

BELYI MAPS AND BICRITICAL POLYNOMIALS

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ABSTRACT

Let K be a number field. We will show that any bicritical polynomial $f(z) \in K[z]$ is conjugate to a polynomial of the form $a\mathcal{B}_{d,k}(z) + c \in \bar{K}[z]$ where $\mathcal{B}_{d,k}(z)$ is a normalized single-cycle Belyi map with combinatorial type $(d; d-k, k+1, d)$. We use results of Ingram [14] to determine height bounds on pairs (a, c) such that $a\mathcal{B}_{d,k}(z) + c$ is post-critically finite. Using these height bounds, we completely describe the set of post-critically finite cubic polynomials of the form $a\mathcal{B}_{d,k}(z) + c \in \mathbb{Q}[z]$, up to conjugacy over \mathbb{Q} . We give partial results for post-critically finite polynomials over \mathbb{Q} of arbitrary degree $d > 3$.

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CHAPTER 1

INTRODUCTION

Arithmetic dynamics is the intersection of number theory and discrete (holomorphic) dynamical systems. In arithmetic dynamics we study number theoretic properties of points under the iteration of morphisms of algebraic varieties. The orbits of the critical points largely determine the dynamical properties of a rational map, thus we find it useful to focus on maps with a fixed number of critical points.

In the early 1900's Fatou and Julia began studying dynamical systems, eventually giving rise to the famous Mandelbrot set. The Mandelbrot set describes the set of points $c \in \mathbb{C}$ such that the orbit of 0 under the map $z^2 + c$ is bounded. This is an example of a unicritical polynomial, i.e. a polynomial with a single finite critical point. Unicritical polynomials have been studied extensively by Milnor, whose work is fundamental in dynamical systems.

A large portion of the work in this thesis is motivated by the question: when is the orbit of a critical point finite? Maps that have that property are called post-critically finite and they are of great interest to complex and arithmetic dynamicists. Post-critically finite unicritical maps have been previously studied, however little is known about families of maps with additional critical points.

A natural next step is to consider polynomials with two critical points, which we call bicritical polynomials. First, we find a normal form for bicritical polynomials, up to conjugacy. We will show that we can use normalized single-cycle Belyi maps to define a normal form for bicritical polynomials. Furthermore, we conjecture that conjugating any post-critically finite bicritical polynomial to this standard form will not change the field over which it is defined. This would allow us to use this standard form to study dynamical properties of all bicritical polynomials over a given number field, including determining post-critically finite bicritical polynomials.

All cubic polynomials are either unicritical or bicritical. In the past, bicritical cubic polynomials were studied using the Hubbard-Branner normal form. While this form is convenient in complex dynamics, it is not ideal for arithmetic dynamics as conjugating to this form does not preserve the field of definition for all post-critically finite cubic polynomials. Assuming our conjecture holds, we will be able to completely describe the set of post-critically finite cubic polynomials over \mathbb{Q} using

our normal form.

1.1 Outline

In Chapter 2 we review the background necessary to navigate the results of this thesis. Chapter 3 presents joint work with Manes and Melamed describing the dessins d'enfants for single-cycle Belyi maps. This work inspired the standard form for bicritical polynomials, which is detailed in Chapter 4. In Chapter 5 we use a result due to Ingram [14] to determine height bounds on the parameters of the standard form which may admit post-critically finite bicritical polynomials over \mathbb{Q} . Finally, using the results from Chapter 5 we are able to list all post-critically finite cubic polynomials with coefficients in \mathbb{Q} (Chapter 6).

CHAPTER 2 BACKGROUND

The author invites the reader to explore some of the main concepts that will be used in the later chapters. This chapter will introduce the reader to the main ideas of arithmetic dynamics, discuss unicritical polynomials, which motivate the results regarding bicritical polynomials, and set the reader up to understand the use of Belyi maps in the latter chapters.

2.1 Dynamical Systems

Classical dynamical systems concern points under iteration of a morphism $f : \mathbb{P}^1(\mathbb{C}) \rightarrow \mathbb{P}^1(\mathbb{C})$. In arithmetic dynamics, we replace \mathbb{C} with any field K of number theoretic interest. Furthermore, we can replace \mathbb{P}^1 with any algebraic variety. For $n \in \mathbb{N}$ (where the natural numbers are positive integers), we define f^n to be the n^{th} iterate of f :

$$f^n(z) = \underbrace{(f \circ f \circ \dots \circ f)}_n(z).$$

For the remainder of the thesis we will assume \mathbb{P}^1 refers to $\mathbb{P}^1(\bar{K})$, where \bar{K} is a fixed algebraic closure of our field K . We can write a morphism $f : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ as a rational map $f(z) = \frac{p(z)}{q(z)} \in K(z)$.

A main goal of dynamics is to classify points based on their behavior under iteration of a dynamical system, defined as their orbit, \mathcal{O} :

$$\mathcal{O}_f(\alpha) = \{f^n(\alpha) : n \in \mathbb{Z}_{\geq 0}\}.$$

Definition 2.1.1. Let α be a point in \mathbb{P}^1 .

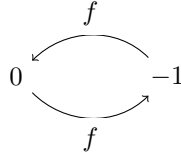
- A point is *periodic* for f if $f^m(\alpha) = \alpha$ for some $m \in \mathbb{N}$. If m is the least such number we say that α has primitive period m .
- If $f(\alpha) = \alpha$, we call α a *fixed point* for f .
- If a point α is eventually periodic, i.e., $f^{m+k}(\alpha) = f^m(\alpha)$ for some $k > 0$ and $m \geq 0$, we call α *pre-periodic of period (m, k)* for f .
- If the orbit of α is infinite, then we say that α is a *wandering point* for f .

We set the following notation:

$$\text{PrePer}(f, K) = \{\alpha \in K : \alpha \text{ is preperiodic for } f\}.$$

Notice that fixed points are periodic of period 1 and periodic points of period n are preperiodic of period $(0, n)$. We can view the orbit structure of a point for a function f as a directed graph encoding the action of f , called a *portrait*. For preperiodic points, the portrait will be a finite graph. The majority of the results in this thesis will concern preperiodic points.

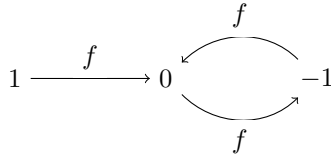
Example 2.1.2. Consider the function $f(z) = z^2 - 1$ with $\alpha = 0$. Below is the portrait of α for f .



So 0 is periodic of period 2 for f and

$$\mathcal{O}_f(0) = \{0, 1\}.$$

Now, consider $\alpha = 1$.



So $f(1) = f^3(1)$, hence 1 is preperiodic for f of period $(1, 2)$, and

$$\mathcal{O}_f(1) = \{1, 0, -1\}.$$

We will omit the labeling of the edges in a portrait when the context is clear.

Example 2.1.3. Let $f(z) = z^2 - 2$ and $\alpha = 2$. Then the portrait of the orbit of α appears below.



Definition 2.1.4. A function $f(z) \in K(z)$ is conjugate to $g(z) \in \bar{K}[z]$ if there exists a linear fractional transformation $\phi \in \text{PGL}_2(\bar{K})$ such that $f^\phi = g$ where $f^\phi = \phi \circ f \circ \phi^{-1}$.

Conjugacy gives an equivalence relation on rational maps, and we write $f \sim g$ if $f^\phi = g$ for some $\phi \in \text{PGL}_2$. Notice that

$$(f^\phi)^n(z) = (f^n)^\phi(z);$$

thus conjugation by ϕ respects dynamical structure. This implies that if α is preperiodic of period (m, k) under f , then $\phi(\alpha)$ is preperiodic of period (m, k) under f^ϕ .

Example 2.1.5. Consider the function $f(z) = 2z^2 - 1$. Conjugating by $\phi(z) = 2z$,

$$f^\phi(z) = z^2 - 2.$$

Notice that -1 is preperiodic of period $(1, 1)$ for f , and $\phi(-1) = -2$ is preperiodic of period $(1, 1)$ for f^ϕ . Similarly, 1 is fixed for f and $\phi(1) = 2$ is fixed for f^ϕ .

In general, if $f(z) = cz^2 + 1$, then $f(z)$ is conjugate to $g(z) = z^2 + c$. Letting $\phi(z) = cz$ we have

$$f^\phi(z) = g(z).$$

2.2 Ramification Points

The orbits of critical points are crucial to understanding a dynamical system. Let $f(z) \in \mathbb{C}(z)$ and consider $\alpha \in \mathbb{P}^1$ such that neither $\alpha = \infty$ nor $f(\alpha) = \infty$. We can then write the Taylor expansion of f around α . For some $c \neq 0$

$$f(z) = f(\alpha) + c(z - \alpha)^e + O\left((z - \alpha)^{e+1}\right). \quad (2.1)$$

Definition 2.2.1. The *ramification index* of f at α is the smallest integer $e \geq 1$ such that (2.1) holds. We write this as $e_f(\alpha)$.

We wish to define ramification index for all points $\alpha \in \mathbb{P}^1$. The following proposition shows that ramification index is well-defined on conjugacy classes, which will allow us to extend the definition.

Proposition 2.2.2. *Let $f \in \mathbb{C}(z)$ and $\phi \in \text{PGL}_2$. If none of $\alpha, f(\alpha), \phi(\alpha), f^\phi(\phi(\alpha))$ are ∞ , then the ramification indices satisfy*

$$e_f(\alpha) = e_{f^\phi}(\phi(\alpha)).$$

Proof. Define $\beta = \phi(\alpha)$. Taking the derivative we get

$$\begin{aligned} (f^\phi)'(\beta) &= \phi'(f(\phi^{-1}(\beta))) \cdot f'(\phi^{-1}(\beta)) \cdot (\phi^{-1})'(\beta) \\ &= \phi'(f(\alpha)) \cdot f'(\alpha) \cdot (\phi^{-1})'(\beta). \end{aligned}$$

Since $(\phi^{-1})(\beta) = \frac{1}{\phi'(\alpha)}$,

$$(f^\phi)'(\beta) = \frac{\phi'(f(\alpha))}{\phi'(\alpha)} \cdot f'(\alpha).$$

Since $\phi \in \text{PGL}_2$ then ϕ' doesn't vanish at finite points, so we can conclude that $f'(\alpha)$ vanishes if and only if $(f^\phi)'(\beta)$ vanishes, independent of choice of f . We now proceed by induction.

Suppose that the n^{th} derivative $f^{(n)}(\alpha) = 0$ for some $n \in \mathbb{N}$, and

$$(f^\phi)^{(n)}(\beta) = \frac{\phi'(f(\alpha))}{\phi'(\alpha)} \cdot f^{(n)}(\alpha).$$

Then

$$\begin{aligned} (f^\phi)^{(n+1)}(\beta) &= \frac{\phi'(\alpha)\phi'(f(\alpha))f^{(n+1)}(\alpha)}{(\phi'(\alpha))^2} \\ &= \frac{\phi'(f(\alpha))}{\phi'(\alpha)} \cdot f^{(n+1)}(\alpha). \end{aligned}$$

Therefore for all $n \in \mathbb{N}$, $(f^\phi)^{(n)}(\beta) = 0$ if and only if $f^{(n)}(\alpha) = 0$.

If $e_f(\alpha)$ is the ramification index of α for f , then $f^{(n)}(\alpha) = 0$ for all $1 \leq n < e_f(\alpha)$, and $f^{(e_f(\alpha))}(\alpha) \neq 0$.

Therefore, $(f^\phi)^{(n)}(\beta) = 0$ for all $1 \leq n < e_f(\alpha)$, and $(f^\phi)^{(e_f(\alpha))}(\beta) \neq 0$, so

$$e_{f^\phi}(\beta) = e_f(\alpha). \quad \square$$

We can now define ramification index for any $\alpha \in \mathbb{P}^1$.

Definition 2.2.3. Let $f \in \mathbb{C}(z)$ and $\alpha \in \mathbb{P}^1$. Choose $\phi \in \text{PGL}_2$ such that $\phi(\alpha) \neq \infty$ and $f^\phi(\phi(\alpha)) \neq \infty$. The ramification index of f at α is defined by:

$$e_f(\alpha) := e_{f^\phi}(\phi(\alpha)).$$

Note that Proposition 2.2.2 shows $e_f(\alpha)$ is well-defined. We say α is a *critical point* of f if $e_f(\alpha) > 1$.

Example 2.2.4. Suppose $f(z) = a_d z^d + a_{d-1} z^{d-1} + \dots + a_1 z + a_0 \in \mathbb{C}[z]$. If $a_0 = 0$, replace f by a conjugate polynomial with non-zero constant term, so we can assume $a_0 \neq 0$. Then

$$e_f(\infty) = e_{f^\phi}(\phi(\infty))$$

for $\phi \in \mathrm{PGL}_2$ such that $\phi(\infty) \neq \infty$. Choose $\phi(z) = \frac{1}{z}$ so

$$e_f(\infty) = e_{f\phi}(0).$$

Conjugating by ϕ ,

$$f^\phi(z) = \frac{z^d}{a_d + a_{d-1}z + \dots + a_1z^{d-1} + a_0z^d}.$$

Since $(f^\phi)^{(n)}(0) = 0$ for all $n < d$, and $(f^\phi)^{(d)}(0) \neq 0$, then

$$f^\phi(z) = cz^d + O(z^{d+1}),$$

so $e_{f^\phi}(0) = d$, hence

$$e_f(\infty) = d.$$

Proposition 2.2.5. *If $f : \mathbb{P}^1 \rightarrow \mathbb{P}^1$, $\deg(f) = d$, and $\alpha \in \mathbb{P}^1$, then $1 \leq e_f(\alpha) \leq d$.*

Proof. If necessary, replace f by f^ψ where $\alpha \neq \infty$ and $\psi(\alpha) \neq \infty$. Let f and α be as above. Choose $\phi \in \mathrm{PGL}_2$ such that $\phi(\alpha) = \alpha$ and $\phi(f(\alpha)) = 0$. Define $g := f^\phi$. Since $g(\alpha) = 0$, then $g(z) = (z - \alpha)^e h(z)$ for some $h(z)$ such that $h(\alpha) \neq 0$ and $e \geq 1$. And since $\deg(f) = \deg(g) = d$, we have $e \leq d$. Clearly $e_g(\alpha) = e$. By Proposition 2.2.2 we know that ramification index is well defined on conjugacy classes, so

$$e_f(\alpha) = e_{f\phi}(\phi(\alpha)) = e_g(\alpha) = e. \quad \square$$

The Reimann-Hurwitz formula (see [30, Theorem 1.1]) gives us a relationship between ramification indices and the degree d of a rational function. For a rational map $f : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ of degree d , over a field K of characteristic 0, we have that

$$\sum_{\alpha \in \mathbb{P}^1} (e_f(\alpha) - 1) = 2d - 2.$$

This tells us that a rational map has $2d - 2$ ramification points, counting multiplicity.

Since we now know that a polynomial f will have a critical point at ∞ with $e_f(\infty) = d$, we will use $\mathrm{Crit}(f)$ to mean all finite critical points.

