ORDERINGS OF SEMIFIELDS

A DISSERTATION SUBMITTED TO THE GRADUATE DIVISION OF THE UNIVERSITY OF HAWAIʻI AT MĀNOA IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

IN

MATHEMATICS

MAY 2015

By

Gretel S. Sia

Dissertation Committee:
Thomas Craven, Chairperson
Ralph Freese
Pavel Guerzhoy
Hugh Michael Hilden
Lynne Wilkens

Keywords: orderings, preorderings, semifields, Boolean space

Dedication

To my son, Kyle Christopher S. Sia

and

In loving memories of the late Dr. Nilo G. Arnaiz, and

of my father, the late Sebastian de Padua Salimbagat

Acknowledgment

I would like to express my gratitude first and foremost to my adviser, Dr. Thomas Craven, for his kindness in assuming the duty as such, and his patience in guiding me along the difficult road of writing a dissertation and helping me complete this part of my academic journey. My sincere thanks go as well to the other members of my committee: Dr. Ralph Freese, Dr. Pavel Guerzhoy, Dr. Hugh Michael Hilden, and Dr. Lynne Wilkens, for their willingness to share their time, energy, and expertise that I may be able to take that high final step towards earning my doctorate degree.

I would also like to thank my good friend, my former colleague, Dr. Xiongzhi (Chee) Chen of Princeton University, who, with his immeasurable kindness, generosity, and selflessness, has always been there for me when help is most needed.

My gratitude should also go to my former professors from the UH Manoa Math department - Dr. Wayne Smith, Dr. Joel Weiner, Dr. David Stegenga, Dr. Eric Guentner, Dr. Greg Brumfiel, Dr. Ron Brown, Dr. David Blecker, Dr. Ralph Freese, Dr. George Csordas, Dr. Thomas Craven, Dr. James B. Nation, and Dr. Michelle Manes, from whom I derived most of my knowledge of mathematics.

Last but not the least, I would like to thank the following staff and faculty members, also of the UH Manoa Math Department - secretaries Ms. Shirley Kikiloi, Ms. Alicia Maedo and Ms. Sue Hasegawa, for their friendship and encouragements along the way; professors Dr. Robert Little, Dr. Heiner Dovermann, Dr. James B. Nation, and Dr. George Wilkens, for giving me all the needed opportunities to earn my doctorate degree from this university.

Above all, I thank the Almighty, whose power and presence define my own existence.

Abstract

A fundamental characterization of orderable fields is given by Artin and Schreier: A field F can be ordered if and only if it is formally real. Knebusch, Rosenberg and Ware proved that the space of orderings of F can be topologized to make a Boolean space (compact, Hausdorff, and totally disconnected) providing a connection between the orderings of a field and the theory of quadratic forms. A semifield is an algebraic structure which is a multiplicative abelian group and an additive commutative semigroup at the same time, and multiplication is distributive with respect to addition. In this paper, we study the orderings of general semifields, with particular attention on idempotent semifields (such as are used in tropical algebra) and subsemifields of formally real fields. We show that some subsemifields completely recapture the ordering structure of a field, while others can have dramatically different orderings. Interestingly though, it turns out that the space of orderings of a semifield is also a Boolean space.

Table of Contents

List of Notations

The sets of natural numbers, integers, rational numbers, and real numbers, shall be denoted by the standard notations $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$, respectively.

ΣF^2	the set of all finite sums of squares of the field F	1
$char\ F$	the characteristic of the field F	1
$sgn_P(\varphi)$	the total signature of the quadratic form φ	1
X_F	the space of orderings of the field F	1
$W(F)$	the Witt ring of F	1
$C(X_F, \mathbb{Z})$	the ring of continuous functions over the topological space X_F with values in $\mathbb{Z}$	1
$W_t(F)$	the torsion ideal of $W(F)$	2
$W_{red}(F)$	the reduced Witt ring of F	2
$F(x)$	the field of rational functions over the field F	2
$\mathbb{R}(x)$	the field of real rational functions	2
$\leq_P$	the total order relation associated with the ordering P	5
F^*	the set $F \setminus \{0\}$	5
$\mathbb{R}^{\#}(x)$	the semifield of rational functions with nonnegative real coefficients	21
$H_{(a,b)}$	a Harrison subbasic open set $\{\leq \in X_S \mid a \leq b\}$	23

CHAPTER 1
Introduction

The theory of ordered fields can be traced back to David Hilbert in his 1899 book "Foundations of Geometry" [6]. In that book, the notion about "betweeness" among points of a line plays a significant role [16, p. 16]. He showed, among others, that the "algebra of segments" constructed to study plane geometry was an ordered field. It is for this reason that his work is considered the origin of the theory of ordered fields. For whatever reason there was on his part, however, he did not pursue further interest in this notion of an ordering in a field. But one problem he posed was this: "For any $f \in \mathbb{R}[X_1, X_2, ..., X_n]$, is it true that if $f \geq 0$ in $\mathbb{R}^n$, then f is a sum of squares of rational functions?" This question is what has later become known as Hilbert's 17th Problem [13, p.11].

It was not until more than a quarter-century later that a systematic development of the subject on orderings of fields began when the series of papers by Emil Artin and Otto Schreier were published in 1926 and 1927. In his paper, Artin proved that the answer to the now celebrated Hilbert's 17th Problem is in the affirmative. He successfully applied his sums of squares criterion to furnish a complete solution to this problem [16, p.17]. What is interesting to note though is while the solution to Hilbert's 17th Problem was a major breakthrough by itself, an important theory was developed in the process of finding a solution to it. This has become known to be the Artin-Schreier Theory. Some of its highlights that are pertinent in this thesis are as follows:

(1) A field F is called **formally real** if and only if $a_1^2 + a_2^2 + ... + a_r^2 = 0$ in F implies $a_i = 0$.

(2) A field F can be ordered if and only if F is formally real.

(3) If F is a field with characteristic $charF \neq 2$, then a nonzero element b is totally positive if and only if b is positive in every ordering of F, and if and only if $b \in \Sigma F^2$, where ΣF^2 denotes the set of all finite sums of squares of elements of F.

The Artin-Schreier Theory has also been applied to the algebraic theory of quadratic forms. The relationship between orderings and quadratic forms dates back to the work by Sylvester with his idea of a **signature** [16, p.72]. This notion was later pursued by Pfister and made more popular with his celebrated Local-Global Principle. For a given ordering P, and a quadratic form φ over a formally real field F, he defined the signature $sgn_P(\varphi)$. The result was a **total signature** of the form φ which is a function $Sgn_\varphi : X_F \to \mathbb{Z}$ defined by $Sgn_\varphi(P) = sgn_P(\varphi)$, where X_F denotes the set of all the orderings of F. This gave rise to a homomorphism from the Witt ring $W(F)$ of the field F to the ring $C(X_F, \mathbb{Z})$ of continuous functions over a topologized space X_F with values in $\mathbb{Z}$. The main result was that the kernel of this homomorphism was precisely the torsion ideal $W_t(F)$ of $W(F)$ [16, p.74]. This essentially showed the connection between the study of the reduced Witt ring $W_{red}(F) = W(F)/\ W_t(F)$ and the space of orderings of the field F.

Topologically, the most precise description of the space of orderings X_F of a formally real field F is that it is a Boolean space (compact, Hausdorff, and totally disconnected). This result originated from the work of David Harrison who observed that X_F can be given a natural topology [16, p.75]. This was generalized by Knebusch, Rosenberg and Ware [9]. In [3], it is shown that, for any Boolean space X, there exists a field F with X_F homeomorphic to X. The Boolean space of orderings on a rational function field $F(x)$ has been studied in [2]. For fields which are real closed, the notion of classifying orderings on $F(x)$ by Dedekind cuts on F dates back to Bourbaki, around 1959, the same being posed as problem no. 24 of section 2 in Chapter 6 of the book [1]. (A detailed solution to this problem was given by this writer's thesis adviser in his doctoral thesis in 1974 at Cornell University.)

Shifting the discussion to another algebraic structure, we note, to begin with, that the theory of semifields is a perspective direction of modern algebra which may be considered as included in the theory of semirings and the theory of groups with an auxiliary operation [21]. A semifield is an algebraic structure which is a multiplicative abelian group, a commutative additive semigroup, and whose multiplication is distributive with respect to addition.

An extensive review of the literature on semirings and semifields is found in [5] and [21]. Semirings were already cited as far back as the end of the 19th century in the work of Dedekind and Hilbert. The classical example is the set of nonnegative integers with respect to the operations of addition and multiplication. The theory of semirings began in the 1950s. The modern theory of semirings finds applications in computer algebra, discrete mathematics, and idempotent analysis.

Semifields have been studied since the 1960s. Hanns Weinert pioneered the investigation of its properties [22, 23, 24, 25]. In 1962, he proved that the class of idempotent semifields coincides with the class of lattice ordered groups which have distributive lattices of congruences. Harry Hutchins and Hanns Weinert investigated general properties of semifield kernels [7]. A.N. Semenov established important properties of lattices of semifield congruences (kernels) [19, 20].

Tropical semifields and other idempotent semifields are the subjects in idempotent analysis. Works related to the development of idempotent analysis are, among others, those of V.P. Maslov. He, for example, investigated the problems of linear algebra and the theory of equations over idempotent commutative semifields. His most notable work, however, that is closely related to our work is the so called Maslov dequantization and valuations which are two constructions relating ordered fields and idempotent semifields [14, 15]. A more detailed discussion of this result is found in section 3.5 of this thesis.

It is in view of all of the foregoing theories and results that this new idea of total orderings in a semifield (as distinguished from partial orderings) is being pursued and becomes the subject of this work. In Chapter 2, we define the basic notions of orderings and preorderings of a field and highlight some of the classical results in the theory of ordered fields, which, substantially is the Artin-Schreier Theory. Chapter 3 is a more detailed discussion of semifields and their orderings. Several examples of semifields are given, and the notion of a semifield ordering is defined. While it shows that the direct product of certain semifields can be ordered, it also shows that a semifield can be constructed in such a way that it cannot have an ordering. Moreover, it is shown in this chapter that there exists a subset of a semifield that has similar properties as that of a positive cone of an ordering in a field, hence it is considered the positive cone analogue in a semifield. The chapter contains the significant finding of this study, that the whole idea about orderings in a field can be recovered from the space of orderings of the semifield sums of squares in the field. It ends with a discussion on tropical and idempotent semifields and shows a particular construction by V. P. Maslov that relate tropical semifields and ordered fields [12].

