HYPERBOLICITY PRESERVING DIFFERENTIAL OPERATORS AND CLASSIFICATIONS OF ORTHOGONAL MULTIPLIER SEQUENCES

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This dissertation is dedicated to my loving wife, whose love, encouragement, and friendship has given me the endurance to complete this work.

Una in perpetuum.

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ABSTRACT

It is well known that every linear operator, $T : \mathbb{R}[x] \to \mathbb{R}[x]$, can be represented in the form,

$$T := \sum_{k=0}^{\infty} Q_k(x)D^k, \quad D := \frac{d}{dx},$$

where $\{Q_k(x)\}_{k=0}^{\infty}$ is a sequence of real polynomials. An outstanding open problem in the theory of distribution of zeros of polynomials is to characterize the sequence, $\{Q_k(x)\}_{k=0}^{\infty}$, such that the operator T is hyperbolicity preserving; i.e., T maps polynomials with only real zeros to polynomials of the same kind.

A large portion of this dissertation focuses on diagonal differential operators; that is,

$$T[B_n(x)] := \left(\sum_{k=0}^{\infty} Q_k(x)D^k \right) B_n(x) = \gamma_n B_n(x), \quad n \in \mathbb{N}_0,$$

where $\{B_n(x)\}_{n=0}^{\infty}$ is a simple sequence of real polynomial eigenvectors and $\{\gamma_n\}_{n=0}^{\infty}$ is the corresponding sequence of real eigenvalues. Our analysis leads to new relations between the eigenvector sequence and the eigenvalue sequence in a diagonal differential operator. In particular, we develop new methods for determining the polynomial coefficients, $\{Q_k(x)\}_{k=0}^{\infty}$, in cases of Hermite, Laguerre, or monomial linear transformations. We establish a new representation of linear operators and demonstrate novel hyperbolicity properties for this representation. We show that every Hermite or Laguerre multiplier sequence can be expressed as a sum of classical multiplier sequences.

Using the Malo-Schur-Szegő Composition Theorem, we present a new proof of J. Borcea and P. Brändén's seminal result on the hyperbolicity preservation of finite order differential operators. The hyperbolicity preservation of order two differential operators is studied in minute detail; this leads to a new Turán-Wronskian inequality. In addition, a new algebraic characterization is given for the class of Hermite multiplier sequences. Moreover, we prove that in the case of a Hermite diagonal differential operator, T, the zeros of $Q_k(x)$ and $Q_{k+1}(x)$ are interlacing, $k \geq 0$. We generalize several results of D. Bleecker, G. Csordas, T. Forgács, and A. Piotrowski, partially answer a number of general open problems of T. Craven, G. Csordas, and S. Fisk, and solve a question of M. Chasse. This dissertation concludes with an outline of possible future research and a list of open questions.

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The following notation will be used within the text. We provide a brief description of each symbol below and cite the page number of first appearance.

PREFACE AND A LIST OF THE MAIN RESULTS

In 1929, S. Bochner [15] provided one of the first demonstrations on the uniqueness of the classical orthogonal polynomials. Up to an affine transformation (see Definition 35), the only polynomial eigenvector sequences, $\{B_n(x)\}_{n=0}^{\infty}$, that satisfy an order two diagonal differential operator,

$$\left(Q_2(x)\frac{d^2}{dx^2} + Q_1(x)\frac{d}{dx} + Q_0(x)\right)B_n(x) = \lambda_n B_n(x),$$

are the Hermite, Laguerre, Jacobi, Bessel, or monomial bases. Hence, Bochner concluded that the only orthogonal polynomials that can serve as the eigenvectors of an order two diagonal differential equation are essentially the Hermite, Laguerre, or Jacobi polynomials. After Bochner, we find many more demonstrations of the uniqueness of the classical orthogonal polynomials (see W. Hahn [60, (1935)], H. Krall [70, (1936)], W. Hahn [61, (1937)], F. Tricomi [102, (1948)], R. Ebert [48, (1964)], C. Cryer [37, (1970)]) (see also [29, pp. 150-153]). A complete characterization of differential equations with classical orthogonal polynomial eigenvectors was recently given by L. Miranian [81, (2005)].

Classical orthogonal polynomials play a pivotal role in many fields of study. In applications, the product of the Hermite polynomials with the Gaussian (Hermite functions), serve as the eigenfunctions to the Schrödinger equation [13, 80]. Moreover, in quantum mechanics, the Wigner distribution function relates to the properties of the Laguerre and Hermite polynomials [59]. In 1785, A. Legendre discovered an important sequence of polynomials, later named the Legendre polynomials, that serve as the coefficients in the expansion of the Newtonian potential [73]. While investigating the famous Riemann hypothesis [91] (see also [38, 40, 43, 44]), in 1954, P. Turán was led to study strip preserving operators and hence studied expansions of entire functions in terms of the Hermite polynomials [103, 104]. Orthogonal polynomials also arise in combinatorics, functional analysis, number theory, complex variables, etc.

In modern times, it has become evident the central role that the classical orthogonal polynomials play in the study of hyperbolicity preserving operators (see Definition 76); as can be found in the recent active research by D. Bleecker, J. Borcea, P. Brändén, M. Chasse, T. Craven, G. Csordas, T. Forgács, L. Grabarek, A. Piotrowski, W. Smith, R. Yoshida, etc. (see [5, 6, 11, 14, 17, 21, 25–27, 32–36, 42, 52–55, 58, 85, 105] and the references therein). For example, it was recently discovered that every B_n-multiplier sequence (Definition 89) is also a classical multiplier sequence [85, Theorem 158]. We extend this result and demonstrate the exact relationship between B_n-diagonal operators and classical diagonal operators (see Definition 89). That

is, by using the differentiation operator, $D := \dfrac{d}{dx}$, with a B_n-diagonal operator, $T : \mathbb{R}[x] \to \mathbb{R}[x]$, we obtain the following unique sum of classical diagonal operators, $\{T_n\}_{n=0}^{\infty}$,

$$T = T_0 + T_1 D + T_2 D^2 + T_3 D^3 + \cdots . \tag{0.0.1}$$

We show that the implication, every B_n-multiplier sequence is a classical multiplier sequence, is equivalent to the assertion that, if T is a hyperbolicity preserving operator, then so is T_0 (Theorem 115 and Corollary 136). Moreover, for several bases, $\{B_n\}_{n=0}^{\infty}$, we establish that each T_n in the representation (0.0.1) is hyperbolicity preserving. Hence, for example, every Hermite or Laguerre multiplier sequence is the unique sum of classical multiplier sequences as in (0.0.1) (see Theorems 196, 197, 215, and 216). This analysis leads to a property that distinguishes classes of Hermite and Laguerre multiplier sequences from affine transformations of the Hermite and Laguerre multiplier sequences (see Examples 127 and 129). The new representation (0.0.1), extends to arbitrary linear operators as a Laurent-type series (Theorem 119).

In addition to developing the new representation of (0.0.1), we also investigate the well known differential representation of linear operators (see J. Peetre [84, (1959)], see also A. Piotrowski [85, Proposition 29]). If $T : \mathbb{R}[x] \to \mathbb{R}[x]$ is an arbitrary linear operator, then there is a unique sequence of real polynomials, $\{Q_k(x)\}_{k=0}^{\infty}$, such that

$$T = \sum_{k=0}^{\infty} Q_k(x) D^k, \quad D := \dfrac{d}{dx}. \tag{0.0.2}$$

This representation is the subject of active, ongoing research [5–7, 11, 14, 16, 17, 20, 21, 24, 25, 27, 32, 46, 51–55, 58, 81, 84, 85]. In particular, the following is an outstanding open problem.

Problem 1. Let $T : \mathbb{R}[x] \to \mathbb{R}[x]$ be a linear operator. Characterize the properties of the polynomial coefficients in (0.0.2) such that T is hyperbolicity preserving.

In order to obtain partial solutions of Problem 1, we completely characterize all order two diagonal differential hyperbolicity preservers (Theorem 182). This provides new Turán-type (Corollary 162 and Theorem 187) and Turán-Wronskian-type (Theorem 181) inequalities. Our results extend theorems of J. Borcea and P. Brändén [16, Lemma 3.2], D. Bleecker and G. Csordas [14, Lemma 2.2], K. Blakeman, E. Davis, T. Forgács, and K. Urabe [11, Proposition 5], T. Forgács and A. Piotrowski [55, Conjecture 5.1], and A. Piotrowski [85, Lemma 67, Proposition 68, Lemma 169, Problem 174]. We provide a new proof of the famous Borcea-Brändén Theorem [16, Theorem 1.3] on the hyperbolicity preservation of finite order differential operators (Theorem 138) using the Malo-Schur-Szegő Composition Theorem, Theorem 69. These results shed light

on the nature of hyperbolicity preservers and prompt many new open questions (see, for example, Problem 156).

We analyze the classical orthogonal polynomials in detail with respect to hyperbolicity preservation. We refine results of L. Miranian [81, Theorem 1] (cf. Theorems 86, 107, 110 and 220) and answer questions concerning the polynomial sequence, $\{Q_k(x)\}_{k=0}^{\infty}$, for linear operators that diagonalize on a classical orthogonal basis. Moreover, we provide new methods of calculating $\{Q_k(x)\}_{k=0}^{\infty}$ for a linear operator (Theorem 81), a diagonal differential operator (Theorem 90), a Hermite diagonal differential operator (Theorems 201 and 203), a Laguerre diagonal differential operator (Theorems 218 and 219), and a monomial transformation operator (Theorem 229) (see the open problem of A. Piotrowski [85, Problem 194]). These results build upon several theorems, P. Brändén and E. Ottergren [21, Theorem 1.1], T. Forgács and A. Piotrowski [54, Theorem 3.1 and Theorem 3.9], and A. Piotrowski [85, Proposition 33]. In addition, we provide a new algebraic characterization of Hermite multiplier sequences (Theorem 208) and answer a question of M. Chasse (Theorem 233) (see [27, Conjecture 222]). The aforementioned results provide partial solutions to general questions of T. Craven and G. Csordas [32, Problem 1.3] and S. Fisk [51, Question 30].

Using a characterization of the eigenvalues of an arbitrary linear operator (Theorem 93), we formulate results concerning the nature of polynomial interpolation with respect to differential representation (Theorems 95, 97, 103, 111, and 112 and Corollaries 98, 113, and 115) (cf. A. Piotrowski [85, Theorems 157 and 158] and S. Fisk [51, Question 115]).

A plethora of conjectures and open questions are provided at the end of the dissertation (Chapter 5). Many of the results in this dissertation have recently been summarized in three notes [5–7] that are currently submitted to arXiv.org; one of which has been accepted for publication [7].

To the author's best knowledge the following results appear to be new.

Chapter 2: Theorems 81, 86, 90, 93, 95, 97, Corollary 98, Theorem 103, Corollary 104, Theorems 107, 109, 110, 111, Corollary 113, Theorems 115, 117, 119, Corollaries 136, 140, 147, Theorems 148, 149, Corollary 150, Theorem 154

Chapter 3: Theorems 163, 164, 165, 166, 167, Lemmas 168, 169, Theorems 170, 172, Lemma 180, Theorems 181, 182, 183, 187

Chapter 4: Theorems 196, 197, 201, 203, 204, 205, 208, Lemma 212, Theorems 215, 216, 218, 219, 220, Corollaries 230, 231, Theorem 233

CHAPTER 1
INTRODUCTION

In Section 1.1, we make some introductory remarks about polynomials and cite several open problems. We recall in Section 1.2, some nomenclature and terminology involving various families of polynomials and transcendental entire functions. After a brief discussion of certain generating functions, we recapitulate several salient facts about orthogonal polynomials (Section 1.2). In Section 1.3, we review the main properties of hyperbolic polynomials and the Laguerre-Pólya class. Section 1.3 closes with several combinatorial facts that will be of use in the sequel. In Section 1.4, we define and discuss classical multiplier sequences and the famous Pólya-Schur Theorem (Theorem 54). We end Chapter 1 with a summary of some very important classical theorems on polynomials that will be used throughout the later chapters (Section 1.5).

1.1 Preliminaries

A function, $p : \mathbb{C} \to \mathbb{C}$, is a polynomial of degree n, if it can be expressed as,

$$p(z) := a_0 + \cdots + a_n z^n,$$

where $a_0, \ldots, a_n \in \mathbb{C}$ and $a_n \neq 0$. The Fundamental Theorem of Algebra, first established by C. Gauss [56, (1799)], shows that each polynomial of degree n has exactly n complex zeros, counting multiplicities. Gauss' result, together with the Euclidean algorithm, yields that there exists $z_1, \ldots, z_n \in \mathbb{C}$ such that

$$p(z) = a_n \prod_{k=1}^{n} (z - z_k), \tag{1.1.1}$$

for every $z \in \mathbb{C}$. The relationship between the coefficients, $a_0, \ldots, a_n$, and the zeros, $z_1, \ldots, z_n$, is a long standing open problem in mathematics.

Problem 2. Characterize the coefficients, $a_0, \ldots, a_n \in \mathbb{C}$, such that $p(z) := a_0 + \cdots + a_n z^n$ has only real zeros.

Equation (1.1.1) shows that if a polynomial has only real zeros, then (up to a constant multiple, namely a_n) that polynomial has only real coefficients. Problem 2 is certainly of current interest. By way of illustration, we state the following ostensibly simple open problem that arose from the CMFT international conference in Turkey [30, (June 2009)].

Problem 3. Show that for each $n \in \mathbb{N}$,

$$p_n(x) := \sum_{k=0}^{n} \frac{(n-k)^k}{k!} x^{n-k},$$

has only real zeros.

It is a tantalizing fact that many open problems involving polynomials are simple to state and are devoid of complicated abstract notions; so that even a freshman can understand them. We cite the celebrated 50-year-old conjecture of B. Sendov (formerly known as the Ilieff-Sendov conjecture) (see, for example, [22, 47, 72, 79]).

Problem 4 (The Sendov Conjecture)**.** If all the zeros of the polynomial

$$p(z) = c \cdot \prod_{k=1}^{n} (z - z_k), \ n \geq 2, \ c \neq 0,$$

lie in the closed unit disk $\overline{D}(0,1) = \{z \in \mathbb{C} : |z| \leq 1\}$, then for every $z_k \ (k = 1, \ldots, n)$ the disk $\overline{D}(z_k, 1) = \{z \in \mathbb{C} : |z - z_k| \leq 1\}$ contains at least one zero of $p'(z)$.

Remark 5. The constant 1, in the above statement, is best possible, as the example $p(z) = z^n - 1 \ (n \geq 2)$ shows. There have been more than 80 papers published on the validity of the Sendov Conjecture. The conjecture has also been established for many small degree polynomials and other special cases. However, to this date, this conjecture has neither been proved or disproved. A solution to the Sendov Conjecture would have significant ramifications in Numerical Analysis (see, for example, [98, pp. 206-227]).

Using the terminology of L. Euler [49, (1748)], a question analogous to Problem 2 can be formulated for polynomials of *infinite degree* (in the current parlance, *entire functions*). However, unlike polynomials, not all transcendental entire functions are of equal interest in our investigations. This will be made clear in the next few statements.

In order to analyze the properties of transcendental entire functions, one standard method is to study the polynomials that converge locally uniformly, on compact subsets of $\mathbb{C}$, to these functions. In Section 1.3, we introduce the *Laguerre-Pólya class*, denoted by $\mathscr{L} - \mathscr{P}$ (see Definition 36),

$$\mathscr{L} - \mathscr{P} = \left\{ \begin{array}{l} \text{Entire functions that are, on } \mathbb{C}, \text{ uniform limits locally of real} \\ \text{polynomials with only real zeros.} \end{array} \right\}.$$

At first glance one might egregiously believe that every real entire function with only real zeros is in $\mathscr{L} - \mathscr{P}$. We will show that this is not the case with the aid of the following result for polynomials that can be proved by an elementary application of Rolle's Theorem [93, (1691)].

Theorem 6. *If a polynomial, $p(x)$, has only real zeros, then its derivative, $p'(x)$, also has only real zeros.*

Since every function in $\mathscr{L} - \mathscr{P}$ can be approximated by polynomials with only real zeros, we apply Hurwitz's Theorem [64, (1932)] to extend Theorem 6 to all functions in $\mathscr{L} - \mathscr{P}$. Thus, we say that $\mathscr{L} - \mathscr{P}$ is closed under differentiation. We now consider the following entire functions,

$$f_1(x) := e^x, \qquad f_2(x) := e^{-x}, \qquad f_3(x) := e^{x^2},$$
$$f_4(x) := e^{-x^2}, \qquad f_5(x) := e^{x^3}, \qquad \text{and} \quad f_6(x) := e^{-x^3}.$$

Despite the fact that $f_n(x)$, $n = 1, 2, 3, 4, 5, 6$, has only real zeros, it is interesting that, $e^x, e^{-x}, e^{-x^2} \in \mathscr{L} - \mathscr{P}$ and $e^{x^2}, e^{-x^3}, e^{x^3} \notin \mathscr{L} - \mathscr{P}$. It is easy to verify that $e^x, e^{-x}, e^{-x^2} \in \mathscr{L} - \mathscr{P}$, since

$$\left(1 + \frac{x}{n}\right)^n \to e^x, \qquad \left(1 - \frac{x}{n}\right)^n \to e^{-x}, \qquad \text{and} \quad \left(1 - \frac{x^2}{n}\right)^n \to e^{-x^2},$$

converge locally uniformly on compact subsets of $\mathbb{C}$. We note that while entire functions in $\mathscr{L} - \mathscr{P}$ can be approximated locally uniformly on $\mathbb{C}$ by polynomials with only real zeros, in general, they can also be approximated locally uniformly on $\mathbb{C}$ by polynomials with non-real zeros. For example, sequence, $\{s_n(x)\}_{n=0}^{\infty}$, where $s_n(x) := \sum_{k=0}^{n} \frac{x^k}{k!}$, also converges to e^x and each $s_n(x)$ has at most one real zero [9, 66]. Thus, if one attempted to establish that $e^{x^2} \notin \mathscr{L} - \mathscr{P}$, then it would not be obvious that the analysis of only the polynomial sequence, $\{p_n(x)\}_{n=0}^{\infty}$, where $p_n(x) := \left(1 + \frac{x^2}{n}\right)^n$, is enough to show that $e^{x^2} \notin \mathscr{L} - \mathscr{P}$ (Theorem 23). In order to confirm that $e^{x^2}, e^{-x^3}, e^{x^3} \notin \mathscr{L} - \mathscr{P}$, it suffices to take consecutive derivatives until one arrives at a function with non-real zeros, a violation of the foregoing argument which generalizes Theorem 6 to $\mathscr{L} - \mathscr{P}$ (see also the recent proofs of the Pólya-Wiman Conjecture [35, 36, (1987)], [68, (1990)] and the Wiman Conjecture [99, (1989)], [10, (2003)]). Hence, we calculate

$$f_3^{(2)}(x) = \frac{d^2}{dx^2} e^{x^2} = 2e^{x^2}(2x^2 + 1),$$
$$f_5^{(2)}(x) = \frac{d^2}{dx^2} e^{x^3} = 3e^{x^3} x(3x^3 + 2), \text{ and}$$
$$f_6^{(2)}(x) = \frac{d^2}{dx^2} e^{-x^3} = 3e^{-x^3} x(3x^3 - 2).$$

A much more careful and detailed analysis [74, p. 331] yields a characterization of the Laguerre-Pólya class, which sheds light on some of the observations made above.

Theorem 7. *A function, $f(x)$, is in $\mathscr{L} - \mathscr{P}$ if and only if there exists, $0 \leq \omega \leq \infty$, $a, b, c \in \mathbb{R}$, $a \geq 0$, $\{x_k\}_{k=1}^{\omega} \subseteq \mathbb{R}$, $x_k \neq 0$, $\sum_{k=1}^{\omega} \dfrac{1}{x_k^2} < \infty$, and a non-negative integer m, such that*

$$f(x) = cx^m e^{-ax^2 + bx} \prod_{k=1}^{\omega} \left(1 + \frac{x}{x_k} \right) e^{-x/x_k}. \tag{1.1.2}$$

From (1.1.2) we observe the unexpected fact that every function in $\mathscr{L} - \mathscr{P}$ is of growth order less than or equal to two. We direct the reader to [74, 76, 90] for a more detailed discussion of order, type, and Weierstrass and Hadamard factorizations, which led to the establishment of Theorem 7. For additional characterizations of the Laguerre-Pólya class, see the papers by T. Craven and G. Csordas [33], G. Csordas and A. Escassut [41], or G. Csordas and A. Vishnyakova [45].

Definition 8. Let $K \subseteq \mathbb{C}$. We define, $A(K)$ to be the set of all polynomials in $\mathbb{C}[x]$ all of whose zeros lie in K. To avoid trivialities, the zero polynomial is also a member of $A(K)$. The set $\mathscr{A}(K)$ denotes the set of all entire functions that can be locally uniformly approximated, on $\mathbb{C}$, by polynomials in $A(K)$.

We see that the restriction of the zeros of locally uniformly convergent sequences of polynomials determines the kinds of entire functions that can be approximated. This is made even more clear by the following observation.

Theorem 9. *If K is a compact subset of $\mathbb{C}$, then*

$$A(K) = \mathscr{A}(K).$$

Proof. Note that certainly, $A(K) \subseteq \mathscr{A}(K)$. Thus, we wish to show $\mathscr{A}(K) \subseteq A(K)$. Suppose $f(x) \in \mathscr{A}(K)$. Then by assumption, there is $\{p_k(x)\}_{k=0}^{\infty} \subseteq A(K)$ such that $p_k(x) \to f(x)$ locally uniformly on $\mathbb{C}$. If there is some $M \in \mathbb{N}_0$ such that $\deg(p_k(x)) \leq M$ for every k, then $f(x)$ must also be a polynomial and hence, since K is closed, $f(x) \in A(K)$. If $\{\deg(p_k(x))\}_{k=0}^{\infty}$ is unbounded, then by Hurwitz's Theorem [31, p. 152] and the fact that K is closed, we conclude that $f(x)$ has an infinite number of zeros in K. Thus, by the identity theorem [1, p. 126], since K is bounded, $f(x)$ is the zero polynomial and whence conclude that $f(x) \in A(K)$. $\qquad \square$

The study of the location of zeros of transcendental entire functions, from the point of view of local uniform convergence of polynomials with restricted zero loci, is only interesting when we consider sets K that are non-compact subsets of the complex plane. The most common subsets of $\mathbb{C}$ studied, are lines (e.g. $\mathbb{R}$), rays (e.g. $\mathbb{R}^+$ or $\mathbb{R}^-$), sectors [27], strips [18], and half planes (e.g. H^+ or H^-) [17], where we define

$$H^+ := \{z \in \mathbb{C} : \operatorname{Im}(z) > 0\} \quad \text{and} \quad H^- := \{z \in \mathbb{C} : \operatorname{Im}(z) < 0\}.$$

The following problem summarizes the study of linear operators from $\mathbb{R}[x] \to \mathbb{R}[x]$ with prescribed zero restrictions.

Problem 10. Let $S_1, S_2 \subseteq \mathbb{C}$. Characterize all linear transformations,

$$T : A(S_1) \to A(S_2).$$

1.2 Families of Polynomials

Polynomials will play a central role throughout our discussions. The goal of this section is to provide the foundational polynomial definitions that will be used in the sequel.

Definition 11. We say a sequence of real polynomials, $\{B_n(x)\}_{n=0}^{\infty}$, is a *simple* sequence if $B_0(x) \not\equiv 0$ and $\deg(B_n(x)) = n$ for each $n \in \mathbb{N}_0$.

Definition 12. A *hyperbolic* polynomial refers to a real polynomial, $p(x)$, with only real zeros; by definition, $p(x) \in \mathscr{L} - \mathscr{P}$. By Pólya and Schur's convention [87], we note that the zero polynomial is also considered a hyperbolic polynomial.

Definition 13. A *stable* polynomial refers to a polynomial, $p(x) \in \mathbb{C}[x]$, with all its zeros in $\overline{H^+} = H^+ \cup \mathbb{R}$ (closed upper half plane) or all its zeros in $\overline{H^-} = H^- \cup \mathbb{R}$ (closed lower half plane). To distinguish, we say that $p(x)$ is *upper stable* or *lower stable* for $\overline{H^+}$ or $\overline{H^-}$, respectively. If we remove $\mathbb{R}$, then we say that $p(x)$ is *strictly stable, strictly upper stable*, or *strictly lower stable*.

Definition 14. A bivariate real polynomial, $p(x, w) \in \mathbb{R}[x, w]$, is also called *stable* if $p(x, w) \neq 0$ for every $x, w \in H^+$.

We note the counter-intuitive definition of Definition 14 compared to Definition 13. The reason for this, as done by Borcea and Brändén, is to simplify the statements of the famous Borcea-Brändén Theorems (Theorems 138 and 142).

Remark 15. In general, univariate strictly stable polynomials are not real-valued. This follows since real polynomials have non-real zeros that come in conjugate pairs. In fact, it is commonly observed that a real polynomial, $p(x)$, is stable if and only if $p(x)$ is hyperbolic. However, a bivariate stable polynomial can certainly be real-valued and have many non-real zeros. For example, $p(x, w) = (x^2 - 1)w^2 - 2xw + 1$ is a real bivariate stable polynomial, since $p(x, w)$ can be written as a product of bivariate stable polynomials, $p(x, w) = ((x - 1)w - 1)((x + 1)w - 1)$ (see also [11, Lemma 5] and Example 134).

Definition 16. Given a polynomial, $p(x) = a_0 + a_1 x + \cdots + a_{n-1}x^{n-1} + a_n x^n$, of degree n, $a_n \neq 0$, we define the *reverse*,

$$p^*(x) := a_n + a_{n-1}x + \cdots + a_1 x^{n-1} + a_0 x^n.$$

The reverse of a polynomial is easily calculated from the formula,

$$p^*(x) = x^{\deg(p(x))} p\left(\frac{1}{x}\right).$$

We caution that reversing polynomials with zeros at the origin will result in losing degree. Hence, for some polynomials $(p^*(x))^* \neq p(x)$. However, if $p(x)$ has no zeros at the origin then certainly $(p^*(x))^* = p(x)$. We also note that, $p(x) \in \mathscr{L} - \mathscr{P}$ if and only if $p^*(x) \in \mathscr{L} - \mathscr{P}$. This follows from the simple observation that $p(\alpha) = 0$ if and only if $p^*(\frac{1}{\alpha}) = 0$ for $\alpha \neq 0$.

Definition 17. Let $\{f_k(x)\}_{k=0}^{\infty}$ be a sequence of functions. We define the formal multivariate *generating function*, $f(x, t)$, of the sequence $\{f_k(x)\}_{k=0}^{\infty}$ to be,

$$f(x, t) := \sum_{k=0}^{\infty} \frac{f_k(t)}{k!} x^k.$$

Convergence here is not necessary, since we only formulate this sum formally. However, when $f(x, t)$ is analytic and the series converges uniformly in a neighborhood of zero, then for every k,

$$f_k(t) = \left. \frac{d^k}{dx^k} f(x, t) \right|_{x=0}.$$

Example 18. From Taylor's Theorem [97, p. 110], we can give the generating function for the derivatives of a differentiable function,

$$f(x+y) = \sum_{k=0}^{\infty} \frac{f^{(k)}(y)}{k!} x^k.$$

Definition 19. Given a formal series,

$$f(x) := \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} x^k,$$

we define the nth *Jensen polynomial* associated with $f(x)$ (or associated with $\{\gamma_k\}_{k=0}^{\infty}$) by,

$$g_n(x) := \sum_{k=0}^{n} \binom{n}{k} \gamma_k x^k.$$

We also define the nth *reversed Jensen polynomial* associated with $f(x)$ (or associated with $\{\gamma_k\}_{k=0}^{\infty}$) by,

$$g_n^*(x) := \sum_{k=0}^{n} \binom{n}{k} \gamma_{n-k} x^k.$$

We point out that a "reversed Jensen polynomial" is not equal to "the reverse of a Jensen polynomial". This is because the nth Jensen polynomial will not be degree n if $\gamma_n = 0$.

Example 20. Similar to the generating function for the derivatives of an entire function (see Example 18), $f(x)$, the reversed Jensen polynomials, $\{g_k^*(x)\}_{k=0}^{\infty}$, of $f(x)$, also have a summation formula,

$$
\begin{aligned}
g_n^*(x+y) &= \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(0)(x+y)^{n-k} \\
&= \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(0) \sum_{j=0}^{n-k} \binom{n-k}{j} x^j y^{n-k-j} \\
&= \sum_{k=0}^{n} \sum_{j=0}^{n} \binom{n}{k} f^{(k)}(0) \binom{n-k}{j-k} x^{j-k} y^{n-j} \\
&= \sum_{j=0}^{n} \left(\sum_{k=0}^{n} \binom{n}{k} \binom{n-k}{j-k} f^{(k)}(0) x^{j-k} \right) y^{n-j} \\
&= \sum_{j=0}^{n} \left(\sum_{k=0}^{n} \binom{n}{j} \binom{j}{k} f^{(k)}(0) x^{j-k} \right) y^{n-j} \\
&= \sum_{j=0}^{n} \binom{n}{j} \left(\sum_{k=0}^{j} \binom{j}{k} f^{(k)}(0) x^{j-k} \right) y^{n-j} \\
&= \sum_{j=0}^{n} \binom{n}{j} g_j^*(x) y^{n-j}.
\end{aligned}
$$

When compared with Example 18, this formula gives the impression that the reversed Jensen polynomials are successive derivatives. This is in fact the case, up to a constant multiple. For every $n \in \mathbb{N}_0$,

$$
\begin{aligned}
\frac{d}{dx} g_n^*(x) &= \frac{d}{dx} \sum_{k=0}^n \binom{n}{k} \gamma_{n-k} x^k \\
&= \sum_{k=0}^n \frac{n!}{(k-1)!(n-k)!} \gamma_{n-k} x^{k-1} \\
&= n \sum_{k=1}^{n-1} \frac{(n-1)!}{k!((n-1)-k)!} \gamma_{(n-1)-k} x^k \\
&= n g_{n-1}^*(x).
\end{aligned}
$$

Hence, using the same definition as P. Appell [3, (1880)], reversed Jensen polynomials are equivalent to *Appell sequences.*

Theorem 21 ([34], [44], [90, p. 133])**.** *The generating function for the Jensen polynomials and reversed Jensen polynomials, associated to the formal function $f(x)$ are,*

$$
e^t f(xt) = \sum_{n=0}^\infty \frac{g_n(x)}{n!} t^n \qquad and \qquad e^{xt} f(t) = \sum_{n=0}^\infty \frac{g_n^*(x)}{n!} t^n. \tag{1.2.1}
$$

Proof. Both of these formulas easily follow by performing a Cauchy product [2, p. 204]. Assuming $f(x) := \sum_{n=0}^\infty \frac{\gamma_n}{n!} x^n$, we have,

$$
e^t f(xt) = \left(\sum_{n=0}^\infty \frac{1}{n!} t^n \right) \left(\sum_{n=0}^\infty \frac{\gamma_n x^n}{n!} t^n \right) = \sum_{n=0}^\infty \left(\sum_{k=0}^n \frac{1}{(n-k)!} \frac{\gamma_k x^k}{k!} \right) t^n = \sum_{n=0}^\infty \frac{g_n(x)}{n!} t^n
$$

and

$$
e^{xt} f(t) = \left(\sum_{n=0}^\infty \frac{x^n}{n!} t^n \right) \left(\sum_{n=0}^\infty \frac{\gamma_n}{n!} t^n \right) = \sum_{n=0}^\infty \left(\sum_{k=0}^n \frac{x^{n-k}}{(n-k)!} \frac{\gamma_k}{k!} \right) t^n = \sum_{n=0}^\infty \frac{g_n^*(x)}{n!} t^n \qquad \square
$$

Example 22. It is interesting to note that, given an entire function, $f(x) \in \mathscr{L} - \mathscr{P}$, if the associated Jensen polynomials of $f(x)$ are orthogonal (see Definition 27), then they must be the Laguerre polynomials [28] (up to an affine transformation, see Definition 35). Similarly, if the associated reversed Jensen polynomials (Appell polynomials) are orthogonal, then they must be the Hermite polynomials [78] (up to an affine

transformation). This follows, by manipulating (1.2.1), yielding,

$$\sum_{n=0}^{\infty} \frac{g_n(x)}{n!} t^n = e^t f(xt) = \left(\frac{d}{dt} \left(e^t f(xt) \big|_{x=w/t} \right) \right) \Big|_{w=xt} = \sum_{n=0}^{\infty} \frac{-x g'_{n+1}(x) + (n+1) g_{n+1}(x)}{(n+1)!} t^n$$

and

$$\sum_{n=0}^{\infty} \frac{g_n^*(x)}{n!} t^n = e^{xt} f(t) = \frac{1}{t} \frac{d}{dx} e^{xt} f(t) = \sum_{n=0}^{\infty} \frac{(g_{n+1}^*)'(x)}{(n+1)!} t^n.$$

Hence, for each $n \in \mathbb{N}$, $x g'_n(x) = n g_n(x) - n g_{n-1}(x)$ and $(g_n^*)'(x) = n g_{n-1}^*(x)$, which are unique (up to an affine transformation) differential recursive formulas, with respect to all orthogonal polynomials, for the Laguerre and Hermite polynomials [28, 78], respectively.

From Section 1.1, if we take $f(x) = e^x$, then the Jensen polynomials associated to $f(x)$, $g_n(x) = (1+x)^n$, are precisely the polynomials that showed $f(x) \in \mathscr{L} - \mathscr{P}$. In particular, $g_n\left(\frac{x}{n}\right) \to f(x)$ locally uniformly on compact subsets of $\mathbb{C}$. This observation was first established by J. Jensen, showing the strong connection between Jensen polynomials and the Laguerre-Pólya class. Jensen's proof relies on what is now called the Hermite-Poulain Theorem (see Theorem 77) and the observation that, since $\frac{d^k}{dx^k} x^n = \binom{n}{k} k! x^{n-k}$ for every $n, k \in \mathbb{N}_0$, then the nth reversed Jensen polynomial can be calculated by,

$$f\left(\frac{d}{dx}\right) x^n = \left(\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} \frac{d^k}{dx^k} \right) x^n = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} \binom{n}{k} k! x^{n-k} = \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(0) x^{n-k} = g_n^*(x).$$

Interestingly, the above can be used to reaffirm the formula of Example 20,

$$g_n^*(x+y) = f\left(\frac{d}{dx}\right) (x+y)^n = f\left(\frac{d}{dx}\right) \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} = \sum_{k=0}^{n} \binom{n}{k} g_k^*(x) y^{n-k}.$$

Theorem 23 (J. Jensen [67, (1913)], [33, Lemma 2.2]). *Given any arbitrary entire function, $f(x)$, then the scaled Jensen polynomials $g_n\left(\frac{x}{n}\right) \to f(x)$ locally uniformly on compact subsets of $\mathbb{C}$. Moreover, $f(x) \in \mathscr{L} - \mathscr{P}$ if and only if $g_n(x) \in \mathscr{L} - \mathscr{P}$ for every $n \in \mathbb{N}_0$.*

For more properties and relations of the Jensen or reversed Jensen polynomials see the works of Y. Cheikh and D. Dimitrov [28], T. Craven and G. Csordas [34], T. Craven and G. Csordas [33], and G. Csordas and J. Williams [46].

Definition 24. Given two polynomials, $f(x)$ and $g(x)$, we define the *Wronskian* of $f(x)$ and $g(x)$ as,

$$W[f(x), g(x)] := f(x)g'(x) - f'(x)g(x).$$

Definition 25. Let $p(x)$ and $q(x)$ be non-zero real polynomials, with real zeros $\alpha_1, \ldots, \alpha_n$ and $\beta_1, \ldots, \beta_m$, respectively. Suppose $p(x)$ and $q(x)$ are within a degree of each other; i.e., $|n - m| \leq 1$. We say that $p(x)$ and $q(x)$ have *interlacing* zeros if one of the following holds:

1. $\beta_1 \leq \alpha_1 \leq \cdots \leq \alpha_n \leq \beta_{n+1}$, $m = n + 1$,

2. $\alpha_1 \leq \beta_1 \leq \cdots \leq \beta_m \leq \alpha_{m+1}$, $n = m + 1$,

3. $\beta_1 \leq \alpha_1 \leq \cdots \leq \beta_n \leq \alpha_n$, $n = m$, or

4. $\alpha_1 \leq \beta_1 \leq \cdots \leq \alpha_n \leq \beta_n$, $n = m$.

We define the zero polynomial to be interlacing with any other hyperbolic polynomial. However, non-zero real constants only interlace with other real constants or real linear polynomials. The phrase, *strict interlacing*, is used when $p(x)$ and $q(x)$ have no zeros in common. Furthermore, if $W[p(x), q(x)] \geq 0$ on all of $\mathbb{R}$, then we say that $p(x)$ and $q(x)$ have *properly interlacing* zeros or are in *proper position* and write $p(x) \ll q(x)$. Again, the zero polynomial has proper interlacing zeros with any other hyperbolic polynomial.

The definition of proper interlacing, Definition 25, comes from the fact that if $p(x)$ and $q(x)$ have interlacing zeros then either $W[p(x), q(x)] \geq 0$ or $W[p(x), q(x)] \leq 0$ on all of $\mathbb{R}$ (see [16, 17]). Hence, proper interlacing can be thought of as a "sign-convention". In particular, if $\deg(p(x)) = n$ and $\deg(q(x)) = n + 1$, then $p(x) \ll q(x)$ if and only if the leading coefficients of $p(x)$ and $q(x)$ are of the same sign and $p(x)$ and $q(x)$ have interlacing zeros. If $p(x)$ and $q(x)$ have the same degree, then proper position is dictated by the sign of the leading coefficients and the polynomial that has the largest zero.

Example 26. To clarify the subtleties of proper position, we present the following examples:

$$
\begin{array}{lll}
0 \ll x + 1, & 0 \ll x^2 - 1, & 0 \ll -x - 1, \\
x + 1 \ll 0, & x^2 - 1 \ll 0, & -x - 1 \ll 0, \\
5 \ll x + 1, & 5 \not\ll x^2 - 1, & 5 \not\ll -x - 1, \\
x \ll x - 1, & x \ll x, & x \not\ll x + 1, \\
x \not\ll -x + 1, & x \ll -x, & \text{and} \quad x \ll -x - 1.
\end{array}
$$

Definition 27. Let $\{p_k(x)\}_{k=0}^{\infty}$ be a simple sequence of real polynomials. Suppose there is a positive real integrable function, $w : [a,b] \to \mathbb{R}^+$, $a < b$ (a or b could be infinity), such that the inner product,

$$\langle p_n(x), p_m(x) \rangle := \int_a^b w(x) p_n(x) p_m(x) dx = 0,$$

for every $n \neq m$. The function $w(x)$ is referred to as a *weight function* for the sequence, $\{p_k(x)\}_{k=0}^{\infty}$, and we call the sequence, $\{p_k(x)\}_{k=0}^{\infty}$, an *orthogonal basis* for $\mathbb{R}[x]$.

Theorem 28 ([89, pp. 16, 200]). *Given an orthogonal basis, $\{p_k(x)\}_{k=0}^{\infty}$, with respect to the weight function $w : [a,b] \to \mathbb{R}^+$, then each $p_n(x)$ has only real simple zeros in the interval (a,b). Furthermore, for each $n \in \mathbb{N}_0$, $p_n(x)$ and $p_{n+1}(x)$ have strictly interlacing zeros.*

Definition 29 ([90]). Given parameters, $\alpha, \beta \in \mathbb{R}$, we define the three *classical orthogonal polynomial* families, the *Hermite*, *Laguerre* (with parameter $\alpha > -1$), and *Jacobi* (with parameters $\alpha, \beta > -1$) polynomials:

$$H_n(x) = \sum_{k=0}^{[n/2]} \binom{n}{k} \binom{n-k}{k} k! (-1)^k (2x)^{n-2k},$$

$$L_n^{(\alpha)}(x) = \sum_{k=0}^{n} \binom{n+\alpha}{k+\alpha} \frac{(-1)^k}{k!} x^k, \quad \text{and}$$

$$P_n^{(\alpha,\beta)}(x) = \sum_{k=0}^{n} \binom{n+\alpha}{n-k} \binom{n+\beta}{k} \left(\frac{x-1}{2}\right)^k \left(\frac{x+1}{2}\right)^{n-k}.$$

When $\alpha = 0$, we denote $L_n^{(\alpha)}(x)$ as $L_n(x)$ and call these the classical Laguerre polynomials. Special cases of the Jacobi polynomials, include the *ultraspherical* (or *Gegenbauer*) ($\alpha = \beta$), *Chebyshev* ($\alpha = \beta = \pm\frac{1}{2}$), and *Legendre* ($\alpha = \beta = 0$, denoted simply as $P_n(x)$). For further study of polynomial families and sequences we refer the reader to several comprehensive treatments, [29, 51, 74, 83, 89, 90, 101, 102] and the references therein.

Throughout the study of these families of polynomials (and analytic functions) what one finds is an assortment of named patterns, like the Rodrigues formulas, the three-term recurrence relations, the differential formulations, the differential recurrence relations, the pure recurrence equations, the generating functions, the orthogonality weight functions, etc. Since our primary focus is in studying the classical orthogonal polynomials, we state a few of the formulas that will be needed in the later chapters. We first provide a few preliminary definitions. In particular, the Bessel functions are particularly important to many of the generating functions for the classical orthogonal polynomials.