2.3 Post-Critically Finite Dynamical Systems

Hubbard writes in [5] “According to Fatou and Julia, and the more recent work of Mañe, Sad, and Sullivan, the main question to ask about a rational map is: *what are the orbits under iteration of*

the critical points?” In the present work we are interested in functions for which all critical points have finite orbit.

Definition 2.3.1. A polynomial f is *post-critically finite* or PCF if $\mathcal{O}_f(\alpha)$ is finite for all critical points α of f .

Example 2.3.2. Let $f(z) = z^2 - 1$. Notice that f has critical points 0 and ∞ . Since

$$\mathcal{O}_f(0) = \{0, -1\}$$

and

$$\mathcal{O}_f(\infty) = \{\infty\}$$

then f is post-critically finite.

Example 2.3.3. Let $f(z) = 4z^3 - 6z^2 + \frac{3}{2}$. The critical points of f are 0, 1, and ∞ . Since

$$\begin{aligned}\mathcal{O}_f(0) &= \left\{0, \frac{3}{2}\right\}, \\ \mathcal{O}_f(1) &= \left\{1, -\frac{1}{2}\right\}\end{aligned}$$

and

$$\mathcal{O}_f(\infty) = \{\infty\},$$

then f is PCF.

The orbits of critical points for f determine many dynamical properties of f on all of \mathbb{P}^1 . The study of post-critically finite maps has been of great interest to arithmetic and complex dynamicists. All quadratic post-critically finite maps over \mathbb{Q} have been found [16]. Furthermore, many cubic post-critically finite maps over \mathbb{Q} have been found [14]. These works serve as part of the motivation for the results described within this thesis.

2.4 Places of Number Fields

Many of the results in the latter chapters concern absolute values of points under iterations of polynomial functions.

Definition 2.4.1. An *absolute value* on a field F is a real-valued function $|\cdot|_v : F \rightarrow \mathbb{R}$ with the following properties for any $\alpha, \beta \in F$:

- $|\cdot|_\nu \geq 0$ with $|\alpha|_\nu = 0$ if and only if $\alpha = 0$,
- $|\alpha|_\nu |\beta|_\nu = |\alpha\beta|_\nu$,
- $|\alpha|_\nu + |\beta|_\nu \geq |\alpha + \beta|_\nu$.

We say that this is *non-archimedean* if it satisfies

$$|\alpha + \beta|_\nu \leq \max\{|\alpha|_\nu, |\beta|_\nu\}.$$

Definition 2.4.2. Let p be a prime. The p -adic absolute value on \mathbb{Q} is defined as follows: for any non-zero $\alpha = p^k \frac{r}{s} \in \mathbb{Q}$ with $p \nmid rs$,

$$|\alpha|_p = p^{-k},$$

and $|0|_p = 0$.

Notice that p -adic absolute values are non-archimedean. Furthermore, for any $\alpha, \beta \in \mathbb{Q}$ such that $|\alpha|_p \neq |\beta|_p$,

$$|\alpha + \beta|_p = \max\{|\alpha|_p, |\beta|_p\}.$$

Theorem 2.4.3. [26, Ostrowski's Theorem] *Every non-trivial absolute value on \mathbb{Q} is equivalent to the standard archimedean absolute value, $|\cdot|_\infty$, or $|\cdot|_p$ for some prime p .*

We can extend this notion to any number field, K .

Definition 2.4.4. Fix a field K and let \mathfrak{p} be a non-zero prime ideal in \mathcal{O}_K . The \mathfrak{p} -adic absolute value on K is defined as follows:

$$|\alpha|_{\mathfrak{p}} = N(\mathfrak{p})^{-\text{ord}_{\mathfrak{p}}(\alpha)}.$$

Theorem 2.4.5. [25, Theorem 2.8.1] *Every non-archimedean absolute value on K is equivalent to $|\cdot|_{\mathfrak{p}}$ for some prime ideal \mathfrak{p} in \mathcal{O}_K .*

Definition 2.4.6. A *place* of a number field, K , is an equivalence class of absolute values on K .

We use the notation M_K to denote the set of places on K . Every place in M_K restricts to a place in $M_{\mathbb{Q}}$ [30, pg 83]. If a finite point α is preperiodic for a polynomial f , then all points in the orbit must be ν -adically bounded for each $\nu \in M_K$. Using tools introduced in the following section, we will be able to find effective bounds for specific polynomials and use these bounds to find a unique representative in each conjugacy class of PCF polynomials.

2.5 Heights on Dynamical Systems

We can define a function that measures the arithmetic complexity of a point $\alpha \in \mathbb{P}^1(\bar{\mathbb{Q}})$.

Definition 2.5.1. Let $\alpha \in \bar{\mathbb{Q}}$ and let K be any number field such that $\alpha \in K$. The *logarithmic height* is defined as:

$$h(\alpha) := \frac{1}{[K:\mathbb{Q}]} \sum_{\nu \in M_K} \log \max\{|\alpha|_{\nu}, 1\}.$$

It follows from the definition that $h(\alpha^d) = dh(\alpha)$. We can extend h to all of $\mathbb{P}^1(\bar{\mathbb{Q}})$ by defining $h(\infty) = 0$.

Definition 2.5.2. Let $f : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ be a rational map defined over $\bar{\mathbb{Q}}$. The *canonical height* function for f is the function

$$\hat{h}_f(\alpha) := \lim_{n \rightarrow \infty} \frac{1}{d^n} h(f^n(\alpha)).$$

By Theorem 3.20 of [30] this function exists and satisfies

- $\hat{h}_f(\alpha) = h_f(\alpha) + O(1)$ and
- $\hat{h}_f(f(\alpha)) = d\hat{h}_f(\alpha)$.

The following theorem tells us the relationship between canonical height and preperiodic points of f .

Theorem 2.5.3. [30, Theorem 3.22] *Let $f : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ be a morphism of degree $d \geq 2$. Then α is preperiodic for f if and only if $\hat{h}_f(\alpha) = 0$.*

When f is a polynomial and $\alpha \neq \infty$ we can write

$$\hat{h}_f(\alpha) := \sum_{\nu \in M_K} n_{\nu} \hat{\lambda}_{f,\nu}(\alpha)$$

where

$$\hat{\lambda}_{f,\nu}(\alpha) = \lim_{n \rightarrow \infty} \frac{1}{d^n} \log \max\{|f^n(\alpha)|_{\nu}, 1\}.$$

The value $\hat{\lambda}_{f,\nu}$ at α is the *local height* of f for ν . In this paper we will use results bounding the local height of a polynomial from below to help determine PCF polynomials.

2.6 Belyi Maps

A significant portion of this section and Chapter 3 were written in collaboration with Michelle Manes and Gabbie Melamed and will appear in [20].

Let X be a smooth projective curve.

Definition 2.6.1. A *Belyi map* $f : X \rightarrow \mathbb{P}^1$ is a finite cover (i.e., a finite morphism) that is ramified only over the points $0, 1$, and ∞ .

If $f_1 : X_1 \rightarrow \mathbb{P}^1$ and $f_2 : X_2 \rightarrow \mathbb{P}^1$ are two Belyi maps, we say they are isomorphic if there exists $\iota : X_1 \rightarrow X_2$ such that $f_1 = f_2 \circ \iota$.

The genus of the Belyi map is the genus of the covering curve X . There are multiple ways to realize a Belyi map of degree d (see, for example, [2, 12, 32]):

- (i) explicitly, as a degree d function between projective curves;
- (ii) combinatorially, as a generating system of degree d ; and
- (iii) topologically, as a dessin d'enfant with d edges.

Definition 2.6.2. A *generating system* of degree d is a triple of permutations $(\sigma_0, \sigma_1, \sigma_\infty) \in S_d^3$ with the property that $\sigma_0\sigma_1\sigma_\infty = 1$ and the subgroup $\langle \sigma_0, \sigma_1 \rangle \subseteq S_d$ is transitive.

We say that two generating systems $(\sigma_0, \sigma_1, \sigma_2)$ and $(\hat{\sigma}_0, \hat{\sigma}_1, \hat{\sigma}_2)$ are isomorphic if there exists $\tau \in S_d$ such that $\tau\sigma_i\tau^{-1} = \hat{\sigma}_i$ for all $i \in \{0, 1, 2\}$.

Definition 2.6.3. A *dessin d'enfant* (henceforth *dessin*) is a connected bipartite graph embedded in an orientable surface. The dessin has a fixed cyclic ordering of its edges at each vertex; this manifests as a labeling.

In general, it is a simple matter to describe a dessin from either a generating system or a function. Given $f : X \rightarrow \mathbb{P}^1$:

- Define a black vertex for each inverse image of 0 .
- Define a white vertex for each inverse image of 1 .
- Define edges by the inverse images of the line segment $(0, 1) \in \mathbb{P}^1$.

This process yields a connected bipartite graph. The labeling of the edges arises from the local monodromy around the vertices.

Similarly, given a generating system $(\sigma_0, \sigma_1, \sigma_\infty) \in S_d^3$, we create a dessin with edges labeled $\{1, 2, \dots, d\}$ via the following recipe:

- Draw a black vertex for each cycle in σ_0 (including the one-cycles). The cycles in σ_0 then give an ordering of edges around each of these vertices.

- Draw a white vertex for each cycle in σ_1 (including the one-cycles). The cycles in σ_1 then give an ordering of edges around each of these vertices.

This determines a bipartite graph. Since σ_0 and σ_1 generate a transitive subgroup of S_d , the graph is connected.

It is equally straightforward to describe a generating system from a dessin. The difficulty in completing the picture is often in giving an explicit function realizing the Belyi map as a covering of \mathbb{P}^1 . For some recent results in this area, see [29, 33].

In some simple cases, however, we can explicitly realize this triple correspondence for an infinite family of Belyi maps. For example, for pure power maps we have the following:

Belyi map

$$f : \mathbb{P}^1 \rightarrow \mathbb{P}^1$$

$$z \mapsto z^d$$

Generating system

$$\sigma_0 = d\text{-cycle}, \quad \sigma_1 = \text{trivial}, \quad \sigma_\infty = d\text{-cycle}.$$

Dessin d'enfant

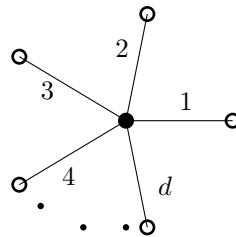


Figure 2.1: The dessin for the degree d power map.

And for Chebyshev polynomials we have:

Belyi map

$$f : \mathbb{P}^1 \rightarrow \mathbb{P}^1$$

$$z \mapsto T_d(z)$$

Generating system

$$\begin{aligned} \sigma_0 &= (23)(45) \cdots ((d-1)d) && \text{or } (23)(45) \cdots ((d-2)(d-1)). \\ \sigma_1 &= (12)(34) \cdots ((d-2)(d-1)) && \text{or } (12)(34) \cdots ((d-1)d). \\ \sigma_\infty &= d\text{-cycle.} \end{aligned}$$

Dessin d'enfant

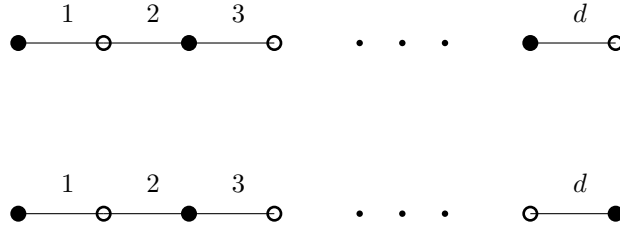


Figure 2.2: The Chebyshev dessins : odd d (top) and even d (bottom).

Consider the case where $X = \mathbb{P}^1$. We say f is a *dynamical Belyi map* if

$$f(\{0, 1, \infty\}) \subseteq \{0, 1, \infty\}.$$

A Belyi map f is *single-cycle* if there is a unique ramification point above each branch point. A single-cycle Belyi map is *normalized* if it has ramification points $\{0, 1, \infty\}$ and each ramification point is fixed.

Single-cycle Belyi maps correspond to generating systems in which each σ_i is a single cycle. Let $e_i = |\sigma_i|$. We describe each conjugacy class of generating systems of single-cycle Belyi maps with the triple (e_0, e_1, e_∞) . We can define the *combinatorial type* of a single-cycle Belyi map of degree d as the tuple

$$(d; e_0, e_1, e_\infty).$$

There is an isomorphism between single-cycle Belyi maps and combinatorial types. Conveniently, if a Belyi map \mathcal{B} has combinatorial type $(d; e_0, e_1, e_\infty)$, then for $\alpha \in \{0, 1, \infty\}$, we have that $e_\alpha = e_{\mathcal{B}}(\alpha)$.

In the case of single-cycle Belyi maps on \mathbb{P}^1 , letting $\iota \in \text{PGL}_2$ we can see that there will exist a normalized Belyi map in each isomorphism class of dynamical Belyi maps. In [1] the authors give a formula for every normalized dynamical Belyi polynomial.

Proposition 2.6.4. [1, Proposition 3.1] *If a normalized Belyi map $f(x)$ has combinatorial type $(d; d - k, k + 1, d)$ then it is given by*

$$f(x) = \frac{1}{k!} \prod_{j=0}^k (d - j) x^{d-k} \sum_{i=1}^k \frac{(-1)^i}{(d - k + i)} \binom{k}{i} x^i.$$

For the remainder of the paper we will use $\mathcal{B}_{d,k}(z)$ to represent the normalized Belyi map of combinatorial type $(d; d - k, k + 1, d)$, as written above. Notice that $d - k$ is the ramification index of 0. By Proposition 2.2.5 we see that $1 \leq d - k \leq d$. However, we can see in Example 2.2.4 that $e_f(\infty) = d$. Since there are 2 ramification points, then by the Reimann-Hurwitz formula, it must be that $2 \leq d - k \leq d - 1$, and therefore

$$d - 2 \geq k \geq 1. \tag{2.2}$$

We will show that we can use the above proposition to determine a unique representative for each conjugacy class of degree d polynomials in $K[z]$ with two critical points for any number field K .

2.7 Unicritical Polynomials

A unicritical polynomial is a polynomial $f \in \mathbb{C}[z]$ such that f has a single finite critical point. A quadratic polynomial is necessarily unicritical, but there are polynomials of any degree that are unicritical. Conjugating to move the critical point to 0 gives us the familiar form $f_c(z) = z^d + c$. The following classical result tells us that a unicritical polynomial of degree d is conjugate to $d - 1$ distinct polynomials of the form $z^d + c$.

Proposition 2.7.1. *Let $f_{d,c} = z^d + c$ with $c \neq 0$, $d \geq 2$. If $f_{d,c_1} \sim f_{d,c_2}$ then $c_2 = \zeta_i c_1$ where ζ_i is a $d - 1^{\text{th}}$ root of unity.*

Proof. A function $f_{d,c} = z^d + c$ has critical points at 0 and ∞ . Since $c \neq 0$, 0 is not fixed, however ∞ is a fixed critical point. Suppose that $f_{d,c_1}^\phi(z) = f_{d,c_2}(z)$ for some $\phi(z) = \frac{az+b}{cz+d} \in \text{PGL}_2(\mathbb{C})$ then ϕ must fix 0 and ∞ , hence $\phi(z) = az$.

$$\begin{aligned} f_{c_2} &= f_{c_1}^\phi(z) \\ &= a \left(\left(\frac{z}{a} \right)^d + c_1 \right) \\ &= \frac{z^d}{a^{d-1}} + c_1 a. \end{aligned}$$

Since f_{c_2} is monic, $a^{d-1} = 1$. Therefore a can be any $d - 1^{\text{th}}$ root of unity, so $c_2 = \zeta_i c_1$ where ζ_i is any $d - 1^{\text{th}}$ root of unity, as desired. \square

Oftentimes, we want to study the dynamics of maps over a field K , and it is convenient to find a standard form for a complete family of functions over K .

