  for every  $u$ .*

In contrast, the next result implies, in particular, that the polynomials  $\psi_{u,k}$  typically do not split completely over  $K$ .

**Theorem 2.2.3.** *If  $M \geq 1$ , then the splitting field of the polynomial  $\psi_{u,k}$  is ramified over  $\mathbb{Q}_p$  for every  $u$  such that  $e(u) \leq a$ .*

In Section 2.3, we state certain congruences between special values of the polynomials  $\psi_u$  and  $\psi_{u,k}$  (Theorem 2.3.1), and derive results from these congruences. Section 2.4 is devoted to the proof of Theorem 2.3.1. This proof involves the techniques of formal groups. In particular, Proposition 2.4.3 claims congruences for the coefficients of series expansions of certain functions on Lubin - Tate formal groups of height 2. The proof of this proposition, which is an adaptation to our setting of an argument invented by Katz [30] (see also [15] for a refinement) is deferred to Section 2.5.

## 2.3 Proofs of the main results

Throughout we assume that  $u \in K$  is a lifting of a supersingular  $j$ -invariant. We derive the main results from the following congruences:

**Theorem 2.3.1.** *Let  $s \in K$  be such that  $\text{ord}_p(\psi_u(s)) \in e(u)\mathbb{Z}$ .*

**a)** *If  $0 < \text{ord}_p(\psi_u(s)) < a + 1$ , then*

$$\text{ord}_p(\psi_{u,k}(s)) = Mp^{a+1} + \text{ord}_p(\psi_u(s)).$$

b) If  $\text{ord}_p(\psi_u(s)) \geq a + 1$ , then

$$\text{ord}_p(\psi_{u,k}(s)) \geq Mp^{a+1} + a + 1.$$

*Proof of Theorem 2.2.1.* If  $e(u) > 1$ , the statement is trivial in view of (2.2.1) and (2.2.2). We thus assume that  $e(u) = 1$ .

We denote by  $r_l \in \overline{\mathbb{Q}}_p$  the roots of the polynomial  $\psi_{u,k}(X)$ :

$$\psi_{u,k}(X) = \prod_l (X - r_l).$$

Choose  $s_1, s_2 \in \mathcal{O}$  such that  $\text{ord}_p(s_1 - u) = a$  and  $\text{ord}_p(s_2 - u) \geq a + 1$ . Since  $K$  is unramified, we have

$$\text{ord}_p(\psi_u(s_1)) = a, \quad \text{and} \quad \text{ord}_p(\psi_u(s_2)) \geq a + 1.$$

If  $\text{ord}_p(r_l - u) \leq a$ , then the ultrametric inequality implies that

$$\text{ord}_p(s_2 - r_l) = \text{ord}_p(r_l - u) \leq \text{ord}_p(s_1 - r_l).$$

We now assume that  $\text{ord}_p(r_l - u) \leq a$  for all roots  $r_l$ , and make use of Theorem 2.3.1 to obtain a contradiction:

$$a + 1 + Mp^{a+1} \leq \text{ord}_p(\psi_{u,k}(s_2)) \leq \text{ord}_p(\psi_{u,k}(s_1)) = a + Mp^{a+1}.$$

Theorem 2.2.1 follows from this observation. □

*Proof of Theorem 2.2.2.* As in the proof of Theorem 2.2.1, we assume that  $e(u) = 1$ , because otherwise the result is immediate from (2.2.1) and (2.2.2).

Choose  $s \in K$  such that  $\text{ord}_p(\psi_u(s)) = 1$ . By Theorem 2.2.1 there is a root  $r_0$  of  $\psi_{u,k}$  and  $\text{ord}_p(r_0 - u) > a \geq 1$ . Therefore  $\text{ord}_p(s - r_0) = 1$ . If we assume that  $\psi_{u,k}$  is irreducible, then because the Galois group preserves distances and all roots are conjugate, we must have  $\text{ord}_p(s - r_l) = \text{ord}_p(s - r_0)$  for all roots  $r_l$ . But this leads



to a contradiction of Theorem 2.3.1,

$$\text{ord}_p(\psi_{u,k}(s)) = \sum_l \text{ord}_p(s - r_l) = M(p^{a+1} + p^a) + 1 > Mp^{a+1} + 1,$$

proving our result. □

*Proof of Theorem 2.2.3.* Let  $s_0 \in K$  be such that  $s_0 \equiv u \pmod{p^{e(u)}}$  and  $s_0$  is not congruent modulo  $p^{e(u)+1}$  to a lifting of a supersingular  $j$ -invariant. By Theorem 2.3.1

$$\text{ord}_p(\psi_{u,k}(s_0)) = Mp^{a+1} + e(u).$$

On the other hand, if we assume that the splitting field of  $\psi_{u,k}$  is unramified, then  $\text{ord}_p(s_0 - r_l) \geq e(u)$ , and we have the contradiction

$$\begin{aligned} \text{ord}_p(\psi_{u,k}(s_0)) &= \sum_l \text{ord}_p(s_0 - r_l) \\ &\geq e(u)(M(p^{a+1} + p^a)/e(u) + 1) \\ &> Mp^{a+1} + e(u). \end{aligned}$$

□

## 2.4 Proof of Theorem 2.3.1

In this section we prove Theorem 2.3.1 with the help of several propositions; one whose proof is postponed to the next section. We derive Theorem 2.3.1 from a certain congruence (see Proposition 2.4.1 below) for Bernoulli - Hurwitz numbers [30, 29]. There are two parallel ways to prove this congruence. First, since the formal group of the elliptic curve with  $j$ -invariant  $s$  has height 2 (the elliptic curve has supersingular reduction at  $p$ ), one can make use of a corollary to Katz' general theorem on formal groups and  $p$ -adic interpolation [29, Corollary 3]. However, the full proof of this theorem has never been published. An alternative approach, which we undertake here, is based on a later observation of Katz [30] (see also [15] for refinements). Namely, one proves that the formal group in question is isomorphic to a Lubin-Tate formal group, and applies an elementary argument which implies the desired congruences.

**Proposition 2.4.1.** *Let  $s \in K$  be such that  $0 < \text{ord}_p(\Psi_{p-1}(s)) \in e(u)\mathbb{Z}$  for some lifting  $u$  of a supersingular  $j$ -invariant. Let  $b \in \mathbb{Z}$  be an integer different from 1 and coprime to  $p$ . Define*

$$T(l) = \frac{(1-b^l)(1-p^{l-2})}{p^{\lfloor (l-2)p/(p^2-1) \rfloor}} \frac{B_l}{l} \Psi_l(s).$$

*Then for some  $\mu \in \mathcal{O}$  such that  $\text{ord}_p(\mu) = 0$  we have the congruences*

$$\mu T(p-1) \equiv T(k) \pmod{p^{a+1}}.$$

*Proof of Theorem 2.3.1.* Let  $l$  be a positive integer that is a multiple of  $p-1$ . (Note that  $(p-1) \mid k$ .) By von Staudt congruences,  $\text{ord}_p(B_l) = -1$ . Fermat's Little Theorem and the Binomial Theorem imply that  $\text{ord}_p(1-b^l) = 1 + \text{ord}_p(l)$ . In order to derive Theorem 2.3.1 from Proposition 2.4.1, we simply equate the  $p$ -orders of the congruences of Proposition 2.4.1 and use the factorization (2.2.3).  $\square$

The proof of Proposition 2.4.1 is more involved, and requires some preliminaries on one-dimensional formal groups. For a formal group  $F$  we denote by  $[p]_F \in \text{End}(F)$  the multiplication by  $p$  map. If  $\alpha \in \mathcal{O}$  is a unit, then the Lubin-Tate lemma [37] implies the existence and uniqueness up to isomorphism of a height two one-parameter formal

group  $G(\alpha)$  over  $\mathcal{O}$  such that

$$[p]_{G(\alpha)}(X) = pX + \alpha X^{p^2}.$$

**Proposition 2.4.2.** *Let  $F$  be the formal group over  $\mathcal{O}$  of the elliptic curve  $E$  defined by the equation*

$$y^2 = 4x^3 - g_2x - g_3, \quad g_2, g_3 \in \mathcal{O} \tag{2.4.1}$$

with  $j$ -invariant

$$s = \frac{1728g_2^3}{g_2^3 - 27g_3^2}.$$

If  $\Psi_{p-1}(s) \equiv 0 \pmod{p}$ , then the discriminant  $\Delta = g_2^3 - 27g_3^2$  is a unit,  $\text{ord}_p(\Delta) = 0$ , and  $F$  is isomorphic to a formal group  $G(\alpha)$  with

$$[p]_{G(\alpha)}(X) = pX + \alpha X^{p^2}.$$

*Proof.* A well-known observation of Deligne (see e.g. [35, p.105] for a proof) is that the modulo  $p$  reduction of the elliptic curve  $\tilde{E}$  (2.4.1) is a supersingular elliptic curve over  $\mathcal{O}/(p)$ . In particular,  $\text{ord}_p(\Delta) = 0$ . It follows (see [26, Table, p.269], [47, Theorem IV.7.4]) that the  $p^2$ -power Frobenius endomorphism of  $\tilde{E}$  factors through the multiplication by  $p$  isogeny,  $\text{Frob} = [p]\alpha^{-1}$ , with a separable isogeny  $\alpha$ . The latter is a multiplication by (a modulo  $p$  reduction of)  $\alpha \in \mathcal{O}$  with  $\text{ord}_p(\alpha) = 0$ . This induces the factorization of the Frobenius endomorphism of the formal group  $\tilde{F}$  of  $\tilde{E}$ , which is the modulo  $p$  reduction of  $F$ . We thus have

$$[p\alpha^{-1}]_F(X) \equiv X^{p^2} \pmod{p} \quad \text{and} \quad [p\alpha^{-1}]_F(X) \equiv p\alpha^{-1}X \pmod{X^2},$$

where the second congruence holds in any formal group. An application of the Lubin - Tate lemma [37] establishes an isomorphism between  $F$  and a formal group  $G'$  over  $\mathcal{O}$  with

$$[p]_{G'}(X) = p\alpha^{-1}X + X^{p^2}.$$

In order to finish the proof we note that both  $G'$  and  $G(\alpha)$  have characteristic polynomial  $t^2 - p\alpha^{-1}$  and are therefore isomorphic (see [24, 25]).  $\square$

If a formal group  $F$  is defined over  $\mathcal{O}$ , then we call a formal power series  $f \in \mathcal{O}[[X]]$  a function on  $F$ . The invariant differentiation  $D$  acts on functions on  $F$ .

**Proposition 2.4.3.** *Let  $f$  be a function on  $G(\alpha)$ . Assume that  $f$  satisfies the difference equation*

$$\sum_{[p](\lambda)=0} f(X \underset{G(\alpha)}{+} \lambda) = 0. \quad (2.4.2)$$

Let

$$L_f(n) = \frac{D^n(f)(0)}{p^{\lfloor np/(p^2-1) \rfloor}}.$$

For all integers  $n, a \geq 0$  with  $n \not\equiv 0, p, 2p, \dots, (p-1)p \pmod{p^2-1}$ , the following congruence holds:

$$(\alpha(p^2-2)!p^{1-p})^{p^a} L_f(n) \equiv L_f(n + p^a(p^2-1)) \pmod{p^{a+1}}.$$

We postpone the proof of Proposition 2.4.3 to the next section.

In order to obtain Proposition 2.4.1 we consider the Laurent series expansion of the Weierstrass  $\wp$ -function associated with the elliptic curve  $E$  defined by (2.4.1)

$$\wp(E, z) = z^{-2} - \sum_{m \geq 1} \frac{B_{2m+2}}{2m+2} (2\pi i)^{2m+2} E_{2m+2} \frac{z^{2m}}{(2m)!}. \quad (2.4.3)$$

Note that  $\wp(E, z) \in K[[z^{-1}, z]]$  since  $(2\pi i)^l E_l$  is a polynomial in  $g_2$  and  $g_3$  with rational coefficients for even  $l \geq 4$ .

The parameter of the formal group corresponding to the elliptic curve  $E$  is  $X = -2\wp(E, z)/\wp'(E, z)$ , and the power series expansion of  $\wp$  in  $X$  belongs to  $\mathcal{O}[[X^{-1}, X]]$  (see [47, Chapter IV, §1]). The series  $\wp$  is not a function on this formal group only due to the pole at zero. This deficiency is, however, easily fixed. For an integer  $N \in \mathbb{Z}$  and a power series  $g \in \mathcal{O}[[X^{-1}, X]]$ , put as in [30, 15]

$$[N]^*g(X) = g([N]X).$$

Note that in terms of the parameter  $z$  we simply have  $[N]^*\wp(E, z) = \wp(E, Nz)$ .

**Proposition 2.4.4.** *Let  $b \in \mathbb{Z}$  be an integer different from 1 and coprime to  $p$ . The power series in  $X$*

$$\wp_{b,p}(E, X) = (1 - [p]^*)(1 - b^2[b]^*)\wp(E, X)$$

*is a function on the formal group of an elliptic curve  $E$ , and satisfies the difference equation (2.4.2).*

*Proof.* We adopt the desired identity to the logarithmic parameter  $z$ , which we consider as the usual complex variable. Let  $\Lambda$  be the period lattice of the elliptic curve  $E$ . The claimed identity becomes

$$\sum_{\lambda} (1 - [p]^*)(1 - b^2[b]^*)\wp(E, z + \lambda) = 0,$$

where the summation is taken over all points  $\lambda$  in the fundamental parallelogram of  $\Lambda$  such that  $p\lambda \in \Lambda$ . In order to check the latter identity it suffices to notice that the function on the left-hand side is  $\Lambda$ -periodic, equals zero at the points of  $\Lambda$ , and has no poles in the fundamental parallelogram of  $\Lambda$ .  $\square$

*Proof of Proposition 2.4.1.* By hypothesis and the factorization (2.2.2), we have  $s \in \mathcal{O}$  such that  $0 < \text{ord}_p(s - u) \in e(u)\mathbb{Z}$ . This allows us to choose  $g_2, g_3 \in \mathcal{O}$  such that  $s$  is the  $j$ -invariant of the elliptic curve (2.4.1) as follows:

If  $s \equiv 0 \pmod{p}$ , then  $u = 0$  and  $\text{ord}_p(s) \in 3\mathbb{Z}$ , so we may write  $s = \nu p^{3k}$  for some unit  $\nu \in \mathcal{O}$  and positive integer  $k$ . Consider the equation

$$s = 1728 \frac{g_2^3}{g_2^3 - 27g_3^2} = 1728 \frac{(g_2/3)^3}{(g_2/3)^3 - g_3^2},$$

with variables  $g_2$  and  $g_3$ . Taking  $g_2 = -\frac{p^k}{4\nu} \in \mathcal{O}$  in the equation, we may rewrite the result as a polynomial equation over  $\mathbb{Z}_p$  with variable  $g_3$ :

$$g_3^2 + \frac{p^{3k}}{1728\nu^3} - \frac{1}{\nu^4} = 0.$$

This polynomial has a pair of simple nonzero roots when considered modulo  $p$ . Therefore a standard application of Hensel's lemma allows us to find a solution  $g_3 \in \mathcal{O}$ . For all other choices of  $s$ , we may choose  $g_2, g_3 \in \mathcal{O}$  in a similar way.

