LOCAL FIELDS, ITERATED EXTENSIONS, AND JULIA SETS

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ABSTRACT

Let K be a field complete with respect to a discrete valuation v of residue characteristic p, and let $f(z) = z^\ell - c \in K[z]$ be a separable polynomial. We explore the connection between the valuation $v(c)$ and the Berkovich Julia set of f. Additionally, we examine the field extensions generated by the solutions to $f^n(z) = \alpha$ for a root point $\alpha \in K$, highlighting the interplay between the dynamics of f and the ramification in the corresponding field extensions.

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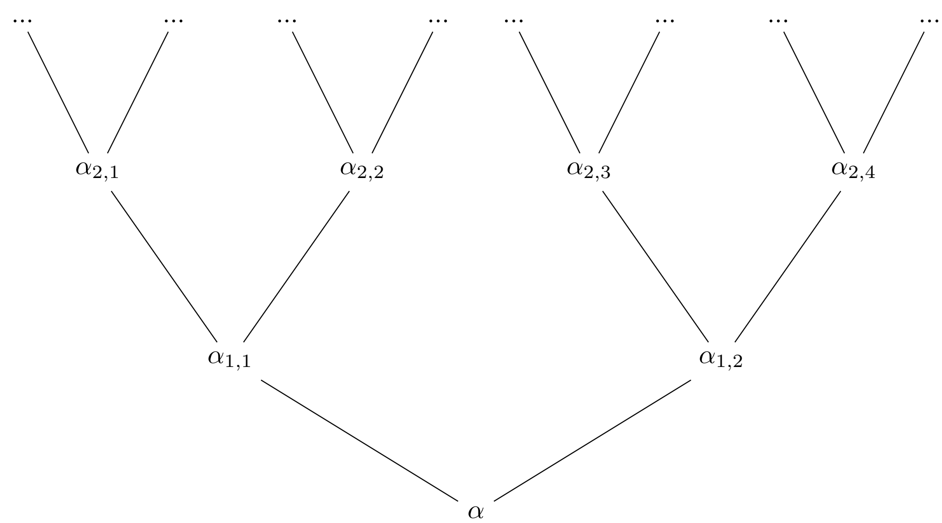
CHAPTER 1
INTRODUCTION

Arithmetic dynamics lies at the intersection of two fields of mathematics, number theory and dynamical systems. Algebraic number theory involves the study of the algebraic numbers, which are the roots of polynomials with coefficients in the rational numbers $\mathbb{Q}$. Adjoining algebraic numbers to $\mathbb{Q}$ yields extension fields of $\mathbb{Q}$, and these extension fields are called number fields if they form a finite dimensional vector space over $\mathbb{Q}$. A fundamental tool of studying a problem over $\mathbb{Q}$ and its extensions is to study the same problem over all the completions of $\mathbb{Q}$. Completing $\mathbb{Q}$ with respect to the usual absolute value leads to the construction of the real numbers $\mathbb{R}$. For every prime number p, we can define a p-adic absolute value, and completing with respect to these p-adic absolute values leads to the construction of the p-adic fields $\mathbb{Q}_p$, which are examples of non-archimedean local fields.

A discrete dynamical system is a self-map of some set, allowing for iteration. A fundamental goal in dynamical systems is to study the long-term behavior of points of the set as the function is iterated. When the underlying set is the complex numbers, there are two subsets that categorize points according to how predictable their dynamical behavior is under a given function: The Fatou set is the set of all points where the iterates of the function act equicontinuously. In other words, it's the set where the dynamics is predictable, where nearby points stay relatively nearby as the function is iterated. The Julia set is the complement of this set, and it is the region of chaos, where nearby points can have wildly different behaviors. We can also define analogously the Fatou and Julia set of a function in the p-adic field setting.

Dynamical systens can generate field extensions of interest to number theorist in the following way: given a polynomial f over a field K, we can think of it as a dynamical system $f\colon K \to K$. One can define the backwards orbit of some root point $\alpha \in K$ under f to be the set of preimages of α under iterations of f. The elements of this backwards orbit form a rooted tree, denoted $T_\infty(\alpha)$, where the nth level of $T_\infty(\alpha)$ corresponds to the preimage of α under the nth iterate of f. Moreover, applying f to any of the nodes (except α) will yield the node directly below it. For example, if f has degree 2, then every point will have 2 preimage points counting multiplicity, and the tree will look like the figure below:



One can then define the field generated by f with the root point α to be the smallest field that contains K and the backwards orbit of f under α, which we denote by K_∞. The absolute Galois group $\mathrm{Gal}(K_\infty/K)$ acts on $T_\infty(\alpha)$ and preserve its tree structure.

In this dissertation, we study the field extensions of p-adic fields generated by the backwards orbits of a particular family of polynomials, specifically $f(z) = z^\ell - c$. It is known that if the base field is instead a number field, one can use the theory of height functions to show that the field extension generated by the backwards orbit has infinite degree, given that the backward orbit is infinite. However, for the case where the base field is local, a recent paper [1] shows that the field extensions generated can actually be finite. In [1], the authors obtained their results by studying the arboreal Galois representations that arises from the action of $\mathrm{Gal}(K_\infty/K)$ on $T_\infty(\alpha)$. In this dissertation, a generalization of p-adic Julia sets serves as a fundamental tool in our investigation.

CHAPTER 2
BACKGROUND

In this chapter we will introduce the background that serves as a foundation for our main results.

2.1 Local Fields

A fundamental tool for studying a problem over the rational numbers $\mathbb{Q}$ and its extensions is to study the same problem over all the completions of $\mathbb{Q}$. Completing $\mathbb{Q}$ with respect to the usual absolute value leads to the construction of the real numbers $\mathbb{R}$. In general, for any field K, an absolute value for K is defined to be a function $|\cdot| : K \longrightarrow \mathbb{R}^{\geq 0}$ that satisfies the following three properties:

(i) $|\alpha| = 0$ if and only if $\alpha = 0$ for all $\alpha \in K$,

(ii) $|\alpha\beta| = |\alpha||\beta|$ for all $\alpha, \beta \in K$,

(iii) $|\alpha + \beta| \leq |\alpha| + |\beta|$ for all $\alpha, \beta \in K$.

Example 2.1.1. The trivial absolute value, $|\cdot|_0 : K \to \mathbb{R}^{\geq 0}$, is given by

$$|\alpha|_0 = \begin{cases} 0 & \text{if } \alpha = 0. \\ 1 & \text{if } \alpha \neq 0. \end{cases}$$

One can verify that $|\cdot|_0$ satisfies all three properties above.

Furthermore, if a non-trivial absolute value $|\cdot|$ satisfies the stronger property

$$\text{(iii')}\ |\alpha + \beta| \leq \max\{|\alpha|, |\beta|\} \leq |\alpha| + |\beta| \text{ for all } \alpha, \beta \in K,$$

we will say that the absolute value is non-archimedean. Any field equipped with a non-archimedean absolute value is said to be a non-arichimedean field. The ring of integers of any non-archimedean field is defined as $\mathcal{O}_K := \{\alpha \in K : |\alpha| \leq 1\}$. This ring has a unique maximal ideal $\mathcal{M} = \{\alpha \in K : |\alpha| < 1\}$.

For every prime number p, we can define a p-adic absolute value for $\mathbb{Q}$, $|\cdot|_p : \mathbb{Q} \longrightarrow \mathbb{R}^{\geq 0}$, given by

$$|\alpha|_p = p^{-v_p(\alpha)},$$

where $v_p : \mathbb{Q} \to \mathbb{Z} \cup \{\infty\}$, called the p-adic valuation for $\mathbb{Q}$, is defined as

$$v_p(\alpha) = \begin{cases} n & \text{if } \; p^{-n}\alpha = \frac{a}{b} \text{ and } (p,a) = (p,b) = 1. \\ \infty & \text{if } \; \alpha = 0. \end{cases}$$

Example 2.1.2. Let $p = 5$, we have

$$v_5(5) = 1 \text{ and } |5|_5 = 5^{-1} = \frac{1}{5}.$$

Similarly,

$$v_5\left(\frac{78}{50}\right) = -2 \text{ and } \left|\frac{78}{50}\right|_5 = 25.$$

Notice that

$$v_5\left(5 + \frac{78}{50}\right) = v_5\left(\frac{328}{50}\right) = -2 \text{ and } \left|5 + \frac{78}{50}\right|_5 = 25 = \max\left\{|5|_5, \left|\frac{78}{25}\right|_5\right\},$$

and one can verify that $|\cdot|_5$ satisfies (iii'), and hence is non-arichimedean.

In general, p-adic absolute values are non-arichimedean and completing $\mathbb{Q}$ with respect to them leads to the construction of the p-adic fields $\mathbb{Q}_p$, which are examples of non-archimedean local fields [10, Chapter 7]. Two absolute values $|\cdot|_a : K \longrightarrow \mathbb{R}^{\geq 0}, |\cdot|_b : K \longrightarrow \mathbb{R}^{\geq 0}$ are called equivalent if there is a $c > 0$ such that $|\cdot|_a = (|\cdot|_b)^c$. I.e., there exist a $c > 0$ such that for all $\alpha \in K$, $|\alpha|_a = (|\alpha|_b)^c$. Ostrowski's theorem says that every non-trivial absolute value for $\mathbb{Q}$ is equivalent to either the usual absolute value or a p-adic absolute value for some prime p [10, Theorem 7.12].

Definition 2.1.3. [10, Definition 7.48] A field K is said to be a **local field** if it is equipped with a non-trivial absolute value such that K is locally compact.

A non-archimedean field K is local if and only if $|K^\times|$ is a discrete subgroup of $\mathbb{R}^{>0}$ and $\mathcal{O}_K/\mathcal{M}$ is finite, where $\mathcal{O}_K$ is the ring of integers of K and $\mathcal{M}$ is the unique maximal ideal of $\mathcal{O}_K$ [10, Proposition 7.46]. The resulting field $\mathcal{O}_K/\mathcal{M}$ is called the residue field of K.

Example 2.1.4. Let $K = \mathbb{Q}_5$. We have

$$\mathcal{O}_K = \{x \in K : |x|_5 \leq 1\},$$

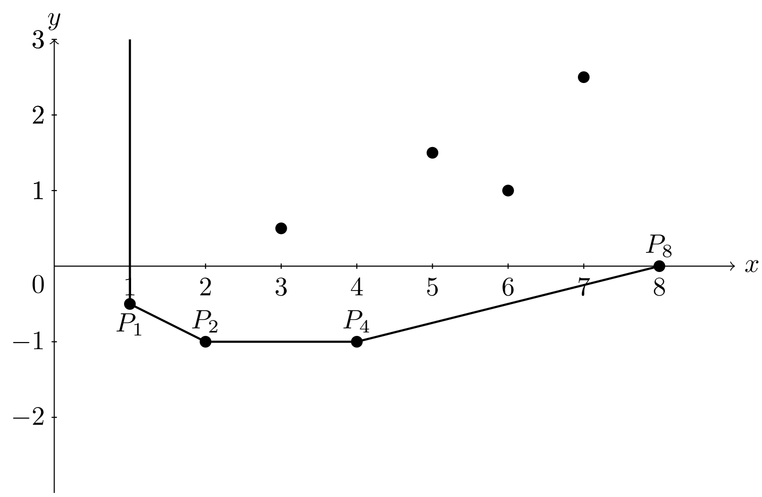
$$\mathcal{M} = \{x \in K : |x|_5 < 1\},$$

and

$$\mathcal{O}_K/\mathcal{M} = \mathbb{Z}/5\mathbb{Z}.$$

Given a polynomial $f(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0 \in \mathbb{Q}_p$, the Newton polygon of f is defined to be the lower convex hull of the points $(i, v_p(a_i))$, ignoring the points where $a_i = 0$, together with a vertical ray above the point $(j, v_p(a_j))$, if $j \neq 0$, $a_j \neq 0$, and $a_k = 0$ for all $k < j$. The p-adic valuation of the roots of f in an algebraic closure of $\mathbb{Q}_p$, which we denote $\overline{\mathbb{Q}_p}$, can then be obtained by looking at the slopes and lengths of the segments of this Newton polygon. In particular, for each segment of length ℓ with slope m, there are ℓ roots of f with p-adic valuation $-m$, counting multiplicity. Moreover, if there is a vertical ray above the point $(j, v_p(a_j))$, then 0 is a root of f with multiplicity j.

