SPARSE ORDINARY GRAPHS

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ABSTRACT

Ordinary graphs are directed graphs that can be viewed as generalizations of symmetric block designs. They were introduced by Fossorier, Jezek, Nation and Pogel in [2] in an attempt to construct new finite projective planes. In this thesis we investigate some special cases of ordinary graphs, most prominently the case where nonadjacent vertices have no common neighbors. We determine all connected graphs of this type that exist.

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Chapter 1

Introduction

1.1 Introduction

In this thesis, we give the definition of ordinary graphs and list some basic facts about them as given in the paper [2] by Fossorier, Jezek, Nation and Pogel. Included are theorems from [2] that explain a connection between partitions of finite projective planes and ordinary graphs. Some results about the special case of ordinary graphs with $a = 0$ were proved by Leif Jørgensen in [11], working in a slightly different context that disallows 2-cycles. We list these results and summarize the proof techniques that were used in [11]. If 2-cycles are not allowed then the only connected ordinary graphs with $a = 0$ are permutation graphs and a type that can be constructed using ordinary tournaments (also known as doubly regular tournaments). When 2-cycles are allowed we can also obtain lexicographic products of one of the above types with a complete graph, but these are the only new graphs that appear. An extension of this result to infinite ordinary graphs is also included. We also compute the eigenvalues of connected ordinary graphs with $a = 0$ and some other interesting ordinary graphs. Using what we have learned about ordinary graphs with $a = 0$, we give two infinite classes of examples of ordinary graphs that have a chance to produce plane partitions. We also show that some of the ordinary graphs that might have been used to construct projectice planes of non-prime-power order do not exist.

1.2 Some basic graph theory

Formally, a *graph* G consists of a set $V(G)$ of *vertices* and a set $E(G)$ of unordered pairs of elements of $V(G)$, called *edges*. A *directed graph*, or *digraph*, also consists of a set of vertices and a set of edges, but here the set of edges is made up of ordered pairs of elements of the vertex

set $V(G)$. In this thesis we consider only directed graphs, and any use of the word graph should be understood to mean directed graph. We often write simply V or E in place of $V(G)$ and $E(G)$, and occasionally we will abuse notation and use the same name for a digraph and its vertex set.

Graphs can be represented in matrix form by the *adjacency matrix*. Given a graph or digraph G with vertex set V and edge set E, the associated adjacency matrix M is defined as follows: For $u, v \in V$, place a 1 in the (u, v) entry of M iff $(u, v) \in E$ and a 0 in the (u, v) entry otherwise.

Given graphs G_1 and G_2 with vertex sets V_1, V_2 and edge sets E_1, E_2 respectively, we desire a notion of when these graphs are essentially the same. G_1 and G_2 are said to be *isomorphic* graphs if there exists a bijection $\phi : V_1 \to V_2$ such that for $u, v \in V_1$, we have $(u, v) \in E_1$ if and only if $(\phi(u), \phi(v)) \in E_2$. Such a map ϕ is called an *isomorphism* from G_1 to G_2.

In this thesis, if we say that a particular graph G "has adjacency matrix M," it should be taken to mean that G can be labeled in such a way that it has adjacency matrix M, i.e., that G is isomorphic to a graph with adjacency matrix M. Note that isomorphisms on a graph correspond precisely to permutations of the rows and columns of the graph's adjacency matrix.

Given a graph G with vertex set V and edge set E, the *complement* of G is the graph $\widetilde{G}$ with vertex set V and edge set $\widetilde{E}$, where $(v, u) \in \widetilde{E}$ if and only if $(u, v) \in E$. A *subgraph* H of G consists of a vertex set $W \subseteq V$ and an edge set $F \subseteq E \cap (W \times W)$. An *induced subgraph* of G is a subgraph with vertex set $W \subseteq V$ whose edge set $F = E \cap (W \times W)$.

A digraph is called *simple* if each ordered pair of vertices (u, v) appears at most once in E and (u, u) does not appear in E for any vertex u. In this thesis we will be working exclusively with simple digraphs.

We will usually write $u \to v$ in place of $(u, v) \in E$. If $S, T \subseteq V$ we will write $S \rightsquigarrow T$ as a shorthand for $(\forall s \in S)\,(\forall t \in T)\, s \to t$. When either S or T consists of a single vertex the singleton set notation will not be used. Finally, the notation $u \rightarrowtail v$ means that $u \to v$ and $v \not\to u$.

We use $\uparrow v$ and $\downarrow v$ to denote the sets of *out-neighbors* and *in-neighbors* of v in a digraph G, which are respectively $\{u \in V : v \to u\}$ and $\{u \in V : u \to v\}$. The *out-degree* and *in-degree* of v are then $d_G^-(v) = |\uparrow v|$ and $d_G^+(v) = |\downarrow v|$. Occasionally the subscripted G will be replaced by a subgraph H to denote the degree of v restricted to H. A digraph is called *k-regular* if the out-degree and in-degree of each vertex is k.

For directed graphs, there are two notions of connectedness because there are two possible interpretations of *path*. For vertices u, v in a digraph G, a *weak u, v-path* is a sequence of vertices

$$u = w_0, w_1, \ldots, w_{k-1}, w_k = v$$

such that for all $0 \leq i \leq k-1$, either $w_i \to w_{i+1}$ or $w_{i+1} \to w_i$. A *strong u, v-path* is a sequence of vertices

$$u = w_0, w_1, \ldots, w_{k-1}, w_k = v$$

such that for all $0 \leq i \leq k-1$, $w_i \to w_{i+1}$. The notion of having a weak path between two given vertices is symmetric, but this is not the case for a strong path. Therefore we define the *out-path component* of a vertex u to be the set of all vertices v of G such that either $v = u$ or there is a strong u, v path in G. The *in-path component* is defined similarly.

A digraph G is *weakly connected* if for each ordered pair of vertices (u, v), G has a weak u, v-path. If for each ordered pair (u, v), G has a strong u, v-path (i.e. G itself is both the out-path component and in-path component of each of its vertices), then G is *strongly connected*. Fortunately, the graphs that we will discuss in this thesis are regular digraphs, and for such graphs these notions of connectedness are equivalent. We include an elementary proof here for completeness. We start with a lemma.

Lemma 1.1. *If G is a regular digraph with vertex set V and S is a subset of V, then the number of arrows into S and out of S are equal.*

Proof. We have the following, due to regularity

$$\sum_{v \in S} d_S^+(v) + \sum_{v \in S} d_{\widetilde{S}}^+(v) = \sum_{v \in S} d_G^+(v) = \sum_{v \in S} d_G^-(v) = \sum_{v \in S} d_S^-(v) + \sum_{v \in S} d_{\widetilde{S}}^-(v).$$

However, in any digraph $\sum_{v \in S} d_S^+(v) = \sum_{v \in S} d_S^-(v)$ holds for any set S, as each is a count of the total number of edges between vertices in S. Subtracting these leaves us with $\sum_{v \in S} d_{\widetilde{S}}^+(v) = \sum_{v \in S} d_{\widetilde{S}}^-(v)$. □

Proposition 1.2. *For regular digraphs, weakly connected and strongly connected are equivalent.*

To prove this, we will show that the out-path component of a vertex u is the same as the in-path component of u.

Proof. Choose a vertex u in G. Let S and T be the out-path and in-path components of G, respectively. Suppose that $v \in T - S$. Then there is an arrow from outside of S that lands in S, so there is an arrow from S to the outside by Lemma 1.1. This contradicts the definition of S and thus $S \subseteq T$. Similarly we can show that $T \subseteq S$.

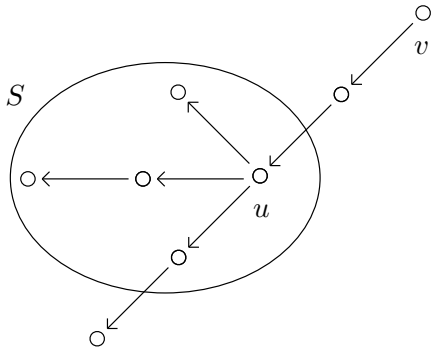

Figure 1.1: The out-path component of u

In particular this argument shows that whenever $u \to v$, G has a strong v, u-path. Now for any weak u, v-path $w_0, w_1, \ldots, w_k$ in G, we can form a strong u, v path in the following way. Begin a sequence of vertices with $u = w_0$. Whenever $w_i \to w_{i+1}$, add w_{i+1} to the sequence. If $w_i \nrightarrow w_{i+1}$ then $w_{i+1} \to w_i$, so there is a strong w_i, w_{i+1}-path $w_i = x_0, x_1 \ldots, x_{j-1}, x_j = w_{i+1}$. In this case add $x_1, \ldots, x_j = w_{i+1}$ to the sequence. The sequence formed in this manner is a strong u, v-path, as demonstrated in Figure 1.2.

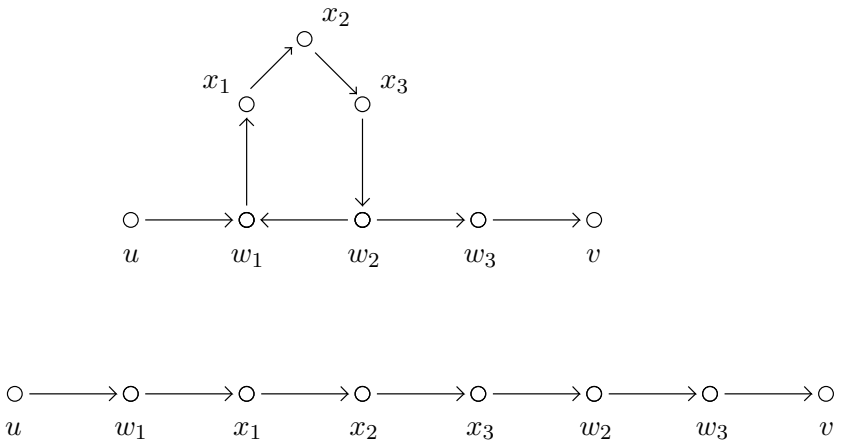

Figure 1.2: Forming a strong u, v-path from a weak path

□

Chapter 2

An introduction to ordinary graphs

We begin by giving the definition of ordinary graph and some important immediate results. Then we provide some examples of ordinary graphs. At the end of this chapter is a discussion of the role that ordinary graphs play in attempting to partition finite projective planes into smaller subplanes, which was the reason for the introduction of ordinary graphs by Fossorier et al. in [2].

2.1 Definitions and basic results

In this section we present the basic definitions and results from [2] that will be needed in this thesis.

On any digraph G we can define the following relations on the vertices so that each pair of vertices is in exactly one of these relations:

1. $u\,A\,v$ if neither $u \to v$ nor $v \to u$
2. $u\,B\,v$ if precisely one of $u \to v$ and $v \to u$ holds
3. $u\,C\,v$ if both $u \to v$ and $v \to u$.

An n-vertex r-regular digraph G is an *ordinary graph* of type $\langle n, r, a, b, c\rangle$ if

1. $u\,A\,v$ implies $|\uparrow u \cap \uparrow v\,| = |\downarrow u \cap \downarrow v\,| = a$
2. $u\,B\,v$ implies $|\uparrow u \cap \uparrow v\,| = |\downarrow u \cap \downarrow v\,| = b$
3. $u\,C\,v$ implies $|\uparrow u \cap \uparrow v\,| = |\downarrow u \cap \downarrow v\,| = c$

We allow any of the parameters a, b, c to have the special value $\times$ if the corresponding relation A,B or C is empty. Note that this is different from a, b or c being equal to 0.

An *ordinary matrix* is the adjacency matrix of an ordinary graph. In other words, we say that an $n \times n$ matrix A is *ordinary* of type $\langle n, r, a, b, c\rangle$ if the following properties hold.

1. A is normal (i.e., $AA^t = A^tA$).
2. Each diagonal entry of AA^t is r.
3. For $i \neq j$, the entry $(AA^t)_{ij}$ is a, b, or c according as $(A + A^t)_{ij}$ is 0, 1, or 2.

One interesting open question about ordinary graphs is whether some parts of this definition are redundant. We will see later that in some special cases we only need conditions on the number of common out-neighbors, and can prove that the in-neighbors satisfy the above equations.

There are several immediate consequences of the definition of ordinary graph.

Lemma 2.1. *Let* G *be an ordinary graph of type $\langle n, r, a, b, c\rangle$ with α, β and γ unordered pairs of vertices in the A, B and C relations respectively. Then*

1. $\alpha + \beta + \gamma = \dfrac{n(n-1)}{2}$
2. $\beta + 2\gamma = nr$
3. $a\alpha + b\beta + c\gamma = n\dfrac{r(r-1)}{2}$.

Proof. The left side of the first equation simply counts all unordered pairs of vertices in G. The left side of the second equation counts all of the arrows in G, which amount to nr. To establish the third equation, consider the set $S = \{(\{u, v\}, w) : u \to w \text{ and } v \to w\}$. If we start by choosing an arbitrary $w \in V$, then there are $\frac{r(r-1)}{2}$ ways of picking the unordered pair $\{u, v\}$. Since there are n choices for w, we have $n\frac{r(r-1)}{2}$ elements in the set S. On the other hand, if we start by fixing an unordered pair $\{u, v\}$ of vertices, then depending on whether $u\,A\,v$, $u\,B\,v$ or $u\,C\,v$, there are a, b or c choices for w. Since there are α ways to choose the pair $\{u, v\}$ that satisfy $v\,A\,w$, β ways that satisfy $v\,B\,w$ and γ ways that satisfy $v\,C\,w$, we count $a\alpha + b\beta + c\gamma$ elements in S. This establishes the third equation. □

Complements of ordinary graphs are again ordinary graphs, the parameters of which can easily be computed from those of the original graph.

Lemma 2.2. *Given an ordinary graph G of type $\langle n, r, a, b, c\rangle$, the complement graph $\widetilde{G}$ is ordinary of type $\langle n, n - r - 1, n + c - 2r, n + b - 2r - 1, n + a - 2r - 2\rangle$. For the complement, $\tilde{a}$, $\tilde{b}$ and $\tilde{c}$ will be $\times$ provided that (respectively) $c = \times$, $b = \times$ or $a = \times$.*

When we take a complement $\tilde{a}$, $\tilde{b}$ and $\tilde{c}$ must not become negative. Therefore we can say the following about the parameters of G.

Corollary 2.3. *Let G be an ordinary graph of type $\langle n, r, a, b, c\rangle$.*

1. *If* $a \neq \times$, *then* $n + a \geq 2r + 2$.
2. *If* $b \neq \times$, *then* $n + b \geq 2r + 1$.
3. *If* $c \neq \times$, *then* $n + c \geq 2r$.

2.2 Some examples of ordinary graphs

At this point we take a brief detour in order to introduce some of the most fundamental examples of ordinary graphs. All of these examples will be useful later when considering ordinary graphs with $a = 0$.

First we have the *trivial graph*, which consists of a single vertex and no edges. This graph is ordinary of type $\langle 1, 0, \times, \times, \times \rangle$. For any set of vertices V with $|V| = n$, if we let $E = \{(u, v) \in V \times V : u \neq v\}$ then we have the *complete graph* on n vertices, which is ordinary of type $\langle n, n-1, \times, \times, n-2 \rangle$. We use K_n to denote this graph. The complement of a complete graph is an *empty graph* and is ordinary of type $\langle n, 0, 0, \times, \times \rangle$. An n-vertex empty graph is denoted by O_n.

Given an n-element vertex set V and any permutation ϕ of the elements of V, put $E = \{(v, \phi(v)) : v \in V\}$. This yields a *permutation graph* and is ordinary of type $\langle n, 1, 0, 0, 0 \rangle$, where any of a, b, c may be $\times$ depending on the permutation ϕ. In particular if ϕ is a cycle then we call the above graph a *cycle* as well.

2.2.1 Ordinary tournaments

A crucial class of ordinary graphs is the class of *ordinary tournaments*. They are a fascinating subject in their own right and will play an important role in the main result proved in this paper, so we have an interest in exploring them in some detail here.

A *tournament* is any directed graph in which precisely one of $u \to v$ or $v \to u$ holds for each pair u, v of distinct vertices of the graph. This immediately gives $A + A^t = J - I$ for the adjacency matrix of any tournament, where I is the identity matrix and J is the matrix with 1 in every entry. Some tournaments also satisfy the requirements to be ordinary graphs, and these can be identified by two simple equations on the graph's adjacency matrix.

Proposition 2.4. *An n-vertex digraph G is an ordinary tournament if and only if its adjacency matrix A satisfies the following equations:*

1. $A + A^t = J - I$
2. $AA^t = kJ + (k+1)I$

where $n = 4k + 3$. If A satisfies these equations then it is an ordinary matrix of type $\langle 4k + 3, 2k + 1, \times, k, \times \rangle$.

With the notation above, G is said to be an ordinary tournament of *order* k.

Proof. In an ordinary tournament, all pairs of vertices are in the B relation. Let us denote the common value of $|\uparrow i \cap \uparrow j|$ by k. Lemma 2.1 shows that $n = 4k + 3$ and $r = 2k + 1$. Thus we have an ordinary graph of type $\langle 4k + 3, 2k + 1, \times, k, \times \rangle$, and therefore the adjacency matrix must satisfy the above equations.

On the other hand, if we have an n-vertex graph G whose adjacency matrix A satisfies the above equations, then A is a tournament by equation 1, and a theorem of H. Ryser (see [7]) shows that $AA^t = A^tA$. Thus G is an ordinary tournament. □

A common terminology already in use for an ordinary tournament is *doubly regular tournament*, and much research has been done on these. Among the results is a theorem of Brown and Reid [10]. A doubly regular tournament with n vertices exists if and only if an $(n + 1) \times (n + 1)$ skew Hadamard matrix exists, where we call an $n \times n$ matrix H with entries 1 and -1 an *Hadamard matrix* of order n if $HH^t = nI$, and an Hadamard matrix is *skew* if $H + H^t = 2I$. It can be shown that the order of an Hadamard matrix must be a multiple of 4. It is unknown whether Hadamard matrices exist for all possible orders, just as it is unknown whether doubly regular tournaments exist for all orders.