In Chapter 4, we discuss the orderings of the semifield of rational functions with nonnegative real coeficicients. In there, it is proved that this semifield does not inherit all of the orderings of the field $\mathbb{R}(x)$. Examples of orderings not induced by the orderings of $\mathbb{R}(x)$ are given. Chapter 5 pertains to the topology of the set of orderings of a semifield. It is proved that as in the case of the space of orderings of a formally real field, the space of orderings of a semifield is Boolean (compact, Hausdorff, and totally disconnected). Owing to the novelty of this work, Chapter 6 is included to contain some possible questions that naturally arise from it.

CHAPTER 2
Orderings of Fields

In this chapter, we introduce some of the basic concepts about a field and its orderings and give examples to illustrate the ideas. We also discuss classical results that we need in understanding the theory we have developed which is now the subject of this thesis.

In studying a certain algebraic structure, some of the pertinent questions to ask would be if it can be ordered, and if so, how does this reflect its own properties, or given its properties, how do these properties determine the presence or absence of an ordering on its elements? The field of real numbers $\mathbb{R}$, and the field of rational numbers $\mathbb{Q}$, for instance, have a natural ordering which is unique, while the field of real rational functions $\mathbb{R}(x)$ has infintiely many orderings. On the other hand, the field of complex numbers $\mathbb{C}$, cannot be ordered in any way. In fact, any finite field, or a field with *characteristic* $p > 0$, cannot be ordered [16, p. 1]. The existence or nonexistence of an ordering on these fields reflect the structure of the fields themselves. In general, the set of orderings of a field can be topologized in such a way that it reflects the algebraic theory of quadratic forms over the field [18].

2.1. Orderings and Preorderings of a Field

We begin this section with the definition of a group, an algebraic structure underlying the structure of a field.

Definition 2.1.1. A group is a nonempty set G on which is defined a binary operation $*$, satisfying the following properties:

(i) Closure: If a, $b \in G$, then $a * b \in G$.

(ii) Associativity: $a * (b * c) = (a * b) * c$ for all a, b, $c \in G$.

(iii) Identity: There exists an element $1 \in G$ such that $1 * a = a * 1 = a$, for all $a \in G$.

(iv) Inverse: For every element $a \in G$, there exists an element $a^{-1} \in G$ such that $a * a^{-1} = a^{-1} * a = 1$.

A group G is **abelian** if the binary operation is commutative, i.e., $a * b = b * a$, for all a, $b \in G$. In this case, the binary operation is often written additively, i.e., $a * b$ is written as $a + b$, and the identity element is written as 0 rather than 1.

For example, the set of integers $\mathbb{Z}$, under addition, is an abelian group.

Definition 2.1.2. A **field** is a nonempty set F equipped with two binary operations, addition $(+)$ and multiplication $(\cdot\ )$, satisfying the following properties:

(i) $(F, +)$ is an abelian group with identity element 0.

(ii) $(F^*, \cdot)$ is a commutative group with identity element 1, where $F^* = F \setminus \{0\}$, and

(iii) $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$ for all a, b, $c \in F$.

Thus, the sets of numbers $\mathbb{R}$, $\mathbb{Q}$, and $\mathbb{C}$, each being equipped with the two operations of addition and multiplication, were being described earlier as fields. Hereafter, instead of writing the raised dot $(\cdot)$ to denote multiplication, we shall indicate the product of two elements a and b of a set either by writing $(a)(b)$, or $a(b)$, or simply ab. Also, we shall denote by F^2 the set of squares of elements of a field F and by ΣF^2 the set of all finite sums of squares of elements of F.

Now, to gain a better understanding about the structure of a field, we will examine some of its features. First, we will introduce the idea of **orderings** of a field, then generalizing this idea, we next introduce the notion of a **preordering**. In the famous book of D. Hilbert, "Foundations of Geometry", the primitive notion of "betweeness" among points on a line plays a fundamental role. Intuitively, the idea of an "ordering" on the elements of a field can be viewed as the algebraic analog of "betweeness" among points of a line [16, p.16]. Subsequent work on this area took place in the 1920's when Artin and Schreier developed the algebraic theory of formally real fields and formulated the basic relationships between formally real fields and fields with orderings. Throughout the following discussion, we shall assume that $char(F) \neq 2$.

Definition 2.1.3. An **ordering** of a field F is a set $P \subsetneq F$ satisfying

(i) $P + P \subseteq P$,

(ii) $P \cdot P \subseteq P$,

(iii) $P \cup -P = F$, and

(iv) $P \cap -P = \{0\}$, where $-P = \{a \in F \mid -a \in P\}$.

This set P is otherwise referred to as "the **positive cone** of the ordering." Thus, $\Sigma\mathbb{R}^2 = \mathbb{R}^2$, the set of all sums of squares of elements of $\mathbb{R}$, is an ordering of $\mathbb{R}$. One can easily verify that properties $(i) - (iv)$ above are satisfied by this set. Similarly, $\Sigma\mathbb{Q}^2$, the set of all sums of squares of the rational numbers, is an ordering of $\mathbb{Q}$. A nonexample of an ordering is the set of all negative real numbers since the set will not be closed under multiplication, and thus, property (ii) will not be satisfied by this set.

Orderings can also be described as order relations. If P is an ordering of F, and a, $b \in F$, we associate with P the order relation $\leq_P$ defined by $a \leq_P b$ if and only if $b - a \in P$. The relation $\leq_P$ is a total order on F. Moreover, it is compatible with the operations on F : if a, b, $c \in F$, and $a \leq_P b$, then $a + c \leq_P b + c$, and if $c \neq 0$, $ac \leq_P bc$. The positive cone P is recovered from $\leq_P$ by defining it to be the set $P = \{a \in F \mid a \geq_P 0\}$. An **ordered field** is a pair $(F, \leq)$, where F is a field and $\leq$ is an ordering on F [10, p.233].

Given such a set P, we shall refer to F as a field **ordered** by P, or simply that F is an **ordered** field. As an immediate consequence of Definition 2.1.3., some other basic properties of an ordered field are as follows:

(a) $\Sigma F^2 \subseteq P$.

Proof: Let $x \in F$. Then by (iii), $x \in P$ or $-x \in P$. If $x \in P$, then $x^2 \in P \cdot P \subseteq P$. Similarly, if $-x \in P$, then $(-x)(-x) = x^2 \in P \cdot P \in P$. Thus, the result follows by (i). ■

In particular, 0 and 1 are in P.

(b) $-1 \notin P$.

Proof: Assume $-1 \in P$. Note that $char(F) \neq 2$. Then for any $a \in F$, we will have

$$a = (\frac{a+1}{2})^2 + (-1)(\frac{a-1}{2})^2 \in P + P \cdot P \subseteq P + P \subseteq P.$$

This contradicts the property that $P \subsetneq F$. ■

In the definition above of an ordering, if we replace the condition that $P \cup -P = F$ by $F^2 \subseteq P$, we get the more general notion of a preordering. This is one of the fundamental concepts in the theory of ordered fields, and it will have a special importance later on in our discussion about semifields. We now define this idea.

Definition 2.1.4. [11, p. 2] A **preordering** of a field F is a set $T \subsetneq F$ satisfying

(i) $T + T \subseteq T$

(ii) $T \cdot T \subseteq T$

(iii) $F^2 \subseteq T$

Consequently, every ordering is a preordering, but not conversely. For instance, the set of all nonnegative real numbers $\Sigma\mathbb{R}^2 = \mathbb{R}^2$, and the set of sums of squares of rational numbers $\Sigma\mathbb{Q}^2$, are preorderings (as well as orderings, as noted previously) of $\mathbb{R}$ and $\mathbb{Q}$, respectively. In fact, for any field F, ΣF^2 is a preordering. Moreover, since for any preordering T of F, $F^2 \subseteq T$, then, $\Sigma F^2 \subseteq T$. Consequently, ΣF^2 is the unique smallest preordering of F. On the other hand, $\Sigma\ (\mathbb{R}(x))^2$ is a preordering but not an ordering. Observe that a polynomial function in x that attains a positive and a negative value cannot possibly be written as plus or minus a sum of squares of rational functions. Hence it is a preordering and not an ordering.

At this point, it is worthwhile to note some classical results which are pertinent in our study. We devote the remainder of this section to these. For instance, by Zorn's Lemma, every preordering of F can be extended to an ordering of F. Aptly called the Extension Lemma, its statement and proof is given below.

Lemma 2.1.5. (Extension Lemma) [10, p.289]**:** Let T be a preordering of F, and let $a \in F \setminus T$. Then there is an ordering $P \subset F$ such that $T \subset P$ and $a \notin P$.

Proof: Let $T' = \{t_1 - at_2 \mid t_1, t_2 \in T\}$. Let $x, y \in T'$. Then $x = t_1 - at_2$ and $y = t_3 - at_4$, for some $t_i \in T$, and we have

$$\begin{aligned} x + y &= (t_1 + t_3) - a(t_2 + t_4) = t_5 - at_6 \in T' \text{ and} \\ xy &= (t_1 t_3 + a^2 t_2 t_4) - a(t_2 t_3 + t_1 t_4) \in T' \end{aligned}$$

Moreover, since $F^2 \subseteq T$, $0, 1 \in T$, and so $T \subseteq T'$ and $-a \in T'$. Thus, we have $F^2 \subseteq T \subseteq T'$. We are left to show that $-1 \notin T'$. Suppose $-1 \in T'$. Then for some $t_i \in T$, $-1 = t_1 - at_2$. If $t_2 = 0$, then $t_1 = -1$, a contradiction since $-1 \notin T$. Thus $t_2 \neq 0$, and $a = t_2^{-1}(1 + t_1) = (t_2^{-1})^2 t(1 + t_1) \in T$, likewise a contradiction.

By Zorn's Lemma, there exists a preordering P, maximal, subject to the conditions that $T' \subset P$ and $-1 \notin P$. We will prove that P is an ordering. Let $b \in F$, and $b \notin P$. Again consider the set $P' = \{(s_1 - bs_2) \mid s_1, s_2 \in P\}$. Then as before, P' is a preordering, and since $P \subset P'$, by the maximality of P, $P = P'$. Moreover, $-b \in P$. Hence, $P \cup -P = F$. Also, if $c \in P \cap -P$, then $c = 0$, for if $c \neq 0$, we would have $-1 = (c^{-1})^2(c)(-c) \in P$, a contradiction. ■

Theorem 2.1.6. [10, p. 288] A field F is formally real if and only if ΣF^2 is a preordering, if and only if it has at least one ordering.