Definition 2.7.2. Let K be a number field with fixed algebraic closure \bar{K} . If $f(z) \in K[z]$ where $[K : \mathbb{Q}]$ is minimal, then we say that K is the *field of definition* of f .

In classical dynamics, we study functions defined over \mathbb{C} , so the family $z^d + c$ for $c \in \mathbb{C}$ provides a representative function in each conjugacy class, though the representative is not necessarily unique if d is odd. In arithmetic dynamics, the field of definition of f is important. If $f \in K[z]$ for a number field K , we would like a normal form that preserves field of definition and gives a representative for each conjugacy class. For $d > 2$, the $z^d + c$ form does not satisfy these criteria. Thus, studying the dynamics of unicritical polynomials over K does not reduce to the study of this family.

Example 2.7.3. Consider the polynomial $f(z) = \frac{4}{3}z^3 - 2z^2 + z - 1 \in \mathbb{Q}[z]$ with critical point $\frac{1}{2}$. Conjugating by $\phi(z) = \frac{2\sqrt{3}}{3}(z - \frac{1}{2})$ we have

$$g(z) = f^\phi(z) = z^3 - \frac{8\sqrt{3}}{9},$$

so $g(z) \in L[z]$, where $L = \mathbb{Q}(\sqrt{3})$.

For any unicritical polynomial $f(z) \in K[z]$ with critical point α we can see that $\alpha \in K$. Conjugating to move the critical point to 0 will not affect the field of definition, however Example 2.7.3 shows that conjugating f to a monic polynomial may affect the field of definition. If $f(z) \in K[z]$ is monic, then conjugating f to be in the form $z^d + c$ will maintain the field of definition. In the case that f is a degree 2 polynomial then conjugating to the monic binomial form preserves the field of definition, and by Proposition 6.0.3, f is conjugate to exactly one polynomial of the form $z^2 + c$. This family of functions has been studied extensively [8–11, 13, 17–19, 21, 22], and studying the dynamics of such functions led to the famous Mandelbrot set, \mathcal{M}_d defined below. For unicritical polynomials of the form $f_{c,d}(z) = z^d + c$, we define the following sets.

$$\mathcal{M}_d = \{c \in \mathbb{C} : \forall n \in \mathbb{N}, |f_{c,d}^n(0)| \leq M \text{ for some } M \in \mathbb{N}\}$$

$$\mathcal{A}_d = \{c \in \mathbb{C} : \#\mathcal{O}_{f_{c,d}}(0) \leq M \text{ for some } M \in \mathbb{N}\}.$$

Notice \mathcal{A}_d is precisely the set of c for which $f_{c,d} \in \mathbb{C}$ is post-critically finite. Since a finite orbit must be bounded, then $\mathcal{A}_d \subseteq \mathcal{M}_d$.

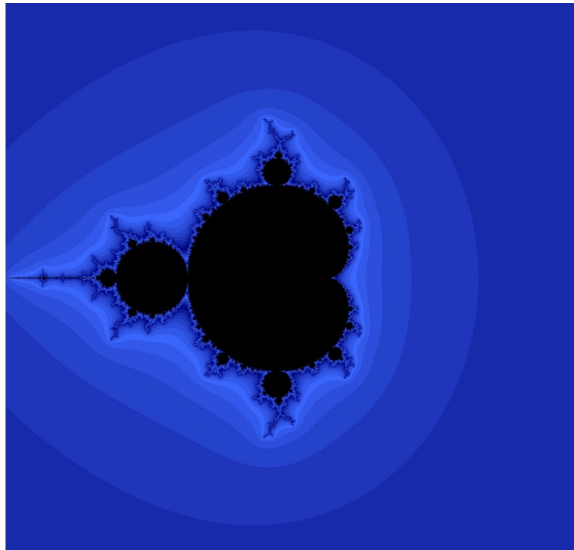


Figure 2.3: \mathcal{M}_2 (image created with FractalStream)

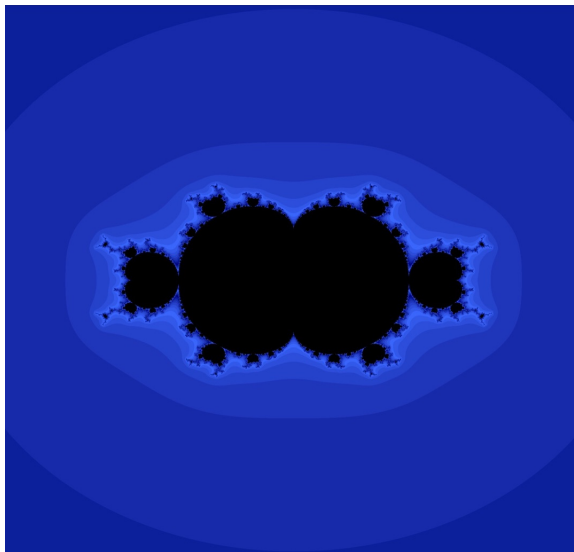


Figure 2.4: \mathcal{M}_3 (image created with FractalStream)

CHAPTER 3

DESSINS D'ENFANTS FOR SINGLE-CYCLE BELYI MAPS

This chapter written in collaboration with Michelle Manes and Gabrielle Melamed and will appear in [20].

Motivated by work in arithmetic dynamics, the authors of [1] study *normalized single-cycle dynamical Belyi maps*. The authors begin with a generating system, and they are able to give explicit formulas for two new infinite families of Belyi maps.

In this note, we give a simple description of the dessins for genus 0 single-cycle dynamical Belyi maps. As an application, we describe the dessins for the two infinite families of maps in [1], completing the triptych in these cases.

The following theorem classifies the dessins d'enfants for all genus 0 single-cycle Belyi maps.

Theorem 3.0.1. *Let $f : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ be a degree- d single-cycle Belyi map with generating system (e_0, e_1, e_∞) . Then f admits a planar dessin d'enfant with:*

- $d - e_1$ white vertices of degree one connected to a black vertex of degree e_0 ,
- $d - e_0$ black vertices of degree one connected to a white vertex of degree e_1 , and
- $e_0 + e_1 - d$ edges connecting the black vertex of degree e_0 and the white vertex of degree e_1 .

See Figure 3.1.

Proof. Let

$$\sigma_0 = (d - e_0 + 1, d - e_0 + 2, \dots, d) \text{ and } \sigma_1 = (1, 2, \dots, e_1).$$

It follows from Riemann-Hurwitz that $e_0 + e_1 + e_\infty = 2d + 1$, so

$$d + 1 \leq e_0 + e_1 \leq 2d - 1.$$

Therefore $\langle \sigma_0, \sigma_1 \rangle$ is transitive since $d - e_0 + 1 \leq e_1$.

The result then follows immediately from the recipe for producing dessins from generating systems described in Section 2.6. □

Recall that the *diameter* of a graph is the maximal number of vertices traversed in a path. The following result gives a restriction on the diameter of the dessins for all single-cycle Belyi maps (not only the genus 0 case).

Proposition 3.0.2. *All single-cycle Belyi maps admit a dessin d'enfant of diameter at most 4.*

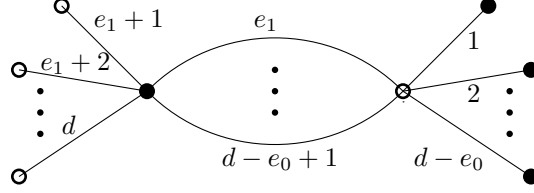


Figure 3.1: The degree d , genus 0 single-cycle dessin with combinatorial type (e_0, e_1, e_∞) .

Proof. Let $f : X \rightarrow \mathbb{P}^1$ be a single-cycle Belyi map. So there is a unique ramification point above 0 and a unique ramification point above 1. Hence there are exactly two vertices in the dessin with degree greater than 1. This implies that the longest path can only include those two vertices and two additional vertices, one black and one white. \square

3.1 New triptychs for single-cycle Belyi maps

Applying Theorem 3.0.1 to the Belyi maps in [1] allows us to describe the three-way correspondence for two new infinite families of Belyi maps.

Let $f : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ be a degree d single-cycle Belyi polynomial, so $(e_0, e_1, e_\infty) = (d - k, k + 1, d)$ for some $1 \leq k < d - 1$. We have the following correspondence:

Belyi map (from [1])

$$f : \mathbb{P}^1 \rightarrow \mathbb{P}^1$$

$$z \mapsto cx^{d-k}(a_0x^k + \dots + a_{k-1}x + a_k),$$

where

$$a_i := \frac{(-1)^{k-i}}{(d-i)} \binom{k}{i} \quad \text{and} \quad c = \frac{1}{k!} \prod_{j=0}^k (d-j).$$

Generating system

$$\sigma_0 = (d - k)\text{-cycle}, \quad \sigma_1 = (k + 1)\text{-cycle}, \quad \sigma_\infty = d\text{-cycle}.$$

Dessin d'enfant See Figure 3.2.

Example 3.1.1. The dessin for the polynomial $f(z) = z^3(6z^2 - 15z + 10)$, which has combinatorial type $(3, 3, 5)$, is shown in Figure 3.3.

We turn now to the second family of Belyi maps described in [1]. These have combinatorial type $(d - k, 2k + 1, d - k)$, meaning that the critical points above 0 and ∞ have the same ramifi-

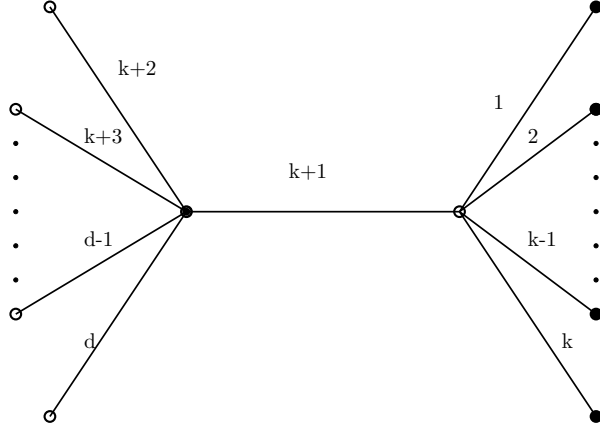


Figure 3.2: The dessin with combinatorial type $(d - k, k + 1, d)$.

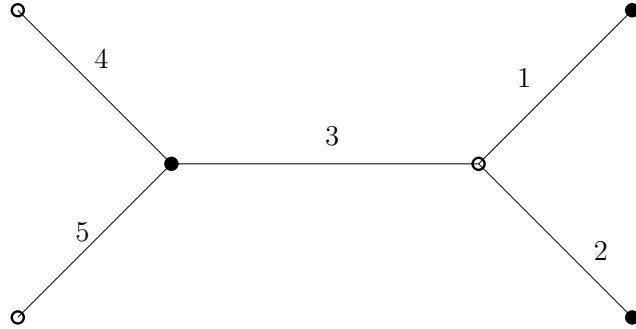


Figure 3.3: The dessin with combinatorial type $(3, 3, 5)$.

cation index. (Here d is the degree of the Belyi map and $1 \leq k < d - 1$.) We have the following correspondence:

Belyi map (from [1])

$$f : \mathbb{P}^1 \rightarrow \mathbb{P}^1$$

$$z \mapsto x^{d-k} \left(\frac{a_0 x^k - a_1 x^{k-1} + \dots + (-1)^k a_k}{(-1)^k a_k x^k + \dots - a_1 x + a_0} \right),$$

where

$$a_i := \binom{k}{i} \prod_{k+i+1 \leq j \leq 2k} (d-j) \prod_{0 \leq j \leq i-1} (d-j) = k! \binom{d}{i} \binom{d-k-i-1}{k-i}.$$

Generating system

$$\sigma_0 = (d - k)\text{-cycle}, \quad \sigma_1 = (2k + 1)\text{-cycle}, \quad \sigma_\infty = (d - k)\text{-cycle}.$$

Dessin d'enfant

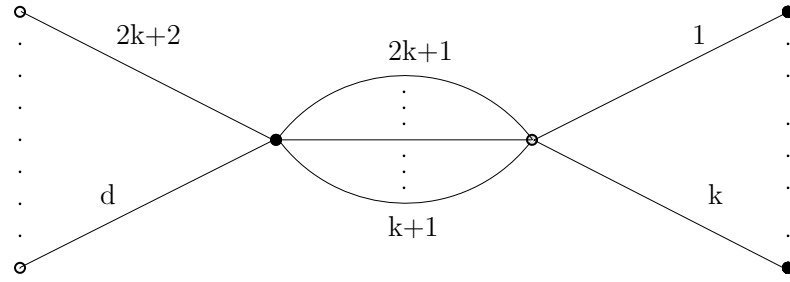


Figure 3.4: The dessin with combinatorial type $(d - k, 2k + 1, d - k)$.

Example 3.1.2. The dessin for the map

$$f(z) = z^8 \left(\frac{42z^2 - 120z + 90}{90z^2 - 120z + 42} \right),$$

which has combinatorial type $(8, 5, 8)$, is shown in Figure 3.5.

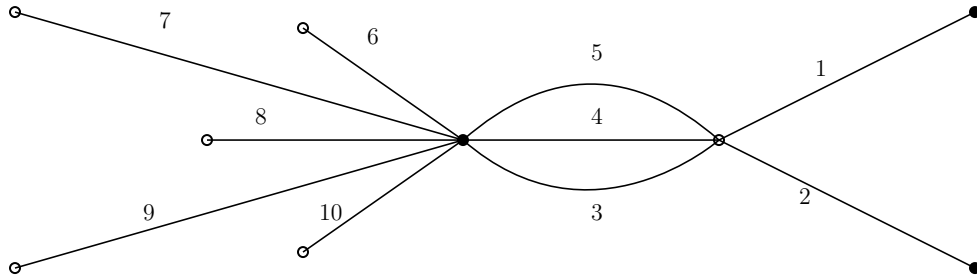


Figure 3.5: The dessin with combinatorial type $(8, 5, 8)$.

CHAPTER 4

A STANDARD FORM FOR BICRITICAL POLYNOMIALS

Definition 4.0.1. Let K be a field. A polynomial $f(z) \in K[z]$ is *bicritical* if there exist $\gamma_1 \neq \gamma_2 \in \bar{K}$ such that $f'(\alpha) = 0$ if and only if $\alpha \in \{\gamma_1, \gamma_2\}$.

We wish to describe all bicritical polynomials of a given degree, up to conjugation. In [1] the authors provide a normal form for single-cycle normalized dynamical Belyi polynomials. These are bicritical polynomials with critical points at 0 and 1, both of which are fixed. We can use this normal form to build a formula for all bicritical polynomials up to conjugacy. For the remainder of this chapter let K be a field of characteristic 0 and \bar{K} be a fixed algebraic closure of K .