Combining Propositions 2.4.4, 2.4.2, and 2.4.3 we obtain the congruences

$$(\alpha(p^2 - 2)!p^{1-p})^{Mp^a} L_{\wp_{b,p}}(n) \equiv L_{\wp_{b,p}}(n + Mp^a(p^2 - 1)) \pmod{p^{a+1}} \quad (2.4.4)$$

where  $n \not\equiv 0, p, 2p, \dots, (p-1)p \pmod{(p^2 - 1)}$  and  $\alpha(p^2 - 2)!p^{1-p}$  is a unit in  $\mathcal{O}$ . We need only consider the case when  $n = p - 3$ .

For all positive even integers  $l$ ,

$$D^l(\wp_{b,p})(0) = -(1 - b^{l+2})(1 - p^l) \frac{B_{l+2}}{l+2} (2\pi i)^{l+2} E_{l+2}.$$

By [18, Proposition 1.17],

$$(2\pi i)^k E_k = \varphi_k(s) \Delta^{\delta(k)} (12g_2)^\epsilon (-216g_3)^\gamma.$$

Combining the above equalities with (2.2.1), we find that

$$T(k) = -L_{\wp_{b,p}}(k-2) \left( \frac{144g_2^2}{\Delta} \right)^\epsilon \left( \frac{-216g_3}{\Delta} \right)^\gamma \Delta^{-\delta(k)}.$$

Therefore, upon multiplying the congruences (2.4.4) by the integral factor

$$-\left( \frac{144g_2^2}{\Delta} \right)^\epsilon \left( \frac{-216g_3}{\Delta} \right)^\gamma \Delta^{-\delta(k)}$$

( $\Delta$  is a unit in  $\mathcal{O}$  by Proposition 2.4.2), and taking

$$\mu := \Delta^{\delta(p-1)-\delta(k)} (\alpha(p^2 - 2)!p^{1-p})^{Mp^a},$$

we obtain the congruences of Proposition 2.4.1. □

## 2.5 Proof of Proposition 2.4.3

In this section we prove Proposition 2.4.3 closely following [15, 30]. Recall that  $p > 3$  (this restriction slightly simplifies the argument).

Let  $\mathcal{O}$  be a commutative ring with identity and  $G$  a one parameter (commutative) formal group over  $\mathcal{O}$  with parameter  $X$  and group law  $F(X, Y) = X +_G Y \in \mathcal{O}[[X, Y]]$ . We will identify the coordinate ring of  $G$  with  $\mathcal{O}[[X]]$ . As in [30] we denote by  $\text{Diff}(G)$  the commutative  $\mathcal{O}$ -algebra of all  $G$ -invariant  $\mathcal{O}$ -linear differential operators of  $\mathcal{O}[[X]]$ . As an  $\mathcal{O}$ -module,  $\text{Diff}(G)$  is free with basis  $D(n), n = 0, 1, 2, \dots$  defined by ‘‘Taylor expansion’’ for all  $f \in \mathcal{O}[[X]]$  by

$$f(X +_G Y) = \sum_{n \geq 0} D(n)(f)Y^n \in \mathcal{O}[[X, Y]].$$

The operator  $D(0)$  is the identity in  $\mathcal{O}[[X]]$ , and  $D(1)$  is the  $G$ -invariant derivation, normalized by  $D(X)(0) = 1$ , which we will denote by  $D$ . Recall that (see [30, Identity 2.4]) for  $0 \leq n \leq p^2 - 1$

$$D(n) = \frac{D^n}{n!}. \quad (2.5.1)$$

Let  $\widehat{\text{Diff}}(G(\alpha))$  be the  $p$ -adic completion of  $\text{Diff}(G(\alpha))$ , then we can define an operator

$$H = \frac{p^2 - 1}{p^2} \sum_{r \geq 2} (-p/\alpha)^r D(r(p^2 - 1)) \in \widehat{\text{Diff}}(G(\alpha)),$$

For convenience of notation, we also define the operator (as in [15])

$$X_o = 1 + H \in \widehat{\text{Diff}}(G(\alpha)).$$

We need the following congruences proved in [15, pp.168-169]

$$DH \equiv 0 \pmod{p \widehat{\text{Diff}}(G(\alpha))}, \quad (2.5.2)$$

and for a non-negative integer  $n$

$$D^n \equiv 0 \pmod{p^{\lfloor np/(p^2-1) \rfloor} \widehat{\text{Diff}}(G(\alpha))}, \quad (2.5.3)$$

$$H^n \equiv 0 \pmod{p^{\lfloor n(1-1/p) \rfloor} \widehat{\text{Diff}}(G(\alpha))}. \quad (2.5.4)$$

We must show that

$$L_f(n + p^a(p^2 - 1)) \equiv (\alpha(p^2 - 2)!p^{1-p})^{p^a} L_f(n) \pmod{p^{a+1}}$$

for  $n \not\equiv 0, p, 2p, \dots, (p-1)p \pmod{p^2 - 1}$ . The difference equation (2.4.2) and the identity (2.5.1) imply

$$X_o = \frac{p^2 - 1}{\alpha p} D(p^2 - 1) = \frac{D^{p^2-1}}{\alpha p(p^2 - 2)!}.$$

It follows that

$$\begin{aligned} & \left( \frac{p^{p-1}}{\alpha(p^2 - 2)!} \right)^{p^a} L_f(n + p^a(p^2 - 1)) - L_f(n) \\ &= (X_o^{p^a} - 1)L_f(n) = D(X_o^{p^a} - 1) \frac{D^{n-1}}{p^{\lfloor np/(p^2-1) \rfloor}}(f)(0). \end{aligned}$$

Since  $\lfloor (n-1)p/(p^2-1) \rfloor = \lfloor np/(p^2-1) \rfloor$  for  $n \not\equiv 0, p, 2p, \dots, (p-1)p \pmod{p^2 - 1}$ , (see [30, §3]), the congruence (2.5.3) implies that  $D^{n-1} \equiv 0 \pmod{p^{\lfloor np/(p^2-1) \rfloor} \widehat{\text{Diff}}(G(\alpha))}$ . It thus suffices to show that  $D(X_o^{p^a} - 1) \equiv 0 \pmod{p^{a+1} \widehat{\text{Diff}}(G(\alpha))}$ . For  $a = 0$ , this coincides with (2.5.2). For  $a \geq 1$ ,

$$\begin{aligned} D(X_o^{p^a} - 1) &= D(1 + H)^{p^a} - D \\ &= p^a DH + \sum_{k=2}^{p^a} \binom{p^a}{k} DH^k \\ &\equiv 0 \pmod{p^{a+1} \widehat{\text{Diff}}(G(\alpha))}. \end{aligned}$$

The latter congruence follows from (2.5.2), the obvious inequality

$$\lfloor (k-1)(1-1/p) \rfloor \geq \text{ord}_p(k)$$



for  $k \geq 2$ , and the following calculation:

$$\begin{aligned} \binom{p^a}{k} DH^k &= \binom{p^a - 1}{k - 1} \frac{p^a}{k} (DH) H^{k-1} \\ &\equiv 0 \pmod{p^{a - \text{ord}_p(k) + 1 + \lfloor (k-1)(1-1/p) \rfloor} \widehat{\text{Diff}}(G(\alpha))} \end{aligned}$$

by (2.5.2) and (2.5.4). □

# Chapter 3

## $p$ -adic coupling of mock modular forms and shadows

The theory of harmonic Maass forms [9, 11, 41], which explains Ramanujan's mock theta functions [41, 51, 52, 53], relies on a correspondence between harmonic Maass forms and cusp forms. For example, Ramanujan's mock theta function

$$f(q) := 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1+q)^2(1+q^2)^2 \cdots (1+q^n)^2} = 1 + q - \cdots - 53q^{24} + \cdots + 19618q^{101} - \cdots$$

is coupled to the weight  $3/2$  cuspidal theta function

$$\Theta(z) := \sum_{n=-\infty}^{\infty} (6n+1)q^{\frac{3}{2}(n+\frac{1}{6})^2} = q^{\frac{1}{24}} - 5q^{\frac{25}{24}} + 7q^{\frac{49}{24}} - 11q^{\frac{121}{24}} + 13q^{\frac{169}{24}} - \dots$$

by Zwegers's [52] harmonic Maass form

$$M_f(z) := q^{-1}f(q^{24}) + \frac{i\sqrt{3}}{3} \int_{-24\bar{z}}^{i\infty} \frac{\Theta(\tau)}{\sqrt{-i(\tau+24z)}} d\tau.$$

Recall from Definition 1.3.3 that we refer to  $q^{-1}f(q^{24})$  as a *mock modular form*, and  $\Theta(z)$  as its *shadow*.

We do not know a simple relationship between the coefficients of  $f(q)$  and  $\Theta(z)$ . More generally, we have the following natural problem.

**Problem.** *Relate the coefficients of a mock modular form to the coefficients of its shadow.*

We solve this problem when the shadow is an integer weight Hecke eigenform. The results of this chapter were published with Pavel Guerzhoy and Ken Ono in [22].

### 3.1 Basic definitions and good Maass forms

For the remainder of the chapter, we fix a normalized (i.e.  $b_g(1) = 1$ ) Hecke eigenform

$$g(z) = \sum_{n=1}^{\infty} b_g(n)q^n \in S_k(N) \quad (3.1.1)$$

with weight  $2 \leq k \in \mathbb{Z}$ . Let  $E_g(z)$  be its *Eichler integral*

$$E_g(z) := \sum_{n=1}^{\infty} b_g(n)n^{1-k}q^n, \quad (3.1.2)$$

and let  $K_g$  be the number field obtained by adjoining to  $\mathbb{Q}$  the coefficients  $b_g(n)$ .

Following [14], we say that a harmonic Maass form  $f(z) \in H_{2-k}(N)$  is *good* for the Hecke eigenform

$$g^c(z) := \overline{g(-\bar{z})} = \sum_{n=1}^{\infty} \overline{b_g(n)}q^n \in S_k(N)$$

if it satisfies the following:

1. The principal part of  $f$  at the cusp  $\infty$  belongs to  $K_g[q^{-1}]$ .
2. The principal parts of  $f$  at other cusps (if any) are constant.
3. We have  $\xi_{2-k}(f) = \frac{g^c}{\langle g^c, g^c \rangle}$ , where  $\langle g^c, g^c \rangle$  is the Petersson product of  $g^c(z)$  with itself.

*Remark.* The existence of an  $f$  which is good for  $g^c$  is guaranteed by Proposition 5.1 of [14]. Moreover, because  $\ker(\xi_{2-k}) = M_{2-k}^1(N) \subset H_{2-k}^*(N)$ , different mock modular forms can cast the same shadow. Hence there are infinitely many harmonic Maass forms  $f$  that are good for  $g^c$ .

## 3.2 Algebraicity of mock modular forms

We now fix an  $f \in H_{2-k}(N)$  which is good for  $g^c$ . Its mock modular form is

$$f^+ = \sum_{n \gg -\infty} c_f(n)q^n. \quad (3.2.1)$$

Bruinier, Rhoades, and Ono [14] proved that  $f^+$  has algebraic coefficients if  $g$  has complex multiplication (CM). Otherwise, we expect a completely different phenomenon. For example, for  $g = \Delta$ , the unique normalized weight 12 cusp form on  $\mathrm{SL}_2(\mathbb{Z})$ , we have [40]

$$11!f^+ \sim 11!q^{-1} - \frac{2615348736000}{691} - 73562460235.68364q - 929026615019.11308q^2 - \dots \quad (3.2.2)$$

After the first two coefficients, the coefficients appear (see [40]) to be transcendental.

**Conjecture.** *Assume the notation and hypotheses above. The mock modular form  $f^+$  has some transcendental coefficients if and only if its shadow  $g$  does not have CM.*

Despite the ambiguity concerning the algebraicity of mock modular forms, we show that  $f^+$  may be regularized in a simple way to obtain an algebraic series.

**Theorem 3.2.1.** *Assume the notation and hypotheses above. If  $\alpha$  is a complex number for which  $\alpha - c_f(1) \in K_g$ , then the coefficients of*

$$\mathcal{F}_\alpha := f^+(z) - \alpha E_g(z) = \sum_{n \gg -\infty} c_f(n)q^n - \alpha \sum_{n=1}^{\infty} b_g(n)n^{1-k}q^n$$

*are in  $K_g$ . In particular, the transcendence degree of  $K_g(c_f(n))$  over  $K_g$  is at most one.*

*Remark.* Obviously, one may always let  $\alpha := c_f(1)$  in Theorem 3.2.1.

**Example 3.2.2.** For  $g = \Delta$ , if we let  $\alpha := c_f(1)$ , then Theorem 3.2.1 implies that  $\mathcal{F}_\alpha$  has  $\mathbb{Q}$ -rational coefficients. Numerically, we indeed find that

$$11!\mathcal{F}_\alpha = 11!q^{-1} - \frac{2615348736000}{691} - 929888675100q^2 - \frac{80840909811200}{9}q^3 - \dots \quad (3.2.3)$$

### 3.3 Coupling the pieces $p$ -adically

Now we fix a complex number  $\alpha$  for which  $\alpha - c_f(1) \in K_g$ . If  $\mathcal{F}_\alpha$  is as in Theorem 3.2.1, then we refer to this  $K_g$ -rational power series as a *regularized mock modular form*. We shall employ these regularizations to couple mock modular forms to their shadows.

