Analyzing Protocols of Information Granularity Allocation to Compute Missing Values in Intuitionistic Reciprocal Preference Relations

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Abstract

In the field of Granular Computing, a fundamental approach focuses on the management of incomplete intuitionistic reciprocal preference relations through the allocation of information granularity and the execution of a corresponding optimization procedure. It facilitates the transformation of numerical models into their granular counterparts, thereby providing a more accurate representation of reality to recover missing information. In decision-making contexts that involve intuitionistic reciprocal preference relations, this approach has proven essential in advancing procedures for estimating incomplete information. Nevertheless, while several protocols for information granularity allocation have been proposed, only one has been actively implemented thus far: the protocol based on a uniform and symmetric allocation of information granularity. To address this limitation, the objective of this study is to assess the effectiveness of existing protocols for allocating information granularity in the estimation of missing values in incomplete intuitionistic reciprocal preference relations. Numerical tests are included to demonstrate the efficacy of the protocols.

Keywords: Consistency, decision-making, missing values, information granularity, intuitionistic reciprocal preference relations.

1. Introduction

The decision-making process is fundamentally concerned with identifying the most appropriate alternative from a wide range of options (Carlsson, 2010). The optimal choice is typically one that has achieved a significant level of acceptance among the stakeholders engaged in the decision (Cabrerizo et al., 2025; Jain & Lim, 2010). In practical scenarios, this process often relies on incomplete intuitionistic preference relations, wherein decision-makers may find it challenging or be reluctant to articulate all pairwise comparisons due to uncertainty or insufficient information (Z. Zhang et al., 2020). These preference relations facilitate the modeling of preferences that exhibit varying preference and non-preference degrees, which range from 0 to 1 (Szmidt & Kacprzyk, 2002). This approach is particularly advantageous when precise information is not readily available.

One of the main challenges in dealing with incomplete preference relations is the presence of missing values, which arise when participants fail to express certain preferences. To address this issue, it is essential to compute the missing information in a way that maintains the coherence of the overall preference structure. In this context, consistency-based imputation procedures have become increasingly relevant. These methods aim to recover missing values by analyzing the internal logic of existing preferences while preserving rational decision-making behavior (Ureña et al., 2015b). There exist two distinct categories of methods for estimating missing values in decision-making contexts. The first category pertains to group decision-making, wherein the estimation of an individual's preferences relies on evaluations provided by other participants employing aggregation procedures. However, these methods are subject to certain limitations, as they necessitate that multiple individuals provide information on one another. This reliance can result in estimations that do not accurately reflect the individual's true preferences due to potential inconsistencies among participant evaluations. In contrast, the second category focuses on estimating missing values based exclusively on the information supplied by the

URI: https://hdl.handle.net/10125/111633
978-0-9981331-9-5

individual. Within this category, consistency-based methods have demonstrated significant efficacy. These methods derive the missing values by assessing the internal coherence of the known preferences without the requirement for external input. The objective of this approach is to complete the preference relation in a manner that upholds logical consistency and faithfully represents the decision-maker's original intent. Such techniques have proven to be effective in both individual and group decision-making scenarios, particularly in situations involving preference relations (see Ureña et al. (2015b) for further discussion).

Recently, an innovative and promising approach has emerged in the form of granular modeling (Pedrycz & Song, 2011). It can be regarded as generalizations of traditional numerical models, providing a higher level of abstraction. By functioning at this advanced level, granular models are more adept at capturing the fundamental characteristics of the systems they represent. This capability allows for a more flexible and robust management of uncertainty within the decision-making process. For example, several models grounded in granular computing have been developed to address critical issues such as consensus, consistency, and incomplete information in the context of decision-making (Qin et al., 2023; Zhang et al., 2022). However, despite the extensive discussion and analysis of various protocols for the allocation of information granularity aimed at building consensus and improving consistency when intuitionistic reciprocal preference relations are used to model the decision-makers' preferences (González-Quesada, Cabrerizo, et al., 2024; González-Quesada, Herrera-Viedma, et al., 2024), only one protocol—a symmetric and uniform allocation of information granularity—has been used for the purpose of estimating missing values (Cabrerizo et al., 2019). To address this deficiency, our objective is to evaluate the effectiveness of existing information granularity allocation protocols and their subsequent optimization in the estimation of missing values within intuitionistic reciprocal preference relations.

This study is structured as follows. Section 2 provides an overview of the relevant background information. In Section 3, we discuss methodologies for estimating missing values in incomplete intuitionistic reciprocal preference relations, utilizing various protocols for the allocation of information granularity. Section 4 includes a numerical experiment that evaluates the effectiveness of each protocol. Section 5 conducts a comparative analysis and Section 6 concludes the study with a summary of the findings, implications, and future research.

2. Preliminaries

This section recalls some concepts and techniques that will be used later on, i.e., an Atanassov's intuitionistic fuzzy set, an intuitionistic reciprocal preference relation, a consistency-based procedure to estimate missing values in intuitionistic reciprocal preference relations, and the Particle Swarm Optimization (PSO) technique.

2.1. Atanassov's intuitionistic fuzzy sets

Atanassov (1986) defined the intuitionistic fuzzy sets as generalizations of the fuzzy sets as follows:

Definition 1 *"An intuitionistic fuzzy set A over a universe of discourse X is represented as $A = \langle \mu_A, \nu_A \rangle$ where $\mu_A : X \to [0,1]$, $\nu_A : X \to [0,1]$, and $0 \leq \mu_A(x) + \nu_A(x) \leq 1$, $\forall x \in X$. For each $x \in X$, the numbers $\mu_A(x)$ and $\nu_A(x)$ are known as the degree of membership and the degree of non-membership of x to A, respectively."*

When $\mu_A(x) = 1 - \nu_A(x)$, $\forall x \in X$, the Atanassov's intuitionistic fuzzy set is equal to a fuzzy set. Nevertheless, an additional value, known as hesitancy degree, is needed to deal with Atanassov's intuitionistic fuzzy sets in the case that there exists a value $x \in X$ such that $\mu_A(x) < 1 - \nu_A(x)$. This degree of hesitancy characterizes the amount of information lacking in determining the degree of membership of x to A and is calculated as $\tau_A(x) = 1 - \mu_A(x) - \nu_A(x)$. The reciprocal relationship between the membership degree and the non-membership degree makes de later irrelevant in the formulation when $\tau_A(x) = 0$ because it can be obtained from the former.

