LINEAR AND NON-LINEAR OPERATORS, AND THE DISTRIBUTION OF ZEROS OF ENTIRE FUNCTIONS

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ABSTRACT

An important chapter in the theory of distribution of zeros of entire functions pertains to the study of linear operators acting on entire functions. This dissertation presents new results involving not only linear, but also some non-linear operators.

If $\{\gamma_k\}_{k=0}^{\infty}$ is a sequence of real numbers, and $Q = \{q_k(x)\}_{k=0}^{\infty}$ is a sequence of polynomials, where $\deg q_k(x) = k$, associate with the sequence $\{\gamma_k\}_{k=0}^{\infty}$ a linear operator T such that $T[q_k(x)] = \gamma_k q_k(x)$, $k = 0, 1, 2, \ldots$. The sequence $\{\gamma_k\}_{k=0}^{\infty}$ is termed a *Q-multiplier sequence* if T is a hyperbolicity preserving operator. Some multiplier sequences are characterized when the polynomial set Q is the set of Jacobi polynomials. In a related question, a family of second order differential operators which preserve hyperbolicity is established. It is shown that a real entire function $\varphi(x)$, expressed in terms of Laguerre-type inequalities, do not require the *a priori* assumptions about the order and type of $\varphi(x)$ to belong to the Laguerre-Pólya class. Recently, P. Brändén proved a conjecture due to S. Fisk, P. R. W. McNamara, B. E. Sagan and R. P. Stanley. The result of P. Brändén is extended, and a related question posed by S. Fisk regarding the distribution of zeros of polynomials under the action of certain non-linear operators is answered.

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INDEX OF NOTATION

The following is the index of notation with a brief description for each entry. Other special notations, which appear locally within statements of results, are not mentioned because of their limited scope.

$Z_c(p(x))$ number of non-real zeros of $p(x)$, counting multiplicities 2

$R(p, p')$ resultant of $p(x)$ 7

$\Delta[p(x)]$ discriminant of $p(x)$ 7

$W[f, g]$ Wronskian of $f(x)$ and $g(x)$ 12

${}_pF_q$ Generalized hypergeometric function 14

$(\alpha)_n$ Pochhammer symbol 14

$\mathscr{L}$-$\mathscr{P}$ Laguerre-Pólya class 24

$\mathscr{L}$-$\mathscr{P}^+$ set of functions in $\mathscr{L}$-$\mathscr{P}$ with non-negative Taylor coefficients 24

$H_n(x)$ n^{th} Hermite polynomial 9

$H_n^\alpha(x)$ n^{th} generalized Hermite polynomial 10

$L_n(x)$ n^{th} Laguerre polynomial 9

$L_n^\alpha(x)$ n^{th} generalized Laguerre polynomial 10

$P_n^{(\alpha,\beta)}(x)$ n^{th} Jacobi polynomial 13

$C_n^\nu(x)$ n^{th} Gegenbauer polynomial 16

$\pi(\Omega)$ polynomials whose zeros lie in Ω 43

$\pi_n(\Omega)$ polynomials of degree $\leq n$ whose zeros lie in Ω 43

CHAPTER 1
INTRODUCTION

1.1 Historical remarks

One of the fundamental open problems in the study of distributions of zeros of entire functions stems from Bernhard Riemann. In 1859, he investigated a problem which involves the *zeta function*, initially defined as

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} \quad \text{where } \operatorname{Re} z > 1.$$

The function $\zeta(z)$ can be extended analytically to the entire complex plane, except for a simple pole at $z = 1$, where the extension is again denoted by $\zeta(z)$. It is conjectured that the non-trivial zeros of $\zeta(z)$ lie on the *critical line* $\{z : \operatorname{Re} z = 1/2\}$. This problem, more commonly known as the *Riemann Hypothesis*, can be equivalently stated in terms of the zeros of an entire function. Let

$$\xi(z) = (z-1)\pi^{-z/2}\Gamma\left(\frac{z}{2}+1\right)\zeta(z), \tag{1.1}$$

where $\Gamma(z)$ denotes the gamma function. Then the Riemann Hypothesis is equivalent to the statement that the function $\xi(1/2 + iz)$ has only real zeros [66]. Investigating the zeros of functions such as $\xi(z)$ in (1.1) is a part of the theory of the location and distribution of zeros of entire functions.

For a sequence of real numbers $\{\gamma_k\}_{k=0}^{\infty}$, we can define a linear operator T on the vector space $\mathbb{R}[x]$ by

$$T[x^n] = \gamma_n x^n \quad (n = 0, 1, 2, \ldots). \tag{1.2}$$

The following problem, suggested by E. Laguerre in 1884, inspired a vast literature on the effect of transformations on entire functions that preserve the location of zeros in a specified region.

Problem 1. *Characterize all real sequences* $\{\gamma_k\}_{k=0}^{\infty}$ *such that*

$$Z_c\left(\sum_{k=0}^{n} \gamma_k a_k x^k\right) \le Z_c\left(\sum_{k=0}^{n} a_k x^k\right), \tag{1.3}$$

where $Z_c(p(x))$ *denotes the number of non-real zeros of* $p(x)$*, counting multiplicities.*

Laguerre [47] and Jensen [44] discovered a number of sequences $\{\gamma_k\}_{k=0}^{\infty}$ whose corresponding operator T defined by (1.2) maps every polynomial which has only real zeros into polynomials with only real zeros. In their 1914 paper [56], G. Pólya and J. Schur completely characterized all sequences such that the corresponding operators maps real polynomials with only real zeros to real polynomials with only zeros.

Investigations of linear operators which preserve hyperbolicity (cf. Definition 89) and stability (cf. Definition 92) are of current interest, and some of the main topics of this disquisition will focus on such operators.

1.2 Synopsis

In Chapter 2, we will present preliminary results on entire functions, investigate problems (Problems 36, 39, and 57) related to the Malo-Schur-Szegő composition theorem (Theorem 34), and establish a new result (Theorem 71) on the generalized Laguerre inequality, based on the Borel-Carathéodory inequality (Theorem 69) and Lindelöf's theorem (Theorem 70).

We investigate various linear operators acting on entire functions in Chapter 3. In the course of our investigation, we revisit Problem 57 from the viewpoint of linear operators (Problems 80 and 82). The new results in Chapter 3 are Theorems 127, 128, 131, 132, and 134. These theorems lead to a complete characterization of certain second order differential operators which preserve hyperbolicity (Theorem 135).

In Chapter 4, we investigate multiplier sequences acting on various polynomial bases. The main results in this chapter (Theorem 150 and Proposition 151) pertain to multiplier sequences for Jacobi polynomials, where we generalize results of T. Forgács et al. [5]. We also establish an affirmative answer to a conjecture of T. Forgács and A. Piotrowski (Proposition 142).

We obtain results in Chapter 5 on non-linear operators acting on the Laguerre-Pólya class which preserve hyperbolicity and stability. The main results in this chapter include extensions of a result of P. Brändén (Propositions 157 and 158), some answers to questions posed by S. Fisk (Theorems 160, 161, and Propositions 170, 174), a result on the location of zeros of a hypergeometric function (Proposition 171), and some results concerning a non-linear operator (Propositions 175 and 176).

Index of results and questions

To the author's best knowledge, the following results and problems posed appear to be new.

Chapter 2:

Problems 36, 39, 57, and

Theorem 71.

Chapter 3:

Problems 80, 82,

Lemmas 75, 121, 122, 125, 126, 133, 130,

Propositions 118, 120, 78,

Theorems 127, 128, 131, 132, 134 and 135.

Chapter 4:

Problems 140, 145, 147, 148,

Lemma 149,

Proposition 142, 151, and

Theorem 150,

Chapter 5:

Problems 162, 163, 167,

Lemmas 159, 169, 173,

Propositions 157, 158, 170, 171, 174, 175, 176,

Theorems 160 and 161.

CHAPTER 2
POLYNOMIALS AND TRANSCENDENTAL ENTIRE FUNCTIONS

This chapter has a three-fold purpose: (i) to present preliminary results on entire functions which will be essential to our subsequent exposition, (ii) to investigate problems (Problems 36, 39, and 57) related to the Malo-Schur-Szegő composition theorem (Theorem 34), and (iii) to establish a new result (Theorem 71) on the generalized Laguerre inequality, based on the Borel-Carathéodory inequality (Theorem 69) and Lindelöf's theorem (Theorem 70).

The sections in this chapter are organized under the following headings: Zeros of polynomials (Section 2.1), Orthogonal polynomials (Section 2.2), Transcendental entire functions (Section 2.4), and Generalized Laguerre inequality (Section 2.5).

2.1 Zeros of polynomials

We will call a complex number z_0 a *zero* of the complex function $f(z)$ if $f(z_0) = 0$, and we will say that z_0 is a *root* of the equation $f(z) = 0$. Among many interesting connections between the zeros of a function and its derivative, we mention Rolle's theorem. Suppose a real-valued function $f(x)$ is differentiable on the interval (a, b), and $f(x)$ is continuous at a and b. If $f(a) = f(b)$, then there exists a number c in the interval (a, b) such that $f'(c) = 0$. In particular, if a and b are zeros of $f(x)$, then there is a zero of $f'(x)$ which lies between a and b. As a consequence of Rolle's theorem, if $f(x)$ has exactly m zeros in the interval $[a, b]$, counting multiplicities, then $f'(x)$ has at least $m-1$ zeros in the interval $[a, b]$, counting multiplicities. In particular, if a polynomial has only real zeros, its derivative also has only real zeros. We adopt a nomenclature recently introduced in the literature.

Definition 2. A polynomial $p(x) \in \mathbb{R}[x]$ whose zeros are all real is said to be *hyperbolic*.

Remark 3. We adopt the convention of G. Pólya and J. Schur [56, footnote, p. 89]; "Hierbei zählen wir die Konstanten zu den Polynomen mit lauter reellen Nullstellen," that is, we count the constant functions to be hyperbolic. This convention becomes convenient when we consider the classes of functions introduced in Section 2.4.1.

In contrast to polynomials, entire functions in general do not always behave well under differentiation.

Example 4. Consider the entire function $f(z) = ze^{z^2}$. The function $f(z)$ has its only zero at $z = 0$. However, its derivative is

$$f'(z) = e^{z^2}(2z^2 + 1),$$

which has non-real zeros. We will return to discuss this function, and entire functions whose zeros remain real under differentiation (cf. Section 2.4.1).

Because of the fundamental role in which Rolle's theorem plays in the theory, many authors such as Schoenberg [61], Sendov [62], J.-Cl. Evard and F. Jafari [33] have investigated complex analogues of the theorem of Rolle. The following theorem is a similar result due to Gauss, that gives the locations of the critical points, although beautiful and relevant in its statement, it is not an exact analogue of Rolle's theorem.

Theorem 5 (Gauss-Lucas Theorem [50, p. 8],[57, Theorem 1.2.1])**.** *If $p(z)$ is a non-constant polynomial, then the zeros of $p'(z)$ belong in the convex hull of the zeros of $p(z)$.* □

The oft quoted theorem that is viewed as the complex analogue of Rolle's theorem is the following (see [33], [50], [61], and [62]).

Theorem 6 (Grace-Heawood Theorem [50, p. 107])**.** *If z_1 and z_2 are any two zeros of an n-th degree polynomial $f(z)$, at least one zero of its derivative $f'(z)$ will lie in the circle C with center at point $[(z_1 + z_2)/2]$ and with a radius of $[(1/2)|z_1 - z_2|(\cot(\pi/n))]$.* □

In consideration of entire functions as the one presented in Example 4, and results such as Theorems 5 and 6, a satisfying complex analogue of Rolle's theorem has not been discovered, even to this day.

2.1.1 Resultants and discriminants

In identifying the zeros of a polynomial, the following notions are quite useful when the polynomial has relatively low degree, or when the coefficients are tractable. The results stated in this section will be employed in Chapter 4.

Definition 7. For a polynomial $p(x) = \sum_{k=0}^{n} a_k x^k$, the *resultant* of $p(x)$ is defined as the $(2n-1) \times (2n-1)$ determinant

$$R(p,p') := \begin{vmatrix} a_n & a_{n-1} & \dots & a_0 & 0 & 0 & \dots & 0 \\ 0 & a_n & a_{n-1} & \dots & a_0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & a_n & a_{n-1} & \dots & a_0 \\ na_n & (n-1)a_{n-1} & \dots & 0\,a_0 & 0 & 0 & \dots & 0 \\ 0 & na_n & (n-1)a_{n-1} & \dots & 0\,a_0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & na_n & (n-1)a_{n-1} & \dots & 0\,a_0 \end{vmatrix}.$$

The *discriminant* of $p(x)$ is defined as

$$\Delta[p(x)] := (-1)^{n(n-1)/2}\frac{1}{a_n}R(p,p').$$

The subscript $\Delta_x[p]$ may be used to clarify the variable of the polynomial.

The discriminant is commonly defined in the following equivalent characterization, which is useful in computing the discriminant, as we will see in Example 13.

Proposition 8 ([2, p. 201], [38, p. 403-404], [57, Theorem 1.3.3])**.** *For a polynomial* $p(x) = \sum_{k=0}^{n} a_k x^k$*,*

$$\Delta[p(x)] = \prod_{i<j}(\alpha_i - \alpha_j)^2 = \prod_{i\neq j}(\alpha_i - \alpha_j),$$

where α_i *are the roots of the polynomial* $p(x)$*.* □

Remark 9. For a quadratic polynomial $p(x) = a_2x^2 + a_1x + a_0$, the discriminant is $a_1^2 - 4a_2a_0$, and the polynomial will have real zeros if and only if the discriminant is non-negative, as one infers from the quadratic formula.

There is a similar characterization for cubic polynomials.

Theorem 10 ([43, p. 154])**.** *Let* $f(x) = ax^3 + bx^2 + cx + d$*,* $a \neq 0$*. Consider the discriminant of* $f(x)$*,* $\Delta := \Delta[f(x)] = b^2c^2 - 4b^3d - 4ac^3 + 18abcd - 27a^2d^2$*. Then*

(i) $\Delta \geq 0$ *if and only if* f *has all real roots, and*

(ii) $\Delta < 0$ *if and only if* f *has one real root and two complex conjugate roots.* □

Remark 11. Given a cubic polynomial $f_a(x) = ax^3 + bx^2 + cx + d$, define a function

$$\delta(a) := b^2c^2 - 4b^3d - 4ac^3 + 18abcd - 27a^2d^2, \tag{2.1}$$

which is the discriminant of $f_a(x)$, dependent on the leading coefficient. If $a = 0$ in (2.1),

$$\delta(0) = b^2c^2 - 4b^3d,$$

and its corresponding polynomial is actually a quadratic, namely, $f_0(x) = bx^2 + cx + d$. The discriminant of $f_0(x)$ is

$$\Delta[f_0(x)] = c^2 - 4bd,$$

which differs from $\delta(0)$ by a factor of b^2.

We conclude the following corollary from Theorem 10, Remarks 9 and 11, that appears to be new, which we will use in in Section 4.3.

Corollary 12. *Let $f(x) = ax^3 + bx^2 + cx + d$. Consider $\Delta := \Delta[f(x)] = b^2c^2 - 4b^3d - 4ac^3 + 18abcd - 27a^2d^2$. Then*

(i) $\Delta \geq 0$ if and only if f has all real roots, and

(ii) $\Delta < 0$ if and only if f has non-real roots,

whether $a = 0$ or not. □

One can easily see from Proposition 8 that the discriminant of any polynomial with only real zeros will be non-negative, but for polynomials of degrees greater than three, the non-negativity of the discriminant does not imply that a polynomial has only real zeros, as seen in the following example.

Example 13. The polynomial $p(x) = x^4 + 1$, has non-real zeros, $x = e^{k\pi/4}$, $k = 1, 3, 5, 7$. By Proposition 8, we compute the discriminant

$$\Delta[p(x)] = \prod_{j<k} (e^{j\pi/4} - e^{k\pi/4})^2 = 2(2e^{\pi/4})(-2)(-2)(2e^{3\pi/4})(-2) = 256,$$

which is positive.

2.2 Orthogonal polynomials

The subject of orthogonal polynomials is a classical one whose origins can be traced to Legendre's work on planetary motion. With important applications to physics and to probability and statistics and other branches of mathematics, the subject flourished through the first third of the 20th century. After the publication of Szegő's well known treatise on the subject [65], mathematicians turned their attention to increasingly greater abstraction. Perhaps as a secondary effect of the computer revolution and the heightened activity in approximation theory and numerical analysis, interest in orthogonal polynomials has gained momentum in recent years.

The fundamental definitions and properties of orthogonal polynomials reviewed in this section will relate to recent investigations discussed in Chapter 4.

Definition 14. A set of polynomials $\{\phi_n(x)\}_{n=0}^{\infty}$ is called a *simple set* if $\phi_n(x)$ is of degree precisely n in x so that the set contains one polynomial of each degree.

Given a simple set of polynomials, any polynomial can be expressed as a unique linear combination of the simple set of polynomials by Definition 14.

Definition 15. Let $\{\phi_n(x)\}_{n=0}^{\infty}$ be a simple set of polynomials. For a strictly positive integrable function $w(x)$ on an interval $a < x < b$, if it is the case that

$$\int_a^b w(x)\phi_n(x)\phi_m(x)dx = 0 \quad \text{for } m \neq n,$$

we say the polynomials $\{\phi_n(x)\}_{n=0}^{\infty}$ are *orthogonal* with respect to the *weight function* $w(x)$ over the interval (a, b).

Example 16. The following classical orthogonal polynomial sequences appear frequently in the literature (see [65], [19], [58], and the references contained therein).

(i) If $a = -\infty$, $b = \infty$, and $w(x) = e^{-x^2}$, then the orthogonal polynomials $\{\phi_n(x)\}_{n=0}^{\infty}$, with respect to $w(x)$ are the *Hermite polynomials* modulo constant factors. For the reader's convenience, we list the first few Hermite polynomials:

$$H_0(x) = 1,$$

$$H_1(x) = 2x,$$

$$H_2(x) = 4x^2 - 2,$$

$$H_3(x) = 8x^3 - 12x,$$

$$H_4(x) = 16x^4 - 48x^2 + 12,$$

$$H_5(x) = 32x^5 - 160x^3 + 120x.$$

The Hermite polynomials can be generalized by replacing the weight function to $w(x) = e^{-x^2/(2\alpha)}$, for $\alpha > 0$ (see Piotrowski [55]), denoted by $H_n^\alpha(x)$.

(ii) If $a = 0$, $b = \infty$, $w(x) = e^{-x}$, then the orthogonal polynomials $\{\phi_n(x)\}_{n=0}^\infty$, with respect to $w(x)$ are the *Laguerre polynomials* modulo constant factors. For the reader's convenience again, we list the first few Laguerre polynomials:

$$L_0(x) = 1,$$

$$L_1(x) = -x + 1,$$

$$L_2(x) = \tfrac{1}{2}(x^2 - 4x + 2),$$

$$L_3(x) = \tfrac{1}{6}(-x^3 + 9x^2 - 18x + 6),$$

$$L_4(x) = \tfrac{1}{24}(x^4 - 16x^3 + 72x^2 - 96x + 24),$$

$$L_5(x) = \tfrac{1}{120}(-x^5 + 25x^4 - 200x^3 + 600x^2 - 600x + 120).$$

The Laguerre polynomials can be generalized by replacing the weight function to $w(x) = e^{-x}x^\alpha$, for $\alpha > -1$ (see Forgács, Piotrowski [36], and Brändén, Ottergren [14]), denoted by $L_n^\alpha(x)$.

(iii) If $a = -1$, $b = 1$, $w(x) = 1$, then the orthogonal polynomials $\{\phi_n(x)\}_{n=0}^\infty$, with respect to $w(x)$ are the *Legendre polynomials* modulo constant factors. For the convenience of the reader once again, we list the first few Legendre polynomials:

$$P_0(x) = 1,$$

$$P_1(x) = x,$$

$$P_2(x) = \tfrac{1}{2}(3x^2 - 1),$$

$$P_3(x) = \tfrac{1}{2}(5x^3 - 3x),$$

$$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3),$$

$$P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x).$$

The Legendre polynomials can also be generalized (see Section 2.2.1). For other orthogonal polynomials, see [65] and [46].

Remark 17. There are several recent investigations pertaining to the location of zeros of polynomials expressed in terms of orthogonal polynomials. In particular, Bleecker and Csordas [6], and Piotrowski [55] investigated the Hermite polynomials; Forgács and Piotrowski [36], as well as Brändén and Ottergren [14] investigated the Laguerre polynomials. Some of their main results will be discussed in Chapter 4.

Some of the well-known properties of orthogonal polynomials, which will be used in Chapter 4, are the following.

Theorem 18 ([58, Theorem 55, p. 149])**.** *Let $w(x) > 0$ on (a, b), and $\{\phi_n\}_{n=0}^{\infty}$ be a simple set of polynomials. If ϕ_n is orthogonal with respect to w, then the zeros of $\phi_n(x)$ are distinct (or simple) and real. In particular, all the zeros lie in the open interval (a, b).* □

Theorem 19 ([65, Theorem 3.3.2, p. 46])**.** *Given a set of orthogonal polynomials $\{\phi_n(x)\}_{k=0}^{\infty}$, let $x_1 < x_2 < \ldots < x_n$ be the zeros of $\phi_n(x)$, $x_0 = a$, $x_{n+1} = b$. Then each interval $[x_\nu, x_{\nu+1}]$, $\nu = 0, 1, 2, \ldots, n$, contains exactly one zero of $\phi_{n+1}(x)$.* □

Theorem 20 ([65, Theorem 3.3.3, p. 46])**.** *Given a set of orthogonal polynomials $\{\phi_n(x)\}_{k=0}^{\infty}$, let $c \in \mathbb{R}$. Then the polynomial*

$$\phi_{n+1}(x) - c\,\phi_n(x)$$

has $n + 1$ distinct real zeros. If $c > 0$ (or when $c < 0$), these zeros lie in the interior of $[a, b]$, with the exception of when $c \leq \phi_{n+1}(b)/\phi_n(b)$ (or when $c < 0$, $c \geq \phi_{n+1}(a)/\phi_n(a)$), the greatest (least) zero which lies in $[a, b]$. □

Remark 21. Given a set of orthogonal polynomials $\{\phi_n(x)\}_{k=0}^{\infty}$, then for all $\gamma, \delta \in \mathbb{R}$,

$$\gamma\,\phi_{n+1}(x) + \delta\,\phi_n(x)$$

has real distinct real zeros.

Proof. If γ or δ is equal to zero, the result follows. If $\gamma, \delta \neq 0$, then the zeros of $\gamma\,\phi_{n+1}(x) + \delta\,\phi_n(x)$ are the same as $\phi_{n+1}(x) + \frac{\gamma}{\delta}\,\phi_n(x)$, and the result follows by Theorem 20. □

The property described in Theorem 19 has the following definition.

Definition 22. Let $f, g \in \mathbb{R}[x]$ with $\deg(f) = n$ and $\deg(g) = m$. We say that f and g have *interlacing zeros*, if f is hyperbolic with zeros $\alpha_1, \ldots, \alpha_n$, g is hyperbolic with zeros $\beta_1, \ldots, \beta_m$, $|n - m| \leq 1$, and one of the following sequences of inequalities hold.

(i) $\alpha_1 \leq \beta_1 \leq \alpha_2 \leq \beta_2 \leq \ldots \leq \alpha_n \leq \beta_m$,

(ii) $\beta_1 \leq \alpha_1 \leq \beta_2 \leq \alpha_2 \leq \ldots \leq \beta_m \leq \alpha_n$,

(iii) $\alpha_1 \leq \beta_1 \leq \alpha_2 \leq \beta_2 \leq \ldots \leq \beta_m \leq \alpha_n$,

(iv) $\beta_1 \leq \alpha_1 \leq \beta_2 \leq \alpha_2 \leq \ldots \leq \alpha_n \leq \beta_m$.

Some situations call for a more precise condition than two polynomials that have interlacing zeros. The following definition is used to clarify such instances.

Definition 23. Given two non-zero polynomials $f, g \in \mathbb{R}[x]$, we say f and g are in *proper position* and write $f \ll g$ if one of the following conditions holds:

(1) f and g have interlacing zeros with form (i) or (iv) in Definition 22 and the leading coefficients of f and g are of the same sign, or

(2) f and g have interlacing zeros with form (ii) or (iii) in Definition 22, and the leading coefficients of f and g are of opposite sign.

Remark 24. By convention, we say that the zeros of any two hyperbolic polynomials of degree 0 or 1 interlace. Also, the zero polynomial is in proper position with any other hyperbolic polynomial f and write $0 \ll f$ or $f \ll 0$.

Definition 25. For any two polynomials $f(x)$ and $g(x)$, the *Wronskian* of $f(x)$ and $g(x)$ is defined by

$$W[f, g] := f(x)g'(x) - f'(x)g(x).$$

It is not difficult to show that if f and g are non-constant polynomials, then $f \ll g$ if and only if $W[g, f] \leq 0$ on the whole real line (see [59, p. 197]). One of the most famous and useful results that involve polynomials with interlacing zeros is the following theorem (see [59, p. 197]).

Theorem 26 (Hermite-Biehler). *Let*

$$f(z) = p(z) + iq(z) = c\prod_{k=1}^{n}(z - \alpha_k) \quad (0 \neq c \in \mathbb{C}),$$

where $p(z)$, $q(z)$ are real polynomials of degree at least 2. Then $p(z)$, $q(z)$ have strictly interlacing zeros if and only if the zeros of $f(z)$ are located in either the open upper half-plane or the open lower half-plane. □

The Hermite-Biehler theorem plays a prominent role in the investigation of the location of zeros of polynomials. In particular, P. Brändén [12] recently utilized the Hermite-Biehler theorem to resolve a conjecture due to S. Fisk, R. P. Stanley, P. R. W. McNamara and B. E. Sagan. We mention parenthetically that the Hermite-Biehler theorem has a transcendental extension, although we will not make use of it in our disquisition (see Levin [48, Chapter VII]).

2.2.1 Jacobi polynomials

Jacobi polynomials are a generalization of the Legendre polynomials (cf. Example 16). Unlike the generalized Hermite and Laguerre polynomials that depend on one parameter, the Jacobi polynomials depend on two parameters (cf. Definition 27), and therefore, often make their investigations more involved (see Chapter 4).

Definition 27. From Definition 15, if $a = -1, b = 1$, $w(x) = (1-x)^{\alpha}(1+x)^{\beta}$, $\alpha > -1$, and $\beta > -1$, then except for a constant factor, the orthogonal polynomial $\phi_n(x)$ with respect to $w(x)$ is the *Jacobi polynomial*, denoted by $P_n^{(\alpha,\beta)}(x)$.

Assurance of the integrability of $w(x)$ is achieved by requiring $\alpha > -1$ and $\beta > -1$; the normalization of $P_n^{(\alpha,\beta)}(x)$ is effected by

$$P_n^{(\alpha,\beta)}(1) = \binom{n+\alpha}{n}. \tag{2.2}$$

The important identity

$$P_n^{(\alpha,\beta)}(x) = (-1)^n P_n^{(\beta,\alpha)}(-x) \tag{2.3}$$

is readily derived by a change of variables.

Rodrigues formula

Some authors use the Rodrigues' formula to define orthogonal polynomials [19, p. 144]. The Rodrigues' formulas are particularly useful for explicit computations that involve orthogonal polynomials. The Rodrigues' formula for the Jacobi polynomials is the following.

Lemma 28 ([65, §4.3])**.** *Given* $\alpha, \beta > -1$*, and* $n = 0, 1, 2, \ldots,$ *we have*

$$(1-x)^{\alpha}(1+x)^{\beta} P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{2^n n!} \left(\frac{d}{dx}\right)^n \left\{(1-x)^{n+\alpha}(1+x)^{n+\beta}\right\}. \tag{2.4}$$

□

We now define some terms which are pertinent to the Jacobi polynomials (cf. [58, §18, §44]).