Proof: For the first "if and only if" statement, we just have to observe that if F is formally real, then $-1 \notin \Sigma F^2$, and so ΣF^2 will satisfy all the conditions for a preordering in Definition 1.1.4. The converse clearly holds since if $-1 \notin \Sigma F^2$, then F must be formally real by definition. For the second "if and only if" statement, we have that if F is formally real, then ΣF^2 is a preordering, and so by the above Lemma, F contains an ordering. Conversely, if F has an ordering, then -1 is not in any ordering and so not in ΣF^2. ■

Definition 2.1.7. Let F be a field and $T \subsetneq F$ a preordering. We define $X_T = \{P \mid P \supseteq T\}$ to be the set of all orderings P in F containing T.

Theorem 2.1.8. [10, p. 289] For any preordering $T \subsetneq F$, $T = \bigcap_{P \in X_T} P$.

Proof: It is immediate that $T \subseteq \bigcap_{P \in X_T} P$. Now let $a \notin T$. Then as in the proof of the Extension Lemma, $T' = T - aT$ is a preordering containing T and $-a$. Let $P \in X_{T'}$ which exists by the Extension Lemma. It follows that $-a \in P$ and thus, $a \notin P$. ■

In the next section, we will see a complicated example of an ordered field which will lead into our most significant example of a semifield.

2.2. Orderings of the Field of Rational Functions over $\mathbb{R}$

We say a field F is **real-closed** if F is formally real, but no proper algebraic extension of F is formally real [10, p. 236]. In [2], it was shown that if F is a real-closed field, then there is a bijective correspondence between the orderings of $F(x)$ and subsets $A \subseteq F$ satisfying the condition

$$a \in A, \ b < a \ \text{ implies } b \in A.$$

The set A corresponds to the ordering of $F(x)$ such that A is the set of elements of F which are less than x. As noted in Chapter 1 above, the idea was originally presented as a Bourbaki exercise (problem 24, Chapter 6, section 2) [1].

Now, let $F = \mathbb{R}$, then we have the field $\mathbb{R}(x)$ of rational functions over $\mathbb{R}$. Like the set A above, we let $C \subseteq \mathbb{R}$ with the property that if $a < b$, a, $b \in \mathbb{R}$, and $b \in C$ then $a \in C$. An ordering P_C on $\mathbb{R}(x)$ is defined as follows: For a nonzero polynomial $f(x) \in \mathbb{R}[x]$, we write $f(x)$ as a product of its irreducible factors, i.e.,

$$f(x) = r(x-\alpha_1)(x-\alpha_2)...(x-\alpha_n)q_1(x)...q_m(x),$$

where r, $\alpha_1, ..., \alpha_n \in \mathbb{R}$, and the polynomials $q_i(x)$ are monic irreducible quadratics. We let $f(x) \in P_C^+$, where $P_C^+ = P_C \setminus \{0\}$, if $r \in P^+ = P \setminus \{0\}$, and the number of $\alpha_i \notin C$ is even, or $r \notin P^+$ and the number of $\alpha_i \notin C$ is odd. For $0 \neq f$, $g \in \mathbb{R}[x]$, $f/g \in P_C^+$ if and only if $fg \in P_C^+$. It is shown that the P_C obtained in this way is an ordering on $\mathbb{R}(x)$, and is the unique ordering on $\mathbb{R}(x)$ with respect to which $C = \{b \in \mathbb{R} \mid b <_p x\}$. [10, p. 239]. As in the general case, there is a one-to-one correspondence between the orderings P_C on $\mathbb{R}(x)$ and all such sets $C \subseteq \mathbb{R}$.

For this special case, we can use the fact that the field $\mathbb{R}$ of real numbers is "cut complete" (or Dedekind complete). This means that for any nonempty sets A, $B \subset \mathbb{R}$, with $A \leq B$ (an abuse of notation, but which means for all $a \in A$ and $b \in B$, $a \leq b$), there is a $c \in \mathbb{R}$ such that $A \leq c \leq B$. Thus, for the choices of the subsets C above, if $C \neq \mathbb{R}$ or $C \neq \varnothing$, C must be $(-\infty, a]$ or $(-\infty, a)$ for some $a \in \mathbb{R}$. Consequently, each real number a gives rise to two orderings on $\mathbb{R}(x)$: $P_{(-\infty,a)}$ will be the ordering in which x is between a and t for any real number $t < a$ while $P_{(-\infty,a]}$ will refer to the ordering in which x is between a and s for any real number $s > a$. [10, p. 239]

CHAPTER 3
Introduction to Semifields and Their Orderings

Forty years after the work of Artin and Schreier witnessed considerable further developments in the theory of ordered fields. The surge of much research done in this area can be attributed to the growing interest in the algebraic theory of quadratic forms. This is because in order to study the quadratic forms over fields, it is important to consider not just one or two given orderings, but all possible orderings at the same time [16, p. 3]. Nevertheless, one of the most natural problems in the theory of ordered fields is to study the behavior of the orderings under field extensions.

This thesis takes the opposite route, and considers the problem of what might happen to the field orderings in its substructures such as the semifields - a problem as natural as the one described earlier, and as T. Y. Lam rightfully observed: "As is almost always the case, the 'natural' problem turns out to be a fruitful one" [16, p. 79].

3.1. Definition and Examples

Definition 3.1.1. A *semifield* is a nonempty set S equipped with two binary operations called addition $(+)$ and multiplication $(\cdot)$ with the following properties:

(i) $(S,+)$ is a commutative semigroup: $a+(b+c)=(a+b)+c$, and $a+b=b+a$, for all a, b, $c \in S$.

(ii) $(S^*,\cdot)$ is an abelian group, where $S^* = S\setminus\{0\}$ if S has a zero element; and

(iii) $a\cdot(b+c)=(a\cdot b)+(a\cdot c)$, for all a, b, $c\in S$.

Thus, a semifield has all the properties of a field, except that it may not have an additive identity (zero element) or its elements may not have additive inverses. Consequently, every field is a semifield but not all semifields are fields. A *proper semifield* is one without a zero element. These will be our primary interest in this thesis.

Examples:

(3.1.1a) The set of nonnegative real numbers and the set of nonnegative rational numbers, together with the usual addition and multiplication operations, are semifields. Under the same operations, the set of positive real numbers and the set of positive rational numbers are proper semifields. More generally, a positive cone, P of an ordering in a formally real field is a semifield.

(3.1.1b) With respect to the total order on $\mathbb{R}$, the set of real numbers, with the operations $\oplus$ and $\odot$ defined respectively by $a\oplus b=\max\{a,b\}$ (alternatively, $a\oplus b=\min\{a,b\}$) and $a\odot b=a+b$, $\{\mathbb{R},\oplus,\odot\}$ is a proper semifield [5, p. 16].

(3.1.1c) With $\odot$ defined as in (2) above, and $a \oplus b = \min\{a, b\}$, $\{\mathbb{R} \cup \{\infty\}, \oplus, \odot\}$ is a semifield with ∞ as the zero element. Similarly, with $a \oplus b = \max\{a, b\}$, $\{\mathbb{R} \cup \{-\infty\}, \oplus, \odot\}$ is a semifield with $-\infty$ as its zero. This will be referred to later as the tropical semifield [5, p. 17].

(3.1.1d) With the usual addition and multiplication on the set of rational functions over the set of real numbers, the subset consisting of nonzero rational functions with nonnnegative real coefficients, which we denote in here by $\mathbb{R}^{\#}(x)$, is a proper semifield. More generally, if S is a proper semifield, and $S(x)$ is the set of all rational expressions over S, and we define addition and multiplication of the elements of $S(x)$ as in the field $\mathbb{R}(x)$ of rational functions over the set of real numbers, then $S(x)$ is a proper semifield.

(3.1.1e) With the addition and multiplication operations in a field F, the set ΣF^2, the set of all finite sums of squares of the elements of F is a semifield. To see this, we only have to observe that ΣF^2 is closed under sum and product. Moreover, if $a \in \Sigma F^2$, $a \neq 0$, then $a = \Sigma a_i^2$, $a_i \in F$ and $a^{-1} = a/a^2 = \Sigma a_i^2/a^2 = \Sigma(a_i/a)^2 \in \Sigma F^2$. Thus, ΣF^2 without 0 is a subgroup of the multiplicative group of F closed under addition. More generally, any preordering T of a formally real field F is a subsemifield.

Definition 3.1.2. Let S be a semifield. An element a of S is *idempotent* if $a + a = a$. We say S is *idempotent* if every element of S is *idempotent.* A nonzero element a of S is cancellable if for any elements b and c of S, $a + b = a + c$ implies $b = c$. We say S is *cancellative* if every nonzero element of S is cancellable.

Examples:

(1) In the examples above, (1) is cancellative, but (2) is not.

(2) The semifield $\{\mathbb{R}, \oplus, \odot\}$, where $a \oplus b = \min\{a, b\}$ (alternatively, $a \oplus b = \max\{a, b\}$) and $a \odot b = a + b$, $\{\mathbb{R}, \oplus, \odot\}$, is idempotent.

In section 3.3, we will see an example of a semifield which is neither cancellative nor idempotent.

3.2. Orderings of a Semifield

Definition 3.2.1. (Semifield Ordering) Let S be a proper semifield. An ordering on S is a binary relation $\leq$ satisfying:

(i) reflexivity: $a \leq a$, $\forall a \in S$

(ii) antisymmetry: If $a \leq b$ and $b \leq a$, then $a = b$, $\forall a, b \in S$.

(iii) transitivity: If $a \leq b$ and $b \leq c$, then $a \leq c$, $\forall a, b, c \in S$.

(iv) totality: For any two elements $a, b \in S$, either $a \leq b$, or $b \leq a$.

(v) compatibility with the operations in $S:$ If $a, b \in S$, and $a \leq b$, then for any $c \in S$, $a + c \leq b + c$, and $ac \leq bc$.

If $a \leq b$ but $a \neq b$, we denote this by $a < b$.

Definition 3.2.2. (Dual Order) Let S be a proper semifield with ordering $\leq$. Then there exists a relation $\leq_r$on S, called the *reverse or the dual order,* having the property that for a, $b \in S$, $a \leq_r b$ if and only if $b \leq a$.