2.2. Intuitionistic reciprocal preference relations

Xu (2007) introduced the notion of an intuitionistic reciprocal preference relation as presented in the following definition:

Definition 2 *"An intuitionistic reciprocal preference relation R on a set of alternatives $X = \{x_1, x_2, \ldots, x_m\}$ $(m \geq 2)$ is characterized by a membership function $\mu_R : X \times X \to [0,1]$ and a non-membership function $\nu_R : X \times X \to [0,1]$, such that $0 \leq \mu_R(x_i, x_j) + \nu_R(x_i, x_j) \leq 1$, $\forall (x_i, x_j) \in X \times X$. The value $\mu_R(x_i, x_j) = \mu_{ij}$ represents the certainty degree up to which the alternative x_i is preferred to the alternative x_j by the decision-maker, whereas the value $\nu_R(x_i, x_j) = \nu_{ij}$ represents the certainty degree up to which the alternative x_i is non-preferred to the alternative x_j by* *the decision-maker. The following conditions are also imposed:"*

$$\mu_{ii} = \nu_{ii} = 0.5 \quad \forall i \in \{1, \ldots, m\} \tag{1}$$

$$\mu_{ji} = \nu_{ij} \quad \forall i, j \in \{1, \ldots, m\} \tag{2}$$

In the existing literature, an intuitionistic reciprocal preference relation R has usually been modeled by a matrix denoted $R = [r_{ij}]$, with $r_{ij} = [\langle \mu_{ij}, \nu_{ij} \rangle]$, $\forall i, j = 1, 2, \ldots, m$.

2.3. Computing missing values in intuitionistic reciprocal preference relations

Ureña et al. (2015a) established the equivalence between intuitionistic reciprocal preference relations and asymmetric fuzzy preference relations. This study provided an isomorphism facilitating the derivation of an asymmetric fuzzy preference relation from an intuitionistic reciprocal preference relation. This methodology addressed the issue of incomplete information within intuitionistic reciprocal preference relations by developing an estimation procedure grounded in consistency with the corresponding equivalent incomplete asymmetric fuzzy preference relation.

Before discussing this procedure, we recall the formal definition of an incomplete intuitionistic reciprocal preference relation (Cabrerizo et al., 2019).

Definition 3 *"A function $g : X \to Y$ is partial when not every element in the set X necessarily maps to an element in the set Y. When every element from the set X maps to one element of the set Y, then we have a total function."*

Definition 4 *"An intuitionistic reciprocal preference relation on a set of alternatives with a partial membership function is an incomplete intuitionistic reciprocal preference relation."*

Assuming two alternatives, x_i and x_j, for which $r_{ij} = [\langle \mu_{ij}, \nu_{ij} \rangle]$ is not known, it is supposed that both the membership degree μ_{ij} and the non-membership degree ν_{ij} are unknown. Based on the reciprocity property, if r_{ij} is not known, then r_{ji} is also unknown. If R is an incomplete intuitionistic reciprocal preference relation, then $S = Q(R) = (q(r_{ij})) = (\mu_{ij})$ is an incomplete asymmetric fuzzy preference relation, being $q : [0,1] \times [0,1] \to [0,1]$ the following function $q(x_1, x_2) = x_1$, and $Q : \mathcal{R} \to \mathcal{L}$ a mapping between the set of intuitionistic reciprocal preference relations, $\mathcal{R}$, and the set of fuzzy preference relations, $\mathcal{L}$ (Ureña et al., 2015a). As a result, S is an incomplete asymmetric fuzzy preference relation if R is an incomplete reciprocal intuitionistic fuzzy preference relation.

Based on Tanino (1984)'s multiplicative transitivity property, Ureña et al. (2015a) demonstrated that the missing value s_{ij} may partially be estimated by means of an intermediate alternative x_k through (3). The following notation was also introduced:

$$A = \{(i, j) \mid i, j \in \{1, 2, \ldots, n\} \land i \neq j\} \tag{4}$$

$$MV = \{(i, j) \in A \mid s_{ij} \text{ is unknown}\} \tag{5}$$

$$EV = A \setminus MV \tag{6}$$

where A represents the set of all pairs of alternatives, MV the set of pairs of alternatives in which the preference degree is unknown, and EV the set of pairs of alternatives whose preference degrees are known.

The estimated value based on global multiplicative transitivity, denoted es_{ij}, was calculated by Ureña et al. (2015a) as follows:

$$es_{ij} = \frac{\sum_{k \in H_{ij}^{01}} es_{ij}^k}{\# H_{ij}^{01}} \tag{7}$$

where $H_{ij}^{01} = \{k \in S_{ij}^{01} \mid (i, j) \in MV \land (i, k), (k, j) \in EV\}$, $\# H_{ij}^{01}$ is the cardinality of H_{ij}^{01}, and $S_{ij}^{01} = \{k \neq i, j \mid (s_{ik}, s_{kj}) \notin \{(1, 0), (0, 1)\}\}$.

Then, to complete both S and R, the approach built by Herrera-Viedma et al. (2007) was used to estimate incomplete information in fuzzy preference relations. See Ureña et al. (2015a) for a more detailed description of the procedure.

