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# AN EXTENDED THEORY OF THE SOCIAL FIELD:

# WITH APPLICATION TO THE BEHAVIOR OF NATIONS

A DISSERTATION SUBMITTED TO THE GRADUATE DIVISION OF THE
UNIVERSITY OF HAWAII IN PARTIAL FULFILLMENT
OF THE REQUIREMENTS FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

IN POLITICAL SCIENCE

DECEMBER 1975

By

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# AN EXTENDED THEORY OF THE SOCIAL FIELD: WITH APPLICATION TO THE BEHAVIOR OF NATIONS

By Paul Robert Williamson

A dissertation submitted to the Graduate Division of the University of Hawaii in partial fulfillment of the requirements for the degree of Doctor of Philosophy

## ABSTRACT

The fundamental question is probed, "To what extent can aspects of social behavior be put in correspondence with concepts of natural phenomena such that mechanical laws become statements of social regularities?" This question is occasioned by the premise that there is a fundamental similarity between the ideas of "social field theory" and the mechanical ideas of physics. In his field theory, Quincy Wright suggested that various characteristics of national actors determine their behavior and, that these characteristics constitute a set of coordinates in a social space (or "field"), for each actor; behavior was thus regarded as a function of the coordinates of the actor in the space. Wright's concept is compared to a physical analogy: actors are like material objects located in a space; various quantitative attributes of actors supply the spatial coordinates

of the objects. The suggestion, by R. J. Rummel, that the relative configuration of social actors in the space gives rise to forces which determine their mutual behavior, resembles the physical idea that the relative configuration of charged particles determines the electromagnetic field. A specialization of the above is Rummel's idea that the mutual behavior of two actors is the result of forces which are a function of their mutual distance in social space. A straightforward extension of this is suggested: that the spatial motion of social actors also gives rise to forces which determine behavior. This "extended" social force is additionally analogous to the electrodynamic force of classical physics in that both are velocity as well as position dependent. The present study chooses to juxtapose this particular combination of social metaphore and physical model -- to regard social force as like electrodynamic force.

The discipline of classical physics provides a formalism treating the idea of forces acting on objects in a space, a part of which is adapted and reinterpreted relative to the above social field concepts. The concepts, themselves, serve as heuristic for the substantive content of this adaptation. A modification of the physical concept of the

vector components of electrodynamic force is interpreted as the series of directed behaviors from a given actor nation to various object nations. Certain alternate empirical interpretations of the resulting formalism imply differing but related predictions of regularities in the behavior of nations. In the first interpretation, the attributes of nations are regarded geometrically as components of their spatial velocity. Given this, the formalism is shown to contain the Rummel equations, connecting directed behavior to national attribute differences. The appropriate geometry of social space is considered. It is shown that Rummel's equations are equally valid for Euclidean and certain non-Euclidean spaces.

The second interpretation considers a different empirical meaning to the velocities of nations. Spatial motion is explicitly defined as a construct based on patterns of over-time covariation among indices of national attributes and behavior. A novel concept of "social time" is introduced, based on over-time variation of behavior. With these definitions the geometry of social space is shown to be of the "Minkowski" (non-Euclidean) type. A modified behavior law is derived, connecting over-time with cross-sectional patterns of behavior and implying a meaningful generalization of the "feudal" interaction model of Johan Galtung's structural theory of imperialism. Given some prior empirical findings

and certain reasonable conjectures about the international system, it is then shown that the modified behavior law directly implies the feudal model. Certain additional lines of inquiry which might broaden the scope of the theory are suggested.