Definition 30 ([4]). The *Pochhammer symbol*, $(\alpha)_n$, is defined as $(\alpha)_n = \alpha(\alpha+1)\cdots(\alpha+(n-1))$. The symbol $\Gamma[x]$ will be used to denote the *Gamma function*. In particular, it is well known that $\Gamma[n] = (n-1)!$ for $n \geq 1$, moreover $(1+\alpha)_n := \dfrac{\Gamma(1+\alpha+n)}{\Gamma(1+\alpha)}$.

Definition 31. For every $\alpha > -1$, we define the *Bessel function* [90, p. 109] of parameter α by

$$J_\alpha(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n+\alpha}n!\Gamma(1+n+\alpha)}x^{2n+\alpha}.$$

In particular,

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{4^n(n!)^2}x^{2n}.$$

It is known that $J_\alpha(x)$ has only real zeros for $\alpha > -1$ (see [28]).

Theorem 32 ([90, Chapter 11]). *The Hermite polynomials satisfy the following:*

1. $\left(-\dfrac{1}{2}D^2 + 2xD\right)H_n(x) = nH_n(x)$, *where* $D := \dfrac{d}{dx}$,

2. $H_n(x) = (-1)^n e^{x^2} D^n e^{-x^2}$ *and* $H_n(x/2) = e^{-D^2}x^n$ *(Rodrigues formula)*,

3. $H_n(x) = 2xH_{n-1}(x) - 2(n-1)H_{n-2}(x)$, *other similar relations hold,*

   (a) $H_n(-x) = (-1)^n H_n(x)$,

   (b) $H'_n(x) = 2nH_{n-1}(x)$,

   (c) $H_n(x) = 2xH_{n-1}(x) - H'_{n-1}(x) = (2x - D)H_{n-1}(x)$,

4. $e^{2xt-t^2} = \sum_{n=0}^{\infty} \dfrac{H_n(x)}{n!}t^n$, *from here,* $H_n(x) = \dfrac{d^n}{dt^n}e^{2xt-t^2}\Big|_{t=0}$, *and*

5. $w(x) = e^{-x^2}$, *from here,* $\displaystyle\int_{-\infty}^{\infty} e^{-x^2}H_n(x)H_m(x)dx = 0$ *for* $n \neq m$.

Theorem 33 ([90, Chapter 12]). *The Laguerre polynomials with parameter $\alpha > -1$ satisfy the following:*

1. $\left(-xD^2 + (x - (1+\alpha))D\right)L_n^{(\alpha)}(x) = nL_n^{(\alpha)}(x)$, *where* $D := \dfrac{d}{dx}$,

2. $L_n^{(\alpha)}(x) = \dfrac{1}{n!}x^{-\alpha}e^x D^n e^{-x}x^{n+\alpha}$ *(Rodrigues formula)*,

3. $nL_n^{(\alpha)}(x) = (2n - 1 + \alpha - x)L_{n-1}^{(\alpha)}(x) - (n - 1 + \alpha)L_{n-2}^{(\alpha)}(x)$, *other similar relations hold,*

   (a) $xDL_n^{(\alpha)}(x) = nL_n^{(\alpha)}(x) - (\alpha + n)L_{n-1}^{(\alpha)}(x)$,

   (b) $DL_n^{(\alpha)}(x) = DL_{n-1}^{(\alpha)}(x) - L_{n-1}^{(\alpha)}(x)$,

(c) $DL_n^{(\alpha)}(x) = -L_0^{(\alpha)}(x) - \cdots - L_{n-1}^{(\alpha)}(x),$

4. $\Gamma(1+\alpha)(xt)^{-\alpha/2}e^t J_\alpha(2\sqrt{xt}) = \sum_{n=0}^{\infty} \dfrac{L_n^{(\alpha)}(x)}{(1+\alpha)_n} t^n,$

5. $\dfrac{1}{(1-t)^{1+\alpha}} e^{(-xt)/(1-t)} = \sum_{n=0}^{\infty} L_n^{(\alpha)}(x) t^n,$ and

6. $w(x) = x^\alpha e^{-x}$, from here, $\displaystyle\int_0^\infty x^\alpha e^{-x} L_n^{(\alpha)}(x) L_m^{(\alpha)}(x) dx = 0$ for $n \neq m$.

In particular, the classical Laguerre polynomials, $L_n(x) = L_n^{(0)}(x)$, satisfy:

1. $\left(-xD^2 + (x-1)D\right) L_n(x) = nL_n(x),$

2. $e^t J_0(2\sqrt{xt}) = \sum_{n=0}^{\infty} \dfrac{L_n(x)}{n!} t^n,$ and

3. $\dfrac{1}{(1-t)} e^{(-xt)/(1-t)} = \sum_{n=0}^{\infty} L_n(x) t^n.$

Theorem 34 ([90, Chapters 10 and 16])**.** *The Jacobi polynomials with parameters $\alpha, \beta > -1$ satisfy the following:*

1. $\left((x^2-1)D^2 + ((2+\alpha+\beta)x - (\beta-\alpha))D\right) P_n^{(\alpha,\beta)}(x) = n(n+1+\alpha+\beta)P_n^{(\alpha,\beta)}(x),$ where $D := \frac{d}{dx},$

2. $P_n^{(\alpha,\beta)}(x) = \dfrac{(-1)^n}{2^n n!}(1-x)^{-\alpha}(1+x)^{-\beta} D^n (1-x)^{n+\alpha}(1+x)^{n+\beta}$ *(Rodrigues formula),*

3. $P_n^{(\alpha,\beta)}(-x) = (-1)^n P_n^{(\alpha,\beta)}(x),$

4. $J_\alpha\left(\dfrac{t(x-1)}{2}\right) J_\beta\left(\dfrac{t(x+1)}{2}\right) = \sum_{n=0}^{\infty} \dfrac{P_n^{(\alpha,\beta)}(x)}{(1+\alpha)_n(1+\beta)_n} t^n,$

5. $\left(\dfrac{1}{\rho}\right)\left(\dfrac{1}{1+t+\rho}\right)^\beta \left(\dfrac{2}{1-t+\rho}\right)^\alpha = \sum_{n=0}^{\infty} P_n^{(\alpha,\beta)}(x)t^n$, $\rho = \sqrt{1 - 2xt + t^2}$, and

6. $w(x) = (1-x)^\alpha(1+x)^\beta$, from here $\displaystyle\int_{-1}^1 (1-x)^\alpha(1+x)^\beta P_n^{(\alpha,\beta)}(x) P_m^{(\alpha,\beta)}(x) dx = 0$ for $n \neq m,$

In particular, the Legendre polynomials, $P_n(x) = P_n^{(0,0)}(x)$, satisfy:

1. $\left((x^2-1)D^2 + 2xD\right) P_n(x) = n(n+1)P_n(x),$

2. $\dfrac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)t^n,$

3. $e^{xt} J_0(t\sqrt{x^2-1}) = \sum_{n=0}^{\infty} \dfrac{P_n(x)}{n!} t^n,$ and

4. $J_0\left(\dfrac{t(x-1)}{2}\right) J_0\left(\dfrac{t(x+1)}{2}\right) = \displaystyle\sum_{n=0}^{\infty} \dfrac{P_n(x)}{(n!)^2}t^n.$

The following concept is of key importance in studying orthogonal polynomials, multiplier sequences, and classes of hyperbolicity preserving operators.

Definition 35. Let $\{p_k(x)\}_{k=0}^{\infty}$ be a simple sequence of real polynomials. Another simple sequence of real polynomials, $\{q_k(x)\}_{k=0}^{\infty}$, is called an *affine transformation* of the sequence $\{p_k(x)\}_{k=0}^{\infty}$, if there exists $\{c_k\}_{k=0}^{\infty} \subseteq \mathbb{R}$, $c_k \neq 0$, $\alpha, \beta \in \mathbb{R}$, $\alpha \neq 0$, such that

$$p_k(x) = c_k \cdot q_k(\alpha x + \beta), \ x \in \mathbb{R}.$$

The properties of reflexivity, symmetry, and transitivity clearly hold. Hence, affine transformations define an equivalence relation on the space of sequences of simple real polynomials.

Throughout the history of polynomial theory, many important theorems are stated "up to an affine transformation", see, for example, [29, 37, 48, 60, 61, 70, 85, 102] and the references therein. In particular, A. Piotrowski shows that multiplier sequence classes are equivalent for bases that are affine transformations [85, Lemma 157, p. 145]. Interestingly, in Chapter 4, we will establish a property that distinguishes multiplier sequence classes that arise from affine transformations of the Hermite basis (see Example 127 and Theorems 196 and 197).

1.3 Laguerre-Pólya Classes

In this section we provide a more comprehensive guide to the many properties of the Laguerre-Pólya class, $\mathscr{L} - \mathscr{P}$.

Definition 36. We define the Laguerre-Pólya class, $\mathscr{L} - \mathscr{P}$, to be the set of entire functions that can be locally uniformly approximated on $\mathbb{C}$ by real polynomials with only real zeros. We define $\mathscr{L} - \mathscr{P}^s$ to be the set of entire functions that can be locally uniformly approximated on $\mathbb{C}$ by real polynomials with only real non-positive zeros. We define $\mathscr{L} - \mathscr{P}^a$ to be the set of entire functions that can be locally uniformly approximated on $\mathbb{C}$ by real polynomials with only non-negative zeros. The notation $\mathscr{L} - \mathscr{P}^{sa}$ denotes $\mathscr{L} - \mathscr{P}^s \cup \mathscr{L} - \mathscr{P}^a$. Given $A \subseteq \mathbb{R}$, we will write $\mathscr{L} - \mathscr{P}A$ (similarly, $\mathscr{L} - \mathscr{P}^s A$, $\mathscr{L} - \mathscr{P}^a A$, $\mathscr{L} - \mathscr{P}^{sa} A$) to denote all functions in $\mathscr{L} - \mathscr{P}$ that have zeros restricted to the set A.

In 1950, P. Turán discovered an important inequality for the Legendre polynomials (see [100, 104]) that is similar in nature to many other important inequalities throughout the literature. These inequalities follow the form, $b^2 - ac \geq 0$, where a, b, c can be functions or scalars with some appropriate ordering. Hence, inequalities of said form have been coined *Turán* or *Turán-type inequalities*. One of the fundamental facts concerning the Laguerre-Pólya class, is the demonstration that every function in $\mathscr{L} - \mathscr{P}$ has Taylor coefficients which satisfy Turán inequalities. More generally, the derivative sequence satisfies the Turán inequalities on all of $\mathbb{R}$, (this is occasionally referred to as a *Laguerre inequality*).

Theorem 37. *If* $f(x) := \sum_{k=0}^{\infty} \dfrac{\gamma_k}{k!} x^k \in \mathscr{L} - \mathscr{P}$, *then* $f^{(k)}(x) - f^{(k+1)}(x)f^{(k-1)}(x) \geq 0$ *for every* $x \in \mathbb{R}$, $k \geq 1$. *In particular,* $\gamma_k^2 - \gamma_{k+1}\gamma_{k-1} \geq 0$ *for every* $k \geq 1$.

Proof. The statement trivially holds for constant and linear polynomials. Let $p(x) = a\prod_{k=1}^{n}(x - r_k) \in \mathscr{L} - \mathscr{P} \cap \mathbb{R}[x]$, $n \geq 2$. Then for each $x \in \mathbb{R}$, where $p(x) \neq 0$,

$$p''(x)p(x) - (p'(x))^2 = p(x)^2 \left(\frac{p'(x)}{p(x)}\right)' = p(x)^2 \left(\sum_{k=1}^{n} \frac{1}{x - r_k}\right)' = p(x)^2 \sum_{k=1}^{n} \frac{-1}{(x - r_k)^2} \leq 0.$$

If $p(x) = 0$, then certainly $p''(x)p(x) - (p'(x))^2 \leq 0$. By Theorem 6 the statement extends to all derivatives. Uniform convergence extends the above to all of $\mathscr{L} - \mathscr{P}$. $\qquad\square$

Although the following theorem is an immediate consequence of the above result, we will provide a slightly different proof that will prove useful in future statements.

Theorem 38. *If* $f(x) = \sum_{k=0}^{\infty} a_k x^k \in \mathscr{L} - \mathscr{P}$, *then* $a_k^2 - a_{k+1}a_{k-1} \geq 0$ *for every* $k \geq 1$.

Proof. Suppose $f(x) = a_0 + \cdots + a_{k-1}x^{k-1} + a_k x^k + a_{k+1}x^{k+1} + \cdots + a_n x^n$ is a polynomial with only real zeros. If $a_{k-1}a_{k+1} < 0$ then the Turán inequality holds trivially. Thus, we suppose $a_{k-1}a_{k+1} \geq 0$. By Theorem 6, the derivative has only real zeros,

$$f^{(k-1)}(x) = a_{k-1}(k-1)! + a_k \frac{k!}{1!}x + a_{k+1}\frac{(k+1)!}{2!}x^2 + \cdots + a_n \frac{n!}{(k-1)!}x^{n-(k-1)}. \qquad (1.3.1)$$

We now reverse (see Definition 16) equation (1.3.1), to obtain another polynomial with only real zeros,

$$p(x) := a_{k-1}(k-1)!x^{n-(k-1)} + a_k \frac{k!}{1!}x^{n-k} + a_{k+1}\frac{(k+1)!}{2!}x^{n-k-1} + \cdots + a_n \frac{n!}{(k-1)!}.$$

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Thus, applying Theorem 6 again,

$$p^{(n-k-1)}(x) = a_{k-1}(k-1)!\frac{(n-(k-1))!}{2!}x^2 + a_k\frac{k!}{1!}\frac{(n-k)!}{1!}x + a_{k+1}\frac{(k+1)!}{2!}(n-k-1)!,$$

has only real zeros; which has a positive discriminant, that is

$$\left(a_k\frac{k!}{1!}\frac{(n-k)!}{1!}\right)^2 - 4\left(a_{k-1}(k-1)!\frac{(n-(k-1))!}{2!}\right)\left(a_{k+1}\frac{(k+1)!}{2!}(n-k-1)!\right) \geq 0.$$

Since by assumption, $a_{k-1}a_{k+1} \geq 0$, then

$$a_k^2 - a_{k-1}a_{k+1} \geq a_k^2 - a_{k-1}a_{k+1}\frac{(n-k+1)(k+1)}{k(n-k)} \geq 0.$$

Thus the Turán inequalities hold for hyperbolic polynomials. Local uniform convergence extends the Turán inequalities to every function in $\mathscr{L} - \mathscr{P}$. $\square$

Corollary 39. *Suppose* $f(x) = a_0 + \cdots + a_k x^k + a_{k+1}x^{k+1} + a_{k+2}x^{k+2} + \cdots + a_{k+m}x^{k+m} + \cdots$ *is a real entire function. If* $a_k \cdot a_{k+2} > 0$ *and* $a_{k+1} = 0$, *then* $f(x) \notin \mathscr{L} - \mathscr{P}$. *If* $a_k \cdot a_{k+m} \neq 0$, $m > 2$, *and* $a_{k+1}, \ldots, a_{k+(m-1)} = 0$, *then* $f(x) \notin \mathscr{L} - \mathscr{P}$.

Proof. The proof follows by employing a similar technique to the proof of Theorem 38 and noting that $Ax^2 + B$ and $Cx^m + D$, $m > 2$, have two non-real zeros for all $A, B, C, D \in \mathbb{R}$, $A \cdot B > 0$, $C \cdot D \neq 0$. Hence, polynomials satisfy the statements above. We now use uniform convergence to extend the result to functions in $\mathscr{L} - \mathscr{P}$. $\square$

Remark 40. It is interesting to note that Corollary 39 tells us that non-constant functions in $\mathscr{L} - \mathscr{P}$ cannot have any negative local maximums or positive local minimums (see, for example, [105, Example 37]).

Remark 41. The "s" in $\mathscr{L} - \mathscr{P}^s$ stands for Taylor coefficients of the *same sign*. Likewise the "a" in $\mathscr{L} - \mathscr{P}^a$ stands for Taylor coefficients of *alternating sign*. This follows from the following result.

Theorem 42. *If* $p(x) = a_0 x^n + a_1 x^{n+1} + \cdots + a_m x^{n+m}$, $a_0 a_m \neq 0$, *is a polynomial in* $\mathscr{L} - \mathscr{P}$, *then*

1. $p(x) \in \mathscr{L} - \mathscr{P}^s$ *if and only if* $a_k \cdot a_{k+1} > 0$ *for* $0 \leq k \leq m-1$, *and*

2. $p(x) \in \mathscr{L} - \mathscr{P}^a$ *if and only if* $a_k \cdot a_{k+1} < 0$ *for* $0 \leq k \leq m-1$.

If $f(x) = x^n \sum_{k=0}^{\infty} a_k x^k$, $a_0 \neq 0$, $n \in \mathbb{N}_0$, *is any transcendental real function in* $\mathscr{L} - \mathscr{P}$, *then*

1. $f(x) \in \mathcal{L} - \mathcal{P}^s$ if and only if $a_k \cdot a_{k+1} > 0$ for $k \geq 0$, and

2. $f(x) \in \mathcal{L} - \mathcal{P}^a$ if and only if $a_k \cdot a_{k+1} < 0$ for $k \geq 0$.

Proof. If $p(x) \in \mathcal{L} - \mathcal{P}^s$ is a polynomial, then multiplying out the factorization of $p(x)$ yields coefficients all of the same sign (either all positive or all negative). This and the fact that, $f(x) \in \mathcal{L} - \mathcal{P}^s$ if and only if $f(-x) \in \mathcal{L} - \mathcal{P}^a$, establishes the theorem in the polynomial cases. The transcendental cases follow with non-strict inequalities from local uniform convergence. The strictness of the inequalities is established by Corollary 39. $\qquad \square$

Example 43. At first, one might suspect that every function in $\mathcal{L} - \mathcal{P}$ with only real non-positive zeros is in $\mathcal{L} - \mathcal{P}^s$. However, this is not the case. The classic example is to first note that

$$\frac{1}{\Gamma(x)} = ze^{\gamma z} \prod_{k=1}^{\infty} \left(1 + \frac{z}{k}\right) e^{-z/k} \in \mathcal{L} - \mathcal{P},$$

where γ is Euler-Mascheroni constant [50, (1734)]. By providing the first few Taylor coefficients,

$$\frac{1}{\Gamma(x)} = x + (0.5772\ldots)x^2 + (-0.6558\ldots)x^3 + \cdots,$$

we can apply Theorem 42 to see that $\dfrac{1}{\Gamma(x)}$ is not in $\mathcal{L} - \mathcal{P}^{sa}$.

The observations above indicate $\mathcal{L} - \mathcal{P}^{sa} \subsetneq \mathcal{L} - \mathcal{P}$; in fact every function in $\mathcal{L} - \mathcal{P}^{sa}$ has growth order less than or equal to one [74]. Similar to Theorem 7 we provide a representation of entire functions in $\mathcal{L} - \mathcal{P}^{sa}$ in the following theorem.

Theorem 44. A function, $f(x)$, is in $\mathcal{L} - \mathcal{P}^s$ if and only if there exists, $0 \leq \omega \leq \infty$, $b, c \in \mathbb{R}$, $b \geq 0$, $\{x_k\}_{k=1}^{\omega} \subseteq \mathbb{R}$, $x_k > 0$, $\sum_{k=1}^{\omega} \dfrac{1}{|x_k|} < \infty$, and a non-negative integer m, such that

$$f(x) = cx^m e^{bx} \prod_{k=1}^{\omega} \left(1 + \frac{x}{x_k}\right).$$

A function, $f(x)$, is in $\mathcal{L} - \mathcal{P}^a$ if and only if there exists, $0 \leq \omega \leq \infty$, $b, c \in \mathbb{R}$, $b \geq 0$, $\{x_k\}_{k=1}^{\omega} \subseteq \mathbb{R}$, $x_k < 0$, $\sum_{k=1}^{\omega} \dfrac{1}{|x_k|} < \infty$, and a non-negative integer m, such that

$$f(x) = cx^m e^{-bx} \prod_{k=1}^{\omega} \left(1 + \frac{x}{x_k}\right).$$

Example 45. It is interesting to note that if $f(x) = \sum_{k=0}^{\infty} a_k x^k \in \mathscr{L} - \mathscr{P}^{sa}$ and $a_n a_m \neq 0$ for some $n, m \in \mathbb{N}_0$, $n < m$, then $a_k \neq 0$ for $n \leq k \leq m$, a property that some functions in $\mathscr{L} - \mathscr{P}$ fail to have. Notice that $e^x, e^{-x}, e^{-x^2}, \sin(x), \cos(x), J_\alpha(x) \in \mathscr{L} - \mathscr{P}$, $\alpha > -1$. We see that $e^{-x^2}, \cos(x), \sin(x)$ serve as simple examples of functions in $\mathscr{L} - \mathscr{P}$ with infinitely many zero Taylor coefficients.

If $p(x) \in \mathscr{L} - \mathscr{P}^s$ and $q(x) \in \mathscr{L} - \mathscr{P}^a$ are non-constant polynomials, then certainly $p(x)q(x) \notin \mathscr{L} - \mathscr{P}^{sa}$ (see Theorem 42). However, the transcendental version of this statement is not so clearly understood. For example, $e^x \in \mathscr{L} - \mathscr{P}^s$, $(x-3)e^{-x} \in \mathscr{L} - \mathscr{P}^a$ and also $e^x((x-3)e^{-x}) \in \mathscr{L} - \mathscr{P}^a$. This leads to the following open problem.

Problem 46. Let $f(x) \in \mathscr{L} - \mathscr{P}^s$ be a transcendental entire function. Are all but a finite number of Taylor coefficients of $e^{\alpha x} f(x)$, $\alpha < 0$, non-zero, where $e^{\alpha x} f(x)$ is transcendental? This question is inspired by a paper of T. Forgács and A. Piotrowski [54].

The study of Taylor coefficients from functions in $\mathscr{L} - \mathscr{P}^{sa}$ has been of interest to many authors. In particular, the recent work of T. Craven and G. Csordas [34] establishes important properties of Taylor coefficients for some functions in $\mathscr{L} - \mathscr{P}^s$.

Theorem 47 ([34, Lemma 2.2]). *Let* $\Phi(x) = \sum_{k=0}^{\infty} \dfrac{\gamma_k}{k!} x^k$ *be a transcendental entire function in the Laguerre-Pólya class. Suppose that the product representation of* $\Phi(x)$ *has the form*

$$\Phi(x) = cx^m e^{\sigma x} \prod_{n=1}^{\omega} \left(1 + \frac{x}{x_n}\right), \qquad 0 \leq \omega \leq \infty,$$

where $\sigma \geq 0$, $x_n > 0$, $c > 0$, $\sum_{n=0}^{\infty} \dfrac{1}{x_n} < \infty$ *and where* m *is a non-negative integer. Then* $\sigma \geq 1$ *if and only if* $0 \leq \gamma_0 \leq \gamma_1 \leq \gamma_2 \leq \cdots$.

Corollary 48. *If* $f(x) = \sum_{k=0}^{\infty} \dfrac{\gamma_k}{k!} x^k \in \mathscr{L} - \mathscr{P}^s$, *where* $\{\gamma_k\}_{k=0}^{\infty}$ *is a non-negative increasing sequence, then* $e^{\alpha x} f(x) \in \mathscr{L} - \mathscr{P}^s$ *for all* $\alpha \geq -1$. *In particular, if* $e^{\alpha x} f(x)$, $\alpha \geq -1$, *is a transcendental entire function, then* $e^{\alpha x} f(x)$ *has only a finite number of Taylor coefficients that are zero (cf. Problem 46).*

Example 49. Consider the modified Bessel function (see Definition 31),

$$f(x) := J_0(2\sqrt{-x}) = \sum_{k=0}^{\infty} \frac{1}{k!k!} x^k \in \mathscr{L} - \mathscr{P},$$

see [90, p. 108]. We calculate,

$$e^{-x} f(x) = \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} \binom{k}{j} \frac{1}{j!} (-1)^{k-j} \right) \frac{x^k}{k!}$$

$$= 1 - \frac{1}{4} x^2 + \frac{1}{9} x^3 - \frac{5}{192} x^4 + \frac{7}{1800} x^5 - \frac{37}{103680} x^6 + \frac{17}{2116800} x^7 + \frac{887}{232243200} x^8 + \cdots .$$

We see that $e^{-x} f(x) \in \mathscr{L} - \mathscr{P} \setminus \mathscr{L} - \mathscr{P}^{sa}$. A special case of Problem 46 is to simply ask, are all but a finite number of Taylor coefficients of $e^{-x} f(x)$ non-zero?

For the reader's convenience, we provide a summary of some basic combinatorics that will be important.

Theorem 50 ([92, p. 49]). *Let $\{\alpha_k\}_{k=0}^{\infty}$ be a sequence of real numbers and set,*

$$\beta_n = \sum_{k=0}^{n} \binom{n}{k} \alpha_k, \quad n \in \mathbb{N}_0.$$

Then, for all $n \in \mathbb{N}_0$,

$$\alpha_n = \sum_{k=0}^{n} \binom{n}{k} \beta_k (-1)^{n-k}.$$

In particular,

$$e^x \sum_{n=0}^{\infty} \frac{\alpha_n}{n!} x^n = \sum_{k=0}^{\infty} \frac{\beta_n}{n!} x^n \quad and \quad e^{-x} \sum_{k=0}^{\infty} \frac{\beta_n}{n!} x^n = \sum_{k=0}^{\infty} \frac{\alpha_n}{n!} x^n.$$

Similarly, if $\{g_k^*(x)\}_{k=0}^{\infty}$ are the reversed Jensen polynomials associated with $\{\gamma_k\}_{k=0}^{\infty}$, then for every $n \in \mathbb{N}_0$,

$$\gamma_n = \sum_{k=0}^{n} \binom{n}{k} g_k^*(-1) \quad and \quad g_n^*(-1) = \sum_{k=0}^{n} \binom{n}{k} \gamma_k (-1)^{n-k}. \tag{1.3.2}$$

1.4 Classical Multiplier Sequences

In the celebrated work of G. Pólya and J. Schur [87, (1914)], a complete characterization was given for sequences of real numbers, $\{\gamma_k\}_{k=0}^{\infty}$, with the property that for every real polynomial, $p(x) = a_0 + a_1 x + \cdots + a_n x^n$, with only real zeros, the polynomial,

$$\gamma_0 a_0 + \gamma_1 a_1 + \cdots \gamma_n a_n x^n,$$

also has only real zeros. In this section, we provide an overview of multiplier sequences and present several important facts and results related to them.

Definition 51. Let $\{\gamma_k\}_{k=0}^\infty$ be a sequence of real numbers. We say that $\{\gamma_k\}_{k=0}^\infty$ is a *multiplier sequence* if for every hyperbolic polynomial, $p(x) = a_0 + a_1 x + \cdots a_n x^n$, the polynomial

$$\gamma_0 a_0 + \gamma_1 a_1 + \cdots + \gamma_n a_n x^n,$$

is also a hyperbolic polynomial.

Example 52. It is not hard to show that $\{r^k\}_{k=0}^\infty$ is a multiplier sequence for any $r \in \mathbb{R}$. Furthermore, constant sequences, $\{\alpha\}_{k=0}^\infty$, $\alpha \in \mathbb{R}$, are also multiplier sequences. Moreover, given $\alpha, \beta \in \mathbb{R}$, the following are also multiplier sequences,

$$\{0, 0, \ldots, 0, \alpha, \beta, 0, \ldots\} \tag{1.4.1}$$

and

$$\{0, 0, \ldots, 0, \alpha, 0, \ldots\}. \tag{1.4.2}$$

The sequences (1.4.1) and (1.4.2) are referred to as *trivial multiplier sequences*.

Theorem 53 ([33], [74, p. 341], [85, Proposition 45, p. 43]). *Let $\{\gamma_k\}_{k=0}^\infty$ be a multiplier sequence. Then each of the following assertions hold:*

1. *if for some $n, m \in \mathbb{N}$, $n < m$, $\gamma_n \gamma_m \neq 0$, then $\gamma_k \neq 0$ for $n \leq k \leq m$,*

2. *$\{|\gamma_k|\}_{k=0}^\infty$, $\{-|\gamma_k|\}_{k=0}^\infty$, $\{(-1)^k|\gamma_k|\}_{k=0}^\infty$, and $\{(-1)^{k+1}|\gamma_k|\}_{k=0}^\infty$ are also multiplier sequences, furthermore one of these is the original sequence, $\{\gamma_k\}_{k=0}^\infty$,*

3. *the sequence $\{\gamma_k\}_{k=m}^\infty$ is also a multiplier sequence, where $m \in \mathbb{N}$,*

4. *$\{\lambda_k \gamma_k\}_{k=0}^\infty$ is also a multiplier sequence, where $\{\lambda_k\}_{k=0}^\infty$ is any other multiplier sequence, and*

5. *the Turán inequalities are satisfied; i.e., $\gamma_{k+1}^2 - \gamma_k \gamma_{k+2} \geq 0$ for every $k \geq 0$.*

Proof. Statement (2) follows from the fact that by definition, $\gamma_{k+2} x^{k+2} - \gamma_k x^k \in \mathscr{L} - \mathscr{P}$ for every $k \in \mathbb{N}_0$, hence $\gamma_k \gamma_{k+2} \geq 0$ for every $k \in \mathbb{N}_0$. Statement (3) follows since applying the sequence $\{\gamma_k\}_{k=0}^\infty$ to the polynomial $p(x)$ is equivalent to applying the sequence $\{\gamma_k\}_{k=m}^\infty$ to the polynomial $x^m p(x)$. The Turán inequalities, item (4), follow because $\gamma_k x^k + 2\gamma_{k+1} x^{k+1} + \gamma_{k+2} x^{k+2} \in \mathscr{L} - \mathscr{P}$, and thus has a non-negative discriminant. Statement (4) is obvious. Statement (1) follows from statement (2) and by applying the sequence to $(1-x)^n$ or $(1+x)^n$ and using Theorem 39. □

Comparison of Theorems 38 and 42 with the above reveals many striking similarities. Indeed, in 1914, G. Pólya and J. Schur [87] gave a remarkable characterization, demonstrating the direct correspondence between entire functions in $\mathscr{L} - \mathscr{P}^{sa}$ and multiplier sequences.

Theorem 54 (G. Pólya and J. Schur [87])**.** *Let $\{\gamma_k\}_{k=0}^{\infty}$ be a sequence of real numbers. Then $\{\gamma_k\}_{k=0}^{\infty}$ is a multiplier sequence if and only if*

$$\sum_{k=0}^{\infty} \frac{\gamma_k}{k!} x^k \in \mathscr{L} - \mathscr{P}^{sa}.$$

By Theorem 53 (2), many results in the literature ignore negative or alternating sequences. However, several of our results establish significant differences between $\mathscr{L} - \mathscr{P}^a$ and $\mathscr{L} - \mathscr{P}^s$ (see, for example, Theorem 204 and Example 206). The sensitivity of the two classes, $\mathscr{L} - \mathscr{P}^s$ and $\mathscr{L} - \mathscr{P}^a$, can also be seen in the following theorem, which holds for sequences arising from $\mathscr{L} - \mathscr{P}^s$, but not for sequences arising from $\mathscr{L} - \mathscr{P}^a$.

Theorem 55 (T. Craven and G. Csordas [34])**.** *Let $\{\gamma_k\}_{k=0}^{\infty}$ be a positive or negative classical multiplier sequence. Then, for each $m \in \mathbb{N}_0$,*

$$\left\{ \sum_{k=0}^{n} \binom{n}{k} \gamma_{m+k} \right\}_{n=0}^{\infty} \quad and \quad \left\{ \sum_{k=0}^{m} \binom{m}{k} \gamma_{n+k} \right\}_{n=0}^{\infty},$$

are also positive or negative classical multiplier sequences, respectively.

Proof. For the first sequence, using a Cauchy product, we calculate

$$\sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \binom{n}{k} \gamma_{m+k} \right) \frac{x^n}{n!} = e^x D^m \sum_{n=0}^{\infty} \frac{\gamma_n}{n!} x^n \in \mathscr{L} - \mathscr{P}^s.$$

For the second sequence, using two Cauchy products, we calculate

$$\sum_{n=0}^{\infty} \left(\sum_{k=0}^{m} \binom{m}{k} \gamma_{n+k} \right) \frac{x^n}{n!} = e^{-x} D^m e^x \sum_{n=0}^{\infty} \frac{\gamma_n}{n!} x^n \in \mathscr{L} - \mathscr{P}^s. \qquad \square$$

Example 56. We show that Theorem 55 does not hold for $\mathscr{L} - \mathscr{P}^a$. Consider the following function in $\mathscr{L} - \mathscr{P}^a$, $J_0(2\sqrt{x}) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{x^k}{k!}$, which is obtained by application of the multiplier sequence $\left\{ \frac{(-1)^k}{k!} \right\}_{k=0}^{\infty}$ to the function e^x. The sequence,

$$\{a_n\}_{n=0}^{\infty} = \left\{ \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{k!} \right\}_{n=0}^{\infty},$$

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has the form,

$$\{a_n\}_{n=0}^{\infty} = \left\{ 1, 0, -\frac{1}{2}, -\frac{2}{3}, -\frac{5}{8}, \ldots, \frac{887}{5760}, \ldots \right\}.$$

Hence, $\{a_n\}_{n=0}^{\infty}$ is not a multiplier sequence (see Theorem 53).

As opposed to analyzing the Taylor coefficients of functions in $\mathscr{L} - \mathscr{P}^{sa}$, a much more natural method of producing sequences is to evaluate a function on the non-negative integers; i.e., $\{f(k)\}_{k=0}^{\infty}$ for some function $f(x)$. To use a function in this way is referred to as *interpolation*. Despite the complete characterization given by G. Pólya and J. Schur, the following is still an unsolved problem in the literature.

Problem 57. Produce a class of real entire functions, M, so that every multiplier sequence can be interpolated by some function in M and every interpolation of a function in M is a multiplier sequence. Furthermore, find a minimal class M with respect to order and type.

Problem 57 has received a great deal of attention in recent years, for example the recent work of M. Chasse [27] provides an in-depth analysis of rational interpolation. See also the fascinating work of T. Craven and G. Csordas [39] on rapidly decreasing sequences. It is interesting to observe the relationship of $\mathscr{L} - \mathscr{P}^{sa}$ to multiplier sequences, as given by Theorem 54, yet the question of interpolation remains a mystery. Let $p(x) \in \mathbb{R}[x]$. Then by Theorem 54 the sequence $\{p(k)\}_{k=0}^{\infty}$ is a multiplier sequence if and only if $\sum_{k=0}^{\infty} \frac{p(k)}{k!} x^k \in \mathscr{L} - \mathscr{P}^{sa}$. It is possible that $p(x)$ need not be hyperbolic and yet $\{p(k)\}_{k=0}^{\infty}$ is a multiplier sequence. For example,

$$\{k^2 + k + 1\}_{k=0}^{\infty},$$

is a multiplier sequence, since $\sum_{k=0}^{\infty} \frac{k^2 + k + 1}{k!} x^k = (x+1)^2 e^x \in \mathscr{L} - \mathscr{P}^{sa}$. While some non-hyperbolic polynomials interpolate a multiplier sequence, the following theorems show that interpolation is still strongly related to $\mathscr{L} - \mathscr{P}^s$.

Theorem 58 ([39, Theorem 1.4]). *If $f(x) \in \mathscr{L} - \mathscr{P}(-\infty, 0]$, then $\{f(k)\}_{k=0}^{\infty}$ is a multiplier sequence.*

Theorem 59 ([39, Proposition 2.2]). *If $p(x) \in \mathbb{R}[x]$ has all its zeros in $(-\infty, n+1)$, for some $n \in \mathbb{N}$, then $\{p(k)k(k-1)(k-2)\cdots(k-n)\}_{k=0}^{\infty}$ is a multiplier sequence.*

Definition 60. We define the nth *Bell polynomial* [94, p. 82] by

$$B_n(x) := e^{-x} \sum_{k=0}^{\infty} \frac{k^n}{k!} x^k = \sum_{n=0}^{n} S_2(n,k) x^k,$$

where $S_2(n,k)$ denotes the Stirling numbers of the second kind (see [88, Section 3, p. 42] or [94, pp. 59-63]).

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In terms of the Bell polynomials, we obtain the following natural characterization of multiplier sequences that can be interpolated by a polynomial.

Theorem 61. *Suppose $p(x) := a_0 + \cdots + a_n x^n$ is a real polynomial with coefficients of the same sign. Then $\{p(k)\}_{k=0}^{\infty}$ is a multiplier sequence if and only if $a_0 B_0(x) + \cdots + a_n B_n(x) \in \mathscr{L} - \mathscr{P}^s$.*

Proof. This theorem follows from the Pólya-Schur Theorem (Theorem 54) and the following calculation,

$$\sum_{k=0}^{\infty} \frac{p(k)}{k!} x^k = \sum_{k=0}^{\infty} \frac{a_0 + a_1 k + \cdots + a_n k^n}{k!} x^k = (a_0 B_0(x) + a_1 B_1(x) + \cdots + a_n B_n(x)) e^x. \qquad \square$$

Problem 62. The above motivates the following open problem. Suppose $T : \mathbb{R}[x] \to \mathbb{R}[x]$ is a linear operator, where $T[B_n(x)] = x^n$ for every $n \in \mathbb{N}$ (by linearity there is only one such operator). Determine the elements of the set,

$$\{T[p(x)] : p(x) \in \mathscr{L} - \mathscr{P}^s\}.$$

If we reverse the operator, $\tilde{T}[x^n] := B_n(x)$, then we can also ask what polynomials does $\tilde{T}$ map to $\mathscr{L} - \mathscr{P}^s$ (see Theorem 229 on calculating the differential representation of $\tilde{T}$)? See also the work of M. Chasse, L. Grabarek, and M. Visontai concerning new properties of the Bell polynomials [26].

1.5   Some Classic Theorems on Polynomials

We now present several classical theorems involving polynomials that have been very influential in our research. Many generalizations and extensions of these theorems can be found in [16, 17, 74, 89].

Theorem 63 (Hermite-Biehler [89, Theorem 6.3.4, p. 197])**.** *Let $f(x)$ and $g(x)$ be real polynomials. Then $f(x) \ll g(x)$ if and only if $f(x) + ig(x)$ is an upper stable polynomial.*

Theorem 64 (Hermite-Kakeya-Obreschkoff [83], [89, Theorem 6.3.8, p. 198])**.** *Let $f(x)$ and $g(x)$ be real polynomials. Then $f(x)$ and $g(x)$ have interlacing zeros if and only if $f(x) + r \cdot g(x) \in \mathscr{L} - \mathscr{P}$ for every $r \in \mathbb{R}$.*

Corollary 65. *Let $f(x)$ and $g(x)$ be real polynomials. Then $f(x) + ig(x)$ is stable if and only if $f(x) + r \cdot g(x) \in \mathscr{L} - \mathscr{P}$ for every $r \in \mathbb{R}$.*

Theorem 66 (Gauss-Lucas [75, p. 22])**.** *Let $f(x)$ be any polynomial in $\mathbb{C}[x]$. If all the zeros of $f(x)$ are in a convex region $K \subseteq \mathbb{C}$, then all the zeros of $f'(x)$ are also in K.*

It seems an extension of Corollary 65 might exist. The author has found the following problem quite interesting while exploring the properties of hyperbolic polynomials.

Problem 67. Let $f(x)$, $g(x)$, and $h(x)$ be real polynomials, where for some $n \in \mathbb{N}$, $\deg(f(x)) = n$, $\deg(g(x)) = n + 1$, and $\deg(h(x)) = n + 2$. Suppose further that $f(x)$, $g(x)$, and $h(x)$ have leading coefficients of the same sign. Show that if $f(x) + g(x) \cdot r + h(x) \cdot r^2 \in \mathscr{L} - \mathscr{P}$ for all $r \in \mathbb{R}$, then $f(x) \ll g(x)$ and $g(x) \ll h(x)$ (see [8]). This question naturally leads to many other intriguing generalizations.

Theorem 68 ([51, Lemma 1.20, p. 13]). *Let $g(x)$ and $f(x)$ be real hyperbolic polynomials, where $\deg(g(x)) < \deg(f(x))$ and $g(x) \ll f(x)$. If $\{a_1, \ldots, a_n\}$ are the zeros of $f(x)$, then there are positive constants, $c_k \geq 0$, $i = 1, 2, \ldots, n$, such that*

$$g(x) = c_1 \frac{f(x)}{x - a_1} + \cdots + c_n \frac{f(x)}{x - a_n}. \tag{1.5.1}$$

Furthermore, every polynomial of the form $(1.5.1)$ is in proper position with $f(x)$.

Theorem 69 (Generalized Malo-Schur-Szegő Composition Theorem [23], [32, Theorem 2.5]). *Let*

$$A(z) = \sum_{k=0}^{m} a_k z^k \quad and \quad B(z) = \sum_{k=0}^{n} b_k z^k$$

be polynomials in $\mathbb{C}[z]$, $a_m b_n \neq 0$, and let

$$C(z) = \sum_{k=0}^{\min\{m,n\}} k! a_k b_k z^k.$$

If $A(z)$ has all its zeros in $S_\alpha := \{z \in \mathbb{C} : \theta_1 < \arg(z) < \theta_1 + \alpha\}$, $\alpha \leq \pi$, $\theta_1 \in \mathbb{R}$, and $B(z)$ has all its zeros in $S_\beta := \{z \in \mathbb{C} : \theta_2 < \arg(z) < \theta_2 + \beta\}$, $\beta \leq \pi$, $\theta_2 \in \mathbb{R}$, then $C(z)$ has all its zeros in $-S_\alpha S_\beta := \{z \in \mathbb{C} : (\theta_1 + \theta_2 + \pi) < \arg(z) < (\theta_1 + \theta_2 + \pi) + \alpha + \beta\}$.