Example: The natural ordering $\leq$ inherited by the semifield of positive real numbers from the field of real numbers is a semifield ordering. Similarly, it inherits the dual order.

Propostion 3.2.3. Let X be a nonempty set and $S = \{f \mid f : X \to \mathbb{R}^+\}$ Then S is a proper semifield equipped with an ordering $\leq$.

Proof: With the usual addition and multiplication operations on S, it can be easily seen as a semifield. Well order X. Define $\leq$ by $f \leq g$ if $f(a) < g(a)$, where $a = \min\{x \in S \mid f(x) \neq g(x)\}$. We will show that $\leq$ is transitive: Let $f \leq g$ and $g \leq h$. Let $a = \min\{x \in S \mid f(x) \neq g(x)\}$ and $b = \min\{x \in S \mid g(x) \neq h(x)\}$. Then exactly one of the following cases holds:

(i) $a = b$. Thus, $f(a) < g(a) < h(a)$, and so $f \leq h$.

(ii) $a < b$. Here we have $f(a) < g(a) = h(a)$, hence, $f \leq h$.

(iii) $b < a$. This implies $f(b) = g(b) < h(b)$, and thus, $f \leq h$.

Compatibility with the semifield operations and totality follow from the properties of addition and multiplication and the natural ordering in $\mathbb{R}^+$. ■

3.3. Orderings on Products of Semifields

As we see in the examples above, various semifields naturally occur as substructures of rings or fields. These are the "nice" ones since their elements behave exactly like the elements of a ring or a field, i.e., they are all the cancellative semifields. On the other hand, given a set of semifields S_i, we can combine them to form a new semifield. The direct product of any number of these S_i, equipped with appropriate operations, is a semifield. [7] Then the natural question to ask is whether this new semifield has an ordering. In this section we will see which products can be ordered and which ones cannot possibly have an ordering.

Lemma 3.3.1. In an ordered cancellative semifield S, if $a < b$, then $a + c < b + c$.

Proof: If $a + c = b + c$ then $a = b$, a contradiction.

Proposition 3.3.2. Let $S_1, S_2, ..., S_n$ be proper cancellative semifields. Then their direct product, $S = S_1 \times S_2 \times ... \times S_n = \{(s_1, s_2, ..., s_n) \mid s_i \in S_i,\ i = 1, 2, ..., n\}$, equipped with component-wise addition and multiplication, is also a proper semifield. If each semifield has an ordering, then S can be ordered lexicographically.

Proof: Both component-wise addition $(+)$ and component-wise multiplication $(\cdot)$ operations on S are easily seen to be commutative and associative following the commutativity and associativity of addition and multiplication operations in each component semifield. Moreover, multiplication distributes over addition, and since each semifield is proper, no element in S will have a zero component, hence, S won't have a zero element either. Also, for every i, s_i^{-1} exists, and thus, every $s \in S$ will have its inverse, $s^{-1} = (s_1^{-1}, s_2^{-1}, ..., s_n^{-1}) \in S$. We therefore have $(S, +)$ a commutative semigroup without a zero, and $(S, \cdot)$ a multiplicative abelian group. Hence, $(S, +, \cdot)$ is a proper semifield.

For each i, let $<_i$denote an ordering of S_i. We define a lexicographic order on S as follows: For any two elements $x = (x_1, x_2, ..., x_n)$, $y = (y_1, y_2, ..., y_n) \in S$, let $x < y$ if $x_1 <_1 y_1$, or $x_1 = y_1$ and there exists $k \in \mathbb{Z}^+$, $k \geq 2$, such that $x_k <_k y_k$, and $x_i = y_i$ for all $i < k$.

We will show that $<$ is transitive. Let $z = (z_1, z_2, ..., z_n) \in S$, and assume that $x < y$ and $y < z$. Then exactly one of the following cases holds:

(i) $x_1 <_1 y_1$ and $y_1 <_1 z_1$, thus, $x_1 <_1 z_1$, and so $x < z$.

(ii) $x_1 <_1 y_1$ and there exists $m \in \mathbb{Z}^+$, $m \geq 2$, such that $y_m <_m z_m$ and $y_i <_i z_i$ for all $i < m$. This implies $z_1 = y_1$, hence, $x_1 <_1 z_1$, and thus, $x < z$.

(iii) There exists $k \in \mathbb{Z}^+$, $k \geq 2$, such that $x_k <_k y_k$, and $x_i = y_i$ for all $i < k$, and $y_1 <_1 z_1$. Since $x_1 = y_1$, then $x_1 <_1 z_1$, and we have $x < z$.

(iv) There exists $k \in \mathbb{Z}^+$, $k \geq 2$, such that $x_k <_k y_k$, and $x_i = y_i$ for all $i < k$, and there exists $m \in \mathbb{Z}^+$, $m \geq 2$, such that $y_m <_m z_m$ and $y_i <_i z_i$ for all $i < m$. Let's consider the following subcases:

(a) $k = m$. It follows that $x_k <_k y_k <_k z_k$, and so $x < z$.

(b) $k < m$. Then $x_k <_k y_k = z_k$, and therefore, $x_k <_k z_k$. Thus, $x < z$.

(c) $m < k$. This implies $x_m = y_m <_m z_m$. Hence, $x < z$.

Thus, $<$ is transitive.

Next, we will show that $<$ is compatible with point-wise addition and multiplication. Let x, y, $z \in S$, with $x < y$, and without loss of generality, assume $x_k <_k y_k$. Then by the above lemma, $(x_k + z_k) <_k (y_k + z_k)$, and $(x_k z_k) <_k (y_k z_k)$. Consequently, we have $x + z < y + z$ and $xz < yz$. Therefore, $\leq$ is an ordering on S. ■

On the other hand, we may get a direct product of semifields that cannot be ordered lexicographically. The following is an example of that.

Example: Consider $T = (\mathbb{R}, \oplus, \odot)$ of example 3.1.1b, and $\mathbb{R}^+ = (\mathbb{R}^+, +, \cdot)$ of example 3.1.1a, in section 3.1 above. Then $T \times \mathbb{R}^+$, equipped with component-wise addition $(+)$ and multiplication $(\cdot)$, is neither a cancellative nor an idempotent semifield. We have

$$\begin{aligned} (a,b) + (c,d) &= (\max\{a,c\},\ b+d) \quad \text{and} \\ (a,b) \cdot (c,d) &= (a+c,\ bd). \end{aligned}$$

In particular, we observe that:

$$\begin{aligned} (2,3) + (3,4) &= (3,7) = (1,3) + (3,4), \quad \text{but} \\ (2,3) &\neq (1,3). \end{aligned}$$

Moreover,

$$(2,3) + (2,3) = (2,6) \neq (2,3).$$

Since this semifield is not cancellative, with respect to the lexicographic order, the compatibility with addition fails, hence, it cannot be ordered in that way. In the next section, we see how we can create a semifield without an ordering.

3.4. Construction of a Semifield with no Ordering

In this section, we see how we can define an addition operation on a lattice-ordered group to make an idempotent semifield and to construct a semifield with no ordering. The lattice ordering on the set plays a key role in both cases.

Definition 3.4.1. [4, p. 3] A partially ordered set (poset) $(L, \preceq)$ is a *lattice* if it satisfies the following two axioms:

(a) *Existence of Binary Joins:* For any two elements $a, b \in L$, the set $\{a, b\}$ has a *join*, $a \vee b$ (also known as the *least upper bound* or the *supremum*).

(b) *Existence of Binary Meets:* For any two elements $a, b \in L$, the set $\{a, b\}$ has a *meet*, $a \wedge b$ (also known as the *greatest lower bound*, or the *infimum*).

Definition 3.4.2. A *lattice-ordered group* or *l-group*, is a group $(G, \cdot)$ on which is defined a binary relation $\preceq$ satisfying $a \preceq b$ implies $a \cdot c \preceq b \cdot c$, for any a, b, $c \in G$, and relative to which the group is a lattice.

Proposition 3.4.3. Let $(G, \cdot, \preceq)$ be a *lattice-ordered group*. Then we can define an addition $(+)$ operation on $(G, \cdot, \vee, \wedge)$ such that $(G, \cdot, +)$ is an idempotent semifield.

Proof: Define addition by $a + b = a \vee b$. Then by the properties of $\vee$, addition is commutative and associative. To see that multiplication distributes over addition, let $a, b \in G$. Then $a \preceq a \vee b$, and $b \preceq a \vee b$. Since G is *lattice-ordered*, $a \cdot c \preceq (a \vee b) \cdot c$ and $b \cdot c \preceq (a \vee b) \cdot c$, for any $c \in G$. Thus, $(a \vee b) \cdot c$ is an upper bound of $a \cdot c$ and $b \cdot c$. Let $a \cdot c \preceq x$ and $b \cdot c \preceq x$. Then $a \preceq x \cdot c^{-1}$ and $b \preceq x \cdot c^{-1}$. This implies that $a \vee b \preceq x \cdot c^{-1}$, and so we have $(a \vee b) \cdot c \preceq x$. This shows that $(a \vee b) \cdot c = (a \cdot c) \vee (b \cdot c)$. Since for any $a \in G$, $a \vee a = a$, then $(G, \cdot, +)$ is an idempotent semifield. ■

Theorem 3.4.4. Let $(S, \cdot, +)$ be an idempotent semifield. Then the group $(S, \cdot)$ is lattice-ordered.

Proof: We define a relation $\preceq$ on S by $a \preceq b$ iff $a + b = b$. It follows that $\preceq$ satisfies the following properties:

(i) $\preceq$ is reflexive: $a \preceq a$, by the idempotency of S;

(ii) $\preceq$ is antisymmetric: If $a \preceq b$ and $b \preceq a$, then $a + b = b$ and $b + a = a$. Thus, by commutativity of addition and transitivity of equality, we have $a = b$.

(iii) $\preceq$ is transitive: If $a \preceq b$ and $b \preceq c$, then $a + b = b$ and $b + c = c$. Hence, $a + c = a + (b + c) = (a + b) + c = b + c = c$, and so $a \preceq c$.

Thus, $(S, \preceq)$ is a poset.