Example 2.1.5. Let $g(z) = (z + \frac{1}{\sqrt{2}})^8 - \frac{1}{2^4} = z^8 + 4\sqrt{2}z^7 + 14z^6 + 14\sqrt{2}z^5 + \frac{35}{2}z^4 + 7\sqrt{2}z^3 + \frac{7}{2}z^2 + \frac{1}{\sqrt{2}}z \in \overline{\mathbb{Q}_2}$. The Newton Polygon of g is the lower convex hull of the points $(1, -0.5)$, $(2, -1)$, $(3, 0.5)$, $(4, -1)$, $(5, 1.5)$, $(6, 1)$, $(7, 2.5)$, $(8, 0)$ and is shown below:



So we conclude that f has:

- Four roots of valuation $-\frac{1}{4}$.
- Two roots of valution 0.
- One root of valuation $\frac{1}{2}$.
- one root at $z = 0$.

2.2 The Berkovich Projective Line

Classical dynamics utilizes topological and algebraic properties of the complex numbers. However, for the fields studied in arithmetic dynamics, a lot of these properties are lacking because they are non-archimedean local fields. For example, $\mathbb{Q}_p$ is not algebraically closed, not compact, and totally disconnected.

One remedy is to embed $\mathbb{Q}_p$ into a topological space that has the desired algebraic and topological properties, and that leads to the construction of the Berkovich projective line. Finite extensions of local fields are local, but infinite extensions are necessarily not. In particular, $\overline{\mathbb{Q}_p}$ is not local. Moreover, $\overline{\mathbb{Q}_p}$ is not complete. Luckily, the completion of $\overline{\mathbb{Q}_p}$ is again algebraically closed, and we call the resulting field $\mathbb{C}_p$. However, $\mathbb{C}_p$ remains totally disconnected and is also not locally compact. Therefore, we embed $\mathbb{C}_p$ into the projective line $\mathbb{P}^1(\mathbb{C}_p)$, and then construct the Berkovich projective line, denoted $\mathbb{P}^1_{\text{an}}$, as a path-connected, Hausdorff compactification of $\mathbb{P}^1(\mathbb{C}_p)$.

The remaining of this section is devoted to describing the points of $\mathbb{P}^1_{\text{an}}$ and also giving a basis for its topology. For a more complete description of the Berkovich projective line, we refer the reader to [3, Chapter 6].

Before we can describe the points in $\mathbb{P}^1_{\text{an}}$, we need to define the following notions.

Definition 2.2.1. Let $a \in \mathbb{C}_p$ and $r > 0$. The open disk and closed disk in $\mathbb{C}_p$ centered at a with radius r, are defined to be

$$D(a, r) := \{x \in \mathbb{C}_p : |x - a| < r\}, \text{ and}$$

$$\overline{D}(a, r) := \{x \in \mathbb{C}_p : |x - a| \leq r\}$$

respectively.

Even though $\mathbb{C}_p$ is a complete and algebraically closed non-archimedean field, it is not spherically complete, meaning that there exists a decreasing sequence of closed disks in $\mathbb{C}_p$ where the intersection is empty. We illustrate the situation by the following example extracted from [3, pg.41].

Example 2.2.2. Let $D_n := \overline{D}(a_n, r_n)$, where

$$a_n := \sum_{j=1}^{n} p^{1-1/j} \text{ and } r_n := |p|_p^{1-1/n}.$$

For example, $a_1 = 1$, $a_2 = 1 + p^{1/2}$, $a_3 = 1 + p^{1/2} + p^{2/3}$, and so on. Similarly, we have $r_1 = 1$, $r_2 = |p|_p^{1/2} = p^{-1/2}$, $r_3 = |p|_p^{2/3} = p^{-2/3}$, and so on.

Note that $|a_{n+1} - a_n|_p = |p^{1-1/(n+1)}|_p = r_{n+1}$. Hence $a_{n+1} \in D_n \cap D_{n+1}$. Also since $r_{n+1} < r_n$, we have $D_{n+1} \subsetneq D_n$. Therefore $\{D_n\}$ a nested sequence of nonempty disks.

We claim that the intersection $\bigcap_{n\geq 1} D_n$ is empty. First note that $\{r_n\}$ is a decreasing sequence such that $\lim_{n\to\infty} r_n = 1/p$. Suppose by contradiction that there is some $b \in \bigcap_{n\geq 1} D_n$. Then for each n, we have $|b - a_n|_p \leq p^{-(n-1)/n}$, i.e. the p-adic expansion of b has the form $1 + p^{1/2} + p^{2/3} + \cdots + p^{(n-1)/n} +$ (higher powers of p) for each n. However that is impossible since the series $\sum_{j=1}^{\infty} p^{1-1/j}$ diverges in $\mathbb{C}_p$. (We have $\lim_{n\to\infty} |p^{1-1/n}|_p = 1/p \neq 0$.)

Definition 2.2.3. Let $\{D_n\}$, $\{E_n\}$ be two decreasing sequences of closed disks with empty intersection. We say that $\{D_n\}$ is equivalent to $\{E_n\}$ if for every $N \geq 1$, there is some $M \geq N$ such that $E_M \subseteq D_N$ and $D_M \subseteq E_N$.

We are now ready to describe the points in $\mathbb{P}^1_{\text{an}}$. In general, the points in $\mathbb{P}^1_{\text{an}}$ are classified into the following four types:

(i) Every point in $\mathbb{P}^1(\mathbb{C}_p)$ corresponds to a point $\zeta \in \mathbb{P}^1_{\text{an}}$ that we call a Type I point.

(ii) Every closed disk $\overline{D}(a, r)$ in $\mathbb{C}_p$ where $r \in p^{\mathbb{Q}}$ corresponds to a point $\zeta \in \mathbb{P}^1_{\text{an}}$ that we call a Type II point and denote $\zeta(a, r)$.

(iii) Every closed disk $\overline{D}(a, r)$ in $\mathbb{C}_p$ where $r \notin p^{\mathbb{Q}}$ corresponds to a point $\zeta \in \mathbb{P}^1_{\text{an}}$ that we call a Type III point and denote $\zeta(a, r)$.

(iv) Every equivalence class of decreasing sequences of closed disks with empty intersection corresponds to a point in $\mathbb{P}^1_{\text{an}}$ that we call a Type IV point.

We will now give a basis for the topological space $\mathbb{P}^1_{\text{an}}$. We define the following notions.

Definition 2.2.4. Let $a \in \mathbb{C}_p$ and $r > 0$. Let

$$D_I(a, r) := \{\zeta \in \mathbb{P}^1_{\text{an}} : \zeta \text{ is Type I and } \zeta \in D(a, r)\},$$

$$D_{II}(a, r) := \{\zeta \in \mathbb{P}^1_{\text{an}} : \zeta \text{ is Type II, } \zeta = \zeta(b, s) \text{ and } \overline{D}(b, s) \subseteq D(a, r)\},$$

$$D_{III}(a, r) := \{\zeta \in \mathbb{P}^1_{\text{an}} : \zeta \text{ is Type III, } \zeta = \zeta(b, s) \text{ and } \overline{D}(b, s) \subseteq D(a, r)\},$$

$$D_{IV}(a, r) := \{\zeta \in \mathbb{P}^1_{\text{an}} : \zeta \text{ is Type IV such that } \zeta \text{ corresponds to some } \{D_n\} \text{ with } D_m \subseteq D(a, r) \text{ for some } m\}.$$

Similarly, let

$$\overline{D}_I(a,r) := \{\zeta \in \mathbb{P}^1_{\text{an}} : \zeta \text{ is Type I and } \zeta \in \overline{D}(a,r)\},$$

$$\overline{D}_{II}(a,r) := \{\zeta \in \mathbb{P}^1_{\text{an}} : \zeta \text{ is Type II, } \zeta = \zeta(b,s) \text{ and } \overline{D}(b,s) \subseteq \overline{D}(a,r)\},$$

$$\overline{D}_{III}(a,r) := \{\zeta \in \mathbb{P}^1_{\text{an}} : \zeta \text{ is Type III, } \zeta = \zeta(b,s) \text{ and } \overline{D}(b,s) \subseteq \overline{D}(a,r)\},$$

$$\overline{D}_{IV}(a,r) := \{\zeta \in \mathbb{P}^1_{\text{an}} : \zeta \text{ is Type IV such that } \zeta \text{ corresponds to some } \{D_n\} \text{ with } D_m \subseteq \overline{D}(a,r) \text{ for some } m\}.$$

The closed Berkovich disk centered at a with radius r, denoted $\overline{D}_{\text{an}}(a,r)$, is then defined by

$$\overline{D}_{\text{an}}(a,r) := \overline{D}_I(a,r) \cup \overline{D}_{II}(a,r) \cup \overline{D}_{III}(a,r) \cup \overline{D}_{IV}(a,r).$$

The open Berkovich disk centered at a with radius r, denoted $D_{\text{an}}(a,r)$, is defined by

$$D_{\text{an}}(a,r) := \begin{cases} D_I(a,r) \cup D_{II}(a,r) \cup D_{III}(a,r) \cup D_{IV}(a,r) & \text{if } \ r \in p^{\mathbb{Q}}, \\ D_I(a,r) \cup D_{II}(a,r) \cup D_{III}(a,r) \cup D_{IV}(a,r) \setminus \{\zeta(a,r)\} & \text{if } \ r \notin p^{\mathbb{Q}}. \end{cases}$$

With the notion of open and closed Berkovich disk, we can finally state the definition of an open $\mathbb{P}^1_{\text{an}}$-disk.

Definition 2.2.5. An open $\mathbb{P}^1_{\text{an}}$-disk is either an open Berkovich disk $D_{\text{an}}(a,r)$ or the complement $\mathbb{P}^1_{\text{an}} \setminus \overline{D}_{\text{an}}(a,r)$ of a closed Berkovich disk.

An open connected Berkovich affinoid, is defined to be a nonempty intersection of finitely many open $\mathbb{P}^1_{\text{an}}$-disks, and the set of open connected Berkovich affinoids form a basis for the Berkovich projective line $\mathbb{P}^1_{\text{an}}$.

The Berkovich projective line $\mathbb{P}^1_{\text{an}}$ can be visualized as an infinite tree in which the branches of the tree corresponds to the Berkovich disks. An example is given by Figure 2.1. In this figure, the Type II point $\zeta(0,1)$ in the middle corresponds to the $\mathbb{C}_p$ disk $\overline{D}(0,1)$. Three Berkovich open disks are highlighted. For example, the bottom branch highlighted is the open Berkovich disk $D_{\text{an}}(0,r)$, with $0 < r < 1$. The leaf at the end of that segment is the $\mathbb{C}_p$ point 0. In general, the Berkovich projective line is a tree with countably many branches at each Type II point, and leaves at the Type I and Type IV points. Type III points are interior points with no branching.

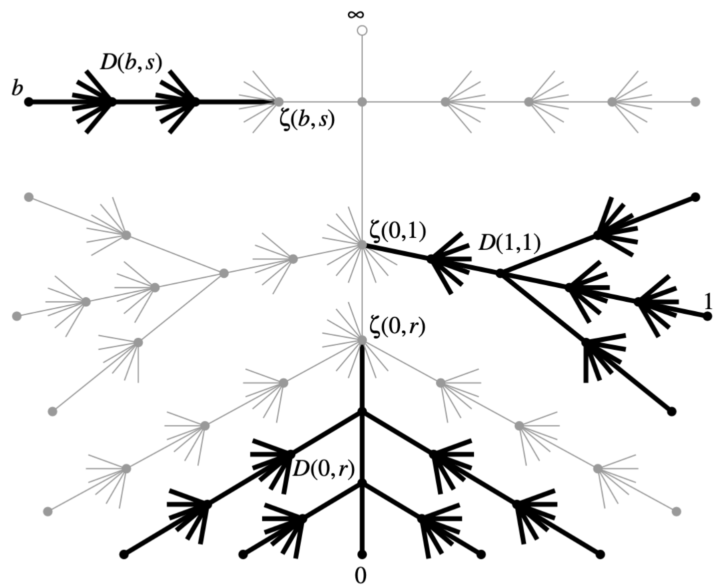


Figure 2.1: Figure 6.2 from [3].