2.4. Particle swarm optimization

It is a stochastic optimization algorithm that tries to solve an optimization problem by emulating the social behavior of bird flocks (Kennedy & Eberhart, 1995). In each iteration of a series of them, the PSO tries to find the most promising candidate solutions according to a certain quality measure, referred to as the fitness function f. By assuming a population

$$es_{ij}^k = \begin{cases} 0 & (s_{ik}, s_{kj}) \in \{(0, 1), (1, 0)\} \\ \dfrac{s_{ik} s_{kj}}{s_{ik} s_{kj} + (1 - s_{ik})(1 - s_{kj})} & \text{otherwise} \end{cases} \tag{3}$$

Algorithm 1 PSO.

for each particle $i = 1, \ldots, n$ **do**
  Initializing its initial position $\mathbf{x}_i^{(0)}$ within the lower and upper limits, l_{inf} and l_{sup}, of the search space with a uniformly distributed random vector: $\mathbf{x}_i^{(0)} \sim U(l_{\text{inf}}, l_{\text{sup}})$
  Initializing $\mathbf{b}_i$ with the initial position: $\mathbf{b}_i \leftarrow \mathbf{x}_i^{(0)}$
  if $(f(\mathbf{b}_i) > f(\mathbf{g}))$ **then**
    Updating $\mathbf{g}$ with the particle's best position: $\mathbf{g} \leftarrow \mathbf{b}_i$
  end if
  Initializing its velocity: $\mathbf{v}_i^{(0)} \sim U(-|l_{\text{sup}} - l_{\text{inf}}|, |l_{\text{sup}} - l_{\text{inf}}|)$
end for
for each iteration $e = 1, \ldots, o$ **do**
  for each particle $i = 1, \ldots, n$ **do**
    for each dimension $h = 1, \ldots, d$ **do**
      Generating two random numbers: $r_h, s_h \sim U(0, 1)$
      Updating its velocity: $v_{i,h}^{(e)} \leftarrow \omega^{(e)} v_{i,h}^{(e-1)} + c_1 r_h (b_{i,h} - x_{i,h}^{(e-1)}) + c_2 s_h (g_h - x_{i,h}^{(e-1)})$
    end for
    Updating its position: $\mathbf{x}_i^{(e)} \leftarrow \mathbf{x}_i^{(e-1)} + \mathbf{v}_i^{(e)}$
    if $(f(\mathbf{x}_i^{(e)}) > f(\mathbf{b}_i))$ **then**
      Updating its best position: $\mathbf{b}_i \leftarrow \mathbf{x}_i^{(e)}$
      if $(f(\mathbf{b}_i) > f(\mathbf{g}))$ **then**
        Updating the global best position: $\mathbf{g} \leftarrow \mathbf{b}_i$
      end if
    end if
  end for
end for
return $\mathbf{g}$

(swarm) of n candidate solutions (particles), $\mathbf{x}_i = (x_{i,1}, \ldots, x_{i,d})$, in a d-dimensional search space, this algorithm optimizes the problem by moving the particles to different locations in the search space through a series of easy mathematical formulas. Each particle is moved to a new location in each iteration according to its velocity, $\mathbf{v}_i = (v_{i,1}, \ldots, v_{i,d})$, whose value is calculated based on its best position so far, $\mathbf{b}_i = (b_{i,1}, \ldots, b_{i,d})$, and the best (global) position reached by any of the other particles in the search space, $\mathbf{g} = (g_1, \ldots, g_d)$. The objective is to guide the particles toward the best solution (best position).

Although various variants of this algorithm may be applicable (Shami et al., 2022; Wang et al., 2018), we have chosen to employ the generic version introduced by Kennedy and Eberhart (1995), with the steps outlined in Algorithm 1. The coefficients c_1 and c_2 serve as acceleration factors that affect the step size of the particle to both its individual best position and the global best position.

Furthermore, ω represents the inertia weight. The assignment of a lower value to ω promotes exploitation, facilitating a local search strategy. Consequently, it is customary to decrease the value of ω as the number of iterations increases. This methodology allows for a comprehensive global search during the initial phases of the algorithm, transitioning to a more localized search in the later stages. Consequently, ω is generally adjusted in accordance with:

$$\omega^{(e)} = (\omega^{(1)} - \omega^{(o)}) \frac{o - e}{o} + \omega^{(o)} \tag{8}$$

In Eq. (8), $\omega^{(1)}$, $\omega^{(e)}$ and $\omega^{(o)}$ denote the initial, current and final values assigned to ω, respectively. Furthermore, the variables o and e denote the maximum number of iterations that will be performed and the current iteration.

This is the algorithm that will be used on the grounds that it has been used in most of the proposals for decision-making problems based on a distribution of information granularity (González-Quesada, Cabrerizo, et al., 2024; Pedrycz & Song, 2011). Nevertheless, any other optimization algorithm such as differential evolution could be applied, as it has also been used in this kind of problems (Cabrerizo et al., 2023; Zhang et al., 2022).

3. Computing missing values via four protocols of information granularity allocation

This section elucidates the methodology for completing incomplete intuitionistic reciprocal preference relations with the highest possible consistency level by employing various established protocols for information granularity allocation. These protocols include:

- A symmetric and uniform allocation, which is denoted as D_{su}.
- An asymmetric but uniform allocation, which is denoted as D_{au}.
- A symmetric but non-uniform allocation, which is denoted as D_{sn}.
- An asymmetric and non-uniform allocation, which is denoted as D_{an}.

To facilitate this process, the concept of an interval-valued (granular) asymmetric fuzzy preference relation is employed. Let $IV(S)$ be an interval-valued asymmetric fuzzy preference relation and $IV(\cdot)$ a family of interval-valued asymmetric fuzzy preference relations (Pedrycz & Song, 2011). This notion serves as a generalization of an asymmetric fuzzy preference relation and is constructed through the implementation of an appropriate information granularity allocation. The idea is to abandon the use of exact numerical values for the estimated values. Instead, we adopt a granular approach by utilizing information granules (intervals) and assigning a predetermined level of granularity to them. This methodology allows us to construct granular asymmetric fuzzy preference relations encompassing a broader range of values that, consequently, facilitate the completion of intuitionistic reciprocal preference relations with a higher consistency level.