Recall that $\mathcal{B}_{d,k}(z)$ is the normalized Belyi map with combinatorial type $(d; d-k, k+1, d)$. A formula for $\mathcal{B}_{d,k}$ is given in Proposition 2.6.4.

Proposition 4.0.2. *Let $g \in K[z]$ be a bicritical polynomial of degree $d \geq 3$. There exists a single-cycle normalized Belyi map $\mathcal{B}_{d,k}$ and an element $\phi \in \text{PGL}_2(\bar{K})$ such that $g^\phi = a\mathcal{B}_{d,k} + c$ for some $a, c \in \bar{K}$.*

Proof. Let $g \in K[z]$ with critical points $\gamma_1, \gamma_2 \in \bar{K}$. Let $k \in \mathbb{N}$ be such that $d-k$ is the ramification index of γ_1 and $k+1$ is the ramification index of γ_2 . Define $\phi(z) = \frac{z-\gamma_1}{\gamma_2-\gamma_1} \in \text{PGL}_2(\bar{K})$, which effectively moves the critical points to 0 and 1, respectively. If $f(z) = g^\phi(z)$, then $f'(z) = \alpha z^{d-k-1}(z-1)^k$, with $\alpha \in \bar{K}$ and $\alpha \neq 0$. Then

$$\begin{aligned}
 f(z) &= \int \alpha z^{d-k-1}(z-1)^k dz \\
 &= \alpha \int z^{d-k-1} \sum_{j=0}^k (-1)^{k-j} \binom{k}{j} z^j dz \\
 &= \alpha \sum_{j=0}^k (-1)^{k-j} \binom{k}{j} \int z^{d-k-1+j} dz \\
 &= \alpha \sum_{j=0}^k (-1)^{k-j} \binom{k}{j} \frac{z^{d-k+j}}{d+j-k} + c \\
 &= \alpha z^{d-k} \left(\sum_{j=0}^k (-1)^{k-j} \binom{k}{j} \frac{z^j}{d+j-k} \right) + c.
 \end{aligned}$$

Changing the limits of summation, we have the following:

$$f(z) = \alpha z^{d-k} \left(\sum_{i=0}^k (-1)^i \binom{k}{k-i} \frac{z^{k-i}}{d-i} \right) + c.$$

Letting $\alpha = a \cdot \frac{(-1)^k}{k!} \prod_{j=0}^k (d-j)$, we have

$$\begin{aligned} f(z) &= a \cdot \frac{(-1)^k}{k!} \prod_{j=0}^k (d-j) z^{d-k} \left(\sum_{i=0}^k (-1)^i \binom{k}{k-i} \frac{z^{k-i}}{d-i} \right) + c \\ f(z) &= a \cdot \frac{1}{k!} \prod_{j=0}^k (d-j) z^{d-k} \left(\sum_{i=0}^k (-1)^{k+i} \binom{k}{k-i} \frac{z^{k-i}}{d-i} \right) + c. \end{aligned} \quad (4.1)$$

Since $f' \in K[z]$ and $\alpha \in \bar{K}$, then $a, c \in \bar{K}$. □

From the proof of Proposition 4.0.2, we see that ϕ is defined over the field $K(\gamma_1, \gamma_2)$, where γ_1 and γ_2 are the critical points of g . So *a priori* ϕ and g^ϕ could be defined over a quadratic extension of K . However, while this may be true of ϕ , the following proposition shows that if g has even degree then the field of definition of g and g^ϕ is the same.

Proposition 4.0.3. *Let K be a number field and $g \in K[z]$ be a bicritical polynomial of even degree. Choose $\phi \in \text{PGL}_2(\bar{K})$ so that g^ϕ has critical points 0 and 1. Then $g^\phi \in K[z]$.*

Proof. Again, suppose $\gamma_1, \gamma_2 \in \bar{K}$ are the critical points of g . Let $k \in \mathbb{N}$ be such that $d-k$ is the ramification index of γ_1 and $k+1$ is the ramification index of γ_2 . Normalize by conjugating g by $\phi(z) = \frac{z-\gamma_1}{\gamma_2-\gamma_1} \in \text{PGL}_2(\bar{K})$. Since $g \in K[z]$, $g' \in K[z]$ also. Since g is bicritical, g' has exactly two roots in \bar{K} . Therefore, $\gamma_1 \in K$ if and only if $\gamma_2 \in K$, and if $\gamma_1, \gamma_2 \in K$ then both ϕ and $g^\phi \in K[z]$.

If $\gamma_1 \notin K$, $g'(z) = \alpha(f(z))^\beta$ where $f \in K[z]$ is an irreducible quadratic polynomial. This implies that $d-k = k+1$, hence $d = 2k+1$, a contradiction since d is even. So it must be that $\gamma_1, \gamma_2 \in K$ and thus $g^\phi \in K[z]$. □

The above proposition need not be true for odd degree polynomials, as shown in the following example.

Example 4.0.4. Consider $f(z) = 4z^3 + 3z^2 + 2z + 1 \in \mathbb{Q}[z]$. Then f has critical points γ_1, γ_2 where

$$\begin{aligned}\gamma_1 &= -\frac{1}{4} - \frac{\sqrt{-15}}{12} \\ \gamma_2 &= -\frac{1}{4} + \frac{\sqrt{-15}}{12}.\end{aligned}$$

Let $\phi(z) = \frac{z-\gamma_1}{\gamma_2-\gamma_1} \in \text{PGL}_2$. Then

$$f^\phi(z) = \frac{5}{6}(-2z^3 + 3z^2) + \frac{1}{12} - \frac{7\sqrt{-15}}{20}$$

has critical points at 0 and 1, however $f^\phi \notin \mathbb{Q}[z]$.

We wish to study the case when the bicritical polynomials are also PCF. In this case, we believe that the field of definition is preserved.

Conjecture 4.0.5. Let K be a number field and $g \in K[z]$ be a post-critically finite bicritical polynomial of odd degree. Choose $\phi \in \text{PGL}_2(\bar{K})$ so that g^ϕ has critical points 0 and 1. Then $g^\phi(z) \in K$.

We normalized by conjugating to set the critical points to 0 and 1, thus we consider $a\mathcal{B}_{d,k}(z) + c$ a normal form for a bicritical polynomial. Assuming Conjecture 4.0.5 is true, studying the dynamics of PCF bicritical polynomials over number fields thus reduces to studying the dynamics of this two parameter family. Notice that we can conjugate a bicritical polynomial in the normal form and maintain normal form simply by swapping the critical points.

Proposition 4.0.6. *Let $f_0 \neq f_1 \in K[z]$ with $f_0(z) = a_0\mathcal{B}_{d,k_0} + c_0$ and $f_1(z) = a_1\mathcal{B}_{d,k_1} + c_1$. The polynomials f_0 and f_1 are conjugate if and only if $k_0 + k_1 = d - 1$, $a_0 = a_1$, and $c_1 = 1 - a_0 - c_0$.*

Proof. First, suppose that $a_0 = a_1$, $c_1 = 1 - a_0 - c_0$, and $k_0 + k_1 = d - 1$. If $\phi(z) = 1 - z$, then $f_0^\phi(z) = 1 - f_0(1 - z)$. Notice that $f_0^\phi(z)$ will still have critical points 0 and 1, with ramification indices $k_0 + 1$ and $d - k_0$. Since $k_0 + k_1 = d - 1$, the ramification index of 0 under f_0^ϕ is $d - k_1$ and the ramification index of 1 is $k_1 + 1$. This implies that $f^\phi = a\mathcal{B}_{d,k_0} + c$ for some $a, c \in K$. Since

$$f_0^\phi(0) = 1 - f_0(1) = 1 - (a_0 + c_0) = c_1$$

and

$$f_0^\phi(1) = 1 - f_0(0) = 1 - c_0 = a_1 + c_1$$

then $f_0^\phi(z) = f_1(z)$.

Now, let $\phi \in \text{PGL}_2(\bar{K})$ be such that $f_0^\phi = f_1$. Since $\phi(\text{Crit}(f_0)) = \text{Crit}(f_1)$, then $\phi(\{0, 1\}) = \{0, 1\}$. If $\phi(z) = z$ then $f_0 = f_1$, so it must be that $\phi(z) = 1 - z$.

If $e_{f_0}(0)$ is the ramification index at 0 under f_0 , then $e_{f_0}(0) = e_{f_0^\phi}(\phi^{-1}(0)) = e_{f_1}(1)$. Since $e_{f_0}(0) = d - k_0$ and $e_{f_1}(1) = k_1 + 1$, then $d - k_0 = k_1 + 1$, hence $k_0 + k_1 = d - 1$, as desired. Furthermore, we know that for $i \in \{0, 1\}$, $f_i(0) = c_i$, and $f_i(1) = a_i + c_i$. Since $f_0^\phi = f_1$, then $f_0^\phi(0) = c_1$, but we also have

$$\begin{aligned} f_0^\phi(0) &= (\phi \circ f_0 \circ \phi^{-1})(0) \\ &= (\phi \circ f_0)(1) \\ &= \phi(a_0 + c_0) \\ &= 1 - a_0 - c_0. \end{aligned}$$

So $c_1 = 1 - a_0 - c_0$. Similarly, $f_0^\phi(1) = a_1 + c_1$, but we also have

$$\begin{aligned} f_0^\phi(1) &= (\phi \circ f_0 \circ \phi^{-1})(1) \\ &= (\phi \circ f_0)(0) \\ &= \phi(c_0) \\ &= 1 - c_0. \end{aligned}$$

So

$$1 - c_0 = a_1 + c_1 = a_1 + 1 - a_0 - c_0,$$

giving us $a_1 - a_0 = 0$, and thus $a_1 = a_0$. □

Recall that $1 \leq k \leq d - 2$. Proposition 4.0.6 tells us that for every degree d , a polynomial $a_0\mathcal{B}_{d,k_0} + c_0 \in K[z]$ is conjugate to $a_1\mathcal{B}_{d,k_1} + c_1 \in K[z]$ with $k_1 \leq \left\lceil \frac{d-2}{2} \right\rceil$.

Example 4.0.7. Consider degree 6 bicritical polynomials over K . By 4.0.6 for any $a, c \in K$, $a\mathcal{B}_{6,4} + c$ is conjugate to $a\mathcal{B}_{6,1} + 1 - a - c$, and $a\mathcal{B}_{6,3} + c$ is conjugate to $a\mathcal{B}_{6,2} + 1 - a - c$. Since

$$\mathcal{B}_{6,1}(z) = -5z^6 + 6z^5 \quad \text{and} \quad \mathcal{B}_{6,2}(z) = 10z^6 - 24z^5 + 15z^4,$$

each conjugacy class of bicritical polynomials in $K[z]$ can be represented by one of the following:

$$f_{6,1}(z) = a(-5z^6 + 6z^5) + c$$

$$f_{6,2}(z) = a(10z^6 - 24z^5 + 15z^4) + c$$

where $a, c \in K$.

Example 4.0.8. Consider cubic bicritical polynomials over K . Since $1 \leq k \leq d - 2$, then the only possibility is $k = 1$. Assuming conjecture 4.0.5, since

$$\mathcal{B}_{3,1}(z) = -2z^3 + 3z^2,$$

each conjugacy class of PCF bicritical cubic polynomials in $K[z]$ can be represented by

$$f(z) = a(-2z^3 + 3z^2) + c$$

where $a, c \in K$.

4.1 Cubic Polynomials

Cubic polynomials have been studied extensively in complex dynamics, e.g. [3–6, 24], and in arithmetic dynamics, e.g. [14]. All of these use the Branner-Hubbard normal form:

$$F(z) = z^3 + Az + B$$

with critical points $\pm\alpha$ where $\alpha = \sqrt{\frac{-A}{3}}$. This form may be preferred in complex dynamics, but is not ideal in arithmetic dynamics because it does not preserve the field of definition.

Given an arbitrary PCF cubic polynomial $f(z) \in K[z]$, conjugating to the Branner-Hubbard form may result in map defined over $L[z]$ where $[L : K] = 2$.

Proposition 4.1.1. *Let $f(z) = a\mathcal{B}_{3,2} + c \in K[z]$, and choose $\phi \in \text{PGL}_2(\bar{K})$ such that f^ϕ is in a Branner-Hubbard form. Then $f^\phi(z) \in L[z]$ where $[L : K] \leq 2$.*

Proof. Let $\phi(z) = \sqrt{-2a}z - \frac{\sqrt{-2a}}{2}$. Then

$$f^\phi(z) = z^3 + \frac{3}{2}az + \sqrt{-2a} \left(\frac{a}{2} + c - \frac{1}{2} \right).$$

So $f^\phi \in L[z]$ where $L = K(\sqrt{-2a})$. If $\sqrt{-2a} \in K$, then $L = K$; otherwise L/K is a quadratic

extension. □

Notice that conjugating by $\psi(z) = -\sqrt{-2a}z + \frac{\sqrt{-2a}}{2}$ will also admit a Branner-Hubbard form,

$$f^\psi(z) = z^3 + \frac{3}{2}az + \sqrt{-2a} \left(\frac{a}{2} + c - \frac{1}{2} \right).$$

However, the field of definition will remain the same.

Example 4.1.2. Consider the PCF polynomial $f \in \mathbb{Q}[z]$ given by $f(z) = -\frac{3}{2}(-2z^3 + 3z^2) + 1$, so $f(z) = a\mathcal{B}_{3,1} + c$ where $(a, c) = (-\frac{3}{2}, 1)$. Conjugating by

$$\phi(z) = \sqrt{-2a}z - \frac{\sqrt{-2a}}{2} = \sqrt{3}z - \frac{\sqrt{3}}{2},$$

we get

$$f^\phi(z) = z^3 - \frac{9}{4}z - \frac{\sqrt{3}}{4}.$$

Notice $f^\phi(z) \in L[z]$ where $L = \mathbb{Q}(\sqrt{3})$, so $[L : \mathbb{Q}] = 2$.

The above proposition implies that results that hold for monic cubic polynomials over K do not necessarily extend to all cubic polynomials over K , since conjugating to the monic form does not preserve field of definition. We will see that the Belyi form will provide a more complete picture of PCF cubic polynomials defined over a number field, allowing us to find many previously unknown cubic PCF polynomials defined over \mathbb{Q} . If Conjecture 4.0.5 holds even just for degree 3, we have found the complete list.

Using the Belyi form, we can look to extend known results for monic cubic polynomials. For degrees $d > 3$, there is not much known about the family of bicritical polynomials. Using the Belyi form, we can look for analogs of results for unicritical polynomials.

CHAPTER 5

POST-CRITICALLY FINITE BICRITICAL POLYNOMIALS IN $\mathbb{Q}[Z]$

We would like to use the Belyi normal form of bicritical polynomials to determine conjugacy classes of post-critically finite bicritical polynomials over \mathbb{Q} . We fix the following notation:

- K is a number field,
- $\nu \in M_K$ is a place of K ,
- $d \in \mathbb{Z}_{\geq 3}$.

5.1 A Helpful Lemma

In this section we will summarize a result of Ingram [14]. By Corollary 2 in [14], we know that for any number field K there are finitely many conjugacy classes of post-critically finite polynomial maps of degree d in $K[z]$.