2.3 Arithmetic Dynamics

A discrete dynamical system is a self-map of some set, allowing for iteration. A fundamental goal in dynamics is to study the long-term behavior of points of the set as the function is iterated. In arithmetic dynamics, the set considered is often the projective line $\mathbb{P}^1(K) := K \cup \{\infty\}$ for some field K, while the map is a rational function $f(z) = \frac{P(z)}{Q(z)} \in K(z)$.

Definition 2.3.1. Let $f \in \mathbb{C}_p(z)$. We define the nth iterate of f, denoted f^n, to be

$$f^n := \underbrace{f \circ f \circ \cdots \circ f}_{n \text{ times}}.$$

In dynamical systems, maps are considered equivalent if they exhibit the same dynamical properties. In arithmetic dynamics, this means that the maps are conjugate, defined as follows.

Definition 2.3.2. Let K be a field and $\overline{K}$ be an algebraic closure. Let $f, g \in K(z)$. We say that f and g are dynamically eqivalent if there exist $\phi \in \mathrm{PGL}(2, \overline{K})$ such that $g = \phi^{-1} \circ f \circ \phi := f^{\phi}$.

Conjugation preserves dynamical properties; in particular, conjugation commutes with iteration, i.e. $(\phi^{-1} \circ f \circ \phi)^n = \phi^{-1} \circ f^n \circ \phi$.

Mathematical objects in $\mathbb{Q}_p$ are said to have **good reduction** if they retain some fundamental properties when reduced modulo p. An example is a monic polynomial f of degree d in $\mathbb{Q}_p$ is said to have good reduction if all the coefficients lie in the unit disk under the p-adic absolute value, so that the reduction of f modulo p is also a polynomial of degree d. Since we usually consider equivalence classes of mathematical objects rather than individual objects, we have the notion of **potential good reduction**. An object has potential good reduction modulo p if some member of the equivalence class has good reduction modulo p. Returning to the example of degree d polynomials, a polynomial f is said to have potential good reduction if it is dynamically equivalent to a map that has good reduction.

Example 2.3.3. Let $f(z) = z^p - c \in \mathbb{C}_p[z]$. Then $f(z)$ has good reduction if and only if

$$|-c|_p = |c|_p \leq 1,$$

or equivalently,

$$v_p(c) \geq 0.$$

When $|c|_p > 1$, f does not have good reduction. However, if we let $b \in \mathbb{C}_p$ be a fixed point of f, i.e. b is a root of $z^p - z - c$, let $\phi(z) = z + b$, and consider $f^\phi = \phi^{-1} \circ f \circ \phi = (z+b)^p - c - b$, then f^ϕ has good reduction if and only if

$$|c|_p \leq p^{\frac{p}{p-1}},$$

or equivalently

$$v_p(c) \geq -\frac{p}{p-1}.$$

Therefore, if $v_p(c) \geq -\frac{p}{p-1}$, f has potential good reduction. It turns out that, by Lemma 3.1.2 in chapter 3, this condition for $v_p(c)$ is necessary for f to have potential good reduction and hence we say that f has persistent bad reduction if $v_p(c) < -\frac{p}{p-1}$.

When K is the complex numbers $\mathbb{C}$, there are two subsets that categorize points according to how predictable their dynamical behavior is under a given function: The **Fatou set** is the set where the dynamics is predictable, where nearby points stay relatively nearby as the function is iterated. The **Julia set** is the complement of this set, and it is the region of chaos, where nearby points can have wildly different behaviors. The definition of these two sets are given below.

Definition 2.3.4. Let $f \in \mathbb{C}(z)$. The Fatou set of f, denoted $\mathcal{F}_f$, is the set of all points $x \in \mathbb{P}^1(\mathbb{C})$ having a neighborhood on which the family of iterates $\{f^n : n \geq 0\}$ is equicontinous. The Julia set of f, denoted $\mathcal{J}_f$, is the complement $\mathcal{J}_f := \mathbb{P}^1(\mathbb{C}) \setminus \mathcal{F}_f$.

When f is a polynomial we have equivalent definitions that can be easier to work with: Let $\mathcal{K}_f \in \mathbb{C}$ be the **filled Julia set**, defined as

$$\mathcal{K}_f := \{\alpha \in \mathbb{C} : \lim_{n\to\infty} f^n(\alpha) \neq \infty\}.$$

Then the Julia set $\mathcal{J}_f$ is the boundary of the filled Julia set: $\mathcal{J}_f = \partial\mathcal{K}_f$ [3, Remark 5.28].

Example 2.3.5. Let $f \in \mathbb{C}[z]$ be the pure power map $f(z) = z^d$, where $d > 1$. Then

$$|\alpha| > 1 \text{ if and only if } \lim_{n\to\infty} f^n(\alpha) = \infty,$$

so $\mathcal{K}_f$ is the unit disc and $\mathcal{J}_f$ is the unit circle.

Example 2.3.6. Let $f \in \mathbb{C}[z]$ be the polynomial $f(z) = z^2 - 1$. Then $\mathcal{J}_f$ is the Basilica Julia set shown in Figure 2.2. Here the black interior is the filled Julia set $\mathcal{K}_f$ and the boundary of it is the Julia set $\mathcal{J}_f$.

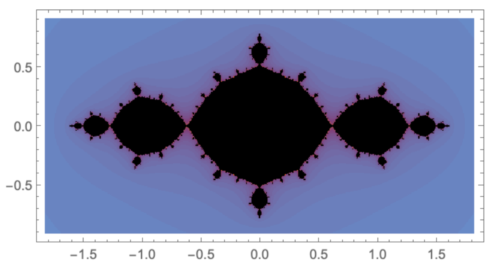


Figure 2.2: Figure generated by Mathematica.

In complex dynamics, complicated Julia sets like Figure 2.2 are much more common than simple descriptions like the one for the pure power map.

We can also define analogously the Fatou and Julia set of a function in $\mathbb{C}_p$, since p-adic absolute values yield a notion of distances between points in $\mathbb{P}^1(\mathbb{C}_p)$. However, striking differences exist between the archimedean and nonarchimedean cases when we study the dynamical properties that arise from iterating rational functions over such fields. An example is that while the Fatou set of a rational function of degree greater than 2 over the complex field may be empty, the Julia set is never empty. However, the exact opposite happens in the nonarchimedean case — the Julia set might be empty while the Fatou set is always non-empty. Moreover, since $\mathbb{P}^1(\mathbb{C}_p)$ is not locally compact and is totally disconnected, the topological properties of the underlying spaces that the Fatou and Julia sets live in are very different for the archimedean and nonarchimedean cases.

One remedy, as mentioned in section 2.1, is to embed $\mathbb{P}^1(\mathbb{C}_p)$ into the Berkovich projective line $\mathbb{P}^1_{\text{an}}$ and consider the dynamics over $\mathbb{P}^1_{\text{an}}$. Rational functions on $\mathbb{C}_p$ can be naturally extended to maps on $\mathbb{P}^1_{\text{an}}$. However, the details are rather technical and we refer the reader to [3, Chapter 7].

Given a rational function $f \in \mathbb{C}_p(z)$, the Berkovich Fatou set, denoted $\mathcal{F}_{f,\text{an}}$, is the set of points which have a neighborhood that is dynamically stable under f, while the Berkovich Julia set $\mathcal{J}_{f,\text{an}}$ is the complement of $\mathcal{F}_{f,\text{an}}$. Once again, if f is a polynomial, then $\mathcal{J}_{f,\text{an}}$ can be described as follows. Define the **Berkovich filled Julia set**, denoted $\mathcal{K}_{f,\text{an}}$, as

$$\mathcal{K}_{f,\text{an}} := \{\zeta \in \mathbb{P}^!_{\text{an}} : \lim_{n\to\infty} f^n(\zeta) \neq \infty\}.$$

We have $\mathcal{J}_{f,\text{an}} = \partial\mathcal{K}_{f,\text{an}}$ [3, Theorem 8.15]. It is this later description that we will adopt as the definition of $\mathcal{J}_{f,\text{an}}$ for a polynomial $f \in \mathbb{C}_p[z]$.

The following proposition from [3, Proposition 8.12] describes the Berkovich Julia sets for rational functions in $\mathbb{C}_p$ with good reduction.

Proposition 2.3.7. *Let $\phi \in \mathbb{C}_p(z)$ a rational function of degree ≥ 2. Then ϕ has good reduction if and only if the Berkovich Julia set is the singleton set*

$$\mathcal{J}_{\phi,an} = \{\zeta(0,1)\}.$$

Hence the Berkovich Julia sets for rational functions in $\mathbb{C}_p$ can be very different from their complex counterparts. For example, the polynomials $f(z) = z^d$ and $g(z) = z^2 - 1$ as polynomials over $\mathbb{C}_p$, would both have the same Berkovich Julia set, i.e. $\mathcal{J}_{f,\text{an}} = \mathcal{J}_{g,\text{an}} = \{\zeta(0,1)\}$; by constrast, if we consider the same polynomials over $\mathbb{C}[z]$, then the Julia set of f is the unit circle and the Julia set of g is the Basilica Julia set, as indicated in Example 2.3.5 and Example 2.3.6.

2.4 Arboreal Galois Representations

Let K be a field and $f \in K[z]$ of degree $d \geq 2$. Given $\alpha \in K$, the preimages of α under iterates of f have a natural tree structure which the absolute Galois group of K acts on. In this section, we will develop the background notions to describe this tree and discuss some recent results on the field extensions when K is a local field. We begin with the following definitions.

Definition 2.4.1. Let K be a field and $\overline{K}$ a fixed algebraic closure. Let $f \in K[z]$ be a polynomial of degree $d \geq 2$ and $\alpha \in K$. For $n \geq 1$, we define the set $V_n(\alpha)$ to be

$$V_n(\alpha) := \{\beta \in \overline{K} : f^n(\beta) = \alpha\},$$

and $V_0(\alpha) := \{\alpha\}$.