Definition 29. For integers $p, q \geq 0$, the *generalized hypergeometric function* is defined as

$${}_pF_q\left(a_1, \ldots, a_p; b_1, \ldots, b_q; x\right) := \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k} \frac{x^k}{k!}, \tag{2.5}$$

where the *Pochhammer symbol* is denoted by

$$\begin{aligned} (\rho)_n &:= \rho(\rho+1)(\rho+2)\cdots(\rho+(n-1)), \quad && n \geq 1, \\ (\rho)_0 &:= 1, && \rho \neq 0. \end{aligned} \tag{2.6}$$

On calculating the nth derivative in (2.4) by Leibniz' rule, we obtain the important representation

$$P_n^{(\alpha,\beta)}(x) = \sum_{\nu=0}^{n} \binom{n+\alpha}{n-\nu} \binom{n+\beta}{\nu} \left(\frac{x-1}{2}\right)^{\nu} \left(\frac{x+1}{2}\right)^{n-\nu}$$

$$= \binom{n+\alpha}{n} \left(\frac{x+1}{2}\right)^n \sum_{\nu=0}^{n} \frac{n(n-1)\cdots(n-\nu+1)}{(\alpha+1)(\alpha+2)\cdots(\alpha+\nu)} \binom{n+\beta}{\nu} \left(\frac{x-1}{x+1}\right)^{\nu}$$

$$= \binom{n+\alpha}{n} \left(\frac{x+1}{2}\right)^n {}_2F_1\left(-n, -n-\beta; \alpha+1; \frac{x-1}{x+1}\right),$$

where ${}_2F_1(a_1, a_2; b; x)$ is defined in (2.5). For the convenience of the reader, we list the first few

Jacobi polynomials. These explicit expressions will be implemented in Theorem 150.

$$P_0^{(\alpha,\beta)}(x) = 1,$$

$$P_1^{(\alpha,\beta)}(x) = (1+\alpha) + \frac{(2+\alpha+\beta)}{2}(x-1),$$

$$\begin{aligned} P_2^{(\alpha,\beta)}(x) = {} & \frac{(2+\alpha)(1+\alpha)}{2} + \frac{(3+\alpha+\beta)(2+\alpha)}{2}(x-1) \\ & + \frac{(4+\alpha+\beta)(3+\alpha+\beta)}{8}(x-1)^2, \end{aligned}$$

$$\begin{aligned} P_3^{(\alpha,\beta)}(x) = {} & \frac{(3+\alpha)(2+\alpha)(1+\alpha)}{6} + \frac{(4+\alpha+\beta)(3+\alpha)(2+\alpha)}{4}(x-1) \\ & + \frac{(5+\alpha+\beta)(4+\alpha+\beta)(3+\alpha)}{8}(x-1)^2 \\ & + \frac{(6+\alpha+\beta)(5+\alpha+\beta)(4+\alpha+\beta)}{16}(x-1)^3, \end{aligned}$$

$$\begin{aligned} P_4^{(\alpha,\beta)}(x) = {} & \frac{(4+\alpha)(3+\alpha)(2+\alpha)(1+\alpha)}{24} \\ & + \frac{(5+\alpha+\beta)(4+\alpha)(3+\alpha)(2+\alpha)}{12}(x-1) \\ & + \frac{(6+\alpha+\beta)(5+\alpha+\beta)(4+\alpha)(3+\alpha)}{16}(x-1)^2 \\ & + \frac{(7+\alpha+\beta)(6+\alpha+\beta)(5+\alpha+\beta)(4+\alpha)}{48}(x-1)^3 \\ & + \frac{(8+\alpha+\beta)(7+\alpha+\beta)(6+\alpha+\beta)(5+\alpha+\beta)}{384}(x-1)^4. \end{aligned}$$

Example 30. When $\alpha = \beta$, the Jacobi polynomials in Definition 27 are called the *ultraspherical polynomials*. They are even or odd polynomials according as n is even or odd. The following are the more well-known cases of ultraspherical polynomials:

$$\begin{aligned} P_n^{(-\frac{1}{2},-\frac{1}{2})}(x) &= \frac{1\cdot 3\cdot\cdots\cdot(2n-1)}{2\cdot 4\cdot\cdots\cdot 2n}T_n(x), \\ P_n^{(\frac{1}{2},\frac{1}{2})}(x) &= 2\frac{1\cdot 3\cdot\cdots\cdot(2n-1)}{2\cdot 4\cdot\cdots\cdot 2n}U_n(x), \end{aligned} \tag{2.7}$$

where $T_n(x)$ and $U_n(x)$ denote the *Tchebichef polynomials*[1] of the first and second kind respectively.

[1]Also known as Chebyshev, Tchebysheff, or other transliteration of Чебышёв

Example 31. As seen in Example 16, the Legendre polynomials are a special case of the ultraspherical polynomials, where $\alpha = \beta = 0$. In establishing properties of Jacobi polynomials, the Legendre polynomials are possibly the simplest case to consider since its parameters α and β are equal to zero.

Remark 32. The *Gegenbauer polynomials*, denoted by $C_n^{\nu}(x)$, are another generalization of the Legendre polynomials (cf. Examples 16, and 31). The n^{th} Gegenbauer polynomials is equal to a constant multiple of the n^{th} ultraspherical polynomial (cf. Example 30),

$$C_n^{\nu}(x) = \frac{(2\nu)_n P_n^{(\nu-1/2,\nu-1/2)}(x)}{(\nu+1/2)_n},$$

$$P_n^{(\alpha,\alpha)}(x) = \frac{(1+\alpha)_n C_n^{\alpha+1/2}(x)}{(1+2\alpha)_n},$$

where $(\rho)_n$ is defined in (2.6).

Differential Equation

The Jacobi polynomials satisfy the following differential equation, a fact that will be invoked later in Section 4.3.

Theorem 33 ([65, Theorem 4.2.1])**.** *The Jacobi polynomials* $y = P_n^{(\alpha,\beta)}(x)$ *satisfy the following linear homogeneous differential equation of the second order:*

$$(1-x^2)y'' + [\beta - \alpha - (\alpha+\beta+2)x]y' + n(n+\alpha+\beta+1)y = 0. \tag{2.8}$$

□

2.3 Composition theorem

The following theorem plays a very important role in the study of the Laguerre-Pólya class of entire functions (cf. 2.4.1), and multiplier sequences (cf. 3.1). It is commonly known as the "Malo-Schur-Szegő composition theorem," or simply the "composition theorem" for short.

Theorem 34 (Malo-Schur-Szegő Theorem [26]). *Let*

$$f(z) = \sum_{k=0}^{n} \binom{n}{k} a_k z^k, \quad g(z) = \sum_{k=0}^{n} \binom{n}{k} b_k z^k, \quad \text{and} \quad h(z) = \sum_{k=0}^{n} \binom{n}{k} a_k b_k z^k.$$

(i) If all the zeros of $f(z)$ lie in a circular region K, then each zero of $h(z)$ has the form $-\zeta_i w$, where ζ_i is a zero of $g(z)$ and $w \in K$.

(ii) If all the zeros of $f(z)$ lie in a convex region K containing the origin, and if all the zeros of $g(z)$ lie in $(-1, 0)$, then the zeros of $h(z)$ also lie in K.

(iii) If the zeros of $f(z)$ lie in $(-a, a)$ and the zeros of $g(z)$ lie in $(-b, 0)$ (or in $(0, b)$), where $a, b > 0$, then the zeros of $h(z)$ also lie in $(-ab, ab)$.

(iv) If the zeros of $p(z) = \sum_{k=0}^{\mu} a_k z^k$ are all real, and if the zeros of $q(z) = \sum_{k=0}^{\nu} b_k z^k$ are all real and of the same sign, then the polynomials

(a) $r(z) = \sum_{k=0}^{m} k! a_k b_k z^k$ and

(b) $s(z) = \sum_{k=0}^{m} a_k b_k z^k$

have only real zeros, where $m = \min(\mu, \nu)$. □

Remark 35. For a hyperbolic polynomial $a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$, with zeros of the same sign, define

$$q(x; \mu) := a_n^{\mu} x^n + a_{n-1}^{\mu} x^{n-1} + \ldots + a_1^{\mu} x + a_0^{\mu},$$

which is also hyperbolic for any positive integer $\mu \geq 1$, by Theorem 34, item (iv) (b). For all hyperbolic polynomials $q(x; 1)$ of degree 2 or less, it is easy to verify that $q(x; \mu)$ is hyperbolic for all real values $\mu \geq 1$. To the best of our knowledge, the following question has not been addressed in the literature.

Problem 36. *For a hyperbolic polynomial $a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$, with zeros of the same sign, $n \geq 3$, is it true that the polynomial*

$$a_n^{\mu} x^n + a_{n-1}^{\mu} x^{n-1} + \ldots + a_1^{\mu} x + a_0^{\mu}$$

is hyperbolic for all real values $\mu \geq 1$?

By the composition theorem, we only need to verify the statement in Problem 36 for $1 < \mu < 2$.

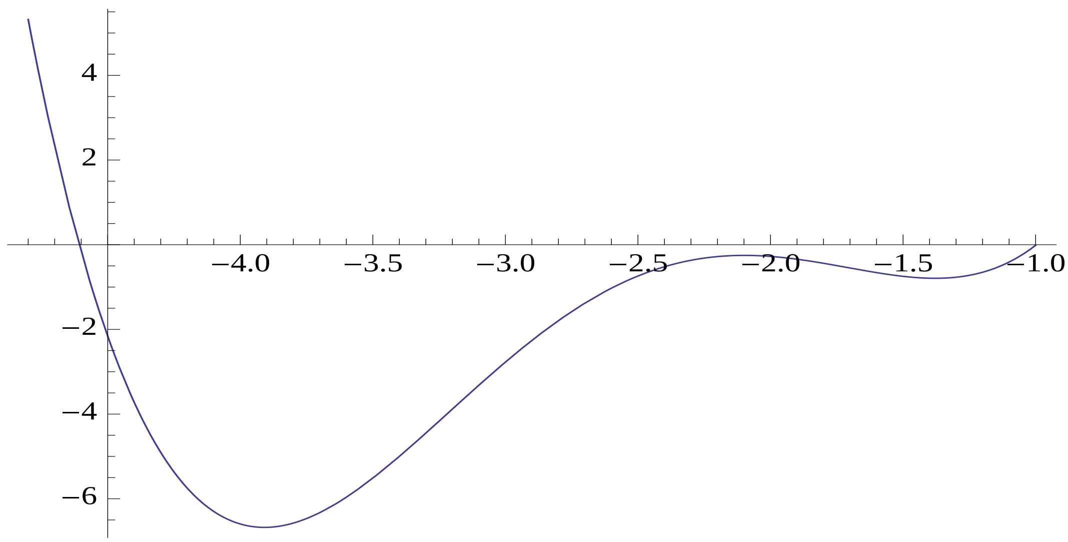
Example 37. Consider the polynomial $x^4 + 8x^3 + 24x^2 + 32x + 16 = (x+2)^4$. Define

$$p(x;\mu) := x^4 + 8^\mu x^3 + 24^\mu x^2 + 32^\mu x + 16^\mu. \tag{2.9}$$

High precision calculation by Mathematica yields the following zeros, two of which are non-real, for the polynomial $p(x;\mu)$ when $\mu = 11/10$,

$$x \approx -2.12227 \pm i\,0.301242, \quad x \approx -4.60733, \quad x \approx -0.997279.$$

The graph of $p(x; 11/10)$ also corroborates this conclusion.



Remark 38. The same results hold if the polynomial $(x+2)^4$ in Example 37 is replaced by $(x+1)^4$. Namely, for

$$p_1(x;\mu) := x^4 + 4^\mu x^3 + 6^\mu x^2 + 4^\mu x + 1, \tag{2.10}$$

a similar calculation which was done in Example 37 by Mathematica yields two non-real zeros for $p_1(x;\mu)$ when $\mu = 11/10$. We used $(x+2)^4$ in Example 37, since the coefficients are more interesting. The polynomial $p(x;\mu)$ in (2.9), as well as $p_1(x;\mu)$ appears to be hyperbolic for $\mu \geq 8/7$. We will revise Problem 36, but we will motivate the revision by considering the following. Define

$$C(x;\mu;n) := \sum_{k=0}^{n} \binom{n}{k}^{\mu} x^k, \tag{2.11}$$

for $\mu \geq 1$, and $n \in \mathbb{N}$. Specifically, $C(x;\mu;4) = p_1(x;\mu)$. The following table describes some computations carried out by Mathematica.

Polynomial	Apparent values of $\mu > 1$ when polynomial is not hyperbolic	Apparent values of $\mu \geq 1$ when polynomial is hyperbolic
$C(x;\mu;4)$	$(1, 8/7]$	$[7/6, \infty)$
$C(x;\mu;5)$	$(1, 9/8]$	$[8/7, \infty)$
$C(x;\mu;6)$	$(1, 6/5]$	$[5/4, \infty)$
$C(x;\mu;10)$	$(1, 5/4]$	$[4/3, \infty)$
$C(x;\mu;14)$	$(1, 4/3]$	$[3/2, \infty)$
$C(x;\mu;34)$	$(1, 3/2]$	$[3/2 + 1/100, \infty)$
$C(x;\mu;36)$	$(1, 3/2 + 1/100]$	$[3/2 + 1/10, \infty)$

The values of $\mu > 1$ which make $C(x;\mu;n)$ not hyperbolic seem to steadily increase to 2 as n increases from $n \geq 5$. Incidentally, the polynomial

$$C(x;\mu;3) = x^3 + 3^\mu x^2 + 3^\mu x + 1$$

is hyperbolic for all $\mu \geq 1$ by Corollary 12, as seen by calculating its discriminant

$$\Delta[C(x;\mu;3)] = (3^\mu + 1)(3^\mu - 3)^3 \geq 0, \quad (\mu \geq 1).$$

A revised version of Problem 36 is the following.

Problem 39. *For any hyperbolic polynomial $a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$, with zeros of the same sign, $n \geq 3$, is it true that*

$$a_n^\mu x^n + a_{n-1}^\mu x^{n-1} + \ldots + a_1^\mu x + a_0^\mu$$

is hyperbolic for all real values $\mu \geq 2$?

A related question motivated by Remark 38 is the following.

Problem 40. *Given any $\mu \in (1, 2)$, does there exist $n \in \mathbb{N}$ such that $C(x;\mu;n) := \sum_{k=0}^{n} \binom{n}{k}^\mu x^k$ is not hyperbolic?*

We will discuss Problem 39 for the case of transcendental entire functions in Section 2.4.1.

In Chapter 3, we will make use of a generalized version of the composition theorem (Theorem 34). In order to state the theorem, we define the following.

Definition 41. Let $S_\alpha(\theta) = \{z : |\operatorname{Arg}(z) - \theta| < \alpha\}$ denote an open sector with vertex at the origin and aperture $\alpha > 0$. The sector $-S_\alpha(\theta)$ is defined as $-S_\alpha(\theta) = \{-w \in \mathbb{C} \,|\, w \in S_\alpha(\theta)\}$, for $\alpha > 0$.

The generalized version of Theorem 34 allows the zeros of the polynomials to be non-real, although the theorem will still apply for hyperbolic polynomials.

Theorem 42 (Generalized Malo-Schur-Szegő Composition Theorem [15])**.** *Given two polynomials $A(z) = \sum_{k=0}^{m} a_k z^k$ and $B(z) = \sum_{k=0}^{n} b_k z^k$, $a_m b_m \neq 0$, let*

$$C(z) = \sum_{k=0}^{\nu} k! a_k b_k z^k, \quad \textit{where} \quad \nu = \min(m, n). \tag{2.12}$$

If $A(z)$ has all its zeros in the sector $S_\alpha(\theta_1)$ $(\alpha \leq \pi)$ and $B(z)$ has all its zeros in the sector $S_\beta(\theta_2)$ $(\beta \leq \pi)$, then $C(z)$ has all its zeros in the sector $-S_{\alpha+\beta}(\theta_1 + \theta_2 + \pi)$. □

2.4 Transcendental entire functions

In this section, we review the standard definitions and properties of entire functions and their growth (see Levin [48, Chapter I]).

An *entire function* is a function of a complex variable holomorphic in the entire plane and consequently represented by an everywhere convergent power series $f(z) = \sum_{k=0}^{\infty} a_k z^k$. The radius of convergence of the series represented by $f(z)$ is infinite, and therefore $\lim_{r\to\infty} \sqrt[n]{|a_n|} = 0$. To characterize the growth of an entire function $f(z)$, we introduce the function $M_f(r) = \max_{|z|=r} |f(z)|$. The rate of growth of the function $M_f(r)$ is an important characteristic of the behavior of an entire function.

Theorem 43 ([48, p. 2, Theorem 1])**.** *If there exists a positive integer n such that*

$$\varliminf_{r\to\infty} \frac{M_f(r)}{r^n} < \infty,$$

then $f(z)$ is a polynomial of degree at most n. □

Definition 44. An entire function $f(z)$ is said to be a *function of finite order* if there exists a positive constant k such that the inequality

$$M_f(r) < e^{r^k}$$

is valid for all sufficiently large values of r ($r > r_0(k)$). The greatest lower bound of such number k is called the *order* of the entire function $f(z)$.

Proposition 45 ([20, p. 285, Proposition 2.15])**.** *Let f be an entire function. Then f has order ρ if and only if*

$$\rho = \varlimsup_{r\to\infty} \frac{\ln\ln M_f(r)}{\ln r}. \tag{2.13}$$

□

For functions of a given order a more precise characterization of the growth is given by the type of the function.

Definition 46. The *type* σ of an entire function $f(z)$ of order ρ is the greatest lower bound of positive numbers A such that for all r sufficiently large,

$$M_f(r) < e^{Ar^\rho}.$$

The function $f(z)$ is said to be of *minimal*, *normal*, and *maximal* type if $\sigma = 0$, $0 < \sigma < \infty$, and $\sigma = \infty$ respectively.

From Proposition 45, one can derive a similar result for the type of an entire function.

Corollary 47. *For an entire function $f(z)$ with order ρ, f has type σ if and only if*

$$\sigma = \varlimsup_{r\to\infty} \frac{\ln M_f(r)}{r^\rho}.$$ □

We shall say that the function $f_2(z)$ is of larger growth than the function $f_1(z)$ if the order of $f_2(z)$ is greater than the order of $f_1(z)$, or if the orders are equal and the type of $f_2(z)$ is larger than the type of $f_1(z)$.

It is easy to see that the order of the sum of two functions is not greater than the larger of the orders of the two summands, and if the orders of the summands and of the sum are all equal, then the type of the sum is not greater than the larger of the types of the two summands. In addition, if one of the two functions is of larger growth than the other, then the sum has the same order and type as the function of larger growth.

The following theorem enables one to determine the order and type of an entire function by the rate of decrease of its Taylor coefficients.

Theorem 48 ([48, p. 4, Theorem 2])**.** *The order and type of an entire function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ are expressed in terms of its Taylor coefficients by the following equations*

$$\rho = \overline{\lim_{n\to\infty}} \frac{n \ln n}{\ln \frac{1}{|a_n|}}, \tag{2.14}$$

$$(\sigma e \rho)^{\frac{1}{\rho}} = \overline{\lim_{n\to\infty}} \left(n^{\frac{1}{\rho}} \sqrt[n]{|a_n|} \right), \quad (\rho > 0). \tag{2.15}$$

□

With the aid of Theorem 48 one can readily construct entire functions of arbitrary order and type.

Example 49. Given an entire function $f(z) = \sum_{n=0}^{\infty} a_n z^n$, if

$$a_n = \left(\frac{\sigma e^{\rho}}{n} \right)^{n/\rho} \qquad (\rho > 0,\ n = 1, 2, 3, \ldots),$$

then $f(z)$ has order ρ and normal type σ. Entire functions of maximal and minimal type can also be constructed. For example, for

$$a_n = \left(\frac{\ln n}{n} \right)^{\frac{n}{\rho}} \qquad (\rho > 0,\ n = 1, 2, 3, \ldots),$$

the entire function $f(z)$ represents a function of order ρ and maximal type, while for

$$a_n = \left(\frac{1}{n \ln n} \right)^{\frac{n}{\rho}} \qquad (\rho > 0,\ n = 2, 3, \ldots)$$

it represents a function of order ρ and minimal type. For $a_n = e^{-n^2}$ we obtain an entire function of order zero. All these assertions follow at once from (2.14) and (2.15). Similarly one can construct entire functions of infinite order.

2.4.1 The Laguerre-Pólya class

The class of entire functions which will be introduced in this subsection consists of limits of sequences of hyperbolic polynomials which converge uniformly on compact subsets of $\mathbb{C}$. We will proceed by introducing the germane terminologies and theorems.

Hurwitz's theorem enables us to make additional conclusions regarding the limit function of a uniformly converging sequence of entire functions.

Theorem 50 (Hurwitz's Theorem [50, p. 4])**.** *Suppose the sequence of entire functions $\{f_n(z)\}_{n=0}^{\infty}$ converges uniformly on compact subsets of $\mathbb{C}$ to the function $f(z)$, where $f(z)$ is not identically zero. If $z_0 \in \mathbb{C}$ is a limit point of the zeros of the functions $f_n(z)$, then z_0 is a zero of $f(z)$. Conversely, if $z_0 \in \mathbb{C}$ is a zero of $f(z)$ of multiplicity m, then for every sufficiently small neighborhood K of z_0, there exists an integer $N = N(K)$ such that K contains exactly m zeros of $f_n(z)$, counting multiplicities, whenever $n \geq N$.* □

Given an entire function, there is a sequence of polynomials that are naturally associated with it.

Definition 51. Let $\varphi(x) = \sum_{k=0}^{\infty} \frac{\alpha_k}{k!} x^k$ be an arbitrary entire function. Then the n^{th} *Jensen polynomial* associated with the function $\varphi(x)$ is defined by

$$g_n(x) = \sum_{k=0}^{n} \binom{n}{k} \alpha_k x^k \quad (n = 0, 1, 2, \ldots).$$

The Jensen polynomials associated with a given entire function satisfy a large number of important properties (see [22]). In particular, Jensen polynomials can be used to approximate entire functions, which is demonstrated by the following lemma.

Lemma 52 (Craven and Csordas [22, Lemma 2.2])**.** *Let*

$$h(z) = \sum_{k=0}^{\infty} \frac{a_k}{k!} z^k \quad (a_k \in \mathbb{C})$$

be an arbitrary entire function. For each fixed non-negative integer p, let

$$g_{n,p}(z) = \sum_{k=0}^{n} \binom{n}{k} a_{k+p} z^k \quad (n = 0, 1, 2, \ldots).$$

Then,

$$\lim_{n\to\infty} g_{n,p}\left(\frac{z}{n}\right) = h^{(p)}(z) \quad (p = 0, 1, 2, \ldots) \tag{2.16}$$

uniformly on compact subsets of $\mathbb{C}$. □

Definition 53. A real entire function φ is said to belong to the *Laguerre-Pólya class*, denoted $\mathscr{L}$-$\mathscr{P}$, if φ is the uniform limit on compact subsets of $\mathbb{C}$, of real polynomials with only real zeros. The subclass $\mathscr{L}$-$\mathscr{P}^+ \subset \mathscr{L}$-$\mathscr{P}$, consists of those functions $\varphi \in \mathscr{L}$-$\mathscr{P}$ that have non-negative Taylor coefficients.

We list some of the properties of the Laguerre-Pólya class (see [48, Chapter VIII], [51], [56], and the references contained therein for a detailed study).

Theorem 54 ([22, Theorem 2.7])**.** *If* $f(x) \in \mathscr{L}$-$\mathscr{P}$*, then its associated Jensen polynomials* $g_n(x)$ *are hyperbolic.* □

Corollary 55. *If* $f(x) \in \mathscr{L}$-$\mathscr{P}$*, then* $f'(x) \in \mathscr{L}$-$\mathscr{P}$*, i.e.,* $\mathscr{L}$-$\mathscr{P}$ *is closed under differentiation.*

Proof. Apply Theorem 50 to the sequence of Jensen polynomials converging to $f(x)$. □

Remark 56. Entire functions in the Laguerre-Pólya class have only real zeros, and by Corollary 55, the derivatives of any of the functions in the class also have only real zeros. In contrast, recall in Example 4 the entire function $f(z) = ze^{z^2}$ has its only zero at $z = 0$, but its derivative $f'(z) = (2z^2+1)e^{z^2}$ has non-real zeros. Thus, *a posteriori*, the function $f(z)$ does not belong in the Laguerre-Pólya class.

Recall the polynomial in Example 37,

$$p(x;\mu) := x^4 + 8^\mu x^3 + 24^\mu x^2 + 32^\mu x + 16^\mu, \quad \mu \in \mathbb{R}.$$

The polynomials $p(x;1), p(x;2) \in \mathscr{L}$-$\mathscr{P}^+$, but $p(x;11/10) \notin \mathscr{L}$-$\mathscr{P}^+$. We pose an extension of Problem 39 to the case of transcendental entire functions.

Problem 57. *For any entire function* $\varphi(x) = \sum_{k=0}^{\infty} a_k x^k \in \mathscr{L}$-$\mathscr{P}^+$*, is it true that*

$$\varphi_\mu(x) := \sum_{k=0}^{\infty} a_k^\mu x^k \in \mathscr{L}\text{-}\mathscr{P}^+$$

for all real values $\mu \geq 2$*?*

Problem 57 will be considered again in Section 3.1. The following theorem gives an indication of why the function $f(z) = ze^{z^2}$ in Remark 56 is not in the Laguerre-Pólya class.

Theorem 58 ([48, Chapter VIII])**.** *A function $\varphi \in \mathscr{L}\text{-}\mathscr{P}$ if and only if it can be expressed as*

$$\varphi(x) = c\,x^n e^{-ax^2+bx} \prod_{j=1}^{\omega} \left(1 + \frac{x}{x_j}\right) e^{-\frac{x}{x_j}}, \quad (0 \le \omega \le \infty), \tag{2.17}$$

where $a \ge 0$, $b, c, \in \mathbb{R}$, $x_j > 0$, $j, n \in \mathbb{N}$, and $\sum_{j=1}^{\infty} x_j^{-2} < \infty$. By convention, if $\omega = 0$, define the empty product to be 1. □

The subclass $\mathscr{L}\text{-}\mathscr{P}^+$ has a similar characterization.

Theorem 59 ([48, Chapter VIII])**.** *A real entire function $\varphi \in \mathscr{L}\text{-}\mathscr{P}^+$ if and only if its Hadamard product representation can be expressed in the form*

$$\varphi(x) = cx^m e^{ax} \prod_{j=1}^{\infty} \left(1 + \frac{x}{x_j}\right),$$

where $a, c \ge 0$, $x_j > 0$, $m, j \in \mathbb{N}$, and $\sum_{j=1}^{\infty} x_j^{-1} < \infty$. □

The Laguerre-Pólya class can also be characterized by means of system of inequalities. The following theorems are some of the known systems of inequalities that characterize the Laguerre-Pólya class.

Theorem 60 (Laguerre inequality: Real version. [25, Theorem 2.2], [22], [52])**.** *Let $\varphi(x) \not\equiv 0$ be a real entire function of the form $e^{-\alpha x^2}\varphi_1(x)$, where $\alpha \ge 0$ and $\varphi_1(x)$ has genus 0 or 1. Set*

$$L_n(\varphi(x)) = \sum_{j=0}^{2n} \frac{(-1)^{j+n}}{(2n)!} \binom{2n}{j} \varphi^{(j)}(x)\varphi^{(2n-j)}(x). \tag{2.18}$$

Then $\varphi(x) \in \mathscr{L}\text{-}\mathscr{P}$ if and only if

$$L_n(\varphi) \ge 0 \quad \text{for all } x \in \mathbb{R} \text{ and all } n = 0, 1, 2, \ldots.$$ □

We remark that $L_1(\varphi(x)) = (\varphi'(x))^2 - \varphi(x)\varphi''(x) \ge 0$, for all $x \in \mathbb{R}$, is referred in the literature as the *Laguerre inequality* (see [22], [39]). The inequalities $L_n(\varphi) \ge 0, (n \ge 1)$ in Theorem 60 are called *extended Laguerre inequalities.* Recently, D. A. Cardon [16] generalized the extended Laguerre inequalities and obtained the following novel characterization of the Laguerre-Pólya class.

Theorem 61 (D. A. Cardon [16]). *Let $f(z) \not\equiv 0$ be a real entire function, $f(z) = e^{-\alpha z^2}\varphi_1(z)$, where $\alpha \geq 0$ and $\varphi_1(z)$ has genus 0 or 1, and $g(z) = \prod_{k=1}^{m}(z+\alpha_k)$ be an even polynomial with non-negative real coefficients and at least one non-real zero. Then for all $z \in \mathbb{R}$ and all non-negative integers k,*

$$A_k(z) := \frac{1}{k!}\left[\frac{d^k}{dt^k}\prod_{j=1}^{m} f(z+\alpha_j t)\right]\Bigg|_{t=0} \geq 0 \tag{2.19}$$

if and only if $f(z) \in \mathscr{L}\text{-}\mathscr{P}$. □

A special case of Theorem 61, when $g(z) = z^2 + 1$, is Theorem 60. For this reason we regard the inequalities in (2.19) as generalizations of the extended Laguerre inequalities of Theorem 60. The functions in the Laguerre-Pólya class can also be characterized by complex versions of the Laguerre inequalities.

Theorem 62 (Laguerre inequality: Complex version I. [29]). *Let $f(z) \not\equiv 0$ be a real entire function, $f(z) = e^{-\alpha z^2}\varphi_1(z)$, where $\alpha \geq 0$ and $\varphi_1(z)$ has genus 0 or 1. Then for all $z := x + iy \in \mathbb{C}$, $x, y \in \mathbb{R}$, $y \neq 0$,*

$$\frac{1}{y}\operatorname{Im}\left(-f'(z)\overline{f(z)}\right) \geq 0 \tag{2.20}$$

if and only if $f(z) \in \mathscr{L}\text{-}\mathscr{P}$. □

Theorem 63 (Laguerre inequality: Complex version II. [44]). *Let $f(z) \not\equiv 0$ be a real entire function, $f(z) = e^{-\alpha z^2}\varphi_1(z)$, where $\alpha \geq 0$ and $\varphi_1(z)$ has genus 0 or 1. Then for all $z \in \mathbb{C}$,*

$$|f'(z)|^2 \geq \operatorname{Re}\left(f(z)\overline{f''(z)}\right) \tag{2.21}$$

if and only if $f(z) \in \mathscr{L}\text{-}\mathscr{P}$. □

The Jensen polynomials (cf. Definition 51) associated with functions in the Laguerre-Pólya class satisfy a Turán-type inequality (see [22], [25], [31], [52], [67]).

Theorem 64 ([22, Theorem 2.7]). *Let $\varphi(x) = \sum_{k=0}^{\infty}\frac{\gamma_k}{k!}x^k$ denote a real entire function, let $g_n(t) := \sum_{k=0}^{n}\binom{n}{k}\gamma_k x^k$, $n = 0, 1, 2, \ldots$, and let*

$$L_n(t) := g_n^2(t) - g_{n-1}(t)g_{n+1}(t), \quad (n = 1, 2, 3, \ldots).$$

Then $\varphi(x) \in \mathscr{L}\text{-}\mathscr{P}$ if and only if the following conditions hold:

(i) $L_n(t) \geq 0$ *for all positive integers* n *and all* $t \in \mathbb{R}$.