A natural question is: how many bicritical PCF polynomials over K in each degree d ? The following proposition says there are always at least two for every degree d and every $1 \leq k \leq d - 2$.

Proposition 5.1.1. *For every degree $d \geq 3$ and value $k \leq d - 2$ there exists at least two non-conjugate PCF polynomials of the form*

$$a\mathcal{B}_{d,k}(z) + c$$

where $a \neq 0$. Namely,

$$\mathcal{B}_{d,k}(z) \text{ and } -\mathcal{B}_{d,k}(z) + 1.$$

Proof. Since $\mathcal{B}_{d,k}$ fixes the ramification points $\{0, 1\}$, it is PCF. Furthermore, that implies that if $f(z) = -\mathcal{B}_{d,k}(z) + 1$ then

$$f(1) = -1 + 1 = 0$$

and

$$f(0) = 1.$$

Thus 0 and 1 are in the same 2-cycle and $f(z)$ is PCF.

By Proposition 4.0.6 the above polynomials are not conjugate. □

Combining the result above with Proposition 4.0.6, we see that there are at least two non-conjugate PCF polynomials for each degree d and each $1 \leq k \leq \lceil \frac{d-2}{2} \rceil$. So there are at least $d-1$ non-conjugate PCF polynomials for degree d for odd d , and at least $d-2$ for even d . Following Ingram [14], we set the following notation:

$$(2d)_\nu = \begin{cases} 1 & \nu \text{ is non-archimedean} \\ 2d & \nu \text{ is archimedean} \end{cases}$$

and

$$C_{f,\nu} = (2d)_\nu \max_{0 \leq i < d} \left\{ 1, \left| \frac{a_i}{a_d} \right|_\nu^{\frac{1}{d-i}}, |a_d|_\nu^{-\frac{1}{d-1}} \right\},$$

where $f(z) = a_d z^d + a_{d-1} z^{d-1} + \dots + a_1 z + a_0$.

Recall from Section 2.5 the definition of local height of f at ν :

$$\hat{\lambda}_{f,\nu}(z) = \lim_{N \rightarrow \infty} d^{-N} \log \max \left\{ 1, |f^N(z)|_\nu \right\}.$$

Lemma 5.1.2. [14, Lemma 4] *Let*

$$f(z) = a_d z^d + a_{d-1} z^{d-1} + \dots + a_1 z + a_0 \in \mathbb{Q}[z]$$

and let $|\cdot|_\nu$ be an absolute value on \mathbb{Q} . If

$$|z|_\nu > C_{f,\nu},$$

then

$$\hat{\lambda}_{f,\nu} = \log |z|_\nu + \frac{1}{d-1} \log |a_d|_\nu + \epsilon(f, z, \nu)$$

where $\epsilon(f, z, \nu) = 0$ if ν is non-archimedean, and

$$-\log 2 \leq \epsilon(f, z, \nu) \leq \log \frac{3}{2}$$

otherwise.

This lemma will give an effective ν -adic bound for preperiodic points of a polynomial $f(z) \in \mathbb{Q}[z]$.

Corollary 5.1.3. *Let $f(z) \in \mathbb{Q}[z]$ be a polynomial of degree $d \geq 2$. For $\alpha \in \mathbb{Q}$, if there exists $\nu \in M_{\mathbb{Q}}$ and $n \in \mathbb{N}$ such that*

$$|f^n(\alpha)|_\nu > C_{f,\nu},$$

then α must be a wandering point for f .

Proof. First, notice that α is a wandering point if and only if $f^n(\alpha)$ is a wandering point for all $n \in \mathbb{N}$, so without loss of generality, assume $|\alpha|_\nu > C_{f,\nu}$ for some $\nu \in M_K$. By Lemma 5.1.2 we know that

$$\hat{\lambda}_{f,\nu}(\alpha) = \log|\alpha|_\nu + \frac{1}{d-1} \log|a_d|_\nu + \epsilon(f, \alpha, \nu).$$

Since $|\alpha|_\nu > C_{f,\nu}$ then

$$\begin{aligned} |\alpha|_\nu &> (2d)_\nu |a_d|_\nu^{-\frac{1}{d-1}} \\ \log|\alpha|_\nu &> \log(2d)_\nu - \frac{1}{d-1} \log|a_d|_\nu. \end{aligned}$$

This implies that

$$\hat{\lambda}_{f,\nu}(\alpha) > \log(2d)_\nu + \epsilon(f, \alpha, \nu).$$

If ν is non-archimedean then $\log(2d)_\nu + \epsilon(f, \alpha, \nu) = 0$, so $\hat{\lambda}_{f,\nu} > 0$. If ν is archimedean then

$$\log(2d)_\nu + \epsilon(f, \alpha, \nu) \geq \log 2d - \log 2 > 0.$$

Therefore, $\hat{\lambda}_{f,\nu}(\alpha) > 0$, so it must be that

$$\hat{h}_f(\alpha) = \frac{1}{[K:\mathbb{Q}]} \sum_{\sigma \in \text{Gal}(K/\mathbb{Q})} \sum_{\nu \in M_{\mathbb{Q}}} \hat{\lambda}_{\sigma(f),\nu}(\sigma(\alpha)) > 0$$

and hence α is a wandering point, as desired. \square

We now specialize the above result to bicritical polynomials.

Corollary 5.1.4. *Let $f(z) = a\mathcal{B}_{d,k} + c \in \mathbb{Q}[z]$ be a bicritical polynomial and let $\alpha \in \mathbb{Q}$. If there exist $\nu \in M_{\mathbb{Q}}$ and $n \in \mathbb{N}$ such that*

$$|f^n(\alpha)|_\nu > C_{f,\nu} = (2d)_\nu \max_{1 \leq i \leq k} \left\{ 1, \left| \frac{\binom{k}{i} d}{d-i} \right|_\nu^{1/i}, \left| \frac{k!}{a \prod_{j=1}^k (d-j)} \right|_\nu^{\frac{1}{d-1}}, \left| \frac{c \cdot k!}{a \prod_{j=1}^k (d-j)} \right|_\nu^{1/d} \right\},$$

then α is a wandering point for f .

Proof. Let $f(z) = a\mathcal{B}_{d,k} + c$ and recall from 4.1

$$\begin{aligned} f(z) &= a \cdot \frac{1}{k!} \prod_{j=0}^k (d-j) z^{d-k} \left(\sum_{i=0}^k (-1)^{k+i} \binom{k}{k-i} \frac{z^{k-i}}{(d-i)} \right) + c \\ &= a \sum_{i=0}^k \frac{(-1)^{k+i} \prod_{j=0}^k (d-j)}{(d-i)k!} \binom{k}{k-i} z^{d-i} + c. \end{aligned}$$

Writing $f(z) = \sum_{\ell=0}^d a_{\ell} z^{\ell}$, we have

$$\begin{aligned} a_0 &= c, \quad a_d = \frac{(-1)^k a \prod_{j=1}^k (d-j)}{k!}, \\ a_{\ell} &= \begin{cases} 0 & 1 \leq \ell \leq d-k-1 \\ \frac{a(-1)^{\ell+k} \binom{k}{d-\ell} \prod_{j=0}^k (d-j)}{\ell \cdot k!} & d-k \leq \ell \leq d-1. \end{cases} \end{aligned}$$

So for $d-k \leq \ell \leq d-1$,

$$\begin{aligned} \left| \frac{a_{\ell}}{a_d} \right|_{\nu}^{\frac{1}{d-\ell}} &= \left| \frac{a \cdot \binom{k}{d-\ell} \prod_{j=0}^k (d-j)}{\ell \cdot k!} \cdot \frac{k!}{a \prod_{j=1}^k (d-j)} \right|_{\nu}^{\frac{1}{d-\ell}} \\ &= \left| \frac{\binom{k}{d-\ell} d}{\ell} \right|_{\nu}^{\frac{1}{d-\ell}}. \end{aligned}$$

Furthermore,

$$\begin{aligned} \left| \frac{a_0}{a_d} \right|_{\nu}^{1/d} &= \left| \frac{c \cdot k!}{a \prod_{j=1}^k (d-j)} \right|_{\nu}^{1/d} \\ |a_d|_{\nu}^{-\frac{1}{d-1}} &= \left| \frac{k!}{a \prod_{j=1}^k (d-j)} \right|_{\nu}^{\frac{1}{d-1}}. \end{aligned}$$

Thus,

$$C_{f,\nu} = (2d)_\nu \max_{d-k \leq \ell \leq d-1} \left\{ 1, \left| \frac{\binom{k}{d-\ell} d}{\ell} \right|_\nu^{\frac{1}{d-\ell}}, \left| \frac{c \cdot k!}{a \prod_{j=1}^k (d-j)} \right|_\nu^{\frac{1}{d}}, \left| \frac{k!}{a \prod_{j=1}^k (d-j)} \right|_\nu^{\frac{1}{d-1}} \right\}.$$

Substituting $i = d - \ell$ we obtain

$$C_{f,\nu} = (2d)_\nu \max_{1 \leq i \leq k} \left\{ 1, \left| \frac{\binom{k}{i} d}{d-i} \right|_\nu^{1/i}, \left| \frac{k!}{a \prod_{j=1}^k (d-j)} \right|_\nu^{\frac{1}{d-1}}, \left| \frac{c \cdot k!}{a \prod_{j=1}^k (d-j)} \right|_\nu^{\frac{1}{d}} \right\} \quad (5.1)$$

as desired. By Corollary 5.1.3 we have that if $|\alpha|_\nu > C_{f,\nu}$ then α must be a wandering point. \square

5.2 Bounds on Post-Critically Finite Polynomials in $\mathbb{Q}[z]$

Using the bound $C_{f,\nu}$ given above, we can find bounds on the absolute values of the parameters a and c of a PCF polynomial of the form $f(z) = a\mathcal{B}_{d,k} + c$.

Remark 5.2.1. Since $f(z) = a\mathcal{B}_{d,k} + c \in \mathbb{Q}[z]$ has critical points $\{0, 1\}$ then by Corollary 5.1.4 we know that if f is PCF then we need every element in the orbits of 0 and 1 to be bounded by $C_{f,\nu}$. In particular, it is necessary that $|f(1)| = |a + c|_\nu \leq C_{f,\nu}$ and $|f(0)| = |c|_\nu \leq C_{f,\nu}$. Thus $\max\{|a|_\nu, |a + c|_\nu\} \leq C_{f,\nu}$ for all $\nu \in M_{\mathbb{Q}}$. At the non-archimedean places, $|a + c|_\nu \leq \max\{|a|_\nu, |c|_\nu\}$ with inequality only if $|a|_\nu = |c|_\nu$, so

$$\max\{|a|_\nu, |a + c|_\nu\} = \max\{|a|_\nu, \max\{|a|_\nu, |c|_\nu\}\} = \max\{|a|_\nu, |c|_\nu\}.$$

If f is PCF, then for every non-archimedean place ν ,

$$\max\{|a|_\nu, |c|_\nu\} \leq C_{f,\nu}.$$

Lemma 5.2.2. *If $f(z) = a\mathcal{B}_{d,k}(z) + c$ is PCF then for non-archimedean $\nu \in M_{\mathbb{Q}}$*

$$C_{f,\nu} = \max_{1 \leq i \leq k} \left\{ 1, \left| \frac{\binom{k}{i} d}{d-i} \right|_\nu^{1/i}, \left| \frac{k!}{a \prod_{j=1}^k (d-j)} \right|_\nu^{\frac{1}{d-1}} \right\}.$$

Proof. Let $f(z) = a\mathcal{B}_{d,k}(z) + c \in \mathbb{Q}[z]$ and $\nu \in M_{\mathbb{Q}}$ be non-archimedean. From (5.1),

$$C_{f,\nu} = \max_{1 \leq i \leq k} \left\{ 1, \left| \frac{\binom{k}{i} d}{d-i} \right|_{\nu}^{1/i}, \left| \frac{k!}{a \prod_{j=1}^k (d-j)} \right|_{\nu}^{\frac{1}{d-1}}, \left| \frac{c \cdot k!}{a \prod_{j=1}^k (d-j)} \right|_{\nu}^{1/d} \right\}.$$

Suppose $C_{f,\nu} = \left| \frac{c \cdot k!}{a \prod_{j=1}^k (d-j)} \right|_{\nu}^{1/d}$ and

$$\left| \frac{c \cdot k!}{a \prod_{j=1}^k (d-j)} \right|_{\nu}^{1/d} > \left| \frac{k!}{a \prod_{j=1}^k (d-j)} \right|_{\nu}^{\frac{1}{d-1}}.$$

This implies the following:

$$\begin{aligned} \left| \frac{c \cdot k!}{a \prod_{j=1}^k (d-j)} \right|_{\nu}^{d-1} &> \left| \frac{k!}{a \prod_{j=1}^k (d-j)} \right|_{\nu}^d, \\ |c|_{\nu}^{d-1} &> \left| \frac{k!}{a \prod_{j=1}^k (d-j)} \right|_{\nu}. \end{aligned}$$

However, since f is PCF, then by Remark 5.2.1

$$|c|_{\nu} \leq C_{f,\nu} = \left| \frac{c \cdot k!}{a \prod_{j=1}^k (d-j)} \right|_{\nu}^{1/d}.$$

Therefore,

$$|c|_\nu^d \leq \left| \frac{c \cdot k!}{a \prod_{j=1}^k (d-j)} \right|_\nu$$

$$|c|_\nu^{d-1} \leq \left| \frac{k!}{a \prod_{j=1}^k (d-j)} \right|_\nu,$$

giving us a contradiction. We conclude that if f is PCF then

$$C_{f,\nu} = \max_{1 \leq i \leq k} \left\{ 1, \left| \frac{\binom{k}{i} d}{d-i} \right|_\nu^{1/i}, \left| \frac{k!}{a \prod_{j=1}^k (d-j)} \right|_\nu^{\frac{1}{d-1}} \right\}$$

as desired. \square

The following lemmas will give us p -adic bounds on the parameters a, c for post-critically finite bicritical polynomials over \mathbb{Q} .

Lemma 5.2.3. *Let $f(z) = a\mathcal{B}_{d,k}(z) + c \in \mathbb{Q}[z]$ be post-critically finite, p be a prime, and let $|\cdot|_p$ be the p -adic absolute value. If $p \nmid \prod_{j=1}^k (d-j)$ then $|a|_p \leq 1$ and $|c|_p^{d-1} \leq |a|_p^{-1}$.*

Proof. Suppose that $f(z) = a\mathcal{B}_{d,k}(z) + c \in \mathbb{Q}[z]$ is PCF and $p \nmid \prod_{j=1}^k (d-j)$. Notice that if $p \mid k!$ then $p \leq k$ so p would appear as a factor in one of every k integers, thus $p \mid \prod_{j=1}^k (d-j)$. Therefore, if $p \nmid \prod_{j=1}^k (d-j)$ then $p \nmid k!$. Since $\binom{k}{i} \mid k!$, then $p \nmid \binom{k}{i}$. From Lemma 5.2.2,

$$C_{f,p} = \max_{1 \leq i \leq k} \left\{ 1, \left| \frac{d \cdot \binom{k}{i}}{d-i} \right|_p^{1/i}, \left| \frac{k!}{a \prod_{j=1}^k (d-j)} \right|_p^{\frac{1}{d-1}} \right\}$$

which simplifies to

$$C_{f,p} = \max_{1 \leq i \leq k} \left\{ 1, |d|_p^{1/i}, |a|_p^{-\frac{1}{d-1}} \right\}.$$

Since $|d|_p \leq 1$, then we can further simplify $C_{f,p}$ to

$$C_{f,p} = \max \left\{ 1, |a|_p^{-\frac{1}{d-1}} \right\}. \quad (5.2)$$

There are two distinct cases:

- (1) $C_{f,p} = 1$, or
- (2) $C_{f,p} = |a|_p^{-\frac{1}{d-1}} > 1$.