Let  $p$  be prime. We fix an algebraic closure  $\overline{\mathbb{Q}}_p$  of  $\mathbb{Q}_p$ , along with an embedding of  $\overline{\mathbb{Q}}$  into  $\overline{\mathbb{Q}}_p$ . This embedding determines an extension of the  $p$ -adic valuation to  $K_g$ . We denote by  $\text{ord}_p : \overline{\mathbb{Q}}_p \rightarrow \mathbb{Q}$  the  $p$ -adic order normalized so that  $\text{ord}_p(p) = 1$ .

*Remark.* It will be clear from our results that the Fourier coefficients of  $\mathcal{F}_\alpha$  may have unbounded negative  $p$ -adic order (also see Remark 5 of [14]).

To relate  $\mathcal{F}_\alpha$  to  $g$ , we use the operator  $D := \frac{1}{2\pi i} \cdot \frac{d}{dz}$ , and we let

$$F_\alpha := D^{k-1} \mathcal{F}_\alpha = \sum_{n \gg -\infty} c_\alpha(n) q^n. \quad (3.3.1)$$

By Theorem 1.1 of [14], combined with the obvious fact that  $D^{k-1} E_g = g$ , it follows that

$$F_\alpha = \sum_{n \gg -\infty} (n^{k-1} c_f(n) - \alpha b_g(n)) q^n$$

is a weight  $k$  weakly holomorphic modular form in  $M_k^!(N)$ . We shall iteratively apply Atkin's  $U := U(p)$  operator to  $F_\alpha$  to couple mock modular forms with their shadows.

To state our result, let  $\beta, \beta'$  be the roots of the  $p$ th Hecke polynomial

$$X^2 - b_g(p)X + p^{k-1} = (X - \beta)(X - \beta') \quad (3.3.2)$$

ordered so that  $\text{ord}_p(\beta) \leq \text{ord}_p(\beta')$ . We then define the cusp form  $\check{g}$  by

$$\check{g} = \sum_{n=1}^{\infty} \check{b}_g(n) q^n := g(z) - \beta^{-1} p^{k-1} g(pz). \quad (3.3.3)$$

If  $p \nmid N$ , then  $\check{g} \in S_k(pN)$ .

We now solve the motivating problem for  $g^c$  by relating the coefficients of  $g$  and  $\check{g}$  to the coefficients of the regularized mock modular form  $\mathcal{F}_\alpha$ .

**Theorem 3.3.1.** *Assume the notation and hypotheses above.*

1. *Suppose that  $p \nmid N$  and  $\text{ord}_p(\beta) \neq (k-1)/2$ , or  $p \mid N$  and  $\beta \neq 0$ . For all but at most one choice of  $\alpha$  with  $\alpha - c_f(1) \in K_g$ , we have that*

$$\check{g} = \lim_{w \rightarrow +\infty} \frac{F_\alpha |_k U(p^w)}{c_\alpha(p^w)}.$$

2. *Suppose that  $g$  has CM. If  $p$  is inert in the field of complex multiplication, then for all but at most one choice of  $\alpha \in K_g$  we have that*

$$g = \lim_{w \rightarrow +\infty} \frac{F_\alpha |_k U(p^{2w+1})}{c_\alpha(p^{2w+1})}.$$

*Remark.* We comment on the limits in Theorem 3.3.1. It can happen that some of the coefficients appearing in the denominators of these formulas vanish. The proof of Theorem 3.3.1 will show that there are at most finitely many  $w$  for which these denominators vanish.

*Remark.* The proof of Theorem 3.3.1 can break down for one exceptional  $\alpha$ . For example, if  $g$  has CM, then  $\alpha = 0$  can be exceptional when  $p$  is a prime which splits in the field of complex multiplication. These exceptional cases are of interest, and they correspond to situations where one directly obtains  $p$ -adic modular forms without iteration.

**Example 3.3.2.** Theorem 3.3.1 implies infinitely many systematic congruences. For  $g = \Delta$  and  $p = 3$ , we have that  $\text{ord}_3(\beta) = 2$ , and also that  $\Delta \equiv \check{\Delta} \pmod{3^9}$ . Using (3.2.2) and (3.2.3), we find that the  $w = 1$  term in Theorem 3.3.1 (1) numerically gives

$$-\frac{D^{11} (f^+ - c_f(1) \sum_{n=1}^{\infty} \tau(n) n^{-11} q^n) |_k U(3)}{39862705122} \equiv \Delta \pmod{27}.$$

In Section 3.4 we prove Theorems 3.2.1 and 3.3.1. We prove these theorems by extending earlier work [14] of Bruinier, Rhoades, and Ono, combined with a careful combinatorial analysis of the action of the Hecke operators  $T(m)$ . This analysis gives the desired implications for the properties of iterations of Atkin's  $U(p)$  operator.

## 3.4 Proof of Theorems 3.2.1 and 3.3.1

Here we prove Theorems 3.2.1 and 3.3.1. We first prove Theorem 3.2.1 by refining earlier work of Bruinier, Rhoades and Ono [14].

### 3.4.1 Proof of Theorem 3.2.1

This result follows from a modification of the proof of Theorem 1.3 of [14]. Recall that we defined

$$F_\alpha(z) := D^{k-1} \mathcal{F}_\alpha(z) = \sum_{n \gg -\infty} c_\alpha(n) q^n.$$

By construction, we have that  $c_\alpha(1)$  is in  $K_g$ .

We use the action of the Hecke operators on  $f(z)$ . Let  $T(m)$  be the  $m$ th Hecke operator for the group  $\Gamma_0(N)$ . Using the same argument as in Lemma 7.4 of [13] (see [14], the proof of Theorem 1.3), we have that

$$f|_{2-k} T(m) = m^{1-k} b_g(m) f + R_m, \quad (3.4.1)$$

where  $R_m \in M_{2-k}^!(N)$  is a weakly holomorphic modular form with coefficients in  $K_g$ . The point is that  $f|_{2-k} T(m)$  and  $m^{1-k} b_g(m) f$  are harmonic Maass forms with equal non-holomorphic part. We apply the differential operator  $D^{k-1}$  to this identity, and we use the commutation relation

$$m^{k-1} D^{k-1} (H|_{2-k} T(m)) = (D^{k-1} H)|_k T(m)$$

which is valid for any 1-periodic function  $H$ . We obtain

$$(D^{k-1} f)|_k T(m) = b_g(m) (D^{k-1} f) + m^{k-1} D^{k-1} R_m.$$

Since  $D^{k-1} f = D^{k-1} f^+$ , and  $D^{k-1} E_g = g$ , and since  $g|_k T(m) = b_g(m) g$ , we conclude that

$$(D^{k-1} (f^+ - \alpha E_g))|_k T(m) = b_g(m) (D^{k-1} (f^+ - \alpha E_g)) + m^{k-1} D^{k-1} R_m. \quad (3.4.2)$$

We claim that the  $q$ -series  $F_\alpha = D^{k-1}(f^+ - \alpha E_g) = D^{k-1}\mathcal{F}_\alpha$  has its coefficients in  $K_g$ . Indeed, we make use of the formula for the action of Hecke operators on Fourier expansions, equate the coefficients of  $q^n$  in (3.4.2), and conclude that for any prime  $m$

$$c_\alpha(mn) + m^{k-1}c_\alpha(n/m) - b_g(m)c_\alpha(n) \in K_g.$$

An inductive argument, using the fact that  $c_\alpha(1)$  is in  $K_g$ , finishes the proof.

### 3.4.2 Proof of Theorem 3.3.1

We assume the notation and hypotheses from the introduction. We require the following elementary proposition.

**Proposition 3.4.1.** *If  $R \in M_{2-k}^!(N)$  has  $K_g$ -coefficients, then there is an  $A$  such that*

$$\text{ord}_p((D^{k-1}R)|_k U(p^n)) \geq n(k-1) - A.$$

*Proof.* The coefficients  $a(n)$  of  $R$  have bounded denominators. In other words, we have that  $A := -\inf_n(\text{ord}_p(a(n))) < \infty$ . Indeed, we can always multiply  $R$  by an appropriate power of  $\Delta$ , and obtain a cusp form of positive integer weight, which has Fourier coefficients with bounded denominators as a linear combination of forms with rational integral Fourier coefficients by Theorem 3.52 of [46]. Dividing back by the power of  $\Delta$  preserves this property since the coefficients of  $1/\Delta$  are integers. The proposition now follows easily from

$$(D^{k-1}R)|_k U(p^n) = \sum_{m \gg -\infty} (p^n m)^{k-1} a(p^n m) q^m. \quad \square$$

We now prove the existence of the limits which appear in Theorem 3.3.1.

**Proposition 3.4.2.** *Assuming the hypotheses in Theorem 3.3.1 (1), we have that*

$$\lim_{w \rightarrow \infty} \beta^{-w} F_\alpha|_k U(p^w) \in \overline{\mathbb{Q}_p}[[q]].$$



*Proof.* We assume that  $p \nmid N$ .

Recall that the weight  $k$  Hecke operator  $T(p)$  acts by

$$F_\alpha(z)|_k T(p) = F_\alpha(z)|_k U(p) + p^{k-1} F_\alpha(pz).$$

Then (3.4.2) with  $m = p$  gives

$$F_\alpha(z)|_k U(p) + \beta\beta' F_\alpha(pz) = (\beta + \beta') F_\alpha(z) + r,$$

where  $r := p^{k-1} D^{k-1} R_p$  is a weakly holomorphic modular form in  $M_k^!(N)$ . We made use of (3.3.2), which implies that

$$\beta + \beta' = b_g(p) \quad \text{and} \quad \beta\beta' = p^{k-1}.$$

Now we let

$$G(z) := F_\alpha(z) - \beta' F_\alpha(pz) \quad \text{and} \quad G'(z) := F_\alpha(z) - \beta F_\alpha(pz). \quad (3.4.3)$$

A simple calculation reveals that

$$G|_k U(p) = \beta G + r \quad \text{and} \quad G'|_k U(p) = \beta' G' + r,$$

and also that

$$F_\alpha|_k U(p) = \frac{\beta}{\beta - \beta'} (\beta G + r) - \frac{\beta'}{\beta - \beta'} (\beta' G' + r).$$

By induction, we find that

$$\begin{aligned} (\beta - \beta') \beta^{-w} F_\alpha|_k U(p^w) &= \left( \beta G + r + \frac{1}{\beta} r|_k U(p) + \dots + \frac{1}{\beta^{w-1}} r|_k U(p^{w-1}) \right) \\ &\quad - (\beta'/\beta)^w \left( \beta' G' + r + \frac{1}{\beta'} r|_k U(p) + \dots + \frac{1}{\beta'^{w-1}} r|_k U(p^{w-1}) \right). \end{aligned}$$

Proposition 3.4.2 now follows from Proposition 3.4.1 and this formula.  $\square$

Now we prove that the limits in Theorem 3.3.1 (2) are well defined.

**Proposition 3.4.3.** *Assuming the hypotheses in Theorem 3.3.1 (2), we have that*

$$\lim_{w \rightarrow \infty} \beta^{-2w} F_\alpha|_k U(p^{2w+1}) \in \overline{\mathbb{Q}}_p[[q]].$$

*Proof.* Since  $p$  is inert in the CM-field, we have  $\beta' = -\beta$ , and so  $\beta^2 = -p^{k-1}$ . As in the proof of Proposition 3.4.2, we rewrite equation (3.4.2) with  $m = p$  :

$$F_\alpha(z)|_k U(p) - \beta^2 F_\alpha(pz) = r,$$

where  $r := p^{k-1} D^{k-1} R_p$  is a weakly holomorphic modular form in  $M_k^!(N)$ . Thus we have that

$$F_\alpha|_k U(p^2) = \beta^2 F_\alpha + r|_k U(p).$$

Acting with the  $U$ -operator on this identity  $2w - 1$  times, we obtain

$$\beta^{-2w} F_\alpha|_k U(p^{2w+1}) = F_\alpha|_k U(p) + \beta^{-2} r|_k U(p^2) + \beta^{-4} r|_k U(p^4) + \dots + \beta^{-2w} r|_k U(p^{2w}). \quad (3.4.4)$$

As in Proposition 3.4.2, we conclude that the  $p$ -adic limit exists.  $\square$

*Proof of Theorem 3.3.1.* Here we prove Theorem 3.3.1 (1). The proof of Theorem 3.3.1 (2) is identical apart from the fact that one applies Proposition 3.4.3 in place of Proposition 3.4.2.

We begin by considering the first Fourier coefficient in Proposition 3.4.2, and put

$$L = L(\alpha) := \lim_{w \rightarrow \infty} \beta^{-w} c_\alpha(p^w).$$

Since  $\lim_{w \rightarrow \infty} \beta^{-w} b_g(p^w) = \beta/(\beta - \beta')$ , and  $c_\alpha(n) = n^{k-1} c_f(n) - \alpha b_g(n)$ , there is at most one choice of  $\alpha \in K_g$  for which  $L(\alpha) = 0$ . For non-exceptional  $\alpha$ , we can then conclude that  $c_\alpha(p^n) \neq 0$  for  $n \gg 0$ .