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# I. INTRODUCTION

In his book, The Study of International Relations, Quincy Wright suggested a conceptual scheme whereby all the objects customary to the study of international relations could, he believed, be assigned a location in a space, which location would be determined by the characteristics, or attributes, of those objects. In Wright's own words

> The analytical approach ... implies that each international organization, national government, association, individual, or other "system of action," or "decision-maker" may be located in a multidimensional field. Such a field may be defined by co-ordinates, each of which measures a political, economic, psychological, sociological, ethical, or other continuum influencing choices, decisions, and actions important for international relations. (Wright, 1955, p. 543)

Wright then proceeded (pp. 543-559) to outline a program of investigation for empirically delineating the international field in terms of the specific variables of explanatory importance to international relations. In geometric terms, Wright was proposing that various characteristics of social objects determine their behavior and, that these characteristics constitute a set of coordinates in a social space (or "field"), for each of the objects. Thus, social behavior was regarded as a function of the location of the behaving object in the space, relative to the coordinate frame. The above statement makes use of two concepts which are unmistakably analogous to physical concepts: First, social actors are regarded as discrete objects located in a space; second, the measures of various attributes of those actors are regarded as supplying the coordinates of the objects

relative to some frame of reference in the space. Turning now to a theory of international behavior proposed by Rummel, we find therein the suggestion that the relative configuration of social actors in the space, specifically their degree of similarity on various attributes, gives rise to forces which determine their mutual behavior (Rummel, 1965, pp. 185, 187-9, 202, 203). This is clearly like the physical idea that the behavior of material objects is due to forces which arise from the relative distances among objects. Moreover, as Sang-Woo Rhee points out, the idea of a "field of behavior" is like the physical idea of a field of force (1970, p. 24). More specifically, the notion that the relative configuration of social actors in the space accounts for behavior is like the physical idea that the relative configuration of electrically charged particles determines the electromagnetic potential (force) field (Feynman, et. al., Vol. II, Chpt. 21, pp. 4-5). Certain related analogies are suggested, thus, between social and natural behavior, involving the idea of "forces" acting upon objects, social or natural, located in a "space."

There is a fundamental question which can be posed about the social field analogies. The ideas of force and space are used in natural inquiry to reflect the known mechanical properties of physical objects. Are there corresponding properties definable with respect to social entities which make the analogy more than a metaphore? Is it possible to view social entities in a perspective which reveals a mechanical aspect to their behavior? This question motivates the present study and underlies the method of approaching the more restricted questions

which will be considered.

It is often the practice in political inquiry to regard analogies metaphorically--to consider them for their heuristic value in suggesting more specific propositions about political behavior--but the above motive calls for a different approach. The specific physical usages of "force" and "space" are represented in a deductive mathematical framework. We choose to pose (and partly answer) the question whether social relations can also be represented in this framework, or a portion thereof.[1] Thus the inquiry moves precisely opposite to the usual direction: it seeks to represent the several field theoretic analogies in a more general abstract framework, itself analogous to part of the physical framework.[2]

The rationale for proceeding according to the above method can be more fully explained by referring to the following idea: Suppose that a particular collection of sentences has the property that, when the sentences are placed in a certain definite (not necessarily unique) order, at least one of them is a logical consequence of some or all of the sentences preceeding it in the given order; and suppose that no two sentences are logically contradictory.[3] Call the above collection

---

[1]That it is naive to assume that because a particular word such as "force" is used in both social and natural discussions there must be a connection between the two usages is a valid remark, but would miss the point of this inquiry which is to try to devise a connection.

[2]And it regards the analogies as constituting, initially, the substantive political content to the study.

[3]This latter property is called "consistency."

of sentences an analytic system (Kemeny, 1959, pp. 14-35). The sentences of an analytic system do not necessarily refer to any empirical matter, and various such references may be selected at will. Call any particular such reference an interpretation of the analytic system. One method of inquiry into a particular empirical topic is to try to devise an analytic system together with an interpretation of it, such that the sentences of the system are not disconfirmed by empirical test. By "disconfirmed" is meant that the interpreted sentences are highly inaccurate of the empirical data to which they refer.[4] In other words, the analytic system is used to try to accurately deduce empirically testable results about the topic of interest. This is the method which will be used in the present discussion.