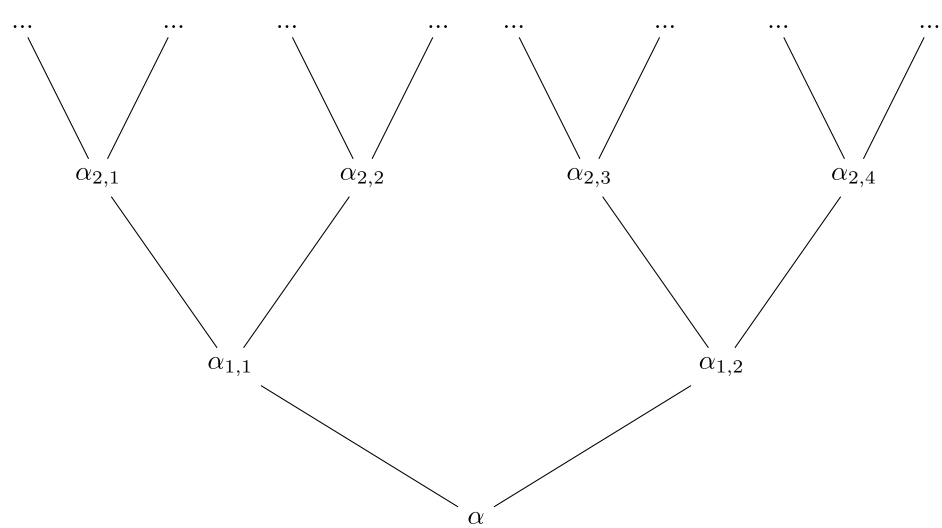
The tree of preimages of α under f, denoted $T_\infty(\alpha)$, is then defined to be the tree with vertices

$$V_\infty(\alpha) := \bigsqcup_{n \geq 0} V_n(\alpha),$$

and edges

$$E_\infty(\alpha) := \{\{\alpha_i, \alpha_j\} \in V_\infty(\alpha) \times V_\infty(\alpha) : f(\alpha_j) = \alpha_i\}.$$

In other words, $T_\infty(\alpha)$ is a rooted tree where the nth level of $T_\infty(\alpha)$ corresponds to the preimage of α under the nth iterate of f, with $f^0(\alpha) := \alpha$. Moreover, applying f to any of the vertices (except α) will yield the node directly below it. The following figure shows an example of one such tree:



We then define a tower of algebraic extensions of K as follows:

Definition 2.4.2. Let K be a field and $\alpha \in K$. Let $f(z) \in K[z]$ be a polynomial of degree $d \geq 2$ and V_n

defined as in Definition 2.4.1. We define

$$K_n := K\left(V_n\right) \quad \text{and} \quad K_\infty := \bigcup_{n \geq 0} K_n.$$

The absolute Galois group $\mathrm{Gal}(\overline{K}/K)$ acts on $T_\infty(\alpha)$ and preserve its tree structure. Studying how $\mathrm{Gal}(\overline{K}/K)$ acts on $T_\infty(\alpha)$ for different polynomials is an active area of research that we call Arboreal Galois representations. See [4, 5, 6, 7, 8, 12, 13, 14, 15] for more details.

If K is a number field, one can use the theory of height functions to show that every such field extension has infinite degree, given that the backward orbit is infinite. However, for the case when K is local, a recent paper [1] shows that the field extensions generated by a polynomial can actually be finite.

More specfically, the authors studied the fields arising from the iterates of the polynomials $f_c(z) = z^\ell - c$ over a local field K with residue characteric p. For the cases where either $p \nmid \ell$ or $p = \ell$, the authors identify a cutoff value

$$\nu_\infty = -\frac{\ell}{\ell - 1} v(\ell)$$

such that:

- for $v(c) < \nu_\infty$, the extension K_∞/K is finite,
- for $v(c) = \nu_\infty$, the extension is infinite but can be finitely ramified, depending on the the valuation of the root point $v(\alpha)$, and
- for $v(c) > \nu_\infty$, the extension is infinite with infinite ramification (infinite wild ramification if $p = \ell$).

In the following chapter, we will extend these results to all possible degrees $\ell \geq 2$.

CHAPTER 3
MAIN RESULTS

The results below come from a joint paper authored by myself with Michelle Manes and Nha Xuan Truong. It has been submitted for publication [9].

The main goal was to extend the results of [1] mentioned in Chapter 2.4 to all possible degrees $\ell \geq 2$. Our main result does exactly this.

Theorem. Suppose that $\ell \geq 2$, $(\ell, p) \neq 1$ and $c \in \overline{K}$. Set $\nu_\infty = \frac{-\ell}{\ell-1} v(\ell)$.

(i) If $v(c) < \nu_\infty$, then K_∞/K is a finite extension

(ii) If $v(c) = \nu_\infty$, then K_∞/K is an infinite extension, and it is finitely ramified if and only if $\ell = p$ and α lies within the closed unit disk centered at a fixed point of f.

(iii) If $v(c) > \nu_\infty$, then K_∞/K is infinitely wildly ramified.

The proofs in [1] were algebraic, but we take a more dynamical approach here. We noted, for example, that for $v(c) \geq \nu_\infty$, the polynomial $f_c(z) = z^\ell - c$ has potential good reduction. We thought this might be related to the extension K_∞/K, and that we could explain this relationship via the Berkovich Julia set of f_c. However, we found that when we consider all degrees $\ell \geq 2$, the story is a bit more complicated.

Writing $\ell = Np^k$ with $(p, N) = 1$, we found a second important cutoff value:

$$\nu_{\text{good}} = \begin{cases} 0 & \text{if } N > 1 \\ -\frac{p}{p-1} & \text{otherwise.} \end{cases}$$

Note that $\nu_\infty \leq \nu_{\text{good}}$, with equality when $p \nmid \ell$ or $p = \ell$ (exactly the cases considered in [1]). We find the following:

- For $v(c) < \nu_\infty$, the extension K_∞/K is finite. This can be explained by a description of the Berkovich Julia set for the polynomial f_c as a Cantor set of Type I points, all of which lie in a finite extension of K.

- For $v(c) \geq \nu_{\text{good}}$, the polynomial f_c has potential good reduction. The Berkovich Julia set is a single Type II point, and the extension $[K_\infty : K]$ is infinite (and infinitely ramified).

- For $\nu_\infty \le v(c) < \nu_{\text{good}}$, the polynomial f_c has persistent bad reduction, but the Berkovich Julia set is a Cantor set containing type II points and possibly type IV points, so it does not lie entirely in a finite extension of K. In this case, the extension K_∞/K is infinite.

We hope that this more dynamical viewpoint may allow future work on other interesting families of functions beyond the unicritical case.

3.1 Criterion for good reduction

We begin with some notation.

K	a non-archimedean field, complete with respect to a valuation v.
$\overline{K}$	a fixed algebraic closure of K.
$\mathcal{O}_K$	the ring of integers $\{\alpha \in K : v(\alpha) \ge 0\}$.
$\mathfrak{p}$	the maximal ideal of $\mathcal{O}_K$.
$\overline{a}$	the image of $a \in \mathcal{O}_K$ under the natural reduction map $\mathcal{O}_K \to \mathcal{O}_K/\mathfrak{p}\mathcal{O}_K$.
$\overline{g}$	the image of a polynomial $g \in \mathcal{O}_K[z]$ obtained by reducing the coefficients.

Let p be the characteristic of the residue field $\mathcal{O}_K/\mathfrak{p}\mathcal{O}_K$, and normalize the valuation so that $v(p) = 1$. The reduction map above induces a reduction map

$$\begin{gathered} \mathbb{P}^1(K) \to \mathbb{P}^1(\mathcal{O}_K/\mathfrak{p}\mathcal{O}_K) \\ [x : y] \mapsto [\bar{x} : \bar{y}], \end{gathered}$$

for $x, y \in \mathcal{O}_K$ chosen so that $\min\{v(x), v(y)\} = 0$. This lets us extend the reduction map to K in a natural way, and points in $\mathbb{P}^1(K) \smallsetminus \mathcal{O}_K$ have image the point at infinity.

Throughout the remainder of this dissertation, we write

$$f_c(z) = z^\ell - c \quad \text{ with } c \in \overline{K} \text{ and } \ell \ge 2.$$

Further we factor the exponent

$$\ell = Np^k, \quad \text{ with } (N, p) = 1,\ N \ge 1, \text{ and } k \ge 0.$$

Finally, we define the two key "cutoff values" for $v(c)$ that are the basis of our investigation:

$$\nu_{\text{good}} = \begin{cases} 0 & \text{if } N > 1 \\ -\frac{p}{p-1} & \text{otherwise.} \end{cases} \tag{3.1.1}$$

$$\nu_\infty = -\frac{\ell}{\ell - 1} v(\ell). \tag{3.1.2}$$

In Section 2.3, we have already introduced the notion of good reduction for monic polynomials in $\mathbb{Q}_p$. Here we rephrase the definition of what it means for a polynomial in $\mathcal{O}_K[z]$ to have good reduction.

Definition 3.1.1. Let $f(z) \in \mathcal{O}_K[z]$. We say that f has **good reduction** if $\deg(\overline{f}) = \deg(f)$, and f has **potential good reduction** if there is some $\phi(z) \in \mathrm{PGL}_2(\overline{K})$, such that $f^\phi := \phi^{-1} \circ f \circ \phi$ has good reduction. If f does not have good reduction, then it has **bad reduction**. If f does not have potential good reduction, we say it has **persistent bad reduction**.

Though the definition considers only polynomials with coefficients $\mathcal{O}_K$, it can be extended to polynomials with coefficients in the field K by considering them as rational functions with homogeneous presentations. (We refer the reader to [2, Section 4.3] for details.) In the case $f_c(z) = z^\ell - c$, we conclude that f_c has good reduction precisely when $v(c) \geq 0$.

We now give necessary and sufficient conditions, based on ℓ and the valuation of the constant term c, for f_c to have potential good reduction. The following lemma of Benedetto allows us to characterize when a polynomial has potential good reduction.

Lemma 3.1.2 ([2, Corollary 4.6])**.** *Let $f \in K[z]$ be a polynomial, and let g be a polynomial conjugate of f such that g is monic and $g(0) = 0$. Then f has potential good reduction if and only if g has good reduction.*

In other words, f has potential good reduction if and only if it is conjugate to a monic polynomial that fixes 0 and has all coefficients with non-negative valuation. Proposition 3.1.4 applies Lemma 3.1.2 to the unicritical polynomials f_c, allowing us to give a characterization of when f_c has potential good reduction solely in terms of the valuation of c. The strategy is to move one of the fixed points of f to 0, resulting in a polynomial conjugate to f and meeting the hypotheses of Benedetto's lemma.

We begin with a useful formula for the p-adic valuation of binomial coefficients, which will help us analyze the coefficients of this conjugate polynomial.

Lemma 3.1.3. *Let $\ell = Np^k$, where $p \nmid N$. For all $n \leq p^k$ we have*

$$v\left(\binom{\ell}{n}\right) = k - v(n).$$

Proof. Using the formula for binomial coefficients, we see that

$$\binom{\ell}{n} = \frac{\ell}{n}\binom{\ell-1}{n-1}.$$

It suffices to show that $v\left(\binom{\ell-1}{n-1}\right) = 0$. By Kummer's Theorem, $v\left(\binom{\ell-1}{n-1}\right)$ is the number of carries when adding $n-1$ and $\ell - n$ written in base p. We observe that in base p, the last k digits of $\ell - 1 = Np^k - 1$ are all $p-1$, so there are no such carries when $n \leq p^k$. $\square$

Proposition 3.1.4. *Let $f(z) \in K[z]$ be a separable polynomial of the form $z^\ell - c$. Then f has potential good reduction if and only if $v(c) \geq \nu_{\text{good}}$.*

Proof. If $v(c) \geq 0$, then $f(z)$ already has good reduction, so assume that $v(c) < 0$.

Let b be a fixed point of $f(z)$, so that $b^\ell - c = b$. We conjugate $f(z)$ by $\phi(z) = z + b$ to $g(z) := \phi^{-1} \circ f \circ \phi$, which satisfies $g(0) = 0$ and is monic. Then $f(z)$ has potential good reduction if and only if $g(z)$ has good reduction.

We have

$$g(z) = (z+b)^\ell - c - b = \sum_{n=1}^{\ell} \binom{\ell}{n} b^n z^{\ell - n}.$$

Thus, $g(z)$ has good reduction if and only if $v\left(\binom{\ell}{n} b^n\right) \geq 0$ for all $n = 1, \ldots, \ell$.

Since $v(c) < 0$, we have

$$0 > v(c) = v(b^\ell - b) \geq \min\{\ell v(b), v(b)\}.$$

We conclude that $v(b) < 0$ and $v(c) = \ell v(b)$.