Let

$$L^{-1} \lim_{w \rightarrow \infty} \beta^{-w} F_\alpha|_k U(p^w) = \sum_{m>0} \check{c}(m) q^m. \quad (3.4.5)$$

It follows from the proof of Proposition 3.4.2 that

$$\left( \lim_{w \rightarrow \infty} \beta^{-w} F_\alpha|_k U(p^w) \right) |_k U(p) = \beta \left( \lim_{w \rightarrow \infty} \beta^{-w} F_\alpha|_k U(p^w) \right).$$

Therefore, by (3.3.3), (3.4.5), and the recursive formula for  $b_g(p^n)$ , we inductively find that

$$\check{c}(p^n) = \beta^n = \check{b}_g(p^n)$$

for all  $n \geq 0$ . For  $m > 0$  such that  $p \nmid m$ , we then have that

$$F_\alpha|_k T(m) = \check{b}_g(m)F_\alpha + r_m$$

with  $r_m = m^{k-1}D^{k-1}R_m$ . Since the operators  $U(p)$  and  $T(m)$  commute, we obtain

$$(F_\alpha|_k U(p^w))|_k T(m) = \check{b}_g(m) (F_\alpha|_k U(p^w)) + r_m|_k U(p^w).$$

We divide this equation by  $c_\alpha(p^w)$ , and then take the limit as  $w \rightarrow +\infty$ . By Proposition 3.4.1, the formulas for Hecke operators, and the property that Fourier coefficients of Hecke eigenforms are the eigenvalues, Theorem 3.3.1 (1) follows easily.  $\square$

# Chapter 4

## Eichler-Shimura theory for mock modular forms

Here we consider fundamental questions concerning periods and harmonic Maass forms. The results of this chapter will be published with Kathrin Bringmann, Pavel Guerzhoy, and Ken Ono in [8].

### 4.1 Introduction and statement of results

Let  $\mathcal{F}(z)$  be a harmonic Maass form with natural decomposition

$$\mathcal{F}(z) = \mathcal{F}^-(z) + \mathcal{F}^+(z),$$

where  $\mathcal{F}^-$  (resp.  $\mathcal{F}^+$ ) is nonholomorphic (resp. holomorphic) on the upper-half of the complex plane  $\mathbb{H}$ . The holomorphic part  $\mathcal{F}^+$  has a Fourier expansion

$$\mathcal{F}^+(z) = \sum_{n \gg -\infty} a_{\mathcal{F}}(n)q^n.$$

Shimura's work [45] on half-integral weight modular forms, for  $k \in 2\mathbb{Z}^+$ , provides maps which interrelate different spaces of modular forms. He defined surjective maps

$$Sh : S_{\frac{k+1}{2}}(4N) \twoheadrightarrow S_k(N),$$

which when combined with the preceding discussion, gives the following diagram:

$$\begin{array}{ccc}
H_{\frac{3-k}{2}}^*(4N) & \xrightarrow{\xi_{\frac{3-k}{2}}} & S_{\frac{k+1}{2}}(4N) \xrightarrow{Sh} S_k(N) \\
& & \uparrow \xi_{2-k} \\
& & H_{2-k}^*(N)
\end{array} \tag{4.1.1}$$

It is natural to study the arithmetic properties of this diagram. Since  $\xi_{\frac{3-k}{2}}$  and  $\xi_{2-k}$  only use the nonholomorphic parts of harmonic Maass forms, the main problem then is that of determining the arithmetic content of the holomorphic parts of these forms. What do they encode?

For Hecke eigenforms  $f \in S_2(N)$ , Bruinier and Ono [13] investigated this problem for the horizontal row of (4.1.1). Using important works of Gross and Zagier [19], of Kohnen and Zagier [33, 34], and of Waldspurger [48], they essentially proved that there is a form  $\mathfrak{F} = \mathfrak{F}^- + \mathfrak{F}^+ \in H_{\frac{1}{2}}^*(4N)$ , satisfying  $Sh(\xi_{\frac{1}{2}}(\mathfrak{F})) = f$ , which has the property that the coefficients of the mock modular form  $\mathfrak{F}^+$  (resp.  $\mathfrak{F}^-$ ) determine the nonvanishing of the central derivatives (resp. values) of the quadratic twist  $L$ -functions  $L(f, \chi_D, s)$ .

Here we study the vertical map in (4.1.1), and we show that forms  $\mathcal{F} \in H_{2-k}^*$  beautifully encode the critical values of  $L$ -functions arising from  $S_k$ . We assume throughout this chapter that  $k \geq 4$  is an even integer and that we are always working with the full modular group,  $SL_2(\mathbb{Z})$ .

For each  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ , we define the  $\gamma$ -mock modular period function for  $\mathcal{F}^+$  by

$$\mathbb{P}(\mathcal{F}^+, \gamma; z) := \frac{(4\pi)^{k-1}}{\Gamma(k-1)} \cdot (\mathcal{F}^+ - \mathcal{F}^+|_{2-k}\gamma)(z), \tag{4.1.2}$$

where  $(g|_w\gamma)(z) := (cz+d)^{-w}g\left(\frac{az+b}{cz+d}\right)$ . The map

$$\gamma \mapsto \mathbb{P}(\mathcal{F}^+, \gamma; z)$$

gives an element in the first cohomology group of  $SL_2(\mathbb{Z})$  with polynomial coefficients, and we shall see that they are intimately related to classical ‘‘period polynomials’’.

For positive  $c$ , let  $\zeta_c := e^{2\pi i/c}$ , and for  $0 \leq d < c$ , let  $\gamma_{c,d} \in \mathrm{SL}_2(\mathbb{Z})$  be any matrix satisfying  $\gamma_{c,d} := \begin{pmatrix} * & * \\ c & d' \end{pmatrix}$ . Here the integers  $0 \leq d' < c'$  are chosen so that  $\frac{d}{c} = \frac{d'}{c'}$  in lowest terms.

**Theorem 4.1.1.** *Let  $\mathcal{F} \in H_{2-k}^*$  and  $f = \xi_{2-k}(\mathcal{F})$ . Then we have that*

$$\overline{\mathbb{P}(\mathcal{F}^+, \gamma_{1,0}; \bar{z})} = \sum_{n=0}^{k-2} \frac{L(f, n+1)}{(k-2-n)!} \cdot (2\pi iz)^{k-2-n}.$$

Moreover, if  $\chi \pmod{c}$  is a Dirichlet character, then

$$\frac{1}{c} \sum_{m \in (\mathbb{Z}/c\mathbb{Z})^\times} \chi(m) \sum_{d=0}^{c-1} \zeta_c^{md} \cdot \overline{\mathbb{P}\left(\mathcal{F}^+, \gamma_{c,d}; \bar{z} - \frac{d}{c}\right)} = \sum_{n=0}^{k-2} \frac{L(f, \chi, n+1)}{(k-2-n)!} \cdot (2\pi iz)^{k-2-n}.$$

Here  $L(f, s)$  (resp.  $L(f, \chi, s)$ ) is the usual  $L$ -function (resp. twisted by  $\chi$ ) for  $f$ .

*Remark.* Theorem 4.1.1, which can be (nontrivially) generalized to arbitrary levels, is related to Manin's observation [38] that twisted  $L$ -values may be given as expressions involving periods. These expressions are typically quite complicated. The theory underlying Theorem 4.1.1 relates the mock modular periods to such periods, and does so in a way which gives nice generating functions.

Theorem 4.1.1 follows from the ‘‘Eichler-Shimura theory’’ we obtain for level 1 weakly holomorphic modular forms and mock modular forms. The pioneering work of Eichler [16] and Shimura [44], expounded upon by Manin [38], is fundamental in the theory of modular forms, and it has deep implications for elliptic curves and critical values of  $L$ -functions. One of the main features of the theory is the Eichler-Shimura isomorphism, which relates spaces of cusp forms to the first parabolic cohomology groups for  $\mathrm{SL}_2(\mathbb{Z})$  with polynomial coefficients. We recall this result now following the discussion in [35, 50].

The matrices  $S$  and  $U$  are defined as in Proposition 1.2.1. Let

$$\mathbf{V} := \mathbf{V}_{k-2}(\mathbb{C}) = \mathrm{Sym}^{k-2}(\mathbb{C} \oplus \mathbb{C})$$

be the linear space of polynomials of degree  $\leq k-2$  in  $z$ . Let

$$\mathbf{W} := \{P \in \mathbf{V} : P + P|_{2-k}S = P + P|_{2-k}U + P|_{2-k}U^2 = 0\}. \quad (4.1.3)$$

The space  $\mathbf{V}$  splits as a direct sum  $\mathbf{V} = \mathbf{V}^+ \oplus \mathbf{V}^-$  of even and odd polynomials. Putting  $\mathbf{W}^\pm := \mathbf{W} \cap \mathbf{V}^\pm$  one obtains the splitting  $\mathbf{W} = \mathbf{W}^+ \oplus \mathbf{W}^-$ .

There are two period maps  $r^\pm : S_k \longrightarrow \mathbf{W}^\pm$

$$\begin{aligned} r^+(f; z) &:= \sum_{\substack{0 \leq n \leq k-2 \\ n \text{ even}}} (-1)^{\frac{n}{2}} \binom{k-2}{n} \cdot r_n(f) \cdot z^{k-2-n}, \\ r^-(f; z) &:= \sum_{\substack{0 \leq n \leq k-2 \\ n \text{ odd}}} (-1)^{\frac{n-1}{2}} \binom{k-2}{n} \cdot r_n(f) \cdot z^{k-2-n}, \end{aligned}$$

where, for each integer  $0 \leq n \leq k-2$ , the  $n$ th *period* of  $f$  is defined by

$$r_n(f) := \int_0^\infty f(it)t^n dt. \quad (4.1.4)$$

Notice that if we let  $r(f; z) := r^-(f; z) + ir^+(f; z)$ , then

$$r(f; z) = \sum_{n=0}^{k-2} i^{-n+1} \binom{k-2}{n} \cdot r_n(f) \cdot z^{k-2-n} = \int_0^{i\infty} f(\tau)(z-\tau)^{k-2} d\tau. \quad (4.1.5)$$

The Eichler-Shimura isomorphism theorem asserts that  $r^-$  (resp.  $r^+$ ) is an isomorphism onto  $\mathbf{W}^-$  (resp.  $\mathbf{W}_0^+ \subseteq \mathbf{W}^+$ , the codimension 1 subspace not containing  $z^{k-2} - 1$ ). Therefore  $\mathbf{W}_0 \subseteq \mathbf{W}$ , the corresponding codimension 1 subspace, represents two copies of  $S_k$ .

We derive a second Eichler-Shimura isomorphism theorem for  $\mathbf{W}_0$ , one involving weakly holomorphic cusp forms. A form  $F \in S_k^!$  has a Fourier expansion of the form

$$F(z) = \sum_{\substack{n \gg -\infty \\ n \neq 0}} a_F(n)q^n.$$

Our work depends on an extension to  $M_k^!$  of the map  $r = r^- + ir^+$ . Since the integrals in (4.1.4) diverge for forms with poles, the extension must be obtained by other means. To define it, suppose that  $F(z) = \sum_{n \gg -\infty} a_F(n)q^n \in M_k^!$ . Its *Eichler integral* [36] is

$$\mathcal{E}_F(z) := \sum_{\substack{n \gg -\infty \\ n \neq 0}} a_F(n)n^{1-k}q^n. \quad (4.1.6)$$

We define the *period function* for  $F$  by

$$r(F; z) := c_k (\mathcal{E}_F - \mathcal{E}_F|_{2-k} S)(z), \quad (4.1.7)$$

where  $c_k := -\frac{\Gamma(k-1)}{(2\pi i)^{k-1}}$ . If  $F$  is a cusp form, then one easily sees that

$$\mathcal{E}_F(z) = \frac{1}{c_k} \cdot \int_z^{i\infty} F(\tau)(z - \tau)^{k-2} d\tau, \quad (4.1.8)$$

and so (4.1.5) implies that (4.1.7) indeed extends the classical period map  $r = r^- + ir^+$ .

*Remark.* A first trivial observation is that  $r(F; z) = \alpha(z^{k-2} - 1)$  if and only if  $\mathcal{E}_F(z) + \frac{\alpha}{c_k}$  is in  $M_{2-k}^!$ .

The period functions  $r(F; z)$  are essentially polynomials in  $z$  with degree  $\leq k-2$ . The contribution from the constant term  $a_F(0)$ , which is a multiple of  $z^{k-1} + \frac{1}{z}$ , poses the only obstruction. Therefore, in analogy with (4.1.5), we define  $r_n(F)$ , the periods of  $F$ , by

$$r(F; z) = \frac{a_F(0)}{k-1} \cdot \left( z^{k-1} + \frac{1}{z} \right) + \sum_{n=0}^{k-2} i^{-n+1} \binom{k-2}{n} \cdot r_n(F) \cdot z^{k-2-n}. \quad (4.1.9)$$

The extended period function  $r$ , restricted to  $S_k^!$ , defines a surjective map:

$$r : S_k^! \twoheadrightarrow \mathbf{W}_0.$$

The ‘‘Eichler-Shimura’’ isomorphism we obtain concerns this map. To compute its kernel, we use the differential operator  $D := \frac{1}{2\pi i} \cdot \frac{d}{dz}$  which, by a well known identity of Bol (see Th. 1.2 of [14]), has the property that the following map is well-defined

$$D^{k-1} : M_{2-k}^! \longrightarrow S_k^!.$$

We prove the following isomorphism theorem.

**Theorem 4.1.2.** *The following sequence is exact*

$$0 \rightarrow D^{k-1}(M_{2-k}^!) \rightarrow S_k^! \rightarrow \mathbf{W}_0 \rightarrow 0.$$



This theorem sheds light on the classical Eichler-Shimura isomorphism. The maps  $r^\pm : S_k \rightarrow \mathbf{W}_0$  each give one copy of  $S_k$ . Theorem 4.1.2 gives two copies of  $S_k$  in a different way, namely one involving the interplay between mock modular forms, weakly holomorphic modular forms, and weakly holomorphic cusp forms. To make this precise, we note that Bol's identity, generalized to harmonic Maass forms, gives the following diagram:

$$\begin{array}{ccccc}
D^{k-1}(M_{2-k}^!) & \hookrightarrow & S_k^! & \xrightarrow{r} & \mathbf{W}_0 & \xleftarrow{r} & S_k \\
\uparrow D^{k-1} & & \uparrow D^{k-1} & & & \nearrow \xi & \\
M_{2-k}^! & \hookrightarrow & H_{2-k}^* & & & & 
\end{array}$$

Moreover, we have that  $D^{k-1}$  only sees the holomorphic parts  $\mathcal{F}^+$  of harmonic Maass forms  $\mathcal{F} \in H_{2-k}^*$  (i.e.  $D^{k-1}(\mathcal{F}) = D^{k-1}(\mathcal{F}^+)$ ). It turns out that the two copies of  $S_k$  arise from the quotient space  $H_{2-k}^*/M_{2-k}^!$  and the inclusion of  $S_k \subseteq S_k^!$ , respectively. In particular, we have

$$\mathbf{W}_0 \cong r(D^{k-1}(H_{2-k}^*)) \oplus r(\xi(H_{2-k}^*)).$$

To make this precise, we relate the three period functions

$$r(\xi_{2-k}(\mathcal{F}^-); z), \quad r(D^{k-1}(\mathcal{F}^+); z), \quad \text{and} \quad \mathbb{P}(\mathcal{F}^+, \gamma_{1,0}; z).$$

We show that these functions are essentially equal up to complex conjugation and the change of variable  $z \rightarrow \bar{z}$ . Strictly speaking, our functions are not defined for  $\bar{z}$ . However, since we apply complex conjugation these period functions are well defined.