First, consider the case where $C_{f,p} = 1$. This implies that

$$1 \geq |a|_p^{-\frac{1}{d-1}},$$

so $|a|_p \geq 1$. However, since f is PCF,

$$|a|_p, |c|_p \leq C_{f,p} = 1.$$

Therefore $|a|_p = 1$, $|a|_p^{-1} = 1$, and $|c|_p^{d-1} \leq 1 = |a|_p^{-1}$.

Now, let us consider the case where $C_{f,p} = |a|_p^{-\frac{1}{d-1}} > 1$. This implies that $|a|_p < 1$, as desired. Furthermore, since f is PCF,

$$|a|_p, |c|_p \leq C_{f,p} = |a|_p^{-\frac{1}{d-1}}.$$

Therefore, $|c|_p^{d-1} \leq |a|_p^{-1}$. □

Lemma 5.2.4. *Let $d \geq 3$ and let $f(z) = a\mathcal{B}_{d,1} + c \in \mathbb{Q}[z]$ be a PCF bicritical polynomial. Then for $p \mid (d-1)$*

$$|(d-1)a|_p \leq 1$$

and

$$|(d-1)c|_p \leq 1.$$

In particular, $(d-1)a \in \mathbb{Z}$.

Proof. Let f be as stated above. Using Lemma 5.2.2 and the fact that $k = 1$ we have

$$C_{f,p} = \max \left\{ 1, \left| \frac{d}{d-1} \right|_p, \left| \frac{1}{a(d-1)} \right|_p^{\frac{1}{d-1}} \right\}. \quad (5.3)$$

If $p \mid (d-1)$, then $p \nmid d$, and thus

$$\left| \frac{d}{d-1} \right|_p = \left| \frac{1}{d-1} \right|_p > 1.$$

So we can simplify

$$C_{f,p} = \max \left\{ |d-1|_p^{-1}, |a(d-1)|_p^{-\frac{1}{d-1}} \right\}.$$

We have two distinct cases:

$$(1) C_{f,p} = |d-1|_p^{-1} \text{ and,}$$

$$(2) C_{f,p} = |a(d-1)|_p^{-\frac{1}{d-1}} > |d-1|_p^{-1}.$$

Consider case (1): Since f is PCF, both $|a|_p, |c|_p \leq C_{f,p} = |d-1|_p^{-1}$. Therefore, both $|a(d-1)|_p$ and $|c(d-1)|_p \leq 1$ as desired.

Consider case (2): $|a(d-1)|_p^{-\frac{1}{d-1}} > |d-1|_p^{-1}$, so

$$\begin{aligned} |a(d-1)|_p^{-1} &> |d-1|_p^{-(d-1)}, \\ |a(d-1)|_p &< |d-1|_p^{d-1}, \\ |a(d-1)|_p &< 1. \end{aligned} \tag{5.4}$$

$$(5.5)$$

We have shown that for both cases, $|a(d-1)|_p \leq 1$, so this must be true for all primes p .

Now, we will use the archimedean absolute value and prove that $|a| < 4$. Suppose $|a| \geq 4, |z_1| \geq |c|$, and $|z_1| \geq 2$. Then

$$\begin{aligned} |f(z_1)| &= \left| az_1^{d-1} (-(d-1)z_1 + d) + c \right| \\ &\geq |a||z_1|^{d-1} |-(d-1)z_1 + d| - |c| \\ &\geq 4 \cdot 2^{d-2} |z_1| |-(d-1)z_1 + d| - |c| \\ &\geq 2^d |z_1| (2(d-1) - d) - |c| \\ &\geq 2^d (d-2) |z_1| - |c| \\ &\geq 2^d |z_1| - |c| \\ &\geq (2^d - 1) |z_1| \\ &> |z_1|. \end{aligned}$$

If $|c| \geq 2$, then this implies that 0 must be a wandering point. If $|c| < 2$, then

$$|a + c| \geq |a| - |c| > 2,$$

so 1 must be a wandering point. Thus, it must be that $|a| < 4$.

Recall that by Lemma 5.2.3 $|a|_p \leq 1$ for all $p \nmid (d-1)$. Since $|a| < 4$, $|(d-1)a|_p \leq 1$ for $p \mid (d-1)$, then

$$a \in \left\{ \frac{n}{d-1} : 1 \leq |n| < 4(d-1) \right\}. \quad (5.6)$$

This implies that $|a|_p \geq \frac{|d-1|_p^{-1}}{4(d-1)-1}$, so then by (5.4),

$$\frac{|d-1|_p^{-1}}{4(d-1)-1} \leq |a|_p < |d-1|_p^{d-2}.$$

Therefore,

$$\begin{aligned} \frac{|d-1|_p^{-1}}{4d-5} &< |d-1|_p^{d-2} \\ \frac{1}{4d-5} &< |d-1|_p^{d-1} \\ 4d-5 &> |d-1|_p^{-(d-1)}. \end{aligned}$$

Since $p \mid (d-1)$ then $|d-1|_p^{-1} > 2$, so then

$$4d-5 > 2^{d-1},$$

which is false if $d \geq 5$, giving a contradiction.

Consider $d = 4$ and $p = 3$. Using (5.6) we can see that $|a|_3 \geq \frac{1}{3}$. So

$$\frac{1}{3} \leq |a|_3 < |d-1|_3^{d-2} = \frac{1}{3^2},$$

giving a contradiction.

Finally, $d = 3$ and $p = 2$. Using (5.6) we can see that $|a|_2 \geq \frac{1}{2}$. So

$$\frac{1}{2} \leq |a|_2 < |d-1|_2^{d-2} = \frac{1}{2},$$

giving a contradiction. Therefore, if f is PCF then $C_{f,p} \neq |a(d-1)|^{-\frac{1}{d-1}} > |d-1|_p^{-1}$. We conclude

that for $p \mid (d-1)$,

$$|(d-1)a|_p \leq 1,$$

and

$$|(d-1)c|_p \leq 1,$$

as desired. By Lemma 5.2.3, $|a|_p \leq 1$ for $p \nmid (d-1)$, thus $a(d-1) \in \mathbb{Z}$. □

5.3 Post-Critically Finite Cubic Polynomials in $\mathbb{Q}[z]$

Note that if $\deg(f) = 3$, then $\deg(f') = 2$. So f is either unicritical or bicritical. Assuming Conjecture 4.0.5, we can use the results above to find all cubic bicritical PCF polynomials (up to conjugacy). In [14], Ingram used a similar strategy to find all monic PCF cubic polynomials over \mathbb{Q} , up to conjugacy. Our result is stronger since the bicritical form preserves the field of definition but the monic form does not (see Section 4.1).

Theorem 5.3.1. *Assuming Conjecture 4.0.5, if $f(z) \in \mathbb{Q}[z]$ is a cubic bicritical post-critically finite polynomial, then $f(z)$ is conjugate to $f_{a,c}(z) = a(-2z^3 + 3z^2) + c$ where*

$$(a, c) \in \left\{ (1, 0), \left(\pm 1, \frac{1}{2} \right), \left(\frac{1}{2}, \pm 1 \right), \left(2, -\frac{1}{2} \right), \left(\frac{3}{2}, 0 \right), (-1, 1), \left(-2, \frac{3}{2} \right), \left(-\frac{3}{2}, 1 \right), \left(-\frac{1}{2}, 0 \right) \right\}.$$

Proof. Assuming Conjecture 4.0.5, by Proposition 4.0.2 we have that any cubic bicritical polynomials over \mathbb{Q} is conjugate to $f(z) = a(-2z^3 + 3z^2) + c$ for some $a, c \in \mathbb{Q}$. By Lemma 5.2.4 $2c \in \mathbb{Z}$ and by 5.2.3 for $p \neq 2$,

$$|c|_p \leq |a|_p^{1/2}.$$

Furthermore, by (5.6) we know that

$$a \in \left\{ \pm \frac{1}{2}, \pm 1, \pm \frac{3}{2}, \pm 2, \pm \frac{5}{2}, \pm 3, \pm \frac{7}{2} \right\}.$$

So for $p \notin \{2, 3, 5, 7\}$, $|a|_p = 1$, thus $|c|_p \leq 1$. For $p \in \{3, 5, 7\}$, $|a| \geq \frac{1}{p}$, so $|a|_p^{-1} \leq p$, so $|c|_p \leq \sqrt{p} < p$. Hence $|c|_p \leq 1$. We conclude that $|2c|_p \leq 1$ for all primes p , so $2c \in \mathbb{Z}$.

We will show that $|c| < \frac{5}{2}$. Suppose that a is contained in the above list and $|\alpha| \geq |c| \geq \frac{5}{2}$.

$$\begin{aligned}
|f(\alpha)| &= \left| a\alpha^2(-2\alpha + 3) + c \right| \\
&\geq |a||\alpha|^2|-2\alpha + 3| - |c| \\
&\geq \frac{1}{2}|\alpha|^2(2|\alpha| - 3) - |\alpha| \\
&\geq \frac{5}{4}|\alpha| \cdot 2 - |\alpha| = \frac{3}{2}|\alpha| \\
&> |\alpha|.
\end{aligned}$$

Hence in such a case, α is a wandering point for f . Since $|\alpha| \geq |c|$ then $|c|$ must be a wandering point for f , in which case f would not be PCF. Therefore, if f is PCF, $|c| < \frac{5}{2}$, so

$$c \in \left\{ 0, \pm 1, \pm \frac{1}{2}, \pm \frac{3}{2}, \pm 2, \pm \frac{5}{2} \right\}.$$

This gives 126 possibilities for (a, c) for which $f(z) = a(-2z^3 + 3z^2) + c$ is post-critically finite. The author used Sage [28] to test all possible pairs (a, c) (see Appendix A for Sage code). \square

5.4 PCF Bicritical Polynomials over \mathbb{Q} of High Degree

Let

$$\mathcal{A}_{d,k} = \left\{ (a, c) \in \mathbb{Q}^2 : f(z) = a\mathcal{B}_{d,k}(z) + c \text{ is PCF} \right\}.$$

Ideally, we wish to completely determine $\mathcal{A}_{d,k}$ for $d \geq 4$ and $1 \leq k \leq d - 2$. By Proposition 4.0.6, we know that every bicritical polynomial of the form $a_0\mathcal{B}_{d,k} + c_0 \in K[z]$ is conjugate to $a_0\mathcal{B}_{d,d-1-k} + (1 - a_0 - c_0)$. Therefore, our task reduces to finding $\mathcal{A}_{d,k}$ for $d \geq 4$, $1 \leq k \leq \left\lfloor \frac{d-2}{2} \right\rfloor$. First, we will strengthen our archimedean restrictions on a .

Proposition 5.4.1. *Let $d \geq 4$. If $f(z) = a\mathcal{B}_{d,k}(z) + c \in \mathbb{Q}[z]$ is PCF then $|a| < 3$.*

Proof. Suppose that $|a| \geq 3$. If $|\alpha| \geq |c|$ and $|\alpha| \geq \frac{3}{2}$, then

$$\begin{aligned}
|f(\alpha)| &= a\mathcal{B}_{d,k}(\alpha) + c \\
&= \left| a\alpha^{(d-1)} (-(d-1)\alpha + d) - c \right| \\
&\geq |a||\alpha|^{d-1} |-(d-1)\alpha + d| - |c| \\
&\geq 3 \cdot \left(\frac{3}{2}\right)^{d-2} |\alpha| |-(d-1)\alpha + d| - |\alpha| \\
&\geq 3 \cdot \left(\frac{3}{2}\right)^{d-2} |\alpha| \cdot \frac{1}{2} - |\alpha| \\
&\geq \left(\frac{3}{2}\right)^{d-1} |\alpha| - |\alpha| \\
&\geq \frac{19}{8} |\alpha| \\
&> |\alpha|.
\end{aligned}$$

Therefore, such an α must be a wandering point. This implies that if $|c| \geq \frac{3}{2}$, then c , and hence 0 , must be a wandering point. If $|c| < \frac{3}{2}$, then $|a + c| \geq \frac{3}{2}$, so $a + c$, and hence 1 , must be a wandering point. Therefore, if $f(z)$ is PCF it must be that $|a| < 3$. \square

Using Proposition 5.4.1 and Lemma 5.2.4 we now have that

$$a \in \left\{ \frac{n}{d-1} : 1 \leq |n| \leq 3(d-1) - 1 \right\}. \quad (5.7)$$

Now, we will see that if $f(z) \in \mathbb{Q}[z]$ is PCF and $f(z) = az^{d-1}(-(d-1)z + d) + c$, then any rational preperiodic point for f must be sufficiently small in the archimedean place.

Proposition 5.4.2. *Let $d \geq 4$ and $f(z) = az^{d-1}(-(d-1)z + d) + c \in \mathbb{Q}[z]$ be PCF. If $\alpha \in \text{PrePer}(f)$ then $|\alpha| < 2$. In particular, $|c| < 2$.*

Proof. We know from (5.7) that $\frac{1}{d-1} \leq |a| < 3$. Suppose that $|\alpha| \geq 2$ and $|\alpha| \geq |c|$. Then

$$\begin{aligned}
|f(\alpha)| &= \left| a\alpha^{d-1}(-(d-1)\alpha + d) + c \right| \\
&\geq |a||\alpha|^{d-1}|-(d-1)\alpha + d| - |c| \\
&\geq \frac{2^{d-2}}{d-1}|\alpha| \cdot |-(d-1)\alpha + d| - |\alpha| \\
&\geq \frac{4}{d-1}|\alpha| \cdot |-(d-1)\alpha + d| - |\alpha| \\
&\geq \frac{4(d-2)}{(d-1)}|\alpha| - |\alpha| \\
&\geq \frac{8}{3}|\alpha| - |\alpha| \\
&> |\alpha|.
\end{aligned}$$

Therefore, $|f(\alpha)| > 2$ and $|f(\alpha)| > |c|$. By induction $|f^n(\alpha)| > |f^{n-1}(\alpha)|$; hence α is a wandering point. Since f is PCF, and $f(0) = c$, then $|c| < 2$. \square

We turn now to determining the sets $\mathcal{A}_{d,k}$. First, consider the case $d = 4$. In this case, every conjugacy class of PCF bicritical polynomials has a representative in $\mathcal{A}_{4,1}$.