The corollary below can be used to provide a simple proof of the Pólya-Schur characterization of multiplier sequences (Theorem 54) (see [74, Theorem 10, p. 346]).

Corollary 70 ([74, Theorem 7, p. 340]). *If $A(z) = \sum_{k=0}^{m} a_k z^k$ has only real negative zeros and $B(z) = \sum_{k=0}^{n} b_k z^k$ has only real zeros, then*

$$C(z) = \sum_{k=0}^{\min\{m,n\}} k! a_k b_k z^k$$

has only real zeros.

<div align="center">

CHAPTER 2
LINEAR OPERATORS ON $\mathbb{R}[X]$

</div>

In this chapter we explore the hyperbolicity preservation properties (see Definition 76) of linear operators on $\mathbb{R}[x]$, providing many new insights and results. Section 2.1 and 2.2 are dedicated to reviewing operator theory as it relates to polynomials and to familiarize the reader with the current literature. In Section 2.3, we establish new methods of calculating the coefficients in a diagonal differential operator (Theorem 90). We also demonstrate the intimate connection between eigenvalues that can be interpolated by a polynomial and the order of differential operators (see Theorems 97). These are used to establish several new results concerning classical orthogonal hyperbolicity preserving diagonal differential operators (see Theorems 103, 86, and 107). Section 2.4 establishes a new representation of differential operators (Theorems 117 and 119) and provides a list of examples demonstrating the nature of hyperbolicity preservation with respect to this new found representation. In Section 2.5 we review a number of known results concerning hyperbolicity preservation. In particular, we provide a new carefully drafted proof of the Borcea-Brändén Theorem concerning finite order hyperbolicity preserving operators (Theorem 138). In essence, Section 2.5 is dedicated to understanding Problem 1 of the Preface. Many examples, lemmas, and corollaries are provided throughout each section.

2.1   Introduction to Operator Theory on $\mathbb{R}[x]$

By studying operators $T : \mathbb{R}[x] \to \mathbb{R}[x]$ one can gain a thorough understanding of how robust hyperbolic polynomials are (see Problem 2), thus shedding light on what polynomials have only real zeros. Consider, for example, differentiation (Theorem 6) or reversing (Definition 16); these operators preserve polynomials with only real zeros. Differentiation preserves linear combinations and is therefore called a *linear operator*, whereas reversing does not preserve linear combinations and is therefore called a *non-linear operator* (see the fascinating work of L. Grabarek [58] for more examples of non-linear operators).

An important issue in studying operators is the notion of *representation*. In order to compare operators and find commonality a similar well understood representation is needed. The word representation is not well defined as there are endless possibilities in representing an operator. We show by example the variety of representations that are present in the literature. The generating function of the derivatives of an entire function, $f(x)$; i.e., the Taylor series expansion,

$$f(x + y) = \sum_{k=0}^{\infty} \frac{f^{(k)}(y)}{k!} x^k,$$

<div align="center">

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</div>

will be useful in analyzing the next few examples (see Example 18).

Example 71. For each $k \in \mathbb{N}_0$, define the linear functionals [96, p. 271], $\alpha_k : \mathbb{R}[x] \to \mathbb{R}$ by $\alpha_k(f) = f^{(k)}(1)$ and $\beta_k : \mathbb{R}[x] \to \mathbb{R}$ by $\beta_k(f) = f^{(k)}(-1)$. Define the sequence of real numbers, $\{\gamma_n\}_{n=0}^{\infty} := \{1, 0, 0, 0, 0, \ldots\}$. We also define the simple multiplication operator and the simple differential operator on polynomials, $f \in \mathbb{R}[x]$,

$$x(f) := x \cdot f(x) \quad \text{and} \quad D(f) := \frac{d}{dx} f(x).$$

Consider the following operators evaluated at a polynomial, $f(x) \in \mathbb{R}[x]$,

$$T_1[f] := \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \alpha_k(f(x)), \quad T_2[f] := \sum_{k=0}^{\infty} \frac{1}{k!} \beta_k(f(x)),$$

$$T_3[f] := \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} x^k D^k f(x), \quad T_4[f] := f(0), \quad \text{and} \quad T_5[x^n] := \gamma_n x^n.$$

Operator T_5 is defined on x^n and then extended to all of $\mathbb{R}[x]$ by linearity. The reason for this example is the surprising non-obvious fact that,

$$T_1 = T_2 = T_3 = T_4 = T_5,$$

where two operators are defined to be equal if they produce the same result for each polynomial in $\mathbb{R}[x]$. Operators T_1 and T_2 are said to be represented by *linear functionals*, operator T_3 is called a *differential operator*, operator T_4 is called an *evaluation operator*, and operator T_5 is referred as a *classical diagonal operator*.

The above example demonstrates that when defining properties of operators, we must first consider the representation to use. For example, notions such as *dimension* (Definition 79) and *order* (Definition 82) only apply to particular representations. We see also that uniqueness of representation is a non-trivial matter. Indeed, in Example 71, the linear functional representation is shown to not be unique. On the other hand, Theorem 80 shows that the differential representation is unique. We give another example of operator representation.

Example 72. For each $k \in \mathbb{N}_0$, define the linear functionals, $\alpha_k : \mathbb{R}[x] \to \mathbb{R}$ by $\alpha_k(f) = f^{(k)}(1)$. We also define the *finite difference operator*, $\Delta : \mathbb{R}[x] \to \mathbb{R}[x]$ by $\Delta(f) = f(x+1) - f(x)$. Consider the following operators,

$$T_1[f] := f(x+1), \quad T_2[f] := \sum_{k=0}^{\infty} \frac{1}{k!} D^k f(x),$$

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$$T_3[f] := (\Delta + 1)f(x), \quad \text{and} \quad T_4[f] := \sum_{k=0}^{\infty} \frac{x^k}{k!} \alpha_k(f(x)).$$

Similar to Example 71, we have,

$$T_1 = T_2 = T_3 = T_4,$$

where T_1 is referred to as the *unit shift operator*. In particular, T_2 motivates the notation e^D for the unit shift operator.

Theorem 73. *Suppose $f(x)$ is an entire function. Then for each $n \in \mathbb{N}_0$,*

$$f(xD)x^n = f(n)x^n.$$

Moreover, for each $p(x) = a_0 + a_1 x + \cdots + a_n x^n \in \mathbb{R}[x]$,

$$f(xD)p(x) = a_0 f(0) + a_1 f(1)x + \cdots + a_n f(n)x^n.$$

Theorem 74 (A. Piotrowski [85, Proposition 33, p. 35]). *Suppose $f(x) := \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} x^k$ is an entire function and let $\{g_k^*(x)\}_{k=0}^{\infty}$ be the reversed Jensen polynomials associated to $f(x)$. If $T : \mathbb{R}[x] \to \mathbb{R}[x]$ is a linear operator such that $T[x^n] = \gamma_n x^n$, then*

$$T = \sum_{k=0}^{\infty} \frac{g_k^*(-1)}{k!} x^k D^k.$$

Remark 75. For $\alpha \in \mathbb{R}$, define $T : \mathbb{R}[x] \to \mathbb{R}[x]$ by $T[f(x)] = f(x + \alpha)$. Then extending the calculations found in Example 72, we can write $T = e^{\alpha D}$.

In the sequel our focus will be on linear operators under a differential representation that preserver hyperbolicity. The majority of our results focus on the relations between hyperbolicity preservers and their representation as a differential operator.

Definition 76. Given an operator, $T : \mathbb{R}[x] \to \mathbb{R}[x]$. We call T a *hyperbolicity preserver* or T is said to be a *hyperbolicity preserving* operator if $T[p(x)] \in \mathscr{L} - \mathscr{P}$ whenever $p(x) \in \mathscr{L} - \mathscr{P} \cap \mathbb{R}[x]$.

Theorem 77 (Hermite-Poulain [83, p. 4], [86, p. 128]). *If $p(x) \in \mathscr{L} - \mathscr{P}$ and $q(x) \in \mathscr{L} - \mathscr{P} \cap \mathbb{R}[x]$, then $p(D)q(x) \in \mathscr{L} - \mathscr{P} \cap \mathbb{R}[x]$.*

Theorem 78 (Laguerre [83, Satz 3.2], [39, Theorem 1.4]). *If $p(x) \in \mathscr{L} - \mathscr{P}(-\infty, 0]$ and $q(x) \in \mathscr{L} - \mathscr{P} \cap \mathbb{R}[x]$, then $p(xD)q(x) \in \mathscr{L} - \mathscr{P} \cap \mathbb{R}[x]$.*

2.2 Differential Operators

When $\mathbb{R}[x]$ is analyzed as a vector space, then commonly linear functionals, $\alpha_k : \mathbb{R}[x] \to \mathbb{R}$, are used to represent the operator. In this sense, the operator, $T : A \to B$, $A, B \subseteq \mathbb{R}[x]$, is understood as changing from one basis in A to another basis in B. Thus, a notion of dimension can be defined.

Definition 79. Let $T : \mathbb{R}[x] \to \mathbb{R}[x]$ be a linear operator. We define the set of linear functionals, $\alpha_k : \mathbb{R}[x] \to \mathbb{R}$ by $\alpha_k(f(x)) = \dfrac{f^{(k)}(0)}{k!}$. In this way,

$$T[f(x)] = \sum_{k=0}^{\infty} \alpha_k(f(x))T[x^k],$$

for every $f(x) \in \mathbb{R}[x]$. If the dimension of span$\{T[1], T[x], \ldots\}$ is infinite, then T is said to be an operator of *infinite dimension*. If span$\{T[1], T[x], \ldots\}$ is not infinite, we define the *dimension* of operator T to be the dimension of the vector space, span$\{T[1], T[x], \ldots\}$.

Theorem 80 ([84], [85, Proposition 29, p. 32]). *Let $T : \mathbb{R}[x] \to \mathbb{R}[x]$ be a linear operator. Then there exists a sequence of unique real polynomials, $\{Q_k(x)\}_{k=0}^{\infty}$, such that*

$$T[f(x)] = \left(\sum_{k=0}^{\infty} Q_k(x)D^k \right) f(x),$$

for every $f(x) \in \mathbb{R}[x]$. Moreover, the polynomial coefficients satisfy the recursive formula,

$$Q_n(x) = \frac{1}{n!} \left(T[x^n] - \sum_{k=0}^{n-1} Q_k(x)D^k x^n \right), \ n \geq 1, \ Q_0(x) = T[1].$$

It is unnecessary to use the standard basis, $\{x^k\}_{k=0}^{\infty}$, in Theorem 80. Any simple sequence of real polynomials (Definition 11) will calculate the same unique polynomial coefficients for the differential representation of a linear operator.

Theorem 81. *Let $T : \mathbb{R}[x] \to \mathbb{R}[x]$, $T = \displaystyle\sum_{k=0}^{\infty} Q_k(x)D^k$, be a linear operator and let $\{B_n(x)\}_{n=0}^{\infty}$ be a simple sequence of real polynomials. Then the polynomial coefficients satisfy the recursive formula,*

$$Q_n(x) = \frac{1}{B_n^{(n)}(x)} \left(T[B_n(x)] - \sum_{k=0}^{n-1} Q_k(x)B_n^{(k)}(x) \right),$$

where $n \geq 1$ and $Q_0(x) = \dfrac{1}{B_0(x)}T[B_0(x)]$.

Proof. From Theorem 80, we know that there exist unique $\{Q_k(x)\}_{k=0}^{\infty} \subseteq \mathbb{R}[x]$ such that $T = \sum_{k=0}^{\infty} Q_k(x)D^k$. For some $n \in \mathbb{N}_0$, evaluate T at $B_n(x)$; hence,

$$T[B_n(x)] = \sum_{k=0}^{n} Q_k(x)B_n^{(k)}(x).$$

Noting that $B_n(x)$ is degree n (i.e., $B_n^{(n)} \neq 0$), we solve the above for $Q_n(x)$ and obtain,

$$Q_n(x) = \frac{1}{B_n^{(n)}(x)} \left(T[B_n(x)] - \sum_{k=0}^{n-1} Q_k(x)B_n^{(k)}(x) \right). \qquad \square$$

Definition 82. Let $T : \mathbb{R}[x] \to \mathbb{R}[x]$ be a linear operator and let $\{Q_k(x)\}_{k=0}^{\infty}$ be the unique sequence of real polynomials in Theorem 81, such that $T[f(x)] = \sum_{k=0}^{\infty} Q_k(x)f^{(k)}(x)$ for every $f(x) \in \mathbb{R}[x]$. If there exists n such that $Q_k(x) \equiv 0$ for $k \geq n$, then T is said to be of *finite order*. If T is a finite order operator, then we define the *order* of T to be the largest k such that $Q_k(x) \not\equiv 0$. If $Q_k(x) \not\equiv 0$ for infinitely many k, then T is said to be of *infinite order*.

Example 83. The only operator, $T : \mathbb{R}[x] \to \mathbb{R}[x]$, that is both finite dimensional and of finite order is the zero operator, where $T[f(x)] = 0$ for all $f(x) \in \mathbb{R}[x]$. In particular, if T is a non-zero finite order operator, then T must be of infinite dimension.

Theorem 84. *Let $T := \sum_{k=0}^{\infty} Q_k(x)D^k$ and $W := \sum_{k=0}^{\infty} R_k(x)D^k$ be linear operators on $\mathbb{R}[x]$. If T is of order n and W is of order m, then TW is of order $n + m$ and*

$$\begin{aligned}
TW =& Q_n(x)R_m(x)D^{n+m} \\
&+ (Q_n(x)(nR_m'(x) + R_{m-1}(x)) + Q_{n-1}(x)R_m(x)) D^{n+m-1} \\
&+ \cdots + (Q_0(x)R_0(x) + \cdots + Q_n(x)R_0^{(n)}(x)).
\end{aligned}$$

We can extend the theorem above and provide a more detailed analysis of the degrees of the polynomial coefficients that arise from products of trinomial differential operators. We introduce the notation for arbitrary polynomials.

Definition 85. The symbol π_n will denote a polynomial of degree n, where $n > 0$. We will use π_0 to denote a non-zero constant polynomial, and for $n < 0$, π_{-n} will denote the zero polynomial. The notation π_n^* will be used to denote a polynomial of degree n or less. The above notation allows one to discuss the

degrees of polynomials without the need of providing specifics of the polynomials themselves. For example, writing $\pi_n + \pi_n = \pi_{n-1}$ indicates that we added two arbitrary polynomials of same degree and produced a polynomial of one less degree. In particular, we note that if π_n is used more than once in an expression, then it should not be assumed that the same polynomial is of discussion for each instance of π_n. In general, this notation will only be used in Theorems 86, 107, 110, and 220, to simplify the statements and demonstrate the form of the operators in question. See [90, p. 152] for a similar discussion of the symbol π_n.

Theorem 86. *For each $n \in \mathbb{N}$, the following hold:*

1. $(\pi_0 D^2 + \pi_1 D + \pi_0)^n$

$$= \pi_0 D^{2n} + \pi_1^* D^{2n-1} + \cdots + \pi_{n-1}^* D^{n+1} + \pi_n^* D^n + \pi_{n-1}^* D^{n-1} + \cdots + \pi_1^* D + \pi_0,$$

2. $(\pi_1 D^2 + \pi_1 D + \pi_0)^n$

$$= \pi_n D^{2n} + \pi_n^* D^{2n-1} + \cdots + \pi_n^* D^{n+1} + \pi_n^* D^n + \pi_{n-1}^* D^{n-1} + \cdots + \pi_1^* D + \pi_0, \text{ and}$$

3. $(\pi_2 D^2 + \pi_1 D + \pi_0)^n$

$$= \pi_{2n} D^{2n} + \pi_{2n-1}^* D^{2n-1} + \cdots + \pi_{n+1}^* D^{n+1} + \pi_n^* D^n + \pi_{n-1}^* D^{n-1} + \cdots + \pi_1^* D + \pi_0.$$

Proof. The proof of this theorem follows a sequence of elementary statements that, in general, provide an overview of *operator symbol* manipulation. We first note several finite order differential operator properties:

1. $\pi_n \pi_m = \pi_{n+m}$,

2. $\pi_n + \pi_n = \pi_n^*$, and

3. $D\pi_n = \pi_n D + \pi_{n-1}$.

We now derive several additional finite order differential operator properties:

1. $\pi_n D^2 \pi_m D^k = \pi_{n+m} D^{k+2} + \pi_{n+m-1}^* D^{k+1} + \pi_{n+m-2} D^k$,

2. $\pi_n D^1 \pi_m D^k = \pi_{n+m} D^{k+1} + \pi_{n+m-1} D^k$, and

3. $\pi_n D^0 \pi_m D^k = \pi_{n+m} D^k$.

Hence, with these operator symbol manipulations, the theorem follows by induction. $\square$

We refer the reader to the work of P. Blasiak and P. Flajolet [12] for a comprehensive guide to differential operator symbol manipulation.

Definition 87. Let $\alpha : \mathbb{R}[x] \to \mathbb{R}$ and $\beta : \mathbb{R}[x] \to \mathbb{R}$ be linear functionals. Let $p(x)$ and $q(x)$ be two polynomials in $\mathbb{R}[x]$. Then the operator,

$$T[f(x)] = \alpha(f(x))p(x) + \beta(f(x))q(x), \ f(x) \in \mathbb{R}[x],$$

is called a *trivial operator*.

Remark 88. Trivial operators will be excluded in several well known characterizations of hyperbolicity preservers (see Theorems 142, 194, and 213). In general, trivial operators are at most of dimension 2. If a trivial operator is of dimension 2, then the operator is hyperbolicity preserving if and only if the $p(x)$ and $q(x)$, in the definition above, have real interlacing zeros (this follows from the Hermite-Kakeya-Obreschkoff Theorem, Theorem 64). Furthermore, every trivial multiplier sequence (see Example 52) is a trivial operator.

2.3 Diagonal Differential Operators

Definition 89. Suppose $T : \mathbb{R}[x] \to \mathbb{R}[x]$ is a linear operator. If there is $\{B_n(x)\}_{n=0}^{\infty}$, a simple sequence of real polynomials, and $\{\gamma_n\}_{n=0}^{\infty}$, a sequence of real numbers, such that for every $n \in \mathbb{N}_0$, $T[B_n(x)] = \gamma_n B_n(x)$, then T is said to *diagonalize* or to be *diagonalizable* and is called a *diagonal differential operator* with respect to the basis, $\{B_n(x)\}_{n=0}^{\infty}$, and eigenvalues, $\{\gamma_n\}_{n=0}^{\infty}$. If the diagonalizing simple basis, $\{B_n(x)\}_{n=0}^{\infty}$, is the Hermite, Laguerre, Jacobi, or standard basis, then T is referred to as a *Hermite diagonal differential operator*, *Laguerre diagonal differential operator*, *Jacobi diagonal differential operator*, or *classical diagonal differential operator*, respectively. Furthermore, if T is hyperbolicity preserving and diagonalizes on the Hermite, Laguerre, Jacobi, or standard basis, then the eigenvalues, $\{\gamma_n\}_{n=0}^{\infty}$, are called a *Hermite multiplier sequence*, *Laguerre multiplier sequence*, *Jacobi multiplier sequence*, or *classical multiplier sequence*, respectively. If the specifics of the basis, $\{B_n(x)\}_{n=0}^{\infty}$, are not known in a diagonal differential hyperbolicity preserver, then the eigenvalues are called a B_n-*multiplier sequence*.

We now extend Theorem 81 for diagonal differential operators (cf. [84] or [85, Proposition 29, p. 32]) (see also [27, Proposition 216, p. 106]).

Theorem 90. *Given a diagonal differential operator, $T : \mathbb{R}[x] \to \mathbb{R}[x]$, with respect to the simple basis, $\{B_n(x)\}_{n=0}^{\infty}$, and the sequence of real numbers, $\{\gamma_n\}_{n=0}^{\infty}$, so that $T[B_n(x)] = \gamma_n B_n(x)$ for every $n \in \mathbb{N}_0$, then*

there is a unique sequence of real polynomials, $\{Q_k(x)\}_{k=0}^{\infty} \subseteq \mathbb{R}[x]$, with $\deg(Q_k(x)) \leq k$, such that

$$T = \sum_{k=0}^{\infty} Q_k(x) D^k, \qquad D := \frac{d}{dx}.$$

In particular, each $Q_n(x)$ is given by the recursive formula,

$$Q_n(x) = \frac{1}{B_n^{(n)}(x)} \left(\gamma_n B_n(x) - \sum_{k=0}^{n-1} Q_k(x) B_n^{(k)}(x) \right), \qquad (2.3.1)$$

where $n \geq 1$ and $Q_0(x) = \frac{1}{B_0(x)} T[B_0(x)]$.

Definition 91. Suppose $T : \mathbb{R}[x] \to \mathbb{R}[x]$, $T = \sum_{k=0}^{\infty} Q_k(x) D^k$, is an arbitrary linear operator. If $\deg(Q_k(x)) \leq k$ for each $k \in \mathbb{N}_0$, then T is said to be a *triangular* operator. If $\deg(T[f(x)]) = n$ for every $f(x) \in \mathbb{R}[x]$, where $\deg(f(x)) = n$, then T is said to be a *degree preserving* operator.

Example 92. Every degree preserving linear operator is a diagonalizable linear operator (see [27, Theorem 182, p. 91]) and every diagonalizable linear operator is a triangular linear operator. However, the converse statements are not true. For example,

$$T := \left(x^2 + \frac{x}{2} \right) D^2 - 2xD + 1,$$

is a triangular linear operator and yet T does not possess a quadratic eigenvector (see [5, Example 25] or [65]). Likewise,

$$W := (x^2 - 1)D^2 + 2xD - 6,$$

diagonalizes with respect to the Legendre polynomials (see Definition 29 and Theorem 34) and eigenvalues, $\{n^2 + n - 6\}_{n=0}^{\infty}$, but W is not degree preserving since $W[x^2] = -2$.

Throughout the literature [62, 69–71, 101, 106], we find most methods for finding eigenvalues of a linear operator involve either special cases or solving systems of equations. Many authors require special constraints or properties before proceeding to discuss the eigenvalues. Hence, while the next theorem is very simple to realize, we have not found its full generality within the current literature nor its many consequences. Here we provide a very simple method of determining every eigenvalue of every triangular operator. Moreover, we will reverse this formula and derive several new results about the relations between operator *order* and *polynomial interpolation*. These notions will provide uniqueness theorems for the classical orthogonal polynomials and

answer many questions concerning the sequence $\{\deg(Q_k(x))\}_{k=0}^{\infty}$ in the representation of a linear operator,

$$T := \sum_{k=0}^{\infty} Q_k(x)D^k.$$

Theorem 93. *Let* $T : \mathbb{R}[x] \to \mathbb{R}[x]$, $T = \sum_{k=0}^{\infty} Q_k(x)D^k$, *be a triangular operator (i.e.,* $\deg(Q_k(x)) \le k$ *for every* $k \in \mathbb{N}_0$*). If* T *has an eigenvector,* $p(x) \in \mathbb{R}[x]$, $\deg(p(x)) = n$, *then*

$$\alpha = \sum_{k=0}^{n} \binom{n}{k} Q_k^{(k)}$$

is the corresponding eigenvalue; i.e., $T[p(x)] = \alpha p(x)$.

Proof. Suppose

$$\left(\sum_{k=0}^{\infty} Q_k(x)D^k \right) p(x) = \alpha p(x).$$

Then differentiating both sides n times yields,

$$\sum_{k=0}^{n} \sum_{j=0}^{n} \binom{n}{j} Q_k^{(j)}(0) p^{(k+n-j)}(0) = \alpha p^{(n)}.$$

Hence, since $p(x)$ is of degree n (that is, $p^{(n)} \neq 0$) and $\deg(Q_k(x)) \le k$,

$$\sum_{k=0}^{n} \binom{n}{k} Q_k^{(k)} = \alpha. \qquad \square$$

Example 94. From Example 92, a diagonalizable operator, T, is also a triangular operator; i.e., if T has a simple sequence of eigenvectors, $\{B_n(x)\}_{n=0}^{\infty}$, then the polynomial coefficients, $\{Q_k(x)\}_{k=0}^{\infty}$, satisfy $\deg(Q_k(x)) \le k$. However, if a linear operator has merely a few eigenvectors, then the linear operator need not be a triangular operator. For example, the non-triangular operator,

$$T := (x^2 - 1)D - x,$$

has the eigenvector of $x - 1$ with the corresponding eigenvalue of 1; i.e., $T[x - 1] = 1(x - 1)$. This example also shows that the condition of "triangular" in Theorem 93 is required, since the eigenvalue of 1 above does not follow from the calculation in Theorem 93.

Theorem 95. *If T is a diagonal differential operator,*

$$T[B_n(x)] := \left(\sum_{k=0}^{\infty} Q_k(x)D^k \right) B_n(x) = a_n B_n(x), \qquad n \in \mathbb{N}_0,$$

then for every $n \in \mathbb{N}_0$,

$$a_n = \sum_{k=0}^{n} \binom{n}{k} Q_k^{(k)}. \tag{2.3.2}$$

We observe that equation (2.3.2) is similar in nature to the formulas in Theorem 50. In particular, if the eigenvalues are known to be interpolated by a polynomial (see Problem 57), then what does that reveal about $Q_k^{(k)}$? The answer to this question is summarized in the following lemma.

Lemma 96. *Let $p(x)$ be a real polynomial and set*

$$\gamma_n = \sum_{k=0}^{n} \binom{n}{k} p(k)(-1)^{n-k}.$$

If $n > \deg(p)$, then $\gamma_n = 0$. If $n = \deg(p)$, then $\gamma_n = p^{(n)} \neq 0$.

Proof. Since

$$\sum_{k=0}^{n} \binom{n}{k} x^k = (1+x)^n,$$

$$\sum_{k=0}^{n} \binom{n}{k} k^m x^k = (xD)^m (1+x)^n, \qquad D := \frac{d}{dx}. \tag{2.3.3}$$

In equation (2.3.3), each differentiation reduces by one the multiplicity of the zero at -1. Thus we have,

$$\sum_{k=0}^{n} \binom{n}{k} k^m (-1)^{n-k} = (-1)^n \, (xD)^m (1+x)^n|_{x=-1} = \begin{cases} 0, & 0 \leq m < n; \\ n!, & m = n \end{cases}. \tag{2.3.4}$$

Calculation (2.3.4) is also found in [57, Equation 1.13, p. 2]. Thus, for arbitrary polynomial, $p(x)$, $\deg(p(x)) \leq n$, $p(x) = a_0 + a_1 x + \cdots + a_n x^n$ (a_n can be zero), we have,

$$\sum_{k=0}^{n} \binom{n}{k} p(k)(-1)^{n-k} = \sum_{k=0}^{n} \binom{n}{k} (a_0 + a_1 k + \cdots + a_n k^n)(-1)^{n-k}$$

$$= a_0 \left(\sum_{k=0}^{n} \binom{n}{k}(-1)^{n-k} \right) + \cdots + a_n \left(\sum_{k=0}^{n} \binom{n}{k} k^n (-1)^{n-k} \right) = a_n n!.$$

The result now follows. $\qquad \square$

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Theorem 97. *Suppose T is a diagonal differential operator,*

$$T[B_n(x)] := \left(\sum_{k=0}^{\infty} Q_k(x) D^k \right) B_n(x) = a_n B_n(x), \qquad n \in \mathbb{N}_0.$$

Then $\{a_n\}_{n=0}^{\infty}$ can be interpolated by a polynomial of degree m if and only if $\deg(Q_m(x)) = m$ and $\deg(Q_k(x)) < k$ for $k > m$. Moreover, $\{a_n\}_{n=0}^{\infty}$ cannot be interpolated by a polynomial if and only if $\deg(Q_k(x)) = k$ for infinitely many $k \in \mathbb{N}_0$.

Proof. By Theorem 95,

$$a_n = \sum_{k=0}^{n} \binom{n}{k} Q_k^{(k)}.$$

Hence, by Theorem 50,

$$Q_n^{(n)} = \sum_{k=0}^{n} \binom{n}{k} a_k (-1)^{n-k}.$$

Thus, if $\{a_k\}_{k=0}^{\infty}$ can be interpolated by a polynomial, $p(x)$, $\deg(p(x)) = n$, then, by Lemma 96, $Q_k^{(k)} = 0$ for $k > n$ and $Q_n^{(n)} \neq 0$. Likewise, if $Q_k^{(k)} = 0$ for $k > n$ and $Q_n^{(n)} \neq 0$ for some $n \in \mathbb{N}_0$, then $\{a_k\}_{k=0}^{\infty}$ is interpolated by the degree n polynomial, $p(x) := \sum_{k=0}^{n} \binom{x}{k} Q_k^{(k)}$. The second statement of the theorem is simply the negation of the first statement. $\qquad\square$

An immediate consequence of Theorem 97 is the following.

Corollary 98. *Suppose T is a diagonal differential operator,*

$$T[B_n(x)] := \left(\sum_{k=0}^{\infty} Q_k(x) D^k \right) B_n(x) = a_n B_n(x), \qquad n \in \mathbb{N}_0.$$

Then each of the following statements hold.

1. *If T is of finite order, then $\{a_n\}_{n=0}^{\infty}$ can be interpolated by a polynomial.*

2. *If $\{a_n\}_{n=0}^{\infty}$ cannot be interpolated by a polynomial, then T is of infinite order.*

3. *If $\{a_n\}_{n=0}^{\infty}$ is a non-constant sequence with a bounded sub-sequence, then T is of infinite order.*

4. *If $\{a_n\}_{n=0}^{\infty}$ is either a non-constant non-negative decreasing sequence or a non-constant non-positive increasing sequence, then T is of infinite order.*

5. *If $\{a_n\}_{n=0}^{\infty}$ is a non-constant alternating sequence, then T is of infinite order.*

Example 99. Consider the classical hyperbolicity preserving diagonal differential operator,

$$T[x^n] = \frac{1}{n!}x^n,$$

see Theorems 54 and 78. Operator T has the following differential representation (see Theorem 80),

$$T = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\sum_{k=0}^{n} \binom{n}{k} \frac{1}{k!} (-1)^{n-k} \right) x^n D^n = 1 - \frac{1}{2}x^2 D^2 + \frac{2}{3}x^3 D^3 - \frac{5}{8}x^4 D^4 + \cdots.$$

Corollary 98 part (3) implies that T is an infinite order differential operator.

Example 100. Consider the following hyperbolicity preserving Hermite diagonal differential operators,

$$T[H_n(x)] = nH_n(x) \quad \text{and} \quad \tilde{T}[H_n(x)] = (-1)^n nH_n(x).$$

We provide the first few polynomial coefficients from Theorem 90,

$$T = xD - \frac{1}{2}D^2,$$

and

$$\tilde{T} = -xD + \left(2x^2 - \frac{1}{2} \right) D^2 + \left(-2x^3 + x \right) D^3 + \cdots.$$

We see that T is a finite order differential operator while Corollary 98 part (5) shows that $\tilde{T}$ is an infinite order differential operator.

Example 101. While the notions of "polynomial interpolation" and "finite order" seem intimately connected (Theorem 97), there exist operators that are of infinite order and have eigenvalues that can be interpolated by a polynomial. Consider the following Legendre diagonal differential operator,

$$T[P_n(x)] = nP_n(x).$$

Using the Legendre differential equation (see Theorem 34), it is clear that,

$$(T^2 + T)[P_n(x)] = (n^2 + n)P_n(x) = ((x^2 - 1)D^2 + 2xD)P_n(x).$$

If T is of finite order, then Theorem 84 reveals that T must be an operator of order one; i.e., $T = A(x)D + B(x)$, where $A(x)$ and $B(x)$ are polynomials. Moreover, Theorem 84 demonstrates $A(x)^2 = x^2 - 1$, an impossibility for any polynomial. Hence, T cannot be of finite order. Thus, T is an infinite order diagonal differential operator with eigenvalues that can be interpolated by a polynomial.

The operator, T, in the above example fails to be hyperbolicity preserving [11, Proposition 2]. Hence, the following question arises (see also [11, 52]).

Problem 102. If $T[B_n(x)] = a_n B_n(x)$ is a hyperbolicity preserving operator and $\{a_n\}_{n=0}^{\infty}$ can be interpolated by a polynomial, then must T be of finite order?

Theorem 103. *Suppose for the simple basis, $\{B_n(x)\}_{n=0}^{\infty}$, there is a finite order differential operator, M, such that $M[B_n(x)] = nB_n(x)$ for $n \in \mathbb{N}_0$. Let T be any diagonal differential operator with respect to $\{B_n(x)\}_{n=0}^{\infty}$ such that*

$$T[B_n(x)] := \left(\sum_{k=0}^{\infty} Q_k(x)D^k \right) B_n(x) = a_n B_n(x), \qquad n \in \mathbb{N}_0.$$

Then $\{a_n\}_{n=0}^{\infty}$ can be interpolated by a polynomial if and only if T is a finite order diagonal differential operator.

Proof. If $\{a_n\}_{n=0}^{\infty}$ is interpolated by a polynomial, $p(x)$, then by uniqueness, $T = p(M)$ (see Theorems 80 and 84). Conversely, if T is of finite order, then Corollary 98 shows that $\{a_n\}_{n=0}^{\infty}$ can be interpolated by some polynomial, $p(x)$. Hence, again $T = p(M)$ and is therefore of finite order, see Theorem 84. $\qquad \square$

Corollary 104. *The Hermite, generalized Laguerre, and classical diagonal differential operators are of finite order if and only if the eigenvalues of the operators can be interpolated by polynomials.*

Corollary 104 provides characterizations of Hermite, Laguerre, and classical multiplier sequences that arise from finite order diagonal differential operators; namely, these multiplier sequences are precisely those that can be interpolated by polynomials (cf. [16, Theorem 1.9]).

In 1938, H. Krall [69] established that orthogonal polynomials do not satisfy any odd order differential equations. Using this, in 2005, L. Miranian [81] established the exact form of classical orthogonal polynomial differential equations. In particular, L. Miranian showed the following relationship between the eigenvalues that can be interpolated by a polynomial in a Jacobi diagonal differential operator and finite order operators that diagonalize on the Jacobi polynomials.

Theorem 105 (L. Miranian [81])**.** *Suppose T is a Jacobi diagonal differential operator, $\alpha, \beta > -1$,*

$$T[P_n^{(\alpha,\beta)}(x)] := \left(\sum_{k=0}^{\infty} Q_k(x) D^k \right) P_n^{(\alpha,\beta)}(x) = a_n P_n^{(\alpha,\beta)}(x), \qquad n \in \mathbb{N}_0.$$

Then T is a finite order operator if and only if there exists a polynomial $p(x)$ such that

$$T = p\left((x^2 - 1)D^2 + 2xD \right) \quad and \quad a_n = p(n^2 + n).$$

Problem 106. Suppose $\{B_n(x)\}_{n=0}^{\infty}$ is a simple sequence of real polynomials with a "minimal" non-trivial finite order differential equation, M, $M[B_n(x)] = a_n B_n(x)$; i.e., if L is any other non-trivial differential operator such that $L[B_n(x)] = b_n B_n(x)$, then the order of M is smaller than that of L. If T is any other finite order differential operator such that $T[B_n(x)] = c_n B_n(x)$, then must there exist a polynomial, $p(x)$, such that $p(M) = T$ and $p(a_n) = c_n$?

Combining Theorems 97 and 86, Corollary 104, and L. Miranian's result (Theorem 105) provides a comprehensive understanding of the classical orthogonal polynomial differential equations (cf. with the hyperbolicity preserving cases, Theorems 110 and 220).

Theorem 107. *Suppose T_H, T_L, and T_P are non-zero diagonal differential operators, $\alpha, \beta > -1$, where $T_H[H_k(x)] = a_k H_k(x)$, $T_L[L_k^{(\alpha)}(x)] = b_k L_k^{(\alpha)}(x)$, and $T_P[P_k^{(\alpha,\beta)}(x)] = c_k P_k^{(\alpha,\beta)}(x)$.*

1. *If T_H is a finite order diagonal differential operator, then there exists a polynomial $p(x)$, $\deg(p(x)) = n$, such that $a_k = p(k)$ for every $k \in \mathbb{N}_0$. In this case T_H is order $2n$ and has the differential form,*

$$\pi_0 D^{2n} + \pi_1^* D^{2n-1} + \cdots + \pi_{n-1}^* D^{n+1} + \pi_n D^n + \pi_{n-1}^* D^{n-1} + \cdots + \pi_1^* D + \pi_0^*.$$

2. *If T_L is a finite order diagonal differential operator, then there exists a polynomial $p(x)$, $\deg(p(x)) = n$, such that $b_k = p(k)$ for every $k \in \mathbb{N}_0$. In this case T_L is order $2n$ and has the differential form,*

$$\pi_n D^{2n} + \pi_n^* D^{2n-1} + \cdots + \pi_n^* D^{n+1} + \pi_n D^n + \pi_{n-1}^* D^{n-1} + \cdots + \pi_1^* D + \pi_0^*.$$

3. *If T_P is a finite order diagonal differential operator, then there exists a polynomial $p(x)$, $\deg(p(x)) = n$, such that $c_k = p(k^2 + k)$ for every $k \in \mathbb{N}_0$. In this case T_P is order $2n$ and has the differential form,*

$$\pi_{2n} D^{2n} + \pi_{2n-1}^* D^{2n-1} + \cdots + \pi_n^* D^n + \cdots + \pi_1^* D + \pi_0^*.$$

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We wish to refine the results of Theorem 107 and thus turn our attention to hyperbolicity preservation. As we will find the hyperbolicity preservation property will force many of the polynomial coefficients, π_k^*, to be exactly degree k, π_k. We first refine Theorem 97 for hyperbolicity preservers.

Theorem 108. *Suppose $\{\gamma_k\}_{k=0}^{\infty}$ is a positive increasing multiplier sequence and let $\{g_k^*(x)\}_{k=0}^{\infty}$ be the associated reversed Jensen polynomials. If $\{\gamma_k\}_{k=0}^{\infty}$ can be interpolated by a polynomial of degree n, then $g_k^*(-1) \neq 0$ for $0 \leq k \leq n$ and $g_k^*(-1) = 0$ for $k > n$. If $\{\gamma_k\}_{k=0}^{\infty}$ cannot be interpolated by a polynomial, then $g_k^*(-1) \neq 0$ for every $k \in \mathbb{N}_0$.*

Proof. Let $T := \sum_{k=0}^{\infty} Q_k(x)D^k$ be the linear operator such that for each $n \in \mathbb{N}_0$, $T[x^n] = \gamma_n x^n$. We note that $Q_k^{(k)} = g_k^*(-1)$ for every $k \in \mathbb{N}_0$ (see Definition 19 and Theorem 95). Suppose $\{\gamma_k\}_{k=0}^{\infty}$ can be interpolated by a polynomial of degree n. By assumption γ_0 is positive, hence $g_0^*(-1) = \gamma_0 > 0$. Also by Theorem 97, $Q_n^{(n)} \neq 0$ and $Q_k^{(k)} = 0$ for $k > n$. Thus, $g_n^*(-1) \neq 0$ and $g_k^*(-1) = 0$ for $k > n$. From Corollary 48 and Theorem 50 we have that

$$e^{-x}\sum_{k=0}^{\infty}\frac{\gamma_k}{k!}x^k = \sum_{k=0}^{\infty}\frac{g_k^*(-1)}{k!}x^k \in \mathscr{L} - \mathscr{P}^s,$$

hence by Theorem 42, $g_k^*(-1) \neq 0$ for $0 \leq k \leq n$. Similar reasoning also establishes the case of when $\{\gamma_k\}_{k=0}^{\infty}$ cannot be interpolated by a polynomial. See also [5]. $\square$

Corollary 109. *Let T be a hyperbolicity preserving diagonal differential operator,*

$$T[B_n(x)] := \left(\sum_{k=0}^{\infty} Q_k(x)D^k\right)B_n(x) = \gamma_n B_n(x), \qquad n \in \mathbb{N}_0,$$

where $0 < \gamma_k \leq \gamma_{k+1}$ for every $k \in \mathbb{N}_0$. If $\{\gamma_k\}_{k=0}^{\infty}$ can be interpolated by a polynomial of degree n, $p(x)$, then $\deg(Q_k(x)) = k$ for $0 \leq k \leq n$. If $\{\gamma_k\}_{k=0}^{\infty}$ cannot be interpolated by a polynomial, then $\deg(Q_k(x)) = k$ for all $k \in \mathbb{N}_0$.

Given that every classical orthogonal diagonal differential operator with positive eigenvalues has increasing eigenvalues (see [11, Theorem 7], [53, Theorem 6], [85, Theorem 152, p. 140], and [55, Theorem 4.6]), we arrive at a refinement of Theorem 107 for hyperbolicity preservers (see also [5]).