Therefore, $g(z)$ has good reduction if and only if

$$v(c) \geq -v\left(\binom{\ell}{n}\right)\frac{\ell}{n} = -\frac{k - v(n)}{n}\ell \quad \text{for all } 1 \leq n \leq \ell. \tag{3.1.3}$$

We observe that

$$\min_{1\le n\le \ell}\left\{\frac{k-v(n)}{n}\right\} = \begin{cases} -\frac{p}{p-1} & \text{if } \ell = p^k, \text{ for } k\ge 1, \\ 0 & \text{if } \ell = Np^k \text{ for } N>1 \text{ and } (N,p)=1. \end{cases}.$$

So the condition in (3.1.3) is equivalent to $v(c) \ge \nu_{\text{good}}$. □

3.2 Newton polygon of $(z+y)^\ell - y^\ell - d$

The tools in this section allow us to extend the "general lemmas" in [1, Section 3] to exponents ℓ where $p \mid \ell$ but $p \ne \ell$. The idea is the same as in the earlier paper: We want to understand $v(x-y)$ when we know that $f_c(x) - f_c(y) = d$. To do this, we consider the Newton polygon of the polynomial $F(z) = (z+y)^\ell - y^\ell - d$ and apply our results to $z = x-y$.

The following technique will help our analysis of the Newton polygon. Suppose that we have a list of points:

$$P_i = (i, y_i) \quad \text{with} \quad y_i \in \mathbb{R}, \text{ for } i = 0,1,\ldots,n.$$

We may *shift them relative to a linear function* $y = ax+b$ (for some $a, b \in \mathbb{R}$), by transforming them into

$$Q_i = (i, y_i + ai + b) \quad \text{for} \quad i = 0,1,\ldots,n.$$

We observe that the x-coordinates of the break-points of the lower convex hulls of $P_i's$ and $Q_i's$ are exactly the same. In particular:

(i) For a fixed i_0, the point P_{i_0} is a vertex of the convex hull of the P_i's if and only if the point Q_{i_0} is a vertex of the convex hull of the Q_i's.

(ii) Let m_{AB} represent the slope of the segment from point A to point B. Then for every $i \ne j$ we have $m_{Q_iQ_j} = m_{P_iP_j} + a$.

Lemma 3.2.1. *Let $\ell = p^k$ with $k \ge 1$. Let NP be the Newton polygon of*

$$F(z) = (z+y)^\ell - y^\ell - d.$$

Let m_1 and m_ℓ be the first and last slope of NP respectively. Set $\lambda_0(y) = \infty$, $\lambda_{k+1}(y) = -\infty$, and $\lambda_n(y) =$

$k-n+\frac{p}{p-1}+\ell v(y)$ *for* $n=1,\ldots,k$. *If* $d\neq 0$, *choose* $0\leq n_0\leq k$ *so that* $\lambda_{n_0+1}(y)\leq v(d)<\lambda_{n_0}(y)$.

(i) The x*-coordinates of the vertices of NP are exactly* $0,p^{n_0},p^{n_0+1},\ldots,p^k$.

(ii) If $n_0\leq k-1$, *then*

$$m_1=\frac{k-n_0+(\ell-p^{n_0})v(y)-v(d)}{p^{n_0}}\quad \textit{and}\quad m_\ell=-\frac{p}{\ell(p-1)}-v(y).$$

(iii) If $n_0=k$, *then* $m_1=m_\ell=-\dfrac{v(d)}{\ell}$.

If $d=0$, *then the vertices of NP have* x*-coordinates* $p,p^2,\ldots,p^k$. *The first slope is* $m_1=\infty$ *and the last slope is* $m_\ell=-\frac{p}{\ell(p-1)}-v(y)$.

Proof. Write

$$F(z)=(z+y)^\ell-y^\ell-d=\sum_{n=1}^{\ell}\binom{\ell}{n}y^{\ell-n}z^n-d.$$

By Lemma 3.1.3, the Newton Polygon (NP) is the lower convex hull of the points

$$\{P_0=(0,v(d)),P_n=(n,k-v(n)+(\ell-n)v(y))\colon 1\leq n\leq\ell\}. \tag{3.2.1}$$

To determine the vertices of NP for the family of functions

$$F(z)=(z+y)^\ell-y^\ell-d,$$

as $v(d)$ and $v(y)$ vary, we shift these points by the linear function $y=v(y)x-\ell v(y)$ to obtain a new set of points:

$$\{Q_0=(0,v(d)-\ell v(y)),Q_n=(n,k-v(n))\colon 1\leq n\leq\ell\}. \tag{3.2.2}$$

As discussed earlier, the x-coordinates of the vertices of NP are precisely the x-coordinates of the vertices of the convex hull of these points. Note that the coordinates of all points, except for the first one in (3.2.2), are independent of $v(y)$ and $v(d)$.

We first consider the points in (3.2.2) excluding the Q_0:

$$\{Q_n=(n,k-v(n))\colon 1\leq n\leq\ell\}. \tag{3.2.3}$$

We claim that the lower convex hull of the points in (3.2.3) consists of

$$Q_1, Q_p, \dots, Q_{p^j}, \dots, Q_{p^k}.$$

To prove this, it suffices to show that for any $p^{n-1} < i < p^n$, the point Q_i lies above the line segment $Q_{p^{n-1}}Q_{p^n}$.

The slope of $Q_{p^{n-1}}Q_{p^n}$ is $\frac{-1}{p^n - p^{n-1}}$, and the slope of $Q_{p^{n-1}}Q_i$ is $\frac{n-1-v(i)}{i-p^{n-1}}$.

For Q_i to lie above $Q_{p^{n-1}}Q_{p^n}$, we need

$$\frac{n-1-v(i)}{i-p^{n-1}} > \frac{-1}{p^n - p^{n-1}}.$$

This inequality holds because $p^{n-1} < i < p^n$ ensures $0 < i - p^{n-1} < p^n - p^{n-1}$, and $v(i) \leq n-1$ implies $n - 1 - v(i) > -1$.

Thus, the convex hull of the points in (3.2.2) consists of

$$Q_0, Q_{p^j}, Q_{p^{j+1}} \dots, Q_{p^{k-1}}, Q_{p^k}, \text{ for some } j = 0, 1, \dots k.$$

Here, j depends on the y-coordinate of Q_0.

Switching back to the points in (3.2.1), we see that the convex hall consists of:

$$P_0, P_{p^j}, P_{p^{j+1}} \dots, P_{p^{k-1}}, P_{p^k}.$$

Consider two points in the set of points (3.2.1) where the x-coordinates are consecutive powers of p:

$$P_{p^{n-1}} = \left(p^{n-1}, k - (n-1) + (\ell - p^{n-1})v(y)\right) \text{ and } P_{p^n} = \left(p^n, k - n + (\ell - p^n)v(y)\right).$$

The y-intercept of the line between these two points is given by:

$$\lambda_n = k - n + \frac{p}{p-1} + \ell v(y).$$

Choose n_0 so that $\lambda_{n_0} \leq v(d) < \lambda_{n_0+1}$.

- If $n_0 < k$, we see that for all $i < n_0$, P_{p^i} is above the line connecting $(0, v(d))$ and $P_{p^{n_0}}$, but $P_{p^{n_0}}$ is below the line connecting $(0, v(d))$ and $P_{p^{n_0+1}}$.

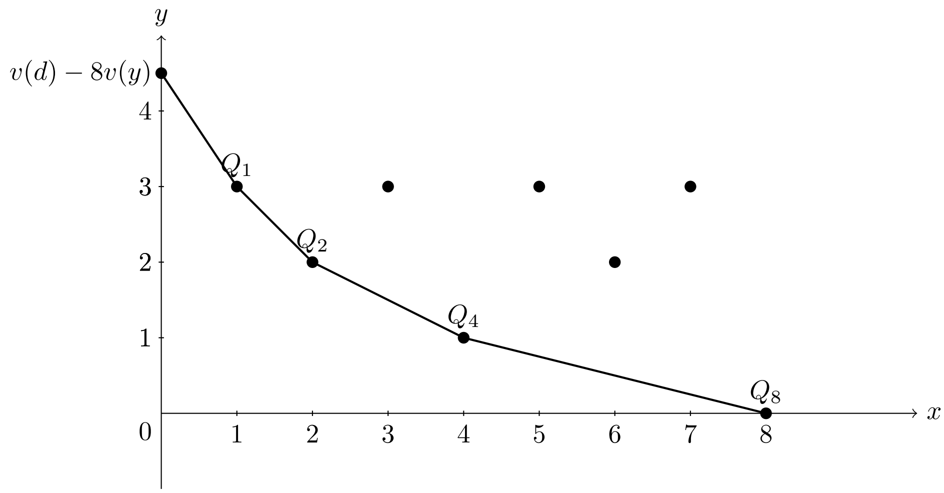
- If $n_0 = k$, then all points P_{p^i} for $i < k$ lie above the line connecting $(0, v(d))$ and P_{p^k}, so NP is a single segment.

In both cases, we see that $j = n_0$ as desired.

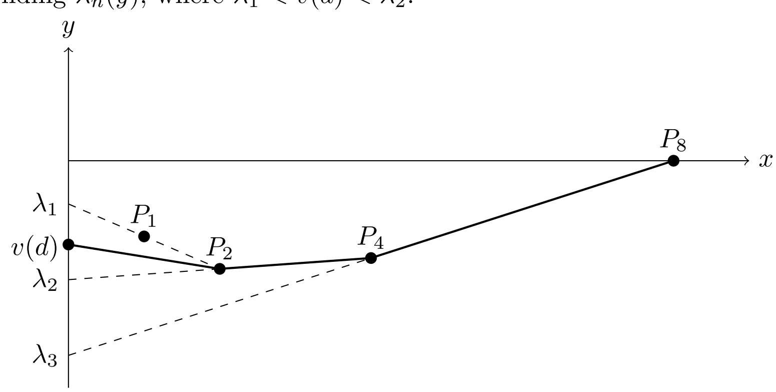
The slopes of NP given in the lemma statement are a straightforward calculation from equation (3.2.1) once we know the x-coordinates of NP.

In the case $d = 0$, we use the same reasoning to see that NP has a vertical line to the point $(p, k-1+(\ell-p)v(b))$. The slope of $m_1 = \infty$ corresponds to the root at $z = 0$. The vertices of the lower convex hull are then the points $(p^n, k-n+(\ell-p^n)v(b))$. $\square$

Example 3.2.2. The following figure illustrates the shifted Newton Polygon of $(z+y)^8 - y^8 + d$.



Example 3.2.3. The following figure illustrates the Newton Polygon of $(z+y)^8 - y^8 + d$ with the corresponding $\lambda_n(y)$, where $\lambda_1 < v(d) < \lambda_2$.



For the remaining case of ℓ, we have a similar result. The proof is essentially the same (just the last endpoint changes), so we omit the details.

Lemma 3.2.4. *Let $\ell = Np^k$ where $p \nmid N$ and $N > 1$. Let NP be the Newton polygon of*

$$F(z) = (z+y)^\ell - y^\ell - d.$$

Let m_1 and m_ℓ be the first and last slope of NP respectively.

Set $\lambda_n(y) = k - n + \frac{p}{p-1} + \ell v(y)$ for $n = 1, \ldots, k$ and $\lambda_{k+1}(y) = \ell v(y)$. Then

(i) If $d = 0$, then the vertices of NP have x-coordinates $p, p^2, \ldots, p^k, Np^k$. The first slope is $m_1 = \infty$ and the last slope is $m_\ell = -v(y)$.

(ii) If $\lambda_{n_0+1}(y) \leq v(d) < \lambda_{n_0}(y)$ for some integer $0 \leq n_0 \leq k$, then the x-coordinates of the vertices of NP are exactly $0, p^{n_0}, p^{n_0+1}, \ldots, p^k, Np^k$. and

$$m_1 = \frac{k - n_0 + (\ell - p^{n_0})v(y) - v(d)}{p^{n_0}};\ m_l = -v(y).$$

(iii) If $v(d) \leq \lambda_{k+1}(y)$, then $m_1 = m_\ell = -\frac{v(d)}{\ell}$.

Similar to Lemma 3.1 in [1], we obtain the following corollary.

Corollary 3.2.5. Let $d, y \in \overline{K}$ and let $f_c(z) = z^\ell - c$. Then

(i) If $v(d) > k - 1 + \frac{p}{p-1} + \ell v(y)$, the equation $f_c(x) - f_c(y) = d$ has at least one root $x \in K(d, y)$.