It will be important for us in this paper that the equations above that characterize an ordinary tournament are equivalent to the following three equations:

1. $A + A^t = J - I$
2. $AA + A^tA^t = (2k + 1)(J - I)$
3. $JA = AJ = (2k + 1)J$

Proposition 2.5. *An n-vertex digraph is an ordinary tournament if and only if its adjacency matrix A satisfies the above equations, where $n = 4k + 3$.*

Proof. If A is the adjacency matrix of an ordinary tournament, then A satisfies the equations of Proposition 2.4, and therefore $AA^t = A^tA$. Now

$$AA + A^tA^t = (A + A^t)^2 - AA^t - A^tA = (A + A^t)^2 - 2AA^t$$

$$= (J - I)^2 - 2kJ - (2k + 2)I = (2k + 1)(J - I)$$

which establishes equation 2. Equation 3 is clear also because A is ordinary with $r = (2k+1)$.

If A satisfies the equations of Proposition 2.5, then

$$AA + AA^t = A(A + A^t) = A(J - I) = (2k+1)J - A$$

and

$$A^tA^t + AA^t = (A^t + A)A^t = (J - I)A^t = (2k+1)J - A^t$$

Adding these together, we have

$$AA + AA^t + A^tA^t + AA^t = (4k+2)J - (A + A^t)$$

$$(2k+1)(J - I) + 2AA^t = (4k+2)J - (J - I)$$

$$AA^t = kJ + (k+1)I$$

as desired. □

The smallest ordinary tournament is the graph with vertices $\{0, 1, 2\}$ where $0 \to 1 \to 2 \to 0$. An infinite class of examples is given by a construction of Szekeres (see Ito [6]): Let q be a prime power such that $q \equiv 5 \pmod 8$ and let F be the finite field of order q. Let Q be the unique subgroup of the multiplicative group of F having index 4. Label the cosets $Q, -Q, R, -R$. The set of vertices is $\{v\} \cup F_1 \cup F_2$ where F_1 and F_2 are isomorphic copies of F. For convenience, when $x \in F$ we denote the associated elements in F_1 and F_2 by x_1 and x_2 respectively. Then the edges are as follows:

For all $x \in F$, we have $v \to x_1 \to x_2 \to v$.

If $y - x \in Q \cup R$, then $x_1 \to y_1$.

If $y - x \in -Q \cup -R$, then $x_2 \to y_2$.

If $y - x \in Q \cup -R$, then $x_1 \to y_2$ and $x_2 \to y_1$.

This defines an ordinary tournament with $2q + 1$ vertices. In the next section we will see another infinite class of examples of ordinary tournaments, which will be extended further in Chapter 4.

2.2.2 Circulant graphs

A very general class of graphs (which are not necessarily ordinary) is the class of *circulant graphs*. These are graphs whose adjacency matrix is obtained by cycling a single row (the matrix is called a *circulant matrix*). We write $C(R)$, where R is a row vector, to denote the circulant graph (and/or its matrix) obtained by cycling the row R. Circulant graphs are a subclass of Cayley graphs, which will be discussed later.

Clearly cycles are examples of circulant graphs, for example the 4-cycle $C(0100) =$

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

An interesting example of a circulant graph is $C(0110100)$, which is the ordinary tournament of order 1. In fact, if $4k+3$ is a prime p and we take the i^{th} entry of R to be 1 if i is a nonzero quadratic residue in the field of order p and 0 otherwise, then $C(R)$ is an ordinary tournament of order k. This is actually a special case of a construction due to Paley, which we will discuss later along with Cayley graphs.

We will also use the circulant notation above for graphs that have block circulant adjacency matrices. Here the entries in the row matrix R are matrix blocks, which are meant to be cycled to produce the matrix $C(R)$. If every block in R is circulant then $C(R)$ will also be circulant, but typically it is not.

2.3 Partitioning projective planes

As we mentioned earlier, the purpose for introducing ordinary graphs in Fossorier et al. [2] was to construct projective planes of nonprime-power order. The idea is to "glue together" several copies of a smaller projective plane. An ordinary graph with the right parameters tells us how to glue the smaller planes together, although it is still necessary to solve a system of equations once the desired ordinary graph is found.

Constructions of several known planes are given as examples in [2]. However, no new planes have been constructed using this method as of yet. Later in this thesis we will give a list of

ordinary graphs with the necessary parameters to attempt the construction of new planes, but the hurdle of solving complicated systems of equations still remains.

In this section, we will present the most general results from [2] about the relationship between ordinary graphs and projective plane partitions. We will also include several of the examples from this paper.

2.3.1 Theorems about ordinary graphs and projective plane partitions

Recall that a *projective plane* Π consists of a set P of points and a set L of lines such that each pair of points determines a line, each pair of lines determines a point, and there exists a set of four points, no three of which are collinear. For $p \in P$ and $l \in L$ we write $p \leq l$ to denote that the point p is on the line l. It can be shown that if Π is a finite projective plane then $|P| = |L| = n^2+n+1$, each point must lie on $n+1$ lines, and each line must contain $n+1$ points. This number n is called the *order* of the plane.

The *incidence matrix* M of a finite projective plane Π with points P and lines L is a matrix where the rows are indexed by P and the columns by L. We place a 1 in the (p, l)-entry if $p \leq l$ and a 0 otherwise. With the correct labeling of points and lines, the incidence matrix for a projective plane of order n, if viewed as the adjacency matrix of a directed graph, is an ordinary matrix of type $\langle n^2+n+1, n+1, 1, 1, 1\rangle$ (where a, b, or c may be $\times$). We must only verify that the points and lines can be labeled with the same index set I in such a way that point i is not on line i for every $i \in I$. A famous theorem of P. Hall is needed (see, for example, Ryser [13]). Recall that a *system of distinct representatives* (SDR) for a family of sets $S_1, S_2, \ldots, S_k$ is a set of elements $x_1, x_2, \ldots, x_k$ such that $x_i \in S_i$ for all $1 \leq i \leq k$.

Theorem 2.6. *(P. Hall) The family of sets $S_1, S_2, \ldots, S_k$ has an SDR if and only if $|\bigcup_{i\in I} S_i| \geq |I|$ for all $I \subseteq \{1, 2, \ldots, k\}$.*

Proposition 2.7. *Every finite projective plane of order n is isomorphic to one with an incidence matrix which is an ordinary matrix of type $\langle n^2+n+1, n+1, 1, 1, 1\rangle$.*

Proof. Let the lines of the plane be $\{l_i : 1 \leq i \leq n^2+n+1\}$ and the points of the plane be $\{p_i : 1 \leq i \leq n^2+n+1\}$. Define $S_i = \{p_j : p_j \not\leq l_i\}$. We must show that the family $\{S_i\}$ has an SDR. Note that each point of the projective plane is not incident with n^2 of the lines, and dually each line has n^2 points not incident upon it. Let $I \subseteq \{1, 2, \ldots, n^2+n+1\}$. Then the set $\{(x_j, S_k) : x_j \in S_k,\ k \in I\}$ has $n^2|I|$ elements, and this set is contained in the set

$\{(x_j, S_k) : x_j \in \bigcup_{i\in I} S_i,\ x_j \in S_k\}$ which has $n^2|\bigcup_{i\in I} S_i|$ elements. Thus $n^2|I| \leq n^2|\bigcup_{i\in I} S_i|$ and therefore $|I| \leq |\bigcup_{i\in I} S_i|$ as required. □

Ordinary graphs also come into play when we attempt to partition a projective plane into several smaller subplanes. Given a projective plane Π of order n with P and L, an *ordinary partition* of Π is a partition of Π into equal sized subplanes Π_i with P_i and L_i such that for each i, whenever there exists $p \in P_i$ and $l \in L_j$ with $p \leq l$ then for every $p \in P_i$ there exists $l \in L_j$ such that $p \leq l$.

If we have a partition of Π into N subplanes Π_i with $0 \leq i \leq N-1$ we can construct a graph G_Π with vertex set $\{0, 1, ..., N-1\}$ and $i \to j$ if $i \neq j$ and there exists $p \in P_i$ and $l \in L_i$ with $p \leq l$. The theorem below, a rephrasing of Theorem 9 in [2], says that G_Π is an ordinary graph provided that the partition is ordinary. See that paper for a proof.

Theorem 2.8. *Given an ordinary partition of a projective plane Π of order n into N order q subplanes Π_i $(0 \leq i \leq N-1)$ the associated graph G_Π is ordinary of type $\langle N, n-q, q^2+q+1, q^2, q^2-q-1\rangle$, where a, b or c may be $\times$.*

Given an ordinary partition and the associated graph as above, for each pair (i, j) with $i \to j$ we can define a bijection λ_{ij} of P_i onto L_j by $\lambda_{ij}(p) = l$ for $p \in P_i$ where $l \in L_j$ and $p \leq l$. Each pair of points in Π must determine a unique line, so we have the following property of the maps λ_{ij} (Lemma 9 in [2]):

Lemma 2.9. *Suppose we have an ordinary partition of a projective plane as in Theorem 2.8. Given $p \in P_i$ and $r \in P_j$ where $i \neq j$, exactly one of the following holds:*

1. *$r \leq \lambda_{ij}(p)$,*
2. *$p \leq \lambda_{ij}(r)$,*
3. *there exists k such that $\lambda_{ik}(p) = \lambda_{jk}(r)$.*

Conversely, if we have structures as are given in the conclusions of Theorem 2.8 and Lemma 2.9 then we can construct a larger projective plane that partitions into the subplanes that are required in the hypotheses of Theorem 2.8 and Lemma 2.9. This is Theorem 11 in [2].

Theorem 2.10. *Assume that $n^2+n+1 = N(q^2+q+1)$ and there exists a projective plane Ψ of order q. Let Π_i for $0 \leq i \leq N-1$ be disjoint copies of Ψ. Suppose that there exists an ordinary graph G of type $\langle N, n-q, q^2+q+1, q^2, q^2-q-1\rangle$ (where a, b or c may be $\times$) and that there exist bijections $\lambda_{ij} : P_i \to L_j$ for those (i, j) where $i \to j$ in G such that the conclusion of Lemma 2.9 holds for each λ_{ij}. Then there exists a projective plane of order n with an ordinary partition into subplanes of order q.*

2.3.2 Examples of plane partitions using ordinary graphs

In order to actually find examples of decompositions of projective planes into subplane partitions, it is much more convenient to consider adjacency matrices of graphs rather than looking at the graphs themselves. Then our goal is to construct a matrix that will be the incidence matrix of a projective plane. Furthermore, it is very difficult to find examples that meet the requirements of the theorems above without limiting the scope of the search. In Fossorier et al. [2], only circulant ordinary graphs are considered, the subplanes are assumed have a circulant incidence matrix, and the maps λij are assumed to be some power of the cyclic permutation $C(0100...0)$. Working in this framework makes it extremely unlikely to construct a nondesarguesian plane, but it has allowed for the construction of several examples, a few of which are listed below after some discussion. See [2] for the full details.

Following the scheme above, suppose we have a circulant ordinary matrix M with parameters $\langle N, n-q, q^2+q+1, q^2, q^2-q-1\rangle$ as given in Theorem 2.10 and a circulant incidence matrix P of a projective plane of order q. Let $S = C(0100...0)$. The matrix M is our blueprint for constructing H, which will be the incidence matrix of our "new" projective plane. Suppose $M = C(R)$ where R is a $0,1$ row vector. Then $H = C(T)$ where $T_0 = P$ and for $1 \leq i \leq N-1$, $T_i = 0$ if $R_i = 0$ and $T_i = S^{x_i}$ if $R_i = 1$, where x_i is an integer with $0 \leq x_i < q^2+q+1$.

Once an appropriate M and P are found, the difficulty is in solving the equations for the x_i that will make H into the incidence matrix of a projective plane. Here are three examples where this construction has produced planes that were already known. The details of solving for the x_i have been omitted.

Example 1. Here we construct a plane of order 9 from planes of order 3. In this example, we have $q = 3$, $n = 9$, and $N = 7$. We use $M = C(0111111)$, an ordinary graph of type $\langle 7, 6, \times, \times, 5\rangle$, $P = C(0110100000100)$, and of course $S = C(0100000000000)$. After solving for the x_i, we have $H = C(P, S^0, S^5, S^{11}, S^{11}, S^5, S^0)$ as the incidence matrix of a projective plane of order 9.

Example 2. Here we construct a plane of order 16 from planes of order 4. The parameters we are working with are $q = 4$, $n = 16$, and $N = 13$. We use $M = C(0111111111111)$, $P = C(0100110000101000000000)$, and $S = C(01000000000000000000)$. It turns out that $H = C(P, S^0, S^2, S^9, S^6, S^{14}, S^{20}, S^{14}, S^6, S^9, S^2, S^0)$ works as the incidence matrix of a projective plane of order 16.

Example 3. If we allow ourselves to consider the matrix $P = C(011)$ to be the incidence matrix of a degenerate plane of order $q = 1$, we can construct larger planes from these. For example, take $M = C(0110100)$ and $S = C(010)$ to make $H = C(P, S^0, S^0, O, S^0, O, O)$, the incidence matrix of a plane of order 4. This technique has been used in [2] to produce planes of orders 7, 13, 16, 19 and 31 as well.

Chapter 3

Uniformity, Cayley graphs, and the lexicographic product on graphs

In this chapter we define for ordinary graphs a property we call uniformity. This property is similar to vertex-transitivity, which is also discussed here. We introduce Cayley graphs and the lexicographic product on graphs, which will be of great importance throughout the rest of this thesis.

3.1 Uniformity

For a vertex v of a digraph G, define $A_v = \{u \in G : v\,A\,u\}$, $B_v = \{u \in G : v\,B\,u\}$ and $C_v = \{u \in G : v\,C\,u\}$. Also define $\alpha_v = |A_v|$, $\beta_v = |B_v|$ and $\gamma_v = |C_v|$. By a similar counting argument to Lemma 2.1, we have:

Lemma 3.1. *For each vertex v of an ordinary graph:*

1. $\alpha_v + \beta_v + \gamma_v = n - 1$
2. $\beta_v + 2\gamma_v = 2r$
3. $a\alpha_v + b\beta_v + c\gamma_v = r(r-1)$.

Proof. Fix $v \in V$. The left side of the first equation counts each vertex in $V - \{v\}$. The left side of the second equation counts each vertex in $\uparrow v \cup \downarrow v$, which has $2r$ elements. To prove the third equality, consider the set $\{(v, u, w) : v \to u \text{ and } w \to u\}$. If we fix such a u there are $r - 1$ distinct choices for w. As there are r choices for u, the set of triples has $r(r-1)$ elements. On the other hand if we first choose any $w \in V - \{v\}$, then depending on whether $v\,A\,w$, $v\,B\,w$, or $v\,C\,w$ there are a, b or c choices for u. Since there are α_v choices for w that satisfy $v\,A\,w$, β_v choices for w that satisfy $v\,B\,w$ and γ_v choices for w that satisfy $v\,C\,w$, we count $a\alpha_v + b\beta_v + c\gamma_v$ elements in the set of triples above. □

We say that a digraph G is *uniform* if α_u, β_u and γ_u are each constant across all vertices u in G. If G is ordinary then the equations in Lemma 3.1 make it clear that if any one of α_u, β_u and γ_u is constant across all u then all three are constant. One way for this to happen is for any of a, b or c to be $\times$, in which case the corresponding α_u, β_u or γ_u is 0 for all vertices u. We also obtain uniformity if the system of equations is linearly independent, since it can then be solved to fix α_u, β_u and γ_u. For the system to be linearly dependent $a + c = 2b$ must hold. In the next chapter, we will investigate graphs with $a = 0$. The following proposition will be useful:

Proposition 3.2. *If G is an ordinary graph with $a = 0$ then either G is uniform or $b = \frac{r-1}{2}$ and $c = r - 1$.*

Proof. If G is not uniform and $a = 0$, then $b, c \neq \times$ and $a + c = 2b$, as noted above. Then $c = 2b$ and it is easy to obtain $b = \frac{r-1}{2}$ and $c = r - 1$ from Lemma 2.1. □

3.2 Graph automorphisms and Cayley graphs

An *automorphism* σ of a graph G is an isomorphism from G to itself–that is, a bijection of the vertex set of G such that $u \to v$ if and only if $\sigma(u) \to \sigma(v)$. The automorphisms of G form a group denoted as $\mathrm{Aut}(G)$, which acts on $V(G)$. $\mathrm{Aut}(G)$ is called a *permutation group*, as it is a subgroup of the group of permutations on $V(G)$. Recall that a group action is *transitive* if there is a single orbit. A transitive permutation group Σ acting on a finite set S is called *regular* if for any $s, t \in S$ there exists a unique $\sigma \in \Sigma$ such that $\sigma(s) = t$. It is unfortunate that we already have a definition of the term "regular" for graphs. In what follows, the context should make it clear which term is being used.

These preliminaries lead us to an important property for graphs. A graph G is called *vertex-transitive* if $\mathrm{Aut}(G)$ acts transitively on the vertices of G. $\mathrm{Aut}(G)$ can also be viewed as acting on the edge set of G, and if $\mathrm{Aut}(G)$ acts transitively on $E(G)$, then G is called *edge-transitive*. Edge-transitivity implies vertex-transitivity (as long as every vertex has an edge), but the converse is not true.

If an ordinary graph is vertex-transitive, then it clearly must be uniform. However, not every uniform ordinary graph is vertex-transitive. For example, there are up to isomorphism two different ordinary tournaments of order 3. If we take as our graph G the disjoint union of these two tournaments, then G is ordinary of type $\langle 30, 7, 0, 3, \times \rangle$ and uniform (since $c = \times$), but is not vertex-transitive.

Earlier we made use of circulant graphs. We can generalize this idea to allow graphs that are generated by any group. The following definition and more facts than are included here can be found in Gross and Yellen [4]. Given a group G and a subset S of G such that $1 \notin S$ and $S = S^{-1}$, we define $\mathrm{Cay}(G; S)$, the *Cayley graph on G with connection set S*, to be the graph with vertex set G such that for $g, h \in G$, $g \to h$ if and only if $h = gs$ for some $s \in S$. With this notation, the circulant graph $C(R)$ where R is a length m row vector is $\mathrm{Cay}(Z_m; S)$ where $S = \{z \in Z_m : R_z = 1\}$.