3.1. $\mathbf{D_{su}}$ protocol

Let S be the incomplete asymmetric fuzzy preference relation associated with the incomplete intuitionistic reciprocal preference relation R expressed by a decision-maker, and let ε be the information granularity level allowed by him or her. This protocol modifies the estimated values, es_{ij}, of the incomplete asymmetric fuzzy preference relation within the interval:

$$iv_{ij} = [\max(0, es_{ij} - 0.5\varepsilon), \min(1, es_{ij} + 0.5\varepsilon)] \quad (9)$$

This means that this protocol substitutes the estimated values of the asymmetric fuzzy preference relation by intervals of the same length that are symmetrically distributed around the estimated values.

Let $\bar{S} \in IV(S)$, this optimization model is built by resolving the next optimization problem:

$$\begin{cases} \max\limits_{\bar{s}_{ij}} CL \\ \text{s.t.} \begin{cases} |\bar{s}_{ij} - es_{ij}| < 0.5\varepsilon & (i,j) \in MV \\ \bar{s}_{ij} = s_{ij} & (i,j) \in EV \end{cases} \end{cases} \quad (10)$$

where CL denotes the consistency level related to the complete intuitionistic reciprocal preference relation $\bar{R}$ derived from $\bar{S}$. Its value is computed as shown by Ureña et al. (2015a).

Model (10) is resolved by using the PSO algorithm. At iteration "e" of the algorithm, let $\mathbf{x}_k(e) = (x_{k,1}(e), x_{k,2}(t), \ldots, x_{k,d}(e))$ be the k-th particle, whose dimension d is $\#MV$, i.e., the number of decision variables in model (10), whose constraints are managed by converting $x_{k,h}$ to a value within the limits of the interval as:

$$\bar{s}_{ij} = y + (z - y)x_{k,h} \quad (11)$$

where z and y correspond to the upper and lower limits of iv_{ij}. We refer to González-Quesada, Cabrerizo, et al. (2024) for a more detailed explanation on how to apply the PSO.

3.2. $\mathbf{D_{au}}$ protocol

Unlike the previous protocol, this approach offers improved flexibility by using intervals of equal length that are asymmetrically distributed around the estimated values. This irregular structure offers increased adaptability, which can be used in the optimization procedure. For the estimated value, es_{ij}, of the incomplete asymmetric fuzzy preference relation, this protocol adjusts their values within the following boundaries:

$$iv_{ij} = [\max(0, es_{ij} - \gamma_{ij}\varepsilon), \min(1, es_{ij} + (1 - \gamma_{ij})\varepsilon)] \quad (12)$$

where $\gamma_{ij} \in [0, 1]$ specifies the asymmetric positioning of the interval with respect to es_{ij} with its length governed by the information granularity level ε. This suggests that the intervals can be arranged asymmetrically among various components, thus enhancing the available flexibility.

Let $\bar{S} \in IV(S)$, this optimization model is built by resolving the next optimization problem:

$$\begin{cases} \max\limits_{\bar{s}_{ij}} CL \\ \text{s.t.} \begin{cases} |\bar{s}_{ij} - es_{ij}| < \varepsilon & (i,j) \in MV \\ \bar{s}_{ij} = s_{ij} & (i,j) \in EV \end{cases} \end{cases} \quad (13)$$

$$\begin{cases} \max\limits_{\bar{s}_{ij}} CL \\ \text{s.t.} \begin{cases} \sum\limits_{(i,j)\in MV} 2|es_{ij} - \bar{s}_{ij}| \leq \#MV \cdot \varepsilon & (i,j) \in MV \\ \bar{s}_{ij} = s_{ij} & (i,j) \in EV \end{cases} \end{cases} \tag{16}$$

The PSO algorithm is applied to solve model (13); however, unlike the D_{su} protocol, the constraints are handled as follows:

$$\bar{s}_{ij} = y + (z - y)x_{k,h} \tag{14}$$

where y and z are computed by $\max(0, es_{ij} - \varepsilon)$ and $\min(1, es_{ij} + \varepsilon)$, respectively. We again refer to González-Quesada, Cabrerizo, et al. (2024) for a more detailed explanation on how to apply the PSO to solve this problem.

3.3. $\mathbf{D_{sn}}$ protocol

Taking the incomplete asymmetric fuzzy preference relation S into account, this protocol substitutes the estimated values with intervals that are symmetrically arranged around them. Nonetheless, in this instance, each interval may vary in length. Consequently, every es_{ij} is linked to a unique information granularity level ε_{ij}. For every estimated value, es_{ij}, of the incomplete asymmetric fuzzy preference relation, this protocol adjusts its value within the boundaries:

$$iv_{ij} = [\max(0, es_{ij} - 0.5\varepsilon_{ij}), \min(1, es_{ij} + 0.5\varepsilon_{ij})] \tag{15}$$

Let $\overline{S} \in IV(S)$, this optimization model is built by resolving the optimization problem presented in (16), where the value of ε is the average level of granularity allowed to the decision-maker. This value is calculated as follows:

$$\varepsilon = \frac{\sum_{(i,j)\in MV} \varepsilon_{ij}}{\#MV} \tag{17}$$

As previously mentioned about the PSO algorithm, and knowing that $x_{k,h}(e) \in [0,1]$, where $h = 1, 2, \ldots, d$, the following fitness function g is utilized in order to control the constraints presented in optimization model (16):

$$g(\mathbf{x}_k(e)) = \begin{cases} CL & p(\mathbf{x}_k(e)) = 1 \\ 0 & \text{otherwise} \end{cases} \tag{18}$$

where CL represents the corresponding consistency level of the complete asymmetric fuzzy preference relation characterized by the vector specified as a parameter that is calculated by using the procedure introduced by Ureña et al. (2015a), while p is a function that checks if the information granularity level meets the requirements, i.e., it is lower than the level allowed by the decision-maker. If the asymmetric fuzzy preference relation created by the given vector has an average information granularity level that exceeds the limit, the function returns 0, indicating that the relation is invalid. Otherwise, it returns 1.