Theorem 110. *Suppose T_H, T_L, and T_P are non-trivial hyperbolicity preserving diagonal differential operators, $\alpha, \beta > -1$, where $T_H[H_k(x)] = a_k H_k(x)$, $T_L[L_k^{(\alpha)}(x)] = b_k L_k^{(\alpha)}(x)$, and $T_P[P_k^{(\alpha,\beta)}(x)] = c_k P_k^{(\alpha,\beta)}(x)$, $a_0, b_0, c_0 \neq 0$, $a_k, b_k, c_k \geq 0$.*

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1. *If T_H is a finite order diagonal differential operator, then there exists a polynomial $p(x)$, $\deg(p(x)) = n$, such that $a_k = p(k)$ for every $k \in \mathbb{N}_0$. In this case T_H is order $2n$ and has the differential form,*

$$\pi_0 D^{2n} + \pi_1^* D^{2n-1} + \cdots + \pi_{n-1}^* D^{n+1} + \pi_n D^n + \pi_{n-1} D^{n-1} + \cdots + \pi_1 D + \pi_0.$$

2. *If T_L is a finite order diagonal differential operator, then there exists a polynomial $p(x)$, $\deg(p(x)) = n$, such that $b_k = p(k)$ for every $k \in \mathbb{N}_0$. In this case T_L is order $2n$ and has the differential form,*

$$\pi_n D^{2n} + \pi_n^* D^{2n-1} + \cdots + \pi_n^* D^{n+1} + \pi_n D^n + \pi_{n-1} D^{n-1} + \cdots + \pi_1 D + \pi_0.$$

3. *If T_P is a finite order diagonal differential operator, then there exists a polynomial $p(x)$, $\deg(p(x)) = n$, such that $c_k = p(k^2 + k)$ for every $k \in \mathbb{N}_0$. In this case T_P is order $2n$ and has the differential form,*

$$\pi_{2n} D^{2n} + \pi_{2n-1} D^{2n-1} + \cdots + \pi_n D^n + \cdots + \pi_1 D + \pi_0.$$

We now begin work on the uniqueness of diagonal differential operators.

Theorem 111. *Suppose $T := \displaystyle\sum_{k=0}^{\infty} Q_k(x) D^k$ is a diagonal differential operator with respect to $\{a_n\}_{n=0}^{\infty}$ and $\{A_n(x)\}_{n=0}^{\infty}$ as well as with respect to $\{b_n\}_{n=0}^{\infty}$ and $\{B_n(x)\}_{n=0}^{\infty}$; that is,*

$$T[A_n(x)] = a_n A_n(x) \quad \text{and} \quad T[B_n(x)] = b_n B_n(x).$$

Then $a_n = b_n$ for all $n \in \mathbb{N}_0$.

Proof. Use Theorem 95 after noting that formula (2.3.2) is independent of the eigenvector sequence. $\square$

Thus, a diagonal differential operator can represent at most one eigenvalue sequence. Upon stating the above theorem, we immediately ask if the B_n's in a diagonal differential operator are also unique. Simple examples demonstrate that this is not the case. However, under additional restrictions on the eigenvalues, we can show that the basis chosen for diagonalization is unique up to a constant multiple.

Theorem 112 ([71, H. Krall and I. Scheffer]). *Suppose $T := \displaystyle\sum_{k=0}^{\infty} Q_k(x) D^k$ is a diagonal differential operator with respect to $\{a_n\}_{n=0}^{\infty}$ and $\{A_n(x)\}_{n=0}^{\infty}$ as well as with respect to $\{a_n\}_{n=0}^{\infty}$ and $\{B_n(x)\}_{n=0}^{\infty}$,*

$$T[A_n(x)] = a_n A_n(x) \quad \text{and} \quad T[B_n(x)] = a_n B_n(x), \qquad n \in \mathbb{N}_0.$$

For a fixed m, suppose $a_m \neq a_k$ for all $0 \leq k < m$. Then there is $\beta \in \mathbb{R}$, $\beta \neq 0$, such that

$$A_m(x) = \beta B_m(x), \quad x \in \mathbb{R}.$$

Proof. Since $\{B_n(x)\}_{n=0}^{\infty}$ is a simple sequence, $A_m(x) = \beta_m B_m(x) + \beta_{m-1} B_{m-1}(x) + \cdots + \beta_0 B_0(x)$, $\beta_m \neq 0$. We now apply T to $A_m(x)$ and calculate $T[A_m(x)]$ in two different ways,

$$T[A_m] = a_m A_m = a_m \beta_m B_m + a_m \beta_{m-1} B_{m-1} + \cdots + a_m \beta_0 B_0, \tag{2.3.5}$$

and

$$T[A_m] = T[\beta_m B_m + \cdots + \beta_0 B_0] = a_m \beta_m B_m + a_{m-1} \beta_{m-1} B_{m-1} + \cdots + a_0 \beta_0 B_0. \tag{2.3.6}$$

Equating coefficients from equation (2.3.5) and (2.3.6), yields, $a_m \beta_{m-1} = a_{m-1} \beta_{m-1}$, $a_m \beta_{m-2} = a_{m-2} \beta_{m-2}$, ..., $a_m \beta_0 = a_0 \beta_0$. By assumption, $a_m \neq a_k$ for $0 \leq k < m$, thus $\beta_k = 0$ for $0 \leq k < m$. Hence, we have $A_m(x) = \beta_m B_m(x)$ as desired. $\qquad\square$

Corollary 113. *Suppose $T := \displaystyle\sum_{k=0}^{\infty} Q_k(x) D^k$ is a diagonal differential operator with respect to $\{a_n\}_{n=0}^{\infty}$ and $\{A_n(x)\}_{n=0}^{\infty}$ as well as with respect to $\{a_n\}_{n=0}^{\infty}$ and $\{B_n(x)\}_{n=0}^{\infty}$,*

$$T[A_n(x)] = a_n A_n(x) \quad and \quad T[B_n(x)] = a_n B_n(x).$$

Also, suppose that $\{a_n\}_{n=0}^{\infty}$ is a non-zero, non-constant, multiplier sequence that can be interpolated by a polynomial. Then there is a sequence of real numbers, $\{\beta_n\}_{n=0}^{\infty}$, $\beta_n \neq 0$, such that

$$A_n(x) = \beta_n B_n(x), \qquad n \in \mathbb{N}_0.$$

Proof. Since $\{a_n\}_{n=0}^{\infty}$ is a multiplier sequence, then by the Turán inequalities (Theorem 53), if $\{|a_n|\}_{n=0}^{\infty}$ starts decreasing then it will continue to decrease indefinitely. The sequence $\{|a_n|\}_{n=0}^{\infty}$ cannot decrease indefinitely since $\{a_n\}_{n=0}^{\infty}$ is interpolated by a non-constant polynomial; i.e., $|a_n| \to \infty$ as $n \to \infty$. Thus, $\{|a_n|\}_{n=0}^{\infty}$ must be a strictly increasing sequence. Hence, $a_n \neq a_m$ for $n \neq m$. Now apply Theorem 112 to obtain the desired result. $\qquad\square$

Example 114. In some sense, Theorem 112 and Corollary 113 are best possible. Consider the following diagonal differential operators,

$$T[x^n] := \left(\sum_{k=0}^{\infty} Q_k(x) D^k \right) x^n = (-1)^n x^n,$$

and

$$W[H_n(x)] := \left(\sum_{k=0}^{\infty} R_k(x) D^k \right) H_n(x) = (-1)^n H_n(x).$$

Using the recursive formula of Theorem 90, by induction, for every $k \in \mathbb{N}_0$,

$$Q_k(x) = R_k(x) = \frac{(-2)^k}{k!} x^k.$$

Hence, $T = W$. Thus, T is a diagonal differential operator that is hyperbolicity preserving and that can be diagonalized with respect to two distinct (the two bases are not related by an affine transformation, see Definition 35) simple bases.

Since the only terms of the Q_k's $(Q_k(x) = \alpha_k x^k + \cdots)$ that affect the eigenvalues of a diagonal differential operator are the leading terms, $\alpha_k x^k$, we conclude that elimination of the remaining terms provides the same eigenvalues as the original operator. In addition, forcing monomial polynomial coefficients in a diagonal differential operator allows a diagonalization with the classical basis (Theorems 50 and 74). Thus, we obtain the following result, which is a generalization of Piotrowski's Theorem that "every B_n-multiplier sequence is also a classical multiplier sequence" [85, Theorem 158, p. 145] (this will be further generalized in Corollary 136).

Theorem 115. *Suppose T is a hyperbolicity preserving diagonal differential operator with respect to the simple sequence $\{B_n(x)\}_{n=0}^{\infty}$,*

$$T[B_n(x)] := \left(\sum_{k=0}^{\infty} Q_k(x) D^k \right) B_n(x) = a_n B_n(x), \qquad n \in \mathbb{N}_0.$$

Then

$$\tilde{T}[x^n] := \left(\sum_{k=0}^{\infty} \frac{Q_k^{(k)}}{k!} x^k D^k \right) x^n = a_n x^n, \qquad n \in \mathbb{N}_0,$$

is also a hyperbolicity preserving diagonal differential operator with respect to the standard basis, $\{x^n\}_{n=0}^{\infty}$.

Proof. We simply note that by Theorem 95 and the definition of the reversed Jensen polynomials (Definition 19), $g_k^*(-1) = Q_k^{(k)}$. Now apply Theorem 50 and [85, Theorem 158, p. 145]. $\qquad\square$

Example 116. In Theorem 115, the condition that T is diagonalizable is required. By the Hermite-Poulain Theorem (Theorem 77) it is not hard to show that

$$T := (x^3 + x^2)D^2 - (x^3 + x^2)$$

is a hyperbolicity preserving operator. However,

$$\tilde{T} := \frac{6x+2}{2!}x^2D^2 - (x^3 + x^2),$$

is not hyperbolicity preserving, since $\tilde{T}[x^2 + 10x + 16] = -x^2(x^3 + 11x^2 + 20x + 14)$.

The concept above inspires a new type of representation for diagonal differential operators. Theorem 115 can be thought of as "slicing" a layer off of the operator T, which has the same hyperbolicity properties that T possesses. Hence, we ask, how does one obtain more slices from the operator T? What hyperbolicity preserving properties will the remaining slices have? This question is addressed and analyzed in the next section. Moreover, in Chapter 4, we will establish that every Hermite and Laguerre diagonal differential operator has "slices" that form additional Hermite and Laguerre multiplier sequences.

2.4 Differential Operator Diagonalizations

In this section we present a new representation of diagonal differential operators (Theorem 117). To provide a little more generality, we note that Theorem 117 will be proved for all triangular operators (see Definition 91). We recall Theorems 80, 81, and 90 which demonstrate how to calculate the Q_k's in differential operators and diagonal differential operators.

Theorem 117. *Let $T : \mathbb{R}[x] \to \mathbb{R}[x]$ be a linear operator,*

$$T = \sum_{k=0}^{\infty} Q_k(x)D^k,$$

where $\deg(Q_k(x)) \le k$ for every $k \in \mathbb{N}_0$. Define the family of sequences,

$$\{b_{n,k}\}_{k=0}^{\infty} := \left\{ \sum_{j=0}^{k} \binom{k}{j} Q_{j+n}^{(j)}(0) \right\}_{k=0}^{\infty}, \qquad n \in \mathbb{N}_0. \tag{2.4.1}$$

For each $n \in \mathbb{N}_0$, define the classical diagonal differential operator,

$$T_n[x^k] := b_{n,k} x^k, \quad k \in \mathbb{N}_0. \tag{2.4.2}$$

Then,

$$T = \sum_{n=0}^{\infty} T_n D^n. \tag{2.4.3}$$

Furthermore, the representation in (2.4.3) is unique.

Proof. Since the operators under consideration are defined on $\mathbb{R}[x]$, the question of convergence is not an issue. By Theorem 50, for every $n \in \mathbb{N}_0$, we know the differential representation of T_n, namely,

$$T_n = \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} \binom{k}{j} b_{n,j} (-1)^{k-j} \right) \frac{1}{k!} x^k D^k = \sum_{k=0}^{\infty} \frac{Q_{k+n}^{(k)}(0)}{k!} x^k D^k.$$

Note that $\dfrac{Q_{k+n}^{(k)}(0)}{k!} x^k$ is precisely the kth term of the polynomial, $Q_{k+n}(x)$. Hence, each summand, in each T_n, is a single term from some $Q_k(x)$. Furthermore, no two T_n's use the same term from a particular $Q_k(x)$. Finally, because $\deg(Q_k(x)) \le k$, we are assured that every term in every $Q_k(x)$ will be present in some T_n. The uniqueness follows from the uniqueness of the polynomial coefficients (Theorem 80). $\qquad \square$

Example 118. Theorem 117 can be best understood with the aid of a concrete illustrative example. Define the differential operator,

$$T := \underbrace{(a_2 x^2 + b_1 x + c_0)}_{Q_2(x)} D^2 + \underbrace{(a_1 x + b_0)}_{Q_1(x)} D + \underbrace{(a_0)}_{Q_0(x)},$$

where $a_2, a_1, a_0, b_1, b_0, c_0 \in \mathbb{R}$. Using Theorem 117, we rewrite T, in terms of T_n's, and obtain

$$T = \left(\frac{Q_2^{(2)}(0)}{2!} x^2 D^2 + \frac{Q_1^{(1)}(0)}{1!} x^1 D^1 + \frac{Q_0^{(0)}(0)}{0!} x^0 D^0 \right) D^0 +$$

$$\left(\frac{Q_2^{(1)}(0)}{1!} x^1 D^1 + \frac{Q_1^{(0)}(0)}{0!} x^0 D^0 \right) D^1 +$$

$$\left(\frac{Q_2^{(0)}(0)}{0!} x^0 D^0 \right) D^2$$

$$= \underbrace{(a_2 x^2 D^2 + a_1 x D + a_0)}_{T_0} + \underbrace{(b_1 x D + b_0)}_{T_1} D + \underbrace{(c_0)}_{T_2} D^2.$$

Theorem 117 can be extended to arbitrary linear operators on $\mathbb{R}[x]$; reminiscent of a Laurent series from complex variables (see [77, p. 222]).

Theorem 119. *whLet $T : \mathbb{R}[x] \to \mathbb{R}[x]$ be an arbitrary linear operator,*

$$T := \sum_{k=0}^{\infty} Q_k(x) D^k.$$

Define the family of sequences,

$$\{b_{n,k}\}_{k=0}^{\infty} := \left\{ \sum_{j=0}^{k} \binom{k}{j} Q_{j+n}^{(j)}(0) \right\}_{k=0}^{\infty}, \qquad n \in \mathbb{Z},$$

where we take $Q_{j+n}^{(j)}(0) = 0$ for $j + n < 0$. For each $n \in \mathbb{Z}$, define the classical diagonal differential operator,

$$T_n[x^k] := b_{n,k} x^k.$$

Then,

$$T = \sum_{n=1}^{\infty} T_{-n} D^{-n} + \sum_{n=0}^{\infty} T_n D^n, \tag{2.4.4}$$

where we define $D \cdot D^{-1} = 1$. Furthermore, the representation in (2.4.4) is unique.

Proof. We first note that for each $n \in \mathbb{N}_0$, $T_n = \sum_{k=0}^{\infty} \dfrac{Q_{k+n}^{(k)}(0)}{k!} x^k D^k$ (see Theorems 50 and 74). Similar to the proof of Theorem 117, each term from the T_n's are in one-to-one correspondence with each term in the Q_k's. Thus, a change of index yields,

$$T = \sum_{n=0}^{\infty} Q_n(x) D^n = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} \frac{Q_k^{(k)}(0)}{k!} x^k \right) D^n = \sum_{n=-\infty}^{\infty} \left(\sum_{k=0}^{\infty} \frac{Q_{k+n}^{(k)}(0)}{k!} x^k \right) D^{k+n} = \sum_{n=-\infty}^{\infty} T_n D^n. \qquad \square$$

Example 120. Similar to Example 118 we provide an example to illustrate Theorem 119. Define the differential operator,

$$T := (a_2 x^2 + b_1 x + c_0) D^2 + (z_1 x^2 + a_1 x + b_0) D + (y_0 x^2 + z_0 x + a_0),$$

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where $y_0, z_1, z_0, a_2, a_1, a_0, b_1, b_0, c_0 \in \mathbb{R}$. Using Theorem 119, we rewrite T in terms of T_n's,

$$
\begin{aligned}
T \quad = \quad & \underbrace{(y_0 x^2 D^2)}_{T_{-2}} & \boxed{D^{-2}} \ + \\
& \underbrace{(z_1 x^2 D^2 + z_0 x D)}_{T_{-1}} & \boxed{D^{-1}} \ + \\
& \underbrace{(a_2 x^2 D^2 + a_1 x D + a_0)}_{T_0} & \boxed{D^0} \ + \\
& \underbrace{(b_1 x D + b_0)}_{T_1} & \boxed{D^1} \ + \\
& \underbrace{(c_0)}_{T_2} & \boxed{D^2} \ .
\end{aligned}
$$

Example 121. It is possible for representation (2.4.4) to be "transcendental" in both directions. Consider the differential operator,

$$
T := \sum_{k=0}^{\infty} (x^{2k} + 1) D^k.
$$

Then for $n \in \mathbb{N}$, $T_{-n} = x^{2n} D^{2n}$ and for $n \in \mathbb{N}_0$, $T_n = 1$. Hence,

$$
\begin{aligned}
T &= \cdots + T_{-2} D^{-2} + T_{-1} D^{-1} + T_0 D^0 + T_1 D^1 + T_2 D^2 + \cdots \\
&= \cdots + (x^4 D^4) D^{-2} + (x^2 D^2) D^{-1} + (1) D^0 + (1) D^1 + (1) D^2 + \cdots .
\end{aligned}
$$

Problem 122. The above example is not hyperbolicity preserving. We ask, do there exist hyperbolicity preservers, $T = \sum_{k=-\infty}^{\infty} T_k D^k$, such that $T_k \not\equiv 0$ for every $k \in \mathbb{N}$? Compare with the open problem on "increasing degree" of A. Piotrowski [85, Problem 197, p. 172].

Upon attaining the representation (2.4.3) in Theorem 117, we direct our attention to the property of hyperbolicity preservation. If T in equation (2.4.3), is hyperbolicity preserving, then what properties do the T_n's possess? One might hope that the T_n's also enjoy the property of hyperbolicity preservation. This hope would certainly be warranted since, in fact, T_0 (see $\tilde{T}$ from the previous section, Theorem 115) always possesses the property of hyperbolicity preservation in a diagonal differential operator (see also [5, 20, 85]). In addition, classical multiplier sequences and operators of the form $f(xD)$ and $f(D)$ trivially have T_n's that are hyperbolicity preserving, by the Hermite-Poulain [83, p. 4] (Theorem 77) and Laguerre Theorems [83, Satz 3.2] (Theorem 78). However, in general, our hope is not justified as the next several examples

demonstrate. The following Turán-type inequality (2.4.5) will be of great use. However, for the sake of clarity of presentation, we will postpone the proof of Theorem 123 till Chapter 3.

Theorem 123 ([7, Theorem 15 and 19]). *Let $a, b, c, r_1, r_2, r_3 \in \mathbb{R}$, $r_1 \neq r_2$. If $Q_2(x) = a(x - r_1)(x - r_2)$, $Q_1(x) = b(x - r_3)$, and $Q_0(x) = c$, then T is hyperbolicity preserving, where*

$$T := Q_2(x)D^2 + Q_1(x)D + Q_0(x),$$

if and only if a, b, c are of the same sign and

$$b^2 \left(\frac{(r_1 - r_3)(r_3 - r_2)}{(r_1 - r_2)^2} \right) - ac \geq 0. \tag{2.4.5}$$

Set $\left(\dfrac{(r_1 - r_3)(r_3 - r_2)}{(r_1 - r_2)^2} \right) = \dfrac{1}{4}$, when $r_1 = r_2 = r_3$. If $r_1 = r_2$ and $r_1 \neq r_3$, then T is not hyperbolicity preserving.

Remark 124. We remark that the condition that a, b, c be of the same sign, in Theorem 123, cannot be removed. For example, the following operator satisfies inequality (2.4.5) but not the necessary sign condition of the leading coefficients. Indeed,

$$T := (x - 1)(x + 1)D^2 - 2xD + 1$$

is not hyperbolicity preserving, as can be seen since $T[x^2] = -x^2 - 2$.

Example 125. Consider the following differential operator,

$$T := (x - 2)(x + 1)D^2 + 3(x + 1/2)D + 1.$$

By an application of Theorem 123, operator T is certainly hyperbolicity preserving,

$$3^2 \left(\frac{(2 - (-1/2))\left((-1/2) - (-1)\right)}{((-1) - 2)^2} \right) - 1 \cdot 1 = \frac{1}{4} \geq 0.$$

However, $T_1 = -xD + \dfrac{3}{2}$ (see (2.4.2)) is not a hyperbolicity preserver, since $T_1[x^2 - 1] = -\dfrac{1}{2}x^2 - \dfrac{3}{2}$.

Example 126. The Legendre polynomials, $\{P_n(x)\}_{n=0}^\infty$, satisfy the differential equation (Theorem 34),

$$((x^2 - 1)D^2 + (2x)D + 1)P_n(x) = (n^2 + n + 1)P_n(x). \tag{2.4.6}$$

Equation (2.4.6) was first verified to be hyperbolicity preserving in [11, Lemma 5]. We provide here an alternate verification that $(x^2 - 1)D^2 + (2x)D + 1$ is a hyperbolicity preserver using Theorem 123,

$$2^2 \left(\frac{(1 - 0)(0 - (-1))}{(-1 - 1)^2} \right) - 1 \cdot 1 = 1 - 1 = 0 \geq 0.$$

Thus, T is hyperbolicity preserving, where $T[P_n(x)] := (n^2 + n + 1)^3 P_n(x)$, and

$$
\begin{aligned}
T =& ((x^2 - 1)D^2 + (2x)D + 1)^3 \\
=& (x^6 - 3x^4 + \boxed{3x^2} - 1)D^6 + \\
& (18x^5 - 36x^3 + \boxed{18x})D^5 + \\
& (101x^4 - 130x^2 + \boxed{29})D^4 + \\
& (208x^3 - 160x)D^3 + \\
& (145x^2 - 57)D^2 + \\
& (26x)D + \\
& 1.
\end{aligned}
$$

Consider the highlighted terms above to calculate T_4 (see (2.4.2)),

$$T_4 = 3x^2 D^2 + 18xD + 29.$$

From Theorem 123 we infer that the operator T_4 fails to be hyperbolicity preserving,

$$18^2 \left(\frac{1}{4} \right) - 3 \cdot 29 = 81 - 87 = -6 < 0.$$

Example 127. By a result of A. Piotrowski, multiplier sequences are invariant under affine transformations (see [85, Lemma 157, p. 145]). Let us consider then an affine transformation of the Hermite polynomials, $\{H_n(x \pm 3)\}_{n=0}^\infty$, and a multiplier sequence for these shifted Hermite polynomials, $\{n^2 + n + 1\}_{n=0}^\infty$ (see Theorem 194). Thus T is hyperbolicity preserving, where $T[H_n(x \pm 3)] = (n^2 + n + 1)H_n(x \pm 3)$. We express

T in the form (see formula (2.3.1)),

$$T = \left(\frac{1}{4}\right) D^4 + \left(\boxed{-x} \mp 3\right) D^3 + \left(x^2 \pm 6x + \boxed{\frac{15}{2}}\right) D^2 + (2x \pm 6) D + (1) . \qquad (2.4.7)$$

Using the highlighted terms in (2.4.7) we set $T_2 = -xD + \dfrac{15}{2}$ (see (2.4.2)) and note that T_2 is not hyperbolicity preserving since $T[2x^8 - 2x^6] = -x^8 - 3x^6$.

Remark 128. It is intriguing to see that while affine transformations have the same multiplier sequences, the T_n's in equation (2.4.3) may not preserve hyperbolicity. Hence, as we will see in Theorems 196 and 197, the Hermite polynomials are distinguished amongst affine transformations of the Hermite polynomials since the Hermite polynomials will have T_n's from equation (2.4.3) that do preserve hyperbolicity.

Example 129. Consider the shifted Laguerre polynomials (see [85, Lemma 157, p. 145]), $\{L_n(x + 2)\}_{n=0}^{\infty}$, and a multiplier sequence for these shifted Laguerre polynomials, $\{n\}_{n=0}^{\infty}$ (see Theorem 213). Thus T is hyperbolicity preserving, where $T[L_n(x + 2)] = nL_n(x + 2)$, and

$$T = \left(\boxed{-x} - 2\right) D^2 + (x + \boxed{1})D + (0) .$$

Consider the operator formed by the highlighted terms, $T_1 = -xD + 1$ (see (2.4.2)). The operator, T_1, fails to preserve hyperbolicity since $T_1[x^2 - 1] = -x^2 - 1$.

Example 130. A more technical example is the following. Using the generalized Malo-Schur-Szegő Composition Theorem [23] (Theorem 69) it can be shown that, with $p(x) = (x + 1)^3$, the operator

$$\begin{aligned}
T :&= -\frac{1}{6}p'''(x)D^3 + \frac{1}{2}p''(x)D^2 - p'(x)D + p(x) \\
&= -D^3 + (\boxed{3x} + 3)D^2 + (-3x^2 - 6x \boxed{-3})D + (x^3 + 3x^2 + 3x + 1)
\end{aligned}$$

is hyperbolicity preserving (see also [105, Proposition 112, p. 47]). Define $T_1 := 3xD - 3$ (see (2.4.2)), and note that $T_1[x^2 - 1] = 3x^2 + 3$ and thus T_1 is not hyperbolicity preserving.

Example 131. We consider another example involving Q_k's, where $\deg(Q_k(x)) > k$ for some $k \in \mathbb{N}_0$. Using the Hermite-Poulain Theorem [83, p. 4] (Theorem 77) it can be shown that the non-diagonal operator,

$$T := (\boxed{x^2} + 2x + 1)D^2 - (x^2 + 2x + \boxed{1}),$$

preservers hyperbolicity. The operator $T_0 = x^2 D^2 - 1$ (see (2.4.2)) is not a hyperbolicity preserver, since $T_0[x^2 - 1] = x^2 + 1$. This example is even more interesting considering the fact that, in general, T_0 is always hyperbolicity preserving, whenever T is any arbitrary diagonal differential hyperbolicity preserver (see Theorem 115 or [5, Theorem 45]).

We remark that Examples 125, 130, 129, 127, 126, 125, and 131 demonstrate the very high sensitivity of the following results; namely, for Hermite or Laguerre multiplier sequences the T_n's in Theorem 117 are hyperbolicity preservers (see Chapter 4). It is surprising that not only will each T_n be hyperbolicity preserving, the family of sequences, $\{b_{n,k}\}_{k=0}^{\infty}$ (see (2.4.1)), turn out to be Hermite or Laguerre multiplier sequences, respectively. In this sense, every Hermite or Laguerre multiplier sequence generates a family of additional Hermite or Laguerre multiplier sequences.

2.5   Hyperbolicity Preserving Differential Operators

In this section we present results pertaining to hyperbolicity preserving differential operators. We will present several observations and examples concerning hyperbolicity preservers. In particular, we provide a new proof of the famous Borcea-Brändén Theorem concerning finite order hyperbolicity preservers (Theorem 138); we show that the Borcea-Brändén Theorem follows from the Malo-Schur-Szegő Composition Theorem (see Theorem 69). The motivation of our investigation stems from the open problem stated in the Preface, Problem 1. We reiterate Problem 1 here, for the reader's convenience.

Problem 132 (Problem 1)**.** Let $T : \mathbb{R}[x] \to \mathbb{R}[x]$ be a linear operator, represented as

$$T := \sum_{k=0}^{\infty} Q_k(x) D^k,$$

where $Q_k(x) \in \mathbb{R}[x]$, $k \in \mathbb{N}_0$. Characterize the polynomial sequences, $\{Q_k(x)\}_{k=0}^{\infty}$, so that T is hyperbolicity preserving.

Our first insight into the nature of polynomial coefficients, $\{Q_k(x)\}_{k=0}^{\infty}$, where T is a hyperbolicity preserver, originates from a preliminary result of P. Brändén.

Theorem 133 (P. Brändén [20, Lemma 2.7])**.** *If the finite order differential operator,* $T : \mathbb{R}[x] \to \mathbb{R}[x]$,

$$T = \sum_{k=0}^{n} Q_k(x) D^k,$$

preserves hyperbolicity, then each $Q_k(x)$ is hyperbolic. Moreover, $Q_{k-1}(x) \ll Q_k(x)$ for $1 \leq k \leq n$.

The condition $Q_{k-1}(x) \ll Q_k(x)$ from Theorem 133 explains the "sign-convention" (see Definition 25) that is present within finite order hyperbolicity preservers (see Theorem 123). However, the following example demonstrates that, in general, the converse of Theorem 133 is false.

Example 134. Consider the following list of finite order differential operators:

$$T_1 := (x^2 - 1)D^2 + 2xD - 1,$$

$$T_2 := (x^2 - 1)D^2 + 2xD + 0,$$

$$T_3 := (x^2 - 1)D^2 + 2xD + 1,$$

$$T_4 := (x^2 - 1)D^2 + 2xD + 2,$$

$$T_5 := (x^2 - 1)D^2 - 2xD - 1,$$

$$T_6 := (x^2 - 1)D^2 - 2xD + 0,$$

$$T_7 := (x^2 - 1)D^2 - 2xD + 1, \text{ and}$$

$$T_8 := (x^2 - 1)D^2 - 2xD + 2.$$

It was shown in [11, Lemma 4 and 5] that T_2 and T_3 are hyperbolicity preserving. The other six examples fail to be hyperbolicity preservers:

$$T_1[x^2 - 1] = 5x^2 + 2,$$

$$T_4[(x - 10)^3] = 2(x - 10)(7x^2 - 50x + 97),$$

$$T_5[x^2] = -3x^2 - 2,$$

$$T_6[x^2] = -2x^2 - 2,$$

$$T_7[x^2] = -x^2 - 2, \quad \text{and}$$

$$T_8[(x - 10)^3] = 2(x - 10)(x^2 + 10x + 97).$$

Theorem 135 ([16, Lemma 4.4]). *Suppose $T : \mathbb{R}[x] \to \mathbb{R}[x]$, $T = \sum_{k=0}^{\infty} Q_k(x)D^k$, is hyperbolicity preserving. Then for $r, \beta, \alpha \in \mathbb{R}$, $\beta \neq 0$, $0 < \alpha < 1$, the operators*

$$\sum_{k=0}^{\infty} Q_k(x + r)D^k, \ \sum_{k=0}^{\infty} Q_k\left(\frac{x}{\beta}\right)\beta^k D^k, \ and \ \sum_{k=0}^{\infty} Q_k(x)\alpha^k D^k,$$

are also hyperbolicity preservers.

Proof. For the first operator, we note that if $f(x)$ is hyperbolic, then $f(x-r)$ is hyperbolic. So $\sum_{k=0}^{\infty} Q_k(x) f^{(k)}(x-r)$ is hyperbolic. Thus, $\sum_{k=0}^{\infty} Q_k(x+r) f^{(k)}(x)$ is hyperbolic and we conclude that $\sum_{k=0}^{\infty} Q_k(x+r) D^k$ is hyperbolicity preserving. Likewise, if $f(x)$ is hyperbolic, then so is $f(\beta x)$. Thus, $\sum_{k=0}^{\infty} Q_k(x) \beta^k f^{(k)}(\beta x)$ is also hyperbolic and so $\sum_{k=0}^{\infty} Q_k\left(\frac{x}{\beta}\right) \beta^k f^{(k)}(x)$ is hyperbolic. We conclude, $\sum_{k=0}^{\infty} Q_k\left(\frac{x}{\beta}\right) \beta^k D^k$ is hyperbolicity preserving.

For the third operator, we follow the outline of J. Borcea and P. Brändén [16, Lemma 4.4]. Let $\alpha \in (0,1)$ and define

$$T := \sum_{k=0}^{\infty} Q_k(x) D^k.$$

Given any hyperbolicity preserver, H, we can apply the Hermite-Kakeya-Obreschkoff Theorem [27, Remark 6, p. 5] and a Lemma of S. Fisk [51, p. 18, Lemma 1.31], to conclude that $H[f'(x)] \ll H[f(x)]$ and $H[f'(x)] \ll xH[f'(x)]$, hence for each $m \in \mathbb{N}_0$,

$$H + \left(\frac{1-\alpha}{\alpha \cdot m}\right) xHD,$$

is hyperbolicity preserving. Iteration m times with the operator T, yields

$$\sum_{k=0}^{m} \binom{m}{k} \frac{k!}{m^k} \left(\frac{1-\alpha}{\alpha}\right)^k \frac{x^k}{k!} TD^k$$

is hyperbolicity preserving. Letting $m \to \infty$, we conclude that

$$\sum_{k=0}^{\infty} \left(\frac{1-\alpha}{\alpha}\right)^k \frac{x^k}{k!} TD^k$$

is hyperbolicity preserving. Thus, if $f(x)$ is a hyperbolic polynomial, then

$$\left[\sum_{k=0}^{\infty} Q_k(x) \alpha^k D^k\right] f(x) = \sum_{k=0}^{\infty} Q_k(x) \alpha^k f^{(k)}(x)$$

$$= \sum_{k=0}^{\infty} Q_k(x) \alpha^k \left(\sum_{j=0}^{\infty} \frac{f^{(k+j)}(\alpha x)}{j!} ((1-\alpha)x)^j\right)$$

$$= \sum_{j=0}^{\infty} \left(\frac{1-\alpha}{\alpha}\right)^j \frac{x^j}{j!} \sum_{k=0}^{\infty} Q_k(x) \alpha^{k+j} f^{(k+j)}(\alpha x)$$

$$= \sum_{j=0}^{\infty} \left(\frac{1-\alpha}{\alpha}\right)^j \frac{x^j}{j!} \left(\sum_{k=0}^{\infty} Q_k(x)D^k\right)(D^j)f(\alpha x)$$

$$= \left[\sum_{j=0}^{\infty} \left(\frac{1-\alpha}{\alpha}\right)^j \frac{x^j}{j!}TD^j\right]f(\alpha x)$$

must also have only real zeros. Hence, T is hyperbolicity preserving. $\qquad\square$

The above theorem allows us to generalize Theorem 115 to arbitrary triangular operators (Definition 91). Compare this with Example 131.

Corollary 136. *Suppose* $T : \mathbb{R}[x] \to \mathbb{R}[x]$ *is a triangular hyperbolicity preserving operator,*

$$T := \sum_{k=0}^{\infty} Q_k(x)D^k.$$

Then $\tilde{T}$ *is also a triangular hyperbolicity preserver and diagonalizes on the standard basis,*

$$\tilde{T}[x^n] := \left(\sum_{k=0}^{\infty} \frac{Q_k^{(k)}}{k!}x^k D^k\right)x^n = \left(\sum_{k=0}^{n} \binom{n}{k}Q_k^{(k)}\right)x^n.$$

Proof. From Theorem 135, we know that

$$T_m := \sum_{k=0}^{\infty} Q_k(mx)\frac{1}{m^k}D^k,$$

is hyperbolicity preserving for every $m \in \mathbb{N}_0$. Thus, letting $m \to \infty$ yields the above. $\qquad\square$

Interestingly, Corollary 136 provides a method of associating a multiplier sequence to every triangular hyperbolicity preserver. Moreover, if a finite order operator is not triangular, then one can compose a D^m (m sufficiently large) so that the operator will be triangular. In this way, every linear operator can be related to a classical multiplier sequence.

Utilizing the work of J. Borcea and P. Brändén [16], we derive the following corollary that will be useful in proving the finite order version of the Borcea-Brändén Theorem. The key ingredient needed is Corollary 98; a useful tool for analyzing finite order hyperbolicity preservers.

Corollary 137. *If* $T := \sum_{k=0}^{n} Q_k(x)D^k$ *is a non-zero hyperbolicity preserver, then for every* $\alpha, \beta \in \mathbb{R}$, $\beta > 0$, *the meromorphic function* $\sum_{k=0}^{n} Q_k(x+\alpha)\left(\frac{\beta}{x}\right)^k$ *is a non-zero hyperbolic meromorphic function.*

Proof. To prove $\sum_{k=0}^{n} Q_k(x + \alpha) \left(\frac{\beta}{x} \right)^k$ is hyperbolic, we note that by Theorem 135,

$$\sum_{k=0}^{n} Q_k \left(\frac{x}{m} + \alpha \right) m^k \left(\frac{\beta}{m} \right)^k D^k$$

is hyperbolicity preserving for $m \in \mathbb{N}_0$, where $m > \beta$. Hence,

$$\lim_{m \to \infty} \frac{1}{x^m} \left(\sum_{k=0}^{n} Q_k \left(\frac{x}{m} + \alpha \right) m^k \left(\frac{\beta}{m} \right)^k D^k \right) x^m = \sum_{k=0}^{n} Q_k(x + \alpha) \left(\frac{\beta}{x} \right)^k$$

is a hyperbolic meromorphic function.

We now show that $\sum_{k=0}^{n} Q_k(x+\alpha) \left(\frac{\beta}{x} \right)^k$ is not the zero function. Let $m \in \mathbb{N}_0$ such that $\deg(Q_k(x)) \leq k+m$ for $0 \leq k \leq n$ and $\deg(Q_k(x)) = k + m$ for some $0 \leq k \leq n$. Thus we calculate the leading coefficient,

$$\sum_{k=0}^{n} Q_k(x + \alpha) \left(\frac{\beta}{x} \right)^k = \left(\sum_{k=0}^{n} \frac{Q_k^{(k+m)}}{(k+m)!} \beta^k \right) x^m + \cdots .$$

To see that $\sum_{k=0}^{n} \frac{Q_k^{(k+m)}}{(k+m)!} \beta^k \neq 0$ we apply Corollary 136 to TD^m (a triangular operator) to conclude that

$$\tilde{T}[x^j] = \left(\sum_{k=0}^{n} \frac{Q_k^{(k+m)}}{(k+m)!} x^{k+m} D^{k+m} \right) x^j := \gamma_j x^j$$

is a classical hyperbolicity preserving operator. Thus by the Pólya-Schur Theorem (Theorem 54),

$$\sum_{k=0}^{\infty} \frac{\gamma_k}{k!} x^k = e^x \left(\sum_{k=0}^{n} \frac{Q_k^{(k+m)}}{(k+m)!} x^{k+m} \right) \in \mathscr{L} - \mathscr{P}^{sa}.$$

However, $\tilde{T}$ is a finite order diagonal differential operator, hence by Corollary 98, the eigenvalues of $\tilde{T}$, $\{\gamma_k\}_{k=0}^{\infty}$, are interpolated by a polynomial; i.e., $\{\gamma_k\}_{k=0}^{\infty}$ cannot be an alternating sequence (see Theorem 42). Thus, $\sum_{k=0}^{n} \frac{Q_k^{(k+m)}}{(k+m)!} x^{k+m} \in \mathscr{L} - \mathscr{P}^s$ and has no positive roots. In particular, since $\beta > 0$,

$$\sum_{k=0}^{n} \frac{Q_k^{(k+m)}}{(k+m)!} \beta^{k+m} \neq 0,$$

and thus we have shown that $\sum_{k=0}^{n} Q_k(x + \alpha) \left(\frac{\beta}{x} \right)^k$ is not the zero function. $\qquad \square$

Corollary 137 and the Malo-Schur-Szegő Composition Theorem (Theorem 69) provide a new interesting proof of the famous Borcea-Brändén Theorem.

Theorem 138 (J. Borcea and P. Brändén [16, Theorem 1.3]). *Let $T : \mathbb{R}[x] \to \mathbb{R}[x]$, where $T = \sum_{k=0}^{n} Q_k(x)D^k$, $Q_k(x) \in \mathbb{R}[x]$, and $T \not\equiv 0$. Then T is hyperbolicity preserving if and only if*

$$\sum_{k=0}^{n} Q_k(x)(-w)^k$$

is a real bivariate stable polynomial.

Proof. One direction follows the proof of J. Borcea and P. Brändén (see [16]). Assume $T = \sum_{k=0}^{n} Q_k(x)D^k$ is hyperbolicity preserving and let $z_0, w_0 \in H^+ := \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$. We will show that $\sum_{k=0}^{n} Q_k(z_0)(-w_0)^k \neq 0$. Let $\alpha, \beta \in \mathbb{R}$, $\beta > 0$, and $v \in H^+$ such that

$$z_0 = v + \alpha \qquad \text{and} \qquad -w_0 = \frac{\beta}{v}.$$

Using Corollary 137, we see that

$$\sum_{k=0}^{n} Q_k(x + \alpha) \left(\frac{\beta}{x} \right)^k$$

must be a non-zero hyperbolic meromorphic function. Hence, since $v \in H^+$,

$$\sum_{k=0}^{n} Q_k(v + \alpha) \left(\frac{\beta}{v} \right)^k \neq 0;$$

that is, $\sum_{k=0}^{n} Q_k(z_0)(-w_0)^k \neq 0$.