Next, we define $\vee$ by $a \vee b = a + b$ and $a \wedge b = (a^{-1} \vee b^{-1})^{-1}$. We will show that for every two elements $a, b \in S$, both $\sup\{a, b\} = a \vee b$ and $\inf\{a, b\} = a \wedge b$ exist. Let $a, b \in S$. Then by the idempotency of S, $a + (a + b) = (a + a) + b = a + b$. This shows that $a \preceq a \vee b$. Similarly, $b \preceq a \vee b$. Hence, $a \vee b$ is an upper bound of a and b. Let x be any other upper bound of a and b. Then $a \preceq x$ and $b \preceq x$, and we have $a + x \preceq x$ and $b + x \preceq x$. Again, by the idempotency of S, we obtain $a + b + x = x$, and so $a \vee b \preceq x$. This shows that $a \vee b = \sup\{a, b\}$.

Now, $(a^{-1} \vee b^{-1})^{-1} + a = [(ab)^{-1}(a + b)]^{-1} + a = [ab + a(a + b)](a + b)^{-1} = (ab + aa)(a + b)^{-1} = a$. Thus, $(a^{-1} \vee b^{-1})^{-1} \preceq a$. Similarly, $(a^{-1} \vee b^{-1})^{-1} \preceq b$. This shows that $(a^{-1} \vee b^{-1})^{-1}$ is a lower bound of a and b. Let y be any other lower bound of a and b. So $y \preceq a$ and $y \preceq b$. This implies that $y + a = a$ and $y + b = b$. Then $y + (a^{-1} + b^{-1})^{-1} = y + [(ab)^{-1}(a + b)]^{-1} = [ya + yb + ab](a + b)^{-1} = [(y + b)a + yb](a + b)^{-1} = [(a + y)b](a + b)^{-1} = ab(a + b)^{-1} = [(ab)^{-1}(a + b)]^{-1} = (a^{-1} + b^{-1})^{-1}$. Thus, $y \preceq (a^{-1} \vee b^{-1})^{-1}$. Therefore, $(a^{-1} \vee b^{-1})^{-1} = a \wedge b$.

To show that multiplication is order-preserving, suppose that $a \preceq b$. Then $a + b, = b$, and for any $c \in S$, $ac + bc = c(a + b) = bc$.Thus, $ac \preceq bc$. ■

Corollary 3.4.5. A totally ordered group can have an addition defined making it an ordered idempotent semifield.

Theorem 3.4.6. There exists a semifield with no total ordering.

Proof: Let $S = \mathbb{Z} \times \mathbb{Z} = \{(a, b) \mid a, b \in \mathbb{Z}\}$ with component-wise addition $+$. Then $(S, +)$ is the usual additive group. Define a relation $\preceq$ on S by $(a, b) \preceq (c, d)$ if $a \leq c$ and $b \leq d$. Let $\wedge$ denote the greatest lower bound and $\vee$ the least upper bound of any two elements on S. Then $(S, \preceq, \wedge, \vee)$ is lattice - ordered. Define addition and multiplication operations on S by $(a, b) \oplus (c, d) = (a, b) \vee (c, d)$ and $(a, b) \odot (c, d) = (a + c, b + d)$, respectively. Then $(S, \oplus, \odot)$ is a proper idempotent semifield with no ordering. Suppose there exists an ordering $\leq_S$ on S. Let $a,\ b, c, b \in \mathbb{Z}$ with $a < c$ and $b > d$, and consider two elements $(a, b), (c, d) \in S$. Then either $(a, b) \leq_S (c, d)$ or $(c, d) \leq_S (a, b)$. Without loss of generality, assume that

$$(1) \qquad (a, b) \leq_S (c, d).$$

Adding (a, b) to both sides of the inequality gives

$$(a, b) \oplus (a, b) \leq_S (c, d) \oplus (a, b)$$

Since S is idempotent, then we have

$$(2) \qquad (a, b) \leq_S (c, d) \oplus (a, b) = (c, d) \vee (a, b) = (c, b).$$

On the other hand, adding (c, d) to both sides of the inequality (1) gives

$$(a, b) \oplus (c, d) \leq_S (c, d) \oplus (c, d).$$

This gives us

$$(3) \qquad (c, b) \leq_S (c, d).$$

Now, consider the elements (a, d), $(c, b) \in S$. Assuming $(a, d) \leq_S (c, b)$ and then adding (c, d) to both sides of the inequality gives

$$(c, d) \leq_S (c, b)$$

contradicting (3). On the other hand, assuming $(c, b) \leq_S (a, d)$ and then adding (a, b) to both sides of the inequality gives

$$(c,b) \leq_S (a,b)$$

contradicting (2). ■

3.5. Tropical Semifields and Idempotent Semifields

We begin this section with a definition.

Definition 3.5.1. [5, p. 21] (Tropical Semifields) Let $T = \mathbb{R} \cup \{-\infty\}$. Define addition $\oplus$ on T by $x \oplus y = \max\{x, y\}$ and multiplication $\odot$ by $x \odot y = x + y$. Then $T_{trop} = (T, \oplus, \odot)$ is a semifield, called the **tropical semifield**.

This semifield T_{trop} is also termed **schedule algebra** having its application in the analysis of industrial processes and finding the critical paths of graphs [5, p. 17]. If $\mathbb{R} \cup \{\infty\}$ is used and $x \oplus y = \min\{x, y\}$, the semifield is more particularly referred to as **optimization algebra** [5, p. 16] having its application in solving shortest-path problems in optimization theory and in solving domination problems in graph theory [5, p. 16]. It is easy to check that T_{trop} is a commutative semigroup with respect to addition, having $-\infty$ as the additive identity, and a commutative group with respect to multiplication, with 0 as its multiplicative identity, and that the distributive law $x \odot (y \oplus z) = (x \odot y) \oplus (x \odot z)$ holds.

The field $\mathbb{R}$ is transformed to a tropical structure when the classical arithmetic operations of addition and multiplication are now defined as tropical operations : $a + b = \max\{a, b\}$ (alternatively, $a + b = \min\{a, b\}$), and $a \cdot b = a + b$. The subject on on tropical semifields is a familiar topic in idempotent mathematics or idempotent analysis as these tropical structures have wide applications in industrial processes. For instance, the tropical semifield ($\mathbb{R} \cup \{\infty\}$, min, $+$) is important in solving the shortest-path problem in optimization while the tropical semifield ($\mathbb{R} \cup \{-\infty\}$, max, $+$), otherwise called the *schedule algebra* or the *max-plus algebra* finds application in the analysis of industrial processes. [5, p. 16]

In [8], it is said that the term "tropical" is borrowed from computer science, where it was introduced to commemorate a Brazilian scientist, Imre Simon. The passing from the classical arithmetic operations to the tropical arithmetic operations actually had appeared earlier in the scholarly work of Litvinov and Maslov relating to idempotent analysis.

As noted previously, a proper semifield lacks the operation "subtraction". It is a field if it has a zero and every element has an additive inverse. Proper semifields are seen naturally as subsets of an ordered field. For example, if $(F, \leq)$ is an ordered field, $F^+ = \{x \in F \mid x > 0\}$ is a proper semifield. Semifields of the form F^+ are cancellative semifields, i.e., $a + b = c + b$ implies $a = c$. Cancellative semifields behave very similarly to rings and fields. On the other extreme are the idempotent semifields where for all $a \in S$, $a + a = a$. These semifields are clearly not cancellable.

It is possible to construct some examples of idempotent semifields starting from an abelian ordered group $(\Lambda, +, <)$. We add to it an extra element $-\infty$ with the property that for all $\lambda \in \Lambda$, $-\infty < \lambda$, and we define a zerosumfree semifield: $(\Lambda \cup \{-\infty\}, \oplus, \odot, -\infty, 0)$ with the tropical operations $\oplus$, and $\odot$ defined by $a \oplus b = \max\{a, b\}$, and $a \odot b = \left\{ \begin{array}{c} a+b \quad \text{if } a,\ b \in \Lambda \\ -\infty \ \ \text{if } a = -\infty \text{ or } b = -\infty \end{array} \right\}$.

It is worthwhile to note that there exists a relation between ordered fields and idempotent semifields. There are actually two constructions relating these two algebraic structures: Maslov dequantization and valuations. The two constructions are actually two different ways for seeing the same thing and they are both described in [8]. We end this section with a description of one of these constructions, the Maslov dequantization. We will see how the process relates ordered fields to idempotent semifields.

Given a number $t \in (0, 1)$, consider the function:

$$z \mapsto \log_{(\frac{1}{t})} z = (-\frac{1}{\log t}) \log z \in \mathbb{R} \cup \{-\infty\}.$$

This function is bijective, with inverse $x \mapsto t^{-z}$, and it preserves the order $\leq$ on $\mathbb{R}$. The operations addition "+" and multiplication "$\cdot$" are transformed via conjugation in the following way:

$$\begin{aligned} x \oplus_t y &= \log_{(\frac{1}{t})}(t^{-x} + t^{-y}), \text{ and} \\ x \odot_t y &= \log_{(\frac{1}{t})}(t^{-x} \cdot t^{-y}) = x + y. \end{aligned}$$

Hence, every t induces a semifield structure on $\mathbb{R} \cup \{-\infty\}$, isomorphic to $\mathbb{R}^{\#}$: $(\mathbb{R} \cup \{-\infty\}, \oplus_t, \odot_t, -\infty, 0)$. In the limit semifield for t tending to zero, we have

$$\lim_{t \to 0} x \oplus_t y = \max\ (x, y).$$

The limit semifield is the tropical semifield T_{trop} [12].

We now see that idempotent semifields can be constructed whenever the tropical arithmetic operations can be conveniently defined on a set. As another example, let $S = \{x_1^{a_1}...x_n^{a_n} \mid a_1, ..., a_n \in \mathbb{Z}\}$, the set of Laurent monomials in the variables $x_1, x_2, ..., x_n$. Define multiplication and addition, respectively, by

$$x_1^{a_1}...x_n^{a_n} \odot x_1^{b_1}...x_n^{b_n} = x_1^{a_1+b_1}...x_n^{a_n+b_n}$$

$$x_1^{a_1}...x_n^{a_n} \oplus x_1^{b_1}...x_n^{b_n} = x_1^{\min\{a_1,b_1\}}...x_n^{\min\{a_n,b_n\}}$$

Then $(S, \oplus, \odot)$ is an idempotent semifield.