Note that the graph $\mathrm{Cay}(G; S)$ is connected if and only if S generates the group.

We say that a graph G is a *Cayley graph* if G is isomorphic to $\mathrm{Cay}(H; S)$ for some group H and subset S of H. This definition is necessary if we want the property of being a Cayley graph to be one that is preserved under isomorphism. For example, there are many ways to label a cycle so that it is not circulant.

We would like a condition on the graph that is invariant under isomorphism that we can use to verify whether the graph is a Cayley graph or not. This is provided by Sabidussi's Theorem [14]. Here we have modified the proof for digraphs.

Theorem 3.3. *A digraph G is a Cayley graph if and only if $\mathrm{Aut}(G)$ contains a regular subgroup.*

Proof. Suppose that G is a Cayley graph. We aim to show that $\mathrm{Aut}(G)$ contains a regular subgroup, and thus we may as well assume that $G = \mathrm{Cay}(G; S)$ since the property of having a regular subgroup is preserved under isomorphism. In particular we are assuming that G is a group. Define the map $\eta : G \to \mathrm{Aut}(G)$ by $\eta(g) = \sigma_g$ where $\sigma_g(x) = gx$ for $x \in G$. Clearly η is one-to-one. Thus $|\mathrm{Im}(\eta)| = |G|$. To show that η is a group homomorphism, let $g, h \in G$. We have $\sigma_g\sigma_h(x) = \sigma_g(hx) = ghx = \sigma_{gh}(x)$, so $\eta(g)\eta(h) = \sigma_g\sigma_h = \sigma_{gh} = \eta(gh)$ as required. Now $\mathrm{Im}(\eta)$ is a subgroup of $\mathrm{Aut}(G)$, and it is regular because for any $g, h \in G$, there is a unique element of $\mathrm{Im}(\eta)$, namely $\eta(hg^{-1}) = \sigma_{hg^{-1}}$ that takes g to h.

On the other hand, suppose that G is a digraph such that $\mathrm{Aut}(G)$ contains the regular subgroup H. Fix a vertex $u \in G$. By regularity of H, for each $v \in G$ there exists precisely one map in H that takes u to v. Thus $H = \{\sigma_v : v \in G\}$ where $\sigma_v(u)$ is the unique element of H such that $\sigma_v(u) = v$. Let $S = \{\sigma_v \in H : v \in {\downarrow}u\}$. We will show that G is isomorphic to the graph $X = \mathrm{Cay}(K; S)$. Define the map $\phi : X \to G$ by $\phi(\sigma_v) = v$. Clearly ϕ is a bijection. It remains to show that $\sigma_{v_1} \to \sigma_{v_2}$ in X if and only if $v_1 \to v_2$ in G. If $\sigma_{v_1} \to \sigma_{v_2}$ in X, then there exists $\sigma_v \in S$ such that $\sigma_{v_1} = \sigma_{v_2}\sigma_v$. Now $v_1 = \phi(\sigma_{v_1}) = \phi(\sigma_{v_2}\sigma_v) = \sigma_{v_2}(v)$. Since $\sigma_v \in S$, we have $v \to u$. But σ_{v_2} is an automorphism of G, so $v_1 = \sigma_{v_2}(v) \to \sigma_{v_2}(u) = v_2$ as desired. If $v_1 \to v_2$ in G,

then $\sigma_{v_2}^{-1}(v_1) \to u$, so $\sigma_{v_2}^{-1}(v_1) \in {\downarrow}u$ and thus there exists $v \in {\downarrow}u$ such that $\sigma_{v_2}^{-1}(v_1) = v$. Therefore $v_1 = \sigma_{v_2}(v)$, and note that $\sigma_v \in S$. Now $\sigma_{v_1}(u) = v_1 = \sigma_{v_2}(v) = \sigma_{v_2}\sigma_v(u)$. By the regularity of H we must therefore have $\sigma_{v_1} = \sigma_{v_2}\sigma_v$. Since $\sigma_v \in S$, we conclude that $\sigma_{v_1} \to \sigma_{v_2}$ in X, as required. This completes the proof that G is isomorphic to X. □

It should be clear that every Cayley graph is vertex-transitive. One might ask if the converse holds as well. It does not, and the well-known Petersen graph is a counterexample. The vertices of this graph are all 2-element subsets of $\{1, 2, 3, 4, 5\}$, and two vertices have an edge iff they are disjoint subsets (see Figure 3.2 below). It is clear that this graph is vertex-transitive from its construction. It can be shown that the automorphism group of the Petersen graph is isomorphic to the symmetric group on 5 objects, and that the automorphism group contains no regular subgroup (see C. D. Godsil [3]). Therefore by Sabidussi's Theorem it is not a Cayley graph. The Petersen graph is the smallest example of a vertex-transitive graphs that is not a Cayley graph. For more examples, see the paper by McKay and Praeger [9]. The Petersen graph is not usually thought of as a digraph, but if we think of each of its edges as a bidirected edge $\leftrightarrow$, then it is an ordinary graph of type $\langle 10, 3, 1, \times, 0\rangle$. This confirms that the additional hypothesis of being an ordinary graph does not alter the result that not every vertex-transitive graph is a Cayley graph.

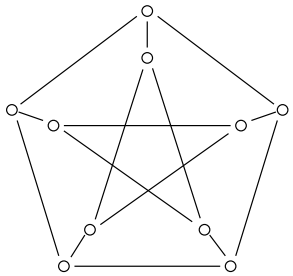

Figure 3.1: The Petersen graph

One important class of Cayley graphs is given by a construction of Paley (see, for example, Jørgensen's paper [12]): Let $V(G)$ be the elements of the finite field of size p^m, and define the edges so that $x \to y$ if $x - y$ is a nonzero quadratic residue in the multiplicative group of the field. When $p^m = 4k + 1$ the construction results in an ordinary matrix of type $\langle 4k + 1, 2k, k, \times, k - 1\rangle$. More importantly, when $p^m = 4k+3$ the construction yields an ordinary tournament of order k. These are called *Paley tournaments*. The case where $m = 1$ was already discussed in the section on circulant graphs.

3.3 The lexicographic product on graphs

There are many ways to define a product of two simple digraphs G_1 and G_2, but one that is particularly useful for generating ordinary graphs is the lexicographic product, which we will denote by $G_1 \circ G_2$. The vertex set of $G_1 \circ G_2$ is the Cartesian product $V(G_1) \times V(G_2)$ and we have $(u_1, u_2) \to (v_1, v_2)$ if and only if $u_1 \to v_1$ or $u_1 = v_1$ and $u_2 \to v_2$.

If we order the vertices of $G_1 \circ G_2$ lexicographically, then the lexicographic product can be expressed very nicely using adjacency matrices. Let A be the adjacency matrix of G, with dimension m, and B be the adjacency matrix of G_2, with dimension n. Then M, the adjacency matrix of $G_1 \circ G_2$ is a dimension mn block matrix with m blocks of size n in each row and column. The blocks are B on the diagonal, the zero matrix where A has a 0, and J where A has a 1. If we have two adjacency matrices M_1 and M_2, we define the lexicographic product $M_1 \circ M_2$ to be the matrix described above. We compute the lexicographic product $M_1 \circ M_2$ below, where $M_1 = C(010)$ and $M_2 = C(01)$.

$$M_1 \circ M_2 = \begin{pmatrix} M_2 & J & 0 \\ 0 & M_2 & J \\ J & 0 & M_2 \end{pmatrix}$$

Writing out the full matrices, we have:

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \circ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

First we discuss some properties of the lexicographic product for general graphs. Proofs of the facts in Proposition 3.4 and others can be found in Imrich and Klavzar [5]. Later we determine precisely when the lexicographic product of two ordinary graphs will again be an ordinary graph.

Proposition 3.4. *The lexicographic product is associative on simple digraphs. The complement and transpose operations distribute over the lexicographic product, and the graph with a single vertex is an identity for the lexicographic product.*

Following [5] (see p.187), we will call a subset S of $V(G)$ or its induced subgraph *externally related* if, for every $u \in V(G) - S$, $u \to s$ for some $s \in S$ implies $u \rightsquigarrow S$ and $s \to u$ for some $s \in S$ implies $S \rightsquigarrow u$. We will call an equivalence relation $\equiv$ a *strong congruence* on G if $\equiv$ partitions G into classes that are externally related and isomorphic as induced subgraphs. If $\equiv$ is a strong congruence on G then $G/\equiv$ is a graph with a well-defined edge relation, because if $[u] \neq [v]$ then $u \to v$ implies $[u] \rightsquigarrow [v]$ as sets in G, so in the graph $G/\equiv$ we have $[u] \to [v]$. The following proposition explains the relationship between strong congruences and the lexicographic product.

Proposition 3.5. *Let G, G_1, and G_2 be simple digraphs.*

1. *If $\equiv$ is a strong congruence on G and all $\equiv$-classes are isomorphic to E, then $(G/\equiv) \circ E$ is isomorphic to G.*
2. *On $G_1 \circ G_2$, define $\equiv$ by the rule $(u_1, u_2) \equiv (v_1, v_2)$ iff $u_1 = v_1$. Then $\equiv$ is a strong congruence with classes isomorphic to G_2, and $(G_1 \circ G_2)/\equiv$ is isomorphic to G_1.*

Proof. Beginning with the first claim, suppose $\equiv$ is a strong congruence on G such that all $\equiv$-classes are isomorphic to E. If v is an arbitrary vertex of G, let $[v]$ be the $\equiv$-class containing v, and let θ_v be the isomorphism from $[v]$ to E. Then define $\phi(v) = ([v], \theta_v(v))$. If $[v]$ is an arbitrary $\equiv$-class in $(G/\equiv)$, let $\lambda_{[v]}$ be the isomorphism from E to $[v]$. Then define $\psi([v], e) = \lambda_{[v]}(e)$. Then ϕ and ψ are inverse maps between G and $(G/\equiv) \circ E$. To show that ϕ respects the edge relation, suppose that $u \to v$ in G. Then there are two possibilities. If $[u] \neq [v]$, then $[u] \to [v]$ in $(G/\equiv)$ and therefore $\phi(u) \to \phi(v)$ in $(G/\equiv) \circ E$. Otherwise $[u] = [v]$. In this case we have $\theta_u = \theta_v$ and $\theta_u(u) \to \theta_u(v)$ in E since θ_u is an isomorphism. Thus $\phi(u) \to \phi(v)$ in $(G/\equiv) \circ E$. To show that ψ respects the edge relation, suppose that $([u], e) \to ([v], f)$ in $(G/\equiv) \circ E$. Then either $[u] = [v]$ and $e \to f$ or $[u] \to [v]$. If $[u] = [v]$ and $e \to f$, then $\lambda_{[u]} = \lambda_{[v]}$ and $\lambda_{[u]}(e) \to \lambda_{[u]}(f)$ since $\lambda_{[u]}$ is an isomorphism. If $[u] \to [v]$, then $[u] \rightsquigarrow [v]$ as subgraphs of G, and thus $\lambda_{[u]}(e) \to \lambda_{[v]}(f)$. In either case we have shown that

$$\psi(([u], e)) = \lambda_{[u]}(e) \to \lambda_{[v]}(f) = \psi([v], f)$$

as desired. We can now conclude that $(G/\equiv) \circ E \cong G$.

Now we prove the second claim. It is clear that the $\equiv$-classes are isomorphic to G_2, so to prove that $\equiv$ is a strong congruence, we must only show that the $\equiv$-classes are externally related. Suppose that $(u_1, u_2) \to (v_1, v_2)$ in $G_1 \circ G_2$ and $[(u_1, u_2)] \neq [(v_1, v_2)]$. Then $u_1 \neq v_1$, so $u_1 \to v_1$. Now let $x \neq u_1$ in G_1 and $y \neq v_1$ in G_1. Then by the definition of $G_1 \circ G_2$, we have $(u_1, u_2) \to (y, v_2)$ and $(x, u_2) \to (v_1, v_2)$. Thus the $\equiv$-classes are externally related.

To show that $(G_1 \circ G_2)/\equiv$ is isomorphic to G_1, for arbitrary $g \in G_1$ define $\phi(g) = [(g, h)]$, where h is any vertex of G_2. For arbitrary $[(g_1, g_2)] \in (G_1 \circ G_2)/\equiv$ define $\psi([(g_1, g_2)]) = g_1$. Then ϕ and ψ are inverse maps between G_1 and $(G_1 \circ G_2)/\equiv$. To show that ϕ respects the edge relation, suppose that $u \to v$ in G_1. Then for any vertices x and y of G_2 we have $(u, x) \to (v, y)$ in $G_1 \circ G_2$, and thus if $\phi(u) = [(u, x)]$ and $\phi(v) = [(v, y)]$ we have $[(u, x)] \to [(v, y)]$ in $(G_1 \circ G_2)/\equiv$. To show that ψ respects the edge relation, suppose that $[(u_1, u_2)] \to [(v_1, v_2)]$ in $(G_1 \circ G_2)/\equiv$. Then in particular $(u_1, u_2) \to (v_1, v_2)$ in $G_1 \circ G_2$, and $u_1 \neq v_1$ because otherwise $[(u_1, u_2)] = [(v_1, v_2)]$. Thus we can conclude that $u_1 \to v_1$, which is as desired. Hence $(G_1 \circ G_2)/\equiv$ is isomorphic to G_1. □

The lexicographic product also behaves nicely for Cayley graphs, and preserves some of the graph properties we previously discussed.

Proposition 3.6. *If G_1 and G_2 are graphs such that $G_1 = \mathrm{Cay}(G_1; S_1)$ and $G_2 = \mathrm{Cay}(G_2; S_2)$ (so that in particular G_1 and G_2 are groups), then the graph $G_1 \circ G_2 = \mathrm{Cay}(G_1 \times G_2; S)$ where $S = \{(s_1, g_2) : s_1 \in S_1, g_2 \in G_2\} \cup \{(e_1, s_2) : s_2 \in S_2\}$ (where e_1 is the identity element of G_1).*

Proof. We must show that in $G_1 \circ G_2$, we have $(u_1, u_2) \to (v_1, v_2)$ if and only if there exists an element $s = (w_1, w_2) \in G_1 \times G_2 \in S$ such that $(u_1, u_2) = (v_1, v_2)(w_1, w_2)$. Suppose first that $(u_1, u_2) \to (v_1, v_2)$. Then either $u_1 \to v_1$ or $u_1 = v_1$ and $u_2 \to v_2$. In the first case, there exists $w_1 \in S_1$ such that $u_1 = v_1 w_1$, so that $(u_1, u_2) = (v_1, v_2)(w_1, v_2^{-1} u_2)$ and $(w_1, v_2^{-1} u_2) \in S$. In the second case, there exists $w_2 \in S_2$ such that $u_2 = v_2 w_2$, so that $(u_1, u_2) = (v_1, v_2)(e_1, w_2)$ and $(e_1, w_2) \in S$. Thus in either case the required element of S exists.

On the other hand, let (w_1, w_2) be an arbitrary element of S. Then either $w_1 \in S_1$ or $w_1 = e_1$ and $w_2 \in S_2$. Now let (u_1, u_2) be an arbitrary vertex of $G_1 \circ G_2$. If $w_1 \in S_1$, then $u_1 \to u_1 w_1$, so $(u_1, u_2) \to (u_1 w_1, u_2 w_2) = (u_1, u_2)(w_1, w_2)$. If $w_1 = e_1$ and $w_2 \in S_2$, then $u_2 \to u_2 w_2$, so that $(u_1, u_2) \to (u_1, u_2 w_2) = (u_1, u_2)(e_1, w_2)$. In either case we have the required edge in $G_1 \circ G_2$, and hence $G_1 \circ G_2 = \mathrm{Cay}(G_1 \times G_2; S)$ as claimed. □

The next proposition is clear, but we state it for completeness.

Proposition 3.7. *If G_1 and G_2 are uniform regular digraphs, then $G_1 \circ G_2$ is uniform and regular as well.*

Theorem 3.8. *For ordinary graphs G_1 of type $\langle n_1, r_1, a_1, b_1, c_1 \rangle$ and G_2 of type $\langle n_2, r_2, a_2, b_2, c_2 \rangle$, the lexicographic product $G_1 \circ G_2$ is ordinary if and only if the following conditions hold:*

1. $a_1 = \times$ *or* $a_2 = \times$, *or* $a_1 = r_1$ *and* $a_2 = 0$
2. $b_1 = \times$ *or* $b_2 = \times$
3. $c_1 = \times$ *or* $c_2 = \times$, *or* $c_1 = r_1 - 1$ *and* $c_2 = 2r_2 - n_2$.

If the conditions are satisfied, $G_1 \circ G_2$ *will be ordinary of type* $\langle N, R, A, B, C\rangle$ *where*

$$\begin{aligned} N &= n_1 n_2 \\ R &= n_2 r_1 + r_2 \\ A &= n_2 r_1 + a_2 = n_2 a_1 \text{ or } \times \\ B &= n_2 r_1 + b_2 = n_2 b_1 + r_2 \text{ or } \times \\ C &= n_2 r_1 + c_2 = n_2 c_1 + 2r_2 \text{ or } \times . \end{aligned}$$

Proof. Clearly there are $n_1 n_2$ vertices in $G_1 \circ G_2$. For an arbitrary vertex (u_1, u_2) we have

$$\uparrow(u_1, u_2) = \{(x, y) \ : \ x \in \uparrow u_1, \ y \in G_2\} \cup \{(u_1, y) \ : \ y \in \uparrow u_2\}$$

and thus $|\uparrow(u_1, u_2)| = n_2 r_1 + r_2$. We get the same count for the in-degree. For a pair of vertices $(u_1, u_2),\ (v_1, v_2)$, if $u_1 = v_1$ we have that

$$\uparrow(u_1, u_2) \cap \uparrow(v_1, v_2) = \{(x, y) \ : \ x \in \uparrow u_1, \ y \in G_2\} \cup \{(u_1, y) \ : \ y \in \uparrow u_2 \cap \uparrow v_2\}.$$

The first set in this union has $n_2 r_1$ elements. The vertices (u_1, u_2) and (v_1, v_2) are A, B, or C-related according to the relation between u_2 and v_2. Depending on this relation, the second set in the union above has either a_2, b_2, or c_2 elements. Thus $\uparrow(u_1, u_2) \cap \uparrow(v_1, v_2)$ is either $n_2 r_1 + a_2$, $n_2 r_1 + b_2$ or $n_2 r_1 + c_2$. Computation of shared in-neighbors gives the same results.