Conversely, suppose that $\sum_{k=0}^{n} Q_k(z)(-w)^k \neq 0$ for every $z, w \in H^+$. Since T is not the zero operator there is $0 \leq v \leq n$ such that $Q_v(x) \not\equiv 0$ and $Q_k(x) \equiv 0$ for $k < v$. By assumption, for each $w \in H^+$,

$$\sum_{k=v}^{n} Q_k(z)(-w)^{k-v}$$

is a univariate lower stable polynomial (see Definition 13). Hence, letting $w \to 0$, we conclude that $Q_v(x)$ is a univariate real stable polynomial; i.e., $Q_v(x)$ is a non-zero hyperbolic polynomial (see Remark 15). Thus,

let $f(x)$ be any hyperbolic polynomial of degree m, $m > v$. We will show that

$$g(x) := \sum_{k=0}^{n} Q_k(x) f^{(k)}(x)$$

has only real zeros. Fix $z_0 \in H^+$. Then for each $\epsilon > 0$,

$$\sum_{k=0}^{n} Q_k(z_0)(w - \epsilon \cdot i)^k := \sum_{k=0}^{n} Q_{\epsilon,k}(z_0) w^k \tag{2.5.1}$$

has zeros only in $S_\pi := \{w \in \mathbb{C} : \theta_1 + 0 < \arg(w) < \theta_1 + \pi\} = H^+$, where $\theta_1 = 0$ (cf. Theorem 69). Moreover, there is $0 < \delta < 1$ such that

$$f(z_0 - w) = \sum_{k=0}^{m} \frac{f^{(k)}(z_0)}{k!}(-w)^k \tag{2.5.2}$$

only has zeros in $S_{\pi-2\delta} := \{w \in \mathbb{C} : \theta_2 < \arg(w) < \theta_2 + (\pi - 2\delta)\}$, where $\theta_2 = \delta$ (cf. Theorem 69). Hence by the Malo-Schur-Szegő Composition Theorem (Theorem 69), equations (2.5.1) and (2.5.2) yield,

$$\sum_{k=0}^{n} Q_{\epsilon,k}(z_0) f^{(k)}(z_0)(-w)^k,$$

has zeros only in

$$-S_\pi S_{\pi-2\delta} = \{w \in \mathbb{C} : (0 + \delta + \pi) < \arg(w) < (0 + \delta + \pi) + (\pi) + (\pi - 2\delta)\}.$$

Letting $\epsilon \to 0$, we conclude that

$$\sum_{k=0}^{n} Q_k(z_0) f^{(k)}(z_0)(-w)^k, \tag{2.5.3}$$

has zeros only in $\{z \in \mathbb{C} : \pi + \delta \leq \arg(z) \leq 3\pi - \delta\} \subseteq \mathbb{C} - (-\infty, 0)$ or (2.5.3) is the zero polynomial. However, since $Q_v(x)$ and $f(x)$ are hyperbolic and $\deg(f(x)) > v$, the vth coefficient of (2.5.3) is not zero; that is, $Q_v(z_0) f^{(v)}(z_0)(-1)^v \neq 0$ (see also Theorem 6). Thus, the former case must hold and in particular for $w = -1 \in \mathbb{C} - (-\infty, 0)$,

$$\sum_{k=0}^{n} Q_k(z_0) f^{(k)}(z_0) \neq 0.$$

Thus, for every $z_0 \in H^+$, $g(z_0) \neq 0$. Therefore $g(x)$ is a univariate real stable polynomial; i.e. $g(x)$ is hyperbolic (see Remark 15). $\qquad\square$

Remark 139. The proof above demonstrates an interesting property of finite order hyperbolicity preservers. Namely, if T is a finite order hyperbolicity preserver, then there is $m \in \mathbb{N}_0$, such that $T[f(x)] \equiv 0$ if $\deg(f(x)) \leq m$ and $T[f(x)] \not\equiv 0$ if $\deg(f(x)) > m$, for $f(x) \in \mathscr{L} - \mathscr{P} \cap \mathbb{R}[x]$. This property is not possessed by non-hyperbolicity preserving operators. For example, if

$$T := x^2 D^2 - 2,$$

then $T[1] = -2$, $T[x] = -2x$, $T[x^2] = 0$, and $T[x^3] = 4x^3$ (see Theorem 184).

With the aid of uniform convergence, we can derive a "transcendental" version of Theorem 138.

Corollary 140. *Suppose there is a sequence of real bivariate stable polynomials, $\{p_n(x, w)\}_{n=0}^{\infty}$, and a real bivariate entire function, $f(x, w) = \sum_{k=0}^{\infty} Q_k(x)w^k$, $Q_k(x) \in \mathbb{R}[x]$, such that*

$$p_n(x, w) \to f(x, w),$$

locally uniformly on compact subsets of $\mathbb{C} \times \mathbb{C}$. Then

$$T := \sum_{k=0}^{\infty} Q_k(x)(-1)^k D^k$$

is a hyperbolicity preserving operator.

Example 141. Surprisingly, not all hyperbolicity preserving operators can be found by uniform limits of real bivariate stable polynomials. Consider the following operator,

$$T := \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} x^k D^k.$$

If $f(x) \in \mathbb{R}[x]$, then $T[f(x)] = f(0)$ (see Example 71). Hence, T is hyperbolicity preserving. However, the corresponding transcendental entire function, $f(x, w)$, from Corollary 140, is

$$f(x, w) = \sum_{k=0}^{\infty} \frac{1}{k!} x^k w^k = e^{xw},$$

a function that fails to be the local uniform limit of real bivariate stable polynomials. This follows because if e^{xw} was approximable by real bivariate stable polynomials, $\{p_n(x, w)\}_{n=0}^{\infty}$, then $e^{x \cdot x}$ would be approximable

by the real univariate stable polynomials, $\{p_n(x,x)\}_{n=0}^{\infty}$, hence e^{x^2} would be in $\mathscr{L} - \mathscr{P}$, which is impossible (see Theorem 6).

At first, the foregoing analysis indicates that hyperbolicity preservers simply have an "alternating" counterpart; maybe a replacement of $f(x,w)$ with $f(x,-w)$ would yield a characterization. However, this is not the case; Example 134 shows that $(x^2-1)D^2+2xD+1$ is hyperbolicity preserving while $(x^2-1)D^2-2xD+1$ is not. Hence, the issue here seems to somehow stem from the triviality of the operator (T is of dimension one) rather than some inherited hyperbolicity property from finite order operators (see Theorem 138, Example 52, and Definitions 87 and 79).

The Borcea-Brändén Theorem (Theorem 138) does extend to all operators of infinite order, although its statement is not as transparent as Corollary 140.

Theorem 142 ([17, Theorem 5]). *Suppose $T : \mathbb{R}[x] \to \mathbb{R}[x]$, where $T = \displaystyle\sum_{n=0}^{\infty} Q_n(x)D^n$ is a non-trivial operator. Then T is hyperbolicity preserving if and only if either*

$$\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} Q_k(x)\frac{x^{n-k}}{(n-k)!}\right)w^n \quad or \quad \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} Q_k(x)\frac{x^{n-k}}{(n-k)!}\right)(-w)^n$$

is the local uniform limit on $\mathbb{C} \times \mathbb{C}$ of real bivariate stable polynomials.

Despite the amazingly comprehensive and far-reaching results of the Borcea-Brändén Theorem, we still find operators where the hyperbolicity preservation property is uncertain. This seems to stem from the fact that very little is known about real bivariate stable polynomials and real bivariate transcendental entire functions that are local uniform limits of real bivariate stable polynomials. In fact, determining said class seems a more difficult task than Problem 2 of the Introduction. However, the Borcea-Brändén Theorems do offer the chance to view hyperbolicity preservers in a new light. In fact, many non-obvious properties of hyperbolicity preservers can be determined from the Borcea-Brändén Theorems.

Problem 143. Consider the following cubic triangular operator,

$$T_{\alpha,\beta} := (x^3 - x)D^3 + \beta(x - .5)(x + .25)D^2 + \beta x D + \alpha.$$

For what α and β is $T_{\alpha,\beta}$ hyperbolicity preserving? Compare with Problem 189.

To better understand hyperbolicity preserving differential operators, we will begin to list methods of finding sequences of polynomials, $\{Q_k(x)\}_{k=0}^{\infty}$, so that the corresponding operator, $T = \displaystyle\sum_{k=0}^{\infty} Q_k(x)D^k$, is

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hyperbolicity preserving. For the reader's convenience, we recall here the following theorem (Theorem 78), stated in the perspective of our current objective (see also Theorems 50 and 73).

Theorem 144 (Laguerre [83, Satz 3.2], [39, Theorem 1.4]). *If $f(x) \in \mathscr{L} - \mathscr{P}(-\infty, 0]$, then the linear operator*

$$T := f(xD) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \binom{n}{k} f(k)(-1)^{n-k} \right) \frac{x^n}{n!} D^n$$

is hyperbolicity preserving.

Theorem 145. *If $T := \sum_{k=0}^{\omega} Q_k(x)D^k$, $0 \le \omega \le \infty$, is a non-trivial hyperbolicity preserving differential operator, then each of the following is also a hyperbolicity preserver:*

1. $\displaystyle\sum_{k=0}^{\omega} Q_k(\alpha x + \beta)\gamma^k D^k$, $\alpha, \beta, \gamma \in \mathbb{R}$, $\alpha, \gamma \ge 0$,

2. $\displaystyle\sum_{k=0}^{\omega} Q_k(-x)(-1)^k D^k$,

3. $\displaystyle\sum_{k=0}^{\omega} Q_k(x)k D^{k-1}$,

4. $\displaystyle\sum_{k=0}^{\omega} Q_k'(x) D^k$,

5. $\displaystyle\sum_{k=0}^{\omega} Q_{\omega-k}(x)(-1)^k D^k$, $\omega < \infty$,

6. $\displaystyle\sum_{k=0}^{\omega} x^k Q_k(D) = \sum_{j=0}^{\infty} \left(\frac{1}{j!} \sum_{k=0}^{\omega} Q_k^{(j)}(0)x^k \right) D^j$, $\omega < \infty$, *and*

7. $\displaystyle\sum_{k=0}^{\omega} \left(\sum_{j=0}^{k} \frac{Q_j(x)}{(k-j)!} \right) D^k$.

Proof. Items (1) and (2) follow from the fact that if $p(x, w)$ is stable then so is $p(\alpha x + \beta, \gamma w)$, $\alpha, \gamma > 0$, $\beta \in \mathbb{R}$. Items (3) and (4) follow from the Gauss-Lucas Theorem (Theorem 66). Item (5) is a polynomial reverse (see Definition 16) and item (6) is simply a statement where x and w have changed roles. Item (7) is Te^D. $\qquad\square$

Of interest is also the following theorem of Borcea-Brändén, that greatly generalizes Theorem 135.

Theorem 146 ([21], [16, Lemma 3.7]). *Suppose $\{\gamma_k\}_{k=0}^{\infty}$ is a positive multiplier sequence. If $T = \sum\limits_{k=0}^{\infty} Q_k(x) D^k$ is a hyperbolicity preserving operator, then*

$$\sum_{k=0}^{\infty} Q_k(x) \gamma_k D^k$$

is also a hyperbolicity preserving operator.

In general, for a multiplier sequence, $\{\gamma_k\}_{k=0}^{\infty}$, the sequence $\{g_k^*(-1)\}_{k=0}^{\infty}$ is not necessarily a multiplier sequence, where $\{g_k^*(x)\}_{k=0}^{\infty}$ are the associated reversed Jensen polynomials of $\{\gamma_k\}_{k=0}^{\infty}$ (cf. [54, Lemma 3.6]). Hence, the following corollary is not immediately obvious and is worth mentioning (cf. Theorem 55).

Corollary 147. *Let $\{\alpha_k\}_{k=0}^{\infty}$ be a positive multiplier sequence and let $\{\gamma_k\}_{k=0}^{\infty}$ be an arbitrary multiplier sequence. If $\{g_k^*(x)\}_{k=0}^{\infty}$ are the associated reversed Jensen polynomials of sequence $\{\gamma_k\}_{k=0}^{\infty}$, then*

$$\left\{ \sum_{k=0}^{n} \binom{n}{k} \alpha_k g_k^*(-1) \right\}_{n=0}^{\infty}$$

is also a multiplier sequence.

Proof. This follows because if $\{\gamma_k\}_{k=0}^{\infty}$ is a multiplier sequence, then $\sum\limits_{k=0}^{\infty} \dfrac{g_k^*(-1)}{k!} x^k D^k$ is hyperbolicity preserving (Theorem 74). Hence, by Theorem 146, $T := \sum\limits_{k=0}^{\infty} \dfrac{\alpha_k g_k^*(-1)}{k!} x^k D^k$ is hyperbolicity preserving. In particular, $T[x^n] = \beta_n x^n$, where $\beta_n = \sum\limits_{k=0}^{n} \binom{n}{k} \alpha_k g_k^*(-1)$ (cf. Theorem 50). $\qquad\square$

Theorem 148. *Suppose $f(x) \in \mathscr{L} - \mathscr{P}^s$ and let $\{g_k(x)\}_{k=0}^{\infty}$ be the associated Jensen polynomials of $f(x)$. Then*

$$\sum_{k=0}^{\infty} \frac{g_k(x)}{k!} D^k, \quad D := \frac{d}{dx},$$

is hyperbolicity preserving.

Proof. The generating function, of the associated Jensen polynomials, is $e^t f(xt)$ (see Theorem 21). Since $f(x) \in \mathscr{L} - \mathscr{P}^s$, then there is $\{p_n(x)\}_{n=0}^{\infty} \subseteq \mathscr{L} - \mathscr{P}^s \cap \mathbb{R}[x]$ such that

$$\left(1 - \frac{t}{n} \right)^n p_n(-xt) \to e^{(-t)} f(x(-t)) = \sum_{k=0}^{\infty} \frac{g_k(x)}{k!} (-t)^k,$$

locally uniformly on $\mathbb{C} \times \mathbb{C}$ (see Corollary 159). Thus by Corollary 140 the result follows. $\qquad\square$

Theorem 149. *Suppose $f(x) \in \mathscr{L} - \mathscr{P}$ and let $\{g_k^*(x)\}_{k=0}^\infty$ be the associated reversed Jensen polynomials of $f(x)$. Then*

$$\sum_{k=0}^\infty \frac{g_k^*(x)}{k!} D^k, \quad D := \frac{d}{dx},$$

is hyperbolicity preserving.

Proof. The generating function, of the associated reversed Jensen polynomials, is $e^{xt}f(t)$ (see Theorem 21). Since $f(x) \in \mathscr{L} - \mathscr{P}$, then there is $\{p_n(x)\}_{n=0}^\infty \subseteq \mathscr{L} - \mathscr{P} \cap \mathbb{R}[x]$ such that

$$\left(1 - \frac{xt}{n}\right)^n p_n(-t) \to e^{x(-t)}f(-t) = \sum_{k=0}^\infty \frac{g_k^*(x)}{k!}(-t)^k,$$

locally uniformly on $\mathbb{C} \times \mathbb{C}$ (see Corollary 159). Thus by Corollary 140 the result follows. $\qquad\square$

Corollary 150. *The following operators are hyperbolicity preserving,*

$$\sum_{k=0}^\infty \frac{H_k(x)}{k!} D^k \quad and \quad \sum_{k=0}^\infty \frac{L_k(-x)}{k!} D^k,$$

where $H_k(x)$ and $L_k(x)$ denote the kth Hermite and Laguerre polynomials, respectively.

Proof. The associated Jensen polynomials of $J_0\left(2\sqrt{-x}\right)$ are $\{L_k(-x)\}_{k=0}^\infty$; i.e., $e^t J_0(2\sqrt{-xt}) = \sum_{k=0}^\infty \frac{L_k(-x)}{k!} t^k$ (see Theorem 33). Similarly, the associated reversed Jensen polynomials of e^{-x^2} are $\{H_k(x/2)\}_{k=0}^\infty$; i.e., $e^{xt}e^{-t^2} = \sum_{k=0}^\infty \frac{H_k(x/2)}{k!} t^k$ (see Theorem 32). Now apply Theorems135, 148 and 149. $\qquad\square$

Example 151. Define the linear operator $T := \sum_{k=0}^\infty \frac{L_k(x)}{k!} D^k$. We show that T is not hyperbolicity preserving. Observe that

$$
\begin{aligned}
T[2x^2 - 4] &= \frac{1}{2}L_2(x)(4) + L_1(x)(4x) + L_0(x)(2x^2 - 4) \\
&= \frac{1}{2}\left(\frac{1}{2}x^2 - 2x + 1\right)(4) + (-x + 1)(4x) + (1)(2x^2 - 4) \\
&= x^2 - 4x + 2 - 4x^2 + 4x + 2x^2 - 4 \\
&= -x^2 - 2.
\end{aligned}
$$

Example 152. We show that the Legendre polynomials (see Theorem 34) fail to possess the property found in Corollary 150; i.e., the Legendre polynomials cannot be used as the Q_k's in a hyperbolicity preserver.

Define the operators,

$$T := \sum_{k=0}^{\infty} \frac{P_k(x)}{k!} D^k \quad \text{and} \quad \tilde{T} := \sum_{k=0}^{\infty} \frac{P_k(-x)}{k!} D^k.$$

We see that $T[2x^2 + 12x + 18] = 9x^2 + 24x + 17$ and $\tilde{T}[2x^2 + 12x + 18] = x^2 + 17$.

Problem 153. If $\{Q_k(x)\}_{k=0}^{\infty}$ is an orthogonal basis such that $\sum_{k=0}^{\infty} \frac{Q_k(x)}{k!} D^k$ is hyperbolicity preserving, must $\{Q_k(x)\}_{k=0}^{\infty}$ be an affine transformation of the Hermite or Laguerre polynomials?

One of the many reasons the Pólya-Schur characterization of multiplier sequences (Theorem 54) is so significant, stems from the fact that G. Pólya and J. Schur associate every hyperbolicity preserver to a particular transcendental entire function in $\mathscr{L} - \mathscr{P}^{sa}$. It is in this same spirit that a characterization of hyperbolicity preservers is desired. Towards that goal we establish the following, which will be of great use in Chapter 3 (see, for example, Theorem 172).

Theorem 154. *If* $\sum_{k=0}^{\infty} Q_k(x)D^k$ *is non-trivial hyperbolicity preserving operator, then for every* $r \in \mathbb{R}$,

$$\sum_{k=0}^{\infty} Q_k(r)x^k \quad \text{and} \quad \sum_{k=0}^{\infty} Q_k(x)r^k,$$

are in $\mathscr{L} - \mathscr{P}$.

Proof. The Borcea-Brändén Theorem (Theorem 142) says that either

$$e^{xw} \sum_{k=0}^{\infty} Q_k(x)w^k \quad \text{or} \quad e^{-xw} \sum_{k=0}^{\infty} Q_k(x)(-w)^k,$$

is the local uniform limit of real bivariate stable polynomials. Suppose the latter case holds; that is, suppose there exists a sequence of real bivariate stable polynomials, $\{p_n(x, w)\}_{n=0}^{\infty}$, such that

$$p_n(x, w) \to e^{-xw} \sum_{k=0}^{\infty} Q_k(x)(-w)^k.$$

Since for each $n \in \mathbb{N}_0$, $p_n(x, w) \neq 0$ for $x, w \in H^+$, then by letting $x \to r$ and $w \to -r$, we conclude that $p_n(r, w), p_n(x, -r) \in \mathscr{L} - \mathscr{P}$ (see Remark 15). Hence,

$$e^{-rw} \sum_{k=0}^{\infty} Q_k(r)(-w)^k \quad \text{and} \quad e^{xr} \sum_{k=0}^{\infty} Q_k(x)r^k$$

are the local uniform limit of hyperbolic polynomials and are therefore in $\mathscr{L} - \mathscr{P}$. Since $\mathscr{L} - \mathscr{P}$ is closed under multiplication and linear compositions, then

$$\sum_{k=0}^{\infty} Q_k(r)w^k \quad \text{and} \quad \sum_{k=0}^{\infty} Q_k(x)r^k,$$

are in $\mathscr{L} - \mathscr{P}$. A similar argument holds for the other case. $\qquad\square$

Example 155. The condition in Theorem 154 that T is not a trivial operator is a necessary condition. Consider the following trivial multiplier sequence (see Example 52),

$$\{\gamma_k\}_{k=0}^{\infty} := \{2, 1, 0, 0, 0, 0, \ldots\}.$$

Define the Hermite diagonal differential operator,

$$T[H_n(x)] := \gamma_n H_n(x).$$

Using Theorem 90, we calculate the differential form of T,

$$T = 2 - xD + \frac{1}{2}D^2 + \left(\frac{x^3}{6} - \frac{x}{4}\right)D^3 + \left(-\frac{x^4}{12} + \frac{1}{16}\right)D^4 + \cdots.$$

Setting $x = 0$ and replacing D with w, yields

$$2 + \frac{1}{2}w^2 + \frac{1}{16}w^4 + \cdots,$$

a function with non-real zeros (see Theorem 38). Operator T also servers as an example where some Q_k's are not hyperbolic (cf. Theorem 133). Again, the lack of "nice" properties in the Q_k's of T seems to follow from the triviality of T (cf. Example 141 and the comments after Example 141).

Problem 156. What additional assumptions on $\{Q_k(x)\}_{k=0}^{\infty}$ would allow the converse of Theorem 154 to also hold?

CHAPTER 3
ORDER TWO DIFFERENTIAL OPERATORS

In their beautiful papers [16, 17], J. Borcea and P. Brändén demonstrate the strong relationship between the hyperbolicity preserving differential operators and two variable stable entire functions (see Theorems 138 and 142). Hence, one is led to ponder the nature of two variable stable polynomials.

Problem 157. Characterize real bivariate stable polynomials, $p(x, w)$; i.e., find all real polynomials, $p(x, w)$, such that $p(x, w) \neq 0$ for $x, w \in H^+$.

In some sense, trying to answer Problem 157 is naturally a more tedious task than trying to answer our original question, Problem 2. However, even if complete analysis of stable polynomials seems a more daunting task, partial analysis yields many insightful observations concerning differential operators and hyperbolic polynomials.

In this chapter, our main goal is to characterize quadratic two variable stable polynomials and thus shed light on the nature of the simplest hyperbolicity preserving operators (cf. Theorems 138 and 142). The purpose of this chapter is to establish a necessary and sufficient condition involving a Turán-Wronskian-type inequality that determines the hyperbolicity preserving property of quadratic differential operators.

3.1 Quadratic Stable Polynomials

Real bivariate stable polynomials (see Definition 13 and 14) are concerned with complex numbers in H^+. Thus, the following lemma will be quite useful throughout our discussions.

Lemma 158. *If we define*

$$H^+ H^+ = \{xw : x \in H^+ \text{ and } w \in H^+\},$$

then $H^+ H^+ = \mathbb{C} - [0, \infty)$. In particular, if $x, w \in H^+$ and $xw \in \mathbb{R}$, then $xw < 0$. Likewise, if $r < 0$ is any real number, then there is $x, w \in H^+$ such that $xw = r$.

Proof. This follows easily by consideration of the arguments for x and w. (cf. with the defined sectors in the Malo-Schur-Szegő Composition Theorem, Theorem 69). □

Corollary 159. *Let $p(x)$ be a real polynomial. Then $p(-xw)$ is a real bivariate stable polynomial if and only if $p(x)$ has only negative real zeros.*

From Section 2.5, we reiterate several of the important statements concerning stable polynomials (see Theorems 133 and 154).

Theorem 160. *Suppose* $f(x, w) = \sum_{k=0}^{n} Q_k(x)(-w)^k$ *is a real bivariate stable polynomial. Then,*

1. $Q_k(x) \ll Q_{k+1}(x)$ *for* $0 \leq k \leq n - 1$,

2. $f(r, w) = \sum_{k=0}^{n} Q_k(r)w^k \in \mathscr{L} - \mathscr{P}$ *for every* $r \in \mathbb{R}$, *and*

3. $f(x, r) = \sum_{k=0}^{n} Q_k(x)r^k \in \mathscr{L} - \mathscr{P}$ *for every* $r \in \mathbb{R}$.

Example 161. It should be noted that the conclusions of Theorem 160 are insufficient to establish that a real bivariate polynomial is stable. For example, the polynomial $x^2 w^2 - 1$ is not a bivariate stable polynomial, however $r^2 w^2 - 1$ and $x^2 r^2 - 1$ are in $\mathscr{L} - \mathscr{P}$ for all $r \in \mathbb{R}$. Furthermore, since the coefficient of w is the zero polynomial, then $x^2 w^2 - 1$ satisfies the interlacing condition of the coefficients; i.e., $Q_k(x) \ll Q_{k+1}(x)$.

Corollary 162. *If* $f(x)(-w)^2 + g(x)(-w) + h(x)$, $f(x), g(x), h(x) \in \mathbb{R}[x]$, *is a real bivariate stable polynomial, then for every* $x \in \mathbb{R}$,

$$g(x)^2 - 4f(x)h(x) \geq 0.$$

We now, case-by-case, start characterizing real bivariate stable polynomials.

Theorem 163. *Let*

1. $f_1(x, w) := b$, $b \in \mathbb{R}$,

2. $f_2(x, w) := a(-w) + b$, $a, b \in \mathbb{R}$, *and*

3. $f_3(x, w) := a(x - r_1)(-w) + b$, $a, b, r_1 \in \mathbb{R}$.

Then $f_1(x, w)$ *and* $f_2(x, w)$ *are real bivariate stable polynomials, and* $f_3(x, w)$ *is real bivariate stable if and only if* $ab \geq 0$.

Proof. The stability of $f_1(x, w)$ and $f_2(x, w)$ is trivial. If $a = 0$, then $f_3(x, w)$ is trivially stable. If $a \neq 0$ and $f_3(x, w)$ is stable, then $(x - r_1)w \neq \frac{b}{a}$ for every $x, w \in H^+$. Hence, from Lemma 158, $ab \geq 0$. Likewise, if $\frac{b}{a} \geq 0$ (i.e., a and b are of the same sign), then $(x - r_1)w \neq \frac{b}{a}$ for every $x, w \in H^+$ (cf. Corollary 159). $\square$

We continue increasing our repertoire of stable polynomials. See also the stable polynomials of [14, Lemma 1.2] and [16, Lemma 3.2].

Theorem 164. *Let*

1. $f_1(x, w) := a(-w)^2 + b(-w) + c, \ a, b, c \in \mathbb{R}$,

2. $f_2(x, w) := a(x - r_1)(-w)^2 + b(-w) + c, \ a, b, c, r_1 \in \mathbb{R}, \ and$

3. $f_3(x, w) := a(x - r_1)(x - r_2)(-w)^2 + b(-w) + c, \ a, b, c, r_1, r_2 \in \mathbb{R}$.

Then $f_1(x, w)$ is a real bivariate stable polynomial if and only if $b^2 - 4ac \geq 0$, $f_2(x, w)$ is a real bivariate stable polynomial if and only if $ab \geq 0$ and $ac = 0$, and $f_3(x, w)$ is a real bivariate stable polynomial if and only if $a = 0$ or $b, c = 0$.

Proof. The case of $f_1(x, w)$ follows from the quadratic formula. We now consider the case of $f_2(x, w)$. By Theorem 160, for each fixed $x \in \mathbb{R}$, $a(x - r_1)(-w)^2 + b(-w) + c$, has only real zeros. Thus, $b^2 - 4a(x - r_1)c \geq 0$ for every $x \in \mathbb{R}$. Since $b^2 - 4a(x - r_1)c$ is linear, we must conclude that the leading coefficient $-4ac$ is zero. Hence, $ac = 0$. Also, Theorem 160 gives, $b \ll a(x - r_1)$, hence $ab \geq 0$. Conversely, if $ab \geq 0$ and $ac = 0$, then $f_2(x, w)$ reduces to a product of stable polynomials from Theorem 163. In the case of $f_3(x, w)$, if $f_3(x, w)$ is stable, then by Theorem 160, $b \ll a(x - r_1)(x - r_2)$. Hence, either $a = 0$ or $b = 0$. If $a \neq 0$ and $b = 0$, then rewriting yields, $w^2 \neq \dfrac{c}{a(x - r_1)(x - r_2)}$ for every $x, w \in H^+$. By Lemma 158, this is equivalent to saying that the meromorphic function $\dfrac{c}{a(x - r_1)(x - r_2)}$ maps H^+ to $[0, \infty)$. We conclude that $c = 0$. Conversely, by Theorem 163, if $a = 0$ or $b, c = 0$ then $f_3(x, w)$ is a stable polynomial. $\square$

Theorem 165. *Let $f(x, w) := a(-w)^2 + b(x - r_3)(-w) + c, \ a, b, c, r_3 \in \mathbb{R}$. Then $f(x, w)$ is a real bivariate stable polynomial if and only if $ab \leq 0$ and $ac \leq 0$.*

Proof. If $b = 0$, then Theorem 163 implies the result. Assume $b \neq 0$. If $f(x, w)$ is stable, then Theorem 160 establishes $c \ll b(x - r_3)$ and $b(x - r_3) \ll a$, so that $ab \leq 0$ and $bc \geq 0$, and thus it also follows that $ac \leq 0$. Conversely, assuming $ab \leq 0$ and $ac \leq 0$, we have $bc \geq 0$. Suppose $w \in H^+$ and $f(x, w) = 0$. Then,

$$x - r_3 = \frac{a}{b}w + \frac{c}{b}\frac{1}{w}.$$

Hence, $x - r_3 \in \overline{H^-}$. Thus, $f(x, w)$ is stable. $\square$

Theorem 166. *Let $f(x, w) := a(x - r_1)(-w)^2 + b(x - r_3)(-w) + c, \ a, b, c, r_1, r_3 \in \mathbb{R}$. Then $f(x, w)$ is a real bivariate stable polynomial if and only if $ac(b^2(r_1 - r_3) - ac) \geq 0$ and $bc \geq 0$.*

Proof. If $b = 0$, Theorem 160 and Corollary 39 imply the result. Thus suppose $b \neq 0$. Assume $f(x, w)$ is stable. Theorem 160 establishes $c \ll b(x - r_3)$ and thus $bc \geq 0$. Since $f(x, w)$ is stable, then by Corollary 162, we also have,

$$(b(x - r_3))^2 - 4a(x - r_1) \geq 0,$$

a quadratic that is never negative. Hence, the discriminant,

$$(2r_3b^2 + 4ac)^2 - 4b^2(b^2r_3^2 + 4acr_1) \leq 0,$$

thus

$$ac(b^2(r_1 - r_3) - ac) \geq 0.$$

Now suppose $ac(b^2(r_1 - r_3) - ac) \geq$ and $bc \geq 0$. Let $w \in H^+$ and suppose $f(x, w) = 0$ for some $x \in \mathbb{C}$, then

$$x - r_1 = \frac{b^2(r_1 - r_3) - ac}{abw - b^2} + \frac{c}{bw}.$$

Hence, if $ab \geq 0$, then $ac \geq 0$, so $b^2(r_1 - r_3) - ac \geq 0$, thus we conclude $x \in \overline{H^-}$. Likewise, if $ab \leq 0$, then $ac \leq 0$, so $b^2(r_1 - r_3) - ac \leq 0$, and again $x \in \overline{H^-}$. We conclude that $f(x, w)$ must be a real bivariate stable polynomial. $\square$

Theorem 167. *Let* $f(x, w) := a(x - r_1)^2(-w)^2 + b(x - r_3)(-w) + c, \ a, b, c, r_1, r_3 \in \mathbb{R}, \ a, b \neq 0$. *The polynomial* $f(x, w)$ *is stable if and only if* $ab > 0$, $bc \geq 0$, $r_1 = r_3$, *and* $b^2 - 4ac \geq 0$.

Proof. Suppose $f(x, w)$ is a stable polynomial. By Theorem 160, $c \ll b(x - r_3)$ and $b(x - r_3) \ll a(x - r_1)^2$, and so $ab \geq 0$, $bc \geq 0$ and $r_1 = r_3$. Also, by Theorem 160,

$$f(1 + r_1, w) = aw^2 - bw + c \in \mathscr{L} - \mathscr{P}.$$

Thus $b^2 - 4ac \geq 0$. Now suppose $ac \geq 0$, $bc \geq 0$, $r_1 = r_3$, and $b^2 - 4ac \geq 0$. With these assumptions, $az^2 - bz + c \in \mathscr{L} - \mathscr{P}^a$. Hence, by Corollary 159,

$$a((x - r_1)w)^2 - b((x - r_1)w) + c,$$

is a real bivariate stable polynomial. $\square$

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In nearly every case above, $f(x, w)$ is a linear polynomial in x. Hence, partial fraction decomposition proved to be a useful tool in analyzing the stability of $f(x, w)$. However, the next case is drastically different; namely, $f(x, w)$ is a non-trivial quadratic in x and w. To better understand this case, we will establish several lemmas below that provide necessary and sufficient conditions for the stability of a *quadratic* real bivariate polynomial (see also [7]).

Lemma 168. *Let A, $B \in \mathbb{C} - \mathbb{R}$ be two non-real complex numbers such that*

1. $0 < \mathrm{Arg}(B) < \mathrm{Arg}(A) < 2\pi$,

2. $\mathrm{Arg}(A) - \mathrm{Arg}(B) < \pi$, *and*

3. $\mathrm{Im}(A) < \mathrm{Im}(B)$.

Then for any r_1, $r_2 \in \mathbb{R}$, $r_1 < r_2$, there exist $x, w \in H^+$ such that $(x + r_1)w = A$ and $(x + r_2)w = B$.

Proof. Case 1: $B \in H^+$. The point B may be located in either quadrant I, on the positive imaginary axis, or in quadrant II, as described in Figure 3.1.1. The hypotheses (1), (2), and (3) implies that point A is located somewhere in the shaded region of the corresponding point B.
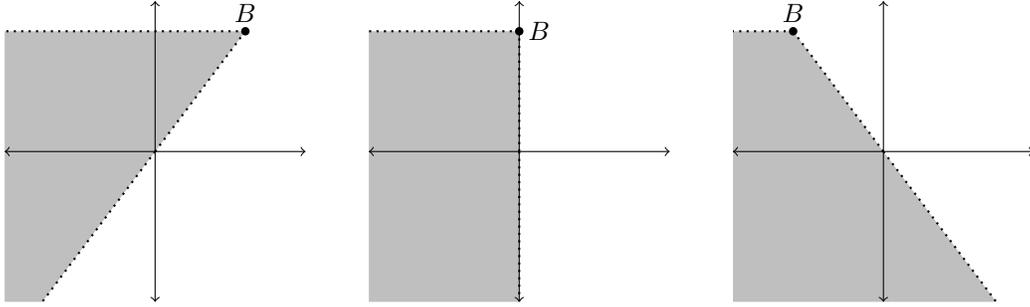

Figure 3.1.1: Various possibilities of B in H^+.

Define the function $f : [0, \mathrm{Arg}(B)] \to \mathbb{R}$ by

$$f(\theta) := \mathrm{Im}(e^{-i\theta}A) - \mathrm{Im}(e^{-i\theta}B).$$

Then $f(0) < 0$ by (3), and $f(\mathrm{Arg}(B)) > 0$ by (2). Thus by continuity, there exist $\theta_0 \in (0, \mathrm{Arg}(B))$ such that $f(\theta_0) = 0$, which implies that $(e^{-i\theta_0}B - e^{-i\theta_0}A) > 0$ by (1). Define the function $g : [0, \infty) \to \mathbb{R}$ by

$$g(k) := k(e^{-i\theta_0}B - e^{-i\theta_0}A).$$

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Notice $g \geq 0$, $g(0) = 0$, and $\lim\limits_{k \to +\infty} g(k) = +\infty$. Thus, there exist $k_0 > 0$ such that $g(k_0) = r_2 - r_1$. Let

$$x = \frac{1}{2}(k_0 e^{-i\theta_0} B + k_0 e^{-i\theta_0} A - r_1 - r_2), \text{ and } w = \frac{1}{k_0} e^{i\theta_0}.$$

It follows that $x, w \in H^+$, $(x + r_1)w = A$, and $(x + r_2)w = B$.

Case 2: $B \in H^-$. Similar to Case 1, the point B may be located in either quadrant III, on the negative imaginary axis, or in quadrant IV, as described in Figure 3.1.2. Point A is located somewhere in the shaded region of the corresponding point B by hypotheses (1), (2), and (3).
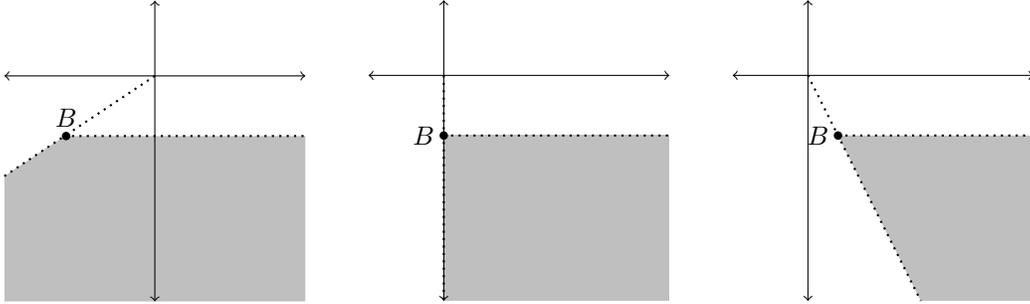

Figure 3.1.2: Various possibilities of B in H^-.

Define the function $f : [0, 2\pi - \operatorname{Arg}(B)] \to \mathbb{R}$ by

$$f(\theta) := \operatorname{Im}(e^{i\theta} A) - \operatorname{Im}(e^{i\theta} B).$$

Then $f(0) < 0$ by (3), and $f(2\pi - \operatorname{Arg}(B)) > 0$ by (2). Thus by continuity, there exist $\theta_0 \in (0, 2\pi - \operatorname{Arg}(B))$ such that $f(\theta_0) = 0$, which implies that $(e^{i\theta_0} B - e^{i\theta_0} A) < 0$ by (1). Define the function $g : (-\infty, 0] \to \mathbb{R}$ by

$$g(k) := k(e^{i\theta_0} B - e^{i\theta_0} A).$$

Then $g \geq 0$, $g(0) = 0$, and $\lim\limits_{k \to -\infty} g(k) = +\infty$. Thus, there exist $k_0 < 0$ such that $g(k_0) = r_2 - r_1$. Let

$$x = \frac{1}{2}(k_0 e^{i\theta_0} B + k_0 e^{i\theta_0} A - r_1 - r_2), \text{ and } w = \frac{1}{k_0} e^{-i\theta_0}.$$

It follows that $x, w \in H^+$, $(x + r_1)w = A$, and $(x + r_2)w = B$. $\qquad\square$

Lemma 169. *Let $a, b, r_1, r_2, r \in \mathbb{R}$, $a, b \geq 0$, and $r_1 \neq r_2$. Set*

$$f(x, w) = ((x + r_1)w - a)((x + r_2)w - b), \quad x, w \in \mathbb{C}. \tag{3.1.1}$$

Then

$$f(x, w) - r \neq 0 \text{ for all } x, w \in H^+ \quad \text{if and only if} \quad r \in [0, ab].$$

Proof. Since the factors of $f(x, w)$ in (3.1.1) are symmetric, we let $r_1 < r_2$. There are three cases to prove necessity. The following is an outline.

Case 1. $r \in (-\infty, 0)$, and $a < b + 2\sqrt{|r|}$.

Case 2. $r \in (-\infty, 0)$, and $a \geq b + 2\sqrt{|r|}$.

Case 3. $r \in (ab, \infty)$.

We show in each case that there exist $x, w \in H^+$ such that $f(x, w) = r$.

Case 1. Consider $r \in (-\infty, 0)$, and $a < b + 2\sqrt{|r|}$. Define $g : [0, \pi/2] \to \mathbb{R}$ by

$$g(\theta) := \left(\sqrt{|r|}e^{i\theta} + b\right) - \left(\sqrt{|r|}e^{i(\pi-\theta)} + a\right)$$
$$= \sqrt{|r|}(2\cos(\theta)) - a + b.$$

The function g is real valued and $g(0) = b + 2\sqrt{|r|} - a > 0$ by assumption. Thus by continuity, there exists $\theta_0 \in (0, \pi/2)$ such that $g(\theta_0) > 0$, which implies the following.

(a) $\text{Im}\left(\sqrt{|r|}e^{i\theta_0} + b\right) - \text{Im}\left(\sqrt{|r|}e^{i(\pi-\theta_0)} + a\right) = 0$,

(b) $\text{Re}\left(\sqrt{|r|}e^{i\theta_0} + b\right) - \text{Re}\left(\sqrt{|r|}e^{i(\pi-\theta_0)} + a\right) > 0$, and

(c) $\left(\sqrt{|r|}e^{i\theta_0} + b\right), \left(\sqrt{|r|}e^{i(\pi-\theta_0)} + a\right) \in H^+$.

By (a), (b), and (c),

$$\text{Arg}\left(\sqrt{|r|}e^{i(\pi-\theta_0)} + a\right) - \text{Arg}\left(\sqrt{|r|}e^{i\theta_0} + b\right) > 0.$$

Define the function $h : (0, 1] \to \mathbb{R}$ by

$$h(k) := \text{Arg}\left(k\sqrt{|r|}e^{i(\pi-\theta_0)} + a\right) - \text{Arg}\left(\frac{\sqrt{|r|}}{k}e^{i\theta_0} + b\right).$$

The function h is real valued, and $h(1) > 0$. Thus by continuity, there exists $k_0 \in (0,1)$ such that $h(k_0)$ remains positive,

$$\mathrm{Arg}\left(k_0\sqrt{|r|}e^{i(\pi-\theta_0)} + a\right) - \mathrm{Arg}\left(\frac{\sqrt{|r|}}{k_0}e^{i\theta_0} + b\right) > 0, \tag{3.1.2}$$

but the imaginary parts satisfy the following strict inequality,

$$\mathrm{Im}\left(k_0\sqrt{|r|}e^{i(\pi-\theta_0)} + a\right) < \mathrm{Im}\left(\frac{\sqrt{|r|}}{k_0}e^{i\theta_0} + b\right). \tag{3.1.3}$$

Let

$$A = k_0\sqrt{|r|}e^{i(\pi-\theta_0)} + a, \quad \text{and} \quad B = \frac{\sqrt{|r|}}{k_0}e^{i\theta_0} + b.$$

Then (3.1.2) and (3.1.3) satisfy items (1), (2), and (3) of Lemma 168 and hence there exist $x, w \in H^+$ such that $(x + r_1)w = A$ and $(x + r_2)w = B$. Thus,

$$f(x, w) = ((x + r_1)w - a)((x + r_2)w - b) = \left(k_0\sqrt{|r|}e^{i(\pi-\theta_0)}\right)\left(\frac{\sqrt{|r|}}{k_0}e^{i\theta_0}\right) = -|r| = r.$$

Case 2: We consider $r \in (-\infty, 0)$, and $b + 2\sqrt{|r|} \leq a$. We will only need $b < a + 2\sqrt{|r|}$. This is easily seen to be true by adding $2\sqrt{|r|}$ to both sides of $b + 2\sqrt{|r|} \leq a$, and observing that $b < b + 4\sqrt{|r|}$. Define the function $g : [0, \pi/2] \to \mathbb{R}$ by

$$g(\theta) := \left(\sqrt{|r|}e^{i(2\pi-\theta)} + a\right) - \left(\sqrt{|r|}e^{i(\pi+\theta)} + b\right) = \sqrt{|r|}(2\cos(\theta)) + a - b.$$

Again, g is real valued, and $g(0) = a + 2\sqrt{|r|} - b > 0$. Thus by continuity, there exists $\theta_0 \in (0, \pi/2)$ such that $g(\theta_0) > 0$, which implies the following:

(a) $\mathrm{Im}\left(\sqrt{|r|}e^{i(2\pi-\theta_0)} + a\right) - \mathrm{Im}\left(\sqrt{|r|}e^{i(\pi+\theta_0)} + b\right) = 0$,

(b) $\mathrm{Re}\left(\sqrt{|r|}e^{i(2\pi-\theta_0)} + a\right) - \mathrm{Re}\left(\sqrt{|r|}e^{i(\pi+\theta_0)} + b\right) > 0$, and

(c) $\left(\sqrt{|r|}e^{i(2\pi-\theta_0)} + a\right), \left(\sqrt{|r|}e^{i(\pi+\theta_0)} + b\right) \in H^-$.