3.4. $\mathbf{D_{an}}$ protocol

This protocol substitutes the estimated values in S with intervals of varying lengths placed asymmetrically around each value. This means that for a given estimated value es_{ij} of S, its value is adjusted within the range:

$$iv_{ij} = [\max(0, es_{ij} - \gamma_{ij}\varepsilon_{ij}), \min(1, es_{ij} + (1 - \gamma_{ij})\varepsilon_{ij})] \tag{19}$$

where $\gamma_{ij} \in [0,1]$ controls the asymmetric location of the interval associated with es_{ij}, and its length is ε_{ij}.

Let $\overline{S} \in IV(S)$, this optimization model is built by resolving the optimization problem defined in (20), where εdenotes the average information granularity level permitted by the decision-maker (see (17)). Similar to the previous protocols, we use the PSO algorithm again to modify the estimated values of the asymmetric fuzzy preference relation S to maximize its consistency level.

$$\begin{cases} \max\limits_{\bar{s}_{ij}} CL \\ \text{s.t.} \begin{cases} \sum\limits_{(i,j)\in MV} |es_{ij} - \bar{s}_{ij}| \leq \#MV \cdot \varepsilon & (i,j) \in MV \\ \bar{s}_{ij} = s_{ij} & (i,j) \in EV \end{cases} \end{cases} \tag{20}$$

$$R_i = \begin{bmatrix} \langle 0.50, 0.50\rangle & \langle 0.10, 0.80\rangle & x & x & \langle 0.30, 0.60\rangle & \langle 0.70, 0.20\rangle & x \\ \langle 0.80, 0.10\rangle & \langle 0.50, 0.50\rangle & \langle 0.30, 0.60\rangle & x & x & \langle 0.70, 0.20\rangle & \langle 0.30, 0.70\rangle \\ x & \langle 0.60, 0.30\rangle & \langle 0.50, 0.50\rangle & \langle 0.30, 0.40\rangle & x & \langle 0.30, 0.50\rangle & x \\ x & x & \langle 0.40, 0.30\rangle & \langle 0.50, 0.50\rangle & x & \langle 0.70, 0.10\rangle & \langle 0.20, 0.50\rangle \\ \langle 0.60, 0.30\rangle & x & x & x & \langle 0.50, 0.50\rangle & \langle 0.90, 0.10\rangle & \langle 0.60, 0.30\rangle \\ \langle 0.20, 0.70\rangle & \langle 0.20, 0.70\rangle & \langle 0.50, 0.30\rangle & \langle 0.10, 0.70\rangle & \langle 0.10, 0.90\rangle & \langle 0.50, 0.50\rangle & x \\ x & \langle 0.70, 0.30\rangle & x & \langle 0.50, 0.20\rangle & \langle 0.30, 0.60\rangle & x & \langle 0.50, 0.50\rangle \end{bmatrix}$$

$$R_c = \begin{bmatrix} \langle 0.50, 0.50\rangle & \langle 0.10, 0.80\rangle & \langle \mathbf{0.37}, \mathbf{0.48}\rangle & \langle \mathbf{0.20}, \mathbf{0.37}\rangle & \langle 0.30, 0.60\rangle & \langle 0.70, 0.20\rangle & \langle \mathbf{0.16}, \mathbf{0.56}\rangle \\ \langle 0.80, 0.10\rangle & \langle 0.50, 0.50\rangle & \langle 0.30, 0.60\rangle & \langle \mathbf{0.29}, \mathbf{0.32}\rangle & \langle \mathbf{0.33}, \mathbf{0.54}\rangle & \langle 0.70, 0.20\rangle & \langle 0.30, 0.70\rangle \\ \langle \mathbf{0.48}, \mathbf{0.37}\rangle & \langle 0.60, 0.30\rangle & \langle 0.50, 0.50\rangle & \langle 0.30, 0.40\rangle & \langle \mathbf{0.25}, \mathbf{0.57}\rangle & \langle 0.30, 0.50\rangle & \langle \mathbf{0.24}, \mathbf{0.42}\rangle \\ \langle \mathbf{0.37}, \mathbf{0.20}\rangle & \langle \mathbf{0.32}, \mathbf{0.29}\rangle & \langle 0.40, 0.30\rangle & \langle 0.50, 0.50\rangle & \langle \mathbf{0.17}, \mathbf{0.41}\rangle & \langle 0.70, 0.10\rangle & \langle 0.20, 0.50\rangle \\ \langle 0.60, 0.30\rangle & \langle \mathbf{0.54}, \mathbf{0.33}\rangle & \langle \mathbf{0.57}, \mathbf{0.25}\rangle & \langle \mathbf{0.41}, \mathbf{0.17}\rangle & \langle 0.50, 0.50\rangle & \langle 0.90, 0.10\rangle & \langle 0.60, 0.30\rangle \\ \langle 0.20, 0.70\rangle & \langle 0.20, 0.70\rangle & \langle 0.50, 0.30\rangle & \langle 0.10, 0.70\rangle & \langle 0.10, 0.90\rangle & \langle 0.50, 0.50\rangle & \langle \mathbf{0.11}, \mathbf{0.66}\rangle \\ \langle \mathbf{0.56}, \mathbf{0.16}\rangle & \langle 0.70, 0.30\rangle & \langle \mathbf{0.42}, \mathbf{0.24}\rangle & \langle 0.50, 0.20\rangle & \langle 0.30, 0.60\rangle & \langle \mathbf{0.66}, \mathbf{0.11}\rangle & \langle 0.50, 0.50\rangle \end{bmatrix}$$