(ii) *If* $\gamma_0 \neq 0$ *and* $\gamma_1^2 - \gamma_0\gamma_2 > 0$, *then*

   (a) $g_{n+1}(t_0) = 0$, *whenever* $L_n(t_0) = 0$, $t_0 \neq 0$, *and*

   (b) $\gamma_{n+1} = 0$, *whenever* $\gamma_n^2 - \gamma_{n-1}\gamma_{n+1} = 0$.

(iii) *If* $\gamma_0 \neq 0$ *and* $\gamma_1^2 - \gamma_0\gamma_2 = 0$, *then* $\varphi(x) = \gamma_0 e^{\frac{\gamma_1}{\gamma_0}x}$.

(iv) *If* $\gamma_0 = 0$, *then* $\varphi(x) = x^r\psi(x)$, *with* $\psi(0) \neq 0$, *where* $\psi(x)$ *satisfies (i), (ii), and (iii) for the appropriately redefined* γ_n, g_n, *and* L_n. □

The following definition will be used for the next characterization of functions in the Laguerre-Pólya class.

Definition 65. Let m be any positive integer. The sequence $\{a_k\}_{k=0}^{\infty}$ is called *m-times positive (totally positive)* if all minors of order m or less (of any order) of the infinite upper triangular matrix

$$\begin{pmatrix} a_0 & a_1 & a_2 & a_3 & \cdots \\ 0 & a_0 & a_1 & a_2 & \cdots \\ 0 & 0 & a_0 & a_1 & \cdots \\ 0 & 0 & 0 & a_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{2.22}$$

are non-negative. These positive sequences are also called *Pólya frequency* sequences. The class of all m-times positive (totally positive) sequences are denoted by PF_m (PF_∞). The classes of corresponding generating functions $f(z) = \sum_{k=0}^{\infty} a_k z^k$ are also denoted by PF_m and PF_∞.

Theorem 66 ([1], [45, Theorem C])**.** *Let* $\varphi(x) = \sum_{k=0}^{\infty} a_k x^k$ *be a real entire function. Then* $\varphi(x) \in \mathscr{L}\text{-}\mathscr{P}^+$ *if and only if* $\varphi(x) \in PF_\infty$. □

For the next characterization of the Laguerre-Pólya class, it is useful to begin with functions whose zeros all lie in the closed strip

$$\overline{S}(A) = \{z \in \mathbb{C} \, : \, |\text{Im}(z)| \leq A\} \, . \tag{2.23}$$

Let the class of entire functions $\mathfrak{S}(A)$, $0 \leq A < \infty$, consist of functions of the form (2.17) with $x_k \in \overline{S}(A) \setminus \{0\}$. Thus, if $f \in \mathfrak{S}(A)$, for some $A \geq 0$, has only real zeros, then $f \in \mathscr{L}\text{-}\mathscr{P}$.

Theorem 67 ([28, p. 344])**.** *Let* $f(z) \in \mathfrak{S}(A)$*. For* $\mu \in \mathbb{R} \setminus \{0\}$*, set* $f_\mu(x) = 2\cos(\mu D)f(x)$*, where* $D = \frac{d}{dx}$*. Then* $f(x) \in \mathscr{L}\text{-}\mathscr{P}$*, that is* $A = 0$*, if and only if for all* $x, \mu \in \mathbb{R}$*,* $\mu \neq 0$*,* $L_1(f_\mu(x)) \geq 0$ *and*

$$(\operatorname{Im}[f'(x + i\mu)])^2 - \operatorname{Im}[f(x - i\mu) \operatorname{Im}[f(x + i\mu)] \geq 0, \tag{2.24}$$

where $L_1(f)$ *is defined in* (2.18)*.* □

2.5 Generalized Laguerre inequality

In the previous section, Theorems 60, 61, 62, and 63 assume that the non-zero real entire function is of the form $e^{-\alpha z^2}\varphi_1(z)$, where $\alpha \geq 0$ and $\varphi_1(z)$ has genus 0 or 1. G. Csordas and A. Vishnyakova [30] showed that the various sufficient conditions for a real entire function $\varphi(x)$, to belong to the Laguerre-Pólya class, expressed in terms of Laguerre-type inequalities, do not require the *a priori* assumptions about the order and type of $\varphi(x)$. In Theorem 71, we establish a related result (Theorem 71), based in part on the Borel-Carathéodory Inequality (Theorem 69). To this end, we first establish an inequality that is known as the Carathéodory inequality.

Proposition 68 (The Carathéodory Inequality [48, p. 17])**.** *Let* $f(z)$ *is any function holomorphic in the disk* $|z| \leq R$*, set* $A(r) = \max_{|z|=r}(\operatorname{Re} f(z))$*. Then for* $0 < r < R$*,*

$$M_f(r) \leq \frac{R + r}{R - r}[A(R) + |f(0)|]. \tag{2.25}$$

Proof. If $f \equiv$ constant, then (2.25) holds. So suppose $f \not\equiv$ constant. First assume $f(0) = 0$. Then it follows that $A(R) > A(0) = 0$, since

$$1 = |e^{f(0)}| \leq |e^{f(z)}| = e^{\operatorname{Re} f(z)} < e^{A(R)}.$$

Let

$$g(z) := \frac{f(Rz)}{2A(R) + f(Rz)},$$

and observe that (i) g is holomorphic in the disk $|z| \leq R$, (ii) $g(0)$=0, and (iii) $|g(z)| \leq 1$, for $|z| \leq 1$. Thus, by (i), (ii), and (iii), we can invoke the Schwarz lemma to conclude that $|g(z)| \leq |z|$, for $|z| < 1$. Hence, solving for f in terms of g,

$$f(Rz) = \frac{2g(z)A(R)}{1-g(z)}.$$

Set $\zeta = Rz$, $|\zeta| = r < R$. Then

$$|f(\zeta)| \le \frac{2\left|\frac{\zeta}{R}\right| A(R)}{1 - \frac{|\zeta|}{R}} = \frac{2rA(R)}{R-r} \le \frac{R+r}{R-r}A(R). \tag{2.26}$$

If $f(0) \neq 0$, apply (2.26) to $f(z) - f(0)$, and obtain

$$|f(z) - f(0)| \le \frac{2r}{R-r} \max_{|z|=R} \operatorname{Re}(f(z) - f(0)) \le \frac{2r}{R-r}(A(R) + |f(0)|).$$

Hence,

$$|f(z)| \le \frac{2r}{R-r}[A(R) + |f(0)|] + |f(0)| \le \frac{R+r}{R-r}[A(R) + |f(0)|]. \qquad \square$$

Theorem 69 (The Borel-Carathéodory Inequality [48, p. 18]). *Let $f : H^+ \to H^+$ be an analytic function, where $H^+ = \{z \in \mathbb{C} : \operatorname{Im}(z) > 0\}$ is the open upper-half plane. Then for $z \in H^+$ and $|z| > 1$, f satisfies the inequalities*

$$\frac{1}{5}|f(i)|\frac{\sin\theta}{r} < |f(z)| < 5|f(i)|\frac{r}{\sin\theta} \quad (z = re^{i\theta},\ 0 < \theta < \pi). \tag{2.27}$$

Proof. Map the upper half-plane $\operatorname{Im} z > 0$ onto the unit disk

$$w = \frac{z-i}{z+i} \quad \text{or} \quad z = -i\frac{w+1}{w-1}.$$

The function

$$F(w) = if\left(-i\frac{w+1}{w-1}\right)$$

is defined in the unit disk and satisfies $\operatorname{Re} F(w) \le 0$. Consequently, $A_F(R) \le 0$, where we denote the maximum of the real part of a function $f(z)$, holomorphic in the disk $|z| \le r$, by $A_f(r)$. From Carathéodory's inequality (Proposition 68) for the unit disk $|w| \le 1$, we have

$$|f(z)| \le |f(i)| - \operatorname{Re}(if(i))\frac{2|z-i|}{|z+i| - |z-i|} \quad (\operatorname{Im} z > 0).$$

Note that for $|z| \geq 1$,

$$|z-i| \leq 2r, \quad |z+i| \leq 2r, \quad \text{and} \quad |z+i|^2 - |z-i|^2 = 4r\sin\theta.$$

Thus,

$$|f(z)| < |f(i)| \left(1 + \frac{4r}{\sin\theta}\right) < 5|f(i)| \frac{r}{\sin\theta}. \tag{2.28}$$

The left side of the inequality is attained by noting that the function $f(z)$ does not vanish in the open upper half-plane. Writing the inequality (2.28) for the function $1/f(z)$, we attain (2.27). □

For the following technical result, we denote by $n(r)$, the number of zeros of $f(z)$ in the closed disk $\{z \in \mathbb{C} : |z| \leq r\}$.

Theorem 70 (Lindelöf's Theorem [7, p. 27])**.** *An entire function $f(z)$ of positive integral order $\rho > 1$ is of normal (finite) type if and only if*

(i) $n(r) = O(r^\rho) \quad (r \to \infty)$ *and*

(ii) the sums

$$|S(r)| := \left| \sum_{\{z: f(z)=0, |z| \leq r, z \neq 0\}} \frac{1}{z^\rho} \right| \tag{2.29}$$

are bounded as $r \to \infty$. □

We proceed to generalize Theorem 62 without the assumptions about the order and type of the real entire function.

Theorem 71. *Let $\varphi(z) \not\equiv 0$, be a real entire function. If*

$$\frac{1}{y} \operatorname{Im} \left\{ -\varphi'(z)\overline{\varphi(z)} \right\} \geq 0 \quad \textit{for all } z : x + iy \in \mathbb{C},\ y \neq 0. \tag{2.30}$$

Then $\varphi(z) \in \mathscr{L}\text{-}\mathscr{P}$.

Proof. First, we show that with the given condition (2.30), the function $\varphi(z)$ has only real zeros. We observe that a calculation shows that for each fixed $x \in \mathbb{R}$,

$$m(y; x) := \frac{1}{2y} \frac{\partial}{\partial y} |\varphi(x+iy)|^2 = \frac{1}{y} \operatorname{Im} \left(-\varphi'(z)\overline{\varphi(z)} \right) \geq 0. \tag{2.31}$$

Suppose that $\varphi(x_0 + iy_0) = 0$ for some y_0. By (2.31), the non-negative function $|\varphi(x_0 + iy)|^2$ is increasing for $y > 0$, and it is decreasing for $y < 0$. Therefore, $|\varphi(x_0+iy)|^2$ attains its minimum only for $y = 0$ (since $\varphi(z)$ is non-constant) and whence $y_0 = 0$. Since $\varphi(z)$ does not vanish on the simply connected domain H^+, $\varphi(z)$ has an analytic logarithm there; that is, $f(z) := \log \varphi(z)$ is analytic in H^+. Now set $u(z) := \operatorname{Re} f(z) = \log |\varphi(z)|$ and $v(z) := \operatorname{Im} f(z)$. Then $f'(z) = \frac{\partial}{\partial x} u(z) + i \frac{\partial}{\partial x} v(z)$ and so by Cauchy-Riemann equations

$$g(z) := -f'(z) = -\frac{\partial}{\partial x} u(z) - i\frac{\partial}{\partial x} v(z) = -\frac{\partial}{\partial y} v(z) + i\frac{\partial}{\partial y} u(z). \tag{2.32}$$

Now for $z \in H^+$, $e^{f(z)} = \varphi(z)$ and $e^{2\operatorname{Re} f(z)} = |\varphi(z)|^2 \geq 0$. Since

$$\frac{1}{2y}\frac{\partial}{\partial y}|\varphi(z)|^2 = \frac{1}{y}\operatorname{Im}\left\{-\varphi'(z)\overline{\varphi(z)}\right\} \geq 0$$

by (2.30), we infer the inequality

$$\begin{aligned} \frac{1}{y}\operatorname{Im}\left\{-\varphi'(z)\overline{\varphi(z)}\right\} &= \frac{1}{2y}\frac{\partial}{\partial y}|\varphi(z)|^2 \\ &= \frac{1}{2y}\frac{\partial}{\partial y}e^{2\operatorname{Re} f(z)} = \frac{1}{y}e^{2u(z)}\frac{\partial}{\partial y}u(z) \geq 0, \ (z \in \mathbb{C}, y > 0). \end{aligned} \tag{2.33}$$

Therefore, if $z = x + iy \in H^+$, then by (2.33), $\frac{\partial}{\partial y} u(z) \geq 0$ and hence, consulting (2.32), $g : H^+ \to H^+$. Thus, by the Borel-Carathéodory inequality (Theorem 69), there exists a positive constant C_1, such that

$$|g(z)| \leq C_1|z| \quad \text{for all} \quad z \in \Omega := \{z \ : \ \operatorname{Im} z \geq |\operatorname{Re} z| + 1\}. \tag{2.34}$$

Integrating $g(z) := -f'(z)$ along a linear segment $[i, z]$, where $z \in \Omega$, we obtain the following upper estimate with a new constant $C_2 > 0$:

$$\begin{aligned} |f(z)| &= \left|\int_{[i,z]} f'(w)dw + f(i)\right| \\ &\leq C_1|z|\left(\int_{[i,z]}|dw|\right) + |f(i)| \leq C_2|z|^2 \quad \text{for all} \quad z \in \Omega. \end{aligned} \tag{2.35}$$

Next, we fix $|z| = r \geq 1$ and note that since φ is a real entire function $|\varphi(x + iy)| = |\varphi(x - iy)|$ for all $x, y, \in \mathbb{R}$. By the maximum modulus principle, and choosing $r \geq 1$, sufficiently large, so that

$x+iy \in \Omega$, we deduce the following estimates:

$$\begin{aligned}
\max_{|x+iy|\le r} |\varphi(x+iy)| &\le \max_{-r\le x\le r} |\varphi(x+ir)| \quad (x+ir \in \Omega) \\
&= \max_{-r\le x\le r} e^{\operatorname{Re} f(x+ir)} \\
&\le \max_{-r\le x\le r} e^{|f(x+ir)|} \\
&\le e^{Cr^2}, \qquad \text{(cf. (2.35))}.
\end{aligned} \tag{2.36}$$

Thus, we have established that the real entire function φ is of order $\rho(\varphi) \le 2$ and that if $\rho(\varphi) = 2$, then φ is of normal type. Accordingly, we consider two cases. In the first place, if $\rho(\varphi) < 2$, then by the Hadamard factorization theorem, the canonical product of the zeros of φ has genus at most 1 (see, for example, [7, p. 22]). Secondly, suppose that $\rho(\varphi) = 2$ and that φ has an infinite number of zeros $\{x_k\}_{k=0}^{\infty}$ ($x_k \ne 0, k \ge 1$). In this case, since φ is of normal type, we can invoke Lindelöf's theorem (Theorem 70) to conclude that the sums

$$|S(r)| := \sum_{|x_k|\le r} \frac{1}{x_k^2} \tag{2.37}$$

are bounded as $r \to \infty$. Therefore, $\sum_{k=1}^{\infty} 1/x_k^2 < \infty$ and the genus of the canonical product of the zeros of φ is again at most 1. Consequently, another appeal to the Hadamard factorization theorem shows that $\varphi(z)$ has the representation

$$\varphi(z) = ce^{az^2+bz} z^m \prod_{k=1}^{\infty} (1 - \frac{z}{x_k}) e^{z/x_k}, \tag{2.38}$$

where $a, b \in \mathbb{R}$, $x_k \in \mathbb{R} \setminus \{0\}$, m is a non-negative integer, c is a non-zero real number, and $\sum_{k=1}^{\infty} 1/x_k^2 < \infty$.

To complete the proof of the theorem, we show that $a \le 0$ in (2.38). The proof hinges on the fact, noted above, that $|\varphi(iy)|$ is an increasing function of y, $y > 0$. Now, a calculation shows that

$$h(y) := |\varphi(iy)| = |c||y|^m e^{-ay^2} \prod_{k=1}^{\infty} \left(1 + \frac{y^2}{x_k^2}\right)^{1/2}, \tag{2.39}$$

where we adhere to the usual convention, whereby the empty product has value 1. Then for $y > 0$, logarithmic differentiation and some algebraic manipulation yield the following expression

$$H(y) := \frac{1}{y}\frac{h'(y)}{h(y)} = \frac{m}{y^2} - 2a + \sum_{k=1}^{\infty} \frac{1}{x_k^2 + y^2}. \tag{2.40}$$

Since $h(y)$ is a positive increasing function $y > 0$, it follows that $H(y) \geq 0$ for all $y > 0$ (cf. the left-handed side of (2.40)). We next demonstrate that the assumption that $a > 0$ is untenable for it leads to a contradiction. Let $0 < \varepsilon < \frac{a}{4}$. Since the series $\sum_{k=1}^{\infty} 1/x_k^2$ converges, there exists a positive integer N such that for all $y > 0$,

$$\sum_{k=N+1}^{\infty} \frac{1}{x_k^2 + y^2} \leq \sum_{k=N+1}^{\infty} \frac{1}{x_k^2} < \frac{\varepsilon}{3}. \tag{2.41}$$

Fixing N, there exists a positive number y_0, y_0 sufficiently large, such that for all $y \geq y_0$,

$$\frac{m}{y^2} < \frac{\varepsilon}{3} \quad \text{and} \quad \sum_{k=1}^{N} \frac{1}{x_k^2 + y^2} < \frac{\varepsilon}{3}. \tag{2.42}$$

Inequalities (2.41) and (2.42) show that, if $a > 0$, then for all $y \geq y_0$,

$$H(y) = \frac{1}{y}\frac{h'(y)}{h(y)} = \frac{m}{y^2} - 2a + \sum_{k=1}^{\infty} \frac{1}{x_k^2 + y^2} < \varepsilon - 2a < \frac{a}{4} - 2a < 0. \tag{2.43}$$

This is the desired contradiction and hence $a > 0$. Thus $\varphi \in \mathscr{L}\text{-}\mathscr{P}$. □

CHAPTER 3
LINEAR OPERATORS ACTING ON ENTIRE FUNCTIONS

In this chapter, we investigate various linear operators acting on entire functions. In the course of our investigation, we revisit Problem 57 from the viewpoint of linear operators (Problems 80 and 82). The new results in this chapter are Theorems 127, 128, 131, 132, and 134, and they lead to a complete characterization of certain second order differential operators which preserve hyperbolicity (Theorem 135).

This chapter is subdivided into three sections under the following headings: Multiplier sequences (Section 3.1), Hyperbolicity and stability and perservers (Section 3.2), and Differntial operators (Section 3.3).

3.1 Multiplier sequences

Definition 72. A sequence of real numbers $\{\gamma_k\}_{k=0}^{\infty}$ is called a *multiplier sequence* if the corresponding (diagonal) linear operator given by $T[x^k] = \gamma_k x^k$, $k \in \mathbb{N} \cup \{0\}$, has the property that

$$T[p(x)] \in \mathscr{L}\text{-}\mathscr{P}, \quad \text{whenever} \quad p(x) \in \mathscr{L}\text{-}\mathscr{P} \cap \mathbb{R}[x],$$

where $\mathscr{L}\text{-}\mathscr{P}$ is defined in Section 2.4.1 (Definition 53). We say the sequence $\{\gamma_k\}_{k=0}^{\infty}$ is *trivial* if there is a non-negative integer k such that $\gamma_n = 0$ for all $n \notin \{k, k+1\}$, or $\gamma_k = c$ for $k = 0, 1, 2, \ldots$.

Multiplier sequences were characterized in a seminal paper by G. Pólya and J. Schur [56].

Theorem 73 (G. Pólya and J. Schur [56])**.** *Let $\{\gamma_k\}_{k=0}^{\infty}$ be a sequence of real numbers, and let $T : \mathbb{R}[x] \to \mathbb{R}[x]$ be the corresponding (diagonal) linear operator defined by $T[x^k] = \gamma_k x^k$, for $k \in \mathbb{N}$. Define $\varphi(x) = T[e^x]$ to be the formal power series*

$$\varphi(x) = \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} x^k.$$

Then the following are equivalent:

(i) $\{\gamma_k\}_{k=0}^{\infty}$ is a multiplier sequence;

(ii) $T[\mathscr{L}\text{-}\mathscr{P}] \subseteq \mathscr{L}\text{-}\mathscr{P}$;

(iii) $\varphi(x)$ *is the uniform limit of polynomials with only real zeros of the same sign on compact subsets of* $\mathbb{C}$*;*

(iv) Either $\varphi(x)$ *or* $\varphi(-x)$ *is an entire function that can be expressed as*

$$Cx^n e^{ax} \prod_{k=1}^{\infty} \left(1 + \frac{x}{x_k}\right),$$

where $n \in \mathbb{N}, C \in \mathbb{R}, a, x_k > 0$ *for all* $k \in \mathbb{N}$ *and* $\sum_{k=1}^{\infty} x_k^{-1} < \infty$*;*

(v) For all non-negative integers n *the polynomial* $T[(1+x)^n]$ *has only real zeros of the same sign.* □

We will refer to the multiplier sequences in the following example for contrasting various sequences in the sequel.

Example 74.

(i) The sequence $\{k\}_{k=0}^{\infty}$, is a multiplier sequence, as it corresponds to the operator $x\dfrac{d}{dx}$ (cf. Corollary 55).

(ii) For a non-zero real number r, the sequence $\left\{r^k\right\}_{k=0}^{\infty}$ is a multiplier sequence, since for any polynomial $p(x) = \sum_{k=0}^{n} a_k x^k \in \mathscr{L}\text{-}\mathscr{P}$, $\sum_{k=0}^{n} r^k a_k x^k = p(rx) \in \mathscr{L}\text{-}\mathscr{P}$.

(iii) By part (ii) of Theorem 73, $\left\{k^2+k+1\right\}_{k=0}^{\infty}$ is a multiplier sequence, since

$$\sum_{k=0}^{\infty} \frac{k^2+k+1}{k!} x^k = (x+1)^2 e^x \in \mathscr{L}\text{-}\mathscr{P}.$$

We will see a related sequence to the following lemma in Section 3.1.1 (cf. Example 87). The following multiplier sequence will be used later in Chapter 5.

Lemma 75. *For* $n, d \in \mathbb{N}$*, the sequence* $\left\{\frac{1}{(n-k+d)!}\right\}_{k=0}^{\infty}$ *(where* $\frac{1}{k!} = 0$ *for* $k < 0$*) is a multiplier sequence.*

Proof. Fix $n \in \mathbb{N}$. By (v) of Theorem 73, it suffices to show that the polynomial

$$f(x) := \sum_{k=0}^{n} \binom{n}{k} \frac{1}{(n-k+d)!} x^k$$

has only real negative zeros. The sequence $\left\{\frac{1}{(k+d)!}\right\}_{k=0}^{\infty}$ is a multiplier sequence, thus

$$p(x) := \sum_{k=0}^{n} \binom{n}{k} \frac{1}{(k+d)!} x^k = \sum_{k=0}^{n} \binom{n}{k} \frac{1}{(n-k+d)!} x^{n-k}$$

has only real negative zeros. Thus it follows that $f(x)$ has only real negative zeros, as desired. □

The following theorem by Laguerre shows that multiplier sequences abound.

Theorem 76 (Laguerre [51, Satz 3.2])**.** *If $\varphi(x)$ is a real polynomial with only non-positive zeros, then $\{\varphi(k)\}_{k=0}^{\infty}$ is a multiplier sequence.* □

Theorem 76 has a stronger version (Theorem 86), which will be stated in Section 3.1.1.

Remark 77. Sequences interpolated by polynomials are often called by their degree designations. For example, *linear, quadratic,* and *cubic* sequences are interpolated by polynomials of degree 1, 2, and 3 respectively. Sequence (iii) in Example 74 is interpolated by the polynomial $x^2 + x + 1$, which has non-real zeros. Thus, Theorem 76 cannot be applied to deduce that $\left\{k^2 + k + 1\right\}_{k=0}^{\infty}$ is a multiplier sequence. This discrepancy is clarified in Proposition 78.

To the best of our knowledge, the following observation has not been documented in the literature.

Proposition 78. *For $m \in \mathbb{N}$, let $q(x) = \sum_{i=0}^{m} a_i \binom{x}{i} i! = \sum_{i=0}^{m} b_i x^i \in \mathbb{R}[x]$, $b_i \geq 0$. Then $\{q(k)\}_{k=0}^{\infty}$ is a multiplier sequence if and only if $p(x) = \sum_{i=0}^{m} a_i x^i \in \mathscr{L}\text{-}\mathscr{P}^+$.*

Proof. By part (ii) of Theorem 73, it suffices to show

$$\sum_{k=0}^{\infty} \frac{q(k)}{k!} x^k = p(x) e^x.$$

To this end, we observe that for $n \in \mathbb{N}$,

$$\sum_{k=0}^{\infty} \binom{k}{n} \frac{n!\, x^k}{k!} = \sum_{k=n}^{\infty} \frac{x^k}{(k-n)!} = \sum_{k=0}^{\infty} \frac{x^{k+n}}{k!} = x^n e^x,$$

where $\binom{k}{n} = 0$ for $k < n$. The result follows by linearity. □

The following properties are necessary conditions for a sequence to be a multiplier sequence.

Proposition 79 ([21], [48, p. 341])**.** *Let $T = \{\gamma_k\}_{k=0}^{\infty}$ be a multiplier sequence.*

(i) *The sequence* $\{\gamma_{k+m}\}_{k=0}^{\infty}$, *is a multiplier sequence for any integer* $m \geq 0$.

(ii) *Suppose there exists an integer* $m \geq 0$ *such that* $\gamma_m \neq 0$. *If* $\gamma_n = 0$ *for any* $n > m$, *then* $\gamma_k = 0$ *for* $k \geq n$.

(iii) *The sequence* $\left\{(-1)^k \gamma_k\right\}_{k=0}^{\infty}$ *is a multiplier sequence.*

(iv) *The Turán inequalities hold; that is,*

$$T_k := \gamma_k^2 - \gamma_{k-1}\gamma_{k+1} \geq 0, \quad k = 1, 2, 3, \dots . \qquad \square$$

Recall Problem 57: For any entire function $\varphi(x) = \sum_{k=0}^{\infty} a_k x^k \in \mathscr{L}\text{-}\mathscr{P}^+$, is it true that $\varphi_\mu(x) := \sum_{k=0}^{\infty} a_k^\mu x^k \in \mathscr{L}\text{-}\mathscr{P}^+$ for all real values $\mu \geq 2$? First we pose a question that may be more natural to ask, as we did before revising the polynomial version of Problem 57 (cf. Problem 36).

Problem 80. *Characterize all multiplier sequences* $\{\gamma_k\}_{k=0}^{\infty}$ *such that*

$$\varphi_\mu(x) := \sum_{k=0}^{\infty} \frac{\gamma_k^\mu \, x^k}{k!} \in \mathscr{L}\text{-}\mathscr{P}^+$$

for all real values $\mu \geq 0$.

After all, when $\mu = 0$, the function $\sum_{k=0}^{\infty} \frac{x^k}{k!} = e^x \in \mathscr{L}\text{-}\mathscr{P}^+$.

Example 81. Consider the entire function

$$\varphi_{1/5}(x) := \sum_{k=0}^{\infty} \frac{x^k}{(k!)^{(6/5)}} = \sum_{k=0}^{\infty} \frac{\gamma_k^\mu \, x^k}{k!}, \quad \left(\gamma_k = \frac{1}{k!},\ \mu = 1/5\right).$$

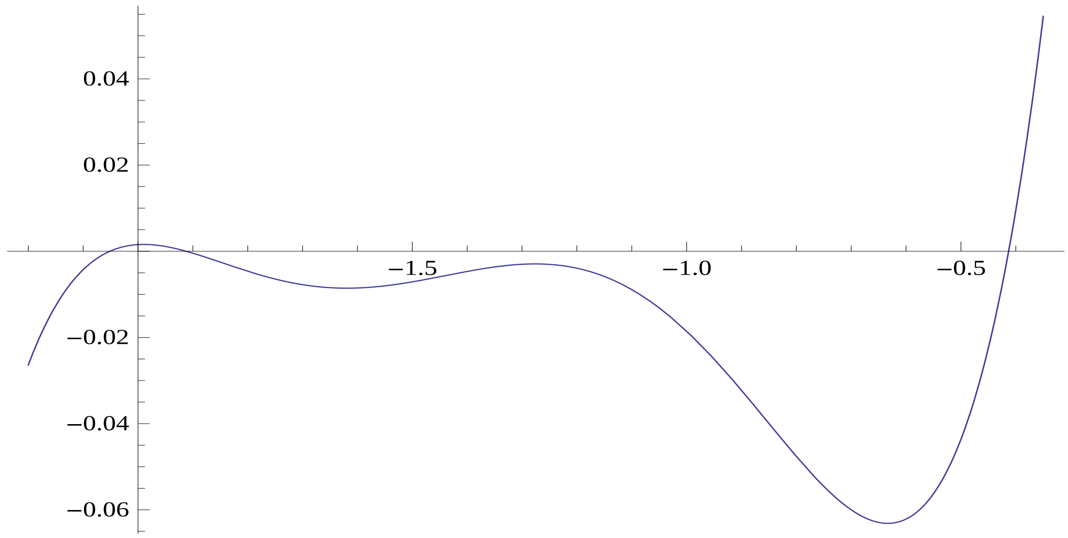
The fifth Jensen polynomial associated with $\varphi_{1/5}(x)$ is

$$g_5(x) = \sum_{k=0}^{5} \binom{5}{k} \left(\frac{1}{k!}\right)^{1/5} x^k.$$

High precision calculation by Mathematica yields two non-real zeros of $g_5(x)$,

$$x \approx -1.261 \pm i\,0.133263, \quad x \approx -0.413044, \quad x \approx -1.91233, \quad x \approx -2.05128.$$

Thus, by Theorem 54, $\varphi(x) \notin \mathscr{L}\text{-}\mathscr{P}^+$. We also deduce that $g_5(x)$ has non-real zeros in observing its graph having a negative local maximum.