That the inquiry seeks to devise a physical analogy to social behavior has a strategic relationship to the above method. By far,

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[4] A more detailed rendering of this idea is as follows: The combination of particular system and interpretation constitutes a particular mathematical theory. If the corresponding empirical relationships are found to hold exactly or, within "satisfactory" limits, to hold approximately, the theory is said not to have been disconfirmed. In this case, the theoretician proceeds to other collections of data within the same class to determine if they, too, satisfy the interpreted relationship. This process continues until some collection of data is found which does not satisfactorily approximate the indicated relationship, or, until it is generally accepted that the number of correspondences between theory and data is sufficient to establish the predictive value of the theory. In the former case, the theory is said to be disconfirmed; in the latter, it is said to be "successful." (Rapoport, 1961)

the most rigorous and successful[5] use of the analytic method has been in the study of natural phenomena. The discipline of classical physics[6] provides an existing analytic system treating the idea of forces acting on objects in a space, and the present discussion will take the form of seeking to adopt and reinterpret a part of that system relative to the above concepts of international relations theory.[7] The technical aspect of the inquiry is thereby less formidable than if one were starting fresh (but still difficult).

The fundamental question which was posed above is "To what extent can aspects of social behavior be put in correspondence with concepts of natural phenomena such that natural laws become statements of social laws?" The importance of a specifically analytic approach to this specific question is evident in several ways:

1. The question is posed whether the field concepts of Wright and others might actually lead to inferences about observable social

---

[5] in the sense described in note 4 above.

[6] as opposed to quantum physics, which is not relevant to the present inquiry.

[7] It should be emphasized that the interest is in those parts of the analytic system of physics which appear useful to social inquiry. The physical interpretation per se of that system relative to natural phenomena is irrelevant. Another point is that none of the conjectures about, or derivations of "social laws" in this discussion are posed with the expectation that their empirical application will have as much accuracy as in physics. As in all social inquiry, the expectation is that the generalizations are at best "usually" or "approximately" true.

relations which could be empirically tested. The present discussion will consider how the analytic framework which is constructed might itself suggest testable propositions. In effect, Wright's original formulation (i.e. the above passage) serves as an heuristic from which is constructed the system from which testable propositions are derived. This question is addressed in Chapter III, where the prior speculations of Rummel are obtained as a result, eq. (3.3), and, in Chapters IV and V, where a modified behavior law, eq. (4.15) is obtained.

2. The idea of visualizing social actors as objects in a space suggests at once the question, What type of space is appropriate? The discussion of space by Wright makes no mention regarding this except, as in the above quotation, to suggest that the social space forms a continuum.[8] Since non-Euclidean spaces seem to be customarily regarded as "special" in some sense, one might conjecture that, in the absence of any qualifying remark, Wright regarded the space of nations to be Euclidean. The Euclidean concept of space informs many of the statistical measures employed in quantitative social analysis (Harman, 1967, Chpt. 4; Guttman, 1968, p. 474). Rummel, in his own work with social-spatial ideas, gives explicitly the Euclidean formula for the magnitude of vectors in the space (Rummel, 1965, pp. 199, 202). Since the Pearson product moment correlation, r, can be interpreted as the angle between two vectors in an Euclidean space (Harman, 1967, pp. 59-60), Rummel chooses to regard various statistical procedures

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[8] By "continuum" is meant here that between any two arbitrary points in the space there is located a third point.

based[9] on r as reflecting various vector relations in a social space (1965, p. 188). In the present discussion it will be assumed that the social space forms a continuum. The system is constructed in a manner which does not, however, require that the space be Euclidean. The analytic system of physics also treats the geometry of (physical) space in a manner which does not require that the space be Euclidean, and the system developed here adapts a portion thereof. It will be shown that the above behavior law posed by Rummel can be obtained independently from whether the space is Euclidean. The analysis considers the question, What would the world of social objects and relations look like, in order that a particular type of space should be its appropriate geometric representation? A "Minkowski" space is then assumed to obtain the modified behavior law, eq. (4.15).