Theorem 5.4.3. *If $f(z) \in \mathbb{Q}[z]$ is a quartic bicritical post-critically finite polynomial, then $f(z)$ is conjugate to $f_{a,c}(z) = a(-3z^4 + 4z^3) + c$ for*

$$(a, c) \in \mathcal{A}_{4,1},$$

where

$$\mathcal{A}_{4,1} = \left\{ (1, 0), (-1, 1), \left(\frac{1}{3}, 1\right), \left(\frac{4}{3}, 1\right), \left(-\frac{1}{3}, \frac{4}{3}\right), \left(-\frac{4}{3}, \frac{4}{3}\right) \right\}.$$

Proof. Let $f(z) \in \mathbb{Q}[z]$ be a quartic bicritical post-critically finite polynomial. By Propositions 4.0.3 and 4.0.6, $f(z)$ is conjugate to a polynomial of the form $a\mathcal{B}_{4,1} + c$. Therefore, every such polynomial is conjugate to

$$f(z) = a(-3z^4 + 4z^3) + c$$

with $(a, c) \in \mathcal{A}_{4,1}$. By Lemma 5.2.4, $|3a|_3 \leq 1$ and $|3c|_3 \leq 1$. Furthermore, by Lemma 5.2.3, for $p \neq 3$, $|a|_p \leq 1$, and $|c|_p \leq |a|_p^{-\frac{1}{d-1}}$. By (5.7) we can see that $|a|_p^{-1} \leq 8$, so

$$|c|_p \leq 8^{\frac{1}{3}} \leq 2.$$

This implies that $|c|_p \leq 1$ for $p \neq 2, 3$, $|c|_3 \leq 3$, and $|c|_2 \leq 2$. Therefore,

$$c \in \left\{ \frac{n}{6} : 0 \leq |n| \leq 11 \right\}.$$

Recall from (5.4.1) that we know

$$a \in \left\{ \frac{n}{3} : 1 \leq |n| \leq 8 \right\}.$$

This gives 368 possible pairs (a, c) that may be contained in $\mathcal{A}_{4,1}$. The author used Sage [28] to test all pairs (see Appendix A for Sage code). \square

For higher degree polynomials, we need to compute $\mathcal{A}_{d,k}$ for values of k up to $\left\lceil \frac{d-2}{2} \right\rceil$. For now, we will focus on the case $k = 1$. For sufficiently high degrees, we can determine $\mathcal{A}_{d,1}$ by determining all rational preperiodic points for

$$f(z) = az^{d-1}(-(d-1)z + d) + c \in \mathbb{Q}.$$

Since $f(0) = c$ and $f(1) = a + c$, then $\{c, a + c\}$ must be contained in the set of preperiodic points, and thus we can determine potential pairs (a, c) that may be contained in $\mathcal{A}_{d,1}$. In [1] the authors determine that for a dynamical Belyi map

$$f(z) = -(d-1)z^d + dz^{d-1}$$

for d satisfying particular divisibility properties, $\text{PrePer}(f, \mathbb{Q}) = \{0, 1, \frac{d}{d-1}\}$. Their methods involved reducing f modulo primes of good reduction and comparing possible periods mod p . Their technique did not allow them to prove this result, for example, for $d = 35$. Using the techniques above, we are able to extend this result to all bicritical polynomials over \mathbb{Q} for degrees $d \geq 11$.

Theorem 5.4.4. *Let $f(z) = az^{d-1}(-(d-1)z + d) + c \in \mathbb{Q}[z]$ be post-critically finite and $d \geq 11$.*

Then

$$\left\{ 0, 1, \frac{d}{d-1} \right\} \subseteq \text{PrePer}(f, \mathbb{Q}) \subseteq \left\{ -1, 0, 1, \frac{d}{d-1} \right\}$$

Proof. Let $f(z) = az^{d-1}(-(d-1)z + d) + c \in \mathbb{Q}[z]$ and let α be a preperiodic point of f . By

Corollary 5.1.4, $|\alpha|_\nu \leq C_{f,\nu}$ for all $\nu \in M_{\mathbb{Q}}$. Consider the non-archimedean places.

$$\begin{aligned} C_{f,p} &= \max \left\{ 1, \left| \frac{d}{d-1} \right|_p, |a(d-1)|_p^{-\frac{1}{d-1}} \right\} \\ &= \max \left\{ |d-1|_p^{-1}, |a(d-1)|_p^{-\frac{1}{d-1}} \right\}. \end{aligned}$$

There are two cases:

- (1) $C_{f,p} = |d-1|_p^{-1}$ or,
- (2) $C_{f,p} = |a(d-1)|_p^{-\frac{1}{d-1}} \neq |d-1|_p^{-1}$.

The first case immediately implies $|\alpha(d-1)|_p \leq 1$. Consider the second case:

$$|\alpha|_p \leq |a(d-1)|_p^{-\frac{1}{d-1}}.$$

By (5.7),

$$a \in \left\{ \frac{n}{d-1} : 1 \leq |n| \leq 3(d-1) - 1 \right\},$$

so

$$a(d-1) \in \{n : 1 \leq |n| \leq 3(d-1) - 1\}.$$

Therefore, $|a(d-1)|_p^{-1} \leq 3(d-1) - 1$, so

$$\begin{aligned} |\alpha|_p^{d-1} &\leq 3(d-1) - 1 \\ |\alpha|_p &\leq (3d-4)^{\frac{1}{d-1}}. \end{aligned}$$

Taking a logarithmic derivative shows the right hand side is decreasing for $d \geq 5$, so for degree $d \geq 5$,

$$|\alpha|_p \leq 11^{1/4} < 2,$$

which implies $|\alpha|_p \leq 1$.

Therefore in both cases, we conclude that $\alpha(d-1) \in \mathbb{Z}$. By Proposition 5.4.2, $|\alpha| \leq 2$, so

$$\alpha \in \left\{ \frac{n}{d-1} \mid 1 \leq |n| \leq 2(d-1) - 1 \right\}.$$

Therefore,

$$(d-1)\alpha = n \in \mathbb{Z} \quad \text{for } 1 \leq n < 2(d-1).$$

If α is preperiodic then so is $f(\alpha)$, hence

$$(d-1)f(\alpha) = (d-1)a\alpha^{d-1}(-(d-1)\alpha + d) + (d-1)c \in \mathbb{Z}.$$

Rewriting this using $n = \alpha(d-1)$, we have

$$(d-1)a\alpha^{d-1}(-n + d) + (d-1)c \in \mathbb{Z}.$$

By Lemma 5.2.3 we know that for $p \nmid (d-1)$, $|c|_p^{d-1} \leq |a|_p^{-1}$. Since $|a|_p^{-1} < 3(d-1)$, using similar arguments to above we can show $|c|_p \leq 1$. Thus with Lemma 5.2.4 we have that $(d-1)c \in \mathbb{Z}$. This implies that

$$(d-1)a\alpha^{d-1}(-n + d) \in \mathbb{Z};$$

hence

$$\left| (d-1)a\alpha^{d-1}(-n + d) \right|_p \leq 1$$

for all primes p . This implies the following:

$$\begin{aligned} |(d-1)a|_p |\alpha|_p^{d-1} |-n + d|_p &\leq 1 \\ |\alpha|_p^{d-1} |-n + d|_p &\leq |(d-1)a|_p^{-1}. \end{aligned}$$

Let us take a closer look at $|-n + d|_p$. Assume that $|-n + d|_p \neq 0$. Since $n \leq 2d - 3$, then

$$\begin{aligned} |-n + d|_p &\geq \frac{1}{|-n + d|} \\ &\geq \frac{1}{|n| + |d|} \\ &\geq \frac{1}{2d - 3 + d} \\ &\geq \frac{1}{3d - 3}. \end{aligned}$$

Also, recall from (5.7) that $|(d-1)a|_p^{-1} \leq 3d-4$. This gives us the following:

$$\begin{aligned} |\alpha|_p^{d-1} \cdot \frac{1}{3d-3} &\leq |(d-1)a|_p^{-1} \\ |\alpha|_p^{d-1} &\leq |(d-1)a|_p^{-1} \cdot (3d-3) \\ |\alpha|_p^{d-1} &\leq (3d-4) \cdot (3d-3) \\ |\alpha|_p &\leq ((3d-4)(3d-3))^{\frac{1}{d-1}}. \end{aligned}$$

Again, a logarithmic derivative shows the right hand side is decreasing for $d \geq 11$. Therefore, if $d \geq 11$,

$$((3d-4)(3d-3))^{\frac{1}{d-1}} < 2,$$

so

$$|\alpha|_p \leq 1.$$

If we allow $|-n+d|_p = 0$, then $n = d$. Hence $\alpha = \frac{d}{d-1}$.

So to restate what we have shown, for $d \geq 11$, if $\alpha \in \text{PrePer}(f, \mathbb{Q})$ for

$$f(z) = az^{d-1}(-(d-1)z+d) + c \in \mathbb{Q}[z]$$

then $\alpha \in \mathbb{Z}$, or $\alpha = \frac{d}{d-1}$. Since $|\alpha| < 2$, then $\alpha \in \left\{0, 1, -1, \frac{d}{d-1}\right\}$.

Since f is post-critically finite, we know that 0 and 1 are pre-periodic. Since $f(0) = c$, then c is pre-periodic. Notice that $f\left(\frac{d}{d-1}\right) = c$, and since c is preperiodic, $\frac{d}{d-1}$ must be as well. \square

The following two theorems describe $\mathcal{A}_{d,1}$ for $d > 4$.

Theorem 5.4.5. *For degrees $d \geq 11$,*

$$\mathcal{A}_{d,1} = \left\{(-1, 1), (1, 0), \left(\frac{d}{d-1}, 0\right), \left(\frac{1}{d-1}, 1\right), \left(-\frac{d}{d-1}, \frac{d}{d-1}\right), \left(-\frac{1}{d-1}, \frac{d}{d-1}\right)\right\}.$$

Proof. By 5.4.4 we know that

$$\text{PrePer}(f, \mathbb{Q}) \subseteq \left\{-1, 1, \frac{d}{d-1}, 0\right\}.$$

Also, if $(a, c) \in \mathcal{A}_{d,1}$, then

$$\{c, a+c\} \subseteq \text{PrePer}(f, \mathbb{Q}).$$

We have four cases:

(1) $c = -1$,

(2) $c = 0$,

(3) $c = 1$,

(4) $c = \frac{d}{d-1}$.

Consider case (1): $c = -1$. Since $a + c \in \{-1, 0, 1, \frac{d}{d-1}\}$ and $a \neq 0$, then $a \in \{1, 2, \frac{2d-1}{d-1}\}$. For each of these, $f(0) = -1$, and $|f(-1)| \geq 4$ so 0 is a wandering point, hence $c \neq -1$.

Consider case (2): $c = 0$. Since $a + c \in \{-1, 0, 1, \frac{d}{d-1}\}$ and $a \neq 0$, then $a \in \{-1, 1, \frac{d}{d-1}\}$. When $c = 0$, $f(1) = a$. Since $|f(-1)| \geq 4$ then 1 is a wandering point for f when $a = -1$. If $a = 1$ or $a = \frac{d}{d-1}$ then $\{c, a + c\} \subseteq \{0, 1, \frac{d}{d-1}\}$ then by 5.4.4 we know that $a + c$ and c are preperiodic, hence

$$\left\{ (1, 0), \left(\frac{d}{d-1}, 0 \right) \right\} \subseteq \mathcal{A}_{d,1}.$$

Consider case (3): $c = 1$. Since $a + c \in \{-1, 0, 1, \frac{d}{d-1}\}$ and $a \neq 0$, then $a \in \{-2, -1, \frac{1}{d-1}\}$. If $a \in \{-1, \frac{1}{d-1}\}$ then $\{a + c, c\} \subseteq \{0, 1, \frac{d}{d-1}\}$ so by 5.4.4 we know that a and $a + c \in \text{PrePer}(f)$. Hence

$$\left\{ (-1, 1), \left(\frac{1}{d-1}, 1 \right) \right\} \subseteq \mathcal{A}_{d,1}.$$

Suppose $a = -2$. Then $f(1) = -1$, and $|f(-1)| > 2$ so 1 must be a wandering point, thus $(-2, 1) \notin \mathcal{A}_{d,1}$.

Finally, consider case (4): $c = \frac{d}{d-1}$. Since $a + c \in \{-1, 0, 1, \frac{d}{d-1}\}$ and $a \neq 0$, then $a \in \{\frac{-2d+1}{d-1}, -\frac{d}{d-1}, -\frac{1}{d-1}\}$. Since $f(0) = f(\frac{d}{d-1}) = \frac{d}{d-1}$ then 0 is preperiodic for f . We need to check that 1 is preperiodic. If $a \in \{-\frac{d}{d-1}, -\frac{1}{d-1}\}$ then $f(1) \in \{0, 1\}$, so

$$\left\{ \left(-\frac{d}{d-1}, \frac{d}{d-1} \right), \left(-\frac{1}{d-1}, \frac{d}{d-1} \right) \right\} \subseteq \mathcal{A}_{d,1}.$$

If $a = \frac{-2d+1}{d-1}$ then $f(1) = -1$ and $|f(-1)| \geq 2$ so 1 is a wandering point, hence $(\frac{-2d+1}{d-1}, \frac{d}{d-1}) \notin \mathcal{A}_{d,1}$. Therefore, for $d \geq 11$,

$$\mathcal{A}_{d,1} = \left\{ (-1, 1), (1, 0), \left(\frac{d}{d-1}, 0 \right), \left(\frac{1}{d-1}, 1 \right), \left(-\frac{d}{d-1}, \frac{d}{d-1} \right), \left(-\frac{1}{d-1}, \frac{d}{d-1} \right) \right\}. \quad \square$$

Theorem 5.4.6. For $5 \leq d \leq 10$,

$$\mathcal{A}_{d,1} = \left\{ (-1, 1), (1, 0), \left(\frac{d}{d-1}, 0 \right), \left(\frac{1}{d-1}, 1 \right), \left(-\frac{d}{d-1}, \frac{d}{d-1} \right), \left(-\frac{1}{d-1}, \frac{d}{d-1} \right) \right\}.$$

Proof. Let $f(z) = az^{d-1}(-(d-1)z + d) + c$ be post-critically finite. By Lemma 5.2.3, for $p \nmid (d-1)$, $|a|_p \leq 1$ and $|c| \leq |a|_p^{-\frac{1}{d-1}}$. Furthermore, by Lemma 5.2.4 $|(d-1)a|_p \leq 1$ and $|(d-1)c|_p \leq 1$ for $p \mid (d-1)$. By Proposition 5.4.1,

$$a \in \left\{ \frac{n}{d-1} : 1 \leq |n| \leq 3(d-1) - 1 \right\},$$

thus $|a|_p^{-1} \leq 3(d-1) - 1$. Hence for $p \nmid (d-1)$,

$$|c|_p \leq (3(d-1) - 1)^{\frac{1}{d-1}} < 2,$$

so $|c| \leq 1$ for such primes. This implies that both $a(d-1), c(d-1) \in \mathbb{Z}$. Since $|c| < 2$ then

$$c \in \left\{ \frac{m}{d-1} : 0 \leq |m| \leq 2(d-1) - 1 \right\}.$$

This gives $(4d-5)(6d-8)$ pairs (a, c) that could potentially be in $\mathcal{A}_{d,1}$ for $5 \leq d \leq 10$. The author used Sage [28] to test all pairs, and determined that for $5 \leq d \leq 10$ (see Appendix A for Sage code),

$$\mathcal{A}_{d,1} = \left\{ (-1, 1), (1, 0), \left(\frac{d}{d-1}, 0 \right), \left(\frac{1}{d-1}, 1 \right), \left(-\frac{d}{d-1}, \frac{d}{d-1} \right), \left(-\frac{1}{d-1}, \frac{d}{d-1} \right) \right\}. \quad \square$$

Corollary 5.4.7. *For $d \geq 4$,*

$$\mathcal{A}_{d,1} = \left\{ (-1, 1), (1, 0), \left(\frac{d}{d-1}, 0 \right), \left(\frac{1}{d-1}, 1 \right), \left(-\frac{d}{d-1}, \frac{d}{d-1} \right), \left(-\frac{1}{d-1}, \frac{d}{d-1} \right) \right\}.$$

Proof. The result follows immediately from 5.4.3, 5.4.6, and 5.4.5. □

CHAPTER 6

POST-CRITICALLY FINITE UNICRITICAL POLYNOMIALS

As discussed in Section 2.7, unicritical polynomials have been extensively studied by dynamicists. We can use the techniques described in Chapter 5 to determine all post-critically finite unicritical polynomials over \mathbb{Q} . In [7], Buff looked at unicritical polynomials from a complex dynamics point of view, and he used that work to answer questions of Milnor and of Baker and DeMarco. Some of his preliminary work — specifically the normal form in 6.0.1 and the bound on $|a|$ in 6.0.2 — overlaps with the work in this chapter. However, because Buff was working over \mathbb{C} , he did not consider questions about field of definition. Therefore, we provide full proofs of these results from a more arithmetic point of view.