By (a), (b), and (c),

$$\mathrm{Arg}\left(\sqrt{|r|}e^{i(2\pi-\theta_0)} + a\right) - \mathrm{Arg}\left(\sqrt{|r|}e^{i(\pi+\theta_0)} + b\right) > 0.$$

Define the function $h : [1, \infty) \to \mathbb{R}$ by

$$h(k) := \mathrm{Arg}\left(k\sqrt{|r|}e^{i(2\pi-\theta_0)} + a\right) - \mathrm{Arg}\left(\frac{\sqrt{|r|}}{k}e^{i(\pi+\theta_0)} + b\right).$$

The function h is real valued, and $h(1) > 0$. Thus by continuity, there exists $k_0 > 1$ such that $h(k_0)$ remains positive,

$$\mathrm{Arg}\left(k_0\sqrt{|r|}e^{i(2\pi-\theta_0)} + a\right) - \mathrm{Arg}\left(\frac{\sqrt{|r|}}{k_0}e^{i(\pi+\theta_0)} + b\right) > 0, \tag{3.1.4}$$

but the imaginary parts satisfy the following strict inequality,

$$\mathrm{Im}\left(k_0\sqrt{|r|}e^{i(2\pi-\theta_0)} + a\right) < \mathrm{Im}\left(\frac{\sqrt{|r|}}{k_0}e^{i(\pi+\theta_0)} + b\right). \tag{3.1.5}$$

Let

$$A = k_0\sqrt{|r|}e^{i(2\pi-\theta_0)} + a, \quad \text{and} \quad B = \frac{\sqrt{|r|}}{k_0}e^{i(\pi+\theta_0)} + b.$$

Then (3.1.4) and (3.1.5) satisfies items (1), (2), and (3) of Lemma 168, hence there exist $x, w \in H^+$ such that $(x + r_1)w = A$ and $(x + r_2)w = B$. Thus,

$$f(x, w) = ((x + r_1)w - a)((x + r_2)w - b) = \left(k_0\sqrt{|r|}e^{i(2\pi-\theta_0)}\right)\left(\frac{\sqrt{|r|}}{k_0}e^{i(\pi+\theta_0)}\right) = -|r| = r.$$

Case 3: We consider $r \in (ab, \infty)$. Since $r > ab$, $r = a'b'$, for some $a' > a$, and $b' > b$. Define the function $g : [\pi/2, \pi] \to [a - a', a] \times [b - b', b]$ by

$$g(\theta) := \left(\mathrm{Re}(a'e^{-i\theta}) + a,\ \mathrm{Re}(b'e^{i\theta}) + b\right).$$

Since $a - a', b - b' < 0$, $g(\pi) = (a - a', b - b')$ has negative coordinates. By continuity, there exists $\theta_0 \in (\pi/2, \pi)$ such that $g(\theta_0)$ has negative coordinates, which implies that $a'e^{-i\theta_0} + a$ is in quadrant three, and $b'e^{i\theta_0} + b$ is in quadrant two. Let

$$A = a'e^{-i\theta_0} + a, \quad \text{and} \quad B = b'e^{i\theta_0} + b.$$

Again, by Lemma 168, there exist $x, w \in H^+$ such that $(x + r_1)w = A$, and $(x + r_2)w = B$. Thus,

$$f(x, w) = ((x + r_1)w - a)((x + r_2)w - b) = \left(a'e^{-\theta_0 i}\right)\left(b'e^{\theta_0 i}\right) = a'b' = r.$$

To prove sufficiency, first consider $r \in (0, ab]$. By way of contradiction, assume there exist $x, w \in H^+$ such that $((x + r_1)w - a)((x + r_2)w - b) = r$. Let $A = ((x + r_1)w - a)$, $B = ((x + r_2)w - b)$. Since $x + r_1, x + r_2 \in H^+$, the rotation by $\mathrm{Arg}(w) \in (0, \pi)$ and the shifts to the left by $a, b > 0$ restrict the location of A and B considerably. Indeed, since AB is a positive real number, $\mathrm{Arg}(A) + \mathrm{Arg}(B) = 2\pi$. In particular, since $r_1 < r_2$, B must be in H^+, which implies

$$0 < \mathrm{Arg}(w) < \mathrm{Arg}((x + r_2)w) < \mathrm{Arg}((x + r_2)w - b) < \pi, \tag{3.1.6}$$

and A must be in H^-, which implies

$$\pi < \mathrm{Arg}((x + r_1)w - a) < \mathrm{Arg}((x + r_1)w) < \pi - \mathrm{Arg}(w) < 2\pi. \tag{3.1.7}$$

The following figure illustrates inequalities (3.1.6) and (3.1.7). We let ϵ and δ be the horizontal distance
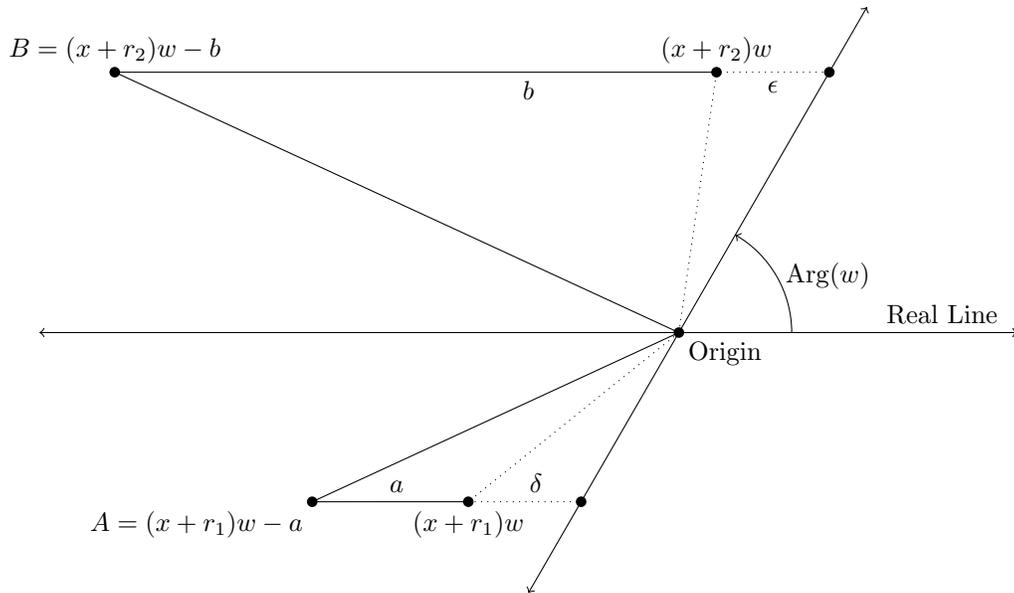

Figure 3.1.3: Picture of inequalities (3.1.6) and (3.1.7).

from $(x + r_1)w$ and $(x + r_2)w$ to the line formed by $\mathrm{Arg}(w)$. In fact, $\delta = \dfrac{\mathrm{Im}(x + r_1)}{\sin(Arg(w))}$, and $\epsilon = \dfrac{\mathrm{Im}(x + r_2)}{\sin(Arg(w))}$, so that $\delta = \epsilon > 0$. We redraw the picture with different labels and examine the resulting geometry. The
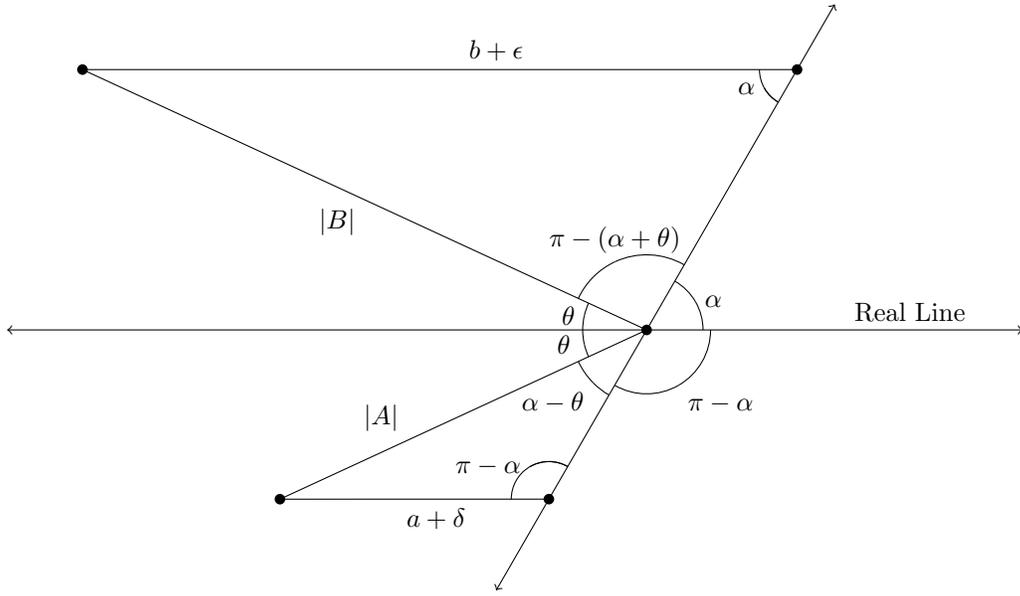
Figure 3.1.4: Geometric representation of the inequalities (3.1.6) and (3.1.7).

inequalities $\alpha - \theta > 0$ and $\pi - (\alpha + \theta) > 0$ imply $0 < \theta < \alpha < \pi - \theta < \pi$, so that

$$\sin(\theta) < \sin(\alpha),$$

since $\sin(\theta) = \sin(\pi - \theta)$. Thus,

$$0 < \left(\frac{\sin(\theta)}{\sin(\alpha)} \right)^2 < 1,$$

and the law of sines yields that

$$(a + \delta)(b + \epsilon) = \frac{|A| \sin(\alpha - \theta)}{\sin(\pi - \alpha)} \cdot \frac{|B| \sin(\pi - (\alpha + \theta))}{\sin(\alpha)} = \left(1 - \left(\frac{\sin(\theta)}{\sin(\alpha)} \right)^2 \right) |AB| < |AB|.$$

Hence we have the contradiction that

$$ab < (a + \delta)(b + \epsilon) < |AB| = r.$$

To finish the proof, consider $r = 0$. By way of contradiction, suppose there are $x, w \in H^+$ such that

$$((x + r_1)w - a)((x + r_2)w - b) = 0.$$

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Thus, $(x + r_1)w = a$, or $(x + r_2)w = b$. However, neither of these can hold, since the product of any two complex numbers in H^+ cannot be a non-negative real number (see Lemma 158). $\square$

Theorem 170. *Let* $f(x, w) := a(x - r_1)(x - r_2)(-w)^2 + b(x - r_3)(-w) + c$, *where* $a, b, c, r_1, r_2, r_3 \in \mathbb{R}$, $r_1 \neq r_2$, $a, b \neq 0$. *Then* $f(x, w)$ *is real bivariate stable polynomial if and only if* $ab \geq 0$, $bc \geq 0$, *and*

$$b^2 \left(\frac{(r_1 - r_3)(r_3 - r_2)}{(r_1 - r_2)^2} \right) - ac \geq 0.$$

Proof. Suppose $f(x, w)$ is stable. By Theorem 160, $c \ll b(x - r_3)$ and $b(x - r_3) \ll a(x - r_1)(x - r_2)$, hence, $ac \geq 0$ and $bc \geq 0$. By Corollary 162,

$$(b(x - r_3))^2 - 4a(x - r_1)(x - r_2)c \geq 0, \tag{3.1.8}$$

hence the discriminant of the quadratic in x from (3.1.8),

$$(-2r_3 b^2 + 4ac(r_1 + r_2))^2 - 4(b^2 - 4ac)(b^2 r_3^2 - 4acr_1 r_2) \leq 0.$$

Rewriting yields,

$$b^2 \left(\frac{(r_1 - r_3)(r_3 - r_2)}{(r_1 - r_2)^2} \right) - ac \geq 0.$$

Now suppose $ac \geq 0$, $bc \geq 0$, and

$$b^2 \left(\frac{(r_1 - r_3)(r_3 - r_2)}{(r_1 - r_2)^2} \right) - ac \geq 0.$$

Thus,

$$\left(\frac{b}{a} \right)^2 \frac{(r_1 - r_3)(r_3 - r_2)}{(r_1 - r_2)^2} - \frac{c}{a} \in \left[0, \left(\frac{b}{a} \frac{r_1 - r_3}{r_1 - r_2} \right) \left(\frac{b}{a} \frac{r_3 - r_2}{r_1 - r_2} \right) \right],$$

and so by Lemma 169,

$$a \left(\left((x - r_1)(-w) + \frac{b}{a} \frac{r_1 - r_3}{r_1 - r_2} \right) \left((x - r_2)(-w) + \frac{b}{a} \frac{r_3 - r_2}{r_1 - r_2} \right) - \left(\left(\frac{b}{a} \right)^2 \frac{(r_1 - r_3)(r_3 - r_2)}{(r_1 - r_2)^2} - \frac{c}{a} \right) \right)$$

is stable. Hence, $f(x, w) = a(x - r_1)(x - r_2)(-w)^2 + b(x - r_3)(-w) + c$ is stable. $\square$

Remark 171. In Theorems 163, 164, 165, 166, 167, and 170 we see a common theme established. Namely, the conditions for stability are given in two parts; coefficient conditions that arise from the proper position

requirement of the polynomial coefficients and then an extra condition that involves addressing part (3) of Corollary 162. To summarize, we can rewrite Theorems 163-170 into the following form.

Theorem 172. *Suppose $Q_2(x), Q_1(x), Q_0(x)$ are real polynomials with $Q_0(x) \ll Q_1(x)$, $Q_1(x) \ll Q_2(x)$, $Q_0(x)Q_1(x)Q_2(x) \not\equiv 0$, and $\deg(Q_k(x)) \leq k$ for $k = 0, 1, 2$. Then $Q_2(x)(-w)^2 + Q_1(x)(-w) + Q_0(x)$ is a real bivariate stable polynomial if and only if*

$$Q_2(r)x^2 + Q_1(r)x + Q_0(r) \in \mathscr{L} - \mathscr{P}$$

for every $r \in \mathbb{R}$.

Example 173. We saw in Theorem 163 part (2), that polynomials of the form $f(x)(-w)^2 + g(x)(-w) + h(x)$, $f(x), g(x), h(x) \in \mathbb{R}[x]$, are not real bivariate stable polynomials, where $\deg(f(x)) = 1$ and $\deg(g(x)) = \deg(h(x)) = 0$, $f(x), g(x), h(x) \not\equiv 0$. Similarly, we consider higher degree polynomial coefficients,

$$F(x, w) := f(x)(-w)^2 + g(x)(-w) + h(x).$$

Surprisingly, if $\deg(f(x)) = n$ and $\deg(g(x)) = \deg(h(x)) = n - 1$, $n > 1$, then $F(x, w)$ is not a real bivariate stable polynomial. This follows by Corollary 162 which shows that

$$g(x)^2 - 4f(x)h(x) \geq 0$$

for every $x \in \mathbb{R}$. However, $\deg(g(x)^2 - 4f(x)h(x)) = 2n - 1$ is of odd degree and cannot be non-negative for every $x \in \mathbb{R}$.

Example 174. Theorem 160, shows that if $f(x)(-w)^2 + g(x)(-w) + h(x)$, $f(x), g(x), h(x) \in \mathbb{R}[x]$, and $f(x), g(x), h(x) \not\equiv 0$, is a real bivariate stable polynomial, where for some $n > 1$, $\deg(f(x)) = n + 1$ and $\deg(g(x)) = n$, then $\deg(h(x))$ is either $n - 1$, n, or $n + 1$. We have examples where $\deg(h(x)) = n - 1$ (see Example 134). Example 173, shows that $\deg(h(x)) \neq n$. Hence, an intriguing question is whether there are real bivariate stable polynomials such that $\deg(h(x)) = n + 1$. Consider the following polynomial,

$$F(x, w) := x^2(-w)^2 + x(-w) - x^2.$$

We show that $F(x, w)$ is a real bivariate stable polynomial. Suppose $w \in H^+$ and $F(x, w) = 0$, $x \neq 0$, rewriting yields $2x = \dfrac{1}{w + 1} + \dfrac{1}{w - 1}$ and hence, $x \in H^-$.

Example 175. Assuming $a, b, c > 0$ and $r_1, r_2, r_3 \in \mathbb{R}$, $r_1 \neq r_2$, we note the following sequence of equivalent calculations,

$$a(r - r_1)(r - r_2)w^2 + b(r - r_3)w + c \in \mathscr{L} - \mathscr{P}, \text{ for every } r \in \mathbb{R},$$

$$(b(r - r_3))^2 - 4a(r - r_1)(r - r_2)c \geq 0, \text{ for every } r \in \mathbb{R},$$

$$(b^2 - 4ac)r^2 - (2r_3 b^2 - (r_1 + r_2)4ac)r + (r_3^2 b^2 - r_1 r_2 4ac) \geq 0, \text{ for every } r \in \mathbb{R},$$

$$(2r_3 b^2 - (r_1 + r_2)4ac)^2 - 4(b^2 - 4ac)(r_3^2 b^2 - r_1 r_2 4ac) \leq 0,$$

$$b^2 \left(\frac{(r_1 - r_3)(r_3 - r_2)}{(r_1 - r_2)^2} \right) - ac \geq 0,$$

$$(2ab(r_1 + r_2 - 2r_3))^2 - 4(a^2((r_1 + r_2)^2 - 4r_1 r_2))(b^2 - 4ac) \leq 0,$$

$$(a^2(r_1 + r_2)^2 - 4a^2 r_1 r_2)r^2 - (2ab(r_1 + r_2) - 4abr_3)r + (b^2 - 4ac) \geq 0, \text{ for every } r \in \mathbb{R},$$

$$(-a(r_1 + r_2)r^2 + br)^2 - 4(ar^2)(ar_1 r_2 r^2 - br_3 r + c) \geq 0, \text{ for every } r \in \mathbb{R}, \text{ and}$$

$$a(w - r_1)(w - r_2)r^2 + b(w - r_3)r + c \in \mathscr{L} - \mathscr{P}, \text{ for every } r \in \mathbb{R}.$$

Hence, the conditions "$Q_2(r)w^2 + Q_1(r)w + Q_0(r) \in \mathscr{L} - \mathscr{P}$ for every $r \in \mathbb{R}$" and "$Q_2(x)r^2 + Q_1(x)r + Q_0(x) \in \mathscr{L} - \mathscr{P}$ for every $r \in \mathbb{R}$" are equivalent characterizations of stability in Theorem 170.

Example 176. Consider the following polynomials,

$$f_1(x, w) := (-w)^2 + (-w) + c,$$

$$f_2(x, w) := x(-w)^2 + (x + 1)(-w) + c,$$

$$f_3(x, w) := x^2(-w)^2 + x(-w) + c,$$

$$f_4(x, w) := (x^2 - 1)(-w)^2 + x(-w) + c, \text{ and}$$

$$f_5(x, w) := -(-w)^2 + x(-w) + c.$$

By Theorems 163, 164, 165, 166, 167, and 170, we observe that f_1, f_2, f_3, f_4 fail to be bivariate stable polynomials for c sufficiently large. However, f_5 is a bivariate stable polynomial for every $c > 0$.

Problem 177. Characterize real bivariate stable polynomials of the form $f(x)(-w)^2 + g(x)(-w) + h(x)$, $f(x), g(x), h(x) \in \mathbb{R}[x]$, where $\deg(f(x)) = \deg(h(x)) = 2$ and $\deg(g(x)) = 1$.

Problem 178. Suppose $f(x), g(x), h(x) \in \mathbb{R}[x]$ have leading coefficients of the same sign, and $\deg(h(x)) = n$, $\deg(g(x)) = n+1$, and $\deg(f(x)) = n+2$, $n > 1$. Show that $f(x)(-w)^2 + g(x)(-w) + h(x)$ is a real bivariate stable polynomial if and only if $f(x)r^2 + g(x)r + h(x) \in \mathscr{L} - \mathscr{P}$ for every $r \in \mathbb{R}$ (cf. Problem 67).

Problem 179. Suppose $p(x), q(x)$ are real hyperbolic polynomials, where $q(x) \ll p(x)$. If $p(x)(-w)^2 + p(x)(-w) + q(x)$ is a real bivariate stable polynomial, then must $q(x) = \alpha p(x)$, $\alpha > 0$?

Properly interlacing polynomial coefficients is a property directly related to the Wronskian (see Definition 24). Thus, it seems a characterization involving the Wronskian might prove useful. Below we establish a peculiar Turán-Wronskian type inequality (see also [7]). We will require the following lemma.

Lemma 180. *For $c_i, r_j \in \mathbb{R}$, $i = 0, 1, 2$, $j = 1, 2, 3$, $c_1, c_2 \neq 0$, let $Q_0(x) = c_0$, $Q_1(x) = c_1(x - r_3)$, $Q_2(x) = c_2(x - r_1)(x - r_2)$. If $f(x, w) = Q_2(x)(-w)^2 + Q_1(x)(-w) + Q_0(x)$ is a real bivariate stable polynomial, then*

$$0 \leq c_1^2 \left(\frac{1}{4} \right) - c_0 c_2.$$

Furthermore, if $r_1 \neq r_2$, then

$$0 \leq c_1^2 \frac{(r_1 - r_3)(r_3 - r_2)}{(r_2 - r_1)^2} - c_0 c_2 \leq c_1^2 \left(\frac{1}{4} \right) - c_0 c_2.$$

Thus, if $c_1^2 - 4c_0 c_2 = 0$, then $2r_3 = r_1 + r_2$.

Proof. Theorem 167 deals with the case of when $r_1 = r_2$, thus it suffices to show

$$0 \leq \frac{(r_1 - r_3)(r_3 - r_2)}{(r_2 - r_1)^2} \leq \frac{1}{4}.$$

The left inequality holds because Q_2 and Q_1 have interlacing zeros by Theorem 160. To show the right inequality, we proceed as follows,

$$0 \leq (2r_3 - (r_1 + r_2))^2,$$

$$4(r_1 r_3 + r_2 r_3) \leq (r_2 + r_1)^2 + 4r_3^2,$$

$$4(r_1 r_3 - r_1 r_2 - r_3^2 + r_2 r_3) \leq r_2^2 - 2r_1 r_2 + r_1^2, \text{ and}$$

$$4(r_1 - r_3)(r_3 - r_2) \leq (r_2 - r_1)^2. \qquad \square$$

Theorem 181. *For $c_i, r_j \in \mathbb{R}$, $i = 0, 1, 2$, $j = 1, 2, 3$, $c_1, c_2 \neq 0$, let $Q_0(x) = c_0$, $Q_1(x) = c_1(x - r_3)$, $Q_2(x) = c_2(x - r_1)(x - r_2)$ with $Q_0(x) \ll Q_1(x)$ and $Q_1(x) \ll Q_2(x)$. Then*

$$f(x, w) := Q_2(x)(-w)^2 + Q_1(x)(-w) + Q_0(x)$$

is a real bivariate stable polynomial if and only if

$$W[Q_0(x), Q_2(x)]^2 - W[Q_0(x), Q_1(x)]W[Q_1(x), Q_2(x)] \leq 0, \quad x \in \mathbb{R}.$$

Proof. Since $Q_0 \ll Q_1$ and $Q_1 \ll Q_2$, the signs of c_0, c_1, c_2 are same, and $r_1 \leq r_3 \leq r_2$. Define

$$
\begin{aligned}
w(x) :=& W[Q_0, Q_2]^2 - W[Q_0, Q_1]W[Q_1, Q_2] \\
=& c_0 c_2 (4 c_0 c_2 - c_1^2) x^2 + 2 c_0 c_2 (-2 c_0 c_2 (r_1 + r_2) + c_1^2 r_3) x \\
& + c_0 c_2 (c_0 c_2 (r_1 + r_2)^2 + c_1^2 (r_1 r_2 - r_1 r_3 - r_2 r_3)).
\end{aligned}
$$

Suppose $r_1 = r_2$, then $w(x) = -c_0 c_2 (c_1^2 - 4 c_0 c_2)(x - r_1)^2$. It is clear that $w(x) \leq 0$ if and only if $0 \leq c_1^2 - 4 c_0 c_2$, thus we apply Theorem 167.

Suppose $0 = c_1^2 - 4 c_0 c_2$ and $r_1 \neq r_2$. By Lemma 180, Theorem 170 can be restated as, "$f(x, w)$ is a real bivariate stable polynomial if and only if $2 r_3 = r_1 + r_2$". We recalculate w, under the assumption that $c_1^2 - 4 c_0 c_2 = 0$,

$$w(x) = 4 c_0^2 c_2^2 (2 r_3 - r_1 - r_2) x + c_0^2 c_2^2 (2(r_1 + r_2)(r_1 + r_2 - 2 r_3) - (r_1 - r_2)^2).$$

We now see that, $w(x) \leq 0$, if and only if, $2 r_3 = r_1 + r_2$.

Thus we may assume $0 \neq c_1^2 - 4 c_0 c_2$ and $r_1 \neq r_2$, in which case the graph of $w(x)$ is a parabola with vertex

$$\left(r_3, \frac{c_0 c_1^2 c_2}{c_1^2 - 4 c_0 c_2} \left(c_0 c_2 (r_1 - r_2)^2 + c_1^2 (r_1 - r_3)(r_2 - r_3) \right) \right).$$

Since w is a quadratic, $w(x) \leq 0$ if and only if the leading coefficient

$$c_0 c_2 (4 c_0 c_1 - c_1^2) < 0,$$

and the y-coordinate of the vertex is

$$\frac{c_0 c_1^2 c_2}{c_1^2 - 4c_0 c_2} \left(c_0 c_2 (r_1 - r_2)^2 + c_1^2 (r_1 - r_3)(r_2 - r_3) \right) \leq 0.$$

Thus, $w(x) \leq 0$ if and only if $0 < c_1^2 - 4c_0 c_1$ and $0 \leq c_1^2 (r_1 - r_3)(r_3 - r_2) - c_0 c_2 (r_1 - r_2)^2$. By Lemma 180 and Theorem 170 those conditions are equivalent to the stability of $f(x, w)$. □

3.2 Higher Order Hyperbolicity Preservers

From the previous section and using Theorems 138 and 172, we immediately establish the following in terms of hyperbolicity preservers instead of real bivariate stable polynomials.

Theorem 182. *Suppose $Q_k(x) \in \mathbb{R}$, $\deg(Q_k(x)) \leq k$, $k = 0, 1, 2$, $Q_0(x)Q_1(x)Q_2(x) \not\equiv 0$, and $Q_k(x) \ll Q_{k+1}(x)$, $k = 0, 1$. Then*

$$Q_2(x)D^2 + Q_1(x)D + Q_0(x)$$

is hyperbolicity preserving if and only if

$$Q_2(r)x^2 + Q_1(r)x + Q_0(r) \in \mathscr{L} - \mathscr{P}$$

for every $r \in \mathbb{R}$.

Many insights can be gained by analyzing the above theorem. We continue to develop observations concerning the nature of hyperbolicity preservers. From Example 173 we gain the following theorem.

Theorem 183. *Suppose $f(x), g(x), h(x)$ are real valued polynomials, $f(x), g(x), h(x) \not\equiv 0$, where $\deg(f(x)) = n$, $\deg(g(x)) = n - 1$, $\deg(h(x)) = n - 1$, $n \geq 1$. Then*

$$f(x)D^2 + g(x)D + h(x)$$

is not hyperbolicity preserving.

Theorem 184. *Suppose $A(x)D^2 + B(x)$ is hyperbolicity preserving, where $A(x), B(x) \in \mathbb{R}[x]$ and $A(x), B(x) \not\equiv 0$. Then $B(x) = \alpha A(x)$ for some $\alpha < 0$.*

Proof. Since $A(x)D^2 + B(x)$ is hyperbolicity preserving, then by Theorem 138, $A(x)w^2 + B(x) \neq 0$ for every $x, w \in H^+$. Also, by Theorem 133, $A(x)$ and $B(x)$ are hyperbolic. We conclude that for every $x, w \in H^+$,

$A(x) \neq 0$ and $-\dfrac{B(x)}{A(x)} \neq w^2$. Since, $H^+ H^+ = \mathbb{C} - [0, \infty)$ (Lemma 158), then the meromorphic function $g(x) := -\dfrac{B(x)}{A(x)}$ maps H^+ to $[0, \infty)$, an impossibility for a non-constant meromorphic function. We conclude that $-\dfrac{B(x)}{A(x)} \equiv \beta$ for some $\beta \in [0, \infty)$. Hence, $B(x) = (-\beta)A(x)$ for every $x \in \mathbb{C}$. $\qquad\square$

Theorem 185. *Suppose $A(x)D^3 + B(x)$ is hyperbolicity preserving, where $A(x), B(x) \in \mathbb{R}[x]$. Then either $A(x) \equiv 0$ or $B(x) \equiv 0$.*

Proof. Suppose $A(x)B(x) \not\equiv 0$. Since $A(x)$ and $B(x)$ are polynomials there is $x_0 \in H^+$ such that $A(x_0) \neq 0$ and $B(x_0) \neq 0$. It is known that there are three distinct non-zero complex numbers, $w_1, w_2, w_3 \in \mathbb{C}$, such that

$$(w_k)^3 = \frac{B(x)}{A(x)}, \quad k = 1, 2, 3,$$

and where $w_1 e^{(2\pi/3)i} = w_2$ and $w_1 e^{(4\pi/3)i} = w_3$. Hence for some $k \in \{1, 2, 3\}$, $w_k \in H^+$ and $A(x)(-w_k)^3 + B(x) = 0$. Thus by Theorem 138, $A(x)D^3 + B(x)$ is not hyperbolicity preserving. $\qquad\square$

Corollary 186. *If $T := A(x)D^n + B(x)D^m$, where $A(x), B(x) \in \mathbb{R}[x]$, $|n - m| \geq 3$ and $A(x)B(x) \not\equiv 0$, then T is not hyperbolicity preserving.*

The above theorems lead us to believe that hyperbolicity preservers are very intimately related to entire functions in $\mathscr{L} - \mathscr{P}$. Indeed, from Theorem 154, we extend Theorems 38 and Corollary 39 to non-trivial hyperbolicity preservers.

Theorem 187. *Let $T := \displaystyle\sum_{k=0}^{\infty} Q_k(x)D^k$ be a non-trivial linear operator on $\mathbb{R}[x]$. If for some $r \in \mathbb{R}$ and $k \in \mathbb{N}_0$, $Q_k(r)Q_{k+3}(r) \neq 0$ and $Q_{k+1}(r), Q_{k+2}(r) = 0$, then T is not hyperbolicity preserving. If for some $r \in \mathbb{R}$ and $k \in \mathbb{N}_0$, $Q_k(r)Q_{k+2}(r) > 0$ and $Q_{k+1}(r) = 0$, then T is not hyperbolicity preserving. If for some $r \in \mathbb{R}$ and $k \in \mathbb{N}_0$, $Q_{k+1}(r)^2 - Q_k(r)Q_{k+2}(r) < 0$, then T is not hyperbolicity preserving.*

Proof. By Theorem 154, if T is hyperbolicity preserving, then $\displaystyle\sum_{k=0}^{\infty} Q_k(r)x^k \in \mathscr{L} - \mathscr{P}$ for every $r \in \mathbb{R}$. We now apply Theorems 38 and 39. $\qquad\square$

Note that the condition of "non-trivial" in Theorem 187 is required, as the hyperbolicity preserver in Example 207 shows. The following example helps clarify the properties found in Theorem 187.

Example 188. Consider the following non-trivial hyperbolicity preserver,

$$T[x^n] = \frac{1}{n!}x^n.$$

Using Theorem 80, we can calculate the differential representation of T,

$$T := \sum_{k=0}^{\infty} Q_k(x)D^k = 1 - \frac{x^2}{4}D^2 + \frac{1}{9}x^3D^3 + \cdots .$$

Since the $Q_0(x) \equiv 1$ and $Q_1(x) \equiv 0$ we conclude, by Theorem 187, that $Q_2(x) \leq 0$ for $x \in \mathbb{R}$ and every zero of $Q_2(x)$ is also a zero of $Q_k(x)$ for $k > 2$, which is indeed the case.

In some sense, the class of hyperbolicity preservers seem to satisfy more stringent Turán-type inequalities for the polynomial coefficients than that of entire functions in $\mathscr{L} - \mathscr{P}$. Indeed, the Turán inequalities for three polynomials, $f(x), g(x), h(x)$, where $g(x)^2 - f(x)h(x) \geq 0$ on all of $\mathbb{R}$, in comparison to the Turán inequalities for three constants, a, b, c, $b^2 - ac \geq 0$ indicates the hyperbolicity preservers are not very common.

Let use consider the next order hyperbolicity preserver. What is the class of *cubic* hyperbolicity preservers? One could easily generate a few cubic hyperbolicity preservers. For example, given any hyperbolic polynomial, $p(x)$,

$$\left(p(x)D^3 + 3p'(x)D^2 + 3p''(x)D + p'''(x)\right)f(x) = \left(D^3p(x)\right)f(x), \quad f(x) \in \mathbb{R}[x],$$

is hyperbolicity preserving. However, this example aligns the interlacing properties of the polynomial coefficients perfectly. Consider a slightly modified cubic hyperbolicity preserver,

$$T_\alpha := (x^3 - x)D^3 + 10(x + .5)(x - .25)D^2 + 10xD + \alpha.$$

For what α is T_α hyperbolicity preserving? The cubic case is very interesting; in light of H. Krall [69], there are no orthogonal bases that satisfy an order three differential equation. To date the only known finite order diagonal differential hyperbolicity preservers are operators that diagonalize with respect to a classical orthogonal basis. Are there any order three hyperbolicity preserving diagonal differential operators (besides the obvious classical basis)? We state this more formally.

Problem 189. Suppose T is an order three diagonal differential hyperbolicity preserver,

$$T[B_n(x)] := \left(Q_3(x)D^3 + Q_2(x)D^2 + Q_1(x)D + Q_0(x)\right) B_n(x) = \gamma_n B_n(x).$$

Must $\{B_n(x)\}_{n=0}^{\infty}$ be an affine transformation (Definition 35) of the standard basis, $\{x^n\}_{n=0}^{\infty}$? Compare with problem 143.

CHAPTER 4
ORTHOGONAL DIAGONAL DIFFERENTIAL OPERATORS

Let $\{Q_k(x)\}_{k=0}^{\infty}$ be a sequence of real polynomials, $\{\gamma_k\}_{k=0}^{\infty}$ be a sequence of real numbers, and $\{B_n(x)\}_{n=0}^{\infty}$ be a simple sequence in $\mathbb{R}[x]$, such that

$$T[B_n(x)] := \left(\sum_{k=0}^{\infty} Q_k(x)D^k \right) B_n(x) = \gamma_n B_n(x). \qquad (4.0.1)$$

We recall that the operator T is a diagonal differential operator with respect to $\{B_n(x)\}_{n=0}^{\infty}$ and $\{\gamma_k\}_{k=0}^{\infty}$ (see Definition 89). In the literature, many questions are unanswered with respect to diagonal differential operators. For example, consider the following open problems.

Problem 190. Given the eigenvalue sequence from equation (4.0.1), $\{\gamma_n\}_{n=0}^{\infty}$, find the sequence $\{Q_k(x)\}_{k=0}^{\infty}$.

Problem 191. If the eigenvalue sequence in equation (4.0.1), $\{\gamma_n\}_{n=0}^{\infty}$, is a B_n-multiplier sequence (see Definition 89), then what properties do the Q_k's possess?

Problem 192. In equation (4.0.1), what assumptions on the polynomial coefficients, $\{Q_k(x)\}_{k=0}^{\infty}$, imply that T must be a hyperbolicity preserving operator?

The question of finding the eigenvalues, $\{\gamma_n\}_{n=0}^{\infty}$, given the polynomial coefficients, $\{Q_k(x)\}_{k=0}^{\infty}$, was addressed and answered in Section 2.3 (see Theorem 93). In this chapter, our goal is to analyze Problems 190-192 in the simplest non-trivial cases, namely when $\{B_n(x)\}_{n=0}^{\infty}$ is a sequence of classical orthogonal polynomials.

4.1 Hermite Diagonal Differential Operators

In studying the Riemann hypothesis [91], it was discovered by P. Turán that Hermite expansions play an important role in determining the zeros within a strip (see [103, 104]). Hence, recent interest has been sparked concerning Hermite diagonal differential operators. Substantial discoveries of Hermite expansions were made by G. Csordas, T. Craven and D. Bleecker [14, 34]. This led to the highly celebrated characterization of Hermite multiplier sequences given by A. Piotrowski in 2007 [85, Theorem 152, p. 140]. Since then many additional properties of Hermite multiplier sequences and Hermite diagonal differential operators have been found [5, 6, 21, 27, 54, 81].

We relate Hermite diagonal differential operators to Theorem 117 and demonstrate that each T_n (see (2.4.2)) is hyperbolicity preserving, thus establishing the first distinct property with respect to affine transformations of Hermite polynomials (see Example 127). We provide a new algebraic characterization of Hermite multiplier sequences (Theorem 208). We find new formulations for the Q_k's in a Hermite diagonal differential operator (see Theorems 201, 203, and 204) (cf. [54, Theorem 3.1]). We extend recent results of T. Forgács and A. Piotrowski concerning the zeros of the polynomial coefficients in Hermite diagonal differential operators (Theorem 205) (cf. [54, Theorem 3.7]).

We begin this section by demonstrating that every hyperbolicity preserving Hermite diagonal differential operator has T_n's (see (2.4.2)) which are also hyperbolicity preserving operators, a non-trivial matter (see Examples 125-131). This will be done in two phases. First we will find a formula for $b_{n,k}$ (see (2.4.1)). Second, we will show that, for each $n \in \mathbb{N}_0$, $\{b_{n,k}\}_{k=0}^{\infty}$ is a Hermite multiplier sequence and hence $\{b_{n,k}\}_{k=0}^{\infty}$ is also a classical multiplier sequence; i.e., each T_n is hyperbolicity preserving (see Remark 195). We will need a few preliminary calculations involving the Hermite polynomials (see Theorem 32).

Lemma 193. *For $k, j \in \mathbb{N}_0$, the kth derivative of the $(k + 2j + 1)^{th}$ and $(k + 2j)^{th}$ Hermite polynomials evaluated at zero is,*

$$H_{k+2j+1}^{(k)}(0) = 0 \quad and \quad H_{k+2j}^{(k)}(0) = \frac{(k+2j)!2^k(-1)^j}{j!}.$$

We provide the complete characterization of the Hermite multiplier sequences due to A. Piotrowski. Compare the following result with Theorem 54.

Theorem 194 ([34], [85, Theorem 152, p. 140]). *Let $\{\gamma_k\}_{k=0}^{\infty}$ be a sequence of real numbers and let $\{g_k^*(x)\}_{k=0}^{\infty}$ be the sequence of reversed Jensen polynomials associated with $\{\gamma_k\}_{k=0}^{\infty}$. The sequence $\{\gamma_k\}_{k=0}^{\infty}$ is a positive or negative non-trivial Hermite multiplier sequence if and only if*

$$e^{-x} \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} x^k = \sum_{k=0}^{\infty} \frac{g_k^*(-1)}{k!} x^k \in \mathscr{L} - \mathscr{P}^s.$$

The sequence $\{\gamma_k\}_{k=0}^{\infty}$ is an alternating non-trivial Hermite multiplier sequence if and only if

$$e^{x} \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} x^k = e^{2x} \sum_{k=0}^{\infty} \frac{g_k^*(-1)}{k!} x^k \in \mathscr{L} - \mathscr{P}^a.$$

Remark 195. We note that the absolute value of any Hermite multiplier sequence will be non-decreasing (see Theorem 47 and [85, Theorem 152, p. 140]). Moreover, every Hermite multiplier sequence is also a classical multiplier sequence (see [85, Theorem 158, p. 145]).