3. The possibility that mechanical- (physical-) like laws describe a portion of social behavior has been raised and skeptically examined by others (Deutsch, 1963, pp. 26-45; Luterbacher, 1973, pp. 1-2; Boulding, 1963, pp. 22-24) but not (to the knowledge of this writer) in a manner which explicitly considered the analytic system employed in physics. The inquiry on which this paper is based was partly motivated by the belief that a reliable critique of the value of a "social physics" framework must include such an explicit examination. This paper may facilitate such a critique since it explores the use of a portion of that analytic system in a model of social behavior.

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[9] viz., canonical analysis and principle component analysis. See Anderson, 1958, Chpts. 11 and 12.

Specifically, it is shown that a modification of the physical concept of the vector components of electrodynamic force can be interpreted as the directed behavior from a given actor nation to various object nations.

4. The method to be followed may help to improve the basic program of study of world politics by improving the clarity of a few of the ideas (including those cited above) in that program.

It is frequently noted (e.g. Levy, 1969, p. 93; Singer, 1965, pp. 5-6) that most current theories of international behavior fail to provide propositions that are operational. What seems to be less frequently noticed is that the logical status of the concepts themselves, i.e. whether basic or derived, whether postulated or deduced, whether mutually consistent or not, is usually unclear in current theory.[10] This means that even when individual elements of a particular theory are empirically interpreted and tested the implications for the theory as a whole are likely to be unclear.[11] That such implications are unclear makes it difficult to decide, on the basis of empirical findings, when a theory is holding up well enough to merit further attention.

Another related problem is that there are, at the present

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[10]The arms race theory of Richardson (1960) and the conflict of interest measure derived by Axelrod (1967) are two of the occasional exceptions to this remark. The above observation is also offered by Rummel (1965, pp. 183-4) and by Levy (1969).

[11]Singer and Small, for example, seem to recognize this limitation, in connection with their research on alliance formation and the incidence of war versus the more vague "balance of power" concepts which inspired that research (1968, p. 285).

juncture, a number of conceptual schemes or theories of international behavior[12] none of which have any logical status relative to the others, which makes it difficult to make contact among them, i.e. to determine questions of consistency and complimentarity. The present inquiry is motivated by the desire to cast a few of these ideas in one possible mathematical framework such that their mutual consistency and complimentarity and their logical implications become capable of mechanical evaluation. This is achieved in one modest respect in Chapter V, where it is shown how a particular formulation of the social field idea might relate to a portion of Galtung's structural theory of imperialism (Galtung, 1971).

It should be noted that the purpose of the present inquiry is not, per se, to exhibit operational propositions but, to make a start on a system which implies several propositions that are potentially operational. This emphasis is properly described in the following remark, which has been taken out of its original context:

> The real content of Newton's laws is this: that the force is supposed to have some independent properties, in addition to

---

[12]The term "theory" is meant here to include any general organizing scheme, be it operational, quantitative, deductive, or not, which purports to discuss a significant portion of international behavior. Within that loose a meaning, it is easy to cite many such "theories." In addition to the above field theory ideas there are, to name a few, "balance of power" (Claude, 1962, pp. 11-93; Singer and Small, 1968; Organski, 1968, pp. 272-299), "power transition" (Organski, 1968, pp. 338-376), "structural imperialism" (Galtung, 1971), and "viability" (Boulding, 1963, pp. 58-79, 227-276) theories.

> the law F = ma; but the specific independent properties that the force has were not completely described by Newton or by anybody else, and therefore the physical law F = ma is an incomplete law. It implies that if we study the mass times the accelleration and call the product the force, i.e., if we study the characteristics of force as a program of interest, then we shall find that forces have some simplicity; the law is a good program for analyzing nature, it is a suggestion that the forces will be simple. (Feynman, et. al., 1963, vol. I, Chapt. 12, pp. 1-2; emphasis in the original)

Similarly, the motivation here is to exhibit a social law which implies an interesting program of social study and which is open ended as to the specific operational meanings (i.e. "independent properties").