If $u_1 \neq v_1$ then the relation between (u_1, u_2) and (v_1, v_2) is determined by the relation between u_1 and v_1. In any case, $\uparrow(u_1, u_2) \cap \uparrow(v_1, v_2)$ contains the set $\{(x, y) : x \in \uparrow u_1 \cap \uparrow v_1,\ y \in G_2\}$, plus more if (u_1, u_2) and (v_1, v_2) are B or C-related. If they are B-related then either $u_1 \to v_1$ or $v_1 \to u_1$ and we also have $\{(u_1, y) : u_2 \to y\}$ or $\{(v_1, y) : v_2 \to y\}$ as common out-neighbors depending on which direction the arrow points. If (u_1, u_2) and (v_1, v_2) are C-related then both of the above sets are common out-neighbors. Therefore, depending on the relation, $\uparrow(u_1, u_2) \cap \uparrow(v_1, v_2)$ is either $n_2 a_1$, $n_2 b_1 + r_2$ or $n_2 c_1 + 2r_2$. As above, computation of shared in-neighbors gives the same results.

In order for $G_1 \circ G_2$ to be ordinary, for each of the A, B and C relations, either at least one of these two cases must not occur or the two possible values for the number of shared neighbors must be equal. Thus the following equations must hold (if both of the parameters in the equation are not $\times$):

1. $n_2 r_1 + a_2 = n_2 a_1$
2. $n_2 r_1 + b_2 = n_2 b_1 + r_2$
3. $n_2 r_1 + c_2 = n_2 c_1 + 2r_2$.

Rewriting these equations allows us to see how difficult they are to satisfy:

1. $n_2(a_1 - r_1) = a_2$
2. $n_2(r_1 - b_1) = r_2 - b_2$
3. $n_2(r_1 - c_1) = 2r_2 - c_2$.

Examining the first equation, we see that since $n_2 > a_2$ we must have $a_1 - r_1 = 0$ and $a_2 = 0$. Since $n_2 > r_2 - b_2$ but $r_1 > b_1$, the second equation is impossible to satisfy. For the third equation, $2n_2 > 2r_2 - c_2$ and $r_1 > c_1$ so we must have $r_1 - c_1 = 1$ and $2r_2 - c_2 = n_2$. If G_1 and G_2 satisfy all three of these requirements, then we have shown that $G_1 \circ G_2$ will be ordinary of the type listed above. □

Chapter 4

Special cases of ordinary graphs

In this chapter we discuss several classes of graphs and designs that have been previously defined, and which turn out to be special cases of ordinary graphs. We can use the work that has already been done on these objects to help us understand ordinary graphs better.

4.1 Symmetric block designs

Here we show how ordinary graphs are a generalization of symmetric block designs and list some results about these designs.

Let X be a set of v elements $x_1, x_2, ..., x_v$ and let $X_1, X_2, ..., X_b$ be distinct subsets of X. These subsets of X are called a *balanced incomplete block design* (BIBD), or equivalently a $\langle b, v, r, k, \lambda \rangle$*-configuration*, if

1. each X_i has k elements,
2. each 2-element subset of X is a subset of exactly λ of the subsets X_i,
3. the parameters v, k, λ satisfy $0 < \lambda$ and $k < v - 1$.

The definition above does not explain the parameter r. It is defined in the proposition below.

Proposition 4.1. *Given a BIBD as above, pick $x \in X$ and let r be the number of subsets X_i that have x as a member. Then r is invariant across all choices of x.*

Proof. We count the number of occurrences in $X_1, X_2, ..., X_b$ of 2-element subsets of X that contain x. Note that there are $v - 1$ of these 2-element subsets in X. By the second equation in the definition of a BIBD, there are λ occurrences of each of these 2-element subsets, so we count $\lambda(v - 1)$ total occurrences. But by the first equation in the definition, for each X_i that contains

x, X_i contains $k-1$ of these 2-element sets. As there are r such sets X_i, we count $r(k-1)$ total occurrences of 2-element sets containing x in $X_1, X_2, ..., X_b$. Therefore we have shown that $r(k-1) = \lambda(v-1)$ for any $x \in X$. Since v, k, and λ are invariants of the design, so is r. □

One can find many more results about these configurations, see for example Ryser [13]. Here we are interested in those configurations for which $b = v$ and $r = k$. These are called *symmetric balanced incomplete block designs*, or more simply *symmetric block designs*. According to Proposition 4.1, for a symmetric block design we have $r(r-1) = \lambda(v-1)$. Next we give the definition of $\langle v, k, \lambda \rangle$-configurations, which turn out to be symmetric block designs.

Given a set X of v elements $x_1, x_2, ..., x_v$ and v distinct subsets $X_1, X_2, ..., X_v$ of X, the subsets are called a $\langle v, k, \lambda \rangle$-*configuration* if

1. each X_i has k elements,
2. $|X_i \cap X_j| = \lambda$ for $i \neq j$,
3. the parameters v, k, λ satisfy $0 < \lambda < k < v-1$.

The *incidence matrix* of the configuration is the matrix A where $a_{ij} = 1$ if $x_j \in X_i$ and $a_{ij} = 0$ otherwise. By the above properties, A satisfies

$$AA^t = (k-\lambda)I + \lambda J.$$

Recall that a matrix A is *normal* if $AA^t = A^tA$. In [13] it is shown that the incidence matrix of a $\langle v, k, \lambda \rangle$-configuration is normal.

Theorem 4.2. *(Ryser) The incidence matrix A of a $\langle v, k, \lambda \rangle$-configuration is normal.*

Proof. Let $AA^t = B = (k-\lambda)I + \lambda J$. To compute $\det(B)$ we perform several column and row operations. First subtract column 1 from each of the other columns. Then add each of the other rows to row 1. The resulting matrix is lower triangular with $k + (v-1)\lambda$ in the $(1,1)$-entry and $k - \lambda$ in the remaining diagonal entries. Thus

$$\det(B) = (k-\lambda)^{v-1}(k + (v-1)\lambda).$$

Since $\lambda < k$ we know that $\det(A)\ \det(A^t) = \det(AA^t) = \det(B) \neq 0$. Thus $\det(A) \neq 0$ and A is invertible. As each X_i has k elements of X, we have $AJ = kJ$. But $k(A^{-1}J) = A^{-1}(kJ) = A^{-1}AJ = J$ so $A^{-1}J = \frac{1}{k}J$. Now

$$AA^tJ = ((k-\lambda)I + \lambda J)J = (k - \lambda + \lambda v)J$$

and multiplying both sides by A^{-1} we get

$$A^t J = \frac{1}{k}(k - \lambda + \lambda v)J.$$

Taking the transpose of both sides yields

$$JA = \frac{1}{k}(k - \lambda + \lambda v)J,$$

and thus

$$JAJ = \frac{v}{k}(k - \lambda + \lambda v)J.$$

However, from earlier we know that

$$JAJ = kvJ,$$

so therefore

$$k - \lambda = k^2 - \lambda v.$$

We can substitute this in an earlier equation to obtain

$$JA = kJ.$$

Armed with this, we can compute that

$$A^t A = A^{-1}(AA^t)A = A^{-1}BA = (k - \lambda)I + \lambda A^{-1}JA$$

$$= (k - \lambda)I + \lambda k A^{-1}J = (k - \lambda)I + \lambda J = B = AA^t.$$

□

Now that we know A is normal, we have $A^t A = (k - \lambda)I + \lambda J$, from which we can deduce that each $x \in X$ is an element of k of the subsets X_i, and that each two-element subset $\{x_j, x_l\}$ is a subset of λ of the subsets X_i. Thus we see that a $\langle v, k, \lambda \rangle$-configuration is equivalent to a symmetric block design–that is, a BIBD with $b = v$ and $r = k$.

Theorem 4.2 has several other consequences for us. It is now clear that the incidence matrix for a finite projective plane of order n is also the incidence matrix for a $\langle v, k, \lambda \rangle$-configuration where $v = n^2 + n + 1$, $k = n + 1$ and $\lambda = 1$. We can also show that the incidence matrix of a $\langle v, k, \lambda \rangle$-configuration is equivalent an ordinary matrix of type $\langle v, k, \lambda, \lambda, \lambda \rangle$ ordinary graph, a

generalization of Proposition 2.7. As was the case earlier, we need Hall's SDR Theorem to allow us to have 0's on the diagonal.

Proposition 4.3. *The elements and sets of a $\langle v, k, \lambda\rangle$-configuration can be labeled in such a way that the incidence matrix of the configuration is an ordinary matrix of type $\langle v, k, \lambda, \lambda, \lambda\rangle$.*

Proof. Recall that $k < v - 1$. Let $S_i = \{x_j : x_j \notin X_i\}$. Then each set S_i has $v - k$ elements and each element x_i is in $v - k$ of the sets S_i. We must show that for any $I \subseteq \{1, 2, \ldots, v\}$ we have $|\bigcup_{i\in I} S_i| \geq |I|$. The set $\{(x_j, S_k) : x_j \in S_k,\ k \in I\}$ has $(v - k)\,|I|$ elements, and this set is contained in the set $\{(x_j, S_k) : x_j \in \bigcup_{i\in I} S_i,\ x_j \in S_k\}$ which has $(v - k)\,|\bigcup_{i\in I} S_i|$ elements. Thus the condition for Hall's Theorem is met, and we have our SDR. □

We can also use Hall's Theorem to put 1's on the diagonal of the incidence matrix of a $\langle v, k, \lambda\rangle$-configuration. If we then change all these 1's to 0's we obtain an ordinary matrix with slightly different parameters.

Proposition 4.4. *There is a $\langle v, k, \lambda\rangle$-configuration if and only if there is an ordinary matrix of type $\langle v, k - 1, \lambda, \lambda - 1, \lambda - 2\rangle$, where any of a, b or c may be $\times$.*

Proof. In a $\langle v, k, \lambda\rangle$-configuration each element is in k sets and each set contains k elements. Thus we can use the argument from the proof of Proposition 4.3 with X_i in place of S_i to give us an SDR that allows us to have 1's on the diagonal of the incidence matrix of the configuration. Denote this matrix by M. Then subtracting the identity matrix from M results in an ordinary matrix of type $\langle v, k - 1, \lambda, \lambda - 1, \lambda - 2\rangle$. Note that if $\lambda = 1$, then the ordinary matrix will have $c = \times$, because having a 1 in both of the entries (i, j) and (j, i) for $i \neq j$ of M means that $\{x_i, x_j\} \subseteq S_i \cap S_j$, which contradicts $\lambda = 1$. This was noticed by Jørgensen in [12].

Conversely if M is an ordinary matrix of type $\langle v, k - 1, \lambda, \lambda - 1, \lambda - 2\rangle$ (with any of a, b or $c = \times$), it is clear that adding the identity matrix to M yields the incidence matrix of a $\langle v, k, \lambda\rangle$-configuration. □

In light of Proposition 4.3, we may expect the quantity $\frac{a\alpha+b\beta+c\gamma}{\binom{n}{2}}$ to play the same role for an ordinary graph G that λ does for a $\langle v, k, \lambda\rangle$ design, since this quantity represents the average number of common neighbors for a pair of vertices in G. In fact by Lemma 2.1,

$$\frac{a\alpha + b\beta + c\gamma}{\binom{n}{2}} = \frac{n\binom{r}{2}}{\binom{n}{2}} = \frac{r(r-1)}{n-1}.$$

As noted above, for a symmetric block design, and thus for a $\langle v, k, \lambda\rangle$ design, $\lambda = \frac{r(r-1)}{v-1}$.

4.2 Strongly regular graphs

In case we have an ordinary graph G with $b = \times$, we may as well consider G as simply a graph rather than a digraph. Such graphs have already been studied extensively (see Shrikhande and Sane [15], Biggs [1] or Gross and Yellen [4]). A regular graph on n vertices with degree r is *strongly regular* with parameters $\langle n, r, \lambda, \mu\rangle$ if

1. each pair of adjacent vertices have λ common neighbors
2. each pair of non-adjacent vertices have μ common neighbors

From this definition we see that an ordinary $\langle n, r, a, \times, c\rangle$ graph is equivalent to a strongly regular graph with $\lambda = c$ and $\mu = a$. Of the many results about strongly regular graphs, we present here the derivation of the eigenvalues of these graphs, following Biggs [1].

Proposition 4.5. *The adjacency matrix A of a strongly regular graph G with parameters $\langle n, r, \lambda, \mu\rangle$ satisfies the equation $A^2 + (\mu - \lambda)A + (\mu - r)I = \mu J$.*

Proof. We count paths of length 2 in G, recalling that the (u, v)-entry of A^2 is the number of paths of length 2 in G. The following counts all come directly from the definition of a strongly regular graph. If u and v are adjacent, then there are λ paths of length 2 from u to v. If u and v are not adjacent, then there are μ such paths. There are r paths of length 2 from a vertex u to itself. Thus $A^2 = \lambda A + \mu(J - I - A) + rI$ and hence $A^2 + (\mu - \lambda)A + (\mu - r)I = \mu J$. □

Using this proposition, we can compute the eigenvalues of a strongly regular graph. Here, we record this fact as a theorem in the language of ordinary graphs.

Theorem 4.6. *An ordinary matrix of type $\langle n, r, a, \times, c\rangle$ has three eigenvalues. They are:*

$$r \text{ with multiplicity } 1,$$

$$\frac{\sqrt{(c-a)^2 + 4(r-a)} + (c-a)}{2}$$

with multiplicity

$$\frac{(n-1)\left(\sqrt{(c-a)^2 + 4(r-a)} - (c-a)\right) - 2r}{2\sqrt{(c-a)^2 + 4(r-a)}},$$

and

$$-\frac{\sqrt{(c-a)^2 + 4(r-a)} - (c-a)}{2}$$

with multiplicity

$$\frac{(n-1)\left(\sqrt{(c-a)^2+4(r-a)}+(c-a)\right)+2r}{2\sqrt{(c-a)^2+4(r-a)}}.$$

Proof. For an the adjacency matrix A of an ordinary graph G of type $\langle n, r, a, \times, c\rangle$, Proposition 4.5 says that $A^2 + (a-c)A + (a-r)I = aJ$. The eigenvalues of the matrix J are n and 0, so the eigenvalues of A are r and the two roots of the polynomial $x^2 + (a-c)\,x + (a-r)$. The multiplicity of r is 1, and we can compute the multiplicities of the other two eigenvalues θ_1 and θ_2. Denote these by m_1 and m_2 respectively. Since A is a real symmetric matrix with trace 0, the eigenvalues with multiplicity sum to 0. Also, the multiplicites must add to n. Thus we have the equations $m_1 + m_2 = n - 1$ and $r + m_1\theta_1 + m_2\theta_2 = 0$, which determine m_1 and m_2. The results of solving are given in the theorem. □

4.3 Normally regular digraphs

In [11], Leif Jørgensen defined normally regular digraphs (NRDs), using different terminology than used in the paper by Fossorier et al. [2] and in this thesis. However, NRDs turn out to be equivalent to ordinary graphs with $c = \times$. In [11], Jørgensen classifies connected NRDs with $\mu = 0$, which are equivalent to ordinary graphs with $a = 0$ and $c = \times$. Jørgensen's proof is omitted here as it is similar to the proof given later in Theorem 5.14. Several other results from [11] are included here, some which we later extend to allow for bidirected edges (2-cycles).

A *normally regular digraph* with parameters $\langle v, k, \lambda, \mu\rangle$ is a simple digraph on v vertices without 2-cycles such that:

1. every vertex has out-degree k,
2. every pair of adjacent vertices has λ common out-neighbors, and
3. every pair of non-adjacent vertices has μ common out-neighbors.

As we will see below, the quantities $k-\mu+(\mu-\lambda)^2$ and $k+\mu-\lambda$ come up often enough that it is convenient to give them names. We will denote them by η and ρ respectively.

If A is the adjacency matrix of a $\langle v, k, \lambda, \mu\rangle$ normally regular digraph then

$$AA^t = kI + \lambda(A + A^t) + \mu(J - I - A - A^t).$$

We can rewrite this in the form

$$(A + (\mu - \lambda)I)(A + (\mu - \lambda)I)^t = (k - \mu + (\mu - \lambda)^2)I + \mu J$$

or

$$(A + (\mu - \lambda)I)(A + (\mu - \lambda)I)^t = \eta I + \mu J$$

We may call a graph *normal* if its adjacency matrix is normal. Jørgensen goes on to prove that normally regular digraphs are normal, with a proof that is similar to that of Theorem 4.2.

Theorem 4.7. *(Jørgensen) Normally regular digraphs are normal.*

Proof. Let A be the adjacency matrix of a $\langle v, k, \lambda, \mu\rangle$-NRD, and let $B = A + (\mu - \lambda)I$. Then $BB^t = \eta I + \mu J$. Using the same argument as in Theorem 4.2, we can show that $\det(BB^t) = \eta^{v-1}(\eta + \mu v)$. Recalling that $\eta = k - \mu + (\mu - \lambda)^2$, and since $k + (\mu - \lambda)^2 \geq k \geq \mu$, we see that $\eta = 0$ would force $k = \mu = \lambda$. The only way $k = \lambda$ is possible is when $k = 0$, in which case A has no 1's and is trivially normal. Furthermore, we have $\eta + \mu v = k + (v - 1)\,\mu + (\mu - \lambda)^2 > 0$, again unless $k = 0$. Thus we may assume that $\det(BB^t) \neq 0$, so that $\det(B) \neq 0$ and therefore B^{-1} exists.