First, we will determine a normal form for the family of unicritical polynomials over $K[z]$ where K is a number field.

Theorem 6.0.1. *Let $f(z) \in K[z]$ be a degree d unicritical polynomial. Then either $f(z)$ is \bar{K} -conjugate to z^d , or to a unique polynomial*

$$az^d + 1 \in K[z].$$

Proof. From Section 2.7 we can assume without loss of generality that $f(z) = bz^d + c \in K[z]$. If $c = 0$ then $f(z) = bz^d$ for $b \in K$. Conjugating by $\phi(z) = b^{\frac{1}{d-1}}z \in \text{PGL}$ we get $f^\phi(z) = z^d$.

Now, assume $c \neq 0$. Conjugating by $\phi(z) = \frac{z}{c} \in \text{PGL}$ we have

$$f^\phi(z) = bc^{d-1}z^d + 1.$$

Since $b, c \in K$, then $bc^{d-1} \in K$. Letting $a = bc^{d-1}$, we have that $f(z)$ is conjugate to $az^d + 1$ for $a \in K$, as desired. Furthermore, ϕ is the only map in PGL fixing 0 and ∞ such that f^ϕ has constant term 1. Therefore, $f(z)$ is conjugate to $az^d + 1 \in K[z]$ for a unique $a \neq 0 \in K$. \square

The above theorem implies that up to conjugacy every cubic unicritical polynomial $f \in K[z]$ is a power map or of the form $az^d + 1$. In both cases $\text{Crit}(f) = \{0\}$, and if f is a power map then $f(0) = 0$, hence PCF. Therefore, in order to completely describe all other PCF cubic unicritical polynomials, we need only consider those of the form $f(z) = az^d + 1$ for $0 \neq a \in K$. Now, we will determine an archimedean bound on the value $a \in K$.

Proposition 6.0.2. *If $f(z) = az^d + 1$ is post-critically finite then $|a| \leq 2$.*

Proof. Suppose $|a| > 2$ and $|\alpha| \geq 1$. Then

$$\begin{aligned} |f(\alpha)| &= |a\alpha^d + 1| \\ &\geq |a||\alpha|^d - 1 \\ &> 2|\alpha| - 1 \\ &> 2|\alpha| - |\alpha| \\ &> |\alpha|. \end{aligned}$$

By induction on iteration of α we can see that α must be a wandering point. Since $\text{Crit}(f) = \{0\}$ and $f(0) = 1$, then 1, and hence 0, must also be a wandering point. Therefore, if $f \in K[z]$ is PCF it must be that $|a| \leq 2$. \square

Theorem 6.0.3. *Let $f(z) = az^d + 1 \in \mathbb{Q}[z]$ and $d \geq 2$. For d even then f is PCF if and only if $a \in \{-2, -1\}$, and for d odd, f is PCF if and only if $a = -1$.*

Proof. In this case, the bound $C_{f,p}$ is given by

$$C_{f,p} = \left\{1, |a|_p^{-1/d}, |a|_p^{-1/(d-1)}\right\}.$$

By Corollary 5.1.4, if $|f^n(\alpha)|_p \geq C_{f,p}$ for any $p \in M_{\mathbb{Q}}$ then α must be a wandering point. We want to check that the critical point 0 is not a wandering point. Notice that $f(0) = 1$, and $f(1) = a + 1$, so we want to check that $|a + 1|_p$ is bounded by $C_{f,p}$ for all primes p . Since $|a + 1|_p \leq \max\{|a|_p, 1\}$, then it is sufficient to require

$$\max\{|a|_p, 1\} \leq C_{f,p}. \tag{6.1}$$

We have two distinct cases:

- (1) $C_{f,p} = 1$,
- (2) $C_{f,p} = |a|_p^{-1/i} > 1$ for $i \in \{d, d-1\}$.

In case (1), applying (6.1) gives $|a|_p \leq 1$. However, if $|a|_p < 1$ then $|a|_p^{-1/d} > 1$, contradicting case (1). So case (1) gives that $|a|_p = 1$.

In case (2), we have that $|a|_p^{-1/i} > 1$, so then $|a|_p < 1$.

Since in both cases $|a|_p \leq 1$, then it must be that $|a|_p \leq 1$ for all primes p , hence $a \in \mathbb{Z}$.

By Proposition 6.0.2 $|a| \leq 2$, so $a \in \{\pm 1, \pm 2\}$. Notice that $|a| \geq 1$. Suppose that $|\alpha| > 2$. Then

$$\begin{aligned} |f(\alpha)| &= |a\alpha^d + 1| \\ &\geq |a||\alpha|^d - 1 \\ &> 2^{d-1}|\alpha| - 1 \\ &> 2|\alpha| - |\alpha| \\ &> |\alpha|. \end{aligned}$$

By induction $|f^{n+1}(\alpha)| > |f^n(\alpha)|$ for all $n \geq 0$, so α must be a wandering point.

Consider the case where $a = 1$. Then $f^3(0) = 2^d + 1 > 2$ so 0 must be a wandering point.

If $a = 2$, then $f^2(0) = 3 > 2$, so 0 must be a wandering point.

If $a = -1$, then $f^2(0) = 0$, so 0 is periodic for f of period 2, hence f is PCF.

Finally, consider $a = -2$. If d is even then $f^2(0) = f^3(0) = -1$, so 0 is preperiodic of period (2, 1), hence f is PCF. If d is odd, then $f^3(0) = 3$, so 0 is a wandering point.

Therefore, we have that if $f(z) = az^d + 1$ is PCF and d is odd, then $a = -1$. For even d , $a \in \{-1, -2\}$. Furthermore, $f(z) = az^d + 1$ is PCF for those a values. \square

6.1 Post-Critically Finite Cubic Polynomials

Note that if $\deg(f) = 3$, then $\deg(f') = 2$. So f is either unicritical or bicritical. In Section 5.3 we determined the complete list of post-critically finite bicritical cubic polynomials of the form $a\mathcal{B}_{3,1}(z) + c \in \mathbb{Q}[z]$, up to conjugacy. Using Theorem 6.0.3 and assuming Conjecture 4.0.5 we know the complete list of all post-critically finite unicritical cubic polynomials over \mathbb{Q} , up to conjugacy.

As in Section 4.1, we can conjugate a cubic polynomial $f(z) \in \mathbb{Q}[z]$ to the Branner-Hubbard normal form of a cubic:

$$F(z) = z^3 - 3a^2z + b.$$

In [14] Ingram lists the post-critically finite Branner-Hubbard cubics in $\mathbb{Q}[z]$, but because of field of definition issues, he does not list all cubic PCF polynomials. Assuming Conjecture 4.0.5, the left column of the following table provides a complete list of PCF cubic polynomials. The right column gives the conjugate PCF cubics found by Ingram in [14].

$f(z) \in \mathbb{Q}[z]$	Branner-Hubbard Normal Form from [14]
z^3	z^3
$-z^3 + 1$	
$-2z^3 + 3z^2 + \frac{1}{2}$	
$-2z^3 + 3z^2$	$z^3 + \frac{3}{2}z$
$-z^3 + \frac{3}{2}z^2 - 1$	
$2z^3 - 3z^2 + 1$	$z^3 - \frac{3}{2}z$
$2z^3 - 3z^2 + \frac{1}{2}$	
$z^3 - \frac{3}{2}z^2$	$z^3 - \frac{3}{4}z + \frac{3}{4}$ or $z^3 - \frac{3}{4}z - \frac{3}{4}$
$-3z^3 + \frac{9}{2}z^2$	
$-4z^3 + 6z^2 - \frac{1}{2}$	$z^3 + 3z$
$4z^3 - 6z^2 + \frac{3}{2}$	$z^3 - 3z$
$3z^3 - \frac{9}{2}z^2 + 1$	

Table 6.1: PCF cubic polynomials and the Hubbard normal forms in \mathbb{Q}

The following is a collection of critical orbits of post-critically finite cubic polynomials $f(z) \in \mathbb{Q}[z]$ given in the table above.

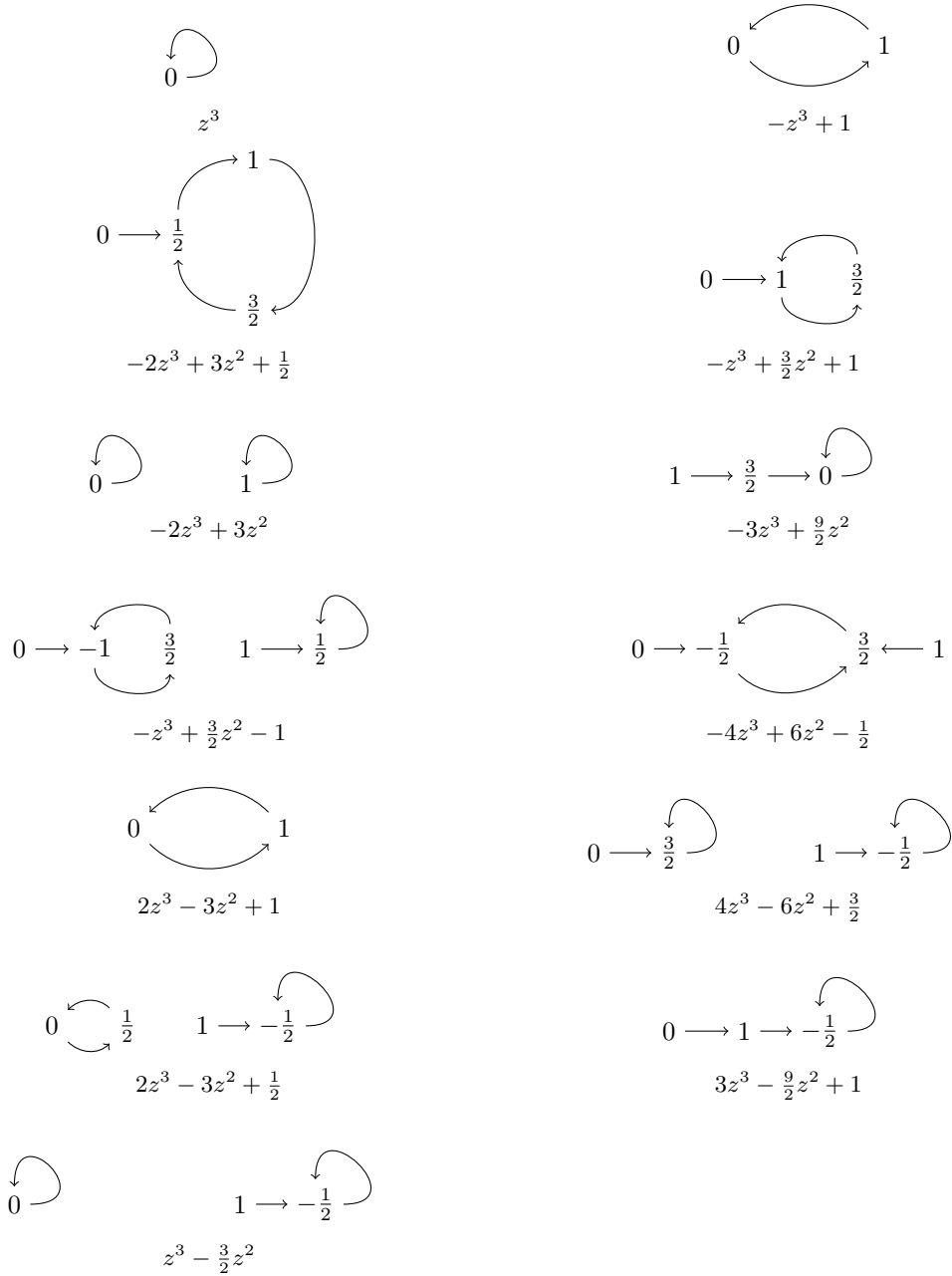


Figure 6.1: Critical portraits of all cubic PCF polynomials over \mathbb{Q}

Appendices

APPENDIX A

SAGE CODE

Algorithm:

- We input the degree d of the polynomial, f .
- Next, we build the list of possible a and c values in \mathbb{Q} given by equations (5.7) and (5.6).
- We use the built-in projective space package to define \mathbb{P}^1 and the space of morphisms on \mathbb{P}^1 .
- We set up a list which will contain all the pairs (a, c) which may admit post-critically finite maps.
- We iterate over all the possible pairs (a, c) . For each pair, we define the polynomial f , and then use the built in function `is_postcritically_finite`. This function works by determining the critical points of the polynomial, and then using a height calculation to determine if the critical points are preperiodic.
- If the function is post-critically finite, we append (a, c) to the list.

```
def pcftest_kis1(d):
    #input degree
    #build the lists of a and c to test.
    List1a = range(1,4*(d-1))
    List2a = [-i for i in List1a]
    Try_a = [i/(d-1) for i in List1a+List2a]
    List1c = range(1, 2*(d-1))
    List2c = [-i for i in List1c]
    List2c.append(0)
    Try_c =[i/(d-1) for i in List1c+List2c]

    P1.<x,y> = ProjectiveSpace(QQ,1);
    H = P1.Hom(P1);
    perlist = [];

    for a in Try_a:
        for c in Try_c: #try each pair (a,c)
```

```
f = H([- (d-1)*a*x^d+d*a*x^(d-1)*y+c*y^d,y^d]); #build polynomial
if f.is_postcritically_finite(): #check if pcf
    perlist.append((a,c));
    #append pairs to list if they admit a pcf map.
print perlist
```

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