Although the inquiry seeks primarily a logical framework for social theory it is certainly also motived by the premise that a scientific understanding of society requires a systematic empirical basis. Consider the following statement from a recent article:

> The report at hand ... [is] designed to help us sort out some of the dominant regularities in the international system and to aid in evaluating a number of equally plausible, but logically incompatible, theoretical formulations. To be quite explicit about it, we suspect that anyone who takes a given model of war (or most other international phenomena) very seriously at this stage of the game has not looked at the referent world very carefully. Just as our colleagues in the physical and biological sciences have found that nature is full of apparent inconsistencies and paradoxes, requiring a constant interplay between theoretical schemes and empirical investigations, we believe that the complexities of war and global politics will require more than mathematical rigor and elegant logical exercises. (Singer, Bremer and Stuckey, 1972, p. 21)

The underlining is added to highlight the role which questions, of the type that abstract theory should be able to answer, play relative to the rationale of empirical inquiry. While this passage was written to justify a decision by the writers to approach their substantive question in an empirical manner, the rationale of the passage is (in

the view of this writer) equally supportive of a decision to approach the material in an abstract-mathematical manner because, as the writers acknowledge, a "constant interplay" is required between the two approaches. A decision by some investigators to focus on the one aspect will advance the theoretical understanding of politics only if other investigators choose to focus on the complimentary aspect. It is this essential complimentarity which gives the present study its relevance to the empirical study of international relations.

## II. ANALYTIC TREATMENT OF THE IDEA OF SOCIAL FORCE

The point of departure is the suggestion made by Wright (1965, pp. 1240-1260, 1484-1488) and also by Rummel (1965) that the behavior of any two social actors (e.g. nations) toward each other is a function of their mutual distance in the social space.[13] A straightforward extension of this idea would be that the behavior between any two social entities is <u>also</u> a function of their motion in the space. As mentioned above, Rummel has suggested that the spatial distance between actors gives rise to "social forces" and that the behavior of nations is the result of the magnitude and direction of these forces acting upon the nations. The corresponding assertion in the above extension of the social field idea is that the motion of social actors in the space also gives rise to forces which act upon them.

In the classical theory of physics, natural objects exhibit behavior corresponding to two kinds of force, one of which is dependent on distance, the other, on both distance and velocity. The first kind, gravitational force, corresponds (in the non-relativistic treatment of gravitation) to an interaction between two material objects which is

---

[13] Wright meant particularly "behavior" in the sense of the probability that war would occur between two nations, while, as mentioned above, Rummel applied the idea to all types of empirically observed behavior. The term "space" is used here in place of the use, by Wright, of the term "field." The latter term has a different usage, introduced in eq. (2.5), below.

inversely proportional to the square of their mutual distance. The second kind, electrodynamic force, corresponds to an interaction between electrically charged objects that is mutual distance dependent, as before, but which also depends on the motion of the objects. It was pointed out in the introduction that there are certain analogies between the social field and physical concepts. The above "extended" idea of social force is additionally analogous to the electrodynamic force in that both are velocity as well as position dependent. We choose, therefore, to juxtapose this particular combination of social metaphore and physical model and to regard social force as like electrodynamic force.[14] In the present section the formal development of the former is therefore begun in parallel to the mathematical treatment of the latter.[15] (In chronological order, the idea of the electrodynamic analogy was first considered, which then suggested the quite reasonable extension of Wright's and Rummel's idea.)

Having chosen the above approach, there is no escaping the need for a somewhat elaborate system of mathematical notation, definitions

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[14]Another reason for choosing the electrodynamic analogy is that its formulation is consistent with the Maxwell equations (Bradbury, 1968, pp. 597-599), which in turn imply the particular non-Euclidean space considered, starting in Chapter IV below. It was originally hoped that a social analog to Maxwell's equations could be devised which would lead directly to social-empirical implications, but this is not accomplished in the present paper.