From the definition of an NRD and the definition of B, every row of B has $\rho = k + \mu - \lambda$ 1's. Thus $BJ = \rho J$ and $B^{-1}J = \frac{1}{\rho}J$. Now

$$B^tJ = B^{-1}(BB^t)J = B^{-1}(\eta I + \mu J)J = \frac{1}{\rho}(\eta + \mu v)J,$$

and taking the transpose of both sides yields

$$JB = \frac{1}{\rho}(\eta + \mu v)J.$$

Thus

$$JBJ = \frac{v}{\rho}(\eta + \mu v)J.$$

However, from earlier we know that

$$JBJ = \rho v J,$$

so therefore

$$\rho = \frac{\eta + \mu v}{\rho}.$$

We can substitute this in an earlier equation to obtain

$$JB = \rho J = BJ.$$

Now we can compute that

$$B^tB = B^{-1}(BB^t)B = B^{-1}(\eta I + \mu J)B = \eta I + \mu J = BB^t.$$

□

Recall that in the definition of ordinary graph we assumed that the graph was normal. Since NRDs have now been shown to be normal, they are equivalent to ordinary graphs with $c = \times$. In the rest of this section, Jørgensen's results about NRDs will be translated into the terminology that we have established for ordinary graphs. The next theorem is critical for our work in this thesis. Jørgensen's proof of it can be found in [11]. Our proof comes later, in Theorem 5.14.

Theorem 4.8. *(Jørgensen) A connected ordinary graph with $a = 0$ or $\times$ and $c = \times$ is either a cycle, an ordinary tournament, or has the adjacency matrix:*

$$\begin{pmatrix} 0 & \bar{1}^t & 0 & \bar{0}^t \\ \bar{0} & T & \bar{1} & T^t \\ 0 & \bar{0}^t & 0 & \bar{1}^t \\ \bar{1} & T^t & \bar{0} & T \end{pmatrix}$$

where T is the adjacency matrix of an ordinary tournament and $\bar{0}$, $\bar{1}$ represent column vectors of all 0s or all 1s.

This third type of graph will be referred to as $D(T)$ throughout this thesis.

We can say something in general about the eigenvalues of ordinary graphs with $c = \times$, and Theorem 4.8 allows us to compute the eigenvalues of a connected ordinary graph with $a = 0$ and $c = \times$. This computation is extended to allow for $c \neq \times$ later.

Theorem 4.9. *(Jørgensen) An ordinary graph of type $\langle n, r, a, b, \times \rangle$ has r as an eigenvalue, and its other eigenvalues lie on a circle in the complex plane with center $b - a$ and radius $\sqrt{\eta} = \sqrt{r - a + (a - b)^2}$.*

Proof. Let A be the adjacency matrix for the graph. Let θ be an eigenvalue of A with eigenvector x. By the normality of A, x is also an eigenvector for A^t with eigenvalue $\bar{\theta}$. Thus $(\theta + a - b)(\bar{\theta} + a - b)$

is an eigenvalue of $(A+(a-b)I)(A+(a-b)I)^t = \eta I + aJ$. But we know that these eigenvalues are $\eta + na$ and η.

It is easy to see that r is an eigenvalue of A, with the all 1's vector as an eigenvector. Multiplying this eigenvector by $\eta I + aJ$ allows us to see that the associated eigenvalue of $\eta I + aJ$ is $\eta + na$. Thus we have deduced that $(r + a - b)^2 = \eta + na$, which can also be deduced from Lemma 2.1 in the case where $c = \times$.

Any other eigenvalue θ of A must be associated with the eigenvalue η of $\eta I + aJ$. Thus $(\theta + a - b)(\bar{\theta} + a - b) = \eta$, which can be rewritten to say

$$|\theta - (b - a)| = \sqrt{\eta}.$$

In other words, all the eigenvalues of A other than r lie on a circle in the complex plane with center $b - a$ and radius $\sqrt{\eta}$. □

Theorem 4.10. *(Jørgensen) The eigenvalues of a connected ordinary graph of type $\langle n, r, a, b, \times\rangle$ where $a = 0$ or $\times$ are:*

1. *the complex n^{th} roots of 1, each with multiplicity 1, if the graph is a cycle;*
2. *r, $-\frac{1}{2}(1+\sqrt{2r+1}i)$, $-\frac{1}{2}(1-\sqrt{2r+1}i)$, with multiplicities $1, r, r$ respectively, if the graph is a tournament;*
3. *-1, r, $\sqrt{r}i$, $-\sqrt{r}i$, with multiplicities r, 1, $\frac{r+1}{2}$, $\frac{r+1}{2}$ respectively, if the graph is $D(T)$ for some ordinary tournament T.*

Proof. For a cycle this is clear. Suppose A is the adjacency matrix of an ordinary tournament. Then we have the equations from Proposition 2.5 for A. Multiply the first by A^t on both sides to obtain $(A^t)^2 = (b+1)(J-I) - A^t$. Substituting this into the second equation gives $A^2 = b(J-I) + A^t$, and substituting using the first equation yields

$$A^2 = bA + (b+1)A^t.$$

Now an eigenvalue θ of A must satisfy

$$\theta^2 = b\theta + (b+1)\bar{\theta}.$$

We can solve this equation to deduce that the possible eigenvalues are:

$$r,\ 0,\ -\frac{1}{2}(1+\sqrt{2r+1}i),\ -\frac{1}{2}(1-\sqrt{2r+1}i).$$

In light of the fact that the eigenvalues sum to zero, the multiplicities are:

$$1,\ 0,\ r,\ r.$$

This means that 0 is not in fact an eigenvalue.

Now suppose that A is the adjacency matrix of $D(T)$ for some ordinary tournament T. We count the paths of length 2 to obtain an equation for A^2. There are zero paths of length 2 from a vertex to itself. If $u \to v$ or $v \to u$ then there are $\frac{r-1}{2}$ paths of length 2 from u to v. If u and v are not adjacent then there are r paths of length 2 from u to v. Thus

$$A^2 = r(J - I - (A + A^t)) + \frac{r-1}{2}(A + A^t),$$

or

$$A^2 + \frac{r+1}{2}(A + A^t) = r(J - I).$$

Therefore if θ is an eigenvalue of A, then either

$$\theta^2 + \frac{r+1}{2}(\theta + \bar{\theta}) = -r$$

or

$$\theta^2 + \frac{r+1}{2}(\theta + \bar{\theta}) = 2r^2 + r.$$

Solving these equations, the possible eigenvalues we get are

$$r,\ -(2r+1),\ -r,\ -1,\ \sqrt{r}i,\ -\sqrt{r}i.$$

However, we know from Theorem 4.9 that all of the eigenvalues (in this case including r) lie on a circle in the complex plane with center $\frac{r-1}{2}$ and radius $\frac{r+1}{2}$. This leaves us with eigenvalues

$$r,\ -1,\ \sqrt{r}i,\ -\sqrt{r}i.$$

Since the eigenvalues sum to 0, the multiplicities are

$$1,\ r,\ \frac{r+1}{2},\ \frac{r+1}{2}$$

as claimed. □

We have seen in this section that finding equations for AA^t in terms of A, A^t, I and J is extremely useful in analyzing graphs and designs. Unfortunately, looking at the definition of ordinary graphs, it seems unlikely that such a nice equation can be found for them. Therefore we are forced to explore other techniques, as we will see in the next chapter.

Chapter 5

Ordinary graphs with $a = 0$

5.1 Motivation

One of the reasons for interest in the $a = 0$ case is that all disconnected ordinary graphs have $a = 0$. Each component of a disconnected ordinary graph must also be ordinary with parameters in agreement with those of the whole graph, so each of these components will have $a = 0$ or $a = \times$. Another reason for interest is found in the following corollary to Lemma 2.1. The restrictions that Lemma 2.1 place on the parameters of an ordinary graph cause a to be small whenever n is large in comparison to r. When n gets large enough, we are forced into the case where $a = 0$.

Corollary 5.1. *Let G be an ordinary graph of type $\langle n, r, a, b, c\rangle$ and k be a positive integer. If $n > \frac{r^2-r}{k} + 2r + 1$ then $a < k$ (and we do not have $a = \times$). In particular, if $n > r^2 + r + 1$ then $a = 0$.*

Proof. Using the equations in Lemma 2.1, we have $\beta+\gamma \leq \beta+2\gamma = nr$, so $n\frac{n-1}{2} = \alpha+\beta+\gamma \leq nr + \alpha$ and thus $\alpha \geq n\left(\frac{n-1}{2} - r\right)$. This tells us that if $n > 2r + 1$ then $\alpha > 0$ so that $a \neq \times$. Our condition $n > \frac{r^2-r}{k} + 2r + 1$ guarantees this. Continuing by using the third equation of Lemma 2.1, we have $nr\frac{r-1}{2} \geq a\alpha \geq an(\frac{n-1}{2} - r)$, so $a \leq \frac{r^2-r}{n-2r-1}$. Hence if $n > \frac{r^2-r}{k} + 2r + 1$, then $a < k$. In particular for $k = 1$ we get that $n > r^2 + r + 1$ implies $a = 0$ □

It is interesting to note that this bound on n that forces $a = 0$ is the same as the *Moore bound* that forces the diameter of a digraph to be greater than 2 (see Gross and Yellen [4]). In general the Moore bound says that if Δ is the maximum vertex degree, n is the number of vertices of the graph, and D is the diameter of the graph, then $n \leq \sum_{i=0}^{D} \Delta^i$. So we see that for an ordinary graph $D > 2$ forces $a = 0$. On the other hand, $a = 0$ does not imply $D > 2$. For example, a 4-cycle has $a = 0$ and $D = 2$.

5.2 The $\overline{C}$ relation

The goal of this section is to show that any connected ordinary graph with $a = 0$, $c \neq \times$ can be expressed as the lexicographic product of a connected ordinary graph with $a = 0$, $c = \times$ and a complete graph. In order to establish this, first we need to prove several general facts about graphs with $a = 0$.

5.2.1 Lemmas about the $a = 0$ case

Here our goal is to establish that if G is connected with $a = 0$ then for any vertex in G there is another in the A relation with it, but nearby (distance 2 away).

Lemma 5.2. *If G is an ordinary graph with $a = 0$ then each vertex of G is in the A relation with at least one other vertex.*

Proof. Since $a \neq \times$ there exist u and v with $u \, A \, v$. We have $|\uparrow u \cap \uparrow v| = 0$, so the sets $\uparrow u$, $\uparrow v$, $\{u\}$ and $\{v\}$ are disjoint. Thus $n \geq 2r + 2$. By Lemma 3.1, to have $\alpha_w = 0$ for any $w \in G$ we must have $\gamma_w = 2r + 1 - n$, and thus $n \leq 2r + 1$. Therefore $\alpha_w > 0$ for all vertices w in G. □

Lemma 5.3. *If $u \, A \, v$ then any minimal directed u, v path P is one-way. In other words, P is an induced subgraph of G.*

Proof. Induct on the following statement: Any minimal directed path of length m whose endpoints are in the A relation is one-way. For $m = 2$ this is clear because any extra arrows would violate the A-related endpoints. Now suppose P is a minimal directed u, v path of length $m > 2$ where $u \, A \, v$. Let v^- be the vertex preceding v and u^+ be the vertex following u in P. If $u \to v^-$ then the path is shortened, and if $v^- \to u$ then u and v have a common neighbor, which is not allowed because $a = 0$. Therefore $u \, A \, v^-$, and thus the arrows in P connecting u and v^- are one-way by the induction hypothesis. Using the same argument on u^+ and v completes the proof. □

Lemma 5.4. *If G is connected with $a = 0$ then for each vertex u in G there is another vertex w that is out-distance 2 away from u and is in the A relation with u.*

Proof. From Lemma 5.2, we know that each vertex u is in the A relation with some other vertex v. Since G is connected we can take a minimal u, v directed path P. Then by Lemma 5.4 $w \nrightarrow u$ for all $w \in P - u$. Thus the vertex that is distance 2 along P from u satisfies our claim. □

5.2.2 Our main lemma

Now we are ready to prove our main lemma.

Lemma 5.5. *If G is a connected ordinary graph with $a = 0$ and u, v are distinct vertices of G, then $A_u = A_v$ if and only if $u \, C \, v$.*

Proof. First, if $u \, C \, v$ and $u \, A \, w$ then both $v \to w$ and $w \to v$ contradict $a = 0$, so we must have $v \, A \, w$ as well. On the other hand, suppose that $A_u = A_v$. By Lemma 5.4 there are x and y such that $u \rightarrowtail x \rightarrowtail y$ and $u \, A \, y$. Let $S = \uparrow u \cap \uparrow x$. Since $u \, B \, x$, we know that $|S| = b$. The fact that $a = 0$ allows us to determine the relationships between all of these vertices we have introduced (see Figure 5.2.2). We have $x \rightsquigarrow S$ and $x \rightarrowtail y$, so y cannot be in the A relation with any vertex in S since $a = 0$. Furthermore $u \rightsquigarrow S$ and $u \, A \, y$, so in fact we must have $S \rightsquigarrow y$, again because $a = 0$. Since $A_u = A_v$, we must have $v \, A \, y$, and no vertex in $S \cup \{x\}$ can be in the A relation with v. In fact, it must be that $v \rightarrowtail x$ because $x \to y$ and $a = 0$. Finally, we can conclude that $v \rightsquigarrow S$ because $S \rightsquigarrow y$ and $a = 0$. But now $|\uparrow u \cap \uparrow v| \geq |S \cup \{x\}| = b + 1$, so u and v can be in neither the A nor the B relation, leaving $u \, C \, v$ as the only remaining possibility. □

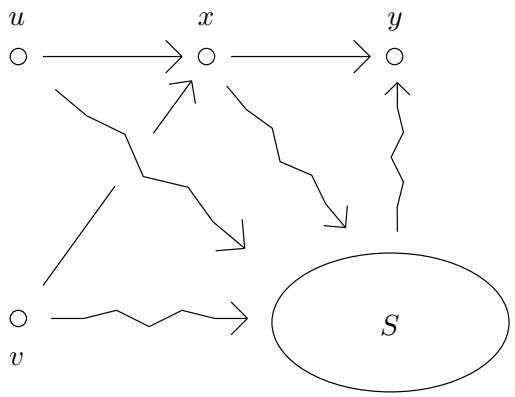

Figure 5.1: Consequences of $u \to x \to y$ and $a = 0$

At this point we introduce the relation $\overline{C}$ on the vertices of a digraph G, where $u \, \overline{C} \, v$ if $u \, C \, v$ or $u = v$. Lemma 5.5 allows us to deduce that the $\overline{C}$ relation is transitive, which makes it an equivalence relation on G.

Corollary 5.6. *If G is a connected ordinary graph with $a = 0$, then the $\overline{C}$ relation on G is an equivalence relation.*

Proof. $\overline{C}$ is already reflexive and symmetric. If $u\,\overline{C}\,v\,\overline{C}\,w$ then by the above lemma $A_u = A_v = A_w$. Using the reverse implication we get $u\,\overline{C}\,w$. Hence $\overline{C}$ is transitive. □

5.2.3 $\overline{C}$ is a strong congruence

We are now ready to show that $\overline{C}$ is a strong congruence on connected ordinary graphs with $a = 0$. This means we can factor out the $\overline{C}$ relation to obtain smaller graphs. Later we will see that the resulting graphs remain ordinary.

Lemma 5.7. *If G is a connected ordinary graph with $a = 0$, then G is uniform and $b = \frac{r-1}{2}$ and $c = r - 1$ (with either one possibly being $\times$).*

Proof. Recall from Proposition 3.2 that for any ordinary graph G with $a = 0$, either G is uniform or $b = \frac{r-1}{2}$ and $c = r - 1$. Now we'll show that if G is also assumed to be connected we can prove either of these by assuming the other, thereby establishing both.

First suppose that G is uniform and $c \neq \times$. Let $u\,C\,v$ and $u \to w$. We'd like to show that $v \to w$ as well. If $u\,C\,w$ then $v\,C\,w$ by Corollary 5.6, so we do have $v \to w$. Therefore we may assume that $w \nrightarrow u$. Then we must have $v\,B\,w$ since $|\downarrow v \cap \downarrow w| > 0$. We know from Lemma 5.5 that $A_u = A_v$, but if $w \to v$ then any other vertex adjacent to w must be adjacent to either u or v since $a = 0$ (see Figure 5.2.3), and in fact must then be adjacent to both since $u\,C\,w$. This means that $\widetilde{A_w} \subset \widetilde{A_u}$, $\widetilde{A_v}$ and thus A_u, $A_v \subset A_w$. The containment must be strict by Lemma 5.5. Then we would have $\alpha_u < \alpha_w$ which contradicts uniformity. Therefore $v \to w$. Since $u \to w$ implies $v \to w$ for any arbitrary w, we have $c = r - 1$. It follows from Lemma 2.1 that if $c = r - 1$ or $c = \times$, then $b = \frac{r-1}{2}$ or $b = \times$.

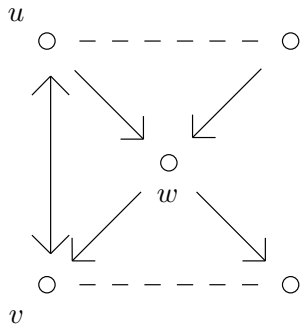

Figure 5.2: Common neighbors of vertices in the C relation

On the other hand, suppose that $b = \frac{r-1}{2}$ and $c = r - 1$. First we'll show that $u \to v$ implies that $\gamma_u \leq \gamma_v$. If $u\,C\,v$, then the transitivity of the C relation gives $\gamma_u = \gamma_v$. Suppose then

that $u\,B\,v$. Then $A_u \neq A_v$, so either $A_u \subset A_v$, which implies $\gamma_u < \gamma_v$, or there exists $w \in A_u - A_v$. In that case we count arrows emanating from u. From the fact that $a = 0$ we can deduce that $v \to w$. Let $S = (\downarrow v \cap \downarrow w) - C_v$ and $T = (\uparrow u \cap \uparrow v) - C_v$. Since $u\,A\,w$ and $a = 0$, and because $S \rightsquigarrow v$, we have $u \rightsquigarrow S$. We also know that $u \rightsquigarrow T$, as well as $u \rightsquigarrow C_u$ and $u \rightsquigarrow C_v \cup v$. All of this is displayed in Figure 5.2.3. The sets S, T, C_u and $C_v \cup v$ are all disjoint as a result of the $a = 0$ hypothesis. Thus $r = |\uparrow u\,| \geq (b - \gamma_v) + (b - \gamma_v) + (\gamma_u) + (\gamma_v + 1)$. Using $b = \frac{r-1}{2}$, this inequality yields $\gamma_u \leq \gamma_v$. We have now established that $u \to v$ implies $\gamma_u \leq \gamma_v$. If u and v are arbitrary vertices of G, then there is a path from u to v and a path from v to u since G is connected. Thus $\gamma_u \leq ... \leq \gamma_v \leq ... \leq \gamma_u$, so $\gamma_u = \gamma_v$. It follows that G is uniform.