Now we pose a revised version of Problem 80.

Problem 82. *Characterize all multiplier sequences* $\{\gamma_k\}_{k=0}^{\infty}$ *such that*

$$\varphi_\mu(x) = \sum_{k=0}^{\infty} \frac{\gamma_k^\mu \, x^k}{k!} \in \mathscr{L}\text{-}\mathscr{P}^+$$

for all real values $\mu \geq 1$.

When multiplier sequences decay sufficiently quickly, they will satisfy Problem 82. We state this more precisely by the following definition (cf. [24, Section 4]).

Definition 83. A sequence of non-negative real numbers $\{s_k\}_{k=0}^{\infty}$ is called *rapidly decreasing*, if the sequence satisfies the following condition

$$s_k^2 \geq \alpha^2 s_{k-1} s_{k+1}, \text{ for } \alpha \geq \max\left\{2, \frac{\sqrt{2}}{2}\left(1+\sqrt{1+s_1}\right)\right\}, \; k = 1, 2, \ldots .$$

For $f(x) = \sum_{k=0}^{\infty} \frac{s_k}{k!} x^k$, where $\{s_k\}_{k=0}^{\infty}$ is a rapidly decreasing sequence, $f(x) \in \mathscr{L}\text{-}\mathscr{P}^+$ ([24, Section 4]). Hence, by Theorem 73, rapidly decreasing sequences are multiplier sequences. Given any rapidly decreasing sequence $\{s_k\}_{k=0}^{\infty}$, the sequence $\{s_k^\mu\}_{k=0}^{\infty}$ is also a rapidly decreasing sequence for any $\mu \geq 1$, since

$$s_k^{2\mu} \geq \alpha^{2\mu} s_{k-1}^\mu s_{k+1}^\mu \geq \alpha^2 s_{k-1}^\mu s_{k+1}^\mu \quad (\alpha \geq 2).$$

Thus, any rapidly decreasing sequence satisfies Problem 82. We will return to consider Problem 82 in a different context in Section 3.2.1.

3.1.1 Complex zero decreasing sequences

A more restricted class of multiplier sequences are defined as follows.

Definition 84. A sequence $\{\gamma_k\}_{k=0}^{\infty}$ is a *complex zero decreasing sequence (CZDS)*, if

$$Z_c\left(\sum_{k=0}^{n}\gamma_k a_k x^k\right) \le Z_c\left(\sum_{k=0}^{n} a_k x^k\right),$$

for any real polynomial $\sum_{k=0}^{n} a_k x^k$, where $Z_c(p(x))$ denotes the number of non-real zeros of $p(x)$, counting multiplicities. (The acronym CZDS will also be used in the plural).

It is immediately clear that a CZDS is a multiplier sequence, but we will see that the converse is not true. It is easy to see that the trivial multiplier sequences are also CZDS. Now we consider the multiplier sequences from Example 74.

Example 85.

(i) The sequence $\{k\}_{k=0}^{\infty}$, is a CZDS, since it corresponds to the operator $x\dfrac{d}{dx}$ (cf. Corollary 55).

(ii) For a non-zero real number r, the sequence $\{r^k\}_{k=0}^{\infty}$ is a CZDS, since for any polynomial $\sum_{k=0}^{n} a_k x^k = p(x)$, $\sum_{k=0}^{n} r^k a_k x^k = p(rx)$.

(iii) The sequence $T = \{k^2+k+1\}_{k=0}^{\infty}$ is *not* a CZDS. Consider the polynomial $p(x) = (x+1)^6(x^2+\frac{1}{2}x+\frac{1}{5})$. Then a calculation shows that

$$T[p(x)] = \frac{1}{10}(x+1)^4(730x^4+785x^3+306x^2+43x+2),$$

which has more non-real zeros than $p(x)$.

A consequence of the following theorem is that complex zero decreasing sequences exist in abundance. Below, we provide a proof of part (iii), which does not seem to appear in the literature.

Theorem 86 (Laguerre's Theorem [51, Satz 3.2])**.**

(i) Let $f(x) = \sum_{k=0}^{n} a_k x^k$ be an arbitrary real polynomial of degree n and let $g(x)$ be a polynomial with only real zeros, none of which lie in the interval $(0, n)$. Then $Z_c\left(\sum_{k=0}^{n} g(k) a_k x^k\right) \le Z_c(f(x))$.

(ii) *Let* $f(x) = \sum_{k=0}^{n} a_k x^k$ *be an arbitrary real polynomial of degree* n*, let* $\varphi(x) \in \mathscr{L}\text{-}\mathscr{P}$*, and suppose that none of the zeros of* φ *lie in the interval* $(0, n)$*. Then* $Z_c\left(\sum_{k=0}^{n} \varphi(k) a_k x^k\right) \leq Z_c(f(x))$*.*

(iii) *If* $\varphi \in \mathscr{L}\text{-}\mathscr{P}(-\infty, 0]$*, then the sequence* $\{\varphi(k)\}_{k=0}^{\infty}$ *is a CZDS.*

Proof. (iii). Consider a polynomial $p(x) = \sum_{k=0}^{n} a_k x^k$ and an entire function $\varphi \in \mathscr{L}\text{-}\mathscr{P}(-\infty, 0]$. There exist a maximal integer $d \geq 0$ such that $\psi(x) = \varphi(x)/x^d$ is entire. Then $\psi \in \mathscr{L}\text{-}\mathscr{P}(-\infty, 0)$, and there exist a sequence of hyperbolic polynomials $\{g_j(x)\}_{j=0}^{\infty}$ which converge to $\psi(x)$ uniformly on compact subsets of $\mathbb{C}$. Consider a closed disk $K = \{z \in \mathbb{C} : |z - (n+1)/2| \leq (n+1)/2\}$, where n is the degree of the polynomial $p(x)$. Then there exist a minimum value $m = \min_{z \in K} |\psi(z)| > 0$. By uniform convergence, for $\epsilon = m/3$, there exist $j_0 \in \mathbb{N}$ such that

$$|\psi(z)| - |g_j(z)| \leq |\psi(z) - g_j(z)| < \epsilon$$

for all $z \in K$ and $j \geq j_0$. Hence, $0 < 2m/3 < |g_j(z)|$ for $j \geq j_0$. Specifically, for $j \geq j_0$, $g_j(x)$ will not vanish for $x \in [0, n]$. By part (i),

$$Z_c\left(\sum_{k=0}^{n} k^d g_j(k) a_k x^k\right) \leq Z_c(p(x))$$

for all $j \geq j_0$. By Hurwitz's theorem (Theorem 50),

$$Z_c\left(\sum_{k=0}^{n} \varphi(k) a_k x^k\right) = Z_c\left(\sum_{k=0}^{n} k^d g_j(k) a_k x^k\right)$$

for large j, thus $\{\varphi(k)\}_{k=0}^{\infty}$ is a CZDS. □

An application of Theorem 86 establishes that certain modified Bessel functions are in $\mathscr{L}\text{-}\mathscr{P}^+$.

Example 87. For each $\mu > 0$, $\{1/\Gamma(k+\mu)\}_{k=0}^{\infty}$ is a CZDS, where $\Gamma(z)$ is the gamma function (see [26, Theorem 4.1]). In particular, for $j \in \mathbb{N}$, $\{1/(k+j)!\}_{k=0}^{\infty}$ is a CZDS. Thus

$$\sum_{k=0}^{\infty} \frac{x^k}{\Gamma(k+\mu)\, k!} \in \mathscr{L}\text{-}\mathscr{P}^+$$

for $\mu > 0$.

The complex zero decreasing sequence version of Problem 82 is the following.

Problem 88. *Characterize all complex zero decreasing sequences* $\{\gamma_k\}_{k=0}^{\infty}$ *such that* $\{\gamma_k^{\mu}\}_{k=0}^{\infty}$ *is CZDS for* $\mu \geq 1$.

3.2 Hyperbolicity and stability preservers

Definition 89. A linear operator $T : \mathbb{R}[x] \to \mathbb{R}[x]$ is said to *preserve hyperbolicity* (or T is a *hyperbolicity preserver*) if $T[p(x)] \in \mathscr{L}\text{-}\mathscr{P}$, whenever $p(x) \in \mathscr{L}\text{-}\mathscr{P} \cap \mathbb{R}[x]$.

By identifying a polynomial $p(x) = \sum_{k=0}^{n} a_k x^k$ with an infinite vector

$$(a_0, a_1, a_2, \ldots, a_n, 0, 0, \ldots),$$

any linear operator T on $\mathbb{R}[x]$ can be represented by an infinite-dimensional matrix M_T, with entries corresponding to $T[x^n]$ as its n^{th} column. The differential operator D, for example, is represented by the matrix

$$M_D = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 2 & 0 & 0 & \ldots \\ 0 & 0 & 0 & 3 & 0 & \ldots \\ 0 & 0 & 0 & 0 & 4 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Multiplier sequences and CZDS are hyperbolicity preserving linear operators. In Definition 72, multiplier sequences were mentioned as diagonal operators. This is because the matrices represented by multiplier sequences are diagonal. Other linear operators which preserve hyperbolicity are not as simple to describe. In 2007, J. Borcea and P. Brändén characterized all linear operators that preserve hyperbolicity.

Theorem 90 (J. Borcea and P. Brändén [8])**.** *A linear operator* $T : \mathbb{R}[x] \to \mathbb{R}[x]$ *preserves hyperbolicity if and only if either*

(i) T *has range of dimension at most two and is of the form* $T[f] = \alpha(f)P + \beta(f)Q$, *where* α *and* β *are linear functionals on* $\mathbb{R}[x]$, *and* P *and* Q *are polynomials with only real interlacing zeros, or*

(ii) $T[e^{-xw}] = \displaystyle\sum_{k=0}^{\infty} \frac{(-w)^n T[x^n]}{n!} \in \overline{A}$, *or*

(iii) $T[e^{xw}] = \sum_{k=0}^{\infty} \frac{w^n T[x^n]}{n!} \in \overline{A}$,

where $A = \{f \in \mathbb{R}[x,w] \,|\, f(x,w) \neq 0,$ *whenever* $\mathrm{Im}(x) > 0$ *and* $\mathrm{Im}(w) > 0\}$*, and* $\overline{A}$ *denotes the set of entire functions in two variables which are uniform limits on compact subsets of polynomials in the set.* □

Theorem 91 (J. Borcea and P. Brändén [9, Theorem 1.3])**.** *Let* $T : \mathbb{R}[x] \to \mathbb{R}[x]$ *be a finite differential linear operator, thus there exists real polynomials* $\{Q_k(x)\}_{k=0}^{n}$ *such that*

$$T = \sum_{k=0}^{n} Q_k(x) D^k.$$

T *is hyperbolicity preserving, if and only if,*

$$\sum_{k=0}^{n} Q_k(x)(-w)^k \neq 0$$

for every $x, w \in H^+$*.* □

We will see in Chapter 5 that hyperbolicity preservation is not limited to linear operators.

3.2.1 Stability preservation

Hyperbolicity preserving operators can be generalized to the following operators.

Definition 92. A polynomial $f(z) \in \mathbb{C}[z]$ is called *stable* if $f(z) \neq 0$ for all $z \in \mathbb{C}$ with $\mathrm{Im}(z) > 0$, and it is called *strictly stable* if $f(z) \neq 0$ for all $z \in \mathbb{C}$ with $\mathrm{Im}(z) \geq 0$. The set of stable polynomials is denoted by $\mathcal{H}_1(\mathbb{C})$.

We see immediately from the definition that a polynomial with real coefficients is stable if and only if it is hyperbolic.

Definition 93. A linear operator T defined on a linear subspace $V \subseteq \mathbb{C}[z]$ is called *stability preserving* on a given subset $M \subseteq V$ if

$$T[\mathcal{H}_1(\mathbb{C}) \cap M] \subseteq \mathcal{H}_1(\mathbb{C}) \cup \{0\}.$$

Note if $V = M = \mathbb{R}[x]$ in the above definition, a stability preserving operator T will be hyperbolicity preserving. The definition of stability can be extended to multivariate polynomials (see [8]).

The classical formulation of stability is commonly known as *Hurwitz stable*, which is when a function has its zeros in the open left half-plane. If a function has its zeros in the closed left half-plane, the function is called *weakly Hurwitz stable.* The following theorem is of a folklore origin and is sometimes attributed to the great Slovakian engineer A. B. Stodola ([37, p. 204], [41]).

Theorem 94 (A. B. Stodola, 1893)**.** *If a real polynomial is Hurwitz stable, then all its coefficients are positive.* □

Remark 95. It may be natural to think the converse of Theorem 94 is true, since a polynomial is made up of product of linear factors $(x + a)$, and if $a > 0$, the zeros are on the left half-plane. The zeros of $x^2 + x + 1$ are also on the left half-plane. However, polynomials such as $(x+3)((x-1)^2+9)$ [50, p. 181], $x^n + 1$ for $n \geq 3$, and $\sum_{k=0}^{d} x^k$ for $d \geq 4$ are polynomials that have zeros on the right half-plane that have positive coefficients.

Notation 96. Rather than specifying the various regions of stability, it is also common to specify a region $\Omega \subset \mathbb{C}$, and denote by $\pi(\Omega)$, the class of all (complex or real) univariate polynomials whose zeros lie in Ω (see [11], and some references therein). Also denote by $\pi_n(\Omega)$, the class of all polynomials (complex or real) of degree $\leq n$ all of whose zeros lie in Ω. The following are some problems in current research that involve $\pi(\Omega)$ and $\pi_n(\Omega)$, $n \in \mathbb{N}$.

Problem 97. *Characterize all linear operators* $T : \pi(\Omega) \to \pi(\Omega) \cup \{0\}$.

Problem 98. *Characterize all linear operators* $T : \pi_n(\Omega) \to \pi(\Omega) \cup \{0\}$ *for* $n \in \mathbb{N}$.

J. Borcea and P. Brändén completely solved Problems 97 and 98 for all closed circular domains and their boundaries, and in [9], they obtained multivariate extensions for all finite order linear differential operators with polynomial coefficients. We emphasize some important remaining open cases of these problems.

Problem 99. *Settle Problems 97 and 98 in the following situations.*

(i) Ω *is an open circular domain,*

(ii) Ω *is a sector or a double sector,*

(iii) Ω *is a strip,*

(iv) Ω *is a half-line.*

Recall Problem 82 in Section 3.1. Characterize all sequences $\{\gamma_k\}_{k=0}^{\infty}$ such that $\varphi_\mu(x) = \sum_{k=0}^{\infty} \frac{\gamma_k^\mu x^k}{k!} \in \mathscr{L}\text{-}\mathscr{P}^+$ for all real values $\mu \geq 1$. We pose the following related problem.

Problem 100. *What is the infimum $\mu > 0$ such that*

$$\varphi_\mu(x) = \sum_{k=0}^{\infty} \frac{\gamma_k^\mu x^k}{k!}$$

is Hurwitz stable for any multiplier sequence $\{\gamma_k\}_{k=0}^{\infty}$?

3.3 Differential operators

Let $D = \dfrac{d}{dx}$ denote differentiation with respect to x. In general, if

$$\psi(y) = \sum_{k=0}^{\infty} Q_k(x) y^k \qquad (Q_k(x) \in \mathbb{C}[x];\ k = 0, 1, 2, \ldots)$$

is a formal power series, then we define the action of the linear operator $\psi(D)$ on an entire function $f(x)$ by

$$\psi(D)[f(x)] = \sum_{k=0}^{\infty} Q_k(x) f^{(k)}(x), \tag{3.1}$$

whenever the right hand side of (3.1) represents an entire function. In the case where each of the polynomials $Q_k(x)$ are real constants, the operator $\psi(D)$ has received a great deal of attention (see [23] and the reference therein). When $f(x)$ is a polynomial, the right hand side of (3.1) is again a polynomial and so the question of convergence does not arise.

We will often think of operators of the form (3.1) as objects in themselves $\psi(D) = \sum_{k=0}^{\infty} Q_k(x) D^k$, where we take D^0 to be the identity operator I. We shall often suppress the symbol I (e.g., the operator $(I + xD)$ will simply be written as $(1 + xD)$). When more than one of these operators are applied in succession, we will adopt the convention of applying the operators in order from right to left. This convention is important since two operators need not commute in general, for example, $(xI)(D)[f(x)] = xf'(x)$, while $(D)(xI)[f(x)] = D[xf(x)] = f(x) + xf'(x)$. There are many different ways to define a linear operator on the vector space of complex polynomials. In the midst of such variety, it is a remarkable fact that no matter how a linear operator $T : \mathbb{C}[x] \to \mathbb{C}[x]$ is defined, it can always be represented formally as a differential operator with complex polynomial coefficients.

Proposition 101 (J. Peetre [53], A. Piotrowski [55, p. 32]). *Let $T : \mathbb{C}[z] \to \mathbb{C}[z]$ be a linear operator. Then there exists a unique sequence of complex polynomials, $\{Q_k(z)\}_{k=0}^{\infty}$, such that*

$$T = \sum_{k=0}^{\infty} Q_k(z) D^k \quad \left(D = \frac{d}{dx} \right). \qquad \square \tag{3.2}$$

M. Chasse has an improvement of Proposition 101, where the polynomials $Q_k(z)$ are expressed in the following form.

Proposition 102 ([18, Proposition 216]). *If $T : \mathbb{C}[z] \to \mathbb{C}[z]$ is a linear operator, then in the representation $T = \sum_{k=0}^{\infty} Q_k(z) D^k$, the polynomials $Q_k(z)$ are given by*

$$Q_k(z) = \frac{1}{k!} \sum_{j=0}^{k} \binom{k}{j} T[z^j](-z)^{k-j}. \qquad \square$$

A problem in current research related to Problems 97, 98, and 99 is the following.

Problem 103 ([11, Problem 10]). *For a linear operator $T = \sum_{k=0}^{\infty} Q_k(z) D^k$, $T : \pi_n(\Omega) \to \pi(\Omega)$, $n \in \mathbb{N}$, characterize the polynomials $Q_k(z)$, where Ω, π, and π_n are defined in Notation 96.*

If the linear operator in Proposition 101 is a diagonal operator, the following proposition gives an explicit representation of the polynomials $Q_k(z)$ in (3.2).

Proposition 104 ([55, Proposition 33]). *Let $\{\gamma_k\}_{k=0}^{\infty}$ be a sequence of real numbers and let*

$$g_n^*(x) = \sum_{k=0}^{n} \binom{n}{k} \gamma_k x^{n-k} \qquad (n = 0, 1, 2, \ldots).$$

Then the linear operator T on the real (or complex) polynomials defined by $T[x^m] = \gamma_m x^m$ $(m = 0, 1, 2, \ldots)$ can be represented as

$$T = \sum_{n=0}^{\infty} \frac{g_n^*(-1)}{n!} x^n D^n. \qquad \square$$

P. Brändén established a result which gives an interesting necessary condition on the coefficient polynomials of a differential operator.

Lemma 105 (P. Brändén [13, Lemma 2.7]). *Suppose the linear operator*

$$T = \sum_{k=M}^{N} Q_k(x) D^k, \tag{3.3}$$

where $Q_k(x) \in \mathbb{R}[x]$ *for* $M \leq k \leq N$*, and* $Q_M(x)Q_N(x) \not\equiv 0$*, preserves hyperbolicity. Then* $Q_j(x) \ll Q_{j+1}(x)$ *for* $M \leq j \leq N-1$*. In particular,* $Q_j(x)$ *is hyperbolic or identically zero for all* $M \leq j \leq N$*.* □

Remark 106. It is worthwhile to note that Lemma 105 cannot be extended to infinite order differential operators. Let T_1 and T_2 be linear transformations given by $T_1[z^k] = \frac{z^k}{k!}$, and $T_2[z^k] = H_k(z)$ for all $k \in \mathbb{N}$, where $H_k(z)$ is the k^{th} Hermite polynomial. If $T = T_2T_1$, then T preserves hyperbolicity, and in the representation

$$T = \sum_{k=0}^{\infty} Q_k(z)D^k,$$

the polynomial $Q_2(z) = -\frac{1}{2}(z^2+1)$ has non-real zeros. M. Chasse has a similar result for a differential operator that preserves stability [18, Proposition 210], which is the following. Suppose the linear operator $T = \sum_{k=0}^{n} Q_k(z)D^k$, is stable (cf. Definition 93), for $n \in \mathbb{N}$. Then the polynomials $Q_k(z)$ are stable (cf. Definition 92).

Proposition 101 guarantees a differential operator representation for any linear operator acting on the space of complex polynomials. The following example lists the differential operator representations for the multiplier sequences in Example 74.

Example 107.

(i) As we have mentioned previously, the complex zero decreasing sequence $T = \{k\}_{k=0}^{\infty}$ has differential operator representation $T = xD$.

(ii) For a non-zero real number r, the complex zero decreasing sequence $T = \{r^k\}_{k=0}^{\infty}$ has differential operator representation $T = \sum_{n=0}^{\infty} \frac{(r-1)^n}{n!} x^n D^n$, which can be derived by Proposition 104.

(iii) The multiplier sequence $T = \{k^2 + k + 1\}_{k=0}^{\infty}$ is represented by the differential operator $T = x^2D^2 + 2xD + 1$.

As hyperbolicity preservers (cf. Section 3.2) generalize the idea of multiplier sequences, the operators such as (i) and (ii) in Example 107 can be generalized in the following definition.

Definition 108. A linear operator $T : \mathbb{R}[x] \to \mathbb{R}[x]$, is a *complex zero decreasing operator (CZDO)*, if

$$Z_c(T[p(z)]) \leq Z_c(p(z)),$$

for any polynomial $p(z) \in \mathbb{R}[z]$, where $Z_c(p(z))$ denotes the number of non-real zeros of $p(z)$, counting multiplicities. (The acronym CZDO will also be used in the plural).

As CZDS are a more restricted class of multiplier sequences, any CZDO will preserve hyperbolicity. The following is a classic theorem regarding CZDO.

Theorem 109 (Hermite-Poulain [51], [54]). *Let $g(x) = \sum_{k=0}^{n} b_k x^k$ be a real polynomial with only real zeros. Then the linear operator $g(D) : \mathbb{R}[x] \to \mathbb{R}[x]$ is CZDO.*

Corollary 110. *If $\varphi(x) \in \mathscr{L}\text{-}\mathscr{P}$, then the linear operator $\varphi(D) : \mathbb{R}[x] \to \mathbb{R}[x]$ is CZDO.* □

Remark 111. Note the CZDO $g(D)$ in Theorem 109 have constant coefficients. In constrast, an application of Proposition 104 to any CZDS yields a CZDO with non-constant coefficients.

In the sequel, we present some of the current results which pertains to differential operators that preserve hyperbolicity, as well as complex zero decreasing operators. The following proposition illustrates an application of the generalized Malo-Schur-Szegő composition theorem (Theorem 42).

Proposition 112 (G. Csordas [27]). *For an arbitrary polynomial $Q(x) \in \mathscr{L}\text{-}\mathscr{P} \cap \mathbb{R}[x]$, $Q(x) \not\equiv 0$, and $t \in (-\infty, 0)$ the operator*

$$T := \sum_{k=0}^{\infty} \frac{t^k}{k!} Q^{(k)}(x) D^k \tag{3.4}$$

preserves hyperbolicity; that is, $T : \mathscr{L}\text{-}\mathscr{P} \cap \mathbb{R}[x] \to \mathscr{L}\text{-}\mathscr{P}$.

Proof. Fix $t \in (-\infty, 0)$ and let $f(x) \in \mathscr{L}\text{-}\mathscr{P} \cap \mathbb{R}[x]$. It suffices to show that (the real polynomial)

$$\varphi(x) := T[f](x) = \sum_{k=0}^{\infty} \frac{t^k}{k!} Q^{(k)}(x) f^{(k)}(x) \in \mathscr{L}\text{-}\mathscr{P}.$$

Fix $z_0 \in H^- = \{z \mid \operatorname{Im} z < 0\}$ and set

$$A(w) := \sum_{k=0}^{\infty} \frac{Q^{(k)}(z_0)}{k!} w^k \quad \text{and} \quad B(w) := \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!} w^k.$$

Since $A(w) = Q(z_0 + w)$, all the zeros of $A(w)$ lie in the open upper half-plane $H^+ = \{z \mid \operatorname{Im} z > 0\}$. (This is an open sector of aperture π.) Similarly, all the zeros of $B(w)$ lie in the open upper half-plane H^+. Thus by Theorem 42, all the zeros of the composition

$$g(w) := \sum_{k=0}^{\infty} Q^{(k)}(z_0) f^{(k)}(z_0) \frac{w^k}{k!}$$

lie in $\mathbb{C} - (-\infty, 0]$. Hence, $g(t) \neq 0$, if $t < 0$. A similar argument shows that if $z_0 \in H^+$, then again $g(t) \neq 0$, if $t < 0$. Therefore $\varphi(z_0) \neq 0$, if $z_0 \in H^- \cup H^+$. □

Proposition 113 (D. Bleecker and G. Csordas [6, Lemma 2.2]). *For $a, b \geq 0$, and $a + b > 0$, the linear operator $T = -aD^2 + xD + b$ preserves hyperbolicity.* □

A. Piotrowski generalized Proposition 113 to the following, and showed the linear operators in Proposition 113 are complex zero decreasing operators.

Proposition 114 ([55, Proposition 68]). *Suppose $p(x) = \sum_{k=0}^{n} a_k x^k$, $a_n \neq 0$, is a real polynomial. If a, b, c, d are real numbers such that $a, b \geq 0$, $a+b > 0$, and $b+cn \geq 0$, then $Z_c(T[p(x)]) \leq Z_c(p(x))$, where $T = -aD^2 + (cx + d)D + b$.* □

Carnicer, Peña, and Pinkus [17] studied linear operators defined by

$$T[x^n] = x^n + \sum_{k=0}^{n-1} b_{n,k} x^k \quad (b_{n,k} \in \mathbb{R};\ n = 0, 1, 2, \ldots), \tag{3.5}$$

and proved the converse of Corollary 110, thereby completely characterizing all CZDO of the form (3.5).

Theorem 115 ([17, p. 5]). *Let the linear operator T be defined by* (3.5). *Then T is CZDO if and only if $T = \varphi(D)$, where $\varphi(x) \in \mathscr{L}\text{-}\mathscr{P}$ and $\varphi(0) = 1$.* □

Forgács and Piotrowski studied the following linear operator while investigating the properties of Laguerre polynomials.

Theorem 116 ([36, Theorem 2.6]). *The linear operator $T = -xD^2 + (x - (a + 1))D + b$ preserves hyperbolicity for $0 \leq b \leq (a + 1)$.* □

The following linear operator was studied by Forgács et al [5] in an investigation of Legendre polynomials.

Proposition 117 ([5, Proposition 5]). *The linear operator $T = (x^2 - 1)D^2 + 2xD + d$ preserves hyperbolicity for $0 \leq d \leq 1$.* □

As it was mentioned in Chapter 2 Example 31, the Legendre polynomials are a special case of the Jacobi polynomials. The following result generalizes Proposition 117 to a linear operator for the general Jacobi polynomials.

Proposition 118. *For $\alpha, \beta \geq -1$, and $0 \leq d \leq (1+\alpha)(1+\beta)$, the linear operator $T = (x^2 - 1)D^2 + [(\alpha+\beta+2)x + \alpha - \beta]D + d$ preserves hyperbolicity.* □

The original proof of Proposition 118 is quite long, so we include it separately in the following subsection (Subsection 3.3.1).

Remark 119. By Lemma 105 we can see that the conditions on α and β in Proposition 118 are optimal. Namely, if $T = (x^2-1)D^2 + [(\alpha+\beta+2)x + \alpha - \beta]D + d$, and T preserves hyperbolicity, then $Q_1(x) = (\alpha+\beta+2)x + \alpha - \beta$ must have its zero between -1 and 1, or equivalently,

$$-1 \leq \frac{\beta - \alpha}{\alpha + \beta + 2} \leq 1,$$

which implies $\alpha \geq -1$ and $\beta \geq -1$.

The differential operator in Proposition 118 for the special case $\alpha = \beta = 0$, $d = 0$ is also a complex zero decreasing operator.

Proposition 120. *For $\lambda > 0$, the differential operator $T = (x^2 - \lambda)D^2 + 2xD$ is a complex zero decreasing operator.*

Proof. Since the derivative operator is a complex zero decreasing operator, if suffices to show that the operator $S := (x^2 - \lambda)D + 2x$ is a complex zero decreasing operator. For any real polynomial $f(x) \in \mathbb{R}[x]$,

$$Z_c(S[f(x)]) = Z_c((x^2 - \lambda)f'(x) + 2xf(x)) = Z_c(D[(x^2-\lambda)f]) \leq Z_c(f(x)). \qquad \square$$

3.3.1 Proof of Proposition 118

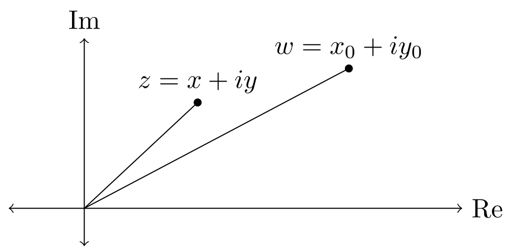
The proof of Proposition 118 requires the following two preliminary results.