[15]The development also diverges in important respects from, and is much less comprehensive than the physical analog.

and deductions in order to carry out the development which is desired. Since many readers may wish to omit some of the technical details of this development, several parts are removed from the main text and are placed, instead, in a Technical Appendix. The following table gives the contents of this appendix. The attention of the reader is directed especially to part A.4 on the summation and indexing conventions adhered to, concerning the meaning of suffixes (i.e. subscripts and superscripts) which occur twice in any one term of an equation.

We are now ready to proceed with the formal discussion of social force. Imagine that each nation in the world possess a definite "position" in an N-dimensional space relative to some reference frame S. Let $x^\eta$ denote the components of this position for an arbitrary nation and assume that each $x^\eta$ comprises a contravariant vector in the space.[16] Imagine also that the position components of each object may be changing with time as a function of a parameter $\tau$. Assume that these functions are each continuous and differentiable as many times as desired. Call the derivatives $v^\eta = dx^\eta/d\tau$ the "velocity relative to S" and call the derivatives $a^\eta = dv^\eta/d\tau = d^2x^\eta/d\tau^2$ the "accelleration relative to S" of the above arbitrary nation. The "force" acting on a nation is defined as $f^\eta = a^\eta$ times a constant which has a particular value for each nation. Since the components $x^\eta$ form a vector in the space so also[17] do the components $v^\eta$, $a^\eta$ and $f^\eta$.

---

[16] See Technical Appendix B.2.

[17] See Technical Appendix C.2.

A. Notational Conventions

1. Kronecker delta, $\delta_{\eta\epsilon}$
2. Gradient, $\partial_{\eta} A_{\epsilon}$
3. nth order quantity, $a^{\sigma\eta \ldots \epsilon}_{\alpha\beta \ldots \gamma}$
4. Summation and indexing conventions
5. Transformation
6. Inverse transformation

B. Definitions and Elementary Properties

1. Contravariant tensor, $a^{\alpha\beta\epsilon \ldots \delta}$
2. Contravariant vector, $a^{\alpha}$
3. Scalar invariant, s
4. Metric tensor of a reference frame, $g_{\epsilon\eta}$
5. Covariant tensor, $a_{\alpha\beta\epsilon \ldots \gamma}$
6. Mixed tensor, $a^{\sigma\eta \ldots \epsilon}_{\alpha\beta \ldots \gamma}$
7. Inner product of two vectors, $g_{\eta\epsilon} a^{\eta} b^{\epsilon}$
8. Mutual orthogonality of two vectors, $g_{\eta\epsilon} a^{\eta} b^{\epsilon} = 0$
9. Lowering of suffixes

C. Miscellaneous Derived Results

1. Orthogonality of force and velocity
2. Transformation law of velocity and acceleration
3. Absolute invariance of velocity
4. Characteristics of the improper reference frame $\varphi$
5. Characteristics of the spatial components $\Delta x^{k}$ of the displacement vector of a nation

Table 1. Contents of Technical Appendix

The immediate purpose of the above conventions and definitions is to display the formal connection between the present deductive system and the deductive system of physics which we seek to adopt. The quantities $x^\eta$, $v^\eta$, $a^\eta$ and $f^\eta$ are called "position," "velocity," and so on because that is what they would mean in a physical interpretation of the developments which follow.[18]

Imagine now that, for some quantity $L$ which is a function of the $x^\eta$ and $v^\eta$, the path $\ell$ of the arbitrary nation as it moves through the space is constrained by the condition that the integral

$$\int L \, d\tau, \quad \tau \text{ a real valued parameter}$$

evaluated over any segment of $\ell$, shall be an extremum[19] of all possible paths connecting the end points of the actual segment. This condition is schematically illustrated in Figure 1.