(ii) If $\ell = p^k$ where $k \geq 1$ and $v(d) < \frac{p}{p-1} + \ell v(y)$, all the roots x of the equation $f_c(x) - f_c(y) = d$ have valuation $\frac{v(d)}{\ell}$.

(iii) If $\ell = Np^k$ where $(N, p) = 1$ and $N > 1$, and $v(d) < \ell v(y)$, all the roots x of the equation $f_c(x) - f_c(y) = d$ have valuation $\frac{v(d)}{\ell}$.

Proof. Let $z = x - y$ then $f(x) - f(y) - d = (z+y)^\ell - y^\ell - d$. The result follows from Lemma 3.2.1 and Lemma 3.2.4. $\square$

3.3 Julia sets of $z^\ell - c$

In this section, we describe the Berkovich filled Julia set of the polynomial $f(z) = z^\ell - c$ for $\ell \geq 2$ following the ideas in [3]. First, we recall the definition of the *Weierstrass degree* of a power series on a closed disk, which is helpful in determining the image of a power series (in our case, the polynomial $f_c(z)$) on a closed disk.

Definition 3.3.1. Let $F(z) = \sum_{n\geq 0} c_n(z-a)^n$ be a nonzero power series converging on a closed disk $\bar{D} = \bar{D}(a,r)$ for $a \in \mathbb{C}_v$ and $r > 0$. The Weierstrass degree $\mathrm{wdeg}_{\bar{D}}(F)$ is the largest integer $d \geq 0$ such that $|c_d|r^d = \max_{n\geq 0} |c_n|r^n$.

Theorem 3.3.2 ([3, Theorem 3.15])**.** *Let $\bar{D} \subset \mathbb{C}_v$ be a closed disk of radius $r > 0$ containing a point $a \in \mathbb{C}_v$, and let $F(z) = \sum_{n\geq 0} c_n(z-a)^n$ be a nonconstant power series converging on $\bar{D}$. Let $d := \mathrm{wdeg}_{\bar{D}}(F-c_0)$, and let $t := |c_d|r^d > 0$. Then $F(\bar{D})$ is a closed disk, centered at c_0, and of radius t. Moreover, $F : \bar{D} \longrightarrow F(\bar{D})$ is everywhere d-to-1, counting multiplicity.*

Lemma 3.3.3 ([3, Lemma 14.8])**.** *Let $\bar{D}, \bar{E} \subset \mathbb{C}_v$ be closed disks, let $a \in \bar{E}$, let F be a power series converging on $\bar{E}$, and suppose that $F(a) \in \bar{D} \subset F(\bar{E})$. Then there is a unique closed disk $\bar{U} \in \bar{E}$ with $a \in \bar{U}$ such that $F(\bar{U}) = \bar{D}$.*

Corollary 3.3.4. Let $F(z) = \sum_{i\geq 1}^{n} c_i z^i$ be a polynomial, and let $\bar{D} = \bar{D}(0,R)$ be a disk centered at 0 with radius $R > 0$. Then in $F^{-1}(\bar{D})$, the disk U containing 0 has radius $r = \left(\frac{R}{|c_m|}\right)^{1/m}$ where $m = \mathrm{wdeg}_{\bar{U}}(F)$.

Proof. Let $\bar{E}$ be a closed disk with $0 \in \bar{E}$ and radius r_E. Since $F(0) = 0$, Theorem 3.3.2 says $F(\bar{E})$ is a closed disk containing 0 of radius $|c_d|r_E^d$ for some $d \geq 1$. So we may choose r_E large enough that $\bar{D} \subseteq F(\bar{E})$. By Lemma 3.3.3, there is a unique closed disk $\bar{U} \subseteq \bar{E}$ with $0 \in \bar{U}$ and $F(\bar{U}) = \bar{D}$. From Theorem 3.3.2, the radius r of $\bar{U}$ must satisfy $R = |c_m|r^m$. ☐

Let $b \in K$ be a fixed point of $f_c(z)$. As in the proof of Theorem 3.1.4, conjugate f_c by a map that moves b to 0 to obtain

$$g(z) = (z+b)^\ell - b - c = \sum_{n=1}^{\ell} a_n z^n \text{ where } a_n = \binom{\ell}{n} b^{\ell-n}. \tag{3.3.1}$$

Since $g(0) = 0$, we have $b^\ell - b = c$, which will be useful in this section.

Notation 3.3.5. Let $r_0 = 0$; for $1 \leq n \leq k$, let $r_n = p^{(-1/(p^n - p^{n-1})) - v(b)}$; and let

$$r_{k+1} = \begin{cases} \infty & \text{if } N = 1 \text{ and } k \geq 1, \\ p^{-v(b)} & \text{if } N > 1. \end{cases}$$

We have the following lemma on the Weierstrass degree of $g(z)$ on disks of various radii. It follows from the notation above that Case (ii) never happens when $N = 1$.

Lemma 3.3.6. *Continue with the notation in 3.3.5 and let $r \in \mathbb{R}_{\geq 0}$.*

(i) If there is some n so that $r_n \leq r < r_{n+1}$, then $\mathrm{wdeg}_{\bar{D}(0,r)}(g) = p^n$.

(ii) If $r_{k+1} \leq r$, then $\mathrm{wdeg}_{\bar{D}(0,r)} = \ell$.

Proof. The Newton polygon of $g(z)$ is described in Lemma 3.2.1, using $y = b$ and $d = 0$. A calculation shows that the slope of the n^{th} segment (between the points with x-coordinates p^{n-1} and p^n) is given by $\log_p r_n$.

Hence for $r_n \leq r < r_{n+1}$, the disk $\bar{D}(0,r)$ contains exactly p^n of the roots of $g(z)$. We apply [3, Theorem 3.13(a)] to conclude that $\mathrm{wdeg}_{\bar{D}(0,r)}(g) = p^n$. If $r_{k+1} \leq r$, all the roots of g would be contained by $\bar{D}(0,r)$, hence $\mathrm{wdeg}_{\bar{D}(0,r)}(g) = \ell$. $\square$

Let F be a polynomial of degree $d \geq 2$. Recall from Chapter 2.3 that the Berkovich filled Julia set of F is

$$\mathcal{K}_{F,\text{an}} = \{\zeta \in \mathbb{P}^1_{\text{an}} : \lim_{n\to\infty} F^n(\zeta) \neq \infty\}.$$

The Berkovich Julia set, denoted $\mathcal{J}_{F,\text{an}}$, is the boundary of $\mathcal{K}_{F,\text{an}}$.

Proposition 3.3.7. *The Berkovich Julia set $\mathcal{J}_{g,\text{an}}$ is the singleton set $\{\zeta(0,1)\}$ if and only if $v(c) \geq \nu_{\text{good}}$.*

Proof. By Proposition 3.1.4, the hypothesis gives that f_c has potential good reduction, so g has good reduction. By [3, Proposition 8.12], the Julia set for a map with good reduction is just the Gauss point. $\square$

It remains to understand the Berkovich Julia set of g when g has bad reduction; that is when $v(c) < \nu_{\text{good}}$.

Let R_0 be the radius of the smallest disk centered at 0 containing all the roots of g. From the Newton polygon of g, we conclude:

$$R_0 = \begin{cases} p^{(-1/(p^k - p^{k-1})) - v(b)} & \text{if } N = 1 \text{ and } k \geq 1, \\ p^{-v(b)} & \text{if } N > 1. \end{cases}$$

By Lemma 3.3.6, for $|z| > R_0$, z^ℓ is the dominant term in $g(z)$, so $|g(z)| = |z|^\ell$. Since $v(c) < \nu_{\text{good}} \leq 0$ and $b^\ell - b = c$, necessarily $v(b) < 0$. It follows that $R_0 > 1$, so $\lim_{n\to\infty} g^n(z) = \infty$ for $|z| > R_0$. Therefore, the Berkovich filled Julia set $\mathcal{K}_{g,\text{an}}$ is contained in the closed disk $\bar{D}(0, R_0) := U_0$.

We consider the preimages of U_0, denoted as $U_m := g^{-m}(U_0)$, for $m \geq 0$. We prove the following properties of U_m.

Lemma 3.3.8. *Suppose $c \in \overline{K}$ with $v(c) < \nu_{\text{good}}$.*

(i) For each $\alpha_m \in g^{-m}(0)$, we have $v(\alpha_m + b) = v(b)$ and $v(\alpha_m) \geq v(b)$.

(ii) Each U_m is the disjoint union of finitely many disks of the same radius.

(iii) U_m *is nested, i.e*

$$U_0 \supsetneq U_1 \supsetneq U_2 \supsetneq \ldots.$$

(iv) The intersection $\bigcap_{i\geq 0} U_i$ *is the disjoint union of disks of radius* R *for some* $R \geq 0$.

Proof. We prove (1) and (2) by induction on m. Since $b^\ell - b = c$ and $v(c) < 0$, we see that $v(b) = v(c)/\ell > v(c)$.

For $\alpha_1 \in g^{-1}(0)$, since $g(\alpha_1) = (\alpha_1 + b)^\ell - b - c = 0$, then

$$v(\alpha_1 + b) \;=\; v(b+c)/\ell \;\geq\; \min\{v(b), v(c)\}/\ell \;=\; v(c)/\ell \;=\; v(b).$$

Thus $v(\alpha_1 + b) = v(b)$ and $v(\alpha_1) \geq v(b)$.

Now let $\alpha_m \in g^{-1}(\alpha_{m-1})$. By the induction hypothesis,

$$v(b) \leq v(\alpha_{m-1}) = v((\alpha_m + b)^\ell - b - c).$$

Since $v(c) < v(b)$, it follows that $v(\alpha_m + b)^\ell = v(c)$, so $v(\alpha_m + b) = v(b)$ and $v(\alpha_m) \geq v(b)$.

For statement (2), since $g^{-m}(0)$ is a finite set, there will be finitely many disjoint disks in $g^{-m}(U_0)$. We need to show that the radii of these disks are all equal. For $m = 0$, the statement is obvious. Now assume it is true for $m - 1$.

For all $0 \leq m$, let R_m be the radius of the disk in $g^{-m}(U_0)$ centered at 0, and let $\alpha_m \in g^{-m}(0)$. We want to show that

$$g(\bar{D}(\alpha_m, R_m)) = \bar{D}(g(\alpha_m), R_{m-1})).$$

We can recenter the polynomial $g(z)$ around α_m

$$g(z) = (z - \alpha_m + \alpha_m + b)^\ell - b - c = \sum_{n=1}^{\ell} \binom{\ell}{n} (\alpha_m + b)^{\ell-n} (z - \alpha_m)^n + g(\alpha_m).$$

By part (1), $v(\alpha_m + b) = v(b)$. Hence, $v\left(\binom{\ell}{n}(\alpha_m + b)^{\ell-n}\right) = v(a_n)$, where a_n is defined as in equation (3.3.1). Let $c_0 := g(\alpha_m)$ and $c_n := \binom{\ell}{n}(\alpha_m + b)^{\ell-n}$ for all $n \geq 1$. We have

$$|a_n| {R_m}^n = |c_n| {R_m}^n$$

for all $n \geq 1$. Therefore

$$\mathrm{wdeg}_{\bar{D}(0,R_m)}(g(z) - a_0) = \mathrm{wdeg}_{\bar{D}(\alpha_m,R_m)}(g(z) - c_0).$$

Therefore by Theorem 4.2,

$$g(\bar{D}(\alpha_m, R_m)) = \bar{D}(g(\alpha_m), R_{m-1})).$$

Thus in the preimage $g^{-m}(U_0)$, the disks centered at 0 and α_m have the same radius.