We obtain the following period relations on  $H_{2-k}$ .

**Theorem 4.1.3.** *If  $\mathcal{F} \in H_{2-k}$ , then we have that*

$$r(\xi_{2-k}(\mathcal{F}); z) \equiv -\frac{(4\pi)^{k-1}}{\Gamma(k-1)} \cdot \overline{r(D^{k-1}(\mathcal{F}); \bar{z})} \pmod{z^{k-2} - 1}.$$

Moreover, there is a function  $\widehat{\mathcal{F}} \in H_{2-k}$  for which  $\xi_{2-k}(\widehat{\mathcal{F}}) = \xi_{2-k}(\mathcal{F})$  and

$$r(\xi_{2-k}(\mathcal{F}); z) = -\frac{(4\pi)^{k-1}}{\Gamma(k-1)} \cdot \overline{r(D^{k-1}(\widehat{\mathcal{F}}); \bar{z})}.$$

*Two remarks*

1) If  $\mathcal{F} \in H_{2-k}^*$  has constant term 0, then we have the following mock modular period identity:

$$r(D^{k-1}(\mathcal{F}); z) = r(D^{k-1}(\mathcal{F}^+); z) = c_k \frac{\Gamma(k-1)}{(4\pi)^{k-1}} \cdot \mathbb{P}(\mathcal{F}^+, \gamma_{1,0}; z). \quad (4.1.10)$$

Moreover, we shall show that there always are forms  $F \in M_{2-k}^!$  for which  $\mathcal{F} + F \in H_{2-k}^*$  has constant term zero.

2) Since  $D^{k-1}$  annihilates constants, one cannot avoid the  $z^{k-2} - 1$  ambiguity in Theorem 4.1.3. Many of the technical difficulties in this chapter arise from the need to carefully take into account the constant terms of Maass-Poincaré series and their corresponding Eisenstein series. This issue is even more complicated in the setting of congruence subgroups. This is why we are content to work in the setting of the full modular group  $\mathrm{SL}_2(\mathbb{Z})$ .

There is a theory of Hecke operators on  $S_k^! / D^{k-1}(M_{2-k}^!)$ . For any positive integer  $m \geq 2$ , let  $T(m)$  be the usual weight  $k$  index  $m$  Hecke operator as introduced in Definition 1.2.9. We say that  $F \in S_k^!$  is a *Hecke eigenform* with respect to  $S_k^! / D^{k-1}(M_{2-k}^!)$  if for every Hecke operator  $T(m)$  there is a complex number  $b(m)$  for which

$$(F |_k T(m))(z) - b(m)F(z) \in D^{k-1}(M_{2-k}^!).$$

This definition includes the usual notion of Hecke eigenforms for (holomorphic) cusp forms as given in Definition 1.2.12. Indeed, in this case we simply have

$$(F |_k T(m))(z) - b(m)F(z) = 0.$$

It is natural to determine the dimension of those subspaces which correspond to a system of Hecke eigenvalues. We prove the following “multiplicity two” theorem.

**Theorem 4.1.4.** *Let  $d = \dim S_k$ , and let  $f_i = \sum b_i(n)q^n \in S_k$  be a basis consisting of normalized Hecke eigenforms. The  $2d$ -dimensional space  $S_k^! / D^{k-1}(M_{2-k}^!)$  splits into a direct sum*

$$S_k^! / D^{k-1}(M_{2-k}^!) = \bigoplus_{i=1}^d \mathbb{T}_i$$

*of two-dimensional spaces  $\mathbb{T}_i$  such that  $f_i \in \mathbb{T}_i$ , and every element of  $\mathbb{T}_i$  is a Hecke eigenform with the same Hecke eigenvalues as  $f_i$ .*

*Remark.* This “multiplicity two” phenomenon also appears in a recent paper by Guerzhoy [20]. Here we give a different proof.

We conclude with a study of Petersson’s inner product, and a related inner product of Bruinier and Funke [11]. The Petersson inner product of cusp forms  $f_1, f_2 \in S_k$  is the hermitian (i.e.  $(f_1, f_2) = \overline{(f_2, f_1)}$ ) scalar product defined by ( $z = x + iy$ )

$$(f_1, f_2) := \int_{\mathbb{H}/\mathrm{SL}_2(\mathbb{Z})} f_1(z) \overline{f_2(z)} y^k \cdot \frac{dx dy}{y^2}. \quad (4.1.11)$$

It is natural to seek an extension of this inner product to  $M_k^!$ . Obviously, one faces problems related to the convergence of the defining integral. Zagier [49, 50] extended the product to Eisenstein series using Rankin’s method. More generally, Borcherds [3] (see [14] for a discussion) defined an extension to  $M_k^!$  using regularized integrals, when at least one of the forms is holomorphic at the cusps. Here we give a closed formula for Borcherds’s extension using periods of weakly holomorphic modular forms.

We relate Petersson’s inner product to the “inner product”  $\{\bullet, \bullet\}$  on  $M_k^!$  which is defined as follows (also see discussions in [11, 14]). If  $F, G \in M_k^!$  have expansions

$$F(z) = \sum_{n \gg -\infty} a_F(n) q^n \quad \text{and} \quad G(z) = \sum_{n \gg -\infty} a_G(n) q^n,$$

then define  $\{F, G\}$  by

$$\{F, G\} := \sum_{\substack{n \in \mathbb{Z} \\ n \neq 0}} \frac{a_F(-n) a_G(n)}{n^{k-1}}. \quad (4.1.12)$$

This antisymmetric (i.e.  $\{F, G\} = -\{G, F\}$ ) pairing dissects  $D^{k-1}(M_{2-k}^!)$  from  $S_k^!$ .

**Theorem 4.1.5.** *Let  $F \in S_k^!$ . The following conditions are equivalent:*

- (i)  $F \in D^{k-1}(M_{2-k}^!)$ ,
- (ii)  $r(F; z) \equiv 0 \pmod{z^{k-2} - 1}$ ,
- (iii)  $\{F, G\} = 0$ , for every  $G \in S_k^!$ .

We now explain how to compute the extended inner product  $(\bullet, \bullet)$ , as given in Definition 4.1.11, in terms of  $\{\bullet, \bullet\}$ . Suppose that  $F, G \in M_k^!$ , and that  $\mathcal{G} \in H_{2-k}$  has the property that  $\xi_{2-k}(\mathcal{G}) = G$ . We shall show that

$$(F, G) = \{F, D^{k-1}(\mathcal{G})\} + a_F(0) \cdot a_{\mathcal{G}}^+(0), \quad (4.1.13)$$

whenever one of the forms  $F$  or  $G$  is holomorphic and where  $a_{\mathcal{G}}^+(0)$  is the constant term of the mock modular form  $\mathcal{G}^+$ . Computing  $(F, G)$  then reduces to the problem of computing  $\{\bullet, \bullet\}$  on  $M_k^!$ .

*Two Remarks.*

- 1) Formula (4.1.13) gives an extension of the Petersson inner product, one which works even when other “regularizations” fail.
- 2) Although there is ambiguity in the choice of  $\mathcal{G} \in H_{2-k}$  such that  $\xi_{2-k}(\mathcal{G}) = G$ , we stress that the right-hand side of (4.1.13) does not depend on this choice.

Generalizing an argument of Kohnen and Zagier [35], we obtain the following closed formula for these products, which is an analog of a classical result of Haberland [23, 35].

**Theorem 4.1.6.** *For  $F, G \in M_k^!$  we have*

$$\begin{aligned} \{F, G\} &= \frac{(2\pi)^{k-1}}{3 \cdot (k-2)!} \sum_{\substack{0 \leq m < n \leq k-2 \\ m \not\equiv n \pmod{2}}} i^{(n+1+m)} \binom{k-2}{n} \binom{n}{m} r_n(F) r_{k-2-m}(G) \\ &+ \frac{2 \cdot (2\pi)^{k-1}}{3 \cdot (k-2)!} \sum_{\substack{0 \leq n \leq k-2 \\ n \equiv 0 \pmod{2}}} i^{(k-n)} \binom{k-1}{n+1} \left( r_n(G) \frac{a_F(0)}{k-1} - r_n(F) \cdot \frac{a_G(0)}{k-1} \right). \end{aligned}$$

In Section 4.3 we derive some fundamental properties of the period functions and certain auxiliary integrals, and we conclude with a proof of Theorem 4.1.3. In Section

4.4, we study Borchers's extension of the Petersson inner product, and we conclude with proofs of Theorems 4.1.2, 4.1.4, 4.1.5, and 4.1.6. In Section 4.5 we recall some crucial analytic number theory which relates Eichler integrals to critical values of  $L$ -functions, and we prove Theorem 4.1.1.

## 4.2 Decomposition of $S_k^!$

The following proposition, whose proof uses Theorem 4.1.3, allows us to decompose a form  $F \in S_k^!$  uniquely into a cusp form and an element in  $D^{k-1}(H_{2-k}^*)$ .

**Proposition 4.2.1.** *Each  $F \in S_k^!$  has a unique representation of the form*

$$F(z) = \phi(z) + \psi(z),$$

where  $\phi \in S_k$  and  $\psi \in D^{k-1}(H_{2-k}^*)$ .

*Proof.* First we show that such a representation, if it exists, is unique. Suppose on the contrary that  $\widehat{\psi}_1, \widehat{\psi}_2 \in H_{2-k}^*$  have the property that

$$F(z) = \phi_1(z) + D^{k-1}(\widehat{\psi}_1)(z) = \phi_2(z) + D^{k-1}(\widehat{\psi}_2)(z),$$

where  $\phi_1, \phi_2 \in S_k$ . Then  $D^{k-1}(\widehat{\psi}_1 - \widehat{\psi}_2)$  is a cusp form, thus the function  $\widehat{\psi}_1 - \widehat{\psi}_2$  has (up to the constant term) no principal part. Since this function is also in  $H_{2-k}^*$  it must be 0.

Now we establish the existence of the desired representation. By modularity, it follows that  $r(F; z) = r^-(F; z) + ir^+(F; z) \in \mathbf{W}_0$ . The classical Eichler-Shimura isomorphism guarantees the existence of cusp forms  $g_1, g_2 \in S_k$  such that

$$r^-(F; z) = r^-(g_1; z) \quad \text{and} \quad r^+(F; z) \equiv r^+(g_2; z) \pmod{z^{k-2} - 1}.$$

By Proposition 1.3.2 (1), the operator  $\xi_{2-k}$  maps  $H_{2-k}^*$  onto  $S_k$ . Therefore there are harmonic Maass forms  $\mathcal{G}_1, \mathcal{G}_2 \in H_{2-k}^*$  for which  $\xi_{2-k}(\mathcal{G}_i) = (2i)^{k-1}g_i^c$ , which one checks are also in  $S_k$ . The fundamental theorem of calculus (with respect to  $\bar{z}$ ) then implies

$$\mathcal{G}_i(z) = \mathcal{G}_i^+(z) + \int_{-\bar{z}}^{i\infty} g_i(\tau)(\tau + z)^{k-2} d\tau,$$

where the  $\mathcal{G}_i^+$  are holomorphic functions on  $\mathbb{H}$ .

The proof of Theorem 4.1.3 (see expression (4.3.10)) then implies that

$$r(D^{k-1}(\mathcal{G}_i); -z) \equiv -c_k \cdot r(g_i; z) \pmod{z^{k-2} - 1}.$$

We let

$$\phi(z) := \frac{g_1(z) + g_2(z)}{2} \quad \text{and} \quad \Psi(z) := \frac{D^{k-1}(\mathcal{G}_1)(z) - D^{k-1}(\mathcal{G}_2)(z)}{2c_k},$$

and obtain that

$$r(F; z) \equiv r(\phi + \Psi; z) \pmod{z^{k-2} - 1}. \quad (4.2.1)$$

Now define

$$h(z) := F(z) - \phi(z) - \Psi(z) \in S_k^!,$$

and observe that by (4.2.1), we have that  $r(h; z) = \alpha(z^{k-2} - 1)$  for some  $\alpha \in \mathbb{C}$ . Of course, this then means that  $\mathcal{E}_h(z) + \frac{\alpha}{c_k} \in M_{2-k}^!$ . Consequently, we then have that

$$h = D^{k-1}(\mathcal{E}_h) = D^{k-1}\left(\mathcal{E}_h + \frac{\alpha}{c_k}\right) \in D^{k-1}(M_{2-k}^!).$$

Letting  $\psi = \Psi + h$  we obtain the desired decomposition. □

## 4.3 Properties of period functions

Here we consider auxiliary functions related to period functions, and we then give some consequences for the period functions of harmonic Maass forms and weakly holomorphic modular forms. We then conclude with the proof of Theorem 4.1.3.

### 4.3.1 Some auxiliary functions related to periods

Here we define auxiliary functions which relate period functions of weakly holomorphic modular forms to Eichler integrals.

Recall again that if  $g \in S_k$ , then

$$c_k \mathcal{E}_g(z) = \int_z^{i\infty} g(\tau)(\tau - z)^{k-2} d\tau.$$

Although such integrals do not converge for  $G \in S_k^!$  with a pole at infinity, for  $\rho := \frac{1+\sqrt{-3}}{2}$  we have the convergent integral

$$\mathcal{E}_G^\rho(z) := \int_z^\rho G(\tau)(\tau - z)^{k-2} d\tau. \quad (4.3.1)$$

An induction argument shows that, for any integer  $n \geq 0$ ,

$$\int_z^\rho G(\tau)(\tau - z)^n d\tau = n! \int_z^\rho \int_{z_n}^\rho \cdots \int_{z_1}^\rho G(z_0) dz_0 \cdots dz_{n-1} dz_n.$$

It follows that

$$D^{k-1}(\mathcal{E}_G^\rho(z)) = c_k G(z),$$

and by (4.1.6) we have that

$$\mathcal{E}_G^\rho(z) = c_k \mathcal{E}_G(z) + q_G(z), \quad (4.3.2)$$

where  $q_G(z)$  is a polynomial of degree  $\leq k - 2$ .