Theorem 196. *Let T be a Hermite diagonal differential operator, $T[H_n(x)] := \gamma_n H_n(x)$, where $\{\gamma_n\}_{n=0}^{\infty}$ is a sequence of real numbers. Then there is a sequence of real polynomials, $\{Q_k(x)\}_{k=0}^{\infty}$, and a sequence of classical diagonal differential operators, $\{T_n\}_{n=0}^{\infty}$, such that*

$$T[H_n(x)] := \left(\sum_{k=0}^{\infty} Q_k(x) D^k \right) H_n(x) = \left(\sum_{k=0}^{\infty} T_k D^k \right) H_n(x) = \gamma_n H_n(x).$$

Moreover, for each $n \in \mathbb{N}_0$,

$$\{b_{2n+1,m}\}_{m=0}^{\infty} = \{0\}_{m=0}^{\infty}$$

and

$$\{b_{2n,m}\}_{m=0}^{\infty} := \left\{ \sum_{k=0}^{m} \binom{m}{k} \frac{(-1)^n}{n!2^n} \left(\sum_{j=0}^{n} \binom{n}{j} \frac{g_{k+n+j}^{*}(-1)}{2^j} \right) \right\}_{m=0}^{\infty},$$

where $T_n[x^m] = b_{n,m} x^m$ for every $n, m \in \mathbb{N}_0$.

Proof. The existence of the sequences $\{Q_k(x)\}_{k=0}^{\infty}$ and $\{T_k\}_{k=0}^{\infty}$ are established by Theorem 90 and 117. We now begin with the remarkable representation formula of T. Forgács and A. Piotrowski that computes the Q_k's in any Hermite diagonal differential operator [54, Theorem 3.1]:

$$Q_k(x) = \sum_{j=0}^{[k/2]} \frac{(-1)^j}{j!(k-2j)!2^{k-j}} g_{k-j}^{*}(-1) H_{k-2j}(x). \tag{4.1.1}$$

This formula yields the following expressions for all $k, n \in \mathbb{N}_0$,

$$Q_{k+2n+1}^{(k)}(0) = 0, \tag{4.1.2}$$

and

$$Q_{k+2n}^{(k)}(0) = \frac{(-1)^n}{n!2^n} \sum_{j=0}^{n} \binom{n}{j} \frac{g_{k+n+j}^{*}(-1)}{2^j}. \tag{4.1.3}$$

Equations (4.1.2) and (4.1.3) could have been calculated using the recursive formula, equation (2.3.1), if one knew, *a priori*, the importance of the $g_{k-j}^{*}(-1)$'s in formula (4.1.1). However, this dependence was not made apparent until formula (4.1.1) was discovered.

Let us now verify (4.1.2) and (4.1.3). Equation (4.1.2) is obvious from formula (4.1.1) and the fact that the Hermite polynomials alternate between even and odd polynomials. We now establish (4.1.3) using

formula (4.1.1) and Lemma 193 as follows:

$$
\begin{aligned}
Q_{k+2n}^{(k)}(0) &= \sum_{j=0}^{[(k+2n)/2]} \frac{(-1)^j}{j!(k+2n-2j)!2^{k+2n-j}} g_{k+2n-j}^*(-1) H_{k+2n-2j}^{(k)}(0) \\
&= \sum_{j=0}^{n} \frac{(-1)^j}{j!(k+2(n-j))!2^{k+n+(n-j)}} g_{k+n+(n-j)}^*(-1) H_{k+2(n-j)}^{(k)}(0) \\
&= \sum_{j=0}^{n} \frac{(-1)^{n-j}}{(n-j)!(k+2j)!2^{k+n+j}} g_{k+n+j}^*(-1) H_{k+2j}^{(k)}(0) \\
&= \sum_{j=0}^{n} \frac{(-1)^{n-j}}{(n-j)!(k+2j)!2^{k+n+j}} g_{k+n+j}^*(-1) \left(\frac{(k+2j)!2^k(-1)^j}{j!} \right) \\
&= \frac{(-1)^n}{n!2^n} \sum_{j=0}^{n} \binom{n}{j} \frac{g_{k+n+j}^*(-1)}{2^j}.
\end{aligned}
$$

We finish the proof by using formula (2.4.1). $\qquad\square$

With the aid of what has been shown thus far, we are now in a position to demonstrate that every Hermite multiplier sequence is the unique sum of classical multiplier sequences. That is, for Hermite multiplier sequences, each T_n in equation (2.4.2) is hyperbolicity preserving. The spirit of the following argument will be the establishment of a Rodrigues-type formula that relates each governing entire function, $\sum_{k=0}^{\infty} \frac{b_{n,k}}{k!} x^k$, of each T_n, with the entire function that defines the hyperbolicity properties of T itself, $\sum_{k=0}^{\infty} \frac{\gamma_k}{k!} x^k$ (see Theorem 54 and 194).

Theorem 197. *Suppose* $\{\gamma_k\}_{k=0}^{\infty}$ *is a Hermite multiplier sequence and let* $\{g_k^*(x)\}_{k=0}^{\infty}$ *be the reversed Jensen polynomials associated with* $\{\gamma_k\}_{k=0}^{\infty}$. *Then, for each* $n \in \mathbb{N}_0$,

$$
\{b_{n,m}\}_{m=0}^{\infty} := \left\{ \sum_{k=0}^{m} \binom{m}{k} \frac{(-1)^n}{n!2^n} \left(\sum_{j=0}^{n} \binom{n}{j} \frac{g_{k+n+j}^*(-1)}{2^j} \right) \right\}_{m=0}^{\infty},
$$

is a Hermite multiplier sequence.

Proof. By assumption, $\{\gamma_k\}_{k=0}^{\infty}$ is a Hermite multiplier sequence. Hence, by Theorem 194, if

$$
f(x) := \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k := \sum_{k=0}^{\infty} \frac{g_k^*(-1)}{k!} x^k,
$$

then, either $f(x) \in \mathscr{L} - \mathscr{P}^s$ or $e^{2x}f(x) \in \mathscr{L} - \mathscr{P}^a$. We wish to show that, $\{b_{n,m}\}_{m=0}^{\infty}$, is a Hermite multiplier sequence; thus using Theorem 194 we must show that if

$$h_n(x) := \sum_{m=0}^{\infty} \left(\sum_{k=0}^{m} \binom{m}{k} b_{n,k}(-1)^{m-k} \right) \frac{x^m}{m!},$$

then either $h_n(x) \in \mathscr{L} - \mathscr{P}^s$ or $e^{2x}h(x) \in \mathscr{L} - \mathscr{P}^a$. We use Theorem 50 and perform the following calculation,

$$
\begin{aligned}
h_n(x) &= \sum_{m=0}^{\infty} \left(\sum_{k=0}^{m} \binom{m}{k} b_{n,k}(-1)^{m-k} \right) \frac{x^m}{m!} \\
&= \sum_{k=0}^{\infty} \left(\frac{(-1)^n}{n!2^n} \left(\sum_{j=0}^{n} \binom{n}{j} \frac{g_{k+n+j}^*(-1)}{2^j} \right) \right) \frac{x^k}{k!} \\
&= \frac{(-1)^n}{n!2^n} \sum_{j=0}^{n} \binom{n}{j} \frac{1}{2^j} \sum_{k=0}^{\infty} \left(\frac{g_{k+n+j}^*(-1)}{k!} \right) x^k \\
&= \frac{(-1)^n}{n!2^n} \sum_{j=0}^{n} \binom{n}{j} \frac{1}{2^j} D^{n+j} f(x) \\
&= \frac{(-1)^n}{n!4^n} D^n \left(\sum_{j=0}^{n} \binom{n}{j} D^j 2^{n-j} \right) f(x) \\
&= \frac{(-1)^n}{n!4^n} D^n (2 + D)^n f(x) \\
&= \frac{(-1)^n}{n!4^n} D^n e^{-2x} D^n e^{2x} f(x). \qquad (4.1.4)
\end{aligned}
$$

Hence, if $f(x) \in \mathscr{L} - \mathscr{P}^s$, then $h_n(x) \in \mathscr{L} - \mathscr{P}^s$ and if $e^{2x}f(x) \in \mathscr{L} - \mathscr{P}^a$, then $e^{2x}h_n(x) \in \mathscr{L} - \mathscr{P}^a$ (see also Corollary 48 and Theorem 194). $\qquad \square$

Equation (4.1.4) yields a little more information than Theorem 197, in particular we derive the recursive formula,

$$h_n(x) = \frac{-1}{4n} D e^{-2x} D e^{2x} h_{n-1}(x), \qquad (n \geq 1, \ h_0(x) := f(x)).$$

Hence, only T_n needs to be diagonalizable with a Hermite multiplier sequence to establish that T_{n+1} is also diagonalizable with a Hermite multiplier sequence.

We note, given a Hermite diagonal differential operator, $T[H_n(x)] = a_n H_n(x)$, $a_n \in \mathbb{R}$, the operator T_0 (see 2.4.2) diagonalizes with the same eigenvalue sequence, namely $T_0[x^n] = a_n x^n$ (see Theorem 115). This indicates that if one assumes each operator T_n yields a Hermite multiplier sequence, then Theorem

197 has a trivial converse, in the sense that if one assumes each T_n diagonalizes with a Hermite multiplier sequence then T itself will also be hyperbolicity preserving. However, what if one only assumes that each T_n is hyperbolicity preserving? Must T be hyperbolicity preserving? We answer this question in the negative, with the following examples.

Example 198. Consider the following Hermite diagonal operator that is not hyperbolicity preserving (see Theorem 194),

$$T[H_n(x)] := \left((-1)^{n+1}(n-1)\right) H_n(x).$$

Thus we calculate,

$$w(x) := \sum_{k=0}^{\infty} \frac{(-1)^{k+1}(k-1)}{k!} x^k = (x+1)e^{-x}.$$

Hence, using equation (4.1.4) (note, $f(x) = e^{-x}w(x)$ (see Theorem 194)), we can calculate the h_n's,

$$h_0(x) := \sum_{k=0}^{\infty} \frac{Q_k^{(k)}(0)}{k!} x^k = (x+1)e^{-2x},$$

$$h_1(x) := \sum_{k=0}^{\infty} \frac{Q_{k+2}^{(k)}(0)}{k!} x^k = \frac{1}{2}e^{-2x}, \quad \text{and}$$

$$h_n(x) := \sum_{k=0}^{\infty} \frac{Q_{k+2n}^{(k)}(0)}{k!} x^k = 0, \quad \text{for } n \geq 2.$$

Hence,

$$T = 1 - xD + \sum_{k=0}^{\infty} \left(\underbrace{\frac{k(-2)^{k+1}}{(k+2)!}}_{h_0^{(k+2)}(0)} x^{k+2} + \underbrace{\frac{-(-2)^{k-1}}{k!}}_{h_1^{(k)}(0)} x^k \right) D^{k+2} = T_0 + T_2 D^2.$$

Thus,

$$T_0[x^n] = \left(1 - xD + \sum_{k=0}^{\infty} \frac{k(-2)^{k+1}}{(k+2)!} x^{k+2} D^{k+2} \right) x^n = \left((-1)^{n+1}(n-1)\right) x^n,$$

$$T_2[x^n] = \left(\sum_{k=0}^{\infty} \left(\frac{-(-2)^{k-1}}{k!} x^k \right) D^k \right) x^n = \left(\frac{1}{2}(-1)^n \right) x^n, \quad \text{and}$$

$$T_{2m}[x^n] = (0) x^n = (0)x^n, \quad \text{for } m \geq 2.$$

We see that for every $n \geq 1$, $h_n(x) \in \mathcal{L} - \mathcal{P}^a$, hence T_{2n} is hyperbolicity preserving (see Theorem 54). However, the original operator T itself is not hyperbolicity preserving, as the following calculation shows,

$$T[4x^2 + 2x - 5] = T[\overbrace{(-3)}^{-3H_0(x)} + \overbrace{(2x)}^{H_1(x)} + \overbrace{(4x^2 - 2)}^{H_2(x)}]$$

$$= 1(-3) + 0(2x) + (-1)(4x^2 - 2) = -4x^2 - 1.$$

Example 199. Consider another Hermite diagonal operator that does not preserve hyperbolicity (see Remark 195), $\{\gamma_k\}_{k=0}^{\infty} = \{(1/2)^k\}_{k=0}^{\infty}$; that is,

$$T[H_n(x)] = \gamma_n H_n(x) := (1/2)^n H_n(x).$$

Using Theorem 117 we write $T = \sum_{n=0}^{\infty} T_n D^n$, where $T_n[x^m] = b_{n,m} x^m$. We rewrite formula (4.1.4) in terms of $b_{n,m}$'s and γ_n's (see Theorem 50),

$$\sum_{k=0}^{\infty} \frac{b_{n,k}}{k!} x^k = \frac{(-1)^n}{n!4^n} e^x D^n e^{-2x} D^n e^x \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} x^k.$$

Since $\sum_{k=0}^{\infty} \frac{\gamma_k}{k!} x^k = e^{x/2}$, then

$$\sum_{k=0}^{\infty} \frac{b_{n,k}}{k!} x^k = \frac{(-1)^n}{n!4^n} \left(-\frac{1}{2}\right)^n \left(\frac{3}{2}\right)^n e^{x/2}.$$

Thus $\sum_{k=0}^{\infty} \frac{b_{n,k}}{k!} x^k \in \mathcal{L} - \mathcal{P}^s$ for every $n \in \mathbb{N}_0$. Hence, T_{2n} is hyperbolicity preserving for every $n \in \mathbb{N}_0$ (see Theorem 54), however, as noted above, T is not hyperbolicity preserving (see Remark 195).

Example 200. To demonstrate the usefulness of Theorem 197, consider the following example. How would one show that

$$\{a_m\}_{m=0}^{\infty} := \{m^{5/2}\}_{m=0}^{\infty}$$

is not a multiplier sequence? Sequence $\{a_m\}_{m=0}^{\infty}$ satisfies the Turán inequalities and is a positive, increasing sequence. Thus some well known methods (Theorem 38, see also [74, p. 341]) do not work. One could apply the sequence to $(1 + x)^5$ to calculate to the fifth associated Jensen polynomial,

$$= (5)x + (56.56\ldots)x^2 + (155.88\ldots)x^3 + (160)x^4 + (55.90\ldots)x^5$$

and verify that this polynomial has non-real zeros, however this can prove to be quite tedious. Instead, we apply Theorem 197 and calculate as summarized in Figure 4.1.1. Hence, after a few simple *numerical*

$$
\begin{array}{rcll}
b_{0,n} & = & 0, & 1, & \cdots \\[4pt]
b_{1,n} & = & -1.41, & -3.65, & \cdots \\[4pt]
b_{2,n} & = & 0.646, & 0.804, & \cdots \\[4pt]
b_{3,n} & = & \boxed{-0.0238}, & \boxed{-0.020}, & \cdots \\[4pt]
& & \vdots & \vdots & \ddots
\end{array}
$$

Figure 4.1.1: Table of Hermite diagonal differential operator eigenvalues.

calculations we arrive at the highlighted portions in Figure 4.1.1 and note that they are negative and increasing, so $\{b_{3,n}\}_{n=0}^{\infty}$ is not a Hermite multiplier sequence (see Remark 195). Thus, the original sequence, $\{a_m\}_{m=0}^{\infty}$, is not a Hermite multiplier sequence. Consequently, since $\{a_m\}_{m=0}^{\infty}$ is an increasing sequence that is not a Hermite multiplier sequence, by Remark 195, we conclude that $\{a_m\}_{m=0}^{\infty}$ cannot be a classical multiplier sequence.

Our next task is to present several relationships between the polynomial coefficients, the Q_k's, and the eigenvalues, the γ_k's, in a Hermite diagonal differential operator,

$$
T[H_n(x)] := \left(\sum_{k=0}^{\infty} Q_k(x) D^k \right) H_n(x) = \gamma_n H_n(x).
$$

In general, in a diagonal differential operator, the relationship between the Q_k's and the γ_k's is not well understood, particularly in the context of hyperbolicity preservation. In special cases direct formulas have been found (see (4.1.1) and Theorem 74). Many recursive formulas have also been established (see Theorem 80, 81 or 90) (see also a non-recursive formula [27, Proposition 216, p. 107]), but a general relation has not been derived that indicates the properties of the Q_k's and the γ_k's for arbitrary hyperbolicity preserving operators. Thus, whenever possible, it is beneficial to present formulas that highlight the nature of the Q_k's in terms of the eigenvalues, the γ_k's. Using calculation (4.1.2) and (4.1.3), in Theorem 201 we can provide another formula for the Q_k's in a Hermite diagonal differential operator (cf. (4.1.1) (see [54]) and (4.2.7)).

Theorem 201. *Let $\{\gamma_n\}_{n=0}^{\infty}$ be a sequence of real numbers and $\{Q_k(x)\}_{k=0}^{\infty}$ be a sequence of real polynomials, such that*

$$T[H_n(x)] := \left(\sum_{k=0}^{\infty} Q_k(x) D^k \right) H_n(x) = \gamma_n H_n(x), \quad n \in \mathbb{N}_0.$$

Then for each $m \in \mathbb{N}_0$,

$$Q_m(x) = \sum_{k=0}^{[m/2]} \frac{(-1)^k}{k! 2^k} \left(\sum_{j=0}^{k} \binom{k}{j} \frac{g_{m-k+j}^*(-1)}{2^j} \right) \frac{x^{m-2k}}{(m-2k)!},$$

where $\{g_k^*(x)\}_{k=0}^{\infty}$ are the associated reversed Jensen polynomials of $\{\gamma_n\}_{n=0}^{\infty}$.

We also derive a complex formulation for the Q_k's in a Hermite diagonal differential operator (Theorem 203). An heuristic argument of the proof of Theorem 203 follows easily by considering the generating function of the Hermite polynomials (see Theorem 32),

$$e^{2xt - t^2} = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n. \tag{4.1.5}$$

We now calculate $T[e^{2xt - t^2}]$ in two ways,

$$T[e^{2xt - t^2}] = \left(\sum_{k=0}^{\infty} Q_k(x) D^k \right) e^{2xt - t^2} = e^{2xt - t^2} \sum_{k=0}^{\infty} Q_k(x)(2t)^k, \quad \text{and}$$

$$T[e^{2xt - t^2}] = T \left[\sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n \right] = \sum_{n=0}^{\infty} \frac{\gamma_n H_n(x)}{n!} t^n.$$

Hence,

$$\sum_{k=0}^{\infty} Q_k(x)(2t)^k = e^{-2xt + t^2} \left(\sum_{n=0}^{\infty} \frac{\gamma_n H_n(x)}{n!} t^n \right). \tag{4.1.6}$$

Thus, performing a Cauchy product on the right hand side of (4.1.6) and comparing the coefficients of t^n on the right and left of (4.1.6), for some $n \in \mathbb{N}_0$, we have,

$$Q_n(x)2^n = \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} \left(\frac{d^{n-k}}{dt^{n-k}} e^{-2xt + t^2} \right) \Big|_{t=0} \left(\frac{d^k}{dt^k} \sum_{j=0}^{\infty} \frac{\gamma_j H_j(x)}{j!} t^j \right) \Big|_{t=0}$$

$$= \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} \frac{d^{n-k}}{dt^{n-k}} \sum_{j=0}^{\infty} \frac{H_j(ix)}{j!} (it)^j \Big|_{t=0} \frac{d^k}{dt^k} \sum_{j=0}^{\infty} \frac{\gamma_j H_j(x)}{j!} t^j \Big|_{t=0}$$

$$= \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} i^{n-k} H_{n-k}(ix) \gamma_k H_k(x).$$

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Remark 202. We must be cautious with the argument above since $T[e^{2xt-t^2}]$ need not converge and hence is only calculated formally. However, even under formal assumptions there is no reason to assume that a differential representation of a linear operator will calculate the same formal series as the operator itself. That is, the calculation,

$$T[e^{2xt-t^2}] = e^{2xt-t^2} \sum_{k=0}^{\infty} Q_k(x)(2t)^k,$$

has not been rigorously established.

Theorem 203. *Let $\{\gamma_n\}_{n=0}^{\infty}$ be a sequence of real numbers and $\{Q_k(x)\}_{k=0}^{\infty}$ be a sequence of real polynomials, such that*

$$T[H_n(x)] := \left(\sum_{k=0}^{\infty} Q_k(x)D^k \right) H_n(x) = \gamma_n H_n(x), \quad n \in \mathbb{N}_0.$$

Then for each $n \in \mathbb{N}_0$,

$$Q_n(x) = \frac{1}{n!2^n} \sum_{k=0}^{n} \binom{n}{k} \gamma_k i^{n-k} H_{n-k}(ix) H_k(x). \tag{4.1.7}$$

Proof. Define

$$T' := \sum_{k=0}^{\infty} Q_k(x)D^k,$$

where we define $Q_k(x)$ from equation (4.1.7). In the spirit of T. Forgács and A. Piotrowski [54, Theorem 3.1], we need only to show that $T'[H_n(x)] = \gamma_n H_n(x)$ for each $n \in \mathbb{N}_0$ (cf. Theorem 80, 81, or 90). We note that for $n, m \in \mathbb{N}_0$, $D^m H_n(x) = 2^m \binom{m}{n} n! H_{n-m}(x)$ ([90, p. 188]). We also note that $\binom{n}{k}\binom{k}{j} = \binom{n}{j}\binom{n-j}{k-j}$ (see [92, p. 3]). Using the generating function of the Hermite polynomials, equation (4.1.5), we now calculate

$$T'[H_n(x)] = \sum_{k=0}^{n} \left(\frac{1}{k!2^k} \sum_{j=0}^{k} \binom{k}{j} \gamma_j i^{k-j} H_{k-j}(ix) H_j(x) \right) \boxed{D^k H_n(x)}$$

$$= \sum_{k=0}^{n} \left(\frac{1}{k!2^k} \sum_{j=0}^{k} \binom{k}{j} \gamma_j i^{k-j} H_{k-j}(ix) H_j(x) \right) \boxed{2^k \binom{n}{k} k! H_{n-k}(x)}$$

$$= \sum_{j=0}^{n} \gamma_j H_j(x) \sum_{k=j}^{n} \binom{n}{k}\binom{k}{j} i^{k-j} H_{k-j}(ix) H_{n-k}(x)$$

$$= \sum_{j=0}^{n} \gamma_j H_j(x) \sum_{k=0}^{n-j} \boxed{\binom{n}{k+j}\binom{k+j}{j}} i^k H_k(ix) H_{(n-j)-k}(x)$$

$$= \sum_{j=0}^{n} \gamma_j H_j(x) \sum_{k=0}^{n-j} \boxed{\binom{n}{j}\binom{n-j}{k}} i^k H_k(ix) H_{(n-j)-k}(x)$$

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$$= \sum_{j=0}^{n} \binom{n}{j} \gamma_j H_j(x) \sum_{k=0}^{n-j} \binom{n-j}{k} \left. \frac{d^k}{dt^k} e^{-2xt+t^2} \right|_{t=0} \cdot \left. \frac{d^{(n-j)-k}}{dt^{(n-j)-k}} e^{2xt-t^2} \right|_{t=0}$$

$$= \sum_{j=0}^{n} \binom{n}{j} \gamma_j H_j(x) \left. \frac{d^{n-j}}{dt^{n-j}} e^{-2xt+t^2} e^{2xt-t^2} \right|_{t=0}$$

$$= \gamma_n H_n(x). \qquad \qquad \square$$

We can establish an interesting relationship between alternating Hermite diagonal differential operators and non-alternating Hermite diagonal differential operators. This will allow us to provide an alternate proof and a non-obvious extension of T. Forgács and A. Piotrowski [54, Theorem 3.7] (cf. Example 206).

Theorem 204. *Let $\{\gamma_k\}_{k=0}^{\infty}$ be a non-trivial sequence of real numbers. Define the Hermite diagonal differential operators*

$$T[H_n(x)] := \left(\sum_{k=0}^{\infty} Q_k(x) D^k \right) H_n(x) = \gamma_n H_n(x)$$

and

$$\tilde{T}[H_n(x)] := \left(\sum_{k=0}^{\infty} \tilde{Q}_k(x) D^k \right) H_n(x) = (-1)^n \gamma_n H_n(x).$$

Then for each $n \in \mathbb{N}_0$,

$$Q_n(x) = \frac{(-2)^n}{n!} \left(\sum_{k=0}^{\infty} \frac{\tilde{Q}_k(x)}{2^k} D^k \right) x^n \qquad (4.1.8)$$

and

$$\tilde{Q}_n(x) = \frac{(-2)^n}{n!} \left(\sum_{k=0}^{\infty} \frac{Q_k(x)}{2^k} D^k \right) x^n.$$

Proof. In light of Remark 202 and Theorem 203, we may conclude that,

$$\sum_{k=0}^{\infty} Q_k(x)(2t)^k = e^{-2xt+t^2} \left(\sum_{n=0}^{\infty} \frac{\gamma_n H_n(x)}{n!} t^n \right)$$

and

$$\sum_{k=0}^{\infty} \tilde{Q}_k(x)(2t)^k = e^{-2xt+t^2} \left(\sum_{n=0}^{\infty} \frac{(-1)^n \gamma_n H_n(x)}{n!} t^n \right);$$

i.e., as formal power series in t, the coefficients are equal (see [82] or [95, p. 130]). Hence, after substitution of $t \to -t$, we have

$$e^{-4xt} \sum_{k=0}^{\infty} Q_k(x)(-2t)^k = e^{-4xt} \left(e^{2xt+t^2} \left(\sum_{n=0}^{\infty} \frac{(-1)^n \gamma_n H_n(x)}{n!} t^n \right) \right)$$

$$= e^{-2xt+t^2} \sum_{n=0}^{\infty} \frac{(-1)^n \gamma_n H_n(x)}{n!} t^n$$

$$= \sum_{k=0}^{\infty} \tilde{Q}_k(x)(2t)^k.$$

Thus,

$$\tilde{Q}_n(x) = \frac{1}{n!2^n} \frac{d^n}{dt^n} \left. e^{-4xt} \sum_{k=0}^{\infty} Q_k(x)(-2t)^k \right|_{t=0}$$

$$= \frac{1}{n!2^n} \sum_{k=0}^{n} \binom{n}{k} (-4x)^{n-k} (-2)^k k! Q_k(x)$$

$$= \frac{(-2)^n}{n!} \sum_{k=0}^{n} \binom{n}{k} \frac{x^{n-k}}{2^k} k! Q_k(x)$$

$$= \frac{(-2)^n}{n!} \left(\sum_{k=0}^{n} \frac{Q_k(x)}{2^k} D^k \right) x^n.$$

By symmetry, equation (4.1.8) also holds. $\qquad\square$

We now extend the work of Forgács and Piotrowski [54, Theorem 3.7] to the alternating Hermite multiplier sequences. Additionally, we show that non-trivial Hermite diagonal differential operators possess interlacing polynomial coefficients.

Theorem 205. *Let $\{\gamma_k\}_{k=0}^{\infty}$ be a non-trivial Hermite multiplier sequence,*

$$T[H_n(x)] := \left(\sum_{k=0}^{\infty} Q_k(x) D^k \right) H_n(x) = \gamma_n H_n(x).$$

Then each $Q_k(x)$ has only real zeros, moreover $Q_k(x)$ and $Q_{k+1}(x)$ have interlacing zeros.

Proof. If $\{\gamma_k\}_{k=0}^{\infty}$ is a Hermite multiplier sequence, then $\{(-1)^k \gamma_k\}_{k=0}^{\infty}$ is also a Hermite multiplier sequence [85, Proposition 119, p. 98]. Hence,

$$\sum_{k=0}^{\infty} \tilde{Q}_k(x) D^k,$$

is a hyperbolicity preserver. Thus, using Theorem 135, we conclude that the operator,

$$\sum_{k=0}^{\infty} \frac{\tilde{Q}_k(x)}{2^k} D^k,$$

is also a hyperbolicity preserver. In particular, by Theorem 204, for each $n \in \mathbb{N}_0$,

$$Q_n(x) = \frac{(-2)^n}{n!} \left(\sum_{k=0}^{\infty} \frac{\tilde{Q}_k(x)}{2^k} D^k \right) x^n,$$

has only real zeros. Moreover, by the Hermite-Kakeya-Obreschkoff Theorem (Theorem 64) (see also [27, Remark 6, p. 5]), since x^n and x^{n+1} have interlacing zeros, then $Q_n(x)$ and $Q_{n+1}(x)$ have interlacing zeros. $\square$

Example 206. The following example demonstrates the difficulties in relating the differential operators of alternating and non-alternating Hermite multiplier sequences. We recall Example 100 and consider the following Hermite diagonal differential operators,

$$T[H_n(x)] := nH_n(x) \quad \text{and} \quad W[H_n(x)] := (-1)^n n H_n(x).$$

Using the recursive formula from Theorem 80, we calculate T and W,

$$T = (x)D + \left(-\frac{1}{2} \right) D^2,$$

and

$$W = (-x)D + \left(2x^2 - \frac{1}{2} \right) D^2 + \left(-2x^3 + x \right) D^3 + \cdots.$$

We see that T is a finite order differential operator while W is an infinite order differential operator (see Corollary 98). The relationship between the polynomial coefficients in T and W is not obvious. We see that Theorem 204 provides new insight into the interplay between alternating and non-alternating Hermite multiplier sequences.

Example 207. The condition that $\{\gamma_n\}_{n=0}^{\infty}$ be a non-trivial sequence is necessary. That is, Theorem 205 does not hold for trivial Hermite multiplier sequences. Consider the Hermite multiplier sequence,

$$\{\gamma_n\}_{n=0}^{\infty} := \{1, 1, 0, 0, 0, \ldots\},$$

and the corresponding diagonal differential operator,

$$T[H_n(x)] := \left(\sum_{k=0}^{\infty} Q_k(x) D^k \right) H_n(x) = \gamma_n H_n(x).$$

Using Theorem 90 we find that,

$$T = 1 + \left(-\frac{x^2}{2} + \frac{1}{4}\right) D^2 + \left(\frac{x^3}{3}\right) D^3 + \left(-\frac{x^4}{8} - \frac{x^2}{8} + \frac{1}{32}\right) + \cdots,$$

and $Q_4(x) = -\frac{1}{8}x^4 - \frac{1}{8}x^2 + \frac{1}{32}$ has non-real zeros (see Theorem 38).

In the spirit of the Pólya-Schur Theorem (Theorem 54), Theorem 204 seems to indicate that only the polynomials $\{x^n\}_{n=0}^{\infty}$ are needed to establish that a Hermite diagonal differential operator is a hyperbolicity preserver. This observation provides us a new algebraic characterization of Hermite multiplier sequences (compare with Pólya-Schur's algebraic characterization, [87] or [85, Theorem 46, p. 46], see also the recent work of P. Brändén, I. Krasikov, and B. Shapiro [19, Proposition 14]).

Theorem 208. *Let $\{\gamma_n\}_{n=0}^{\infty}$ be a non-zero, positive, multiplier sequence of real numbers and let T be a Hermite diagonal differential operator, where $T[H_n(x)] := \gamma_n H_n(x)$ for every $n \in \mathbb{N}_0$. Then T is hyperbolicity preserving if and only if*

$$T[x^n] \in \mathscr{L} - \mathscr{P},$$

for every $n \in \mathbb{N}_0$.

Proof. In order to establish the non-trivial direction, we wish to show T is hyperbolicity preserving; i.e., $\{\gamma_n\}_{n=0}^{\infty}$ is a Hermite multiplier sequence. By assumption, for each $n \geq 2$, the following polynomial has only real zeros (see [90, p. 194] for the Hermite expansion of x^n),

$$
\begin{aligned}
D^{n-2}T[x^n] = D^{n-2}T\left[\frac{n!}{2^n}\sum_{k=0}^{[n/2]}\frac{1}{k!(n-2k)!}H_{n-2k}(x)\right] \\
= D^{n-2}\frac{n!}{2^n}\sum_{k=0}^{[n/2]}\frac{\gamma_{n-2k}}{k!(n-2k)!}H_{n-2k}(x) \\
= \frac{n!}{2^n}\left(\gamma_n\frac{2^{n-2}}{2!}H_2(x) + \gamma_{n-2}\frac{2^{n-2}}{1!}H_0(x)\right) \\
= n!\left(\frac{\gamma_n}{8}(4x^2 - 2) + \frac{\gamma_{n-2}}{4}(1)\right) \\
= \frac{n!\gamma_n}{4}\left(2x^2 + \left(\frac{\gamma_{n-2}}{\gamma_n} - 1\right)\right).
\end{aligned}
$$

Hence, $\frac{\gamma_{n-2}}{\gamma_n} \leq 1$ for every $n \geq 2$. Following the outline of A. Piotrowski [85, Theorem 127, p. 107], since $\{\gamma_n\}_{n=0}^{\infty}$ is assumed to be a multiplier sequence, then the Turán inequalities hold, $\gamma_{n-1}^2 - \gamma_{n-2}\gamma_n \geq 0$ for

every $n \geq 2$. Hence, for each $n \geq 2$,

$$1 \leq \frac{\gamma_n}{\gamma_{n-2}} \leq \left(\frac{\gamma_{n-1}}{\gamma_{n-2}} \right)^2.$$

Thus, $\gamma_{n-2} \leq \gamma_{n-1}$ for $n \geq 2$, and therefore $\{\gamma_n\}_{n=0}^{\infty}$ is a Hermite multiplier sequence (see Remark 195). $\qquad \square$

Remark 209. Finding *minimal* sets that determine hyperbolicity preservation are of extreme importance throughout the literature. In particular, we point out that the set,

$$C := \{T[x^n] : n \in \mathbb{N}_0\}$$

from Theorem 208 is of such *minimal* nature that C cannot be used to determine the hyperbolicity preservation properties of a classical diagonal differential operator. For example,

$$T[x^n] := (n-1)x^n$$

is not hyperbolicity preserving and yet $C \subseteq \mathcal{L} - \mathcal{P}$. Instead the *minimal* set used for classical diagonal differential operators is the set,

$$\{T[(1+x)^n] : n \in \mathbb{N}_0\}$$

also known as G. Pólya and J. Schur's algebraic characterization of classical multiplier sequences [87] (cf. Theorem 54).

Problem 210. Let $\{B_n(x)\}_{n=0}^{\infty}$ be a simple sequence of real polynomials. Find a *minimal* (see Remark 209) set for determining B_n-multiplier sequences (Definition 89).

4.2   Laguerre Diagonal Differential Operators

The main objective of this section is to establish that diagonalizations of Laguerre diagonal differential hyperbolicity preservers have classical diagonal differential operator coefficients that also preserve hyperbolicity (see Theorem 117). We provide a few preliminary remarks for Laguerre multiplier sequences, we then find a formula for the $b_{n,k}$'s (see (2.4.1)), and finally we show that the $b_{n,k}$'s that arise from a Laguerre multiplier sequence yield more Laguerre multiplier sequences. The sensitivity of the proceeding results can be seen in Examples 125, 130, 129, 127, 126, 125, and 131, particularly Example 129. In addition to these results on diagonalizations of Laguerre diagonal differential operators, we also provide new formulas for

finding the polynomial coefficients in Laguerre diagonal differential operators. Before preceding we will need several preliminary calculations (see Theorem 33).

Lemma 211. *For $k, n \in \mathbb{N}_0$, the kth derivative of the nth Laguerre polynomial evaluated at zero is,*

$$L_n^{(k)}(0) = \binom{n}{k}(-1)^k.$$

Lemma 212. *Let n, m, and p be integers. We then have the following combinatorial identity,*

$$\sum_{k=0}^{n}\sum_{j=0}^{m}\binom{m}{j}\binom{k-j}{p-j}\binom{p}{k-j}\binom{n+1}{k+m-j} = \binom{n+1}{p}\binom{n+1}{m} - \binom{n+1-m}{p-m}\binom{p}{n+1-m}.$$

Proof. We first note that $\binom{n+1-m}{p-m}\binom{p}{n+1-m}$ can be added to the summation, hence, we wish to show,

$$\sum_{k=0}^{n+1}\sum_{j=0}^{m}\binom{m}{j}\binom{k-j}{p-j}\binom{p}{k-j}\binom{n+1}{k+m-j} = \binom{n+1}{p}\binom{n+1}{m}. \tag{4.2.1}$$

We perform a substitution of $l = k - j$ on the left side of (4.2.1) and then apply two Vandermonde identities [92, pp. 9, 15],

$$\sum_{k=0}^{n+1}\sum_{j=0}^{m}\binom{m}{j}\binom{k-j}{p-j}\binom{p}{k-j}\binom{n+1}{k+m-j} = \sum_{l=0}^{n+1}\binom{p}{l}\binom{n+1}{m+l}\sum_{j=0}^{m}\binom{m}{j}\binom{l}{p-j}$$

$$= \sum_{l=0}^{n+1}\binom{p}{l}\binom{n+1}{m+l}\binom{m+l}{p}$$

$$= \sum_{j=0}^{n+1}\binom{p}{j-m}\binom{n+1}{j}\binom{j}{p}$$

$$= \sum_{j=0}^{n+1}\binom{j}{p}\binom{p}{j-m}\binom{n+1}{j}$$

$$= \binom{n+1}{p}\binom{n+1}{m}. \qquad \square$$

Theorem 213 ([21, Theorem 1.1]). *Let $\{\gamma_k\}_{k=0}^{\infty}$ be a sequence of real numbers and let $\{g_k^*(x)\}_{k=0}^{\infty}$ be the reversed Jensen polynomials associated with $\{\gamma_k\}_{k=0}^{\infty}$. Sequence $\{\gamma_k\}_{k=0}^{\infty}$ is a positive or negative Laguerre multiplier sequence if and only if*

$$\sum_{k=0}^{\infty} g_k^*(-1)x^k \in \mathbb{R}[x] \cap \mathscr{L} - \mathscr{P}^s[-1, 0].$$

Remark 214. It is known that all Laguerre multiplier sequences can be interpolated by a polynomial and hence, are increasing or decreasing (see [21]). Hence, there are no alternating Laguerre multiplier sequences (Corollary 98) (cf. [21]). Furthermore, all Laguerre multiplier sequences are known to be Hermite multiplier sequences [55, Theorem 4.6], and thus all Laguerre multiplier sequences are also classical multiplier sequences (see [85, Theorem 158, p. 145]) (cf. Remark 195).

Theorem 215. *Let T be a Laguerre diagonal differential operator, $T[L_n(x)] := \gamma_n L_n(x)$, where $\{\gamma_n\}_{n=0}^{\infty}$ a sequence of real numbers. Then there is a sequence of real polynomials, $\{Q_k(x)\}_{k=0}^{\infty}$, and a sequence of classical diagonal differential operators, $\{T_n\}_{n=0}^{\infty}$, such that*

$$T[L_n(x)] := \left(\sum_{k=0}^{\infty} Q_k(x)D^k\right) L_n(x) = \left(\sum_{k=0}^{\infty} T_k D^k\right) L_n(x) = \gamma_n L_n(x).$$

Moreover, for each $n \in \mathbb{N}_0$,

$$\{b_{n,m}\}_{m=0}^{\infty} := \left\{\sum_{k=0}^{m} \binom{m}{k}\frac{(-1)^n}{n!}\left(\sum_{j=0}^{n}\binom{n}{j}\frac{(k+j)!}{((k+j)-n)!}g_{k+j}^*(-1)\right)\right\}_{m=0}^{\infty},$$

where $T_n[x^m] = b_{n,m}x^m$ for every $n, m \in \mathbb{N}_0$.

Proof. The existence of the sequences $\{Q_k(x)\}_{k=0}^{\infty}$ and $\{T_k\}_{k=0}^{\infty}$ are established by Theorem 90 and 117. Recall from Theorem 117 that,

$$\{b_{n,m}\}_{m=0}^{\infty} = \left\{\sum_{k=0}^{m} \binom{m}{k}Q_{k+n}^{(k)}(0)\right\}_{m=0}^{\infty}.$$

Hence, we wish to verify that,

$$Q_{k+n}^{(k)}(0) = \frac{(-1)^n}{n!}\left(\sum_{j=0}^{n}\binom{n}{j}\frac{(k+j)!}{((k+j)-n)!}g_{k+j}^*(-1)\right). \tag{4.2.2}$$

To ease the verification process, we first rewrite formula (4.2.2) as follows,

$$Q_n^{(m)}(0) = \sum_{p=0}^{n}(-1)^{n-m}\binom{n-m}{p-m}\binom{p}{n-m}g_p^*(-1). \tag{4.2.3}$$

We will now verify formula (4.2.3), *tour de force*, by induction. Suppose for every $m \in \mathbb{N}_0$ and $k \in \{0, 1, \ldots, n\}$, formula (4.2.3) holds for $Q_k^{(m)}(0)$. We now calculate $Q_{n+1}^{(m)}(0)$ using the recursive formula

$$Q_{n+1}^{(m)}(0) = \frac{1}{L_{n+1}^{(n+1)}} \left(\boxed{\gamma_{n+1}} \boxed{L_{n+1}^{(m)}(0)} - \sum_{k=0}^{n} \frac{d^m}{dx^m} \left[Q_k(x) L_{n+1}^{(k)}(x) \right] \Big|_{x=0} \right)$$

$$= (-1)^{n+1} \left(\boxed{\sum_{p=0}^{n+1} \binom{n+1}{p} g_p^*(-1)} \boxed{\binom{n+1}{m}(-1)^m} - \sum_{k=0}^{n} \sum_{j=0}^{m} \binom{m}{j} \boxed{Q_k^{(j)}(0)} \boxed{L_{n+1}^{(k+m-j)}(0)} \right)$$

$$= (-1)^{n+1} \left(\sum_{p=0}^{n+1} \binom{n+1}{p} g_p^*(-1) \binom{n+1}{m}(-1)^m \right.$$

$$\left. - \sum_{k=0}^{n} \sum_{j=0}^{m} \binom{m}{j} \left(\boxed{\sum_{p=0}^{n+1} \binom{k-j}{p-j}\binom{p}{k-j}(-1)^{k-j}g_p^*(-1)} \right) \left(\boxed{\binom{n+1}{k+m-j}(-1)^{k+m-j}} \right) \right)$$

$$= \sum_{p=0}^{n+1} \left((-1)^{n+1-m} \left(\binom{n+1}{p}\binom{n+1}{m} - \sum_{k=0}^{n}\sum_{j=0}^{m} \binom{m}{j}\binom{k-j}{p-j}\binom{p}{k-j}\binom{n+1}{k+m-j} \right) \right) g_p^*(-1)$$

$$= \sum_{p=0}^{n+1} (-1)^{n+1-m} \binom{n+1-m}{p-m}\binom{p}{n+1-m} g_p^*(-1). \qquad \square$$

Similar to the Hermite case (see Theorem 197) the following theorem establishes a Rodrigues type formula between $h_n(x)$ ($n \in \mathbb{N}_0$) and $f(x)$. This formula then relates the hyperbolicity preservation of T with each T_n ($n \in \mathbb{N}_0$).