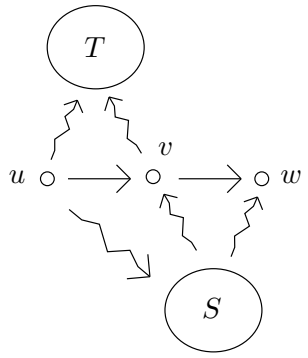

Figure 5.3: Counting the vertices in $\uparrow u$ in the case $c \neq \times$

□

The conclusions of Lemma 5.7 are very nice because of the following lemma.

Lemma 5.8. *If G is a uniform ordinary graph with $c = r - 1$ then the $\overline{C}$ relation on G is a strong congruence.*

Proof. Since $c = r - 1$, each $\overline{C}$ equivalence class is externally related. In addition, uniformity of G ensures that the $\overline{C}$ classes are all the same size. Thus they are isomorphic to the same complete graph. □

Corollary 5.9. *The $\overline{C}$ relation is a strong congruence on any connected ordinary graph with $a = 0$.*

5.2.4 Factoring out the $\overline{C}$ relation

When $\overline{C}$ is a strong congruence on G, we may consider the graph $G/\overline{C}$. If G is a connected ordinary graph with $a = 0$, then $G/\overline{C}$ turns out to be a connected ordinary graph H with

$a = 0$ and $c = \times$. This shows us that all connected ordinary graphs with $a = 0$ and $c \neq \times$ are lexicographic products of such graphs H, and is a large step towards classifying all connected ordinary graphs with $a = 0$.

Theorem 5.10. *Let G be a uniform ordinary graph of type $\langle n, r, a, b, c\rangle$ with $c = r - 1$. Let $m = \gamma_u + 1$ for any $u \in G$ (γ_u is constant across all u). Then G is isomorphic to the lexicographic product $H \circ K_m$ where H is a uniform ordinary graph of type $\langle \frac{n}{m}, \frac{r-m+1}{m}, \frac{a}{m}, \frac{b-m+1}{m}, \times\rangle$. If G is connected, then H is connected as well.*

Proof. Since $\overline{C}$ is a strong congruence on G we know from Proposition 3.5 that $H = G/\overline{C}$ is a well-defined graph. The $\overline{C}$ classes are all complete graphs of size m so $G \cong H \circ K_m$ by Proposition 3.5. Let $\hat{n}$ be the number of $\overline{C}$ classes, so that $m\hat{n} = n$ and $\hat{n}$ is the number of vertices in H. H is uniform since $c = \times$ in H, and clearly H will be connected if G is connected. We now show that H is ordinary.

If v is a vertex in G, then let $[v]$ denote the the $\overline{C}$ equivalence class of v in G, which is also a vertex in H. Note that $u\,A\,v$ in G if and only if $[u]\,A\,[v]$ in H and $u\,B\,v$ in G if and only if $[u]\,B\,[v]$ in H. Each vertex in G has $r - (m - 1)$ arrows to (and from) vertices outside of its $\overline{C}$ equivalence class and each equivalence class has m members, so the out-degree (and in-degree) of each vertex in H will be $\frac{r-(m-1)}{m}$. If $[u]\,A\,[v]$ in H then $u\,A\,v$ in G and all elements of $\uparrow u \cap \uparrow v$ and $\downarrow u \cap \downarrow v$ are outside of $[u]$ and $[v]$, so $|\uparrow[u] \cap \uparrow[v]| = |\downarrow[u] \cap \downarrow[v]| = \frac{a}{m}$ in H. If $[u]\,B\,[v]$ in H then $u\,B\,v$ in G. We may suppose without loss of generality that $u \to v$. Then each member of $[u] - \{u\}$ is in $\downarrow u \cap \downarrow v$. Any other member of $\downarrow u \cap \downarrow v$ lies in a different $\overline{C}$ class from u and v, and every other member of this class must be in $\downarrow u \cap \downarrow v$ as well. Thus $|\downarrow[u] \cap \downarrow[v]| = \frac{b-(m-1)}{m}$. We obtain the same for the shared out-neighbors. Hence we have shown that H is ordinary of the type listed above. □

Corollary 5.11. *If G is a connected ordinary graph of type $\langle n, r, 0, b, c\rangle$, then $G = H \circ K_m$, where H is a connected ordinary graph of type $\langle \frac{n}{m}, \frac{r-m+1}{m}, 0, \frac{k-1}{2}, \times\rangle$.*

Proof. This follows from Theorem 5.10 where we have $a = 0$ and $b = \frac{r-1}{2}$. □

From Theorem 5.10 we get another corollary about ordinary graphs with $a = \times$. These are of interest to us when we study ordinary graphs with $a = 0$, because disconnected ordinary graphs have $a = 0$ and each of the connected components have $a = 0$ or $a = \times$. Classifying all ordinary graphs with $a = \times$ is difficult, but we can at least say the following:

Theorem 5.12. *Let G be an ordinary graph of type $\langle n, r, \times, b, c\rangle$.*

1. *If $b = \times$, then G is a complete graph K_{c+2}.*
2. *If $c = \times$, then G is an ordinary tournament of order b.*
3. *If b, $c \neq \times$, then $b = \dfrac{r-1}{2}$ iff $c = r - 1$. If these hold, then $G \cong T \circ K_m$ where T is an ordinary tournament of order k and $n = (4k+3)m$ and $r = 2(k+1)m - 1$.*

Proof. The first two claims and the first part of the third are easy consequences of Lemma 2.1. If G is an ordinary graph of type $\langle n, r, \times, \frac{r-1}{2}, r-1\rangle$, then G is uniform because $a = \times$. Thus, by Theorem 5.10, $G \cong H \circ K_m$ where H is ordinary with $a = \times$ and $c = \times$, i.e., H is an ordinary tournament. The computations for n and r come from Theorem 3.8. □

5.3 Graphs with $a = 0$ and $c = \times$

We have reduced the problem of classifying connected ordinary graphs with $a = 0$ to just looking at those with $c = \times$. It turns out that in this case, for each vertex v the vertex 2 away and in the A relation (as given by Lemma 5.4) is unique. With this, we can show that there are only two possibilities for connected ordinary graphs with $a = 0$ and $c = \times$. They can be either a cycle or $D(T)$ for some ordinary tournament T.

Lemma 5.13. *If G is a connected ordinary graph with $a = 0$ and $c = \times$ then for each $u \in G$ there exists u^*, the unique vertex out-distance 2 from u and such that $\uparrow u \rightsquigarrow u^*$. If $r \geq 2$ then $u^{**} = u$.*

Proof. Choose $u \in G$, and let $u \to v \to w$. Denote $\downarrow v \cap \downarrow w$ by S and $\uparrow u \cap \uparrow v$ by T. Similarly to the situation in Theorem 3.2, we have $u \rightsquigarrow S$ because both u and S have arrows to v, and $T \rightsquigarrow w$ because v has arrows to w and T (see Figure 5.3). Since $b = \frac{r-1}{2}$, the arrows emanating from u are all accounted for by the disjoint sets S, T and v. Every vertex in each of these sets has an arrow to w. Thus $\uparrow u \rightsquigarrow w$. If any other vertex w' were out-distance 2 from u then we could apply the same argument to get $\uparrow u \rightsquigarrow w'$, but then $|\downarrow w \cap \downarrow w'| \geq r$ which is impossible since $a = 0$ and $b = \frac{r-1}{2}$. Now w has the properties claimed above, and for each $u \in G$ we will refer to this unique vertex as u^*.

If $r \geq 2$, let $x, y \in \uparrow u$ and without loss of generality $x \to y$. Then $x, y \to u^*$ so $u^* \to x^*, y^*$. Also $y \to x^*$ so $x^* \to y^*$. We know that $x^* \neq u^*$, so x^* cannot be out-distance 2 from u. However, we have $u \to y \to x^*$ (see figure 5.3). Therefore either $u \to x^*$ which gives $|\uparrow u \cap \uparrow u^*| > 0$, a contradiction, or $x^* \to u$. In that case we have $u^* \to x^* \to u$ so u is out-distance 2 from u^*, and hence $u^{**} = u$.

□

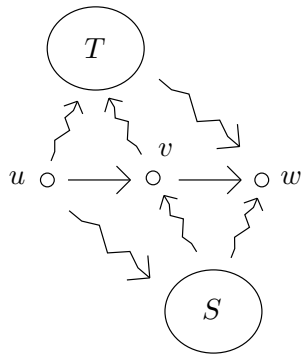

Figure 5.4: Counting the vertices in $\uparrow u$ in the case $c = \times$

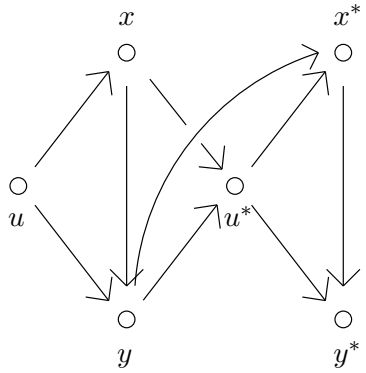

Figure 5.5: Structure of graphs with $c = \times$ and $r \geq 2$

Now we are ready to classify connected ordinary graphs with $a = 0$ and $c = \times$. This theorem is originally due to Leif Jørgensen [11], and was stated earlier in this thesis as Theorem 4.8.

Theorem 5.14. *If G is a connected ordinary graph with $a = 0$ and $c = \times$ then either G is a cycle (if $r = 1$) or G is $D(T)$ for some ordinary tournament T (if $r > 1$). Conversely, if T an ordinary tournament, then $D(T)$ is a connected ordinary graph with $a = 0$ and $c = x$.*

Proof. If G is a connected ordinary graph with $r = 1$ and $c = \times$, then G must be a cycle. If $r > 1$ then we have $x^{**} = x$ for each vertex in G from Lemma 5.13. Pick one vertex in G and designate it 0. Then we also have 0^*, which we will designate with the label $r + 1$. By Lemma 5.13, $\uparrow 0 = \downarrow 0^*$ and $\downarrow 0 = \uparrow 0^*$. Any vertex adjacent to any of the vertices in $\downarrow 0$ or $\uparrow 0$ must then also be adjacent to either 0 or 0^* since $a = 0$. Since G is connected, this means that G must have exactly $2r + 2$ vertices. Therefore G has parameters $\langle 2r + 2, r, 0, \frac{r-1}{2}, \times \rangle$. Now designate the vertices of $\uparrow 0$ with the labels 1 through r and designate the vertices of $\downarrow 0$ with the labels $r + 2$ through $2r + 1$ so that

if we choose a vertex v by its label m with $0 \leq m \leq r$, then the vertex v^* is labeled $m + (r + 1)$. Let T be the subgraph of G induced by $\uparrow 0$. We will show that T must be an ordinary tournament.

First, note that T must be a tournament in order to satisfy the $a = 0$ requirement. Choose u, v in T such that $u \to v$. Then $v \to u^* \to v^* \to u$ as a result of Lemma 5.13. Therefore the adjacency matrix M of G with the labels we have given to the vertices of G appears as shown here:

$$\begin{pmatrix} 0 & \bar{1}^t & 0 & \bar{0}^t \\ \bar{0} & T & \bar{1} & T^t \\ 0 & \bar{0}^t & 0 & \bar{1}^t \\ \bar{1} & T^t & \bar{0} & T \end{pmatrix}$$

Using the fact that M is ordinary, we will show that T is ordinary as well. We have $M + M^t =$

$$\begin{pmatrix} 0 & \bar{1}^t & 0 & \bar{1}^t \\ \bar{1} & T + T^t & \bar{1} & T + T^t \\ 0 & \bar{1}^t & 0 & \bar{1}^t \\ \bar{1} & T + T^t & \bar{1} & T + T^t \end{pmatrix}$$

and $MM^t =$

$$\begin{pmatrix} r & \frac{r-1}{2} \cdot \bar{1}^t & 0 & \frac{r-1}{2} \cdot \bar{1}^t \\ \frac{r-1}{2} \cdot \bar{1} & TT^t + T^tT + J & \frac{r-1}{2} \cdot \bar{1} & TT + T^tT^t \\ 0 & \frac{r-1}{2} \cdot \bar{1}^t & r & \frac{r-1}{2} \cdot \bar{1}^t \\ \frac{r-1}{2} \cdot \bar{1} & TT + T^tT^t & \frac{r-1}{2} \cdot \bar{1} & TT^t + T^tT + J \end{pmatrix}$$

Since M is ordinary of type $\langle 2r+2, r, 0, \frac{r-1}{2}, \times \rangle$, we have $T + T^t = J - I$, $TT + T^tT^t = b(J - I)$ and $TT^t + T^tT + J = bJ + (b+1)I$. Recall from Proposition 2.5 that this is enough to deduce that T is ordinary of type $\langle r, \frac{r-1}{2}, \times, \frac{r-3}{4}, \times \rangle$. On the other hand, if T is an ordinary tournament of type $\langle 4b+3, 2b+1, \times, b, \times \rangle$, then the matrix M is ordinary because T satisfies the above equations. □

Chapter 6

Eigenvalues of ordinary matrices

In this chapter we will present some further results to add on to those obtained by Leif Jørgensen in [11]. His results were listed above in more detail, but to summarize we have:

1. Except for the eigenvalue r, the eigenvalues of an ordinary matrix of type $\langle n, r, a, b, \times \rangle$ all lie on a circle in the complex plane with center $b - a$ and radius $\sqrt{\eta} = \sqrt{r - a + (a - b)^2}$.
2. The eigenvalues of the matrix for an n-cycle are the n complex roots of unity.
3. The eigenvalues of an ordinary tournament with vertex degree r are: r, $-\frac{1}{2}(1 + \sqrt{2r+1}i)$, $-\frac{1}{2}(1 - \sqrt{2r+1}i)$, with multiplicities $1, r, r$ respectively.
4. The eigenvalues of $D(T)$ of type $\langle n, r, 0, b, \times \rangle$ are -1, r, $\sqrt{r}i$ and $-\sqrt{r}i$, with multiplicities r, 1, $\frac{r+1}{2}$ and $\frac{r+1}{2}$.

6.1 Eigenvalues of connected ordinary graphs with $a = 0$

As seen earlier, the ability write matrix equations for AA^t and A^2 in terms of A, A^t, I and J makes it easy to compute the eigenvalues of A. Unfortunately, for ordinary graphs in general there does not seem to be such a nice equation that we can use. For the special case of a connected ordinary graph with $a = 0$ and $r > 0$, we can write:

$$AA^t = rI + \frac{r-1}{2}(A + A^t)$$

since in this case we have shown that $b = \frac{r-1}{2}$ and $c = r - 1$. From this equation we can deduce that all of the eigenvalues of such a matrix (including r in this case) lie on a circle in the complex plane with center $\frac{r-1}{2}$ and radius $\frac{r+1}{2}$, following the method detailed in Theorem 4.10. However, even for this special case, a supporting equation such as the one for A^2 in Theorem 4.10 seems difficult

to find. Therefore we instead develop a method for computing the characteristic polynomial of lexicographic products $A \circ B$ using the characteristic polynomial of A. Then we will be able to compute the eigenvalues of all connected ordinary graphs with $a = 0$ since we already know that they must be the lexicographic product of either a cycle or a graph of the form $D(T)$ with a complete graph.

We require the following fact from a paper by Kovacs, Silver and Williams [8] about determinants of block matrices:

Theorem 6.1. *Let R be a commutative ring. Assume that M is a k by k block matrix of blocks $A^{(i,j)} \in Mat_n(R)$ that commute pairwise. Let $D(M)$ denote the determinant of M viewed as a k by k matrix over $Mat_n(R)$. Then*

$$\det(M) = \det(D(M)) = \det\left(\sum_{\pi \in S_k} (\text{sgn } \pi) A^{(1,\pi(1))} A^{(2,\pi(2))} \cdots A^{(k,\pi(k))}\right)$$

Recall that if A has dimension m and B has dimension n then the adjacency matrix for $A \circ B$ is an mn by mn block matrix M, with m blocks of size n in each row and column. The blocks are B on the diagonal, 0 where A has a 0 and J where A has a 1. Note that ordinary matrices commute with the matrix J. Thus all of the blocks of M commute pairwise as required by Theorem 6.1 above. Here then is our method for computing the characteristic polynomial of $M = A \circ B$.

1. Begin with the characteristic polynomial $p(w)$ of A.
2. Substitute $-\dfrac{x}{y}$ for w in $p(w)$.
3. Multiply the result by y^m
4. Substitute $B - wI$ for x and J for y
5. Compute the determinant of the resulting matrix

When we follow this method, we are computing the characteristic polynomial of $yA+xI$. However, $\det(yA+xI) = y^m \det(A+\frac{x}{y}I) = y^m p(-\frac{x}{y})$. Using x and y allows us to keep track of the matrices we need to multiply together to compute the determinant as in Theorem 6.1.

We are most interested in using this method when $B = K_n$ for some n. Therefore it will be useful to have the following two facts about matrices of the form $sI + tJ$, where I and J have dimension n.

1. $\det(sI + tJ) = s^{n-1}(s + nt)$

2. $(sI + tJ)^m = s^m I + \sum_{i=0}^{m-1} \binom{m}{i} s^i t^{m-i} n^{m-i-1} J$.

The first of these was discussed in the proof of Theorem 4.2 and the second comes from the binomial theorem. Now we are ready to compute the eigenvalues of all connected ordinary graphs with $a = 0$.