$$\overline{R}_{su} = \begin{bmatrix} \langle 0.50, 0.50\rangle & \langle 0.10, 0.80\rangle & \langle \mathbf{0.37}, \mathbf{0.48}\rangle & \langle \mathbf{0.26}, \mathbf{0.38}\rangle & \langle 0.30, 0.60\rangle & \langle 0.70, 0.20\rangle & \langle \mathbf{0.23}, \mathbf{0.56}\rangle \\ \langle 0.80, 0.10\rangle & \langle 0.50, 0.50\rangle & \langle 0.30, 0.60\rangle & \langle \mathbf{0.29}, \mathbf{0.32}\rangle & \langle \mathbf{0.33}, \mathbf{0.54}\rangle & \langle 0.70, 0.20\rangle & \langle 0.30, 0.70\rangle \\ \langle \mathbf{0.48}, \mathbf{0.37}\rangle & \langle 0.60, 0.30\rangle & \langle 0.50, 0.50\rangle & \langle 0.30, 0.40\rangle & \langle \mathbf{0.25}, \mathbf{0.57}\rangle & \langle 0.30, 0.50\rangle & \langle \mathbf{0.24}, \mathbf{0.42}\rangle \\ \langle \mathbf{0.38}, \mathbf{0.26}\rangle & \langle \mathbf{0.32}, \mathbf{0.29}\rangle & \langle 0.40, 0.30\rangle & \langle 0.50, 0.50\rangle & \langle \mathbf{0.26}, \mathbf{0.41}\rangle & \langle 0.70, 0.10\rangle & \langle 0.20, 0.50\rangle \\ \langle 0.60, 0.30\rangle & \langle \mathbf{0.54}, \mathbf{0.33}\rangle & \langle \mathbf{0.57}, \mathbf{0.25}\rangle & \langle \mathbf{0.41}, \mathbf{0.26}\rangle & \langle 0.50, 0.50\rangle & \langle 0.90, 0.10\rangle & \langle 0.60, 0.30\rangle \\ \langle 0.20, 0.70\rangle & \langle 0.20, 0.70\rangle & \langle 0.50, 0.30\rangle & \langle 0.10, 0.70\rangle & \langle 0.10, 0.90\rangle & \langle 0.50, 0.50\rangle & \langle \mathbf{0.18}, \mathbf{0.66}\rangle \\ \langle \mathbf{0.56}, \mathbf{0.23}\rangle & \langle 0.70, 0.30\rangle & \langle \mathbf{0.42}, \mathbf{0.24}\rangle & \langle 0.50, 0.20\rangle & \langle 0.30, 0.60\rangle & \langle \mathbf{0.66}, \mathbf{0.18}\rangle & \langle 0.50, 0.50\rangle \end{bmatrix}$$

4. Numerical experiment

We begin by performing a numerical experiment to evaluate the effectiveness and performance of the information granularity allocation protocols described in Section 3. The goal is to complete an incomplete intuitionistic reciprocal preference relation with the highest possible level of consistency. Let us suppose the incomplete intuitionistic reciprocal preference relation R_i composed of seven alternatives ($m = 7$) and 18 missing values.

To estimate the missing values of R_i, denoted as x, we apply the approach presented in Section 2.3. After that, the complete intuitionistic reciprocal preference relation R_c is obtained (the estimated values are represented in bold). Using the procedure presented by Ureña et al. (2015a), R_c has a consistency level equal to 0.592. Once we have the complete intuitionistic reciprocal preference relation, we can proceed to analyze the different results obtained by the various protocols discussed in Section 3. This is done by assuming that the information granularity level is 0.2, i.e, $\varepsilon = 0.2$ and by applying the PSO with the values: $\omega^1 = 0.9$, $\omega^o = 0.4$, the acceleration coefficients are set to 2, and the number of generations is 100. These values are used because they have usually been used to solve similar problems (González-Quesada, Herrera-Viedma, et al., 2024).

4.1. Result of the $\mathbf{D_{su}}$ protocol

For each estimated value of R_c, this protocol will generate an interval of 0.2, which will be symmetrically distributed around the estimated value to complete the information. For example, for the preference degree of the estimated value r_{14}, the constructed interval is $[0.10, 0.30]$ because $\mu_{14} = 0.20$. After solving the model (10), we obtain $\overline{R}_{su}$. Considering the approach presented by Ureña et al. (2015a), we find that the consistency level of $\overline{R}_{su}$ is 0.643.

4.2. Result of the $\mathbf{D_{au}}$ protocol

For each estimated value of R_c, this protocol will generate an interval of 0.2, which will be distributed asymmetrically, unlike the previous protocol, around the estimated value to complete the information. After

$$\overline{R}_{au} = \begin{bmatrix} \langle 0.50, 0.50\rangle & \langle 0.10, 0.80\rangle & \langle \mathbf{0.44}, \mathbf{0.48}\rangle & \langle \mathbf{0.21}, \mathbf{0.37}\rangle & \langle 0.30, 0.60\rangle & \langle 0.70, 0.20\rangle & \langle \mathbf{0.22}, \mathbf{0.56}\rangle \\ \langle 0.80, 0.10\rangle & \langle 0.50, 0.50\rangle & \langle 0.30, 0.60\rangle & \langle \mathbf{0.29}, \mathbf{0.32}\rangle & \langle \mathbf{0.42}, \mathbf{0.54}\rangle & \langle 0.70, 0.20\rangle & \langle 0.30, 0.70\rangle \\ \langle \mathbf{0.48}, \mathbf{0.44}\rangle & \langle 0.60, 0.30\rangle & \langle 0.50, 0.50\rangle & \langle 0.30, 0.40\rangle & \langle \mathbf{0.30}, \mathbf{0.57}\rangle & \langle 0.30, 0.50\rangle & \langle \mathbf{0.25}, \mathbf{0.42}\rangle \\ \langle \mathbf{0.37}, \mathbf{0.21}\rangle & \langle \mathbf{0.32}, \mathbf{0.29}\rangle & \langle 0.40, 0.30\rangle & \langle 0.50, 0.50\rangle & \langle \mathbf{0.24}, \mathbf{0.50}\rangle & \langle 0.70, 0.10\rangle & \langle 0.20, 0.50\rangle \\ \langle 0.60, 0.30\rangle & \langle \mathbf{0.54}, \mathbf{0.42}\rangle & \langle \mathbf{0.57}, \mathbf{0.30}\rangle & \langle \mathbf{0.50}, \mathbf{0.24}\rangle & \langle 0.50, 0.50\rangle & \langle 0.90, 0.10\rangle & \langle 0.60, 0.30\rangle \\ \langle 0.20, 0.70\rangle & \langle 0.20, 0.70\rangle & \langle 0.50, 0.30\rangle & \langle 0.10, 0.70\rangle & \langle 0.10, 0.90\rangle & \langle 0.50, 0.50\rangle & \langle \mathbf{0.16}, \mathbf{0.66}\rangle \\ \langle \mathbf{0.56}, \mathbf{0.22}\rangle & \langle 0.70, 0.30\rangle & \langle \mathbf{0.42}, \mathbf{0.25}\rangle & \langle 0.50, 0.20\rangle & \langle 0.30, 0.60\rangle & \langle \mathbf{0.66}, \mathbf{0.16}\rangle & \langle 0.50, 0.50\rangle \end{bmatrix}$$