Theorem 216. *Suppose $\{\gamma_k\}_{k=0}^{\infty}$ is a Laguerre multiplier sequence and let $\{g_k^*(x)\}_{k=0}^{\infty}$ be the reversed Jensen polynomials associated with $\{\gamma_k\}_{k=0}^{\infty}$. Then, for each $n \in \mathbb{N}_0$,*

$$\{b_{n,m}\}_{m=0}^{\infty} := \left\{ \sum_{k=0}^{m} \binom{m}{k} \frac{(-1)^n}{n!} \left(\sum_{j=0}^{n} \binom{n}{j} \frac{(k+j)!}{((k+j)-n)!} g_{k+j}^*(-1) \right) \right\}_{m=0}^{\infty},$$

is a Laguerre multiplier sequence.

Proof. By assumption, $\{\gamma_k\}_{k=0}^{\infty}$ is a Laguerre multiplier sequence. Hence, by Theorem 213,

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k := \sum_{k=0}^{\infty} g_k^*(-1) x^k \in \mathbb{R}[x] \cap \mathscr{L} - \mathscr{P}^s[-1,0]. \qquad (4.2.4)$$

To show that, $\{b_{n,m}\}_{m=0}^{\infty}$ is a Laguerre multiplier sequence we must show that,

$$h_n(x) := \sum_{m=0}^{\infty} \left(\sum_{k=0}^{m} \binom{m}{k} b_{n,k}(-1)^{m-k} \right) x^m \in \mathbb{R}[x] \cap \mathscr{L} - \mathscr{P}^s[-1,0].$$

We use Theorem [50] and perform the following calculations,

$$h_n(x) = \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} \binom{k}{j} b_{n,j} (-1)^{k-j} \right) x^k$$

$$= \sum_{k=0}^{\infty} \left(\frac{(-1)^n}{n!} \left(\sum_{j=0}^{n} \binom{n}{j} \frac{(k+j)!}{((k+j)-n)!} g^*_{k+j}(-1) \right) \right) x^k$$

$$= \frac{(-1)^n}{n!} \sum_{j=0}^{n} \binom{n}{j} \sum_{k=0}^{\infty} \left(\frac{(k+j)!}{((k+j)-n)!} g^*_{k+j}(-1) \right) x^k$$

$$= \frac{(-1)^n}{n!} \sum_{j=0}^{n} \binom{n}{j} \sum_{k=0}^{\infty} \frac{f^{(k+j)}(0)}{((k+j)-n)!} x^k$$

$$= \frac{(-1)^n}{n!} \sum_{j=0}^{n} \binom{n}{j} x^{n-j} D^n f(x)$$

$$= \frac{(-1)^n}{n!} (1+x)^n D^n f(x). \tag{4.2.5}$$

Hence, if $f(x) \in \mathbb{R}[x] \cap \mathscr{L} - \mathscr{P}^s[-1,0]$, then $h_n(x) \in \mathbb{R}[x] \cap \mathscr{L} - \mathscr{P}^s[-1,0]$. $\qquad \square$

Similar to the Hermite case, equation (4.2.5) also provides a recursive formula,

$$h_n(x) := \frac{-1}{n} (x+1)^n D(x+1)^{1-n} h_{n-1}(x), \qquad (n \geq 1, \ h_0(x) := f(x)).$$

Thus, again, the hyperbolicity preservation of T_n with a Laguerre multiplier sequence, is enough to establish that T_{n+1} is hyperbolicity preserving with a Laguerre multiplier sequence.

Example 217. We show, similar to Examples [198] and [199], that it is possible for T_n to be hyperbolicity preserving for every n and yet T fail to be hyperbolicity preserving. Consider the following eigenvalue sequence for a Laguerre diagonal differential operator that fails to be a Laguerre multiplier sequence (see (4.2.6) and Theorem [213]),

$$\{a_n\}_{n=0}^{\infty} := \{2, 3, 4, 5, 6, \ldots\},$$

where

$$T[L_n(x)] := a_n L_n(x).$$

From Theorem 117, we obtain $T = \sum_{n=0}^{\infty} T_n D^n$, where $T_n[x^m] = b_{n,m} x^m$ (see (2.4.1)) are classical diagonal differential operators. We calculate $f(x)$ from equation (4.2.4),

$$f(x) := \sum_{k=0}^{\infty} g_k^*(-1) x^k = x + 2. \tag{4.2.6}$$

Hence by formula (4.2.5),

$$h_0(x) = \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} \binom{k}{j} b_{0,j}(-1)^{k-j} \right) x^k = x + 2,$$

$$h_1(x) = \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} \binom{k}{j} b_{1,j}(-1)^{k-j} \right) x^k = -x - 1,$$

and

$$h_n(x) = \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} \binom{k}{j} b_{n,j}(-1)^{k-j} \right) x^k = 0, \qquad \text{for } n \geq 2.$$

We see that $h_0(x) \notin \mathbb{R}[x] \cap \mathscr{L} - \mathscr{P}^s[-1,0]$, hence $\{b_{0,k}\}_{k=0}^{\infty}$ is not a Laguerre multiplier sequence (see Theorem 213). However, if we define a classical multiplier sequence, $W[x^m] := \dfrac{1}{m!} x^m$, then

$$\sum_{k=0}^{\infty} \frac{b_{n,k}}{k!} x^k = e^x W[h_n(x)] \in \mathscr{L} - \mathscr{P}^s.$$

Hence for each $n \in \mathbb{N}_0$, $\{b_{n,k}\}_{k=0}^{\infty}$ is a classical multiplier sequence (see Theorem 54). In addition, for $n \geq 1$, $h_n(x) \in \mathbb{R}[x] \cap \mathscr{L} - \mathscr{P}^s[-1,0]$. Thus, each T_n is hyperbolicity preserving (see Remark 214), each T_n $(n \geq 1)$ diagonalizes with a Laguerre multiplier sequence, but T itself is not a hyperbolicity preserver.

From the calculation of (4.2.3) we can also provide a formula for the Q_k's in a Laguerre differential operator (cf. Theorem 201 and 203).

Theorem 218. *Let $\{\gamma_n\}_{n=0}^{\infty}$ be a sequence of real numbers and $\{Q_k(x)\}_{k=0}^{\infty}$ be a sequence of real polynomials, such that*

$$T[L_n(x)] := \left(\sum_{k=0}^{\infty} Q_k(x) D^k \right) L_n(x) = \gamma_n L_n(x).$$

Then for each $n \in \mathbb{N}_0$,

$$Q_n(x) = \sum_{k=0}^{n} \left(\sum_{p=0}^{n} (-1)^{n-k} \binom{n-k}{p-k} \binom{p}{n-k} g_p^*(-1) \right) x^k, \tag{4.2.7}$$

where $\{g_k^*(x)\}_{k=0}^{\infty}$ are the associated reversed Jensen polynomials of $\{\gamma_n\}_{n=0}^{\infty}$.

Similar to Theorem 203, we provide another formula for the Q_k's in a Laguerre diagonal differential operator.

Theorem 219. *Let $\{\gamma_n\}_{n=0}^{\infty}$ be a sequence of real numbers and $\{Q_k(x)\}_{k=0}^{\infty}$ be a sequence of real polynomials, such that*

$$T[L_n(x)] := \left(\sum_{k=0}^{\infty} Q_k(x) D^k \right) L_n(x) = \gamma_n L_n(x).$$

Then for each $n \in \mathbb{N}_0$,

$$Q_n(x) = \sum_{k=0}^{n} \frac{(-x)^k}{k!} \sum_{j=0}^{n-k} \binom{n-k}{j} (-1)^j \gamma_j L_j(x). \tag{4.2.8}$$

Proof. The proof is very similar to the proof of Theorem 203. Define

$$T' := \sum_{n=0}^{\infty} Q_n(x) D^n,$$

where $Q_n(x)$ is defined from equation (4.2.8). We will establish the result by showing that $T'[L_m(x)] = \gamma_m L_m(x)$ for every $m \in \mathbb{N}_0$. Define the evaluation operator,

$$W := \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n D^n.$$

Note that $W[f(x)] = f(0)$ for every polynomial $f(x)$ (See Example 71). Using Theorem 211 and formula $\binom{n}{k}\binom{k}{j} = \binom{n}{j}\binom{n-j}{k-j}$ (see [92, p. 3]), we now evaluate T' at $L_m(x)$,

$$T'[L_m(x)] = \sum_{n=0}^{m} \left(\sum_{k=0}^{n} \frac{(-x)^k}{k!} \sum_{j=0}^{n-k} \binom{n-k}{j} (-1)^j \gamma_j L_j(x) \right) L_m^{(n)}(x)$$

$$= \sum_{j=0}^{m} (-1)^j \gamma_j L_j(x) \sum_{k=0}^{m} \sum_{n=0}^{m} \binom{n-k}{j} \frac{(-x)^k}{k!} L_m^{(n)}(x)$$

$$= \sum_{j=0}^{m} (-1)^j \gamma_j L_j(x) \sum_{k=0}^{m} \sum_{n=0}^{m} \binom{k}{j} \frac{(-x)^{n-k}}{(n-k)!} L_m^{(n)}(x)$$

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$$= \sum_{j=0}^{m}(-1)^j \gamma_j L_j(x) \sum_{k=0}^{m}\binom{k}{j}\sum_{n=0}^{m}\frac{(-x)^n}{n!}L_m^{(n+k)}(x)$$

$$= \sum_{j=0}^{m}(-1)^j \gamma_j L_j(x) \sum_{k=0}^{m}\binom{k}{j}\boxed{W[L_m^{(k)}(x)]}$$

$$= \sum_{j=0}^{m}(-1)^j \gamma_j L_j(x) \sum_{k=0}^{m}\binom{k}{j}\boxed{L_m^{(k)}(0)}$$

$$= \sum_{j=0}^{m}(-1)^j \gamma_j L_j(x) \sum_{k=0}^{m}\boxed{\binom{k}{j}\binom{m}{k}}(-1)^k$$

$$= \sum_{j=0}^{m}(-1)^j \gamma_j L_j(x) \sum_{k=0}^{m}\boxed{\binom{m}{j}\binom{m-j}{k-j}}(-1)^k$$

$$= \sum_{j=0}^{m}(-1)^j \gamma_j L_j(x) \binom{m}{j}(-1)^j \sum_{k=0}^{m}\left(\binom{m-j}{k}(-1)^k\right)$$

$$= \sum_{j=0}^{m}\binom{m}{j}\gamma_j L_j(x) \sum_{k=0}^{m}\binom{m-j}{k}(-1)^k$$

$$= \gamma_m L_m(x). \qquad \square$$

We are now in a position to refine results concerning the degrees of the polynomial coefficients within a Hermite and Laguerre hyperbolicity preserving diagonal differential operator (this "refinement" was already established for hyperbolicity preserving Jacobi diagonal differential operators, see Theorem 110). In general, we know that the polynomial coefficients of Hermite and Laguerre diagonal differential operators cannot exceed a prescribed degree (see Theorem 107). We also note that in the case of hyperbolicity preservation the degree sequence, $\{\deg(Q_k(x))\}_{k=0}^{\infty}$, follows an even more precise format, namely $\deg(Q_k(x)) = k$ up to the degree of the polynomial that interpolates the eigenvalues (see Theorem 220).

Notice that in Theorems 86 and 107 we have successfully characterized the coefficients of the polynomial coefficients from the Hermite and Laguerre diagonal differential operators. This allows us to provide a further refinement of Theorem 110. See also Definition 85 for the definition of the symbol π_n and π_n^*.

Theorem 220. *Suppose T_H and T_L are hyperbolicity preserving diagonal differential operators, where $T_H[H_k(x)] = a_k H_k(x)$ and $T_L[L_k(x)] = b_k L_k(x)$, $a_0, b_0 \neq 0$, $a_k, b_k \geq 0$.*

1. *If T_H is a finite order diagonal differential operator, then there exists a polynomial $p(x)$, $\deg(p(x)) = n$, such that $a_k = p(k)$ for every $k \in \mathbb{N}_0$. In this case T_H is order $2n$ and has the differential form,*

$$\pi_0 D^{2n} + \pi_1 D^{2n-1} + \cdots + \pi_{n-1}D^{n+1} + \pi_n D^n + \pi_{n-1}D^{n-1} + \cdots + \pi_1 D + \pi_0.$$

2. *If T_L is a finite order diagonal differential operator, then there exists a polynomial $p(x)$, $\deg(p(x)) = n$, such that $b_k = p(k)$ for every $k \in \mathbb{N}_0$. In this case T_L is order $2n$ and has the differential form,*

$$\pi_n D^{2n} + \pi_n D^{2n-1} + \cdots + \pi_n D^{n+1} + \pi_n D^n + \pi_{n-1} D^{n-1} + \cdots + \pi_1 D + \pi_0.$$

Proof. Case 1: We wish to show that T_H has the form given above. From Theorem 110, we already know that $\deg(Q_k(x)) = k$ for $0 \le k \le n$. Hence, we need to show that $\deg(Q_k(x)) = 2n - k$ when $n \le k \le 2n$. From Theorem 86, we have $\deg(Q_k(x)) \le 2n - k$ for $n \le k \le 2n$. Thus, we need only show that $Q_k^{(2n-k)}(0) \ne 0$ for $n \le k \le 2n$. By Theorem 108 and Remark 195, we know that $g_k^*(-1) = 0$ for $k > n$ and $g_n^*(-1) \ne 0$, hence from Theorem 196, calculation (4.1.3), we have,

$$
\begin{aligned}
Q_k^{(2n-k)}(0) &= Q_{(2n-k)+2(k-n)}^{(2n-k)}(0) \\
&= \frac{(-1)^{k-n}}{(k-n)!2^{k-n}} \sum_{j=0}^{k-n} \binom{k-n}{j} \frac{g_{(2n-k)+(k-n)+j}^*(-1)}{2^j} \\
&= \frac{(-1)^{k-n}}{(k-n)!2^{k-n}} \sum_{j=0}^{k-n} \binom{k-n}{j} \frac{g_{n+j}^*(-1)}{2^j} \\
&= \frac{(-1)^{k-n}}{(k-n)!2^{k-n}} g_n^*(-1) \\
&\ne 0.
\end{aligned}
$$

Case 2: Likewise we wish to show that T_L has the form given above. Hence, it suffices to show that $Q_k^{(n)}(0) \ne 0$ for $n < k \le 2n$. By Theorem 108 and Remark 214, we know that $g_k^*(-1) = 0$ for $k > n$ and $g_n^*(-1) \ne 0$, hence from Theorem 215, calculation (4.2.2), we have,

$$
\begin{aligned}
Q_k^{(n)}(0) &= Q_{n+(k-n)}^{(n)}(0) \\
&= \frac{(-1)^{k-n}}{(k-n)!} \left(\sum_{j=0}^{k-n} \binom{k-n}{j} \frac{(n+j)!}{((n+j)-(k-n))!} g_{n+j}^*(-1) \right) \\
&= \frac{(-1)^{k-n}}{(k-n)!} \frac{n!}{(2n-k)!} g_n^*(-1) \\
&\ne 0.
\end{aligned}
$$

$\square$

4.3 Jacobi Diagonal Differential Operators

The Jacobi polynomials are of a very different nature than other classical orthogonal polynomials, namely the Hermite and Laguerre polynomials. In this section we present a small discussion of what is known about Jacobi multiplier sequences. A class of quadratic multiplier sequences is revealed (Theorem 223), a method for calculating differential operators of monomial transformations is discovered (Theorem 229), and a question of M. Chasse is answered (Theorem 233).

The difficultly in analyzing the Jacobi polynomials seems to stem from the following observation; the Hermite and Laguerre diagonal differential operators are finite order diagonal differential operators if and only if the operators eigenvalues are interpolated by a polynomial (see Corollary 104). This is not the case with the Jacobi polynomials (see Example 101). Furthermore, Jacobi polynomials do not enjoy many of the other properties possessed by the Hermite and Laguerre polynomials (see Examples 22, 101, 126, 152, and 232).

Problem 221. In Example 126, we also see that the hyperbolicity preservation property is not preserved in the classical diagonal differential operator coefficients from the diagonalization representation (see Theorem 117). In Example 127 and Example 129 it is revealed that many non-trivial affine transformations of the Hermite and Laguerre bases also fail to have hyperbolicity preserving classical diagonal differential operator coefficients from the diagonalization representation. Since affine transformations seem to play a significant role in diagonalization representation, then it is natural to ask if there is an affine transformation of the Legendre polynomials (or more generally, Jacobi polynomials) such that statements like Theorems 196, 197, 215, and 216 hold?

Very little is known of Jacobi multiplier sequences. Indeed, we provide to date the only known characterizations of Jacobi multiplier sequences. See also several similar and related properties for the Legendre polynomials [11, 52].

Theorem 222 (R. Yoshida [105, Theorem 150]). *For every $c \in \mathbb{R}$, the sequence $\{k+c\}_{k=0}^{\infty}$ is not a multiplier sequence for the Jacobi polynomial basis, $\left\{ P_k^{(\alpha,\beta)}(x) \right\}_{k=0}^{\infty}$, $\alpha, \beta > -1$.*

Theorem 223 (R. Yoshida [105, Theorem 151]). *Let $d \in \mathbb{R}$ and define $\{\gamma_k\}_{k=0}^{\infty} := \{k^2 + (\alpha+\beta+1)k + d\}_{k=0}^{\infty}$. Sequence $\{\gamma_k\}_{k=0}^{\infty}$ is a multiplier sequence for the Jacobi basis, $\left\{ P_k^{(\alpha,\beta)}(x) \right\}_{k=0}^{\infty}$, $\alpha, \beta > -1$, if and only if $0 \le d \le (1+\alpha)(1+\beta)$.*

Proof. By Theorem 34 the Jacobi polynomials satisfy the differential equation,

$$((x^2 - 1)D^2 + ((2 + \alpha + \beta)x - (\beta - \alpha))D + d)P_k^{(\alpha,\beta)}(x) = (k(k + 1 + \alpha + \beta) + d)P_k^{(\alpha,\beta)}(x).$$

Hence, $\{\gamma_k\}_{k=0}^{\infty}$ is a Jacobi multiplier sequence if and only if

$$T := (x - 1)(x + 1)D^2 + (2 + \alpha + \beta)\left(x - \frac{\beta - \alpha}{2 + \alpha + \beta}\right)D + d$$

is hyperbolicity preserving. We see that the proper interlacing conditions are met if $d \geq 0$ (see Remark 171). Thus, by combining Theorem 138 and Theorem 170 (see also Theorem 123) we conclude that T is hyperbolicity preserving if and only if

$$(2 + \alpha + \beta)^2 \left(\frac{\left(1 - \frac{\beta - \alpha}{2 + \alpha + \beta}\right)\left(\frac{\beta - \alpha}{2 + \alpha + \beta} - (-1)\right)}{(1 - (-1))^2}\right) - (1)(d) \geq 0.$$

Since $d \geq 0$ and $\alpha, \beta > -1$, this expression reduces to $0 \leq d \leq (1 + \alpha)(1 + \beta)$. $\qquad\square$

The above characterizations prompt many questions concerning the nature of hyperbolicity preservers.

Problem 224. Are the only quadratic Jacobi multiplier sequences, those that are characterized in Theorem 223? Specifically, if $\{An^2 + (B + \alpha + \beta)n + C\}_{n=0}^{\infty}$, $A, B \neq 0$, is a Jacobi multiplier sequence, then must $A = B$ (cf. [11, Proposition 4])?

Problem 225. Find a characterization of Jacobi multiplier sequences that is similar in nature to Theorems 54, 194 and 213. Specifically, classify all Legendre, Gegenbauer, and Chebyshev multiplier sequences (see Definition 29).

Problem 226. Find a formulation for the Q_k's in a Jacobi diagonal differential operator that is similar in nature to Theorems 201, 203, 218, or 219.

Problem 227. Considering the fact that there are no Legendre linear or cubic multiplier sequences, one naturally asks, are there any cubic Jacobi multiplier sequences (cf. [11, 52])? Are there any Jacobi multiplier sequences that are interpolated by an odd degree polynomial?

Problem 228. The Laguerre polynomials were found to not have any multiplier sequences that arise from an infinite order diagonal differential operator (see [21], see also Corollary 98) (note, however, that the

Hermite polynomials have many multiplier sequences that cannot be interpolated by a polynomial, (see [85, Corollary 129, p. 108]). We ask if the same is true of the Jacobi multiplier sequences; are there any Jacobi multiplier sequences that arise from an infinite order diagonal differential operator? Are there any Jacobi multiplier sequences that are not interpolated by a polynomial?

Despite the many open problems above, we do make some progress towards understanding hyperbolicity preservers that involve a simple sequence of Legendre polynomials. In the next few results will restrict ourselves to a monomial transformations and find a formula for calculating the polynomial coefficients in the corresponding differential representation. This will allow us to answer a question of M. Chasse [27, Conjecture 222] concerning the Legendre polynomials.

Theorem 229 (M. Chasse [27, Proposition 216, p. 107]). *Let $\{B_n(x)\}_{n=0}^{\infty}$ be a simple sequence and define $T : \mathbb{R}[x] \to \mathbb{R}[x]$ by $T[x^n] = B_n(x)$. Then*

$$T = \sum_{k=0}^{\infty} Q_k(x) D^k,$$

where $Q_n(x) = \dfrac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} B_k(x)(-x)^{n-k}$.

Proof. Fix $x \in \mathbb{R}$, $x \neq 0$. By assumption, for every $n \in \mathbb{N}_0$,

$$\sum_{k=0}^{n} Q_k(x) \binom{n}{k} k! x^{n-k} = B_n(x).$$

Hence,

$$\sum_{k=0}^{n} \binom{n}{k} \frac{Q_k(x)}{x^k} k! = \frac{B_n(x)}{x^n}.$$

Reversing this formula (see Theorem 50), yields,

$$\frac{Q_n(x)}{x^n} n! = \sum_{k=0}^{n} \binom{n}{k} \frac{B_k(x)}{x^k} (-1)^{n-k}.$$

Thus for every $n \in \mathbb{N}_0$ and $x \in \mathbb{R}$, $x \neq 0$,

$$Q_n(x) = \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} B_k(x)(-x)^{n-k}.$$

If the above holds on $(-\infty, 0) \cup (0, \infty)$ then by the uniqueness theorem it also holds for $x = 0$. $\qquad\square$

The formula for Q_k above looks like a familiar Cauchy product of two entire functions (for example, see Theorem 21). Indeed,

$$e^{-xw}\left(\sum_{k=0}^{\infty}\frac{B_k(x)}{k!}w^k\right) = \sum_{n=0}^{\infty}\left(\frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}B_k(x)(-x)^{n-k}\right)w^n.$$

Hence, after consideration of Corollary 140, an interesting relation of generating functions and hyperbolicity preservers can be derived.

Corollary 230. *Let $\{B_n(x)\}_{n=0}^{\infty}$ be a simple sequence of real polynomials with generating function,*

$$G(x,w) := \sum_{k=0}^{\infty}\frac{B_k(x)}{k!}w^k.$$

If $e^{-xw}G(x,w)$ can be approximated locally uniformly on $\mathbb{C}\times\mathbb{C}$ by real bivariate stable polynomials, then the operator $T:\mathbb{R}[x]\to\mathbb{R}[x]$ by $T[x^n] = B_n(x)$ is hyperbolicity preserving.

A simple analysis of the generating functions for the Laguerre and Hermite polynomials immediately provide the following hyperbolicity preservers (see Theorems 32 and 33, and Corollary 159) (cf. Corollary 150).

Corollary 231. *The monomial operators $T[x^n] = H_n(x)$ and $W[x^n] = L_n(x)$ are hyperbolicity preservers, where $H_n(x)$ and $L_n(x)$ denote the nth Hermite and Laguerre polynomials, respectively.*

Proof. Observe that

$$e^{-xw}\sum_{k=0}^{\infty}\frac{H_k(x/2)}{k!}w^k = e^{-w^2}\quad\text{and}\quad e^{-xw}\sum_{k=0}^{\infty}\frac{L_k(x)}{k!}w^k = e^{-xw+w}J_0(2\sqrt{xw}),$$

are approximatable by sequences of stable polynomials,

$$\left\{\left(1-\frac{w^2}{n}\right)^n\right\}_{n=0}^{\infty}\quad\text{and}\quad\left\{\left(1-\frac{xw}{n}\right)^n\left(1+\frac{w}{n}\right)^n L_n\left(\frac{xw}{n}\right)\right\}_{n=0}^{\infty},$$

respectively (see Theorems 32 and 33, and Corollary 159). Thus, by Corollary 230, $T_1[x^n] = H_n(x/2)$ and $T_2[x^n] = L_n(x)$ are hyperbolicity preservers. Hence, T and W are hyperbolicity preservers. $\qquad\square$

The hyperbolicity preservation of the Hermite monomial operator, $T[x^n] = H_n(x)$, can also be established by the Hermite-Poulain Theorem (Theorem 77). Since for each $n\in\mathbb{N}_0$, $e^{-D^2}x^n = H_n(x/2)$, then $T = W\circ e^{-D^2}$ is hyperbolicity preserving, where $W[f(x)] = f(2x)$ for each $f\in\mathbb{R}[x]$.

Example 232. We provide, yet another property that the Hermite and Laguerre polynomials possess (Corollary 231) and the Jacobi polynomials fail to possess. Define the monomial Legendre operator, $T[x^n] = P_n(x)$ (see Theorem 34) (see also [25]). Notice

$$T\left[2x^2 + 8x + 8\right] = 2P_2(x) + 8P_1(x) + 8P_0(x) = 2\left(\frac{3}{2}x^2 - \frac{1}{2}\right) + 8(x) + 8(1) = 3x^2 + 8x + 7.$$

Hence, T fails to be hyperbolicity preserving.

With the tools established above, we are now in a position to provide a simple proof for a conjecture made by M. Chasse.

Theorem 233 (M. Chasse [27, Conjecture 222]). *If $T := \sum\limits_{k=0}^{\infty} Q_k(z)D^k$ is the monomial to Legendre polynomial basis transformation, then the $Q_k(z)$ have only real zeros. Furthermore,*

$$Q_{2k}(z) = C_{2k}(z^2 - 1)^k \quad and \quad Q_{2k+1}(z) \equiv 0 \quad k = 0, 1, 2, \ldots,$$

where $C_0 = 1$ and

$$C_{2k} = \frac{(4k+1)!!}{((2k+1)!)^2} - \sum_{j=0}^{k-1} \frac{C_{2j}}{(2k-2j+1)!}.$$

Proof. We first note that by induction $C_k = \frac{(-i)^k}{k!}P_k(0)$. We now invoke a summation formula for the Legendre polynomials [51, Example 14.30, p. 477] (see also [101, formula (4.10.25), p. 99]) to conclude,

$$\frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}P_k(x)(-x)^{n-k} = \frac{(-1)^n}{n!}(1-x^2)^{n/2}P_n(0).$$

Hence, by Theorem 229,

$$T = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!}(1-x^2)^{k/2}P_k(0)D^k = \sum_{k=0}^{\infty} C_k(x^2-1)^k D^k. \qquad \qquad \square$$

CHAPTER 5
POSSIBLE FUTURE RESEARCH

Throughout this dissertation, we provided a number of open problems for further study. In this chapter we expound on several ideas that warrant possible further research. In Section 5.1, we discuss the many bases for which the multiplier sequences have not yet been characterized. Section 5.2 and 5.3 represent additional topics that the author has found of interest while studying hyperbolicity preserving operators. Section 5.4 is dedicated to a list of a few additional problems not found within Chapters 1, 2, 3, and 4.

5.1   Non-Orthogonal Diagonal Differential Operators

In addition to characterizing the remaining classical orthogonal polynomial sequences; the Legendre, Gegenbauer, Chebyshev, or more generally the Jacobi sequencces (see Problem 225), we are also interested in characterizing several other bases that seem to have relevance in polynomial theory.

Problem 234. Characterize multiplier sequences (cf. Theorems 54, 194, and 213) for the following bases:

1. the Bell polynomials (see Definition 60),

2. the Bessel polynomials,

3. the Pochhammer polynomials; i.e., $\{(x)_n\}_{n=0}^\infty := \{1, x, x^2 + x, x^3 + 3x^2 + 2x, \ldots\}$,   and

4. the cyclotomic polynomials; i.e., $\{1, x+1, x^2 + x + 1, x^3 + x^2 + x + 1, \ldots\}$.

Problem 235. Given a transcendental entire function, $f(x) \in \mathscr{L} - \mathscr{P}^s$, $f(0) \neq 0$, characterize multiplier sequences for the associated Jensen polynomials, $\{B_n(x)\}_{n=0}^\infty := \{g_n(x)\}_{n=0}^\infty$, or the associated reversed Jensen polynomials, $\{B_n^*(x)\}_{n=0}^\infty := \{g_n^*(x)\}_{n=0}^\infty$ (see Example 22).

Problem 236. Given any simple sequence of real polynomials, $\{B_n(x)\}_{n=0}^\infty$, what is the relationship between B_n-multiplier sequences and B_n^*-multiplier sequences?

Problem 237. Suppose $\{B_n(x)\}_{n=0}^\infty$ is a simple sequence of real polynomials. Define $T[B_n(x)] = \gamma_n B_n(x)$ and $\tilde{T}[B_n(x)] = (-1)^n \gamma_n B_n(x)$. What is the relationship between T and $\tilde{T}$? Moreover, can Theorem 204 be extended to arbitrary sequences, $\{B_n(x)\}_{n=0}^\infty$?

Not many multiplier sequences have been studied for non-orthogonal bases. Here we call attention to the recent work of T. Forgács, J. Tipton, and B. Wright [53], and A. Piotrowski [85, Remark 164, p. 152] concerning the generalized Hermite polynomials.

5.2   Convergence of Multiplier Sequences

Suppose $f(x) := \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} x^k \in \mathscr{L} - \mathscr{P}^s$, where $0 < \gamma_0 \leq \gamma_1 \leq \cdots$. We note that $\{\gamma_k\}_{k=0}^{\infty}$ is a Hermite multiplier sequence (cf. Remark 195). Furthermore, Corollary 48 and Theorem 108 show that $e^{-x}f(x) \in \mathscr{L} - \mathscr{P}^s$. Hence, there is $\{p_k(x)\}_{k=0}^{\infty} \subseteq \mathscr{L} - \mathscr{P}^s \cap \mathbb{R}[x]$, where $p_k(x) \to e^{-x}f(x)$ $(k \to \infty)$ locally uniformly on $\mathbb{C}$. There is a corresponding sequence, $\{\tilde{p}_k(x)\}_{k=0}^{\infty} \subseteq \mathbb{R}[x]$, such that

$$e^x p_n(x) = \sum_{k=0}^{\infty} \frac{\tilde{p}_n(k)}{k!} x^k, \quad n \in \mathbb{N}_0.$$

Hence, for each $n \in \mathbb{N}_0$, $\{\tilde{p}_n(k)\}_{k=0}^{\infty}$ is a Hermite multiplier sequence and furthermore,

$$\lim_{n \to \infty} \tilde{p}_n(k) = \gamma_k, \quad k \in \mathbb{N}_0.$$

In this way, the associated finite order Hermite diagonal differential hyperbolicity preservers of $\{\tilde{p}_n(k)\}_{k=0}^{\infty}$ (i.e., $W_n[H_k(x)] := \tilde{p}_n(k)H_k(x)$ for each $n \in \mathbb{N}_0$) *converge* to the Hermite diagonal differential hyperbolicity preserver associated with the sequence $\{\gamma_k\}_{k=0}^{\infty}$ (i.e., $W[H_k(x)] := \gamma_k H_k(x)$). These observations inspire several new questions.

Problem 238. Let $\{B_n(x)\}_{n=0}^{\infty}$ be a simple sequence of real polynomials. Is every B_n-multiplier sequence, that can be interpolated by a polynomial, a finite order diagonal differential hyperbolicity preserver (cf. Example 101)?

Problem 239. Let $\{B_n(x)\}_{n=0}^{\infty}$ be a simple sequence of real polynomials. Classify all B_n-multiplier sequences that have differential representation, $\sum_{k=0}^{\infty} Q_k(x)D^k$, where $Q_k(x) \ll Q_{k+1}(x)$, $k \geq 0$. Classify all B_n-multiplier sequences that have differential representation, $\sum_{k=0}^{\infty} Q_k(x)D^k$, where $Q_k(x)$ and $Q_{k+1}(x)$ have interlacing zeros, $k \geq 0$. Classify all B_n-multiplier sequences that have differential representation, $\sum_{k=0}^{\infty} Q_k(x)D^k$, where $Q_k(x) \in \mathscr{L} - \mathscr{P}$. Similar queries are raised by T. Forgács and A. Piotrowski [54, Problems 5.1 and 5.2].

5.3   Transcendental Diagonal Differential Operators

A large portion of current literature is dedicated to operators that map from real polynomials to real polynomials. A somewhat new topic in the field of polynomial theory is to extend the study to operators that

diagonalize on a sequence, $\{F_n(x)\}_{n=0}^{\infty}$, of transcendental entire functions. The following natural question immediately arises.

Problem 240. Given a sequence of entire functions, $\{F_n(x)\}_{n=0}^{\infty}$, classify all F_n-multiplier sequences.

We note that extending our discussions to sequences of entire functions requires a large portion of known theorems to be extended and re-proved (if possible). What classes of functions can be approximated by a transcendental basis? Will $\mathbb{R}[x]$ be in said class? Will the differential representation still be unique? How does one determine the coefficients for a differential representation of an F_n-multiplier sequence? Hermite functions are particularly good candidates for further discussions.

Problem 241. Define $F_n(x) := e^{-x^2} H_n(x)$ for each $n \in \mathbb{N}_0$. Classify all F_n-multiplier sequences.

In order to provide an intuitive understanding of differential operators that diagonalize with a transcendental entire sequence, consider a sequence of real polynomials, $\{B_n(x)\}_{n=0}^{\infty}$, and define the sequence of transcendental entire functions, $\{F_n(x)\}_{n=0}^{\infty} := \{e^{G(x)} B_n(x)\}_{n=0}^{\infty}$, where $G(x)$ is a non-constant entire function. Furthermore, let $\{\gamma_n\}_{n=0}^{\infty}$ be a sequence of real numbers and define

$$T[B_n(x)] := \gamma_n B_n(x), \quad \text{and} \quad \tilde{T}[F_n(x)] := \gamma_n F_n(x).$$

If the differential representation of T is $T = \sum_{k=0}^{\infty} Q_k(x) D^k$, then

$$\tilde{T} = \sum_{k=0}^{\infty} Q_k(x) (-G'(x) + D)^k,$$

in the sense that, $\tilde{T}$ works *formally* on specifically the sequence $\{F_n(x)\}_{n=0}^{\infty}$,

$$\begin{aligned}
\tilde{T}[F_n(x)] &= \left(\sum_{k=0}^{\infty} Q_k(x) (-G'(x) + D)^k \right) e^{G(x)} B_n(x) \\
&= \left(\sum_{k=0}^{\infty} Q_k(x) e^{G(x)} D^k e^{-G(x)} \right) e^{G(x)} B_n(x) \\
&= e^{G(x)} \left(\sum_{k=0}^{\infty} Q_k(x) D^k \right) B_n(x) \\
&= \gamma_n F_n(x).
\end{aligned}$$

The issues of convergence cannot be underestimated. For example, consider the transcendental sequence, $\{\cos(nx)\}_{n=0}^{\infty}$. No polynomial can be approximated locally uniformly on $\mathbb{C}$ by a series of this transcendental

basis. This follows because every finite and infinite combination of $\{\cos(nx)\}_{n=0}^{\infty}$ is periodic with constant 2π, a property that polynomials fail to have.

Interestingly, if we specify a *fixed point* and *neighborhood*, then $\{\cos(nx)\}_{n=0}^{\infty}$ can approximate many polynomials uniformly in that neighborhood. Consider the following polynomials and their trigonometric representations,

$$\sum_{n=1}^{\infty} \frac{\sin(nx)}{n} = -\frac{1}{2}x + \frac{\pi}{2}, \ x \in (0, 2\pi),$$

$$\sum_{n=1}^{\infty} \frac{\cos(nx)}{n^2} = \frac{1}{4}x^2 - \frac{\pi}{2}x + \frac{\pi^2}{6}, \ x \in (0, 2\pi),$$

$$\sum_{n=1}^{\infty} \frac{\sin(nx)}{n^3} = \frac{1}{12}x^3 - \frac{\pi}{4}x^2 + \frac{\pi^2}{6}x, \ x \in (0, 2\pi),$$

$$\sum_{n=1}^{\infty} \frac{\cos(nx)}{n^4} = -\frac{1}{48}x^4 + \frac{\pi}{12}x^3 - \frac{\pi^2}{12}x^2 + \zeta(4) \ x \in (0, 2\pi),$$

$$\vdots$$

Given any neighborhood of $\mathbb{R}$ in which the series above converge uniformly, then they will converge uniformly to a shift of the polynomial on the right hand sides. Hence, it makes sense to define the left hand series above as "hyperbolic" if their right hand side polynomial is hyperbolic. For clarity, we provide numerically approximated graphs of several series in the above form, see Figure 5.3.1.
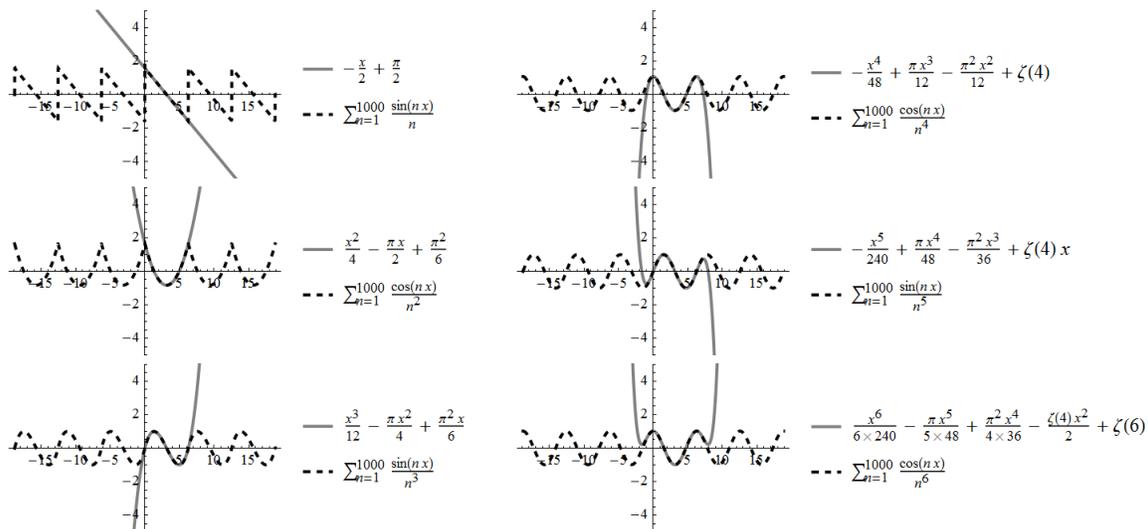
Figure 5.3.1: Graphs of trigonometric series representations for various polynomials.

Problem 242. Classify all multiplier sequences for $\{\sin(nx)\}_{n=0}^{\infty}$ and $\{\cos(nx)\}_{n=0}^{\infty}$. For related work see, for example, P. Hallum [63, Theorem 2.18].

5.4   List of Open Problems

For the reader's convenience we list the following open problems from Chapters 1, 2, 3, and 4: Problems 1, 2, 3, 4, 10, 46, 62, 67, 102, 106, 122, 143, 153, 156, 157, 177, 178, 179, 189, 190, 191, 192, 210, 224, 225, 227, and 228. In addition, we also provide several more related open problems.

Problem 243. Suppose $\{B_n(x)\}_{n=0}^{\infty}$ is an orthogonal basis of real polynomials that is not the Hermite, Laguerre, or Jacobi polynomials. Are there any B_n-multiplier sequences that can be interpolated by a polynomial? Are there any positive B_n-multiplier sequences that cannot be interpolated by a polynomial?

Problem 244. Suppose $\{B_n(x)\}_{n=0}^{\infty}$ is a simple sequence of real polynomials. If every multiplier sequence, that can be interpolated by a polynomial, is also a B_n-multiplier sequence, then must $\{B_n(x)\}_{n=0}^{\infty}$ be an affine transformation of the either the Hermite polynomials or the standard basis?

Problem 245. From Theorems 86, 107, 110, and 220, we see that the degree sequence $\{\deg(Q_k)\}_{k=0}^{\infty}$ is well understood for classical diagonal differential hyperbolicity preservers. In particular, $\{\deg(Q_k)\}_{k=0}^{\infty}$ is always either increasing, decreasing, or uni-modal. Does this observation hold for all finite order hyperbolicity preservers? Does this observation hold for all hyperbolicity preservers with eigenvalues that can be interpolated by a polynomial? We note that, in general, this observation fails for infinite order hyperbolicity preservers, see Example 99.

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