Lemma 121. *If $0 < c \leq d$, $0 < r$, and $0 < \operatorname{Arg}(z) < \frac{\pi}{2}$, then $0 < \operatorname{Arg}(\sqrt{z^2 + r} + d) < \operatorname{Arg}(z + c)$, where the square root of a complex number $w = re^{i\theta}$ is taken to be $\sqrt{w} = r^{1/2}e^{i\theta/2}$, $r \geq 0$, $\theta \in [0, 2\pi)$, and Arg is defined on $[0, 2\pi)$. Similarly, if $0 < c \leq d$, $0 < r$, and $\frac{\pi}{2} < \operatorname{Arg}(z) < \pi$, then $\operatorname{Arg}(-z + c) < \operatorname{Arg}(-\sqrt{z^2 + r} + d) < 2\pi$.*

Proof. If $0 < \operatorname{Arg}(z) < \frac{\pi}{2}$, then $0 < \operatorname{Arg}(\sqrt{z^2 + r}) < \operatorname{Arg}(z)$. Thus, if $|\sqrt{z^2 + r}| \leq |z|$, the result follows. Consider the case $|\sqrt{z^2+r}| > |z|$. It suffices to show

$$\mathrm{Im}(\sqrt{z^2+r}) < \mathrm{Im}(z).$$

Suppose this is false, that is, $\mathrm{Im}(\sqrt{z^2+r}) \geq \mathrm{Im}(z)$. Let $z = x+iy$, and $w = \sqrt{z^2+r} = x_0+iy_0$, so that $y_0 \geq y$. Considering $\mathrm{Arg}(\sqrt{z^2+r}) < \mathrm{Arg}(z)$, $|\sqrt{z^2+r}| > |z|$, and $\mathrm{Im}(\sqrt{z^2+r}) \geq \mathrm{Im}(z)$ together imply that $x_0 > x$. We illustrate z and w in the plane.



From the above, $w^2 = x_0^2 - y_0^2 + 2x_0y_0i$, and

$$x^2 - y^2 + 2xyi = z^2 = w^2 - r = x_0^2 - y_0^2 + 2x_0y_0i.$$

The equality $x_0y_0 = xy$ is given by the imaginary part of the above equation, and hence

$$1 \leq \frac{y_0}{y} = \frac{x}{x_0}.$$

However, this implies that $x_0 \leq x$, a contradiction. A similar argument, *mutatis mutandis*, establishes the case when $\frac{3\pi}{2} < \mathrm{Arg}(z) < 2\pi$.

A shorter second proof, is due to R. Bates [3]. To show $\mathrm{Im}(z) < \mathrm{Im}(\sqrt{z^2+r})$, take a point in the first quadrant, say $z_0 = 1+i$, and verify that $\mathrm{Im}(z_0) < \mathrm{Im}(\sqrt{z_0^2+r})$, for any positive r. We show that $\mathrm{Im}(z) = \mathrm{Im}(\sqrt{z^2+r})$ cannot happen. Suppose for some z in the first quadrant, $\mathrm{Im}(z) = \mathrm{Im}(\sqrt{z^2+r})$. Then for some real number δ, $z+\delta = \sqrt{z^2+r}$. Squaring both sides,

$$z^2 + 2\delta z + \delta^2 = z^2 + r.$$

Solving for z, we find that $z = (r-\delta^2)/(2\delta)$, a real number, which is a contradiction. Thus by continuity, we cannot have $\mathrm{Im}(z) > \mathrm{Im}(\sqrt{z^2+r})$. □

Lemma 122. *For a complex number* $0 < \operatorname{Arg} z < \frac{\pi}{2}$*, let* $f(t) = \sqrt{z^2 + t}$*, for* $t \in \mathbb{R}$*, where the square root of a complex number* $w = re^{i\theta}$ *is taken to be* $\sqrt{w} = r^{1/2}e^{i\theta/2}$*,* $r \geq 0$*,* $\theta \in [0, 2\pi)$*, and* Arg *is defined on* $[0, 2\pi)$*. For* $\xi < \eta$*, let*

$$\Delta(\xi, \eta) := \frac{\operatorname{Im}(f(\xi)) - \operatorname{Im}(f(\eta))}{\operatorname{Re}(f(\xi)) - \operatorname{Re}(f(\eta))}.$$

Then

$$\Delta(s, t) < \Delta(s, u) < \Delta(t, u) < 0,$$

for all $0 \leq s < t < u$*. Similarly, for* $\frac{\pi}{2} < \operatorname{Arg} z < \pi$*, similar results hold true, that is,*

$$0 < \Delta(t, u) < \Delta(s, u) < \Delta(s, t),$$

for all $0 \leq s < t < u$*, where* $\Delta(\xi, \eta)$*,* $f(t) = \sqrt{z^2 + t}$*, and the square root of a complex number is defined exactly in the same way as* $0 < \operatorname{Arg} z < \frac{\pi}{2}$*.*

Proof. Let $z = a + ib$ for $a, b \in \mathbb{R}$. Then

$$\begin{aligned} f(t) &= \frac{\sqrt{\sqrt{4a^2b^2 + (a^2 - b^2 + t)^2} + (a^2 - b^2 + t)}}{\sqrt{2}} \\ &\quad + i\,\frac{\sqrt{\sqrt{4a^2b^2 + (a^2 - b^2 + t)^2} - (a^2 - b^2 + t)}}{\sqrt{2}}, \end{aligned}$$

The function $f(t)$ is a parametric equation in the complex plane. Denote the real coordinate of $f(t)$ by x, and the imaginary coordinate of $f(t)$ by y. Then the parametric derivative of $f(t)$ is

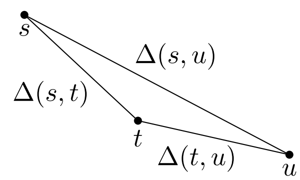
$$\frac{dy}{dx} = -\frac{\sqrt{\sqrt{4a^2b^2 + (a^2 - b^2 + t)^2} - (a^2 - b^2 + t)}}{\sqrt{\sqrt{4a^2b^2 + (a^2 - b^2 + t)^2} + (a^2 - b^2 + t)}},$$

and its second derivative is

$$\frac{d^2y}{dx^2} = \frac{2\sqrt{2}\sqrt{\sqrt{4a^2b^2 + (a^2 - b^2 + t)^2} - (a^2 - b^2 + t)}}{\sqrt{4a^2b^2 + (a^2 - b^2 + t)^2} + (a^2 - b^2 + t)}.$$

The parametric derivative is negative, and the second derivative is positive for $t \geq 0$. Thus, the result follows since $f(t)$ is decreasing and concave up for $t \geq 0$. We illustrate points s, t, and u, for

$0 \le s < t < u$,



A similar argument, *mutatis mutandis*, establishes the case for $\frac{\pi}{2} < \operatorname{Arg} z < \pi$. □

Remark 123. We remark that Lemma 122 implies Lemma 121. The first proof of Lemma 121 is geometric, and the second proof is topological. The proof of Lemma 122 is elementary, albeit the details are somewhat involved.

Proof of Proposition 118. Note that if α or $\beta = -1$, then $d = 0$. By Theorem 90, it suffices to show that $T[e^{-zw}]$ is stable. We evaluate

$$T[e^{-xw}] = e^{-xw}\left[(x^2-1)w^2 - ((\alpha+\beta+2)x + (\alpha-\beta))w + d\right].$$

Since e^{-xw} is nowhere zero, the zeros of $T[e^{-xw}]$ are those of

$$\begin{aligned} f(x,w) &:= (x^2-1)w^2 - ((\alpha+\beta+2)x + (\alpha-\beta))w + d \\ &= w^2x^2 - (\alpha+\beta+2)wx + (-(\alpha-\beta+w)w + d). \end{aligned} \tag{3.6}$$

Consider the expressions

$$\xi = \frac{\left(\frac{\alpha}{2}+\frac{\beta}{2}+1\right) \pm \sqrt{\left(w+\frac{(\alpha-\beta)}{2}\right)^2 + (1+\alpha)(1+\beta) - d}}{w}, \tag{3.7}$$

$$\xi_1 = \frac{\left(\frac{\alpha}{2}+\frac{\beta}{2}+1\right) + \sqrt{\left(w+\frac{(\alpha-\beta)}{2}\right)^2 + (1+\alpha)(1+\beta) - d}}{w}, \quad \text{and} \tag{3.8}$$

$$\xi_2 = \frac{\left(\frac{\alpha}{2}+\frac{\beta}{2}+1\right) - \sqrt{\left(w+\frac{(\alpha-\beta)}{2}\right)^2 + (1+\alpha)(1+\beta) - d}}{w}, \tag{3.9}$$

where the square root of a complex number $v = re^{i\theta}$ is taken to be $\sqrt{v} = r^{1/2}e^{i\theta/2}$, $r \ge 0$, $\theta \in [0, 2\pi)$. Clearly, ξ, ξ_1, and ξ_2 are the solutions to (3.6) for any given $w \in \mathbb{C} \setminus 0$. Let $\operatorname{Im}(w) > 0$, and define

$$z := w + \frac{\alpha-\beta}{2}. \tag{3.10}$$

Case 1: Suppose that

$$z = w + \frac{\alpha - \beta}{2} = ki,$$

or simply, $w = -\frac{(\alpha-\beta)}{2} + ki$ for $k > 0$. Then from (3.7),

$$\xi = \frac{\left(\frac{\alpha}{2} + \frac{\beta}{2} + 1\right) \pm \sqrt{-k^2 + (1+\alpha)(1+\beta) - d}}{-\frac{(\alpha-\beta)}{2} + ik}. \tag{3.11}$$

It is always true that

$$\left(\frac{\alpha}{2} + \frac{\beta}{2} + 1\right)^2 - (1+\alpha)(1+\beta) = \left(\frac{\alpha - \beta}{2}\right)^2 \geq 0, \tag{3.12}$$

which implies that

$$\left(\frac{\alpha}{2} + \frac{\beta}{2} + 1\right)^2 \geq (1+\alpha)(1+\beta) > (1+\alpha)(1+\beta) - k^2 - d. \tag{3.13}$$

If $(-k^2 + (1+\alpha)(1+\beta) - d) \geq 0$, then the square root of both sides of (3.13) is

$$\left(\frac{\alpha}{2} + \frac{\beta}{2} + 1\right) > \sqrt{-k^2 + (1+\alpha)(1+\beta) - d}.$$

Thus the numerator of (3.11) is positive, so that $\text{Im}(\xi) < 0$. If $(-k^2 + (1+\alpha)(1+\beta) - d) < 0$, or $(k^2 - (1+\alpha)(1+\beta) + d) > 0$, then the imaginary part of ξ, up to a positive constant is

$$-\left(\frac{\alpha}{2} + \frac{\beta}{2} + 1\right) k \pm \left(\frac{\beta - \alpha}{2}\right) \sqrt{k^2 - (1+\alpha)(1+\beta) + d}. \tag{3.14}$$

From (3.12),

$$\left(\frac{\alpha}{2} + \frac{\beta}{2} + 1\right)^2 - \left(\frac{\alpha - \beta}{2}\right)^2 = (1+\alpha)(1+\beta) \geq 0,$$

which implies the inequality

$$\left(\frac{\beta - \alpha}{2}\right)^2 k^2 \leq \left(\frac{\alpha}{2} + \frac{\beta}{2} + 1\right)^2 k^2.$$

With the hypothesis $d \leq (1+\alpha)(1+\beta)$, or $-(1+\alpha)(1+\beta) + d \leq 0$, and because we are in the case that $(k^2 - (1+\alpha)(1+\beta) + d) > 0$,

$$\left(\frac{\beta-\alpha}{2}\right)^2 (k^2 - (1+\alpha)(1+\beta) + d) < \left(\frac{\alpha}{2} + \frac{\beta}{2} + 1\right)^2 k^2,$$

which implies that (3.14) is negaive. Thus $\mathrm{Im}(\xi) < 0$.

For the ease of notation, we define $\zeta := z^2 + (1+\alpha)(1+\beta)$ for Case 2 and Case 3.

Case 2: If $0 < \mathrm{Arg}(z) < \frac{\pi}{2}$, where z is defined in (3.10), then

$$0 < \mathrm{Arg}\left(\sqrt{\zeta}\right) < \mathrm{Arg}\left(\sqrt{\zeta - d}\right) < \mathrm{Arg}(z) < \tfrac{\pi}{2}, \tag{3.15}$$

and similarly,

$$\pi < \mathrm{Arg}\left(-\sqrt{\zeta}\right) < \mathrm{Arg}\left(-\sqrt{\zeta - d}\right) < \mathrm{Arg}(-z) < \tfrac{3\pi}{2}, \tag{3.16}$$

where Arg is defined on $[0, 2\pi)$.

First consider (3.8),

$$\xi_1 = \frac{(\frac{\alpha+\beta+2}{2}) + \sqrt{\zeta - d}}{z + \frac{\beta-\alpha}{2}}.$$

If $\beta - \alpha < 0$, then the argument of $z + \frac{\beta-\alpha}{2}$ only grows larger. Thus, by (3.15),

$$0 < \mathrm{Arg}(\sqrt{\zeta - d} + \tfrac{\alpha+\beta+2}{2}) < \mathrm{Arg}(z + \tfrac{\beta-\alpha}{2}) < \pi.$$

Thus $\mathrm{Im}(\xi_1) < 0$. If $\beta - \alpha \geq 0$, then by Lemma 121,

$$0 < \mathrm{Arg}(\sqrt{\zeta - d} + \tfrac{\alpha+\beta+2}{2}) < \mathrm{Arg}(z + \tfrac{\beta-\alpha}{2}) < \frac{\pi}{2},$$

which implies that $\mathrm{Im}(\xi_1) < 0$.

Now consider (3.9),

$$\xi_2 = \frac{(\frac{\alpha+\beta+2}{2}) - \sqrt{\zeta - d}}{z + \frac{\beta-\alpha}{2}}.$$

Adding $c := \frac{\alpha+\beta+2}{2}$ to each of the expressions in (3.16) may change the inequalities. If c is sufficiently small, the inequalities remain unchanged, that is,

$$\pi < \mathrm{Arg}\left(c - \sqrt{\zeta}\right) < \mathrm{Arg}\left(c - \sqrt{\zeta - d}\right). \tag{3.17}$$

Subtracting $\text{Arg}\left(z + \frac{\beta-\alpha}{2}\right)$ from each term in (3.17), we will demonstrate a lower bound

$$\pi < \text{Arg}\left(\frac{(\frac{\alpha+\beta+2}{2}) - \sqrt{\zeta}}{z + \frac{\beta-\alpha}{2}}\right). \tag{3.18}$$

We will need the identity

$$\frac{(\frac{\alpha+\beta+2}{2}) - \sqrt{\zeta}}{z + \frac{\beta-\alpha}{2}} = -\frac{z - \frac{\beta-\alpha}{2}}{(\frac{\alpha+\beta+2}{2}) + \sqrt{\zeta}}, \tag{3.19}$$

and (3.15). Consider (3.15),

$$0 < \text{Arg}(\sqrt{\zeta}) < \text{Arg}(z) < \frac{\pi}{2},$$

and the right hand side of (3.19),

$$-\frac{z - \frac{\beta-\alpha}{2}}{(\frac{\alpha+\beta+2}{2}) + \sqrt{\zeta}}.$$

If $\beta - \alpha > 0$,

$$0 < \text{Arg}((\tfrac{\alpha+\beta+2}{2}) + \sqrt{\zeta}) < \text{Arg}(z - \frac{\beta-\alpha}{2}) < \pi,$$

and inequality (3.18) follows. If $\beta - \alpha \leq 0$,

$$0 < \text{Arg}((\tfrac{\alpha+\beta+2}{2}) + \sqrt{\zeta}) < \text{Arg}(z + \tfrac{\alpha-\beta}{2}) < \frac{\pi}{2}$$

by Lemma 121, so inequality (3.18) holds. If c is larger, Lemma 122 implies that the inequalities in (3.16) changes to either one of the following cases:

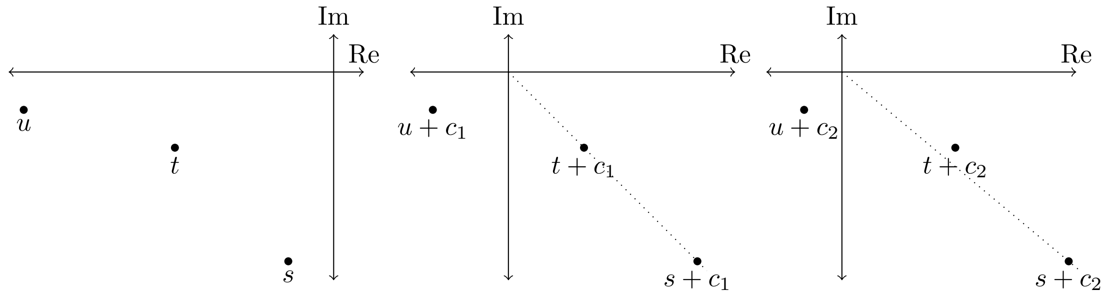
$$\pi < \text{Arg}\left(c - \sqrt{\zeta}\right) < \text{Arg}(c-z) \leq \text{Arg}\left(c - \sqrt{\zeta - d}\right),$$

$$\pi < \text{Arg}(c-z) \leq \text{Arg}\left(c - \sqrt{\zeta}\right) < \text{Arg}\left(c - \sqrt{\zeta - d}\right),$$

or

$$\pi < \text{Arg}(c-z) < \text{Arg}\left(c - \sqrt{\zeta - d}\right) \leq \text{Arg}\left(c - \sqrt{\zeta}\right).$$

We illustrate the inequalities as follows.



where $s = -z$, $t = -\sqrt{\zeta - d}$, and $u = -\sqrt{\zeta}$. In any case, c is so large that

$$\operatorname{Arg}(c - z) \leq \operatorname{Arg}\left(c - \sqrt{\zeta - d}\right). \tag{3.20}$$

On a side note, we need not worry about the case when equality holds in (3.20), since even if $\operatorname{Arg}(c - z) = \operatorname{Arg}\left(c - \sqrt{\zeta - d}\right)$,

$$\operatorname{Arg}\left(c - \sqrt{\zeta}\right) < \operatorname{Arg}\left(c - \sqrt{\zeta - d}\right),$$

which coincides with the case (3.17) already considered. Subtracting $\operatorname{Arg}\left(z + \frac{\beta-\alpha}{2}\right)$ from each term in (3.20) yields

$$\operatorname{Arg}\left(\frac{c - z}{z + \frac{\beta-\alpha}{2}}\right) \leq \operatorname{Arg}\left(\frac{c - \sqrt{\zeta - d}}{z + \frac{\beta-\alpha}{2}}\right).$$

We need only to verify that

$$\pi < \operatorname{Arg}\left(\frac{c - z}{z + \frac{\beta-\alpha}{2}}\right). \tag{3.21}$$

Inequality (3.21) holds, since

$$-\frac{z - c}{z + \frac{\beta-\alpha}{2}} = -\frac{z - \frac{\alpha+\beta+2}{2}}{z + \frac{\beta-\alpha}{2}} = -\frac{z + \frac{\beta-\alpha}{2} - \frac{\beta-\alpha}{2} - \frac{\alpha+\beta+2}{2}}{z + \frac{\beta-\alpha}{2}} = -1 + \frac{\beta + 1}{z + \frac{\beta-\alpha}{2}}. \tag{3.22}$$

<u>Case 3:</u> If $\frac{\pi}{2} < \operatorname{Arg}(z) < \pi$ where z is defined in (3.10), then

$$\frac{\pi}{2} < \operatorname{Arg}(z) < \operatorname{Arg}(\sqrt{\zeta - d}) < \operatorname{Arg}(\sqrt{\zeta}) < \pi, \tag{3.23}$$

and similarly,

$$\frac{3\pi}{2} < \mathrm{Arg}(-z) < \mathrm{Arg}(-\sqrt{\zeta - d}) < \mathrm{Arg}(-\sqrt{\zeta}), \tag{3.24}$$

where Arg is again defined on $[0, 2\pi)$.

First consider (3.8),

$$\xi_1 = \frac{(\frac{\alpha+\beta+2}{2}) + \sqrt{\zeta - d}}{z + \frac{\beta-\alpha}{2}}.$$

Similar to the previous case, the inequalities in (3.23) will change by adding various sizes of $c := \frac{\alpha+\beta+2}{2}$ to each term inside the argument. If c is sufficiently small, the inequalities do not change, i.e.,

$$0 < \mathrm{Arg}(c + z) < \mathrm{Arg}\left(c + \sqrt{\zeta - d}\right) < \mathrm{Arg}\left(c + \sqrt{\zeta}\right) < \pi. \tag{3.25}$$

Adding $\mathrm{Arg}\left(1/(z + \frac{\beta-\alpha}{2})\right)$ to each term in (3.25) may change change some inequalities if any of the arguments exceed 2π. This does not happen because identity (3.19)

$$\frac{(\frac{\alpha+\beta+2}{2}) + \sqrt{\zeta}}{z + \frac{\beta-\alpha}{2}} = \frac{-z + \frac{\beta-\alpha}{2}}{(\frac{\alpha+\beta+2}{2}) - \sqrt{\zeta}},$$

and the use of Lemma 121 on expression (3.24) implies

$$\pi < \mathrm{Arg}\left(-z + \frac{\beta - \alpha}{2}\right) < \mathrm{Arg}\left((\tfrac{\alpha+\beta+2}{2}) - \sqrt{\zeta}\right),$$

so that

$$\pi < \mathrm{Arg}\left(\frac{(\frac{\alpha+\beta+2}{2}) + \sqrt{\zeta}}{z + \frac{\beta-\alpha}{2}}\right).$$

(Namely, let $\pi + a = \mathrm{Arg}\left(-z + \frac{\beta-\alpha}{2}\right)$, $\pi + b = \mathrm{Arg}\left((\frac{\alpha+\beta+2}{2}) - \sqrt{\zeta}\right)$, where $0 < a < b < \pi$. Then $\mathrm{Arg}\left(\frac{-z+\frac{\beta-\alpha}{2}}{(\frac{\alpha+\beta+2}{2})-\sqrt{\zeta}}\right) = a - b$, and $-\pi < -b < (a - b) < 0$.) The inequality

$$\pi < \mathrm{Arg}\left(\frac{z + c}{z + \frac{\beta-\alpha}{2}}\right)$$

holds, since

$$\frac{z + c}{z + \frac{\beta-\alpha}{2}} = \frac{z + \frac{\alpha+\beta+2}{2}}{z + \frac{\beta-\alpha}{2}} = \frac{z + \frac{\beta-\alpha}{2} - \frac{\beta-\alpha}{2} + \frac{\alpha+\beta+2}{2}}{z + \frac{\beta-\alpha}{2}} = 1 + \frac{\alpha + 1}{z + \frac{\beta-\alpha}{2}}. \tag{3.26}$$

Thus the inequalities

$$\operatorname{Arg}\left(\frac{z+c}{z+\frac{\beta-\alpha}{2}}\right) < \operatorname{Arg}\left(\frac{\sqrt{\zeta-d}+c}{z+\frac{\beta-\alpha}{2}}\right) < \operatorname{Arg}\left(\frac{\sqrt{\zeta}+c}{z+\frac{\beta-\alpha}{2}}\right)$$

are preserved, so $\operatorname{Im}(\xi_1) < 0$ for sufficiently small c. For larger values of c, Lemma 122 implies one of the three following cases:

$$0 < \operatorname{Arg}\left(c+\sqrt{\zeta-d}\right) \leq \operatorname{Arg}(c+z) < \operatorname{Arg}\left(c+\sqrt{\zeta}\right) < \pi,$$

$$0 < \operatorname{Arg}\left(c+\sqrt{\zeta-d}\right) < \operatorname{Arg}\left(c+\sqrt{\zeta}\right) \leq \operatorname{Arg}(c+z) < \pi,$$

or

$$0 < \operatorname{Arg}\left(c+\sqrt{\zeta}\right) \leq \operatorname{Arg}\left(c+\sqrt{\zeta-d}\right) < \operatorname{Arg}(c+z) < \pi.$$

Adding $\operatorname{Arg}\left(1/(z+\frac{\beta-\alpha}{2})\right)$ to each term in any of the above cases implies that π is a lower bound. Since

$$\pi < \operatorname{Arg}\left(\frac{z+c}{z+\frac{\beta-\alpha}{2}}\right),$$

by equation (3.26), it follows that $\operatorname{Im}(\xi_1) < 0$.

Finally, consider (3.9),

$$\xi_2 = \frac{(\frac{\alpha+\beta+2}{2}) - \sqrt{\zeta-d}}{z+\frac{\beta-\alpha}{2}}.$$

Lemmas 121, 122, and expression (3.24) imply that

$$\frac{3\pi}{2} < \operatorname{Arg}(c-z) < \operatorname{Arg}(c-\sqrt{\zeta-d}) < \operatorname{Arg}(c-\sqrt{\zeta}) \tag{3.27}$$

where $c := \frac{\alpha+\beta+2}{2}$ as before. Subtracting $\operatorname{Arg}(z+\frac{\beta-\alpha}{2})$ from each term in (3.27) will only decrease the argument. Equation (3.22) yields the same necessary lower bound

$$\pi < \operatorname{Arg}\left(\frac{c-z}{z+\frac{\beta-\alpha}{2}}\right),$$

and thus $\operatorname{Im}(\xi_2) < 0$, which concludes the proof. □

3.3.2 Quadratic differential operators

The main results in this subsection are based on the joint work with R. Bates [4].

Definition 124. A second order differential linear operator of the form

$$T = Q_2(x)D^2 + Q_1(x)D + Q_0(x),$$

where $D = \frac{d}{dx}$, and the polynomials $Q_i(x)$ are of degree i, will be called a *quadratic differential* operator.

For the case of first order differential operators (i.e. differential operators of the form $T = Q_1(x)D + Q_0(x)$), proper positioning of the coefficient polynomials in Lemma 105 is necessary and sufficient for T to preserve hyperbolicity. For more general differential operators, proper positioning of the coefficient polynomials are not sufficient to preserve hyperbolicity. For example, the linear operator $T = (x^2 - 1)D^2 + 2xD + 2$ does not preserve hyperbolicity, since

$$T[(x-10)^3] = 2(x-10)(7x^2 - 50x + 97),$$

and $7x^2 - 50x + 97$ has non-real zeros. Our investigation of the quadratic differential operators will require two preliminary lemmas.

Lemma 125 ([4]). *Let $A,\ B \in \mathbb{C} \setminus \mathbb{R}$ be two non-real complex numbers such that*

(i) $0 < \mathrm{Arg}(B) < \mathrm{Arg}(A) < 2\pi$*,*

(ii) $\mathrm{Arg}(A) - \mathrm{Arg}(B) < \pi$*, and*

(iii) $\mathrm{Im}(A) < \mathrm{Im}(B)$.

Then for any $r_1,\ r_2 \in \mathbb{R}$, $r_1 < r_2$, there are $x, w \in H^+$ such that $(x + r_1)w = A$ and $(x + r_2)w = B$.

Proof. Consider the following cases.

Case 1: $B \in H^+$. The point B may be located in either quadrant I, on the imaginary axis, or in quadrant II, as described in Figure 3.1. The hypotheses (i), (ii), and (iii) implies that point A is located somewhere in the displayed shaded region (see Figure 3.1). Define the function $f : [0, \mathrm{Arg}(B)] \to \mathbb{R}$ by

$$f(\theta) := \mathrm{Im}(e^{-i\theta}A) - \mathrm{Im}(e^{-i\theta}B). \tag{3.28}$$

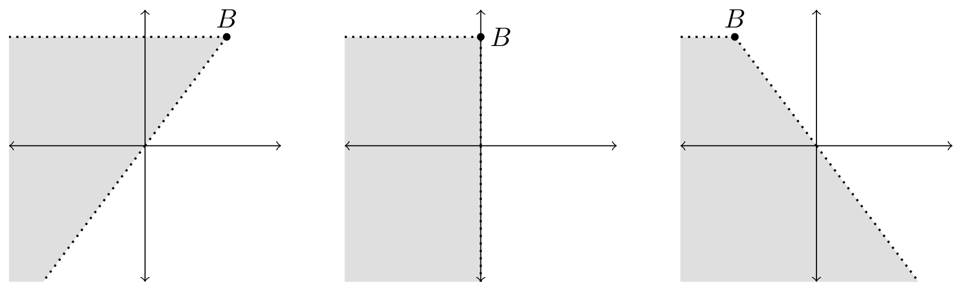


Figure 3.1:

Then $f(0) < 0$ by (iii), and $f(\text{Arg}(B)) > 0$ by (ii). Thus by continuity, there exist $\theta_0 \in (0, \text{Arg}(B))$ such that $f(\theta_0) = 0$, which implies that $(e^{-i\theta_0}B - e^{-i\theta_0}A) > 0$ by (i). Define the function $g : [0, \infty) \to \mathbb{R}$ by

$$g(k) := k(e^{-i\theta_0}B - e^{-i\theta_0}A). \tag{3.29}$$

Notice $g \geq 0$, $g(0) = 0$, and $\lim_{k \to +\infty} g(k) = +\infty$. Thus, there exist $k_0 > 0$ such that $g(k_0) = r_2 - r_1$. Let

$$x = \frac{1}{2}(k_0 e^{-i\theta_0}B + k_0 e^{-i\theta_0}A - r_1 - r_2), \text{ and } w = \frac{1}{k_0}e^{i\theta_0}. \tag{3.30}$$

It follows that $x, w \in H^+$, $(x + r_1)w = A$, and $(x + r_2)w = B$.