$$\overline{R}_{sn} = \begin{bmatrix} \langle 0.50, 0.50\rangle & \langle 0.10, 0.80\rangle & \langle \mathbf{0.40}, \mathbf{0.48}\rangle & \langle \mathbf{0.27}, \mathbf{0.37}\rangle & \langle 0.30, 0.60\rangle & \langle 0.70, 0.20\rangle & \langle \mathbf{0.26}, \mathbf{0.56}\rangle \\ \langle 0.80, 0.10\rangle & \langle 0.50, 0.50\rangle & \langle 0.30, 0.60\rangle & \langle \mathbf{0.30}, \mathbf{0.32}\rangle & \langle \mathbf{0.40}, \mathbf{0.54}\rangle & \langle 0.70, 0.20\rangle & \langle 0.30, 0.70\rangle \\ \langle \mathbf{0.48}, \mathbf{0.40}\rangle & \langle 0.60, 0.30\rangle & \langle 0.50, 0.50\rangle & \langle 0.30, 0.40\rangle & \langle \mathbf{0.33}, \mathbf{0.57}\rangle & \langle 0.30, 0.50\rangle & \langle \mathbf{0.30}, \mathbf{0.42}\rangle \\ \langle \mathbf{0.37}, \mathbf{0.27}\rangle & \langle \mathbf{0.32}, \mathbf{0.30}\rangle & \langle 0.40, 0.30\rangle & \langle 0.50, 0.50\rangle & \langle \mathbf{0.25}, \mathbf{0.41}\rangle & \langle 0.70, 0.10\rangle & \langle 0.20, 0.50\rangle \\ \langle 0.60, 0.30\rangle & \langle \mathbf{0.54}, \mathbf{0.40}\rangle & \langle \mathbf{0.57}, \mathbf{0.33}\rangle & \langle \mathbf{0.41}, \mathbf{0.25}\rangle & \langle 0.50, 0.50\rangle & \langle 0.90, 0.10\rangle & \langle 0.60, 0.30\rangle \\ \langle 0.20, 0.70\rangle & \langle 0.20, 0.70\rangle & \langle 0.50, 0.30\rangle & \langle 0.10, 0.70\rangle & \langle 0.10, 0.90\rangle & \langle 0.50, 0.50\rangle & \langle \mathbf{0.17}, \mathbf{0.66}\rangle \\ \langle \mathbf{0.56}, \mathbf{0.26}\rangle & \langle 0.70, 0.30\rangle & \langle \mathbf{0.42}, \mathbf{0.30}\rangle & \langle 0.50, 0.20\rangle & \langle 0.30, 0.60\rangle & \langle \mathbf{0.66}, \mathbf{0.17}\rangle & \langle 0.50, 0.50\rangle \end{bmatrix}$$

$$\overline{R}_{an} = \begin{bmatrix} \langle 0.50, 0.50\rangle & \langle 0.10, 0.80\rangle & \langle \mathbf{0.37}, \mathbf{0.48}\rangle & \langle \mathbf{0.21}, \mathbf{0.37}\rangle & \langle 0.30, 0.60\rangle & \langle 0.70, 0.20\rangle & \langle \mathbf{0.17}, \mathbf{0.56}\rangle \\ \langle 0.80, 0.10\rangle & \langle 0.50, 0.50\rangle & \langle 0.30, 0.60\rangle & \langle \mathbf{0.29}, \mathbf{0.32}\rangle & \langle \mathbf{0.35}, \mathbf{0.54}\rangle & \langle 0.70, 0.20\rangle & \langle 0.30, 0.70\rangle \\ \langle \mathbf{0.48}, \mathbf{0.37}\rangle & \langle 0.60, 0.30\rangle & \langle 0.50, 0.50\rangle & \langle 0.30, 0.40\rangle & \langle \mathbf{0.26}, \mathbf{0.57}\rangle & \langle 0.30, 0.50\rangle & \langle \mathbf{0.25}, \mathbf{0.46}\rangle \\ \langle \mathbf{0.37}, \mathbf{0.21}\rangle & \langle \mathbf{0.32}, \mathbf{0.29}\rangle & \langle 0.40, 0.30\rangle & \langle 0.50, 0.50\rangle & \langle \mathbf{0.17}, \mathbf{0.43}\rangle & \langle 0.70, 0.10\rangle & \langle 0.20, 0.50\rangle \\ \langle 0.60, 0.30\rangle & \langle \mathbf{0.54}, \mathbf{0.35}\rangle & \langle \mathbf{0.57}, \mathbf{0.26}\rangle & \langle \mathbf{0.43}, \mathbf{0.17}\rangle & \langle 0.50, 0.50\rangle & \langle 0.90, 0.10\rangle & \langle 0.60, 0.30\rangle \\ \langle 0.20, 0.70\rangle & \langle 0.20, 0.70\rangle & \langle 0.50, 0.30\rangle & \langle 0.10, 0.70\rangle & \langle 0.10, 0.90\rangle & \langle 0.50, 0.50\rangle & \langle \mathbf{0.13}, \mathbf{0.67}\rangle \\ \langle \mathbf{0.56}, \mathbf{0.17}\rangle & \langle 0.70, 0.30\rangle & \langle \mathbf{0.46}, \mathbf{0.25}\rangle & \langle 0.50, 0.20\rangle & \langle 0.30, 0.60\rangle & \langle \mathbf{0.67}, \mathbf{0.13}\rangle & \langle 0.50, 0.50\rangle \end{bmatrix}$$

solving the model (13), we obtain $\overline{R}_{au}$. Considering the approach presented by Ureña et al. (2015a), we find that the consistency level of $\overline{R}_{au}$ is 0.690.