Finally, the subsemiring $(\mathbb{Z} \cup \{-\infty\}, \max, +)$ of the schedule algebra $(\mathbb{R} \cup \{-\infty\}, \max, +)$ and the subsemiring $(\mathbb{Z} \cup \{\infty\}, \min, +)$ of the optimization algebra $(\mathbb{R} \cup \{\infty\}, \min, +)$ are also idempotent semifields. [5, p. 17]

3.6. Subsets Whose Elements are Greater than or Equal to 1

This section presents an alternative way of viewing orderings. Taking into account the peculiar feature of a proper semifield in that it has "no subtraction" among its elements owing to the absence of additive inverses, we cannot work with positive cones as in fields. However, we can obtain a subset of the proper semifield S with properties analogous to that of a positive cone in an ordered field.

Theorem 3.6.1. Let S be a proper semifield, and M a proper subset of S with the following properties:

(i) $M + M \subseteq M$

(ii) $M \cdot M \subseteq M$

(iii) $M \cup M^{-1} = S$

(iv) $M \cap M^{-1} = \{1\}$, and

(v) $m \in M$ and $s \in S$ implies $(1 + s)^{-1}(m + s) \in M$.

Then there exists an ordering $\leq$ with $M = \{s \mid s \geq 1\}$.

Proof: Define $\leq$ on S by $a \leq b$ iff $a^{-1}b \in M$. Then the following properties hold:

(1) $\leq$ is reflexive: For all $a \in S$, $a^{-1}a = 1 \in M$, so $a \leq a$.

(2) $\leq$ is antisymmetric: For any $a, b \in S$, if $a \leq b$ and $b \leq a$, then $a^{-1}b \in M$ and $b^{-1}a \in M$. Since $a^{-1}b = (b^{-1}a)^{-1}$, then $a^{-1}b = b^{-1}a = 1$. Hence, $a = b$.

(3) $\leq$ is transitive: For $a, b, c \in S$, suppose $a \leq b$ and $b \leq c$. Then $a^{-1}b \in M$ and $b^{-1}c \in M$. Since M is closed under multiplication, then $a^{-1}c = (a^{-1}b)(b^{-1}c) \in M$, and thus, $a \leq c$.

(4) $\leq$ is compatible with the operations on S. Suppose $a \leq b$. Then for any $c \in S$, $(ac)^{-1}(bc) = (c^{-1}a^{-1})(bc) = (a^{-1}c^{-1})(cb) = a^{-1}b \in M$, and so we have $ac \leq bc$. Also, $a + c \leq b + c$, for otherwise, if $b+c \leq a+c$, then multiplying this last inequality by a^{-1} would give $(a^{-1}b+a^{-1}c) \leq (1+a^{-1}c)$. Consequently, $(a^{-1}b + a^{-1}c)^{-1}(1 + a^{-1}c) \in M$, but $(1 + a^{-1}c)^{-1}(a^{-1}b + a^{-1}c) \notin M$, a contradiction to the hypothesis since $a^{-1}b \in M$ and $a^{-1}c \in S$.

(5) $\leq$ is total: For any two elements a,$b \in S$, if $a \neq b$, then either $a^{-1}b \in M$, or $b^{-1}a = (a^{-1}b)^{-1} \in M$, thus, $a \leq b$ or $b \leq a$. Therefore, $\leq$ is an ordering on S. ■

Theorem 3.6.2. Let S be a proper semifield. Then an ordering $\leq$ on S in which $1 \leq 2$ determines a set M satisfying the conditions of Theorem 3.6.1.

Proof: Let $M = \{x \in S \mid x \geq 1\}$. We will show that M satisfies the conditions of Definition 2.3. If $x \in M$, then $x + x \geq 1 + 1 = 2 > 1$, and so condition (i) holds. Condition (ii) is easily seen to be satisfied since $x \geq 1$ would imply that $x^2 \geq 1$, hence, $1 \leq x^2 \in M \cdot M \subseteq M$. Now, $M^{-1} = \{y \in S \mid y = x^{-1}$, for some $x \in M\} = \{y \in S \mid y \leq 1\}$. Thus, $M \cup M^{-1} = S$, and (iii) holds. Moreover, if $x \in M \cap M^{-1}$, then $1 \leq x \leq 1$. This implies x must be 1. Finally, since $\leq$ is an ordering on S, then for any $s \in S$, we have $1 + s \leq x + s$. Consequently, $1 \leq (1 + s)^{-1}(x + s) \in M$. ■

3.7. Relationship to the Theory of Formally Real Fields

This section is important as this presents the connection between formally real fields and semifield orderings. As we saw in an earlier section, the set of all finite sums of squares ΣF^2 of a formally real field, together with the field operations, is a semifield. Theorem 3.7.2, together with the idea presented in Chapter 5 of this paper will enable us to recover not only all the orderings of F via the orderings of the sums of squares of elements of F, but also the reduced Witt ring of quadratic forms over F [9, chap. 1].

We shall need a generalization of the positive cone of an ordering due to Prestel [17].

Definition 3.7.1. A semiordering on a field F is a subset Q of F satisfying $1 \in Q$, $Q \cup -Q = F$, $Q \cap -Q = \{0\}$, $Q + Q \subseteq Q$, and $F^2Q = Q$. If T is a preordering of F and $TQ \subseteq Q$, we call Q a $T-$semiordering. [10, p. 75]

Note that if $Q \cdot Q \subseteq Q$, then Q is an ordering containing T. In the discussion that follows, for any subset $A \subseteq F$, we will let $A^* = A \backslash \{0\}$.

Theorem 3.7.2. Let F be a formally real field, and $T \subseteq F$ a preordering. Then there is a one-to-one correspondence between the set of semifield orderings $\leq$ in T^* where $1 < 2$ and the sets Q^*, where Q is a T-semiordering of F.

Proof: Let $\leq$ be a semifield ordering of T^*. Let $Q^* = \{x | x = b - a$, where $a, b \in T^*$ and $a < b\}$. We note that Q^* is well-defined, for suppose that $x = b - a = d - c$, but $d < c$. Since $\leq$ is a semifield ordering, then by property (v), we have $a + d < b + c$. But since $b - a = d - c$, then $a + d = b + c$, a contradiction. Observe that $0 \notin Q^*$, and since $1 < 2$, $1 \in Q$. Let $x, y \in Q^*$, where $x = b - a$ and $y = d - c$, $a, b, c, d \in T^*$. Since $\leq$ is a semifield ordering of T^*, and T^* is closed under addition, then $x + y = (b + d) - (a + c) \in Q^*$. This shows $Q^* + Q^* \subseteq Q^*$. Now, let $t \in T^*$. Then $t(b - a) \in T^*Q^*$. But since $ta < tb$, then $t(b - a) = (tb - ta) \in Q^*$. Consequently, $T^*Q^* \subseteq Q^*$. Also, since $(F^2)^* \subseteq T^*$, then $(F^2)^*Q^* \subseteq Q^*$. Let $x \in F$, $x \neq 0$. Then $x = a^2 - b^2$ for some $a, b \in F$. Suppose $x \notin Q^*$. Then $a^2 \not< b^2$ but $b^2 < a^2$. It follows that $-x = -(b^2 - a^2) = a^2 - b^2 \in Q^*$, and thus, $Q^* \cup -Q^* = F^*$. Finally, we show that $Q^* \cap -Q^* = \varnothing$. For suppose $b - a = x \in Q^* \cap -Q^*$. Then $b - a = -(b - a) = a - b$, which implies $a < b$ and $b < a$, but $a \neq b$, a contradiction.

Now, consider Q^*, where Q is a $T-$semiordering of F. For $a, b \in T^*$, define $\leq$ by $a \leq b$ if $a = b$ or $b - a \in Q^*$. It follows immediately that $\leq$ is reflexive, antisymmetric and transitive. Moreover, if $a \not\leq b$, and $a \neq b$, then $b - a \notin Q^*$ but $a - b = -(b - a) \in Q^*$. Hence, $b \leq a$. Now, let $a \leq b$ and $c \in T^*$. For $a = b$, the result follows immediately. Assume $a \neq b$. Then $(b + c) - (a + c) = (b - a) \in Q^*$, and thus, $a + c \leq b + c$. Also, for any $t \in T^*$, we have $tb - ta = t(b - a) \in T^*Q^* \subseteq Q^*$, and thus, $ta \leq tb$. ■

CHAPTER 4

Orderings on the Rational Functions over $\mathbb{R}$ with Nonnegative Real Coefficients

We begin this chapter with a definition.

Definition 4.1. Let F be a formally real field and $S \subseteq F$ a semifield. Let P be an ordering of F. We say P **induces** a semifield order on S if $S \subseteq P$.

Chapter 2 shows how preorderings have controlled the set of semifield orderings. We now look at a semifield of a field with vastly more orderings that are not induced by the orderings of the field. Throughout the discussion, we will denote the proper semifield of rational functions with nonnegative real coefficients by $\mathbb{R}^{\#}(x)$. It has the following orderings:

1. Orderings Induced by the Orderings of the Rational Functions over $\mathbb{R}$, $\mathbb{R}(x)$

Theorem 4.2: The orderings of $\mathbb{R}(x)$ which induce semifield orderings on $\mathbb{R}^{\#}(x)$ are those orderings P_C on $\mathbb{R}(x)$ in which $x > 0$.

Proof: Let $f \in \mathbb{R}^{\#}(x)$. If $x > 0$, then $f(x)$ is positive in any ordering of $\mathbb{R}(x)$, so multiplication by $f(x)$ preserves inequalities. So we obtain semifield orderings of $\mathbb{R}^{\#}(x)$. On the other hand, if $x < 0$ then multiplication by x reverses the inequalities and so compatibility with multiplication fails. ■

2. Lexicographic Ordering

Theorem 4.3: (Ordering $\mathbb{R}^{\#}(x)$ on $\mathbb{Z}^{+}$) : For any rational functions f, g over $\mathbb{R}^{\#}$, define a relation $\preceq$ on the set $\mathbb{R}^{\#}(x)$ by $f \preceq g$ if $f(a) < g(a)$ on the first $a \in \mathbb{Z}^{+}$ at which f and g differ. Then $\preceq$ is an ordering on $\mathbb{R}^{\#}(x)$.