Case 2: $B \in H^-$. Similar to Case 1, the point B may be located in either quadrant III, on the imaginary axis, or in quadrant IV, as described in Figure 3.2. Point A is located somewhere in the shaded region of the corresponding point B by hypotheses (i), (ii), and (iii).

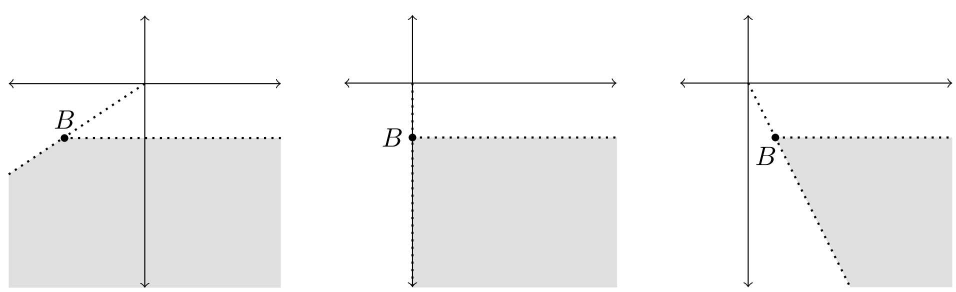


Figure 3.2:

Define the function $f : [0, 2\pi - \text{Arg}(B)] \to \mathbb{R}$ by

$$f(\theta) := \text{Im}(e^{i\theta}A) - \text{Im}(e^{i\theta}B). \tag{3.31}$$

Then $f(0) < 0$ by (iii), and $f(2\pi - \text{Arg}(B)) > 0$ by (ii). Thus by continuity, there exist $\theta_0 \in (0, 2\pi - \text{Arg}(B))$ such that $f(\theta_0) = 0$, which implies that $(e^{i\theta_0}B - e^{i\theta_0}A) < 0$ by (i). Define the function $g : (-\infty, 0] \to \mathbb{R}$ by

$$g(k) := k(e^{i\theta_0}B - e^{i\theta_0}A). \tag{3.32}$$

Then $g \geq 0$, $g(0) = 0$, and $\lim_{k\to-\infty} g(k) = +\infty$. Thus, there exist $k_0 < 0$ such that $g(k_0) = r_2 - r_1$. Let

$$x = \frac{1}{2}(k_0 e^{i\theta_0}B + k_0 e^{i\theta_0}A - r_1 - r_2), \text{ and } w = \frac{1}{k_0}e^{-i\theta_0}. \tag{3.33}$$

It follows that $x, w \in H^+$, $(x + r_1)w = A$, and $(x + r_2)w = B$. □

Lemma 126 ([4]). *Let $a, b, r_1, r_2, r \in \mathbb{R}$, $a, b \geq 0$, and $r_1 \neq r_2$. Set*

$$f(x, w) = ((x + r_1)w - a)((x + r_2)w - b), \quad x, w \in \mathbb{C}. \tag{3.34}$$

Then

$$f(x, w) \neq r \;\; \forall\, x, w \in H^+ \quad \textit{if and only if} \quad r \in [0, ab].$$

Proof. Since the factors of $f(x, w)$ in (3.34) are symmetric, we let $r_1 < r_2$. There are three cases to prove necessity:

Case 1. $r \in (-\infty, 0)$, and $a < b + 2\sqrt{|r|}$;

Case 2. $r \in (-\infty, 0)$, and $a \geq b + 2\sqrt{|r|}$;

Case 3. $r \in (ab, \infty)$.

We show in each case that there exist $x, w \in H^+$ such that $f(x, w) = r$.

Case 1. Consider $r \in (-\infty, 0)$, and $a < b + 2\sqrt{|r|}$. Define $g : [0, \pi/2] \to \mathbb{R}$ by

$$\begin{aligned} g(\theta) &:= \left(\sqrt{|r|}e^{i\theta} + b\right) - \left(\sqrt{|r|}e^{i(\pi-\theta)} + a\right) \\ &= \sqrt{|r|}(2\cos(\theta)) - a + b. \end{aligned} \tag{3.35}$$

The function g is real valued and $g(0) = b + 2\sqrt{|r|} - a > 0$ by assumption. Thus by continuity, there exists $\theta_0 \in (0, \pi/2)$ such that $g(\theta_0) > 0$, which implies the following.

(a) $\text{Im}\left(\sqrt{|r|}e^{i\theta_0} + b\right) - \text{Im}\left(\sqrt{|r|}e^{i(\pi-\theta_0)} + a\right) = 0$,

(b) $\text{Re}\left(\sqrt{|r|}e^{i\theta_0} + b\right) - \text{Re}\left(\sqrt{|r|}e^{i(\pi-\theta_0)} + a\right) > 0$, and

(c) $\left(\sqrt{|r|}e^{i\theta_0}+b\right)$, $\left(\sqrt{|r|}e^{i(\pi-\theta_0)}+a\right) \in H^+$.

By (a), (b), and (c),

$$\operatorname{Arg}\left(\sqrt{|r|}e^{i(\pi-\theta_0)}+a\right)-\operatorname{Arg}\left(\sqrt{|r|}e^{i\theta_0}+b\right)>0. \tag{3.36}$$

Define the function $h:(0,1]\to\mathbb{R}$ by

$$h(k):=\operatorname{Arg}\left(k\sqrt{|r|}e^{i(\pi-\theta_0)}+a\right)-\operatorname{Arg}\left(\frac{\sqrt{|r|}}{k}e^{i\theta_0}+b\right). \tag{3.37}$$

The function h is real valued, and $h(1)>0$. Thus by continuity, there exists $k_0<1$ such that

$$\operatorname{Arg}\left(k_0\sqrt{|r|}e^{i(\pi-\theta_0)}+a\right)-\operatorname{Arg}\left(\frac{\sqrt{|r|}}{k_0}e^{i\theta_0}+b\right)>0, \tag{3.38}$$

such that

$$\operatorname{Im}\left(k_0\sqrt{|r|}e^{i(\pi-\theta_0)}+a\right)<\operatorname{Im}\left(\frac{\sqrt{|r|}}{k_0}e^{i\theta_0}+b\right). \tag{3.39}$$

Let

$$A=k_0\sqrt{|r|}e^{i(\pi-\theta_0)}+a, \quad \text{and} \quad B=\frac{\sqrt{|r|}}{k_0}e^{i\theta_0}+b. \tag{3.40}$$

Then (3.38) and (3.39) satisfy items (i), (ii), and (iii) of Lemma 125, hence there exist $x,w\in H^+$ such that $(x+r_1)w=A$ and $(x+r_2)w=B$. Thus,

$$\begin{aligned} f(x,w) &= ((x+r_1)w-a)((x+r_2)w-b) \\ &= \left(k_0\sqrt{|r|}e^{i(\pi-\theta_0)}\right)\left(\frac{\sqrt{|r|}}{k_0}e^{i\theta_0}\right)=-|r|=r. \end{aligned} \tag{3.41}$$

Case 2: We consider $r\in(-\infty,0)$, and $b+2\sqrt{|r|}\le a$. We will only need $b<a+2\sqrt{|r|}$. This is easily seen to be true by adding $2\sqrt{|r|}$ to both sides of $b+2\sqrt{|r|}\le a$, and observing $b<b+4\sqrt{|r|}$. Define the function $g:[0,\pi/2]\to\mathbb{R}$ by

$$\begin{aligned} g(\theta) &:= \left(\sqrt{|r|}e^{i(2\pi-\theta)}+a\right)-\left(\sqrt{|r|}e^{i(\pi+\theta)}+b\right) \\ &= \sqrt{|r|}(2\cos(\theta))+a-b. \end{aligned} \tag{3.42}$$

Again, g is real valued, and $g(0)=a+2\sqrt{|r|}-b>0$. Thus by continuity, there exists $\theta_0\in(0,\pi/2)$

such that $g(\theta_0) > 0$, which implies the following:

(a) $\operatorname{Im}\left(\sqrt{|r|}e^{i(2\pi-\theta_0)} + a\right) - \operatorname{Im}\left(\sqrt{|r|}e^{i(\pi+\theta_0)} + b\right) = 0$,

(b) $\operatorname{Re}\left(\sqrt{|r|}e^{i(2\pi-\theta_0)} + a\right) - \operatorname{Re}\left(\sqrt{|r|}e^{i(\pi+\theta_0)} + b\right) > 0$,

(c) $\left(\sqrt{|r|}e^{i(2\pi-\theta_0)} + a\right),\ \left(\sqrt{|r|}e^{i(\pi+\theta_0)} + b\right) \in H^-$.

By (a), (b), and (c),

$$\operatorname{Arg}\left(\sqrt{|r|}e^{i(2\pi-\theta_0)} + a\right) - \operatorname{Arg}\left(\sqrt{|r|}e^{i(\pi+\theta_0)} + b\right) > 0. \tag{3.43}$$

Define the function $h : [1, \infty) \to \mathbb{R}$ by

$$h(k) := \operatorname{Arg}\left(k\sqrt{|r|}e^{i(2\pi-\theta_0)} + a\right) - \operatorname{Arg}\left(\frac{\sqrt{|r|}}{k}e^{i(\pi+\theta_0)} + b\right). \tag{3.44}$$

The function h is real valued, and $h(1) > 0$. Thus by continuity, there exists $k_0 > 1$ such that

$$\operatorname{Arg}\left(k_0\sqrt{|r|}e^{i(2\pi-\theta_0)} + a\right) - \operatorname{Arg}\left(\frac{\sqrt{|r|}}{k_0}e^{i(\pi+\theta_0)} + b\right) > 0, \tag{3.45}$$

so that

$$\operatorname{Im}\left(k_0\sqrt{|r|}e^{i(2\pi-\theta_0)} + a\right) < \operatorname{Im}\left(\frac{\sqrt{|r|}}{k_0}e^{i(\pi+\theta_0)} + b\right). \tag{3.46}$$

Let

$$A = k_0\sqrt{|r|}e^{i(2\pi-\theta_0)} + a, \quad \text{and} \quad B = \frac{\sqrt{|r|}}{k_0}e^{i(\pi+\theta_0)} + b. \tag{3.47}$$

Then (3.45) and (3.46) satisfies items (i), (ii), and (iii) of Lemma 125, hence there exist $x, w \in H^+$ such that $(x + r_1)w = A$ and $(x + r_2)w = B$. Thus,

$$\begin{aligned} f(x, w) &= ((x + r_1)w - a)((x + r_2)w - b) \\ &= \left(k_0\sqrt{|r|}e^{i(2\pi-\theta_0)}\right)\left(\frac{\sqrt{|r|}}{k_0}e^{i(\pi+\theta_0)}\right) = -|r| = r. \end{aligned} \tag{3.48}$$

Case 3: We consider $r \in (ab, \infty)$. Since $r > ab$, $r = a'b'$, for some $a' > a$, and $b' > b$. Define the function $g : [\pi/2, \pi] \to [a - a', a] \times [b - b', b]$ by

$$g(\theta) := \left(\operatorname{Re}(a'e^{-i\theta}) + a,\ \operatorname{Re}(b'e^{i\theta}) + b\right). \tag{3.49}$$

Since $a - a', b - b' < 0$, $g(\pi) = (a - a', b - b')$ has negative coordinates. By continuity, there exists $\theta_0 \in (\pi/2, \pi)$ such that $g(\theta_0)$ has negative coordinates, which implies that $a'e^{-i\theta_0} + a$ is in quadrant three, and $b'e^{i\theta_0} + b$ is in quadrant two. Let

$$A = a'e^{-i\theta_0} + a, \quad \text{and} \quad B = b'e^{i\theta_0} + b. \tag{3.50}$$

By Lemma 125, there exist $x, w \in H^+$ such that $(x + r_1)w = A$, and $(x + r_2)w = B$. Thus,

$$f(x, w) = ((x + r_1)w - a)((x + r_2)w - b) = \left(a'e^{-\theta_0 i}\right)\left(b'e^{\theta_0 i}\right) = a'b' = r. \tag{3.51}$$

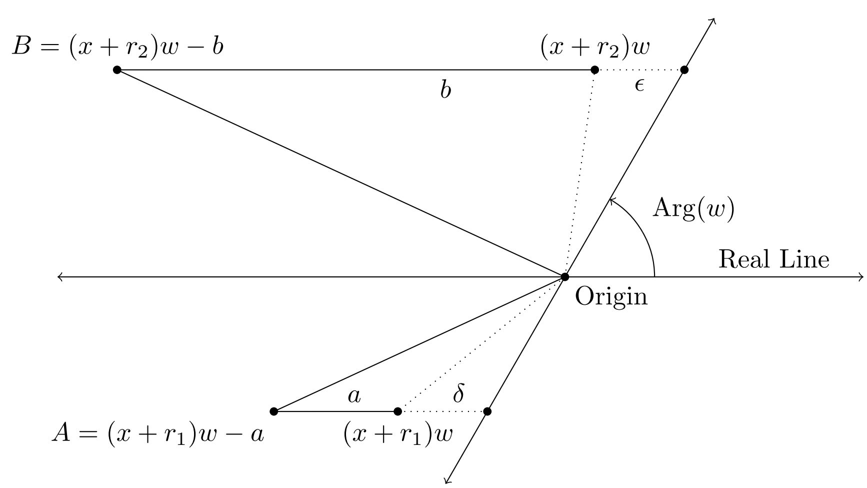
To prove sufficiency, first consider $r \in (0, ab]$. By way of contradiction, assume there exist $x, w \in H^+$ such that $((x+r_1)w-a)((x+r_2)w-b) = r$. Let $A = ((x+r_1)w-a)$, $B = ((x+r_2)w-b)$. The points $x + r_1, x + r_2 \in H^+$ are rotated by $\text{Arg}(w) \in (0, \pi)$, and shifted to the left $a, b > 0$ to attain the location of A and B. Since AB is a positive real number, $\text{Arg}(A) + \text{Arg}(B) = 0$ (mod 2π). In particular, as $r_1 < r_2$, B must be in H^+, which implies

$$0 < \text{Arg}(w) < \text{Arg}((x + r_2)w) < \text{Arg}((x + r_2)w - b) < \pi, \tag{3.52}$$

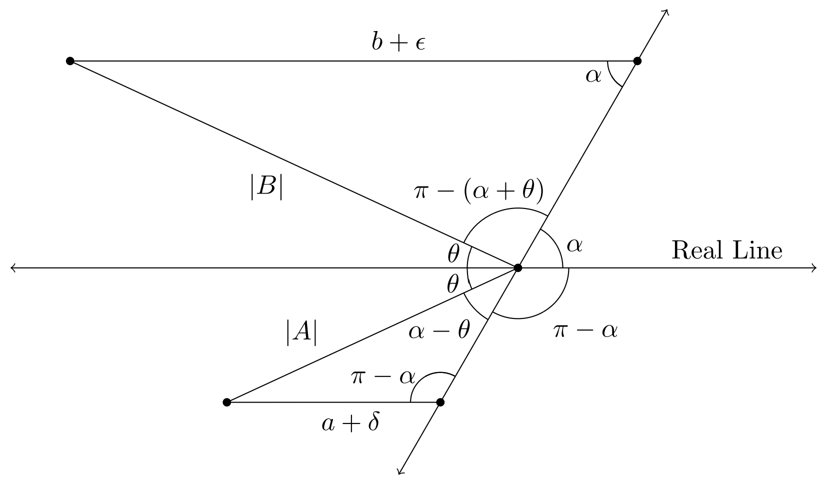
and A must be in H^-, which implies

$$\pi < \text{Arg}((x + r_1)w - a) < \text{Arg}((x + r_1)w) < \pi - \text{Arg}(w) < 2\pi. \tag{3.53}$$

The figure illustrate the inequalities as follows.



We let ϵ and δ be the horizontal distance from $(x+r_1)w$ and $(x+r_2)w$ to the line formed by $\text{Arg}(w)$. In fact, $\delta = \frac{\text{Im}(x+r_1)}{\sin(\text{Arg}(w))}$, and $\epsilon = \frac{\text{Im}(x+r_2)}{\sin(\text{Arg}(w))}$, so that $\delta = \epsilon > 0$ (this can be seen by projecting a perpendicular line from $(x+r_i)w$ to the line created by $\text{Arg}(w)$, with δ and ϵ being the hypotenuse, and the angle opposite of the leg with length $\text{Im}(x+r_i)w$ is $\text{Arg}(w)$). We redraw the picture with different labels and examine the points geometrically.



The inequalities $\alpha-\theta>0$ and $\pi-(\alpha+\theta)>0$ imply $0<\theta<\alpha<\pi-\theta<\pi$, so that

$$\sin(\theta) < \sin(\alpha),$$

since $\sin(\theta)=\sin(\pi-\theta)$. Thus,

$$0 < \left(\frac{\sin(\theta)}{\sin(\alpha)}\right)^2 < 1, \tag{3.54}$$

and the law of sines yield that

$$\begin{aligned} (a+\delta)(b+\epsilon) &= \frac{|A|\sin(\alpha-\theta)}{\sin(\pi-\alpha)} \cdot \frac{|B|\sin(\pi-(\alpha+\theta))}{\sin(\alpha)} \\ &= \left(1-\left(\frac{\sin(\theta)}{\sin(\alpha)}\right)^2\right)|AB| < |AB|. \end{aligned} \tag{3.55}$$

Hence we have the contradiction that

$$ab < (a+\delta)(b+\epsilon) < |AB| = r. \tag{3.56}$$

To finish the proof, consider $r = 0$. By way of contradiction, suppose there are $x, w \in H^+$ such that

$$((x + r_1)w - a)((x + r_2)w - b) = 0.$$

Thus, $(x + r_1)w = a$, or $(x + r_2)w = b$. However, neither of these can hold, since the product of any two complex numbers in H^+ cannot be a non-negative real number. ☐

We will now begin to prove the main results of this chapter.

Theorem 127 ([4])**.** *Let $a, b \geq 0$, $r_1, r_2, R \in \mathbb{R}$, and $r_1 \neq r_2$. Then,*

$$T := (x + r_1)(x + r_2)D^2 + (b(x + r_1) + a(x + r_2))\, D + R,$$

preserves hyperbolicity if and only if $R \in [0, ab]$.

Proof. By Theorem 91, it suffices to show for every $x, w \in H^+$,

$$(x + r_1)(x + r_2)w^2 - (a(x + r_1) + b(x + r_2))\, w + R \neq 0,$$

which is the same as

$$((x + r_1)w - b)((x + r_2)w - a) \neq ab - R.$$

By Lemma 126, this is true if and only if $R \in [0, ab]$. ☐

Theorem 128 ([4])**.** *For $c_i, r_j \in \mathbb{R}$, $i = 0, 1, 2$, $j = 1, 2, 3$, $c_2 \neq 0$, $r_1 \neq r_2$, let $Q_0(x) = c_0$, $Q_1(x) = c_1(x - r_3)$, $Q_2(x) = c_2(x - r_1)(x - r_2)$. Then*

$$0 \leq c_1^2 \left(\frac{(r_1 - r_3)(r_3 - r_2)}{(r_2 - r_1)^2} \right) - c_0 c_2,$$

and c_0, c_1, c_2 are of the same sign if and only if

$$T := Q_2(x)D^2 + Q_1(x)D + Q_0(x),$$

preserves hyperbolicity.

Proof. To prove sufficiency, if T preserves hyperbolicity, then by Lemma 105, c_i, $i = 0, 1, 2$ are of the same sign. Since

$$T = c_2\left((x-r_1)(x-r_2)D^2 + \frac{c_1}{c_2}(x-r_3)D + \frac{c_0}{c_2}\right) \tag{3.57}$$

$$= c_2\Bigg((x-r_1)(x-r_2)D^2$$

$$+ \frac{c_1}{c_2}\left[\frac{(r_1-r_3)}{(r_1-r_2)}(x-r_2) + \frac{(r_3-r_2)}{(r_1-r_2)}(x-r_1)\right]D + \frac{c_0}{c_2}\Bigg), \tag{3.58}$$

then by Theorem 127,

$$\frac{c_0}{c_2} \in \left[0,\ \left(\frac{c_1}{c_2}\right)^2 \frac{(r_1-r_3)(r_3-r_2)}{(r_1-r_2)^2}\right],$$

and

$$0 \le c_1^2\left(\frac{(r_1-r_3)(r_3-r_2)}{(r_2-r_1)^2}\right) - c_0c_2.$$

To prove necessity, suppose c_i, $i = 0, 1, 2$ are of the same sign, and

$$0 \le c_1^2\left(\frac{(r_1-r_3)(r_3-r_2)}{(r_2-r_1)^2}\right) - c_0c_2. \tag{3.59}$$

We want to conclude that

$$\frac{c_1}{c_2}\frac{(r_1-r_3)}{(r_1-r_2)}, \frac{c_1}{c_2}\frac{(r_3-r_2)}{(r_1-r_2)} \ge 0. \tag{3.60}$$

To this end, if $c_1 = 0$, then (3.60) holds immediately. Suppose $c_1 \neq 0$, and that $r_1 < r_2$. Then (3.59) implies $0 \le (r_1 - r_3)(r_3 - r_2)$, and we conclude that $r_1 \le r_3 \le r_2$ (i.e., $r_3 < r_1 < r_2$ cannot hold, since it implies $(r_1 - r_3)(r_3 - r_2) < 0$, and also if $r_1 < r_2 < r_3$, then $(r_1 - r_3)(r_3 - r_2) < 0$), and hence, (3.60) holds. By symmetry, the same conclusion is true if $r_2 < r_1$. Thus by Theorem 127, T preserves hyperbolicity. □

The equalities of (3.57) and (3.58) use a fact established in [35, p. 13, Lemma 1.20]. For the sake of completeness, we state the result.

Lemma 129 (Fisk [35, p. 13, Lemma 1.20])**.** *Assume that f is a polynomial of degree n, with positive leading coefficient, and with zeros $\{a_1, \ldots, a_n\}$. Suppose that g is a polynomial with positive leading coefficient. If g has degree $n-1$, and we write*

$$g(x) = c_1\frac{f(x)}{x-a_1} + \ldots + c_n\frac{f(x)}{x-a_n},$$

then f and g have interlacing zeros if and only if all $c_i \geq 0$ for $i = 1, 2, \ldots, n$. □

We now remove the condition of Q_2 having distinct zeros. We begin with a lemma that is analogous to Lemma 126.

Lemma 130. *Let $a, r \in \mathbb{R}$, $a \geq 0$. Set*

$$f(z) := z^2 - az + r, \quad z \in \mathbb{C}.$$

Then

$$f(z) \neq 0 \quad \forall\, z \in \mathbb{C} - [0, \infty) \quad \textit{if and only if} \quad r \in \left[0, \frac{a^2}{4}\right].$$

Proof. The zeros of f are $\frac{1}{2}\left(a \pm \sqrt{a^2 - 4r}\right)$. To prove necessity, we consider two cases.

Case 1. If $r < 0$, then one of the zeros of f is a negative real number, thus there exist $z_0 \in \mathbb{C} - [0, \infty)$ such that $f(z_0) = 0$.

Case 2. If $r > a^2/4$, then f has two imaginary zeros, thus the zeros of f are in $\mathbb{C} - [0, \infty)$.

To prove sufficiency, suppose $0 \leq r \leq a^2/4$. Then f has two non-negative zeros, so that f never vanishes in $\mathbb{C} - [0, \infty)$. □

Theorem 131 ([4])**.** *Let $a \geq 0$, $r, R \in \mathbb{R}$. Then,*

$$R \in \left[0, \frac{a^2}{4}\right]$$

if and only if

$$T := (x + r)^2 D^2 + a(x + r)D + R$$

is hyperbolicity preserving.

Proof. (⇒) Assume $R \in [0, a^2/4]$. By Theorem 91, it suffices to show for every $x, w \in H^+$,

$$(x + r)^2 w^2 - a(x + r)w + R \neq 0. \tag{3.61}$$

We assume on the contrary that (3.61) is false for some $x, w \in H^+$. Let $z = (x + r)w$ in (3.61), so that $z \in \mathbb{C} - [0, \infty)$, and

$$z^2 - az + R = 0. \tag{3.62}$$

This is impossible by Lemma 130, a contradiction.

($\Leftarrow$) Suppose T is hyperbolicity preserving. By Theorem 91, for every $x, w \in H^+$,

$$(x+r)^2 w^2 - a(x+r)w + R \neq 0. \tag{3.63}$$

Let $z = (x+r)w$ in (3.63), so that $z \in \mathbb{C} - [0, \infty)$, and

$$z^2 - az + R \neq 0, \quad \forall\, z \in \mathbb{C} - [0, \infty), \tag{3.64}$$

which implies that $R \in [0, a^2/4]$ by Lemma 130. □

The analogous statement of Theorem 128 is the following, and its proof follows, *mutatis mutandis*, from the proof of Theorem 128.

Theorem 132. *For $r, c_i \in \mathbb{R}$, $i = 0, 1, 2$, $c_2 \neq 0$, let $Q_0(x) = c_0$, $Q_1(x) = c_1(x - r)$, $Q_2(x) = c_2(x - r)^2$. Then*

$$0 \leq c_1^2 \left(\frac{1}{4}\right) - c_0 c_2$$

and c_0, c_1, c_2 are of the same sign, if and only if

$$T = Q_2(x)D^2 + Q_1(x)D + Q_0(x)$$

preserves hyperbolicity. □

We now wish to find a condition that combines the statements of Theorem 132 and Theorem 128. To this end, we first prove the following lemma.

Lemma 133 ([4])**.** *For $c_i, r_j \in \mathbb{R}$, $i = 0, 1, 2$, $j = 1, 2, 3$, $c_2 \neq 0$, let $Q_0(x) = c_0$, $Q_1(x) = c_1(x - r_3)$, $Q_2(x) = c_2(x - r_1)(x - r_2)$. If $T = Q_2(x)D^2 + Q_1(x)D + Q_0(x)$ is hyperbolicity preserving, then*

$$0 \leq c_1^2 - 4c_0c_2.$$

Furthermore, if $r_1 \neq r_2$ then

$$0 \leq c_1^2 \frac{(r_1 - r_3)(r_3 - r_2)}{(r_2 - r_1)^2} - c_0c_2 \leq c_1^2 \frac{1}{4} - c_0c_2.$$

Thus, if $0 = c_1^2 - 4c_0c_2$, then $2r_3 = r_1 + r_2$.

Proof. If $r_1 = r_2$, the result follows by Theorem 132. Thus, we suppose $r_1 \neq r_2$, and it suffices to show

$$0 \leq \frac{(r_1 - r_3)(r_3 - r_2)}{(r_2 - r_1)^2} \leq \frac{1}{4}. \tag{3.65}$$

The left inequality holds by Lemma 105, since Q_2 and Q_1 have interlacing zeros. To show that the right inequality holds, we proceed as follows,

$$0 \leq (2r_3 - (r_1 + r_2))^2,$$

$$4(r_1 r_3 + r_2 r_3) \leq (r_2 + r_1)^2 + 4r_3^2,$$

$$4(r_1 r_3 - r_1 r_2 - r_3^2 + r_2 r_3) \leq r_2^2 - 2r_1 r_2 + r_1^2,$$

$$4(r_1 - r_3)(r_3 - r_2) \leq (r_2 - r_1)^2.$$

□

It is interesting that a Turán-type inequality which involve Wronskians appears in the characterization of quadratic differential operators which preserve hyperbolicity (Theorems 134 and 135).

Theorem 134 ([4]). *For $c_i, r_j \in \mathbb{R}$, $i = 0, 1, 2$, $j = 1, 2, 3$, $c_2 \neq 0$, let $Q_0(x) = c_0$, $Q_1(x) = c_1(x - r_3)$, $Q_2(x) = c_2(x - r_1)(x - r_2)$ with $Q_0(x) \ll Q_1(x)$ and $Q_1(x) \ll Q_2(x)$. Then*

$$T = Q_2(x)D^2 + Q_1(x)D + Q_0(x)$$

preserves hyperbolicty if and only if

$$W[Q_0, Q_2]^2 - W[Q_0, Q_1]W[Q_1, Q_2] \leq 0.$$

Proof. Since $Q_0 \ll Q_1$ and $Q_1 \ll Q_2$, the signs of c_0, c_1, c_2 are same, and $r_1 \leq r_3 \leq r_2$. Define

$$\begin{aligned} w(x) &:= W[Q_0, Q_2]^2 - W[Q_0, Q_1]W[Q_1, Q_2] \\ &= c_0 c_2 (4c_0 c_2 - c_1^2)x^2 + 2c_0 c_2(-2c_0 c_2(r_1 + r_2) + c_1^2 r_3)x \\ &\quad + c_0 c_2 (c_0 c_2 (r_1 + r_2)^2 + c_1^2 (r_1 r_2 - r_1 r_3 - r_2 r_3)). \end{aligned}$$

Suppose $r_1 = r_2$, then $w(x) = -c_0 c_2 (c_1^2 - 4c_0 c_2)(x - r_1)^2$. It is clear that $w(x) \leq 0$ if and only if $0 \leq c_1^2 - 4c_0 c_2$, thus we apply Theorem 132.