4.3. Result of the $\mathrm{D_{sn}}$ protocol

For each estimated value of R_c, this protocol constructs intervals of different lengths, symmetrically distributed around the estimated value. This is because the value ε ensures that the average length of the intervals is 0.2. After solving the model (16), we get $\overline{R}_{sn}$. Considering the approach presented by Ureña et al. (2015a), its level of consistency is 0.665.

4.4. Result of the $\mathrm{D_{an}}$ protocol

For each estimated value of R_c, this protocol constructs intervals of different lengths distributed asymmetrically around the estimated value. This is because the value $\varepsilon = 0.2$, as in the previous protocol, ensures that the mean length of the intervals is 0.2. After solving the model (20), we obtain $\overline{R}_{an}$. Considering the approach presented by Ureña et al. (2015a), we find that the level of consistency of $\overline{R}_{an}$ is 0.715.

5. Comparative analysis

Although the permissible average level of information granularity has been set at a low value, the outcomes derived from each protocol indicate that the allocation of an information granularity level can significantly improve the consistency of complete intuitionistic reciprocal preference relations when compared to the use of exact numerical values. Notably, the numerical approach referenced in Ureña et al. (2015a), which attained a consistency score of 0.592, has been surpassed by its granular alternatives, namely D_{su}, D_{au}, D_{sn}, and D_{an}, resulting in consistency scores of 0.643, 0.690, 0.665, and 0.715, respectively.

An experimental study was also conducted to more thoroughly evaluate the efficiency of the protocols. This investigation involved the creation of 200 incomplete intuitionistic reciprocal preference relations, characterized by a range of alternatives ranging from 10 to 20 and the inclusion of missing values. With the assumption of $\varepsilon = 0.2$, the results indicated a notable improvement in the consistency level of the complete intuitionistic reciprocal preference relations when granular (interval) values were used to characterize the estimated values. The average improvements achieved through the protocols D_{su}, D_{au}, D_{sn}, and D_{an} were 6.1%, 9.8%, 7.5%, and 13.3%, respectively. Consequently, it can be concluded that the application of an allocation of information granularity serves as a valuable strategy to improve the consistency of complete intuitionistic reciprocal preference relations. Specifically, greater advances were achieved when using a protocol based on an asymmetric allocation of information granularity, as this strategy affords an additional degree of flexibility during the maximization process.

To examine the impact of ε on the effectiveness of the protocol, experiments were carried out by adjusting ε to 0.3 and 0.4. Table 1 demonstrates the enhancement of consistency scores across various granule sizes. The findings clearly indicate that with an increase in ε, the probability of attaining greater consistency values (CL) also grows. This pattern is clear in Table 1, where a significant increase in CL values aligns with elevated ε levels. This effect occurs because, as a higher value is assigned to ε, the intervals within which the preference and non-preference degrees can be placed expand, increasing the potential for maximizing the consistency. Additionally, regardless of the average level of permitted information granularity, the protocol that employs an asymmetric and non-uniform granularity allocation achieves the highest consistency values. This is reasonable, as this protocol, among those evaluated, offers the highest level of flexibility to complete the intuitionistic reciprocal preference relations.

Finally, it should be noted that the estimation procedure outlined in Section 2.3 is capable of estimating all missing values within an incomplete reciprocal preference relation provided that a set of $n-1$ non-leading diagonal preference values (both preference and non-preference degrees) are known and that each alternative is compared at least once (Herrera-Viedma et al., 2007). However, in these cases, we can also apply the granular approach with any of the protocols by positing that missing values, which cannot be estimated using the estimation procedure detailed in Section 2.3, can take any value within the unit interval. In this context, the granularity level is determined to be 2.

Table 1. Values of CL for chosen values of ε.

	D_{su}	D_{au}	D_{sn}	D_{an}
$\varepsilon = 0.2$	0.643	0.690	0.665	0.715
$\varepsilon = 0.3$	0.652	0.700	0.685	0.722
$\varepsilon = 0.4$	0.685	0.714	0.699	0.730

6. Concluding remarks

This study has investigated four techniques for allocating information granularity to enhance consistency in incomplete intuitionistic reciprocal preference relations during the process of estimating their missing values. Although this research does not propose a new protocol, estimation technique, or optimization algorithm, this type of comparative analysis is essential to provide practical guidance to researchers and practitioners on selecting the most effective protocol. To do so, numerical experiments have been conducted to show the efficacy and efficiency of each protocol.

Compared to protocols based on symmetric allocations, those using asymmetric allocations achieved higher consistency scores. Among them, the protocol using a non-uniform allocation of information granularity produced the best results. This is associated with the protocol's flexibility, allowing a more efficient allocation of resources by providing increased detail to the estimated values that require it the most. It is important to emphasize that completing the intuitionistic reciprocal preference relations using the consistency-based approach ensures a more coherent and accurate preference relation, facilitating better decisions.

Future research may explore two directions. First, the performance of metaheuristic algorithms like PSO is often sensitive to parameter tuning. A brief sensitivity analysis or a more robust theoretical justification for the chosen parameters would strengthen the experimental design. In addition, because the use of PSO may pose scalability issued in larger datasets, its computational complexity should be discussed. Second, the application of these protocols to real-world group decision-making scenarios or case studies would greatly enhance the practical relevance of this research.

Acknowledgments

This work was supported by the grant PID2022-139297OB-I00 funded by MCIN / AEI / 10.13039/501100011033 and by ERDF/EU. Moreover, it is part of the project C-ING-165-UGR23, co-funded by the Regional Ministry of University, Research and Innovation and by the European Union under the Andalusia ERDF Program 2021-2027.

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