Proof: It follows immediately that $\prec$ is reflexive and antisymmetric. We now verify that $\prec$ is transitive. Let $f \preceq g$ and $g \preceq h$. Let $a, b \in \mathbb{Z}^{+}$ be the first integers at which $f \neq g$ and $g \neq h$, respectively. Let $c \in \mathbb{Z}^{+}$ be the first integer at which $f \neq h$. We will show that $f(c) < h(c)$. By the hypothesis, we have $f(a) < g(a)$ and $g(b) < h(b)$. We consider two cases: when $a < b$ and when $b < a$. If $a < b$, then $f(a) < g(a) = h(a)$, so $f(a) < h(a)$ and $a = c$, for if $c < a$, then $f(c) = g(c) = h(c)$, a contradiction to the hypothesis. Hence, $f(c) < h(c)$. If $b < a$, then $f(b) = g(b) < h(b)$, so $f(b) < h(b)$, and $c = b$, for if $c < b$, then $f(c) = g(c) = h(c)$, a similar contradiction. Consequently, we have $f \preceq h$, and $\preceq$ is transitive.

The relation $\prec$ can also be readily seen to be compatible with the operations on $\mathbb{R}^{+}(x)$: If $f \preceq g$, and $a \in \mathbb{Z}^{+}$ is the first integer at which $f(a) < g(a)$ then for any $h \in \mathbb{R}^{+}(x)$, we have $f(a) + h(a) < g(a) + h(a)$, and so $f + h \preceq g + h$. Similarly, $f(a)h(a) < g(a)h(a)$, and so $fh \preceq gh$.

Finally, we observe that $\preceq$ is a total order on $\mathbb{R}^{\#}(x)$, for if $f \neq g$, then for some $x \in \mathbb{Z}^{+}$, $f(x) \neq g(x)$. Let $a \in \mathbb{Z}^{+}$ be the first integer at which $f(a) \neq g(a)$. Then either $f(a) < g(a)$ or $g(a) < f(a)$. Hence, either

$f \preceq g$ or $g \preceq f$. ■

We can extend and generalize the above process of ordering $\mathbb{R}^{\#}(x)$ on the positive integers to any sequence of positive reals.

Theorem 4.4: The rational functions over $\mathbb{R}$ with nonnegative real coefficients, $\mathbb{R}^{\#}(x)$, can be ordered on any sequence of distinct positive reals.

Proof: Let $\{r_1, r_2, r_3, ...\}$ be a sequence of positive reals. For any f, $g \in \mathbb{R}^{\#}(x)$, define $<$ by $f < g$ if $f(r_1) < g(r_1)$, or $\exists\ k \in \mathbb{Z}^+$ such that $f(r_k) < g(r_k)$ and $f(r_i) = g(r_i)$, $\forall\ i < k$. We only have to verify that $<$ is transitive. Let f, g, $h \in \mathbb{R}^{\#}(x)$ with $f < g$ and $g < h$. This can happen in several ways:

(1) $f(r_1) < g(r_1) < h(r_1)$. Then $f < h$.

(2) $f(r_1) < g(r_1)$, and $\exists\ k \in \mathbb{Z}^+$, $k > 1$, such that $g(r_k) < h(r_k)$. Then $f(r_1) < h(r_1)$ since $\forall\ i < k$, $g(r_i) = h(r_i)$. In particular, $g(r_1) = h(r_1)$, and thus, $f < h$.

(3) $\exists\ k \in \mathbb{Z}^+$, $k > 1$ such that $f(r_k) < g(r_k)$ and $f(r_i) = g(r_i)$, $\forall\ i < k$, and $g(r_1) < h(r_1)$. Then $f(r_1) = g(r_1) < h(r_1)$, and so $f < h$.

(4) $\exists\ k \in \mathbb{Z}^+$ such that $f(r_k) < g(r_k)$ and $\exists\ n \in \mathbb{Z}^+$ such that $g(r_n) < h(r_n)$. If $k = n$, then we have $f(r_k) < g(r_k) < h(r_k)$, and thus, $f < h$. If $k < n$, then $f(r_k) < g(r_k) = h(r_k)$, and so $f < h$. If $k > n$, then $f(r_n) = g(r_n) < h(r_n)$, hence, $f < h$.

To see that $<$ is compatible with the semifield operations, we let $f < g$ and assume, without loss of generality, that for some $k \in \mathbb{Z}^+$, $f(r_k) < g(r_k)$ and $f(r_i) = g(r_i)$, $\forall\ i < k$. Then by the natural order on $\mathbb{R}$, for any $h \in \mathbb{R}^{\#}(x)$, we have $f(r_k) + h(r_k) < g(r_k) + h(r_k)$ and $f(r_k)h(r_k) < g(r_k)h(r_k)$. Thus, we have $f + g < g + h$ and $fh < gh$. ■

3. Ordering $\mathbb{R}^{\#}(x)$ on a positive real, r

Theorem 4.5: Let $S = \{s_k\}_{k=1}^{\infty}$, where $s_1 = 1$, and $s_k = \pm 1$ for $k \geq 2$. Let's define a relation $\leq$ on $\mathbb{R}^{\#}(x)$ of rational functions with nonnegative real coefficients as follows: For $r \in \mathbb{R}$, $r > 0$, we let $f \leq g$ if $f(r) < g(r)$. If $f(r) = g(r)$, we write $(g - f)(x) = (x - r)^k p(x)$, and let $f \leq g$ if $s_k p(r) > 0$. Then $\leq$ is an ordering on $\mathbb{R}^{\#}(x)$.

Proof: The relation $\leq$ is easily seen to be reflexive and antisymmetric. We will show that $\leq$ is transitive. Let f, g, $h \in \mathbb{R}^{\#}(x)$ with $f \leq g$ and $g \leq h$. We will show that $f \leq h$. We need to consider four cases. The first three are trivial and the conclusion immediately follows. Thus, we only have to verify the last of these cases.

(1) $f(r) < g(r)$ and $g(r) < h(r)$

(2) $f(r) < g(r)$ and $g \leq h$ with $g(r) = h(r)$

(3) $f(r) = g(r)$ and $g(r) < h(r)$

(4) $f \leq g$ and $g \leq h$, with $(g-f)(x) = (x-r)^k p(x)$ and $s_k p(r) > 0$, $(h-g)(x) = (x-r)^l q(x)$, and $s_l q(r) > 0$. Then

$$\begin{aligned}(h-f)(x) &= (h-g)(x) + (g-f)(x) = (x-r)^l q(x) + (x-r)^k p(x),\ k < l \\ &= (x-r)^k [p(x) + (x-r)^{l-k} q(x)] \\ &= (x-r)^k t(x),\ \text{where } t(x) = p(x) + (x-r)^{l-k} q(x),\ t(x) \in \mathbb{R}^+(x)\end{aligned}$$

Since $t(r) = p(r)$, then $s_k t(r) = s_k p(r) > 0$, and thus, $f \leq h$. ■

From the foregoing results, we note the following:

(1) For the sequence $1, 2, 3, \ldots,$ x is infinitesimally close to 1, i.e., $1 < x < 1 + \varepsilon$, for some small $\varepsilon > 0$. Observe that at 1, $x = 1$ since x takes on the value of 1, but at 2, $x > 1$ since x takes on the value of 2. On the other hand, at 1, $x < 1 + \varepsilon$ for any $\varepsilon > 0$. Now, let's take $\varepsilon = \frac{1}{2}$. So, with respect to the orderings of $\mathbb{R}(x)$, we have $(x-1)(x-2) = x^2 - 3x + 2 < 0$. So $x^2 + 2 < 3x$. However, with respect to the ordering of $\mathbb{R}^{\#}(x)$, $x^2 + 2 > 3x$ since while $x^2 + 2 = 3x$ at 1 and at 2, we have $(3)^2 + 3 = 11 > 9 = 3(3)$ at 3. Hence, this is not one of the two orderings induced from $\mathbb{R}(x)$ with x infinitesimally close to 1.

(2) For s_k given by $\{1, -1, \ldots\}$, $r = 1$, we again have x infinitesimally close to 1. But then we will have $x^2 + 1 < 2x$, which is not possible in $\mathbb{R}(x)$ since $(x-1)^2 > 0$ always. Moreover, this differs from the ordering in (1) since in there $x^2 + 1 > 2x$.

With the exception of the orderings described in Theorem 4.2 above, we then observe that the other orderings of $\mathbb{R}^{\#}(x)$ do not come from $\mathbb{R}(x)$.

CHAPTER 5
The Topology on the Space of Orderings of a Semifield

By an $(n - ary)$ **quadratic form** over a field F, we mean a polynomial f in n variables over F that is homogeneous of degree 2. It has the general diagonal form

$$f(X_1, ..., X_n) = \Sigma_{i=1}^{n} a_i X_i^2 \in F[X_1, ..., X_n] = F[X].$$

Moreover, the **signature** of a quadratic form q is defined to be

$$n_+ - n_- = n_+ - (n - n_+) = 2n_+ - n,$$

where n_+ is the number of positive a_i and n_- is the number of negative a_i [10, p. 34].

The relationship between orderings of a formally real field and quadratic forms was discussed by Albrecht Pfister in the proof of his celebrated Local-Global Principle. For a given ordering P and a quadratic form φ over a formally real field F, he defined the signature $sgn_P(\varphi)$, and showed that the set of all orderings X_F gives rise to a total signature of the quadratic form φ, given by the function $Sgn_\varphi : X_F \longrightarrow \mathbb{Z}$, defined by $Sgn_\varphi(P) = sgn_P(\varphi)$. By assigning to every quadratic form φ its total signature, a homomorphism exists from the Witt ring $W(F)$ of the field F to the ring $C(X_F, \mathbb{Z})$ of integral-valued continuous functions defined over X_F. Pfister proved that the kernel of this homomorphism is the torsion ideal $W_t(F)$ of $W(F)$. Consequently, the study of the reduced Witt ring $W(F)/W_t(F)$ is closely tied to the study of the space of orderings X_F of the field F.

It is important to review at this point some basic concepts in topology. Let X be a topological space. We say X is **compact** if every open cover of X has a finite subcover [26, p. 116]. This means that if X is the union of a family of open sets, then there is a finite subfamily of open sets whose union is X. Two points x and y of X can be separated by neighborhoods if there exists a neighborhood U of x and a neighborhood V of y such that $U \cap V = \varnothing$. The space X is **Hausdorff** if any two distinct points in it can be separated by neighborhoods [26, p. 86]. It is a **totally disconnected** space if it has no nontrivial connected subsets, i.e., only the empty set and the one-point sets are connected. A **Boolean space** is a topological space that is compact, Hausdorff, and totally disconnected.

Proposition 5.1. Let A be a set. Denote by $\{-1, 1\}^A$ the space of functions on A whose values are either 1 or -1. Then $\{-1, 1\}^A$ is a Boolean space.