*Remark.* The discussion above holds if  $\rho$  is replaced by any point in  $\mathbb{H}$ . However, the subsequent discussion will make important use of the fact that  $\rho$  is an elliptic fixed point. We could have chosen  $\rho^2$  or  $i$  in its place.

We also require the auxiliary function

$$H_G(z) := \int_{\rho^2}^\rho G(\tau)(z - \tau)^{k-2} d\tau. \quad (4.3.3)$$

We note that  $z$  in this setting is not required to be an element of  $\mathbb{H}$ . In the next proposition we record some properties of the functions  $r(G; z)$ ,  $q_G$ , and  $H_G$  involving the action of the matrices  $S$  and  $T$  as defined in Proposition 1.2.1.

**Proposition 4.3.1.** *Suppose that  $G \in S_k^!$ . Then the following are true:*

1. *We have that*

$$H_G(z) = (\mathcal{E}_G^\rho|_{2-k}(1-S))(z) = (\mathcal{E}_G^\rho|_{2-k}(1-T))(z).$$

2. *We have that*

$$(H_G|_{2-k}(1+S))(z) = 0.$$

3. *We have that*

$$H_G(z) = (q_G|_{2-k}(1-T))(z) = r(G; z) + (q_G|_{2-k}(1-S))(z).$$

4. *We have that*

$$r(G; z) = (q_G|_{2-k}(S-T))(z).$$

*Proof.* Claim (1) follows from the fact that  $-\rho^{-1} = \rho - 1 = \rho^2$ , and claim (2) follows by (1). Claim (3) is obtained by applying  $(1-S)$  and  $(1-T)$  to (4.3.2), and (4) follows immediately from (3).  $\square$

We also require a nonholomorphic analog of  $\mathcal{E}_G^\rho$ , namely the function

$$\Phi_G(z) := \int_{-\bar{z}}^{\rho} G(\tau)(\tau+z)^{k-2} d\tau. \quad (4.3.4)$$

**Proposition 4.3.2.** *Suppose that  $G \in S_k^!$ . Then the following are true:*

1. *We have that*

$$(\Phi_G|_{2-k}T^{-1})(z) = (\Phi_G|_{2-k}S)(z) = \int_{-\bar{z}}^{\rho^2} G(\tau)(\tau+z)^{k-2} d\tau.$$

2. *We have that*

$$H_G(-z) = (\Phi_G|_{2-k}(1-T^{-1}))(z) = (\Phi_G|_{2-k}(1-S))(z).$$

*Proof.* Claim (1), which follows by substitution, immediately implies (2).  $\square$



### 4.3.2 The role of harmonic Maass forms

Here we obtain relations between  $\Phi_G$  and harmonic Maass forms. As in the proof of Proposition 4.2.1, we make use of an involution on  $M_k^!$  which preserves the space  $S_k^!$ . If  $G \in M_k^!$ , then it is defined by

$$G^c(z) := \overline{G(-\bar{z})}. \quad (4.3.5)$$

Therefore, we find that that  $\mathcal{E}_{G^c}(z) = \overline{\mathcal{E}_G(-\bar{z})}$ , which in turn implies that

$$r(G^c; z) = -\overline{r(G; -\bar{z})}. \quad (4.3.6)$$

Suppose that  $G \in S_k^!$  is fixed. By Proposition 1.3.2 (1), let  $\mathcal{F} \in H_{2-k}$  be a harmonic Maass form for which  $\xi_{2-k}(\mathcal{F})(z) = (2i)^{k-1}G^c(z)$ . The fundamental theorem of calculus (with respect to  $\bar{z}$ ), then implies that

$$\mathcal{F}(z) = \int_{-\bar{z}}^{\rho^2} G(\tau)(\tau + z)^{k-2} d\tau + C_G(z), \quad (4.3.7)$$

where  $C_G$  is holomorphic on  $\mathbb{H}$ . The next proposition relates  $\Phi_G$  and  $C_G$ .

**Proposition 4.3.3.** *Assume the notation and hypotheses above. Then the following are true:*

1. *We have that*

$$\Phi_G(z) = \mathcal{F}(z) - (C_G|_{2-k}T)(z) = \mathcal{F}(z) - (C_G|_{2-k}S)(z).$$

2. *We have that*

$$(C_G|_{2-k}T)(z) = (C_G|_{2-k}S)(z).$$

*Proof.* By (4.3.7) and Proposition 4.3.2 (1), we have that

$$(\Phi_G|_{2-k}T^{-1})(z) = (\Phi_G|_{2-k}S)(z) = \mathcal{F}(z) - C_G(z).$$

We obtain (1) by applying  $T$  and  $S$  to  $\mathcal{F}$ , and (2) follows immediately from (1).  $\square$

To prove Theorem 4.1.3, we shall make use of the following elementary proposition.

**Proposition 4.3.4.** *For polynomials  $p(z)$  of degree at most  $-\ell \in 2\mathbb{N}$ , let  $\tilde{p}(z) := p(-z)$ . Then*

$$\left(\widetilde{p|_l S}\right)(z) = (\tilde{p}|_l S)(z) \quad \text{and} \quad \left(\widetilde{p|_l T}\right)(z) = (\tilde{p}|_l T^{-1})(z).$$

### 4.3.3 The proof of Theorem 4.1.3

We require the following proposition.

**Proposition 4.3.5.** *There are forms in  $M_{2-k}^!$  with nonzero constant terms.*

*Proof.* We begin with the following statement of Petersson, which was proven independently by Borcherds using Serre duality on Riemann surfaces in [4]. There exists a weakly holomorphic form  $F \in M_{2-k}^!$  with prescribed principal part

$$\sum_{-\infty \ll n < 0} a_F(n)q^n$$

if and only if

$$\{D^{k-1}(F), f\} = 0 \tag{4.3.8}$$

for all cusp forms  $f \in S_k$ .

Now, if  $F \in M_{2-k}^!$ , then  $F \cdot E_k \in M_2^!$ , and all such forms are derivatives of polynomials in the  $j$ -invariant, so the constant term of  $F \cdot E_k$  is zero. We can express this as the following:

$$a_F(0) = \{D^{k-1}(F), E_k\}. \tag{4.3.9}$$

Let  $d = \dim(M_k)$  and  $f_1, \dots, f_{d-1} \in S_k$  be a basis for  $S_k$  with  $q$ -expansions given by  $f_i(z) = \sum_{n>0} c_i(n)q^n$ . Also, write the  $q$ -expansion of  $E_k(z)$  as  $E_k(z) = \sum_{n \geq 0} c_d(n)q^n$ . To construct a form  $F \in M_{2-k}^!$  with nonzero constant term  $a_F(0)$ , it suffices to find a solution  $\{a_F(n)\}_{n \neq 0}$ , for a large enough integer  $m$ , of the following

linear system coming from (4.3.8) and (4.3.9):

$$\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} c_1(1) & c_1(2) & \dots & c_1(m) \\ c_2(1) & c_2(2) & \dots & c_2(m) \\ \vdots & \vdots & & \vdots \\ c_{d-1}(1) & c_{d-1}(2) & \dots & c_{d-1}(m) \\ c_d(1) & c_d(2) & \dots & c_d(m) \end{bmatrix} \cdot \begin{bmatrix} a_F(-1) \\ a_F(-2) \\ \vdots \\ a_F(1-m) \\ a_F(-m) \end{bmatrix}.$$

A solution exists if and only if the  $d \times m$  matrix has rank  $d$ . For a large enough integer  $m$ , the row rank is  $d$  because the forms  $E_k, f_1, \dots, f_{d-1}$  are linearly independent.  $\square$

We now prove Theorem 4.1.3.

*Proof of Theorem 4.1.3.* We begin by proving the first claim in Theorem 4.1.3. We continue using the notation and hypotheses on  $\mathcal{F}$  and  $G$  from the previous subsection. Namely, we assume that  $G \in S_k^!$ , and  $\mathcal{F} \in H_{2-k}$  satisfy

$$\xi_{2-k}(\mathcal{F}) = (2i)^{k-1} G^c(z).$$

Now let  $F := D^{k-1}(\mathcal{F})$ .

It suffices to prove that

$$\widetilde{r(F; z)} \equiv -c_k \cdot r(G; z) \pmod{z^{k-2} - 1}. \quad (4.3.10)$$

Let  $p_G$  be the holomorphic function given by

$$p_G(z) := C_G(z) - \mathcal{E}_F(z).$$

Since we have that

$$D^{k-1}(p_G)(z) = D^{k-1}(C_G(z)) - D^{k-1}(\mathcal{E}_F(z)) = F(z) - F(z) = 0,$$

it follows that  $p_G$  is a polynomial of degree  $\leq k-2$ . By definition (4.1.7), we obtain, by applying  $S$  to the definition of  $p_G$ , that

$$(p_G|_{2-k}(1-S))(z) = (C_G|_{2-k}(1-S))(z) - \frac{1}{c_k} r(F; z).$$

Moreover, applying  $T$  to the definition of  $p_G$  gives

$$(p_G|_{2-k}(1-T))(z) = (C_G|_{2-k}(1-T))(z). \quad (4.3.11)$$

By Proposition 4.3.3 (2), we then find that

$$\frac{1}{c_k}r(F; z) = (p_G|_{2-k}(S-T))(z). \quad (4.3.12)$$

We now relate the polynomials  $H_G$  and  $p_G$ . Combining Proposition 4.3.2 (2) and Proposition 4.3.3 (1) with the modularity of  $\mathcal{F}$  and (4.3.11), we find that

$$\widetilde{H}_G(z) = (\Phi_G|_{2-k}(1-T^{-1}))(z) = (C_G|_{2-k}(1-T))(z) = (p_G|_{2-k}(1-T))(z).$$

Proposition 4.3.4 then implies that

$$H_G(z) = (\widetilde{p}_G|_{2-k}(1-T^{-1}))(z) = -(\widetilde{p}_G|_{2-k}T^{-1}(1-T))(z),$$

and Proposition 4.3.1 (3) in turn implies that

$$((q_G + \widetilde{p}_G|_{2-k}T^{-1})|_{2-k}(1-T))(z) = 0.$$

This means that the polynomial  $(q_G + \widetilde{p}_G|_{2-k}T^{-1})(z)$  is a constant, say  $\alpha$ . Applying  $TS$  to the resulting identity

$$q_G(z) = -(\widetilde{p}_G|_{2-k}T^{-1})(z) + \alpha, \quad (4.3.13)$$

we obtain

$$(q_G|_{2-k}TS)(z) = -(\widetilde{p}_G|_{2-k}S)(z) + \alpha z^{k-2}. \quad (4.3.14)$$

We now compare  $c_k \cdot r(G; z)$  and  $r(\widetilde{F}; z)$ . By (4.3.12) and Proposition 4.3.4, we have

$$\frac{1}{c_k}r(\widetilde{F}; z) = (\widetilde{p}_G|_{2-k}(S-T^{-1}))(z).$$

Combining this with Proposition 4.3.1 (4), and making use of (4.3.13) and (4.3.14),

we then obtain

$$\begin{aligned}
\frac{1}{c_k} \widetilde{r(F; z)} + r(G; z) &= (\widetilde{p_G|_{2-k}}(S - T^{-1}))(z) + (q_G|_{2-k}(S - T))(z) \\
&= (q_G|_{2-k}(1 - TS))(z) + \alpha(z^{k-2} - 1) + (q_G|_{2-k}(S - T))(z) \\
&= (q_G|_{2-k}(1 - T)(1 + S))(z) + \alpha(z^{k-2} - 1).
\end{aligned}$$

Since Proposition 4.3.1 gives the identities

$$(q_G|_{2-k}(1 - T))(z) = H_G(z) \quad \text{and} \quad (H_G|_{2-k}(1 + S))(z) = 0,$$

we conclude that

$$\frac{1}{c_k} \widetilde{r(F; z)} + r(G; z) = \alpha(z^{k-2} - 1).$$

This proves (4.3.10), and it completes the proof of the first claim of the theorem.

To prove the second claim, it suffices to produce a weakly holomorphic form  $\mathcal{W} \in M_{2-k}^!$  with nonzero constant term. Since  $\xi_{2-k}(\mathcal{W}) = 0$  and  $r(D^{k-1}(\mathcal{F}); z) - r(D^{k-1}(\mathcal{F} + \mathcal{W}); z)$  is a nonzero multiple of  $z^{k-2} - 1$ , the claimed second identity follows easily. The existence of such a form is guaranteed by Proposition 4.3.5.  $\square$

## 4.4 The extended Petersson inner product

We now apply the results of the last section to prove Theorems 4.1.2, 4.1.4, 4.1.5, and 4.1.6.

### 4.4.1 General considerations

We first define the extension of  $(\bullet, \bullet)$  to  $M_k^!$ , and we obtain a closed formula for it in terms of periods. Denote by  $D_T$  the truncated fundamental domain  $(\tau = x + iy)$

$$D_T := \left\{ \tau \in \mathbb{H} : |\tau| \geq 1, |x| \leq \frac{1}{2}, y \leq T \right\}. \quad (4.4.1)$$

Write  $F, G \in M_k^!$  as

$$F(z) = \sum_{n \gg -\infty} a_F(n)q^n \quad \text{and} \quad G(z) = \sum_{n \gg -\infty} a_G(n)q^n.$$

Then we may define an extension of Petersson's inner product as

$$(F, G) = \lim_{T \rightarrow \infty} \left( \int_{D_T} F(\tau) \overline{G(\tau)} y^{k-2} dx dy - \frac{a_F(0) \overline{a_G(0)}}{k-1} T^{k-1} \right) \quad (4.4.2)$$

when the limit exists.

Identity (4.1.13) is a result of the following proposition.