To prove statement (3), it suffices to consider those disks centered at 0. Recall that $U_0 = \bar{D}(0, R_0)$ with

$$v(R_0) = \begin{cases} \frac{p}{l(p-1)} + \frac{v(c)}{\ell} & \text{if } N = 1 \text{ and } k \geq 1, \\ \frac{v(c)}{\ell} & \text{if } N > 1. \end{cases}$$

Let $D(0,r)$ be a disk in $U_1 = g^{-1}(U_0))$ such that $g(D(0,r)) = U_0$. Assume that $r \geq R_0$, then by Lemma 3.3.6, $\mathrm{wdeg}_{\bar{D}(0,r)}(g) = \ell$, hence $R_0 = r^\ell$ by Theorem 3.3.2. The condition $v(c) < \nu_{\mathrm{good}}$ implies that $R_0 > 1$, then $R_0 = r^\ell > r$. This contradiction shows that $U_1 \subsetneq U_0$. For each $k \geq 1$, we observe that $U_{k+1} = f^{-k}(U_1), U_k = f^{-k}(U_0)$, and $U_1 \subsetneq U_0$. Therefore, $U_{k+1} \subsetneq U_k$. This proves (3).

For part (4), let R_m denote the radius of each disk in U_m. By part (3), the sequence $\{R_m\}$ is decreasing and convergent, with $R = \lim_{m\to\infty} R_m$.

In the Berkovich space, every point of $\bigcap_{m\geq 1} U_m$ is either a type II point with radius R or a type IV point. In particular, since $0 \in \bigcap_{m\geq 1} U_m$, we know $\bigcap_{m\geq 1} U_m$ contains $\overline{D}(0, R)$. $\square$

The value of the radius R depends on the value of $v(c)$. We use the following notation.

Notation 3.3.9. Let $c_0 = -\infty$; for $n = 1, \ldots, k$, let $c_n := \frac{-\ell}{\ell-1}(k - n + \frac{p^n-1}{p^n-p^{n-1}})$; in the case $N > 1$, let $c_{k+1} = 0$.

Note that $c_0 < c_1 < \cdots < c_k < c_{k+1}$, and we have

$$\nu_{\mathrm{good}} = \begin{cases} c_k & \text{if } N = 1, \\ c_{k+1} & \text{if } N > 1. \end{cases}$$

The values of c_n are chosen due to the following lemma.

Lemma 3.3.10. *Let $\{a_n\}$ be the coefficients of $g(z)$ in equation* (3.3.1) *and let R be the radius R in Lemma 3.3.8*(4)*. Suppose $c_n \leq v(c) < c_{n+1}$ for some $n = 0, 1, \ldots, k$.*

(i) If $n \geq 1$, then $R = |a_{p^n}|^{-1/(p^n-1)}$.

(ii) If $n = 0$, then $R = 0$.

Proof. Define the function $T : \mathbb{R}^+ \to \mathbb{R} \cup \{\infty\}$ by

$$T(s) = ||g(z)||_{\bar{D}(0,s)} = \max_{n \geq 0} |a_n| s^n.$$

Continuing with the notation in the proof of Lemma 3.3.8, let R_m be the radius of each disjoint disk in $g^{-m}(U_0)$. By Corollary 3.3.4, we have

$$T(R_m) = R_{m-1}.$$

Since $T(s)$ is a continuous function, taking limit on both sides yields $T(R) = R$.

Using Notation 3.3.5, let n_0 be the integer such that $r_{n_0} \leq R < r_{n_0+1}$. Then

$$R = T(R) = |a_{p^{n_0}}| R^{p^{n_0}}.$$

If $n_0 \neq 0$, then $R = |a_{p^{n_0}}|^{-1/(p^{n_0}-1)}$. We prove that for $n \geq 1$

$$r_n \leq |a_{p^n}|^{-1/(p^n-1)} < r_{n+1} \quad \text{if and only if} \quad c_n \leq v(c) < c_{n+1}, \tag{3.3.2}$$

which will prove part (1).

The left inequalities in (3.3.2) are equivalent to

$$v(r_n) \leq \frac{v(a_{p^n})}{p^n - 1} < v(r_{n+1}). \tag{3.3.3}$$

By Lemma 3.1.3, we have

$$v(a_{p^n}) = v\left(\binom{\ell}{p^n} b^{\ell - p^n}\right) = k - n + \frac{(\ell - p^n)}{\ell} v(c).$$

Combining this with the definition of r_n and r_{n+1} for $n < k$, equation (3.3.3) becomes

$$-\frac{p^n - 1}{p^n - p^{n-1}} - \frac{p^n - 1}{\ell} v(c) \leq k - n + \frac{(\ell - p^n)}{\ell} v(c) < -\frac{p^n - 1}{p^{n+1} - p^n} - \frac{p^n - 1}{\ell} v(c).$$

$$-\frac{\ell}{\ell-1}\left(k-n+\frac{p^n-1}{p^n-p^{n-1}}\right) \leq v(c) < -\frac{\ell}{\ell-1}\left(k-n+\frac{p^n-1}{p^{n+1}-p^n}\right).$$

By Notation 3.3.9, if $n < k$ this is equivalent to $c_n \leq v(c) < c_{n+1}$.

If $n = k$, the inequalities in (3.3.3) become

$$-\frac{p^k-1}{p^k-p^{k-1}} - \frac{p^k-1}{\ell}v(c) \leq \frac{(\ell-p^k)}{\ell}v(c) < -\frac{p^k-1}{\ell}v(c),$$

which is equivalent to $c_k \leq v(c) < c_{k+1} = 0$.

We are left with the case $n_0 = 0$, so that $R = |a_1|R$. We need to show that $|a_1| \neq 1$. Note that if $v(c) \geq c_1$, then there is some $n_0 > 0$ such that $c_{n_0} \leq v(c) < c_{n_0+1}$, which is impossible by the chain of equivalences above. So we must have $v(c) < c_1 = -\frac{\ell k}{\ell-1}$. Therefore

$$v(a_1) = k - 0 + \frac{(\ell-1)}{\ell}v(c) < k - k = 0.$$

We conclude that $|a_1| \neq 1$, and thus $R = 0$. □

By Lemmas 3.3.8 and 3.3.10, we deduce that the set $\mathcal{K}_{g,\mathrm{an}}$ is homeomorphic to a Cantor set of type I points if and only if $n_0 = 0$. In this case, we have $\mathcal{K}_{g,\mathrm{an}} = \mathcal{J}_{g,\mathrm{an}}$ We sum up our result in the following theorem.

Theorem 3.3.11. *Suppose that $\ell \in \mathbb{Z}$ and $c \in \bar{K}$. Set*

$$\nu_\infty = \frac{-\ell}{\ell-1}v(\ell) \quad \text{and} \quad \nu_{\mathrm{good}} = \begin{cases} -\frac{p}{p-1} & \text{if } \ell = p^k, \text{ for } k > 1, \\ 0 & \text{if } \ell = Np^k \text{ for } N > 1. \end{cases}$$

(i) If $v(c) < \nu_\infty$, then $\mathcal{J}_{g,\mathrm{an}}$ is homeomorphic to a Cantor set of type I points.

(ii) If $\nu_\infty \leq v(c) < \nu_{\mathrm{good}}$, then $\mathcal{J}_{g,\mathrm{an}}$ is homeomorphic to a Cantor set containing a Cantor set containing type II points and possibly type IV points.

(iii) If $v(c) \geq \nu_{\mathrm{good}}$, then $\mathcal{J}_{g,\mathrm{an}}$ is a single type II point.

Remark 3.3.12. (i) When $v(c) < \nu_\infty$ the Berkovich Julia set coincides with the classical Julia set. We show in the next section that it is necessary and sufficient condition that the extension K_∞/K is finite.

(ii) When $v(c) \geq \nu_{\mathrm{good}}$, the polynomial $g(z)$ has good reduction as mentioned in previous section.

(iii) In [1], the authors consider only the cases when $p \nmid \ell$ or $\ell = p$ (so $k = 1$). In these cases, the cutoff valuations coincide, i.e $\nu_\infty = \nu_{\text{good}}$.

3.4 The extension K_∞

In this section, we describe how the extension K_∞/K varies with $v(c)$. In [1, Corollary 4.4], the authors prove that if $p \nmid \ell$ or $p = \ell$ and $v(c) < \nu_\infty$, then K_∞ is a finite extension of K. The authors prove this by examining directly the valuation $v(\alpha_n)$ such that $f^n(\alpha_n) = \alpha$.

Here we use the results of Section 3.3 to recover this result and extend to all values of $\ell \geq 2$. For simplicity, in this section, we prove the case $\ell = p^k$. The general case $\ell = Np^k$ is similar, requiring only the selection of the last nonzero slope.

3.4.1 Sufficiently Negative Values: $v(c) < v_\infty$

We begin by extending the result that for sufficiently negative values of $v(c)$, the extension K_∞/K is in fact finite. For this, we prove that the Julia set $\mathcal{J}_{g,\text{an}}$ is contained in a finite extension of K, and then we apply Krasner's Lemma [11] to deduce the same is true for K_∞.

Lemma 3.4.1 (Krasner's Lemma). *Let K be a complete non-archimedean field and K^{sep} a separable closure of K. Given an element $\alpha \in K^{sep}$, denote its Galois conjugates by $\alpha = \alpha_1, \alpha_2, \dots, \alpha_n$. Then if $\beta \in K^{sep}$ satisfies $v(\alpha - \beta) > v(\alpha - \alpha_i)$ for $i \neq 1$ then $K(\alpha) \subseteq K(\beta)$.*

Proposition 3.4.2. *Suppose that $\ell \geq 2$ and $c \in \overline{K}$. If $v(c) < \nu_\infty$, then $\mathcal{J}_{f,\text{an}}$ is contained in a finite extension of K.*

Proof. Assume $v(c) < \nu_\infty$ and let b be a fixed point of f. Since $g(z) = f_c(z+b) - b$, the Julia sets of f and g are just translates of each other. We consider the field extensions $K_n = K(f^{-n}(b))$, and $K_\infty = \cup_{n\geq 0} K_n$. (So here we take $\alpha = b$ to construct the field extension K_∞.)

By Theorem 3.3.11(1), the Julia set $\mathcal{J}_{f,\text{an}}$ contains only type I points. Let R_m be the radius of the disk in $g^{-m}(U_0)$ centered at 0, as in the proof of Lemma 3.3.8. Since $\lim_{m\to\infty} R_m = 0$, we choose n such that $R_n < r_1$.

By Lemma 3.3.6(1) and Lemma 3.3.8, we have $\text{wdeg}_{\bar{D}(\alpha_m, R_m)}(g) = 1$ for all $m \geq n$, $\alpha_m \in g^{-m}(0)$.

We prove by induction that $K_m = K_n$ for all $m > n$. Let $\alpha_{n+1} \in K_{n+1}$. Then $\alpha_{n+1} \in U_n$, and hence $\alpha_{n+1} \in \bar{D}({\alpha_n}', R_n)$ for some ${\alpha_n}' \in K_n$.

Since $\mathrm{wdeg}_{\bar{D}(\alpha_n', R_n)} = 1$, Theorem 3.3.2 implies g maps $\bar{D}(\alpha_n', R_n)$ one-to-one onto its image. Therefore, α_{n+1} is the unique root of the polynomial $g(z) - g(\alpha_{n+1}) \in K_n[z]$ in the disk $\bar{D}(\alpha_n', R_n)$. By the uniqueness of α_{n+1}, we have

$$v(\alpha_{n+1} - \alpha_n') > v(\alpha_{n+1} - \alpha_{n+1,i})$$

for all Galois conjugates $\alpha_{n+1,i}$ of α_{n+1} over the complete non-Archimedean field K_n.

By Krasner's Lemma, we have $\alpha_{n+1} \in K_n(\alpha_n') = K_n$.

Let $m > n$ and assume that $K_m = K_n$. Let $\alpha_{m+1} \in K_{m+1}$. Repeating the argument above with n replaced by m, we conclude that $K_m(\alpha_{m+1}) = K_m$, and hence $K_{m+1} = K_m$.

By induction, $K_m = K_n$ for all $m > n$, and thus $K_\infty = K_n$.