Proof: Endow $\{-1, 1\}$ with the discrete topology and $\{-1, 1\}^A$ the product topology. This product topology is the coarsest topology for which all projections $\pi(a) : \{-1, 1\}^A \longrightarrow \{-1, 1\}$ are continuous. A subbasis for this topology consists of all sets of the form

$$\begin{aligned} B_{a,1} &= \{f \in \{-1,1\}^A \mid f(a) = 1\} \text{ and} \\ B_{a,-1} &= \{f \in \{-1,1\}^A \mid f(a) = -1\} \end{aligned}$$

A set of either form is the complement of a set of the other form. Every subbasic open set is therefore clopen. Moreover, all open sets of $\{-1,1\}^A$ may be formed from the finite intersection of elements in the subbasis and then from the arbitrary unions of elements of the base. Since $\{-1,1\}$ has a finite topology, then it is compact. Thus, by Tychonoff's Theorem which states that the product of any collection of compact topological spaces is compact with respect to the product topology, $\{-1,1\}^A$ is compact. Now, to show that the space $\{1,-1\}^A$ is Hausdorff, let $f,\ g \in \{1,-1\}^A$, $f \neq g$. Then for some $a \in X$, $f(a) \neq g(a)$, say $f(a) = 1 \neq -1 = g(a)$. Then f belongs to some open set of the form $B_{a,1}$ and g belongs to some open set of the form $B_{a,-1}$. Since $B_{a,1} \cap B_{a,-1} = \varnothing$, then $\{-1,1\}^A$ is a Hausdorff space. Finally, for all $a \in A$, we have

$$\{1,-1\}^A = B_{a,1} \bigsqcup B_{a,-1},$$

and so $\{1,-1\}^A$ is totally disconnected. Therefore, $\{1,-1\}^A$ is a Boolean space. ■

Definition 5.2. Let S be a proper semifield. Let X_S denote the topological space of orderings of S generated by the subbasis consisting of sets of the form $H_{(a,b)} = \{\leq \in X_S \mid a \leq b\}$. The collection of all such subsets of X_S is called the Harrison subbasis which will be denoted by $H(S)$ and the topology generated by $H(S)$ is called the Harrison topology on S.

As in the case of ordered fields, the set of orderings of a semifield can also be topologized. We will now see the nature of this space of orderings of the semifield S.

Theorem 5.3. Let S be a proper semifield. Then the space of orderings of S, X_S, endowed with the Harrison topology, is a Boolean space.

Proof: Let $\leq \in X_S$. Define $f \in \{1,-1\}^{S\times S}$ by

$$f(a,b) = \begin{cases} 1 & \text{if } a \leq b \\ -1 & \text{if } b < a. \end{cases}$$

Then $\leq$ determines a map: $S \times S \longrightarrow \{1,-1\}$. Thus, there exists an embedding $X_S \hookrightarrow \{1,-1\}^{S\times S}$. By Proposition 5.1, $\{1,-1\}^{S\times S}$ endowed with the natural product topology, is a Boolean space. Consequently, there is an induced subspace topology on X_S. For $\{1,-1\}^{S\times S}$, a subbasis for its topology consists of all sets of the forms $B_{(a,b),1} = \{f \in \{1,-1\}^{S\times S} | f(a,b) = 1\}$ and $B_{(a,b),-1} = \{f \in \{1,-1\}^{S\times S} | f(a,b) = -1\}$. Since $B_{(a,b),-1} = B_{(b,a),1}$, a Harrison subbasic open set for X_S is given by $H(a,b) = B_{(a,b),1} \cap X_S = \{\leq \in X_S | a \leq b\}$. We will show that X_S is a closed subspace of $\{1,-1\}^{S\times S}$. Let $g \in \{1,-1\}^{S\times S}$, with g not determined by an ordering in X_S. Since g is not an image of an ordering in S, then at least one of the following must hold:

(1) If reflexivity fails, then $g \in B_{(a,a),-1}$.

(2) If antisymmetry fails, then there exist $a, b \in S$ such that $g \in B_{(a,b)-1}$, and so $g \in B_{(a,b),1} \cap B_{(a,b)-1}$.

(3) If transitivity fails, then there exist a, b, $c \in S$ such that $g \in B_{(b,c),1}$ and $g \in B_{(a,c),-1}$, hence, $g \in B_{(a,b),1} \cap B_{(b,c),1} \cap B_{(a,c),-1}$.

(4) If totality fails then no $g \in \{1,-1\}^{S\times S}$.

(5a) If addition fails, then there exist a, b, $c \in S$ such that $B_{(e,f),1}$, where $e = a + c$ and $f = b + c$, hence,

$$g \in B_{(a,b),1} \cap B_{(e,,f),-1}.$$

(5b) If multiplication fails, then then there exist a, b, $c \in S$ such that $g \in B_{(g,h),-1}$, where $g = ac$, and $h = bc$, hence, $g \in B_{(a,b),1} \cap B_{(g,h,),-1}$.

Thus, in any case, we can separate g from the image of X_S. ■

We now obtain our main result for a formally real field F as an immediate corollary. We denote by $W(F)$ the Witt ring of equivalence classes of anisotropic quadratic forms over F [9]. Modulo torsion one obtains the reduced Witt ring $W_{red}(F)$ [9, p. 292], and an embedding into the ring of continuous functions from the space of orderings X_F to the ring of integers with the discrete topology, $C(X_F, \mathbb{Z})$.

Corollary 5.4. Let F be a formally real field. Let $S = \Sigma F^2$. Then $W_{red}(F) \hookrightarrow C(X_F, \mathbb{Z}) = C(X'_S, \mathbb{Z})$, where X'_S is the subspace of X_S of semifield orderings with $1 < 2$.

CHAPTER 6
Open Questions

From the novelty of this work, many questions can be posed on the basis of which further investigations or research may be done, among others are the following:

1. While an additive identity, 0, can easily be added to a proper semifield, it creates some problems in studying the orderings. In particular, products behave differently. How should the theory be modified if 0 is allowed in the semifield?

2. Is there any class of semifields where M can be used to develop more of Artin-Schreier Theory, in particular, a criterion for the existence of an ordering.

3. How else may we create an idempotent semifield? A semifield with no ordering?

4. How can we characterize, in general, orderings of a semifield that are not inherited from that of a field? Does Chapter 4 include all orderings of $R^{\#}(x)$?

5. How may we order the set of positive rational functions in n variables over $\mathbb{R}$?

6. How can the theory be extended to noncommutative semifields?

7. Does every cancellative semifield have an ordering?

Bibliography

[1] N. Bourbaki, *Elements of Mathematics, Algebra II*, Chapters $4-7$, Springer Verlag, Berlin, 1990.

[2] T. Craven, *The Topological Space of Orderings of a Rational Function Field,* Duke Mathematical Journal, vol. 41, no. 2. (1974), pp. $339-347$.

[3] T. Craven, *The Boolean Space Orderings of a Field,* Trans. Amer. Math. Soc. 209 (1975), pp. $225-235$.

[4] L. Fuchs, *Partially Ordered Algebraic Systems,* Addison-Wesley Publishing Company, Inc., Reading, MA, 1963.

[5] J. S. Golan, *Semirings and Affine Equations over Them,* Kluwer Academic Publisher, The Netherlands, 2003.

[6] D. Hilbert, *The Foundations of Geometry,* The Open Court Publishing Company, La Salle, 1950.

[7] H. C. Hutchins and H. J. Weinert, *Homomorphisms and Kernels of Semifields,* Period. Mat., 21, No. 2 (1990), pp. $113-152$.

[8] I. Itenberg, G. Mikhalkin, E. Shustin, *Tropical Algebraic Geometry,* 2nd ed., Birkhauser, Basel, 2009.

[9] M. Knebusch, A. Rosenberg, and R. Ware, *Signatures on Semilocal Rings,* J. Algebra 26 (1973), pp. $208-250$.

[10] T. Y. Lam, *Introduction to Quadratic Forms Over Fields*, Graduate Studies in Mathematics, vol. 67, American Mathematical Society, Rhode Island, 2005.

[11] T. Y. Lam, *Orderings, Valuations and Quadratic Forms*, The American Mathematical Society, Rhode Island, 1983.

[12] G. L. Litvinov and S.N. Sergeev, *Tropical and Idempotent Mathematics,* Contemporary Mathematics (2007). International Workshop Tropical - 07, Independent University of Moscow and Laboratory J. - V. Poncelet (online).

[13] M. Marshall, *Positive Polynomials and Sums of Squares*, The American Mathematical Society, Rhode Island, 2008.

[14] V. P. Maslov and V. N. Kolokolcov, *Idempotent Analysis and Its Application in an Optimal Management,* Nauka, Moscow, 1994.

[15] V. P. Maslov and S. N. Samborskii, *Idempotent Analysis,* Adv. Sov. Math., vol 13, American Mathematical Society, Rhode Island, 1992.

[16] B.R. McDonald, *Ring Theory and Algebra III*, Marcel Dekker, Inc., New York, 1980.

[17] A. Prestel, *Lectures on Formally Real Fields,* 2nd ed., Springer-Verlag, 1988.

[18] W. Scharlau, *Quadratic and Hermitian Forms*, Springer Verlag, Berlin, 1985.

[19] A. N. Semenov, *On Lattice of Semifield Congruences,* Vestn. VyatGGU, No. 9 (2003), pp. $92-95$.

[20] A. N. Semenov, *On Structure of Semifields,* Vestn. VyatGGU, No. 8 (2003), pp. $105-107$.

[21] E.M. Vechtomov and A.V. Cheraneva, *Semifields and Their Properties,* Journal of Mathematical Sciences, vol. 163, no. 6 (2009), pp. $625-661$.

[22] H.J. Weinert, *Űber Halbringe and Halbkorper I,* Acta Math. Hung. 13 (1962), pp. $365-373$.

[23] H. J. Weinert, *Űber Halbringe and Halbkorper II,* Acta Math. Hung. 14 (1963), pp. $209-227$.

[24] H. J. Weinert, *Űber Halbringe and Halbkorper III,* Acta Math. Hung. 15 (1964), pp. $177-194$.

[25] H. J. Weinert, *Ein Struktursatz fur idempotente Halbkorper,* Acta Math. Hung. 15 (1964), pp. $289-295$.

[26] S. Willard, *General Topology*, Dover Publications, New York, 1970.