**Proposition 4.4.1.** *In the following cases*

- (i)  $F \in M_k$  and  $G \in M_k^!$
- (ii)  $F \in M_k^!$  and  $G \in M_k$

the extended Petersson product is well defined, and is given by

$$(F, G) = \text{constant term of } F\mathcal{G}^+,$$

where  $\mathcal{G} \in H_{2-k}$  such that  $\xi_{2-k}(\mathcal{G}) = G$ . Moreover, we have that

$$\begin{aligned} (F, G) &= \frac{1}{3 \cdot 2^{k-1}} \sum_{\substack{0 \leq m < n \leq k-2 \\ m \not\equiv n \pmod{2}}} i^{(n+1-m)} \binom{k-2}{n} \binom{n}{m} r_n(F) \overline{r_{k-2-m}(G)} \\ &+ \frac{2}{3 \cdot 2^{k-1}} \sum_{\substack{0 \leq n \leq k-2 \\ n \equiv 0 \pmod{2}}} i^{(k-n)} \binom{k-1}{n+1} \left( \overline{r_n(G)} \frac{a_F(0)}{k-1} + r_n(F) \frac{\overline{a_G(0)}}{k-1} \right). \end{aligned} \quad (4.4.3)$$

*Proof.* The existence of an appropriate harmonic Maass form  $\mathcal{G}$  in every case follows from Proposition 1.3.2 (1). That (4.4.2) is well defined can be proved using the same argument as in Proposition 3.5 of [11]. It is easy to see that the restrictions imposed

in [11] may be relaxed to obtain

$$\begin{aligned} (F, G) &= \lim_{T \rightarrow \infty} \left( \int_{D_T} F(\tau) \overline{G(\tau)} y^{k-2} dx dy - \frac{a_F(0) \overline{a_G(0)}}{k-1} T^{k-1} \right) \\ &= \text{constant term of } F\mathcal{G}^+. \end{aligned}$$

To complete the proof, we need to prove formula (4.4.3). Due to the linearity of Petersson's scalar product, it suffices to consider the following three cases:

- Case (1):  $F = G = E_k$ .
- Case (2):  $F \in S_k^!$  and  $G \in S_k$ .
- Case (3):  $F \in S_k^!$  and  $G = E_k$ .

For Case (1), we begin by recalling the values of the periods for  $E_k$  (see page 240 of [35]):

$$\begin{aligned} r_0(E_k) &= \frac{k}{B_k} \frac{(-1)^{\frac{k}{2}} (k-2)!}{(2\pi)^{k-1}} \cdot \zeta(k-1), \\ r_{k-2}(E_k) &= \frac{k}{B_k} \frac{(k-2)!}{(2\pi)^{k-1}} \cdot \zeta(k-1), \\ r_n(E_k) &= 0 && (\text{for } 0 < n < k-2, n \text{ even}), \\ r_n(E_k) &= -\frac{k}{B_k} (-1)^{\frac{n+1}{2}} \frac{B_{n+1}}{n+1} \frac{B_{k-1-n}}{k-1-n} && (\text{for } 0 < n < k-2, n \text{ odd}). \end{aligned}$$

We substitute these values into the right hand side of (4.4.3), and make use of Euler's identity for Bernoulli numbers

$$\sum_{m=2}^{k-2} \binom{k}{m} B_m B_{k-m} = -(k+1)B_k$$

(for integers  $k \geq 4$ ). Noting that  $\mathcal{G} = \frac{P_{E_k}}{k-1}$  now easily gives the claim computing the constant term of  $E_k\mathcal{G}^+$  using Theorem 1.3.6. We note that this result matches Zagier's calculation [49] for  $(E_k, E_k)$ .

Cases (2) and (3) are proven similarly by modifying an argument of Kohnen and Zagier (see pp. 244-246 of [35]) which they used to prove the Haberland identity for cusp forms. We only prove Case (2) here for brevity.

Consider the given contour integral, and observe that

$$(F, G) = - \lim_{T \rightarrow \infty} \int_{\partial D_T} F(\tau) \mathcal{G}(\tau) d\tau = - \lim_{T \rightarrow \infty} \int_{\partial D_T} F(\tau) \mathcal{G}^-(\tau) d\tau,$$

since the function  $F\mathcal{G}^+$  is holomorphic on  $D_T$ . Therefore we have that

$$\int_{\partial D_T} F(\tau) \mathcal{G}^+(\tau) d\tau = 0.$$

The function  $F\mathcal{G}^-$  is periodic with period 1 in  $x$ , because both  $F$  and  $\mathcal{G}^-$  are. Thus the integrals along the vertical lines cancel. Moreover, as in the proof of Proposition 3.5 in [11], we can show that

$$\lim_{T \rightarrow \infty} \int_{-1/2}^{1/2} F(x + iT) \mathcal{G}^-(x + iT) dx = 0,$$

and so

$$(F, G) = - \int_C F(\tau) \mathcal{G}^-(\tau) d\tau,$$

where  $C$  is the arc of the unit circle from  $\rho^2$  to  $\rho$  which bounds the fundamental domain from the bottom. Note that  $F\mathcal{G}^- d\tau$  is not invariant under  $S$ , so this integral may be non-zero. Also,  $S$  maps  $C$  into itself with orientation reversed, so we have

$$2(F, G) = - \int_C F(\tau) (\mathcal{G}^-|_{2-k}(1-S))(\tau) d\tau.$$

Now, because  $\mathcal{G} = \mathcal{G}^+ + \mathcal{G}^-$  is of weight  $2 - k$ , by Theorem 4.1.3 we have

$$\begin{aligned} (-\mathcal{G}^-|_{2-k}(1-S))(z) &= (\mathcal{G}^+|_{2-k}(1-S))(z) \equiv \frac{1}{c_k} r(D^{k-1}(\mathcal{G}); z) \\ &\equiv -\frac{1}{c_k} \frac{(k-2)!}{(4\pi)^{k-1}} \overline{r(G; \bar{z})} \pmod{z^{k-2} - 1}. \end{aligned}$$

Thus we have that

$$2(2i)^{k-1}(F, G) = - \int_{\rho^2}^{\rho} F(\tau) \overline{r(G; \bar{\tau})} d\tau. \quad (4.4.4)$$



This follows since

$$\int_{\rho^2}^{\rho} F(\tau)(\tau^{k-2} - 1) d\tau = 0,$$

which in turn follows since  $F$  is modular of weight  $k$  without a constant term. We now proceed as in [35] and define a pairing on polynomials in  $\mathbf{V}$  (degree at most  $k-2$ ) as follows

$$\left\langle \sum_{n=0}^{k-2} a_n z^n, \sum_{n=0}^{k-2} b_n z^n \right\rangle := \sum_{n=0}^{k-2} (-1)^n \binom{k-2}{n}^{-1} a_n b_{k-2-n}.$$

This pairing is symmetric and  $\mathrm{SL}_2(\mathbb{Z})$ -invariant (i.e. for all  $p, q \in \mathbf{V}$  and  $\gamma \in \mathrm{SL}_2(\mathbb{Z})$  we have  $\langle p|_{2-k}\gamma, q|_{2-k}\gamma \rangle = \langle p, q \rangle$ ).

We may rewrite (4.4.4) as

$$2(2i)^{k-1}(F, G) = - \left\langle H_F(z), \overline{r(G; \bar{z})} \right\rangle \quad (4.4.5)$$

where  $H_F$  is defined in (4.3.3). Making use of Proposition 4.3.1 (2), (3), and (4), along with the relations defining the space  $\mathbf{W}$ , and the properties of the pairing  $\langle \bullet, \bullet \rangle$ , we have the following:

$$\begin{aligned} \left\langle H_F(z), \overline{r(G; \bar{z})} \right\rangle &= \left\langle (q_F|_{2-k}(1-T))(z), \overline{r(G; \bar{z})} \right\rangle \\ &= \left\langle q_F(z), \overline{r(G; \bar{z})} |_{2-k}(1-T^{-1}) \right\rangle \\ &= \left\langle q_F(z), \overline{r(G; \bar{z})} |_{2-k}(1+ST^{-1}) \right\rangle \\ &= \left\langle q_F(z), \overline{r(G; \bar{z})} |_{2-k}(1+U^2) \right\rangle \\ &= \frac{1}{3} \left\langle q_F(z), \overline{r(G; \bar{z})} |_{2-k}(U^2-U)(1-U^{-1}) \right\rangle \\ &= \frac{1}{3} \left\langle (q_F|_{2-k}(1-U))(z), \overline{r(G; \bar{z})} |_{2-k}(U^2-U) \right\rangle \\ &= \frac{1}{3} \left\langle -r(F; z), \overline{r(G; \bar{z})} |_{2-k}(ST^{-1}-TS) \right\rangle \\ &= \frac{1}{3} \left\langle r(F; z)|_{2-k}(T-T^{-1}), \overline{r(G; \bar{z})} \right\rangle. \end{aligned}$$

Identity (4.4.3) follows by combining the above calculation with (4.4.5) to obtain

$$-6(2i)^{k-1}(F, G) = \left\langle r(F; z)|_{2-k}(T-T^{-1}), \overline{r(G; \bar{z})} \right\rangle. \quad \square$$

### 4.4.2 Proof of Theorem 4.1.5

Let  $F \in S_k^!$  be given.

To prove (i)  $\rightarrow$  (ii), we assume that  $F = D^{k-1}(\mathcal{F})$  where  $\mathcal{F} \in M_{2-k}^!$  has constant term  $\alpha \in \mathbb{C}$ . Then by (4.1.6) we have  $\mathcal{F} - \alpha = \mathcal{E}_F$  and it follows by modularity of  $\mathcal{F}$  and (4.1.7) that  $r(F; z) = \alpha c_k(z^{k-2} - 1)$ .

For the implication (ii)  $\rightarrow$  (iii), we have that  $r(F; z) = \alpha(z^{k-2} - 1)$  for some  $\alpha \in \mathbb{C}$ . By (4.1.7),  $\mathcal{E}_F + \alpha/c_k \in M_{2-k}^!$ . This implies that for  $G \in S_k^!$ , the scalar product  $\{F, G\}$  equals the constant term of the weight 2 weakly holomorphic modular form  $-(\mathcal{E}_F + \alpha/c_k)G$ , and vanishes, because every such form is a derivative of a polynomial in the  $j$ -function.

We now prove (iii)  $\rightarrow$  (i). By Proposition 4.2.1, we may write

$$F = \phi + \psi$$

with  $\phi \in S_k$  and  $\psi = D^{k-1}(\mathcal{G})$  where  $\mathcal{G} \in H_{2-k}^*$ . By our hypothesis and Proposition 4.4.1 (i), for every  $h \in S_k$ ,

$$0 = \{h, F\} = \{h, \psi\} = (h, \xi_{2-k}(\mathcal{G})).$$

We conclude that  $\xi_{2-k}(\mathcal{G}) = 0$ . Therefore  $\mathcal{G} \in M_{2-k}^!$  and  $\psi \in D^{k-1}(M_{2-k}^!)$ . Now for every  $h \in S_k$  there exists  $\mathcal{G}_h \in H_{2-k}^*$  such that  $\xi_{2-k}(\mathcal{G}_h) = h$ . Since  $D^{k-1}(\mathcal{G}_h) \in S_k^!$ , we can use our hypothesis and Proposition 4.4.1 (i) again to conclude that for every  $h \in S_k$ ,

$$0 = \{F, D^{k-1}(\mathcal{G}_h)\} = \{\phi, D^{k-1}(\mathcal{G}_h)\} + \{\psi, D^{k-1}(\mathcal{G}_h)\} = \{\phi, D^{k-1}(\mathcal{G}_h)\} = (\phi, h).$$

The third equality holds by following the proofs of implications (i)  $\rightarrow$  (ii)  $\rightarrow$  (iii). Therefore  $\phi = 0$  and so  $F = \psi \in D^{k-1}(M_{2-k}^!)$  as required.

### 4.4.3 Proof of Theorem 4.1.2

The injectivity of the embedding  $D^{k-1}(M_{2-k}^!) \rightarrow S_k^!$  is obvious, and the exactness in  $S_k^!$  follows immediately from Theorem 4.1.5. Therefore, it suffices to establish surjectivity.

The argument closely follows our proof of Proposition 4.2.1. The Eichler-Shimura isomorphism allows us to write an arbitrary polynomial  $r \in \mathbf{W}_0$  as

$$r = r^-(g_1) + ir^+(g_2)$$

with  $g_1, g_2 \in S_k$ . Using the notation from the proof of Proposition 4.2.1, we then find that  $F(z) := \phi(z) + \Psi(z) \in S_k^!$ , and  $r(F; z) = r \in \mathbf{W}_0$ .

### 4.4.4 Proof of Theorem 4.1.4

Let  $d = \dim(S_k)$ , and for  $1 \leq i \leq d$ , let

$$f_i(z) = \sum_{n>0} b_i(n)q^n \in S_k$$

be a basis of normalized Hecke eigenforms for  $S_k$ . For each  $i$ ,  $\{b_i(n)\}_{n>0}$  is a system of Hecke eigenvalues, and  $f_i \in S_k^!/D^{k-1}(M_{2-k}^!)$ .

Let  $\mathcal{F}_i \in H_{2-k}^*$  such that  $\xi_{2-k}(\mathcal{F}_i) = f_i$ . The differential operator  $\xi_{2-k}$  and the Hecke operator  $T(m)$  obey the following commutation relation

$$(\xi_{2-k}(\mathcal{F}_i |_{2-k} T(m)))(z) = m^{1-k} (\xi_{2-k}(\mathcal{F}_i) |_k T(m))(z).$$

Because  $\xi_{2-k}(\mathcal{F}_i |_{2-k} T(m) - m^{1-k}b_i(m)\mathcal{F}_i) = 0$ , it follows that there is some  $r_m(z) \in M_{2-k}^!$  such that

$$(\mathcal{F}_i |_{2-k} T(m))(z) = m^{1-k}b(m)\mathcal{F}_i(z) + r_m(z).$$

We apply the operator  $D$  to this identity  $k-1$  times and use Bol's identity to find that

$$(D^{k-1}(\mathcal{F}_i) |_k T(m))(z) = b(m)D^{k-1}(\mathcal{F}_i)(z) + m^{k-1}D^{k-1}(r_m)(z).$$

It is clear that  $F_i = D^{k-1}(\mathcal{F}_i) \in S_k^!$  is a weakly holomorphic Hecke eigenform in  $S_k^!/D^{k-1}(M_{2-k}^!)$ .