Theorem 6.2. *Let G be a connected ordinary graph with $a = 0$.*

1. *If $G = C_m \circ K_n$, then the eigenvalues of G are the roots of $(w+1)^{m(n-1)}((w-n+1)^m - n^m)$;*
2. *If $G = D(T) \circ K_n$ where $D(T)$ has vertex degree r, then the eigenvalues of G are*

$$-1,\ n(r+1)-1,\ n(1+\sqrt{r}i)-1,\ n(1-\sqrt{r}i)-1$$

with respective multiplicities

$$(2n-1)(r+1)-1,\ 1,\ \frac{r+1}{2},\ \frac{r+1}{2}.$$

Proof. To prove the first assertion, we start with $(-1)^m(w^m - 1)$, the characteristic polynomial of an m-cycle. Now we substitute $-\frac{x}{y}$ for w and multiply by y^m to obtain

$$x^m - (-1)^m y^m.$$

We substitute $J - (w+1)I$ for x and J for y:

$$(-1)^m \left(((w+1)I - J)^m - J^m\right)$$

$$= (-1)^m \left((w+1)^m I + \sum_{i=0}^{m-1} \binom{m}{i} (w+1)^i (-1)^{m-i} n^{m-1-i} J - n^{m-1} J\right)$$

$$= (-1)^m \left((w+1)^m I + \left(\left(\sum_{i=0}^{m-1} \binom{m}{i} (w+1)^i (-1)^{m-i} n^{m-1-i}\right) - n^{m-1}\right) J\right).$$

Now we compute the determinant of this matrix using the fact above, and obtain the characteristic polynomial of the lexicographic product $C_m \circ K_n$:

$$(-1)^{mn}(w+1)^{m(n-1)} \left((w+1)^m + \left(\sum_{i=0}^{m-1} \binom{m}{i} (w+1)^i (-n)^{m-i}\right) - n^m\right)$$

$$= (-1)^{mn}(w+1)^{m(n-1)} \left((w+1)^m + (w-n+1)^m - (w+1)^m - n^m\right)$$

$$= (-1)^{mn}(w+1)^{m(n-1)}((w-n+1)^m - n^m)$$

which is as claimed.

To prove the second assertion, we begin with the characteristic polynomial of such a matrix A. In Theorem 4.10 this was found to be:

$$(w+1)^r(w-r)(w^2+r)^{\frac{r+1}{2}}.$$

Now we substitute $-\frac{x}{y}$ in for w and multiply the result by y^{2r+2} to get

$$(-x+y)^r(-x-ry)(x^2+ry^2)^{\frac{r+1}{2}}.$$

Substituting $J-(w+1)I$ in for x and J in for y and simplifying slightly, we obtain

$$((w+1)^rI)((w+1)I-(r+1)J)((w+1)^2I+(n-2w-2)J+nrJ)^{\frac{r+1}{2}},$$

which is equal to

$$((w+1)^rI)((w+1)I-(r+1)J)((w+1)^2I+(n(r+1)-2(w+1))J)^{\frac{r+1}{2}}.$$

Now we compute the determinant, using the fact that the determinant is multiplicative. This gives us the characteristic polynomial of $A \circ K_n$:

$$(w+1)^{rn}((w+1)^{n-1}(w+1-n(r+1))((w+1)^{2(n-1)}((w+1)^2+n^2(r+1)-2n(w+1))^{\frac{r+1}{2}},$$

which is equal to

$$(w+1)^{rn+(n-1)+(n-1)(r+1)}(w-n(r+1)+1)(w-n+1\pm n\sqrt{r}i)^{\frac{r+1}{2}},$$

and finally:

$$(w+1)^{(2n-1)(r+1)-1}(w-n(r+1)+1)(w-n(1\pm\sqrt{r}i)+1)^{\frac{r+1}{2}}.$$

□

6.2 Eigenvalues of other ordinary graphs

Theorem 6.2 gives us the eigenvalues for any connected ordinary graph with $a = 0$. We are also interested in ordinary tournaments, and can use the above method to compute the eigenvalues for several interesting lexicographic products of ordinary tournaments. We may take further products than the ones listed below, but doing so no longer results in ordinary graphs. We also list the parameters of each of these graphs as obtained from Theorem 3.8. To give proper credit, the result on the product of an ordinary tournament with an empty graph is included in Jørgensen's paper [11].

Theorem 6.3. *Let T_r be an ordinary tournament with vertex degree r, and let O_n and K_m be the empty graph on n vertices and the complete graph on m vertices. Then:*

1. *$T_r \circ O_n$ is an ordinary graph of type $\langle n(2r+1), nr, nr, n\frac{r-1}{2}, \times\rangle$ and its adjacency matrix has eigenvalues*

$$0,\ nr,\ -\frac{1}{2}n(1+\sqrt{2r+1}i),\ -\frac{1}{2}n(1-\sqrt{2r+1}i)$$

   with respective multiplicities

$$(n-1)(2r+1),\ 1,\ r,\ r.$$

2. *$T_r \circ K_m$ is an ordinary graph of type $\langle m(2r+1), m(r+1)-1, \times, m\frac{r+1}{2}-1, m(r+1)-2\rangle$ and its adjacency matrix has eigenvalues*

$$-1,\ m(r+1)-1,\ \frac{1}{2}m(1+\sqrt{2r+1}i)-1,\ \frac{1}{2}m(1-\sqrt{2r+1}i)-1$$

   with respective multiplicities

$$(m-1)(2r+1),\ 1,\ r,\ r.$$

3. *$(T_r \circ O_n) \circ K_m$ is an ordinary graph of type $\langle mn(2r+1), m(nr+1)-1, mnr, m(n\frac{r-1}{2}+1)-1, m(nr+1)-2\rangle$ and its adjacency matrix has eigenvalues*

$$0,\ -n,\ n(m(r+1)-1),\ \frac{1}{2}m(1+\sqrt{2r+1}i)-n,\ \frac{1}{2}m(1-\sqrt{2r+1}i)-n$$

with respective multiplicities

$$m(n-1)(2r+1),\ (m-1)(2r+1),\ 1,\ r,\ r.$$

4. *$(T_r \circ K_m) \circ O_n$ is an ordinary graph of type $\langle mn(2r+1), mn(r+1)-1, mn(r+1)-1, mn\frac{r+1}{2}-1, mn(r+1)-2\rangle$ and its adjacency matrix has eigenvalues*

$$-1,\ m-1,\ m(rn+1)-1,\ m-1-\frac{1}{2}mn(1+\sqrt{2r+1}i),\ m-1-\frac{1}{2}mn(1-\sqrt{2r+1}i)$$

with respective multiplicities

$$(m-1)n(2r+1),\ (n-1)(2r+1),\ 1,\ r,\ r.$$

Proof. We proceed as in Theorem 6.2. The characteristic polynomial of T_r is (up to a constant multiple)

$$(w-r)(w^2+w+\frac{r+1}{2})^r$$

Now substitute $-\frac{x}{y}$ for w and multiply by y^{2r+1} to obtain

$$(-x-ry)(x^2-xy+\frac{r+1}{2}y^2)^r.$$

From here we substitute $-wI$ in for x and J in for y to compute the eigenvalues of $T_r \circ O_n$. We make a different substitution later to get the eigenvalues of $T_r \circ K_m$.

$$(wI-rJ)(w^2I+wJ+n\frac{r+1}{2}J)^r$$

$$=(wI-rJ)(w^2I+(w+n\frac{r+1}{2})J)^r$$

Taking the determinant of this matrix gives us

$$w^{n-1}(w-rn)(w^{2(n-1)}(w^2+wn+n^w\frac{r+1}{2}))^r$$

$$=w^{(n-1)(2r+1)}(w-rn)(w+\frac{1}{2}(n\pm n\sqrt{2r+1}i))^r$$

$$=w^{(n-1)(2r+1)}(w-rn)(w+\frac{1}{2}n(1\pm\sqrt{2r+1}i))^r$$

as claimed.

Continuing from $(-x-ry)(x^2-xy+\frac{r+1}{2}y^2)^r$, this time we substitute $J-(w+1)I$ for x and J for y to obtain the eigenvalues of $T_r \circ K_m$.

$$((w+1)I-(r+1)J)((w+1)^2I+(n-2(w+1))J+(w+1)J-nJ+n\frac{r+1}{2}J)^r$$

$$=((w+1)I-(r+1)J)((w+1)^2I-(w+1-n\frac{r+1}{2})J)^r$$

Taking the determinant of this matrix, we get

$$(w+1)^{n-1}(w-n(r+1)+1)((w^{2(n-1)}((w+1)^2-nw-n+\frac{n^2(r+1)}{2}))^r$$

$$=(w+1)^{(n-1)+2r(n-1)}(w-n(r+1)+1)(w^2+(2-n)w+\frac{n^2(r+1)}{2}-n+1)^r$$

$$=(w+1)^{(n-1)(2r+1)}(w-n(r+1)+1)(w+1-\frac{1}{2}n(1\pm\sqrt{2r+1}i))^r$$

as desired.

To obtain the eigenvalues of $(T_r \circ O_n) \circ K_m$ and $(T_r \circ K_m) \circ O_n$, we start with the polynomials we just derived and proceed in exactly the same manner. The details are omitted here as they are entirely similar. For $(T_r \circ O_n) \circ K_m$ we obtain

$$(w+1)^{(2r+1)(m-1)n}(w-m+1)^{(2r+1)(n-1)}(w+1-(rn+1)m)(w+m+1+\frac{1}{2}mn(1\pm\sqrt{2r+1}i))^r,$$

and for $(T_r \circ K_m) \circ O_n$ we obtain

$$w^{(2r+1)(n-1)m}(w+n)^{(m-1)(2r+1)}(w-n(m(r+1)-1))(w+n-\frac{1}{2}m(1\pm\sqrt{2r+1}i))^r.$$

□

We have shown in this section that the eigenvalues of ordinary matrices which can be expressed as lexicographic products of some matrix A with either O_n or K_m are not difficult to compute using this procedure. The process will be more difficult for products $A \circ B$ where B is more complicated because we will no longer have easy expressions for the determinant as we had with matrices of the form $sI+tJ$.

Chapter 7

Further results on ordinary graphs

7.1 Ordinary graphs with $a = r$

We can use some of the same techniques that worked for ordinary graphs with $a = 0$ to investigate ordinary graphs with $a = r$. This allows us to extend a theorem of Jørgensen, which was stated in a different form in [11].

Theorem 7.1. *(Jørgensen) If G is an ordinary graph of type $\langle n, r, r, b, \times\rangle$, then $G \cong T \circ O_m$ where $m = n - 2r$ and T is an ordinary tournament of order $\frac{b}{m}$.*

Here is the generalization:

Theorem 7.2. *If G is a uniform ordinary graph of type $\langle n, r, r, b, c\rangle$, then $G \cong H \circ O_m$ where H is an ordinary graph of type $\langle \frac{n}{m}, \frac{r}{m}, \times, \frac{b}{m}, \frac{c}{m}\rangle$ and $m = n - 2r + \gamma_v$ where v is any vertex of G.*

Proof. Let $x\,A\,y\,A\,z$ in G. Then $|\uparrow x \cap \uparrow y| = |\uparrow y \cap \uparrow z| = r$, so $\uparrow x \cap \uparrow z = \uparrow x \cap \uparrow y = \uparrow y \cap \uparrow z$ and therefore $x\,A\,z$ since b, $c < r$ always. Thus the A relation is transitive on G. Say that $x\,\overline{A}\,y$ if $x\,A\,y$ or $x = y$. Then $\overline{A}$ is an equivalence relation on G and since $a = r$, each $\overline{A}$ class is externally related in G. Each $\overline{A}$ class is the same size because G is uniform. Thus by Proposition 3.5, we have that $G \cong H \circ O_m$ where m is the size of an $\overline{A}$ class. It is easy to see from Lemma 3.1 that $m = \alpha_v + 1 = n - 2r + \gamma_v$ for any $v \in G$. The parameters of H are also straightforward to compute. □

Unlike the case where $a = 0$, ordinary graphs with $a = r$ are not always uniform. For an example, let T_1 be an ordinary tournament of order 1 and T_2 be an ordinary tournament of order 2. Consider the block matrix with $T_1 \circ O_3$ and $T_2 \circ O_2$ on the diagonal and 1's in every other entry. This is an ordinary graph of type $\langle 43, 31, 31, 25, 19\rangle$ which is clearly not uniform.

7.2 $D(T)$ as a Cayley graph

In this section we discuss some properties of the graph $D(T)$. Theorems 7.3, 7.4, and 7.5 are due to Jørgensen (see [11] and [12]).

Theorem 7.3. *For any ordinary tournament T, the graph $D(T)$ is not a Cayley digraph of an abelian group.*

Proof. Suppose that $D(T) = \mathrm{Cay}(G; S)$ for some abelian group G and a subset $S \subset G$. Let k be the order of T (i.e. T is of type $\langle 4k+3, 2k+1, \times, k, \times\rangle$). Then $|G| = 8k+k$ and $|S| = 4k+3$. Let $H = \{x + x : x \in G\}$, and let $y \in S$. Suppose $x \in S$ and $x \to y$. Then there exists $z \in S$ such that $-x + y = z$, so $-z + y = x$ and thus $z \to y$. Therefore, $|\downarrow y|$ must be even unless $y \in H$. But we know that $|\downarrow y|$ is odd because as a graph $\downarrow y$ is an ordinary tournament (which has degree $2k+1$). Thus $y \in H$. Since y was arbitrary in S, we conclude that $S \subseteq H$. We can use the same argument to show that $-S \subseteq H$, and thus $|H| \geq 8k+6 > \frac{1}{2}|G|$, which is impossible since $|H| \leq |G|$ from algebra. Thus we have a contradiction to $D(T) = \mathrm{Cay}(G; S)$ for G abelian. □

It is easy to notice that for $T = C(010)$, $D(T) = \mathrm{Cay}(Q; \{i, j, k\})$ where Q is the order 8 quaternion group. In fact, similar groups often give rise to graphs of the form $D(T)$. First we need some definitions and theorems from [12].

Given a tournament T and a vertex x, define T_x to be the tournament obtained by reversing every edge that joins any two of the three sets $\{x\}$, $\uparrow x$ and $\downarrow x$. We say that T has the *weak switching property* if $T_x \cong T$ for every $x \in T$, and we say that T has the *strong switching property* if for every $x \in T$ we have $T_x \cong T$ by an isomorphism that fixes x.

Theorem 7.4. *The Paley tournaments and the Szekeres tournaments have the strong switching property.*

Recall that a *generalized quaternion group* is a group with generators $\{x, y\}$ such that $x^{2m} = 1$, $x^m = y^2$ and $xyx = y$.

Theorem 7.5. *If T is an ordinary tournament with the strong switching property, where $|T| = 2^n - 1$ for some n, then $D(T)$ is a Cayley digraph of the generalized quaternion group of order 2^{n+1}.*

Theorem 3.6 on lexicographic products and Cayley graphs allows us to extend this result to ordinary graphs with $a = 0$ and $c \neq \times$.

Theorem 7.6. *If $G = D(T) \circ K_m$ where T is an ordinary tournament with the strong switching property such that $|T| = 2^n - 1$, then G is a Cayley digraph of $Q \times Z_m$ where Q is the generalized quaternion group of order 2^{n+1} and Z_m is the additive group of integers modulo m.*

7.3 Infinite ordinary graphs

In this section we consider extending the definition of ordinary graphs to allow infinitely many vertices. There are two possibilities to consider: graphs with infinitely many vertices but finite vertex degree (called *locally finite graphs*) and graphs with infinitely many vertices and infinite vertex degree. Those of the first type admit a generalization of our result classifying finite connected ordinary graphs with $a = 0$. We do not consider graphs with infinite vertex degree here.

We keep the same definition of ordinary graph as before, where n is now allowed to be infinite. The only drawback to this is that complements of ordinary graphs are not necessarily ordinary anymore. However, if we only want to consider locally finite graphs, their complements will not be locally finite either, so this does not seem to be much of a loss.

One example of an infinite ordinary graph is an infinite chain. Let V be the set of integers and let $x \to y$ iff $y = x + 1$. This is an ordinary graph of type $\langle \omega, 1, 0, 0, \times \rangle$. In fact, it is up to isomorphism the only connected infinite permutation graph. We will soon see that it is also the only connected infinite, locally finite ordinary graph with $c = \times$.

Notice that the graph in the example above has $a = 0$. In fact, any infinite, locally finite ordinary graph must have $a = 0$ since $n > 2r + 1$. This inspires us to attempt to apply the results of the previous section to these graphs. Not only does this work–it works a little too well in that it eliminates any analogue to graphs of the form $D(T)$ and leaves us with only infinite chains and their lexicographic products with finite complete graphs.

We find that almost all of the argument that we used to classify connected ordinary graphs with $a = 0$ goes through for locally finite graphs as well. Only some minor modifications are necessary to extend the arguments. We discuss them below.

The first adjustment to make is to note that α will be infinite and for each vertex v, α_v will also be infinite. On the other hand, we know that $a = 0$ as discussed above. Therefore we have limited versions of Lemma 2.1 and Lemma 3.1, which we list below. The proofs of Lemma 2.1 and Lemma 3.1 suffice.

Lemma 7.7. *Let* G *be an infinite, locally finite ordinary graph of type* $\langle n, r, a, b, c\rangle$. *If* $r > 0$ *then at least one of* β *and* γ *is infinite. If* $r > 1$ *then either* β *is infinite and* $b > 0$ *or* γ *is infinite and* $c > 0$ *or both. Also, for each vertex* v *of G,* $\beta_v + 2\gamma_v = 2r$ *and* $b\beta_v + c\gamma_v = r(r-1)$.

Before we can extend our theorem, we also need to slightly adjust two of our definitions. For an infinite graph to be *connected* in any of the senses discussed earlier, we require the path that witnesses the connectedness to be finite. Notice that everything we proved in the section on connectivity of graphs did not depend on having a finite number of vertices or edges. In particular it is still true that for regular infinite digraphs, weak and strong connectivity are equivalent. Since it is impossible for a locally finite infinite graph to have $a = \times$, every disconnected locally finite graph is the disjoint union of components that are either infinite and have $a = 0$, finite with $a = 0$ or finite with $a = \times$.