Suppose $0 = c_1^2 - 4c_0c_2$ and $r_1 \neq r_2$. By Lemma 133, Theorem 128 can restated as, "T is hyperbolicity preserving if and only if $2r_3 = r_1 + r_2$". We recalculate w, under the assumption that $0 = c_1^2 - 4c_0c_2$,

$$w(x) = 4c_0^2c_2^2(2r_3 - r_1 - r_2)x + c_0^2c_2^2(2(r_1 + r_2)(r_1 + r_2 - 2r_3) - (r_1 - r_2)^2).$$

We now see that, $w(x) \leq 0$, if and only if, $2r_3 = r_1 + r_2$.

Thus we may assume $0 \neq c_1^2 - 4c_0c_2$ and $r_1 \neq r_2$, in which case the graph of $w(x)$ is a parabola with vertex

$$\left(r_3, \frac{c_0c_1^2c_2}{c_1^2 - 4c_0c_2}\left(c_0c_2(r_1 - r_2)^2 + c_1^2(r_1 - r_3)(r_2 - r_3)\right)\right). \tag{3.66}$$

Since w is a quadratic, $w(x) \leq 0$ if and only if the leading coefficient

$$c_0c_2(4c_0c_1 - c_1^2) < 0,$$

and y-coordinate of the vertex

$$\frac{c_0c_1^2c_2}{c_1^2 - 4c_0c_2}\left(c_0c_2(r_1 - r_2)^2 + c_1^2(r_1 - r_3)(r_2 - r_3)\right) \leq 0. \tag{3.67}$$

Thus, we can say that $w(x) \leq 0$ if and only if $0 < c_1^2 - 4c_0c_1$ and $0 \leq c_1^2(r_1 - r_3)(r_3 - r_2) - c_0c_2(r_1 - r_2)^2$. By Lemma 133 and Theorem 128 those conditions are equivalent to T preserving hyperbolicity. $\square$

In Theorem 134, (i) it is unnecessary to assume that the polynomial coefficients of T have real zeros, and (ii) if Q_2 is a quadratic, then Q_1 cannot be a non-zero constant, both of which follows by Lemma 105. Thus, we state Theorem 134 with a little more generality. Its proof follows, *mutatis mutandis*, from the proof of Theorem 134.

Theorem 135 ([4]). *Suppose $Q_2(x), Q_1(x), Q_0(x)$ are real polynomials such that $deg(Q_2(x)) = 2$, $deg(Q_1(x)) \leq 1$, $deg(Q_0(x)) = 0$. Then*

$$T = Q_2(x)D^2 + Q_1(x)D + Q_0(x)$$

preserves hyperbolicity if and only if

$$W[Q_0(x), Q_2(x)]^2 - W[Q_0(x), Q_1(x)]W[Q_1(x), Q_2(x)] \leq 0,$$

$$Q_0(x) \ll Q_1(x), \quad \textit{and} \quad Q_1(x) \ll Q_2(x). \qquad \square$$

CHAPTER 4
MULTIPLIER SEQUENCES WITH VARIOUS POLYNOMIAL BASES

In this chapter, we investigate multiplier sequences acting on various polynomial bases. The main results in this chapter (Theorem 150 and Proposition 151) pertain to multiplier sequences for Jacobi polynomials. Here we generalize the results of K. Blakeman, E. Davis, T. Forgács, and K. Urabe [5]. We also resolve a conjecture of T. Forgács and A. Piotrowski (Proposition 142). This chapter contains three sections on multiplier sequences for various bases: General polynomial base (Section 4.1), Orthogonal polynomial base (Section 4.2), and Jacobi polynomial base (Section 4.3).

4.1 General polynomial base

In this section, we define multiplier sequences (Definition 72) for a general polynomial base, and some of their known properties.

Definition 136. Let $Q = \{q_k(x)\}_{k=0}^{\infty}$ be a simple set of real polynomials, and let $T = \{\gamma_k\}_{k=0}^{\infty}$, $\gamma_k \in \mathbb{R}$. Define $T[q_k(x)] = \gamma_k q_k(x)$, for $k = 0, 1, 2, \ldots$. If

$$T[p(x)] = \sum_{k=0}^{n} \gamma_k a_k q_k(x) \in \mathscr{L}\text{-}\mathscr{P},$$

whenever $p(x) = \sum_{k=0}^{n} a_k q_k(x) \in \mathscr{L}\text{-}\mathscr{P}$, then we say that the sequence $\{\gamma_k\}_{k=0}^{\infty}$ is a *multiplier sequence for the simple set* Q, or for brevity, a Q*-multiplier sequence*. Moreover, if

$$Z_c(T[p(x)]) \leq Z_c(p(x)),$$

where $Z_c(p(x))$ denotes the number of non-real zeros of $p(x)$, counting multiplicities, then we say that the sequence $\{\gamma_k\}_{k=0}^{\infty}$ is a *complex zero decreasing sequence for the simple set* Q, or for brevity, a Q*-CZDS* (The acronym CZDS will also be used in the plural).

Notation 137. We shall include the adjective *classical* to refer to a complex zero decreasing sequence or a multiplier sequence that corresponds with the simple set of real polynomials $Q = \{x^k\}_{k=0}^{\infty}$. The adjective may be omitted if the context is clear.

The following theorem of A. Piotrowski relates Q-multiplier sequences to classical multiplier sequences.

Theorem 138 (A. Piotrowski [55, Theorem 158]). *Let $Q = \{q_k(x)\}_{k=0}^{\infty}$ be a simple set of polynomials. If the sequence $\{\gamma_k\}_{k=0}^{\infty}$ is a Q-multiplier sequence, then the sequence $\{\gamma_k\}_{k=0}^{\infty}$ is a classical multiplier sequence.* □

Theorem 138 is remarkable in that a Q-multiplier sequence of an arbitrary simple set of polynomials $Q = \{q_k(x)\}_{k=0}^{\infty}$ is also a classical multiplier sequence. M. Chasse [18] proved a property of Q-multiplier sequences whose simple set of polynomials have distinct zeros.

Theorem 139 (M. Chasse [18, Theorem 193]). *Let $Q = \{q_k(x)\}_{k=0}^{\infty}$ be a simple set of polynomials with polynomials that have simple zeros, and positive leading coefficients. If $\{\gamma_k\}_{k=0}^{\infty}$ is a non-negative Q-multiplier sequence with at least 3 non-zero terms, then $\gamma_k \geq \gamma_{k-1}$ for all $k \in \mathbb{N}$.* □

Theorem 139 will be used in Section 4.2, since orthogonal polynomials have disctinct zeros by Thoerem 18.

4.2 Orthogonal polynomial base

For any $a, b \in \mathbb{R}$, and any set of orthogonal polynomial basis, the sequence

$$\{\ldots, 0, 0, a, b, 0, 0, \ldots\} \tag{4.1}$$

is a multiplier sequence by Remark 21. Also, for any constant $c \in \mathbb{R}$, the sequence $\{c\}_{k=0}^{\infty}$ is also a multiplier sequence for any simple set of real polynomials. In the context of an orthogonal polynomial set, we call the sequence of the form (4.1) and the constant sequence $\{c\}_{k=0}^{\infty}$ for $c \in \mathbb{R}$, a *trivial sequence*.

In the case when the simple set of polynomials $Q = \{q_k(x)\}_{k=0}^{\infty}$ have interlacing zeros for consecutive $k \in \mathbb{N}$, it can be shown that a sequence of the form (4.1) will be a Q-multiplier sequence. It is easy to construct a simple set of polynomials where a trivial sequence (4.1) is not a multiplier sequence. For example, if $Q = \left\{\sum_{k=0}^{n} x^k\right\}_{n=0}^{\infty}$, then

$$x^2 + 2x + 1 = (x^2 + x + 1) + (x + 1) - 1,$$

so that the sequence $\{0, 1, 1, 0, 0, \ldots\}$ yields $x^2 + 2x + 2$, and thus, the sequence is not a Q-multiplier sequence.

Problem 140. *Classify the simple set of polynomials such that the sequences of the form* (4.1) *are not multiplier sequences.*

The same three sequences which were considered in Examples 74, 85, and 107 have significantly different characteristics when the simple set of polynomials are changed to various orthogonal polynomials from the classical simple polynomial set $\left\{x^k\right\}_{k=0}^{\infty}$.

Example 141.

(i) The sequence $\{k\}_{k=0}^{\infty}$ is a Hermite multiplier sequence (in fact it is also a Hermite CZDS [55, Theorem 101]), it is also a Laguerre multiplier sequence [55, Proposition 170], but it is not a Legendre multiplier sequence (see T. Forgács et al. [5]).

(ii) The sequence $\left\{r^k\right\}_{k=0}^{\infty}$, for $r \in \mathbb{R}$ is a Hermite CZDS for $|r| \geq 1$, but it is not even a Hermite multiplier sequence for $0 < r < 1$. This sequence is a Laguerre multiplier sequence only for $r = 1$ [36, Proposition 2.2], and a Legendre multiplier sequence only for $|r| = 1$. [5, Theorem 12].

(iii) The sequence $\left\{k^2 + k + 1\right\}_{k=0}^{\infty}$ [24, Example 1.8] is a Hermite multiplier sequence (cf. [6, 55]), and it is a Legendre multiplier sequence (cf. T. Forgács et al. [5]) as well. T. Forgács and A. Piotrowski posed a conjecture for a related sequence.

Proposition 142 ([36, Conjecture 5.1])**.** *The sequence* $\left\{k^2 + ak + b\right\}_{k=0}^{\infty}$ *is an* $\mathscr{L}_0$*-multiplier sequence if and only if* $-1 \leq a \leq 3$ *and* $\max\{0, a-1\} \leq b \leq \frac{1}{8}(1+a)^2$*, where* $\mathscr{L}_\alpha$ *is the generalized Laguerre polynomials (cf. Example 16).*

The resolution of this conjecture can be settled by the following recent result of P. Brändén and E. Ottergren [14], which completely characterizes the generalized Laguerre multiplier sequences.

Theorem 143 (P. Brändén and E. Ottergren [14, Theorem 1.1])**.** *Suppose* $\alpha > -1$*,* $p(y) = \sum_{k=0}^{\infty} \binom{k+\alpha}{k} a_k y^k$ *is a formal power series, and* $\{\lambda_n\}_{n=0}^{\infty}$ *is a non-trivial sequence defined by* $\lambda_n := \sum_{k=0}^{n} a_k \binom{n}{k}$*. Then* $\{\lambda_n\}_{n=0}^{\infty}$ *is an* $\mathscr{L}_\alpha$*-multiplier sequence if and only if* $p(y)$ *is a hyperbolic polynomial with all its zeros contained in the interval* $[-1, 0]$*.* □

Proof of Proposition 142. Consider the sequence $L = \left\{n^2 + an + b\right\}_{n=0}^{\infty}$, $a, b \in \mathbb{R}$. Then by Theorem 143, L is an $\mathscr{L}_0$-multiplier sequence if and only if $p(y) = \sum_{k=0}^{\infty} a_k y^k$ is a hyperbolic polynomial with all its zeros contained in the interval $[-1, 0]$. Let the sequence generated by the coefficients of $p(y)$ be $\Lambda = \{\lambda_n\}_{n=0}^{\infty}$. Comparing Λ and L, we obtain that $a_0 = b$, $a_1 = a+1$, $a_2 = 2$, and $a_k = 0$ for $k \geq 3$. For $p(y) = 2y^2 + (a+1)y + b$ to be hyperbolic with zeros contained in the interval $[-1, 0]$, it must be true that

$$-1 \leq \frac{-(a+1) \pm \sqrt{(a+1)^2 - 8b}}{4} \leq 0.$$

The discriminant implies that $b \leq \frac{(a+1)^2}{8}$. The inequalities

$$\frac{-(a+1) \pm \sqrt{(a+1)^2 - 8b}}{4} \leq 0$$

imply $a \geq -1$ and $b \geq 0$. The inequalities

$$-1 \leq \frac{-(a+1) \pm \sqrt{(a+1)^2 - 8b}}{4}$$

imply $a \leq 3$ and $(a-1) \leq b$. □

Example 144. The sequence $\{1/k!\}_{k=0}^{\infty}$ is a classical CZDS by Laguerre's theorem (Theorem 76), but it is not a Q-multiplier sequence for any orthogonal polynomial set Q by Theorem 139.

Problem 145. *Classify the classical CZDS that are Q-multiplier sequences (Q-CZDS) for a given orthogonal polynomial set Q.*

Theorem 86 says that there are classical multiplier sequences and complex zero decreasing sequences which are interpolated by functions in the Laguerre-Pólya class. The following generalization says that this is also true for the Hermite polynomials.

Theorem 146 (Turán [67, p. 289], D. Bleecker and G. Csordas [6, Theorem 2.7])**.** *If $\varphi \in \mathscr{L}\text{-}\mathscr{P}^+$, then $\{\varphi(k)\}_{k=0}^{\infty}$ is a Hermite multiplier sequence. In particular if $\varphi(x)$ is a real polynomial with only real negative zeros, then $\{\varphi(k)\}_{k=0}^{\infty}$ is a Hermite multiplier sequence.* □

The above theorem prompts a similar question regarding other polynomials bases.

Problem 147. *Given an orthogonal polynomial set Q, does there exist a class of entire functions that interpolate Q-multiplier sequences (Q-CZDS)?*

Unlike the classical multiplier sequences, not much are known regarding the properties of Q-multiplier sequences, even when Q is a set of orthogonal polynomials. For example, in Proposition 79 (i), given a classical multiplier sequence $\{\gamma_k\}_{k=0}^{\infty}$, we know that the sequence $\{\gamma_{k+m}\}_{k=0}^{\infty}$, is also a classical multiplier sequence for any integer $m \geq 0$. However, it appears that the following question is an open problem.

Problem 148. *Given an orthogonal polynomial set Q, and a Q-multiplier sequence (Q-CZDS) $\{\gamma_k\}_{k=0}^{\infty}$, when is it true that $\{\gamma_{k+m}\}_{k=0}^{\infty}$, is a Q-multiplier sequence (Q-CZDS) for an integer $m \geq 1$?*

4.3 Jacobi polynomial base

Sequences interpolated by a linear polynomial, such as $\{k+a\}_{k=0}^{\infty}$, $a \in \mathbb{R}$, are classical multiplier sequences for any $a \in \mathbb{R}$. This is mainly due to the sequence $\{k\}_{k=0}^{\infty}$ arising from the differential operator (cf. Examples 74, 85, and 107). The sequence $\{k+a\}_{k=0}^{\infty}$, $a \geq 0$, are also Hermite multiplier sequences by Theorem 146. Linear Laguerre multiplier sequences are restricted to $\{k+b\}_{k=0}^{\infty}$, $0 \leq b \leq 1$ ([36, Theorem 2.6]). Forgács et al. [5, Proposition 2] proved that there surprisingly does not exist any linear Legendre multiplier sequences. We generalize this result to all Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$, $\alpha, \beta > -1$, in this section. We begin with a lemma.

Lemma 149. *The function*

$$h(\alpha,\beta) := \alpha^2(1+2\beta) + \alpha(17+24\beta+4\beta^2) + 48 + 71\beta + 23\beta^2 + 2\beta^3 \tag{4.2}$$

is positive for $-1 < \alpha \leq \beta$.

Proof.

Case 1. If $\alpha, \beta \geq 0$, then $h(\alpha,\beta)$ is clearly positive.

Case 2. For $-1 < \alpha \leq 0$ and $0 \leq \beta$. Since the term $\alpha^2(1+2\beta)$ is non-negative, it suffices to show that

$$\alpha(17+24\beta+4\beta^2) + 48 + 71\beta + 23\beta^2 + 2\beta^3 > 0$$

or

$$2\beta^3 + (23+4\alpha)\beta^2 + (71+24\alpha)\beta + (48+17\alpha) \tag{4.3}$$

is positive. Thus

$$\begin{aligned} 2\beta^3 + (23+4\alpha)\beta^2 + (71+24\alpha)\beta + (48+17\alpha) & \\ &\geq 2\beta^3 + (23-4)\beta^2 + (71-24)\beta + (48-17) \\ &= 2\beta^3 + 19\beta^2 + 47\beta + 31 > 0. \end{aligned}$$

Case 3. For $-1 < \alpha \leq 0$, $-1/2 \leq \beta \leq 0$. Similar to case 2, it suffices to show (4.3) is positve in the region. The minimum is attained at the boundary or its critical points. Thus for $\alpha = -1$

$$\begin{aligned} 2\beta^3 &+ (23+4\alpha)\beta^2 + (71+24\alpha)\beta + (48+17\alpha) \\ &= 2\beta^3 + 19\beta^2 + 47\beta + 31 > 0. \end{aligned}$$

for $-1/2 \leq \beta \leq 0$. If $\alpha = \beta$, then

$$\begin{aligned} 2\beta^3 &+ (23+4\alpha)\beta^2 + (71+24\alpha)\beta + (48+17\alpha) \\ &= 6\beta^3 + 47\beta^2 + 88\beta + 48 > 0 \end{aligned}$$

for $-1/2 \leq \beta \leq 0$. If $\beta = -1/2$, then

$$\begin{aligned} 2\beta^3 &+ (23+4\alpha)\beta^2 + (71+24\alpha)\beta + (48+17\alpha) \\ &= 6\alpha + 18 > 0 \end{aligned}$$

for $-1 \leq \alpha \leq -1/2$. If $\beta = 0$, then

$$\begin{aligned} 2\beta^3 &+ (23+4\alpha)\beta^2 + (71+24\alpha)\beta + (48+17\alpha) \\ &= 17\alpha + 48 > 0 \end{aligned}$$

for $-1 \leq \alpha \leq 0$. The derivative of (4.3) with respect to α is

$$4\beta^2 + 24\beta + 17,$$

and its zeros are out side of $-1/2 \leq \beta \leq 0$, this implies that there are no critical points in the region. Thus the minimum is a positive number.

Case 4. For $-1 < \alpha \leq \beta \leq -1/2$. The minimum of $h(\alpha, \beta)$ is attained at the boundary or its critical points. Thus for $\alpha = -1$

$$h(-1, \beta) = 2\beta^3 + 19\beta^2 + 49\beta + 32 > 0$$

for $-1 < \beta \leq 0$, and $h(-1,-1) = 0$. If $\alpha = \beta$,

$$h(\beta, \beta) = 8\beta^3 + 48\beta^2 + 88\beta + 48 > 0$$

for $-1 < \beta \leq -1/2$.If $\beta = -1/2$,

$$h(\alpha, -1/2) = 6\alpha + 18 > 0$$

for $-1 \leq \alpha \leq -1/2$. The derivative of $h(\alpha, \beta)$ with respect to α is

$$2\alpha + 4\alpha\beta + 4\beta^2 + 24\beta + 17, \tag{4.4}$$

and the derivative of $h(\alpha, \beta)$ with respect to β is

$$2\alpha^2 + \alpha(8\beta + 24) + 6\beta^2 + 46\beta + 71. \tag{4.5}$$

Setting (4.4) and (4.5) equal to zero, solving (4.4) for a, and substituting a into (4.5) yields

$$\frac{-52\beta^2 - 52\beta + 23}{2(1 + 2\beta)^2} = 0. \tag{4.6}$$

The solution to (4.6) lie outside $-1 < \beta < -1/2$, thus there are no critical points in the region. The result follows. □

We prove in the following theorem that no linear multiplier sequences exist for the Jacobi polynomial base, which generalizes the result of Forgács et al [5, Proposition 2].

Theorem 150. *For all $c \in \mathbb{R}$, $\{k + c\}_{k=0}^{\infty}$ is not a multiplier sequence for the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$, $\alpha, \beta > -1$.*

Proof.

Case 1. Fix (α, β) such that $-1 < \alpha \leq \beta$. Let Γ_c be the operator defined by $\Gamma_c[P_n^{(\alpha,\beta)}(x)] = (n+c)P_n^{(\alpha,\beta)}(x)$ for $n = 0, 1, 2, \ldots$, and consider the function $f(x) = (1+x)^3$ expanded in the Jacobi basis:

$$f(x) = a_3 P_3^{(\alpha,\beta)}(x) + a_2 P_2^{(\alpha,\beta)}(x) + a_1 P_1^{(\alpha,\beta)}(x) + a_0 P_0^{(\alpha,\beta)}(x),$$

where

$$a_3 = \frac{48}{(4+\alpha+\beta)(5+\alpha+\beta)(6+\alpha+\beta)},$$

$$a_2 = \frac{48(3+\beta)}{(3+\alpha+\beta)(4+\alpha+\beta)(6+\alpha+\beta)},$$

$$a_1 = \frac{24(3+\beta)(2+\beta)}{(2+\alpha+\beta)(4+\alpha+\beta)(5+\alpha+\beta)},$$

$$a_0 = \frac{8(3+\beta)(2+\beta)(1+\beta)}{(2+\alpha+\beta)(3+\alpha+\beta)(4+\alpha+\beta)}.$$

The discriminant (Definition 7, Chapter 2) of $\Gamma_c[f(x)]$ is a quadratic in c, multiplied by the factor

$$\frac{-6912(2+\beta)(3+\beta)^2}{(2+\alpha+\beta)^2(3+\alpha+\beta)^2(4+\alpha+\beta)^3(5+\alpha+\beta)^3(6+\alpha+\beta)^4}.$$

(Note that $\Gamma_c[f(x)]$ can reduce in degree if $c = -3$, but because of Corollary 12, it makes no difference.) It will suffice for the quadratic polynomial

$$g(c) := \frac{(2+\alpha+\beta)^2(3+\alpha+\beta)^2(4+\alpha+\beta)^3(5+\alpha+\beta)^3(6+\alpha+\beta)^4}{-6912(2+\beta)(3+\beta)^2}\Delta_x[\Gamma_c[f(x)]]$$

to be positive for all $c \in \mathbb{R}$. To this end, we calculate the discriminant of $g(c)$

$$\begin{aligned}\Delta_c[g(c)] = &-64(2+\beta)(3+\beta)^2(2+\alpha+\beta)(3+\alpha+\beta)(6+\alpha+\beta)^2\\ &\times(48+71\beta+23\beta^2+2\beta^3+\alpha^2(1+2\beta)+\alpha(17+24\beta+4\beta^2))^3.\end{aligned}$$

The last product in the above term

$$\begin{aligned}&(48+71\beta+23\beta^2+2\beta^3+\alpha^2(1+2\beta)+\alpha(17+24\beta+4\beta^2))\\ &\quad=\alpha^2(1+2\beta)+\alpha(17+24\beta+4\beta^2)+48+71\beta+23\beta^2+2\beta^3\end{aligned}$$

is positive by Lemma 149. We observe that the leading coefficient of $g(c)$

$$36(1+\beta)^2(2+\beta)(4+\alpha+\beta)(5+\alpha+\beta)(6+\alpha+\beta)^2$$

is positive for $\alpha, \beta > -1$, thus $g(c)$ is positive for all $c \in \mathbb{R}$. This implies that $\Delta[\Gamma_c[f(x)]]$ is negative, so that the sequence $\{k+c\}_{k=0}^{\infty}$ is not a multiplier sequence for the Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$, $-1 < \alpha \leq \beta$.

Case 2. If $-1 < \beta \leq \alpha$, we consider the function $f(x) = (-1+x)^3$ expanded in the Jacobi basis:

$$(-1+x)^3 = f(x) = a_3 P_3^{(\alpha,\beta)}(x) + a_2 P_2^{(\alpha,\beta)}(x) + a_1 P_1^{(\alpha,\beta)}(x) + a_0 P_0^{(\alpha,\beta)}(x),$$

where

$$a_3 = \frac{48}{(4+\alpha+\beta)(5+\alpha+\beta)(6+\alpha+\beta)},$$

$$a_2 = \frac{-48(3+\alpha)}{(3+\alpha+\beta)(4+\alpha+\beta)(6+\alpha+\beta)},$$

$$a_1 = \frac{24(3+\alpha)(2+\alpha)}{(2+\alpha+\beta)(4+\alpha+\beta)(5+\alpha+\beta)},$$

$$a_0 = \frac{-8(3+\alpha)(2+\alpha)(1+\alpha)}{(2+\alpha+\beta)(3+\alpha+\beta)(4+\alpha+\beta)}.$$

Similar to Case 1, the discriminant of $\Gamma_c[f(x)]$ is a quadratic in c, multiplied by the factor

$$\frac{-6912(2+\alpha)(3+\alpha)^2}{(2+\alpha+\beta)^2(3+\alpha+\beta)^2(4+\alpha+\beta)^3(5+\alpha+\beta)^3(6+\alpha+\beta)^4}.$$

(Again, $\Gamma_c[f(x)]$ can reduce in degree if $c = -3$, but because of Corollary 12, it makes no difference.) It will suffice for the quadratic polynomial

$$g(c) := \frac{(2+\alpha+\beta)^2(3+\alpha+\beta)^2(4+\alpha+\beta)^3(5+\alpha+\beta)^3(6+\alpha+\beta)^4}{-6912(2+\alpha)(3+\alpha)^2} \Delta_x[\Gamma_c[f(x)]]$$

to be positive for all $c \in \mathbb{R}$. To this end, we calculate the discriminant of $g(c)$

$$\begin{aligned}\Delta_c[g(c)] = &-64(2+\alpha)(3+\alpha)^2(2+\alpha+\beta)(3+\alpha+\beta)(6+\alpha+\beta)^2 \\ &\times(48+71\alpha+23\alpha^2+2\alpha^3+\beta^2(1+2\alpha)+\beta(17+24\alpha+4\alpha^2))^3.\end{aligned}$$

The last product in the above term is exactly $h(\beta, \alpha)$ from equation (4.2), and thus it is positive by

Lemma 149. We observe that the leading coefficient of $g(c)$

$$36(1+\alpha)^2(2+\alpha)(4+\alpha+\beta)(5+\alpha+\beta)(6+\alpha+\beta)^2$$

is positive for $\alpha, \beta > -1$, thus $g(c)$ is positive for all $c \in \mathbb{R}$. This implies that $\Delta[\Gamma_c[f(x)]]$ is negative, so that the sequence $\{k+c\}_{k=0}^{\infty}$ is not a multiplier sequence for the Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$, $-1 < \beta \leq \alpha$. □

The level of complexity increases very quickly when investigating properties of Jacobi multiplier sequences because of the two parameters of the Jacobi polynomials. Even in the case of linear sequences, the equations often required an analysis of a multivariate nature. The following is our result on quadratic sequences for Jacobi polynomials. Its proof depends heavily on Proposition 118, which spans an entire subsection (cf. Subsection 3.3.1).

Proposition 151. *The sequence* $\left\{n^2+(\alpha+\beta+1)n+d\right\}_{n=0}^{\infty}$*, for* α, $\beta > -1$*, is a Jacobi multiplier sequence for* $0 \leq d \leq (1+\alpha)(1+\beta)$*.*

Proof. The Jacobi polynomials satisfy the differential equation (2.8)

$$(1-x^2)\,y'' + [\beta-\alpha-(\alpha+\beta+2)x]\,y' + n(n+\alpha+\beta+1)\,y = 0,$$

where $y = P_n^{(\alpha,\beta)}(x)$, for all $n \in \mathbb{N}$. It follows that the linear operator S defined by $S[P_n^{(\alpha,\beta)}(x)] := (n(n+\alpha+\beta+1)+d)P_n^{(\alpha,\beta)}(x)$ has the differential operator representation $T = (x^2-1)D^2 + [(\alpha+\beta+2)x+\alpha-\beta]D + d$. The differential operator T preserves hyperbolicity by Proposition 118. Thus the result follows. □

CHAPTER 5
NON-LINEAR OPERATORS ACTING ON ENTIRE FUNCTIONS

In contrast to Chapter 3, here we will be looking at non-linear operators acting on the Laguerre-Pólya class which preserve hyperbolicity and stability. The main results in this chapter include extensions of a result of P. Brändén (Propositions 157 and 158), some answers to questions posed by S. Fisk (Theorems 160, 161, and Propositions 170, 174), a result on the location of zeros of a hypergeometric function (Proposition 171), and some results on a non-linear operator (Propositions 175 and 176). This chapter is a modified version of [68], presented in Macau, August 2010.

5.1 Non-linear operators preserving stability

In 2009, P. Brändén [12] proved the following theorem, a conjecture due to S. Fisk, R. P. Stanley, P. R. W. McNamara and B. E. Sagan.

Theorem 152 (P. Brändén [12])**.** *If a real polynomial $\sum_{k=0}^{n} a_k x^k$ has only real negative zeros, then the associated polynomial $\sum_{k=0}^{n}(a_k^2 - a_{k-1}a_{k+1})x^k$, also has only real negative zeros, where $a_{-1} = a_{n+1} = 0$.* □

S. Fisk [34] posed a problem related to Theorem 152, which may be formulated as follows.

Problem 153. *Let $r \in \mathbb{N}$. If a real polynomial $\sum_{k=0}^{n} a_k x^k$ has only real negative zeros, then does the associated polynomial $\sum_{k=0}^{n}(a_k^2 - a_{k-r}a_{k+r})x^k$, where $a_t = a_s = 0$ for $t < 0$ and $s > n$, have only real negative zeros?*

To state Theorem 152 and Problem 153 in terms of operators, we follow the exposition of P. Brändén [12, Section 4].