Finally, by [3, Proposition 5.23],

$$\mathcal{J}_{g,\mathrm{an}} = \overline{\cup_{n\geq 0} g^{-n}(0)} \subset \overline{K_n}.$$

Since K is complete and K_n is a finite extension of K, K_n is complete, and hence $\overline{K_n} = K_n$. Therefore,

$$\mathcal{J}_{f,\mathrm{an}} \subset K_n(b),$$

which is a finite extension of K. □

Proposition 3.4.2 says that if our root point α is in the Julia set of f, then K_∞/K is a finite extension. It remains to extend this to an arbitrary root point $\alpha \in K$.

Theorem 3.4.3. *Suppose that $\ell \geq 2$ and $c \in \overline{K}$. If $v(c) < \nu_\infty$, then K_∞/K is a finite extension.*

Proof. Let $\alpha \in K$ and let $K_n = K(f^{-n}(\alpha))$. To show that $K_\infty = \cup_{n\geq 0} K_n$ is a finite extension over K, it is sufficient for us to show that all the solutions of $g^n(z) = \alpha - b$ lie in $D(0, R_0)$ for sufficiently large n.

By Lemma 3.1, if $v(\alpha - b) < \lambda_k(b)$, then for any x such that $g(x) = \alpha - b$, $v(x) = \frac{v(\alpha-b)}{\ell}$, hence $v(x) \geq \lambda_k(b)$ for sufficiently large n and hence all solutions lie within $D(0, R_0)$, i.e. all solutions of $f^n(z) = \alpha$ lie within $D(b, R_0)$. □

3.4.2 Insufficiently Negative Values: $v_\infty \leq v(c) < 0$

. We recall the following notation from Section 3.2:

$$\lambda_n(y) = k - n + \frac{p}{p-1} + \ell v(y),$$

which we will apply here in the case that $y = b$ (a fixed point of f_c) so $v(y) = v(b) = v(c)/\ell$.

The following lemma says that we can further assume that $v(\alpha_n) = v(c)/\ell$ for all $n \geq 1$ and all $\alpha_n \in f^{-n}(\alpha)$.

Lemma 3.4.4 ([1, Lemma 3.2])**.** *Suppose that $v(c) < 0$ and $\alpha_n \in f^{-n}(\alpha)$. If n is sufficiently large, then $v(\alpha_n) = v(c)/\ell$. If $v(\alpha) > v(c)$, then this conclusion holds for all $n \geq 1$.*

Theorem 3.4.5. *Suppose that $\ell \geq 2$ and $\ell \neq p$. Then if $\nu_\infty \leq v(c) < 0$, then K_∞/K is infinitely wildly ramified.*

Proof. First, we begin with the case that $\nu_\infty \leq v(c) < -\frac{p}{p-1}$. By Lemma 3.3.10 , we have

$$\lim_{m\to\infty} R_m = |a_{p^n}|^{-1/(p^n-1)} \text{ for some } n \geq 1.$$

Let $\alpha_0 = \alpha$ and choose $\alpha_1 \in f^{-1}(\alpha_0)$ to be the furthest from α_0. Then for $n \geq 2$, we choose inductively $\alpha_{n+1} \in f^{-1}(\alpha_n)$ to be the closest to α_n. Let $d_n = \alpha_{n+1} - \alpha_n$.

Since $v(c) < -\frac{p}{p-1}$, it follows that $\lambda_k(b) < 0$. We observe that if $v(d_0) \geq \lambda_k(b)$, then we are in Case (2) of Lemma 3.2.1. We conclude that the furthest preimage α_1 has valuation given by the first slope m_ℓ, so

$$v(d_1) = \frac{p}{\ell(p-1)} + \frac{v(c)}{\ell}.$$

Otherwise, if $v(d_0) < \lambda_k(b)$, then by Corollary 3.2.5,

$$v(d_1) = \frac{v(d_0)}{\ell}.$$

Since $v(d_0) < \lambda_k(b) < 0$, we see that $v(d_n) > \lambda_k(b)$ for n large enough. Then

$$v(d_{n+1}) = \frac{p}{\ell(p-1)} + \frac{v(c)}{\ell}.$$

By relabeling, we can assume that $v(d_0) = \frac{p}{\ell(p-1)} + \frac{v(c)}{\ell}$. Note that $v(d_0) = v(R_0)$ where R_0 is defined in Lemma 3.3.8.

We will prove by induction that $v(d_n) = v(R_n)$ for all $n \in \mathbb{Z}$. We already know

$$v(d_0) = v(R_0) = \frac{p}{\ell(p-1)} + \frac{v(c)}{\ell}.$$

Assume that $v(d_m) = v(R_m)$ for all $0 \leq m < n$. we make the following 2 claims:

(i) $-v(d_n)$ is the slope of the first segment of the Newton Polygon of the polynomial $(z+\alpha_n)^\ell - {\alpha_n}^\ell - d_{n-1}$.

(ii) $-v(R_n)$ is the slope of the first segment of the Newton Polygon of the polynomial $(z+b)^\ell - b^\ell - R_{n-1}$.

To prove (1), notice that for all $\beta \in f^{-1}(\alpha_n)$, the difference $\beta - \alpha_n$ satisfies the polynomial

$$(z+\alpha_n)^\ell - \alpha_n^\ell - d_{n-1}.$$

Since α_{n+1} is the root of $f(z) = \alpha_n$ closest to α_n, the slope $-v(d_n)$ corresponds to the first segment of the Newton Polygon of the polynomial $(z+\alpha_n)^\ell - \alpha_n^\ell - d_{n-1}$.

For (2), we write

$$(z+b)^\ell - b^\ell - R_{n-1} = \sum_{n=1}^{\ell} a_n z^n - R_{n-1}, \text{ where } a_n = \binom{\ell}{n} b^{\ell-n}.$$

The Newton Polygon of this polynomial is the lower convex hull of the points

$$\{(n, v(a_n)) : 1 \leq n \leq \ell\} \cup \{(0, v(R_{n-1}))\}.$$

Therefore the slope of the first segment of the Newton Polygon is given by

$$\min_{1\leq m\leq \ell} \frac{v(a_m) - v(R_{n-1})}{m} = -\max_{1\leq m\leq \ell} v((\frac{R_{n-1}}{a_m})^{\frac{1}{m}}) = -v(R_n).$$

The last equality follows from Corollary 3.3.4.

Since $v(\alpha_n) = v(b) = \frac{v(c)}{\ell}$ and $v(d_{n-1}) = v(R_{n-1})$, the polynomials $(z+\alpha_n)^\ell - {\alpha_n}^\ell - d_{n-1}$ and $(z+b)^\ell - b^\ell - R_{n-1}$ have the same Newton Polygon and hence

$$v(d_n) = v(R_n).$$

Then by Lemma 3.3.10, we have

$$\lim_{n\to\infty} v(d_n) = -\frac{v(a_{p^{n_0}})}{p^{n_0} - 1} \text{ for some } n_0 = 1, 2, \ldots, k.$$

and there is an integer m such that for $n \geq m$ we have

$$\lambda_{n_0+1} \leq v(d_n) < \lambda_{n_0}.$$

Setting $q = p^{n_0}$ and using Lemma 3.2.1 , we compute $v(d_n)$ recursively and deduce that

$$\begin{aligned} v(d_{m+n}) &= \frac{v(d_m)}{q^n} - \left(\frac{1}{q} + \frac{1}{q^2} + \cdots + \frac{1}{q^n}\right) v(a_q) \\ &= \frac{v(d_m)}{q^n} - \frac{q^n - 1}{q^n(q-1)} v(a_q). \end{aligned}$$

Since

$$\lim_{n\to\infty} v(d_{m+n}) = -\frac{a_q}{q-1},$$

the exponent of p in the denominator of $v(d_{m+n})$ in the reduced form is unbounded as n increases, hence K_∞/K is infinitely wildly ramified.

For the case $-\frac{p}{p-1} \leq v(c) < 0$. Consider two subcases:

(i) If $v(d_0) \leq \lambda_k$, then by Corollary 3.2.5, it is easy to show that for any $n \geq 0$, we have

$$v(d_n) \leq \lambda_k \text{ and } v(d_n) = \frac{v(d_0)}{\ell^n}.$$

(ii) If $v(d_0) > \lambda_k$, then $v(d_1) = \frac{p}{\ell(p-1)} + \frac{v(c)}{\ell} \leq \frac{p}{p-1} + v(c) = \lambda_k$. Then in this case we have

$$v(d_n) = \frac{v(d_1)}{\ell^{n-1}}, \text{ for } n \geq 1.$$

In both cases, the extension K_∞/K is infinitely wildly ramified. □

Theorem 3.4.6. *Suppose that $\ell \geq 2$, $(\ell, p) \neq 1$, and $c \in \overline{K}$. If $v(c) = \nu_\infty$, then K_∞/K is an infinite extension, and it is finitely ramified if and only if $\ell = p$ and α lies in the closed unit disk centered at a fixed point of f.*

Proof. The result for $k = 1$ is proved in [1, Theorem 1.3]. Suppose $k \geq 2$, then $\lambda_k = \frac{p}{p-1} + v(c) = \frac{p}{p-1} - \frac{k\ell}{\ell-1} < 0$. Using the same argument in Theorem 3.4.5, we deduce that K_∞/K is infinitely wildly ramified for all α. □

3.4.3 Nonnegative Valuation: $v(c) \geq 0$

In this case, we follow the argument presented in [1]. For completeness, we provide the proof below.

Theorem 3.4.7. *Let $\ell \geq 2$ and $\ell \neq p$. If $v(c) \geq 0$, then the extension K_∞/K is infinitely wildly ramified.*

Proof. Fix a sequence (α_n) such that $\alpha_0 = \alpha$ and $\alpha_{n+1} \in f^{-1}(\alpha_n)$. Select $\beta_0 = \alpha$, and $\beta_{n+1} \in f^{-1}(\beta_n)$. We choose $\beta_1 \in f^{-1}(\beta_0)$ to be the element furthest from α_1. Set $d_n = \beta_n - \alpha_n$.

First we consider the case $v(\alpha) \neq v(c)$. If $\min(v(\alpha), v(c)) \neq 0$, then $v(\alpha_1) = \min\{v(\alpha), v(c)\}/\ell$. By induction, $v(\alpha_n) = \min\{v(\alpha), v(c)\}/\ell^n$. Since $(\ell, p) \neq 1$, it follows that K_∞/K is infinitely wildly ramified.

Now, suppose $\min(v(\alpha), v(c)) = 0$. We first consider the cases $v(c) > v(\alpha) = 0$ and $v(\alpha) > v(c) = 0$.

If $v(c) > v(\alpha) = 0$, then $v(\alpha_n) = v(\beta_n) = 0$ for all n. Applying Lemma 3.2.1 with $d = 0$ and $y = \beta_1$, we deduce that

$$v(d_1) = \frac{p}{\ell(p-1)} < \lambda_k.$$

By induction, $v(d_n) < \lambda_k$ for all $n \geq 1$, and thus

$$v(d_n) = \frac{v(d_1)}{\ell^n} = \frac{p}{\ell^{n+1}(p-1)}.$$

Consequently, K_∞/K is infinitely wildly ramified.

In the case $v(\alpha) > v(c) = 0$, we have $v(\alpha_1) = 0$. Relabeling, we may assume $v(\alpha) = 0$ and consider this case with the remaining case $v(\alpha) = v(c) \geq 0$.

Assume $v(\alpha_n) = v(c)$ for all n. If this does not hold, we reduce to one of the previous cases. Since $d_0 = 0$, Lemma 3.2.1 implies

$$v(d_1) = \frac{p}{\ell(p-1)} + v(c) < \lambda_k = \frac{p}{p-1} + \ell v(c).$$

By induction, $v(d_n) < \lambda_k$ for all $n \geq 1$. Hence,

$$v(d_n) = \frac{v(d_1)}{\ell^n} = \left(\frac{p}{\ell(p-1)} + v(c)\right)/\ell^n.$$

It follows that K_∞/K is infinitely wildly ramified. $\square$

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