We also need to slightly modify the definition of uniformity. We will say that the infinite, locally finite graph G is *uniform* if β_v and γ_v are each constant across all vertices v in G. Note that this also causes $|\widetilde{A_v}|$ to be constant across all v as well, which will be important in what follows. Once again, the two equations in Lemma 7.7 are linearly independent unless $2b = c$, which implies (assuming $r > 0$) that $b = \frac{r-1}{2}$ and $c = r - 1$. Now we are ready to extend our previous theorem.

Theorem 7.8. *The only connected infinite, locally finite ordinary graphs are infinite chains and lexicographic products of infinite chains with finite complete graphs.*

Proof. Let G be an infinite, locally finite connected ordinary graph. As discussed above it is clear that $a = 0$. In the original proof it took us some work to establish the fact that in a graph with $a = 0$, $\alpha_v > 0$ for each vertex v. Here with an infinite number of vertices and finite degree for each vertex this is obvious.

Just as in the finite case, we know that either G is uniform or that $b = \frac{r-1}{2}$ and $c = r - 1$. Once again we seek to prove that these conditions imply each other. Following our proof that uniformity of G implies $c = r - 1$, we get that $A_i = A_j \subsetneq A_k$ if $i\,C\,j$ and $i \to k \to j$. Then $\widetilde{A_k}$ is smaller than $\widetilde{A_i}$ which contradicts the uniformity of G. As before, we have shown that $i\,C\,j$ and $i \to k$ imply that $j \to k$, so $c = r - 1$. From this we conclude that $b = \frac{r-1}{2}$ using Lemma 7.7. The proof of the converse goes through with no modifications.

As in the finite case, we can show that G is isomorphic to the lexicographic product $G = H \circ K$ where H is a connected ordinary graph with $a = 0$ and $c = \times$ and K is a complete graph. Given the parameters of G, we know that K must be the complete graph on $\gamma_i + 1$ elements

(where i is any vertex of G), which forces H to be infinite but locally finite in order that G is also. Now we follow our work in Theorem 5.14 above to determine what the possibilities are for such H.

If H has $r = 1$ then it must be a permutation graph, and as we discussed above this means that H must be an infinite chain. Following our proof of Theorem 5.14, we can show that if $r > 1$ then H has $2r + 2$ vertices. As r is finite this means that H cannot be infinite. The $D(T)$ construction that uses ordinary tournaments to obtain ordinary graphs with $a = 0$ cannot result in an infinite, locally finite ordinary graph. Thus all that remains is for H to be an infinite chain. □

7.4 Plane partitions revisited

As discussed earlier, the motivation for study of ordinary graphs was an attempt to construct new finite projective planes, using ordinary graphs as a guide for how to partition a large plane into several smaller isomorphic planes. In this section we show that some of the ordinary graphs that are sought for this purpose are the complements of ordinary graphs with $a = 0$. We can use our study of these graphs in this thesis to provide examples of some graphs that could work for constructing planes, and to show that for some desired parameter sets, no such ordinary graph exists.

Proposition 7.9. *Let $n^2 + n + 1 = N(q^2 + q + 1)$. An ordinary graph G of type $\langle N, n - q, q^2 + q + 1, q^2, q^2 - q - 1\rangle$ (where any of a, b or c may be $\times$) is the complement of an ordinary graph with $a = 0$ if and only if G is a complete graph on $q^2 - q + 1$ vertices or $n = q^2 + 2q + 1$. If $n = q^2 + 2q + 1$, then the complement is of type $\langle q^2 + 3q + 3, 2q + 1, 0, q, 2q\rangle$. An ordinary graph of type $\langle q^2 + 3q + 3, 2q + 1, 0, q, 2q\rangle$ has $a \neq \times$ if $q > 1$.*

Proof. Suppose $n^2 + n + 1 = N(q^2 + q + 1)$ for some integer N, and that we have an ordinary graph of type

$$\langle N, n - q, q^2 + q + 1, q^2, q^2 - q - 1\rangle.$$

Denote the parameters of its complement by $\tilde{r}$, $\tilde{a}$, $\tilde{b}$ and $\tilde{c}$. By Lemma 2.2, the complement will be of type

$$\langle N, N-(n-q)-1, N+(q^2-q-1)-2(n-q), N+q^2-2(n-q)-1, N+(q^2+q+1)-2(n-q)-2\rangle,$$

i.e., of type

$$\langle N, N + q - 1 - n, N + q^2 + q - 1 - 2n, N + q^2 + 2q - 1 - 2n, N + q^2 + 3q - 1 - 2n\rangle.$$

If $\tilde{a} = 0$, then

$$\begin{aligned} N + q^2 + q - 1 - 2n &= 0 \\ N &= 2n - q^2 - q + 1 \\ \frac{n^2 + n + 1}{q^2 + q + 1} &= 2n - q^2 - q + 1 \\ n^2 + n + 1 &= (2n - q^2 - q + 1)(q^2 + q + 1) \\ n^2 + n + 1 &= 2(q^2 + q + 1)n - (q^2 + q)^2 + 1 \\ n^2 - 2(q^2 + q - 1)n + q^2 + q &= 0 \\ (n - (q^2 + q))^2 &= n. \end{aligned}$$

Using the quadratic formula, the equation $(n - y)^2 = n$ has solutions

$$n = \frac{2y + 1 \pm \sqrt{4y + 1}}{2}.$$

If we let $y = q^2 + q$, we obtain the solutions

$$n = \frac{2q^2 + 2q + 1 \pm \sqrt{4q^2 + 4q + 1}}{2} = \frac{2q^2 + 2q + 1 \pm (2q + 1)}{2},$$

so $n = q^2 + 2q + 1$ or $n = q^2$. However, $n = q^2$ gives

$$\begin{aligned} \tilde{r} &= N - (n - q) - 1 = \frac{n^2 + n + 1}{q^2 + q + 1} - (n - q) - 1 \\ &= \frac{q^4 + q^2 + 1}{q^2 + q + 1} - (q^2 - q) - 1 = 0. \end{aligned}$$

This is the complement of a complete graph of type

$$\langle q^2 - q + 1, q^2 - q, \times, \times, q^2 - q - 1 \rangle.$$

These graphs can indeed be used to partition planes, as in Examples 1 and 2 of Section 2.3.2. However, the more interesting graphs occur when we take $n = q^2 + 2q + 1$. Then the parameters of the original graph are $\langle q^2 + 3q + 3, q^2 + q + 1, q^2 + q + 1, q^2, q^2 - q - 1 \rangle$ and the parameters of its complement are $\langle q^2 + 3q + 3, 2q + 1, 0, q, 2q \rangle$. For a graph of this type, if $q > 1$ we must

have $\tilde{a} = 0$ and not $\tilde{a} = \times$ because in general for $\langle n, r, a, b, c\rangle$ ordinary graphs, having $n > 2r + 1$ implies $a \neq \times$, and it is clear that $q^2 + 3q + 3 > 2(2q + 1) + 1$ when $q > 1$. □

Proposition 7.9 shows that we can investigate some ordinary graphs of type $\langle N, n-q, q^2+q+1, q^2, q^2-q-1\rangle$ by studying ordinary graphs of type $\langle q^2+3q+3, 2q+1, 0, q, 2q\rangle$. In the remainder of this section we show how to construct some graphs of this type and show that it is impossible to construct others. Due to our earlier work in this thesis (see Theorems 5.14 and 5.10), we know that such a graph must either be disconnected, must be $C_m \circ K_k$ for some cycle and complete graphs, or must be $D(T) \circ K_k$ for some ordinary tournament T. The parameters for $C_m \circ K_k$ are $\langle mk, 2k-1, 0, k-1, 2k-2\rangle$. For these to match up we would need to have $2k - 1 = 2q + 1$, so $k = q + 1$ and therefore $m = \frac{q^2+3q+3}{q+1}$. Note, however, that $\frac{q^2+3q+3}{q+1} = q + 2 + \frac{1}{q+1}$, which is never an integer except for $q = 0$. The parameters of $D(T) \circ K_k$ where T is an ordinary tournament of order b are $\langle 8(b+1)k, 4(b+1)k-1, 0, 2(b+1)k-1, 4(b+1)k-2\rangle$. For these to work we would need $4(b + 1)k - 1 = 2q + 1$ and $8(b + 1)k = q^2 + 3q + 3$. This is only possible when $q^2 - q - 1 = 0$, which has no integer solutions.

We conclude based on the above observations that the only graphs that fit the parameters for which we are looking are disconnected graphs. These are disjoint unions of graphs with $a = 0$ and graphs with $a = \times$. There are many graphs with $a = \times$, but we require $b = \frac{r-1}{2}$ or $\times$ and $c = r - 1$ or $\times$, which limits the possibilities as noted in Theorem 5.12. The following is a complete list of the graphs we can use in our attempts to construct graphs of type $\langle q^2+3q+3, 2q+1, 0, q, 2q\rangle$. In the list below, C_m is a cycle of length $m > 1$, T is an ordinary tournament of order b, and K_k is a complete graph of size k. For simplicity we allow trivial lexicographic products with K_1, the graph with a single vertex.

- K_k, with parameters $\langle k, k-1, \times, \times, k-2\rangle$
- $C_m \circ K_k$, with parameters $\langle mk, 2k-1, 0, k-1, 2k-2\rangle$
- $T \circ K_k$, with parameters $\langle (4b+3)k, 2(b+1)k-1, \times, (b+1)k-1, 2(b+1)k-2\rangle$
- $D(T) \circ K_k$, with parameters $\langle 8(b+1)k, 4(b+1)k-1, 0, 2(b+1)k-1, 4(b+1)k-2\rangle$

Fixing the parameters so that each component will have the correct vertex degree, the sizes of components that we may build with are as follows:

- K_k must have $2(q + 1)$ vertices ($k = 2(q + 1)$)
- $C_m \circ K_k$ must have $m(q + 1)$ vertices ($m > 1$, $k = q + 1$)

- $T \circ K_k$ must have $\frac{(4b+3)(q+1)}{b+1}$ vertices ($k = \frac{q+1}{b+1}$)
- $D(T) \circ K_k$ must have $4(q+1)$ vertices ($k = \frac{q+1}{2(b+1)}$)

From this list, we note that if we take $T \circ K_{q+1}$ where T has order $b = 0$, then we obtain a component with $3(q+1)$ vertices. Pairing components of this type with components that are complete graphs with $2(q+1)$ vertices allows us to produce graphs with any multiple greater than one of $q+1$ vertices. Since components of the forms $C_m \circ K_k$ and $D(T) \circ K_k$ also have numbers of vertices that are multiples of $q+1$, we can exclude these forms from our search when we consider the sets of parameters that can be achieved.

Proposition 7.10. *If an ordinary graph G of type $\langle q^2+3q+3, 2q+1, 0, q, 2q\rangle$ (where b,c are possibly $\times$) exists, then there exists an ordinary graph G' of the same type which is a disjoint union of complete graphs and lexicographic products of ordinary tournaments with complete graphs. The graphs G and G' need not be isomorphic.*

Of course we are interested in finding as many nonisomorphic graphs of this type as possible, but this proposition narrows our search for values of q that work.

For $q \equiv 1 \pmod 4$ and $q \equiv 3 \pmod 4$ there are infinite classes of examples of graphs that fit the parameters we are seeking. For $q \equiv 1 \pmod 4$ we take the disjoint union of

- one copy of $T_b \circ K_k$ where $b = 1$ and $k = \dfrac{q+1}{2}$, which has $7\,\dfrac{q+1}{2}$ vertices, and
- $\frac{q-1}{4}$ copies of $T_b \circ K_k$ where $b = \dfrac{q-1}{2}$ and $k = 2$, which has $4q+2$ vertices.

The resulting graph has $7\frac{q+1}{2} + (4q+2)\frac{q-1}{4} = q^2+3q+3$ vertices, and has the proper parameters. For $q \equiv 3 \pmod 4$, where $q \geq 11$ we take the disjoint union of

- one copy of K_k with $k = 2q+2$ vertices,
- one copy of $T_b \circ K_k$ with $b = 1$ and $k = \dfrac{q+1}{2}$, which has $7\,\dfrac{q+1}{2}$ vertices,
- one copy of $T_b \circ K_k$ with $b = 3$ and $k = \dfrac{q+1}{4}$, which has $15\,\dfrac{q+1}{4}$ vertices,
- one copy of $T_b \circ K_k$ with $b = \dfrac{q-1}{2}$ and $k = 2$, which has $4q+2$ vertices, and
- $\dfrac{q-11}{4}$ copies of T_b with $b = q$, which has $4q+3$ vertices.

The resulting graph has $(2q+2)+7\frac{q+1}{2}+15\frac{q+1}{4}+(4q+2)+(4q+3)\frac{q-11}{4} = q^2+3q+3$ vertices, and has the proper parameters.

The above examples give us ordinary graphs of the right type for all odd $q \geq 11$. Examples with q even are more difficult to come by. We do have the $\langle 3, 1, \times, 0, \times\rangle$ graph when $q = 0$, but we have not been able to find any more examples where q is even. Below we do some work that culminates in showing that there are no examples when $q+1$ is prime.

Proposition 7.11. *There exists an ordinary graph of type $\langle q^2+3q+3, 2q+1, 0, q, 2q\rangle$ (with b,c possibly being $\times$) if and only if there exists an integer solution $\{x_1, \ldots, x_m, y\}$ to the equation*

$$\sum_{m \mid q+1} (4(q+1)-m)\, x_m + 2(q+1)\, y = q^2+3q+3.$$

Proof. Recall from Proposition 7.10 that in order to get a desired set of parameters, we only need to consider unions of graphs of the form K or $T \circ K$. Also, if we use $T \circ K_k$ where T has order b, then $k = \frac{q+1}{b+1}$. This limits our choices for b to those where $b+1$ divides $q+1$. If we have such a value of b, let m be such that $(b+1)m = q+1$. Then $T \circ K_k$ has $\frac{4b+3}{b+1}(q+1) = 4(q+1)-m$ vertices. Therefore, we can construct a graph of type $\langle q^2+3q+3, 2q+1, \times, q, 2q$ if and only if we can solve the above equation. □

As we see in Proposition 7.11, the more divisors $q+1$ has, the more likely it is that we can construct the graph we want. The examples that were shown above when $q \equiv 1 \pmod 4$ and $q \equiv 3 \pmod 4$ work because we have enough divisors of $q+1$. However, if $q+1$ is prime there are so few options available to construct our graph that it is impossible.

Proposition 7.12. *If $q+1$ is an odd prime, then there is no ordinary graph of type $\langle q^2+3q+3, 2q+1, 0, q, 2q\rangle$, and thus there is no ordinary graph of type $\langle q^2+3q+3, q^2+q+1, q^2+q+1, q^2, q^2-q-1\rangle$ (any of a,b,c may be $\times$).*

Proof. If $q+1$ is prime, then its only divisors are 1 and $q+1$, so we can rewrite the equation from Proposition 7.11 in the form

$$(3q+3)\, x_1 + (4q+3)\, x_2 + (2q+2)\, y = q^2+3q+3$$

We can combine terms to write

$$(4q+3)\, x_2 + (q+1)\,(3x_1+2y) = q^2+3q+3.$$

Renaming variables so that $s = x_2$ and $t = 3x_1 + 2y$, we have

$$(4q + 3)\, s + (q + 1)\, t = q^2 + 3q + 3$$

where $t \neq 1$. We also make a variable substitution $w = q + 1$. We can then write this as a quadratic in w:

$$w^2 + (1 - 4s - t)\, w + (s + 1) = 0.$$

The solutions are

$$w = \frac{4s + t - 1 \pm \sqrt{(4s + t - 1)^2 - 4(s + 1)}}{2}.$$

We require that w be an integer, so the term under the square root must be a square. In general, if $k > 0$ and $z^2 - k$ is a square, then $z^2 - k \leq (z - 1)^2$, so $k \geq 2z - 1$. Therefore, since $4(s+1) > 0$, we must have $4s + 2t \leq 7$. It is easy to check that the only such solutions are when $t = 3$ and $s = 0$, which leads to $q = 0$, and $t = 0$, $s = 1$ which leads to either $q = 0$ or $q = 1$. In these cases, $q + 1$ is not an odd prime, so the claim holds. □

Note that the solutions that we do get in the proof of Proposition 7.12 correspond to graphs of types $\langle 3, 1, \times, 0, \times\rangle$ and $\langle 7, 3, \times, 1, \times\rangle$.

Chapter 8

Conclusions

8.1 Conclusions

In this thesis, we have presented ordinary graphs as a generalization of several types of graphs and designs. Several theorems of Leif Jørgensen have been extended to allow for bidirected edges in the graph. Among these are a characterization of all connected ordinary graphs with $a = 0$, and some work on ordinary graphs with $a = \times$. The theorem listing all connected ordinary graphs with $a = 0$ was extended to locally-finite, infinite ordinary graphs. We also computed the eigenvalues of the most important graphs discussed in this thesis. In much of the work, the lexicographic product of graphs was key.

We also discussed the connection between ordinary graphs and subplane partitions of projective planes, introduced in Fossorier et al. [2]. Several known planes can be partitioned using ordinary graphs, but this technique has yet to lead to the discovery of new projective planes. In this thesis, we presented two infinite classes of ordinary graphs that have the necessary parameters to possibly partition a plane. However, to find a subplane partition it is still necessary to solve a complicated system of equations. We also presented a proof that rules out the existence of some sought-for ordinary graphs.

8.2 Further research

There are several questions that are left unanswered in this thesis. The foremost outstanding problem is to attempt to solve the equations necessary to partition a new projective plane using some of the ordinary graphs presented here. This will be difficult, considering that in [2], the ordinary graphs used to generate examples we always circulant in order to make these equations

easier to solve. Unfortunately, the graphs presented here are nonuniform, and therefore not vertex-transitive. Thus they cannot be Cayley graphs of any group. This means that we probably cannot do much to make the equations any easier to solve.

Another question is, how much of the definition of an ordinary graph is unnecessary? We saw that in some of the special cases of ordinary graphs it was not necessary to require the number of common out-neighbors and in-neighbors to be the same. We still seek either a proof that we can relax our definition or an example of a graph that shows that its full strength is required.

There is also room for a more complete investigation of infinite ordinary graphs, where the vertex degree is allowed to be infinite. An analogue to graphs of the form $D(T)$ should exist in this case, which will lead to a larger class of infinite ordinary graphs.

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