To complete the proof, we show that the forms  $F_i(z)$  are linearly independent. Assume that  $\sum_{i=0}^d c_i F_i(z) = 0$ . Then for each  $j$  such that  $0 \leq j \leq d$ , we make use of Proposition 4.4.1 to obtain

$$0 = \left\{ f_j, \sum_{i=0}^d c_i F_i \right\} = \sum_{i=0}^d c_i \{f_j, F_i\} = \sum_{i=0}^d c_i (f_j, f_i).$$

Because each  $f_i$  is a Hecke eigenform, we know that  $(f_j, f_i) \neq 0$  if and only if  $i = j$ . Therefore each  $c_i = 0$  and the forms  $F_i(z)$  are linearly independent.

Now, we may use a dimension argument by combining Proposition 4.2.1 and Theorem 4.1.2. This shows that the set of forms  $f_i(z)$  together with the set of forms  $F_i(z)$  form a basis of  $S_k^!/D^{k-1}(M_{2-k}^!)$ , proving the theorem.

#### 4.4.5 Proof of Theorem 4.1.6

We make use of Proposition 4.2.1 to obtain the decomposition

$$G = a_G(0)E_k + \phi_G + \psi_G,$$

with  $\phi_G \in S_k$ , and  $\psi_G \in D^{k-1}(H_{2-k}^*)$ . Also, let  $F_0 := F - a_F(0)E_k$  and  $G_0 := G - a_G(0)E_k$ . By the obvious linearity we obtain

$$\{F, G\} = a_F(0)a_G(0)\{E_k, E_k\} + a_F(0)\{E_k, G_0\} + a_G(0)\{F_0, E_k\} + \{F_0, \phi_G\} + \{F_0, \psi_G\},$$

and we now need to prove the required identity for each of the five terms separately.

We begin by letting  $\mathcal{E}_k = -\frac{(4\pi)^{k-1}}{(k-1)!}P_{E_k} \in H_{2-k}$ ,  $\mathcal{F} \in H_{2-k}$ , and  $\mathcal{G} \in H_{2-k}^*$  so that  $E_k = D^{k-1}(\mathcal{E}_k)$ ,  $F_0 = D^{k-1}(\mathcal{F})$ , and  $\psi_G = D^{k-1}(\mathcal{G})$ . It follows that

$$\begin{aligned} \{E_k, E_k\} &= 0 = (E_k, \xi(\mathcal{E}_k)) - \text{constant term of } E_k \mathcal{E}_k^+, \\ \{E_k, G_0\} &= -\{G_0, E_k\} = -\text{constant term of } G_0 \mathcal{E}_k^+ = -(G_0, \xi(\mathcal{E}_k)), \\ \{F_0, E_k\} &= \text{constant term of } F_0 \mathcal{E}_k^+ = (F_0, \xi(\mathcal{E}_k)), \\ \{F_0, \phi_G\} &= -\{\phi_G, F_0\} = -\text{constant term of } \phi_G \mathcal{F}^+ = -(\phi_G, \xi(\mathcal{F})), \text{ and} \\ \{F_0, \psi_G\} &= \text{constant term of } F_0 \mathcal{G}^+ = (F_0, \xi(\mathcal{G})). \end{aligned}$$

In each of these cases, one of the conditions of Proposition 4.4.1 holds. The desired identity now almost immediately follows from (4.4.3) and Theorem 4.1.3. The only difficulty is that Theorem 4.1.3 leaves ambiguity in the 0th and  $(k-2)$ nd periods. However, this ambiguity vanishes because of the relations (4.1.3) defining the space  $\mathbf{W}^-$ . In particular, the sum of the coefficients of a polynomial in  $\mathbf{W}^-$  is zero.

## 4.5 The period polynomial principle and the proof of Theorem 4.1.1

Here we prove Theorem 4.1.1 using the principle that “period polynomials” encode critical values of  $L$ -functions. We choose this perspective, instead of working directly with period integrals of cusp forms, to highlight the role that Bol’s identity plays in relating pairs of functional equations. This is the analytic process by which one obtains critical  $L$ -values. This concept was the subject of unpublished work by Razar in the 1970s.

### 4.5.1 Period polynomials and critical values of $L$ -functions

If  $f$  is a weight  $k$  cusp form, then its *critical values* are the numbers

$$C(f) := \{L(f, 1), L(f, 2), \dots, L(f, k - 1)\},$$

where  $L(f, s)$  is the usual analytically continued  $L$ -function. Here we show that such values arise naturally as the coefficients of “period polynomials”, functions in  $z$  which measure the obstruction to modularity.

**Theorem 4.5.1.** *Suppose that*

$$A(z) = \sum_{n=1}^{\infty} \alpha(n) q^{\frac{n}{\lambda}},$$

$$B(z) = \sum_{n=1}^{\infty} \beta(n) q^{\frac{n}{\lambda}}$$

are holomorphic functions on  $\mathbb{H}$  where  $|\alpha(n)|, |\beta(n)| = O(n^\delta)$ , where  $\lambda, \delta > 0$ . If

$$A(z) = z^{-k} B(-1/z),$$

where  $k \geq 2$  is even, then

$$\mathbb{E}_{A,k}(z) - z^{k-2} \mathbb{E}_{B,k}(-1/z) = \sum_{j=0}^{k-2} \frac{L(A, k-1-j)}{j!} \cdot \left(\frac{2\pi iz}{\lambda}\right)^j.$$

Here  $L(A, s)$  is the analytic continuation of

$$L(A, s) := \sum_{n=1}^{\infty} \frac{\alpha(n)}{n^s},$$

and

$$\mathbb{E}_{\phi,k}(z) := \sum_{n=1}^{\infty} \nu(n) n^{1-k} q^{\frac{n}{\lambda}},$$

when  $\phi(z) = \sum_{n=1}^{\infty} \nu(n) q^{\frac{n}{\lambda}}$ .

*Sketch of the proof.* The proof depends on the relationship between functional equations for  $L$ -functions, Mellin transforms, and inverse Mellin transforms. Since these notions are standard in analytic number theory, here we provide just a brief sketch of the proof.

Since  $A(z) = z^{-k}B(-1/z)$ , the analytically continued Dirichlet series for  $A(z)$  and  $B(z)$ , say  $L(A, s)$  and  $L(B, s)$ , satisfy the functional equation

$$\Lambda_A(s) = i^k \Lambda_B(k - s). \quad (4.5.1)$$

As usual, we have that

$$\begin{aligned} \Lambda_A(s) &:= \left(\frac{2\pi}{\lambda}\right)^{-s} \Gamma(s) L(A, s), \\ \Lambda_B(s) &:= \left(\frac{2\pi}{\lambda}\right)^{-s} \Gamma(s) L(B, s). \end{aligned}$$

Differentiating  $A(z)$  has the effect of taking  $L(A, s)$  to  $L(A, s - 1)$ . Unfortunately, it also results in a much more complicated functional equation than (4.5.1). However, by Bol's identity, we find that we can differentiate  $k - 1$  times and obtain a nice functional equation, one which effectively produces the critical strip. We obtain the functional equation

$$\widehat{\Lambda}_A(s) = (-1)^{k-1} \cdot i^k \cdot \widehat{\Lambda}_B(2 - k - s),$$

where

$$\begin{aligned} \widehat{\Lambda}_A(s) &:= \left(\frac{2\pi}{\lambda}\right)^{-s} \Gamma(s) L(A, s + k - 1), \\ \widehat{\Lambda}_B(s) &:= \left(\frac{2\pi}{\lambda}\right)^{-s} \Gamma(s) L(B, s + k - 1). \end{aligned}$$

By the assumptions on  $A(z)$  and  $B(z)$ , there is a rational function  $\widehat{\Psi}(s)$  for which  $\widehat{\Lambda}_A(s) - \widehat{\Psi}(s)$  is holomorphic and bounded in vertical strips. A simple residue calculation then indeed shows that

$$\mathbb{E}_{A,k}(z) - z^{k-2} \mathbb{E}_{B,k}(-1/z) = \sum_{j=0}^{k-2} \frac{L(A, k-1-j)}{j!} \cdot \left(\frac{2\pi iz}{\lambda}\right)^j. \quad \square$$

We now apply Theorem 4.5.1 to modular forms. Throughout this subsection, we suppose that  $f(z) = \sum_{n=1}^{\infty} a(n)q^n \in S_k$ . A direct calculation for  $0 \leq n \leq k-2$  gives that

$$L(f, n+1) = \frac{(2\pi)^{n+1}}{n!} \cdot r_n(f). \quad (4.5.2)$$

These are the *critical values*. The following immediate application of Theorem 4.5.1 provides a proof of (4.5.2), and it also motivates the definition of the period function  $r(f; z)$  in (4.1.9).

**Corollary 4.5.2.** *We have that*

$$\begin{aligned} \mathcal{E}_f(z) - z^{k-2}\mathcal{E}_f(-1/z) &= \sum_{n=0}^{k-2} \frac{L(f, n+1)}{(k-2-n)!} \cdot (2\pi iz)^{k-2-n} \\ &= \frac{1}{c_k} \cdot \sum_{n=0}^{k-2} i^{1-n} \binom{k-2}{n} \cdot r_n(f) \cdot z^{k-2-n} = \frac{1}{c_k} \cdot r(f; z). \end{aligned}$$

If  $1 \leq d, c$  are coprime integers, then define the *twisted L-function*

$$L(f, \zeta_c^{-d}, s) := \sum_{n=1}^{\infty} \frac{a(n)\zeta_c^{-dn}}{n^s}. \quad (4.5.3)$$

Corollary 4.5.2 has the following generalization for these  $L$ -functions.

**Corollary 4.5.3.** *If  $1 \leq d < c$  are coprime, then let  $\gamma = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ . Then we have that*

$$\mathcal{E}_f(z) - (\mathcal{E}_f|_{2-k}\gamma)(z) = \sum_{n=0}^{k-2} \frac{L(f, \zeta_c^{-d}, n+1)}{(k-2-n)!} \cdot (2\pi i)^{k-2-n} \cdot \left(z + \frac{d}{c}\right)^{k-2-n}.$$

*Proof.* If  $\eta = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$  is a matrix with  $C \neq 0$ , then let

$$f(\eta; z) := f\left(\frac{z}{|C|} - \frac{D}{C}\right).$$

By modularity, it follows that

$$f(\gamma; z) = z^{-k} f\left(\gamma^{-1}; -\frac{1}{z}\right).$$



We now apply Theorem 4.5.1 with

$$A(z) := f(\gamma; z) = \sum_{n=1}^{\infty} a(n) \zeta_c^{-dn} q^{\frac{n}{c}},$$

$$B(z) := f(\gamma^{-1}; z) = f\left(\frac{z}{c} + \frac{a}{c}\right) = \sum_{n=1}^{\infty} a(n) \zeta_c^{an} q^{\frac{n}{c}}.$$

Letting  $z \rightarrow cz + d$  in the conclusion of Theorem 4.5.1 gives

$$\begin{aligned} \mathbb{E}_{A,k}(cz + d) - (cz + d)^{k-2} \mathbb{E}_{B,k}\left(-\frac{1}{cz + d}\right) &= \sum_{j=0}^{k-2} \frac{L(A, k-1-j)}{j!} \cdot \left(\frac{2\pi i(cz + d)}{c}\right)^j \\ &= \sum_{n=0}^{k-2} \frac{L(f, \zeta_c^{-d}, n+1)}{(k-2-n)!} \cdot (2\pi i)^{k-2-n} \cdot \left(z + \frac{d}{c}\right)^{k-2-n}. \end{aligned}$$

The claim now follows, using the following two identities

$$\begin{aligned} \mathbb{E}_{A,k}(cz + d) &= \mathcal{E}_f(z), \\ (\mathcal{E}_f|_{2-k}\gamma)(z) &= (cz + d)^{k-2} \cdot \mathbb{E}_{B,k}\left(-\frac{1}{cz + d}\right). \end{aligned}$$

□

## 4.5.2 Proof of Theorem 4.1.1

We prove Theorem 4.1.1 using Corollaries 4.5.2 and 4.5.3. Suppose that  $f \in S_k$  and  $\mathcal{F} \in H_{2-k}^*$  have the property that  $\xi_{2-k}(\mathcal{F}) = f$ . In Theorem 4.1.3, the constant term of  $\mathcal{F}$  is the only obstacle which keeps us from obtaining equality between the two period polynomials. The problem is that both polynomials depend upon  $\mathcal{F}$  after differentiation, but this operation annihilates the constant term and there is no way to recover it. By working with  $\mathcal{F}$  before differentiation, Theorem 4.1.3 actually implies that

$$c_k \cdot \overline{\mathbb{P}(\mathcal{F}^+, \gamma_{1,0}; \bar{z})} = r(f; z). \quad (4.5.4)$$

The first claim in Theorem 4.1.1 now follows from Corollary 4.5.2.

To prove the second claim, we apply Corollary 4.5.3 using the fact that similarly to (4.5.4) we have, for any matrix  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ , the identity

$$\overline{\mathbb{P}(\mathcal{F}^+, \gamma_{c,d}; \bar{z})} = (\mathcal{E}_f - \mathcal{E}_f|_{2-k} \begin{pmatrix} a & b \\ c & d \end{pmatrix})(z)$$

We require the standard orthogonality relation which asserts that

$$\sum_{d=0}^{c-1} \zeta_c^{-m_1 d} \cdot \zeta_c^{m_2 d} = \begin{cases} c & \text{if } m_1 \equiv m_2 \pmod{c}, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, if  $\mathrm{gcd}(m, c) = 1$ , we have that

$$\frac{1}{c} \sum_{d=0}^{c-1} \zeta_c^{md} \cdot L(f, \zeta_c^{-d}, s) = \sum_{\substack{n \geq 1 \\ n \equiv m \pmod{c}}} \frac{a(n)}{n^s}.$$

Summing in  $m$ , combined with the discussion above, gives the second claim in the theorem.

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