Definition 154. Let $\alpha = \{\alpha_k\}_{k=0}^{\infty}$ be a fixed sequence of complex numbers and given a finite sequence, $\{a_k\}_{k=0}^{n}$, define two new sequences $\{b_k(\alpha)\}_{k=0}^{\infty}$ and $\{c_k(\alpha)\}_{k=0}^{\infty}$, where

$$b_k(\alpha) := \sum_{j=0}^{\infty} \alpha_j a_{k-j} a_{k+j} \quad \text{and} \quad c_k(\alpha) := \sum_{j=0}^{\infty} \alpha_j a_{k-j} a_{k+1+j},$$

and $a_j = 0$ if $j \notin \{0, 1, \ldots, n\}$. Also define two non-linear operators acting on polynomials, U_α, V_α :

$\mathbb{C}[x] \to \mathbb{C}[x]$, by

$$U_\alpha\left(\sum_{k=0}^{n} a_k x^k\right) := \sum_{k=0}^{n} b_k(\alpha) x^k \quad \text{and} \quad V_\alpha\left(\sum_{k=0}^{n} a_k x^k\right) := \sum_{k=0}^{n} c_k(\alpha) x^k. \tag{5.1}$$

P. Brändén extends the non-linear operators U_α and V_α from $\mathscr{L}\text{-}\mathscr{P}^+ \cap \mathbb{R}[x]$ to $\mathscr{L}\text{-}\mathscr{P}^+$, and proves the following theorems [12, Theorem 5.7, Theorem 5.8].

Theorem 155 (P. Brändén [12])**.** *If $\alpha = \{\alpha_k\}_{k=0}^{\infty}$ is a sequence of real numbers, then the following are equivalent.*

(i) $U_\alpha[\mathscr{L}\text{-}\mathscr{P}^+_{\mathbb{N}}] \subseteq \mathscr{L}\text{-}\mathscr{P}^+$.

(ii) $U_\alpha[e^x] \in \mathscr{L}\text{-}\mathscr{P}^+ \cup \{0\}$*; that is,*

$$\sum_{k=0}^{\infty}\left(\sum_{j=0}^{k} \frac{\alpha_j}{(k+j)!(k-j)!}\right) x^k \in \mathscr{L}\text{-}\mathscr{P}^+.$$

(iii) $U_\alpha[\mathscr{L}\text{-}\mathscr{P}^+] \subseteq \mathscr{L}\text{-}\mathscr{P}^+$. □

Theorem 156 (P. Brändén [12])**.** *If $\alpha = \{\alpha_k\}_{k=0}^{\infty}$ is sequence of real numbers, then the following are equivalent.*

(i) $V_\alpha[\mathscr{L}\text{-}\mathscr{P}^+ \cap \mathbb{R}[x]] \subseteq \mathscr{L}\text{-}\mathscr{P}^+$.

(ii) $V_\alpha[e^x] \in \mathscr{L}\text{-}\mathscr{P}^+$*; that is,*

$$\sum_{k=0}^{\infty}\left(\sum_{j=0}^{k} \frac{\alpha_j}{(k+1+j)!(k-j)!}\right) x^k \in \mathscr{L}\text{-}\mathscr{P}^+.$$

(iii) $V_\alpha[\mathscr{L}\text{-}\mathscr{P}^+] \subseteq \mathscr{L}\text{-}\mathscr{P}^+$. □

For $r \in \mathbb{N}$, define

$$S_r := U_\alpha \quad \text{and} \quad \widetilde{S}_r := V_\alpha, \tag{5.2}$$

where $\alpha = \{\alpha_k\}_{k=0}^{\infty}$, $\alpha_0 = 1$, $\alpha_r = -1$, and $\alpha_k = 0$ if $k \notin \{0, r\}$. P. Brändén proved $S_r[\mathscr{L}\text{-}\mathscr{P}^+] \subseteq \mathscr{L}\text{-}\mathscr{P}^+$, and $\widetilde{S}_r[\mathscr{L}\text{-}\mathscr{P}^+] \subseteq \mathscr{L}\text{-}\mathscr{P}^+$, if $r = 0, 1, 2, 3$ (case $r = 1$ is Theorem 152). The following propositions are extensions of the aforementioned results of P. Brändén.

Proposition 157. $S_4[\mathscr{L}\text{-}\mathscr{P}^+] \subseteq \mathscr{L}\text{-}\mathscr{P}^+$*, where S_4 is defined in* (5.2).

Proof. By definition, $S_4[e^x] = \sum_{k=0}^{\infty}[a_k^2 - a_{k-4}a_{k+4}]x^k$, where $a_k = 1/k!$, and $a_k = 0$ for $k < 0$. By Theorem 155, it suffices to show that

$$S_4[e^x] = \sum_{k=0}^{\infty} \frac{8(2k+1)(k^2+k+3)}{k!(k+4)!} x^k \in \mathscr{L}\text{-}\mathscr{P}^+.$$

To this end, consider

$$f(x) := \sum_{k=0}^{\infty} \frac{8(2k+1)(k^2+k+3)(5+k)(6+k)(7+k)}{k!} x^k = p(x)e^x,$$

where the polynomial

$$p(x) = 5040 + 35280x + 52920x^2 + 29400x^3 + 6360x^4 + 552x^5 + 16x^6$$

has only real negative zeros. This assertion can be verified by using Mathematica in conjunction with the intermediate value theorem. Thus the entire function $f(x) \in \mathscr{L}\text{-}\mathscr{P}^+$, and by Theorem 73, the sequence

$$\left\{8(2k+1)(k^2+k+3)(5+k)(6+k)(7+k)\right\}_{k=0}^{\infty}$$

is a multiplier sequence. Next we apply the multiplier sequence $\{1/(k+7)!\}_{k=0}^{\infty}$ (cf. Example 87) to the entire function $f(x)$ to obtain

$$\sum_{k=0}^{\infty} \frac{8(2k+1)(k^2+k+3)(5+k)(6+k)(7+k)}{k!(k+7)!} x^k = \sum_{k=0}^{\infty} \frac{8(2k+1)(k^2+k+3)}{k!(k+4)!} x^k$$

$$= S_4[e^x] \in \mathscr{L}\text{-}\mathscr{P}^+$$

by Theorem 73. □

A similar argument, *mutatis mutandis*, establishes the following proposition.

Proposition 158. $\widetilde{S}_4[\mathscr{L}\text{-}\mathscr{P}^+] \subseteq \mathscr{L}\text{-}\mathscr{P}^+$, *where* $\widetilde{S}_4$ *is defined in* (5.2). □

For our next result, we consider

$$S_6[e^x] = \sum_{k=0}^{\infty}[a_k^2 - a_{k-6}\, a_{k+6}]x^k, \tag{5.3}$$

and

$$\widetilde{S}_6[e^x] = \sum_{k=0}^{\infty}[a_k a_{k+1} - a_{k-6}\, a_{k+7}]x^k, \tag{5.4}$$

where $a_k = 1/k!$, and $a_k = 0$ for $k < 0$.

Lemma 159. *Let $f(x) := S_6[e^x] = \sum_{k=0}^{\infty} b_k x^k$, its partial sum $f_n(x) := \sum_{k=0}^{n} b_k x^k$, and $E_n(x) := f(x) - f_n(x)$. If $x_0 = -43$, then*

$$|E_{30}^{(j)}(x_0)| < 5 \times 10^{-18},$$

where $E_n^{(j)}(x)$ denotes the j-th derivative for $j = 0, 1, 2$.

Proof. The infinite sum obtained by the power series $f(x) = \sum_{k=0}^{\infty} b_k x^k$ evaluated at $x_0 = -43$ is

$$\sum_{k=0}^{\infty} \frac{(720 + 1884k + 1350k^2 + 960k^3 + 90k^4 + 36k^5)}{k!(6+k)!}(-43)^k := \sum_{k=0}^{\infty}(-1)^k c_k.$$

An elementary computation yields $c_k \geq c_{k+1}$ for $k \geq 7$. Hence, $E_k(x_0)$ is an alternating series for $k \geq 7$, and for $j = 0, 1, 2$,

$$|E_{30}^{(j)}(x_0)| \leq |E_{28}(x_0)| \leq |b_{29}| < 5 \times 10^{-18}.$$

□

Theorem 160. *If $x_0 = -43$, then*

$$(f'(x_0))^2 - f(x_0)f''(x_0) < 0, \tag{5.5}$$

where $f(x) = S_6[e^x]$.

Proof. With the notation of Lemma 159, $f_n(x) := \sum_{k=0}^{n} a_k x^k$, and $E_n(x) := f(x) - f_n(x)$. Using Mathematica, for $x_0 = -43$,

$$\begin{aligned} f(x_0) &= f_{30}(x_0) + E_{30}(x_0) \\ &= -5.354465\ldots \times 10^{-2} + E_{30}(x_0), \end{aligned}$$

$$\begin{aligned} f'(x_0) &= f_{30}^{(1)}(x_0) + E_{30}^{(1)}(x_0) \\ &= 7.536322\ldots \times 10^{-5} + E_{30}^{(1)}(x_0), \qquad \text{and} \end{aligned}$$

$$\begin{aligned} f''(x_0) &= f_{30}^{(2)}(x_0) + E_{30}^{(2)}(x_0) \\ &= -3.954149\ldots \times 10^{-3} + E_{30}^{(2)}(x_0). \end{aligned}$$

Hence,

$$\begin{aligned} &(f'(x_0))^2 - f(x_0)f''(x_0) \\ &= (7.536322\ldots \times 10^{-5} + E_{30}^{(1)}(x_0))^2 \\ &\quad -(-5.354465\ldots \times 10^{-2} + E_{30}(x_0))(-3.954149\ldots \times 10^{-3} + E_{30}^{(2)}(x_0)). \end{aligned}$$

By Lemma 159, a calculation show that

$$(f'(x_0))^2 - f(x_0)f''(x_0) \ < \ -2.1 \times 10^{-4}. \qquad \square$$

A similar argument, *mutatis mutandis*, establishes the following theorem.

Theorem 161. *If $x_0 = -56$, then*

$$(g'(x_0))^2 - g(x_0)g''(x_0) < 0, \tag{5.6}$$

where $g(x) = \widetilde{S}_6[e^x]$. $\square$

By Theorem 60, Theorems 160 and 161 imply that $S_6[e^x], \widetilde{S}_6[e^x] \notin \mathscr{L}\text{-}\mathscr{P}^+$. In particular, by Theorem 155 and Theorem 156,

$$S_6[\mathscr{L}\text{-}\mathscr{P}^+ \cap \mathbb{R}[x]] \nsubseteq \mathscr{L}\text{-}\mathscr{P}^+$$

and

$$\widetilde{S}_6[\mathscr{L}\text{-}\mathscr{P}^+ \cap \mathbb{R}[x]] \nsubseteq \mathscr{L}\text{-}\mathscr{P}^+.$$

We pose some questions regarding the operator S_r (similar questions could be considered for the operator $\widetilde{S}_r$).

Problem 162. *Find all $r \in \mathbb{N}$ such that $S_r[\mathscr{L}\text{-}\mathscr{P}^+] \subseteq \mathscr{L}\text{-}\mathscr{P}^+$.*

Problem 163. *Characterize the entire functions $f(x) \in \mathscr{L}\text{-}\mathscr{P}^+$ such that $S_r[f(x)] \in \mathscr{L}\text{-}\mathscr{P}^+$ for all $r \in \mathbb{N}$.*

The existence of entire functions that satisfy Problem 163 is a consequence of the following theorem, which requires rapidly decreasing sequences (cf. Definition 83).

Theorem 164 ([42])**.** *A power series $f(x) = \sum_{k=0}^{\infty} s_k x^k$ whose coefficients form a rapidly decreasing sequence $\{s_k\}_{k=0}^{\infty}$ belong in $\mathscr{L}$-$\mathscr{P}^+$.* □

Example 165. The sequence

$$\left\{\frac{1}{2^{k^2}}\right\}_{k=0}^{\infty} := \{a_k\}_{k=0}^{\infty}$$

satisfies $a_k^2 \geq 4a_{k-1}a_{k+1}$ for $k \in \mathbb{N}$. For $r \in \mathbb{N}$, define the sequence

$$\{t_{k,r}\}_{k=0}^{\infty} := \left\{a_k^2 - a_{k-r}a_{k+r}\right\}_{k=0}^{\infty}.$$

Then the sequence $\{t_{k,r}\}_{k=0}^{\infty}$ also satisfies the condition $t_{k,r}^2 \geq 4t_{k-1,r}t_{k+1,r}$ for $k \in \mathbb{N}$. Thus $f(x) = \sum_{k=0}^{\infty} \frac{x^k}{2^{k^2}}, S_r[f(x)] \in \mathscr{L}$-$\mathscr{P}^+$ for all $r \in \mathbb{N}$ by Theorem 164.

Remark 166. The doctoral dissertation of L. Grabarek [40] investigates various non-linear operators related to the operators discussed in this section.

5.2 Related results

In [34, Question 3], S. Fisk raised the following question.

Problem 167. *Let $d \in \mathbb{N}$, and let $f(x) = \sum_{k=0}^{n} a_k x^k \in \mathscr{L}$-$\mathscr{P}^+ \cap \mathbb{R}[x]$. Form*

$$F_d[f(x)] := \sum_{k=0}^{n} \begin{vmatrix} a_k & \cdots & a_{k+d-1} \\ a_{k-1} & \cdots & a_{k+d-2} \\ \vdots & & \vdots \\ a_{k-d+1} & \cdots & a_k \end{vmatrix} x^k, \text{ where } a_k = 0 \text{ for } k < 0 \text{ and } k > n. \tag{5.7}$$

Is it true that $F_d[f(x)] \in \mathscr{L}$-$\mathscr{P}^+$ for all $f(x) \in \mathscr{L}$-$\mathscr{P}^+ \cap \mathbb{R}[x]$?

We will establish an affirmative answer to Fisk's question (Proposition 170) when the coefficients a_k are the binomials $\binom{n}{k}$. Given a sequence of complex numbers $\{a_k\}_{k=0}^{\infty}$, we consider the infinite matrix

$$\begin{pmatrix} a_0 & a_1 & a_2 & a_3 & \dots \\ a_{-1} & a_0 & a_1 & a_2 & \dots \\ a_{-2} & a_{-1} & a_0 & a_1 & \dots \\ a_{-3} & a_{-2} & a_{-1} & a_0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{5.8}$$

Furthermore, we define the $d \times d$ principal minor, starting at column k of (5.8), by

$$D_k^{(d)} := \det\left(a_{k-i+j}\right), \quad \text{for} \quad 0 \le i, j \le d-1. \tag{5.9}$$

As an application of P. A. MacMahon's Master Theorem [49, Section 495], R. P. Stanley [64, Theorem 18.1] proved the following result.

Theorem 168 (R. P. Stanley [64])**.** *Let $d, n \in \mathbb{N}$, $a_k := \binom{n}{n-k}$, and $a_k := 0$ for $k < 0$ and $k > n$. Then for $0 \le k \le n$,*

$$D_k^{(d)} = \prod_{j=0}^{d-1} \frac{\binom{n+j}{k+j}}{\binom{n-k+j}{n-k}}$$

where $D_k^{(d)}$ is defined in (5.9). □

Lemma 169. *For $d, n \in \mathbb{N}$, the polynomial*

$$B(x) := \sum_{k=0}^{n} \binom{n}{k} \frac{\binom{n+d}{k+d}}{\binom{n-k+d}{n-k}} x^k = \sum_{k=0}^{n} \binom{n}{k} \left[\frac{(n+d)!d!}{(k+d)!(n-k+d)!}\right] x^k,$$

has only real negative zeros.

Proof. Two proofs will be given.

Proof 1. The numerator in the summand of $B(x)$, $(n+d)!d!$, are fixed constants. As noted before (cf. Example 87), $\left\{\frac{1}{(k+d)!}\right\}_{k=0}^{\infty}$ is a multiplier sequence. By Lemma 75, the sequence $\left\{\frac{1}{(n-k+d)!}\right\}_{k=0}^{\infty}$ (where $\frac{1}{k!} = 0$ for $k < 0$) is also a multiplier sequence. By Theorem 73, applying these multiplier sequences to $\sum_{k=0}^{n} \binom{n}{k} x^k$ implies that $B(x)$ has only real negative zeros.

Proof 2. K. Driver and K. Jordaan [32, Theorem 3.2] proved that the hypergeometric polynomial ${}_2F_1(-n, -(n+d); d; x) = B(x)$ has only real negative zeros. □

Using Theorem 168, Theorem 34, Lemma 169, and Lemma 75, a partial answer to Problem 167 is given in the following proposition.

Proposition 170. *For $d, n \in \mathbb{N}$, the polynomial $F_d[(1+x)^n]$ has only real negative zeros, where F_d is defined in* (5.7).

Proof. Fix $n \in \mathbb{N}$. By Theorem 168, $F_d[(1+x)^n] = \sum_{k=0}^{n} \left[\prod_{j=0}^{d-1} \frac{\binom{n+j}{k+j}}{\binom{n-k+j}{n-k}}\right] x^k$. We will complete the proof of the proposition by induction on d.

$$F_1[(x+1)^n] = \sum_{k=0}^{n} \binom{n}{k} x^k = (1+x)^n \in \mathscr{L}\text{-}\mathscr{P}^+ \cap \mathbb{R}[x].$$

Suppose $A(x) := F_d[(1+x)^n] = \sum_{k=0}^{n} \left[\prod_{j=0}^{d-1} \frac{\binom{n+j}{k+j}}{\binom{n-k+j}{n-k}}\right] x^k$ has only real negative zeros. Consider $B(x) = \sum_{k=0}^{n} \binom{n}{k} \frac{\binom{n+d}{k+d}}{\binom{n-k+d}{n-k}} x^k$ from Lemma 169, which has only real negative zeros. By Theorem 34, the composition of $A(x)$ and $B(x)$ is

$$C(x) = \sum_{k=0}^{n} \left[\prod_{j=0}^{d} \frac{\binom{n+j}{k+j}}{\binom{n-k+j}{n-k}}\right] x^k = F_{d+1}[(1+x)^n],$$

which has only real negative zeros. □

Proposition 170 can be generalized to the following result regarding hypergeometric polynomials (Definition 29).

Proposition 171. *For a finite subset $P \subseteq \mathbb{N}$, denote by $|P|$ the number of elements in P. Then the hypergeometric polynomial*

$${}_{|P|+1}F_{|P|}(-n, -(n+\alpha_1), \ldots, -(n+\alpha_{|P|});\ \ \alpha_1, \ldots, \alpha_{|P|};\ \ (-1)^{|P|+1}x)$$

$$= 1 + \sum_{k=1}^{\infty} \frac{\prod_{i=1}^{|P|+1} (-(n+\alpha_{i-1}))_k}{\prod_{j=1}^{|P|} (\alpha_j)_k} \frac{x^k}{k!} = \sum_{k=0}^{n} \binom{n}{k} \left[\prod_{\alpha_i \in P \subseteq \mathbb{N}} \frac{\binom{n+\alpha_i}{k+\alpha_i}}{\binom{n-k+\alpha_i}{n-k}}\right] x^k$$

has only real negative zeros, where $\alpha_0 = 0$, and $(m)_j$ is the Pochhammer symbol (cf. Definition 29).

Proof. The proof is analogous to the proof of Proposition 170. Instead of $B(x) = \sum_{k=0}^{n} \binom{n}{k} \frac{\binom{n+d}{k+d}}{\binom{n-k+d}{n-k}} x^k$, we consider

$$B_{\alpha_j}(x) = \sum_{k=0}^{n} \binom{n}{k} \frac{\binom{n+\alpha_j}{k+\alpha_j}}{\binom{n-k+\alpha_j}{n-k}} x^k \qquad (\alpha_j \in P),$$

which is hyperbolic by Lemma 169. The result is obtained by a repeated application of Theorem 34.

□

Notation 172. Given a function $f(x) = \sum_{k=0}^{\infty} a_k x^k \in \mathscr{L}\text{-}\mathscr{P}^+$, define the associated matrix formed by the sequence $\{a_k\}_{k=0}^{\infty}$ of coefficients of $f(x)$ as in (5.8), where $a_k = 0$ for $k < 0$. Regard the transformation F_d as a non-linear operator on $\mathscr{L}\text{-}\mathscr{P}^+$, where

$$F_d[f(x)] := \sum_{k=0}^{\infty} \begin{vmatrix} a_k & \dots & a_{k+d-1} \\ a_{k-1} & \dots & a_{k+d-2} \\ \vdots & & \vdots \\ a_{k-d-1} & \dots & a_k \end{vmatrix} x^k \quad (a_k = 0 \text{ for } k < 0). \tag{5.10}$$

By the Cauchy-Hadamard formula, $F_d[f(x)]$ is an entire function.

The next lemma will be used to apply the operator F_d to the transcendental function $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$.

Lemma 173. *If $d \in \mathbb{N}$, and the sequence $\{a_k\}_{k=0}^{\infty} := \left\{\frac{1}{k!}\right\}_{k=0}^{\infty}$, with $a_k = 0$ for $k < 0$, then*

$$D_k^{(d)} = \prod_{j=0}^{d-1} \frac{j!}{(k+j)!},$$

where $D_k^{(d)}$ is defined in (5.9).

Proof. A proof will be given by inducting on d.

If $d = 1$, then

$$D_k^{(1)} = \prod_{j=0}^{0} \frac{1}{(k+j)!} = \frac{1}{k!} = a_k.$$

Suppose that

$$D_k^{(d)} = \prod_{j=0}^{d-1} \frac{j!}{(k+j)!} \qquad \text{holds true for all integers } 0 \le k \le d.$$

Consider the $(d+1)\times(d+1)$ principal minor of (5.8) at column k.

$$M_k := \begin{pmatrix} \frac{1}{k!} & \frac{1}{(k+1)!} & \cdots & \frac{1}{(k+d)!} \\ \frac{1}{(k-1)!} & \frac{1}{k!} & \cdots & \frac{1}{(k+d-1)!} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{(k-d)!} & \frac{1}{(k-d+1)!} & \cdots & \frac{1}{k!} \end{pmatrix}.$$

Multiply M_k by $(k+d)!$

$$\begin{pmatrix} (k+1)\cdots(k+d) & (k+2)\cdots(k+d) & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ (k-d+2)\cdots(k+d) & (k-d+3)\cdots(k+d) & \dots & (k+2)\dots(k+d) \\ (k-d+1)\cdots(k+d) & (k-d+2)\cdots(k+d) & \dots & (k+1)\cdots(k+d) \end{pmatrix}.$$

First, row reduce the last row by multiplying the second to last row by $-(k-d+1)$, and adding to the last row to obtain

$$\begin{pmatrix} (k+1)\cdots(k+d) & (k+2)\cdots(k+d) & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ (k-d+2)\cdots(k+d) & (k-d+3)\cdots(k+d) & \dots & (k+2)\cdots(k+d) \\ 0 & (1)(k-d+3)\cdots(k+d) & \dots & (d)(k+2)\cdots(k+d) \end{pmatrix}.$$

Then, row reduce the second to last row by multiplying the third to last row by $-(k-d+2)$, and adding to the last row gives

$$\begin{pmatrix} (k+1)\cdots(k+d) & (k+2)\cdots(k+d) & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ (k-d+3)\cdots(k+d) & (k-d+2)\cdots(k+d) & \dots & (k+3)\cdots(k+d) \\ 0 & (1)(k-d+2)\cdots(k+d) & \dots & (d)(k+3)\cdots(k+d) \\ 0 & (1)(k-d+3)\cdots(k+d) & \dots & (d)(k+2)\cdots(k+d) \end{pmatrix}.$$

Continue this process until the second row is reduced by multiplying first row by k, to obtain

$$\begin{pmatrix} (k+1)\cdots(k+d) & (k+2)\cdots(k+d) & \dots & 1 \\ 0 & (1)(k+2)\cdots(k+d) & \dots & (d) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & (1)(k-d+2)\cdots(k+d) & \dots & (d)(k+3)\cdots(k+d) \\ 0 & (1)(k-d+3)\cdots(k+d) & \dots & (d)(k+2)\cdots(k+d) \end{pmatrix}.$$

The determinant of the lower right $d \times d$ minor is $(d!)D_{k+1}^{(d)} = (d!)\prod_{j=0}^{d-1} \frac{j!}{(k+1+j)!}$.

Thus

$$\det M_k = \frac{1}{(k+d)!}\left[(k+1)\cdots(k+d)(d!)\prod_{j=0}^{d-1}\frac{j!}{(k+1+j)!}\right] = \prod_{j=0}^{d}\frac{j!}{(k+j)!} = D_k^{(d+1)}$$

as desired. □

Using Lemma 173, the following result is attained.

Proposition 174. *For $d \in \mathbb{N}$, $F_d[e^x] \in \mathscr{L}\text{-}\mathscr{P}^+$, where F_d is defined in (5.10).*

Proof. Fix $d \in \mathbb{N}$.

$$F_d[e^x] = \sum_{k=0}^{\infty} \begin{vmatrix} a_k & \dots & a_{k+d-1} \\ a_{k-1} & \dots & a_{k+d-2} \\ \vdots & & \vdots \\ a_{k-d-1} & \dots & a_k \end{vmatrix} x^k,$$

where $a_k = \frac{1}{k!}$, and $a_k = 0$ for $k < 0$. Then by Lemma 173,

$$F_d[e^x] = \sum_{k=0}^{\infty}\left(\prod_{j=0}^{d-1}\frac{j!}{(k+j)!}\right)x^k.$$

Since $\left\{\frac{1}{(k+j)!}\right\}_{k=0}^{\infty}$ is a multiplier sequence for $j = 0, 1, \dots, d-1$, $F_d[e^x] \in \mathscr{L}\text{-}\mathscr{P}^+$. □

5.3 Applications

For a sequence of positive real numbers $\{a_k\}_{k=0}^{\infty}$, D. K. Dimitrov [31] defined the *higher order Turán inequalities* as

$$4(a_k^2 - a_{k-1}a_{k+1})(a_{k+1}^2 - a_k a_{k+2}) - (a_k a_{k+1} - a_{k-1}a_{k+2})^2 \geq 0. \tag{5.11}$$

For a polynomial $\sum_{k=0}^{n} a_k x^k$, we define the non-linear operator J acting on $\mathscr{L}\text{-}\mathscr{P}^+ \cap \mathbb{R}[x]$ by

$$J\left[\sum_{k=0}^{n} a_k x^k\right] := \sum_{k=0}^{n}\left[4\left(a_k^2 - a_{k-1}a_{k+1}\right)\left(a_{k+1}^2 - a_k a_{k+2}\right) - \left(a_k a_{k+1} - a_{k-1}a_{k+2}\right)^2\right] x^k,$$

where $a_k = 0$ for $k < 0$ and $k > n$. The operator J has the following property.

Proposition 175. *If $n \in \mathbb{N}$, then $J[(1+x)^n] \in \mathscr{L}\text{-}\mathscr{P}^+ \cap \mathbb{R}[x]$.*

Proof. $J[(1+x)^n]$

$$= \sum_{k=0}^{n}\left[4\left(\binom{n}{k}^2 - \binom{n}{k-1}\binom{n}{k+1}\right)\left(\binom{n}{k+1}^2 - \binom{n}{k}\binom{n}{k+2}\right) - \left(\binom{n}{k}\binom{n}{k+1} - \binom{n}{k-1}\binom{n}{k+2}\right)^2\right] x^k$$

$$= (4n!(n+1)!(n+2)!)\sum_{k=0}^{n}\binom{n}{k}\left[\frac{x^k}{(k+1)![(k+2)!]^2(n-k-1)![(n-k+1)!]^2}\right].$$

By Example 87 and Lemma 75,

$$\left\{\frac{1}{(k+1)!}\right\}_{k=0}^{\infty}, \quad \left\{\frac{1}{(k+2)!}\right\}_{k=0}^{\infty}, \quad \left\{\frac{1}{(n-k-1)!}\right\}_{k=0}^{\infty}, \quad \text{and} \quad \left\{\frac{1}{(n-k+1)!}\right\}_{k=0}^{\infty}$$

(where $\frac{1}{m!} = 0$ for $m < 0$) are multiplier sequences. Thus $J[(1+x)^n] \in \mathscr{L}\text{-}\mathscr{P}^+ \cap \mathbb{R}[x]$. □

For $f(x) = \sum_{k=0}^{\infty} a_k x^k \in \mathscr{L}\text{-}\mathscr{P}^+$, extend the operator J from $\mathbb{R}[x]$ to $\mathscr{L}\text{-}\mathscr{P}^+$ as the operator F_d was extended in (5.10). Thus

$$J[f(x)] := \sum_{k=0}^{\infty}\left[4(a_k^2 - a_{k-1}a_{k+1})(a_{k+1}^2 - a_k a_{k+2}) - (a_k a_{k+1} - a_{k-1}a_{k+2})^2\right] x^k. \tag{5.12}$$

By the Cauchy-Hadamard formula, $J[f(x)]$ is an entire function.

Proposition 176. *If $J[e^x]$ is defined by (5.12), then $J[e^x] \in \mathscr{L}\text{-}\mathscr{P}^+$.*

Proof.

$$J[e^x] = \sum_{k=0}^{\infty} \left(4 \left[\left(\frac{1}{k!} \right)^2 - \frac{1}{(k-1)!} \frac{1}{(k+1)!} \right] \left[\left(\frac{1}{(k+1)!} \right)^2 - \frac{1}{k!} \frac{1}{(k+2)!} \right] \right.$$

$$\left. - \left[\frac{1}{k!} \frac{1}{(k+1)!} - \frac{1}{(k-1)!} \frac{1}{(k+2)!} \right]^2 \right) x^k$$

$$= \sum_{k=0}^{\infty} \frac{4}{k!(k+1)![(k+2)!]^2} x^k$$

Since $\left\{ \frac{1}{(k+j)!} \right\}_{k=0}^{\infty}$ is a multiplier sequence for $j = 0, 1, 2,$ $J[e^x] \in \mathscr{L}\text{-}\mathscr{P}^+$. $\square$

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