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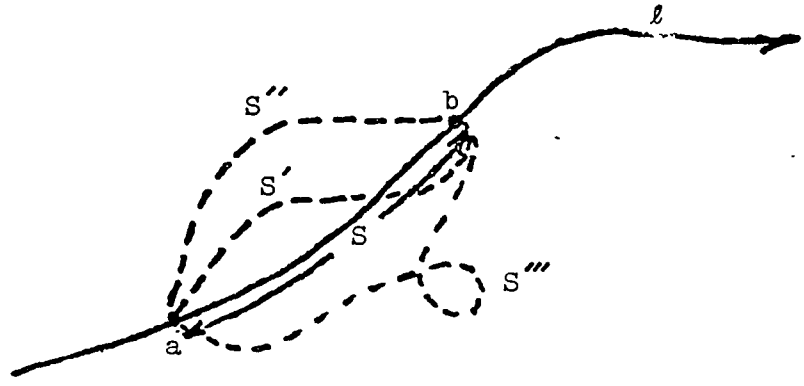
[18] Thus we do not at once fix an empirical meaning to the above quantities; their possible social or political interpretations are speculated upon in due course but the immediate preoccupation is to construct an abstract framework which will serve as a context for such speculations.

[19] i.e. a minimum or maximum.




Either 1) $\int_S L\, d\tau \geq \int_{S'} L\, d\tau, \int_{S''} L\, d\tau, \int_{S'''} L\, d\tau, \ldots$

or 2) $\int_S L\, d\tau \leq \int_{S'} L\, d\tau, \int_{S''} L\, d\tau, \int_{S'''} L\, d\tau, \ldots,$

where a and b are arbitrary points on $\ell$.

Figure 1: Defining Constraint on the Path of Objects in the Nation Space

This constraint implies (Weinstock, 1952, pp. 20-24) that

$$\frac{d}{d\tau} \frac{\partial L}{\partial v^\eta} - \frac{\partial L}{\partial x^\eta} = 0 \qquad (2.1)$$

The particular force law affecting the object is determined by stipulating the function L and L is then called the Lagrangian of that force law. One formulation of the Lagrangian in the case of electrodynamic force is

$$L = -\tfrac{1}{2}\, mg_{\eta\sigma} v^{\eta} v^{\sigma} + kg_{\eta\sigma} v^{\eta} A^{\sigma} \ ,$$

$$A^{\sigma} = A^{\sigma}(x^{\epsilon}) \qquad (2.2)$$

where $m$ and $k$ are constants particular to the object, the $A^{\sigma}$ are functions of the spatial position of the object, and $g_{\eta\sigma}$ is the metric[20] of the particular reference frame $S$. Since we are approaching the idea of social force in analogy to electrodynamic force, we choose to postulate that the Lagrangian of the social force is given by eq. (2.2).[21] By lowering of indices[22], eq. (2.2) can also be written as

$$L = -\tfrac{1}{2}\, mv^{\eta} v_{\eta} + kv^{\eta} A_{\eta} \qquad (2.3)$$

Let $A_{\eta}$ be called the "covariant vector potential" or just "covariant potential" of the space. Substitution of $L$ from eq. (2.3) into (2.1) shows (Bergmann, 1942, p. 119) that

$$mdv_{\eta}/d\tau - kv^{\epsilon}(\partial_{\eta} A_{\epsilon} - \partial_{\epsilon} A_{\eta}) = 0$$

or

$$mdv_{\eta}/d\tau = ma_{\eta} = kv^{\epsilon}(\partial_{\eta} A_{\epsilon} - \partial_{\epsilon} A_{\eta}) \qquad (2.4)$$

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[20] See Technical Appendix B.4. The clearest discussion of the force concept discussed here was found in Feynman, et. al. (1964, Vol. II, especially Chpts. 19, 25 and 26).

[21] The possibility of a Lagrangian for social behavior is also suggested by Luterbacher (1973, pp. 6-7).

[22] See Technical Appendix B.9.

Since $m$ is particular to the nation having the path $\ell$ in the space, the expression $ma_\eta$ is of the form of a "force" acting on that nation. Thus we choose to define $f^\sigma = ma^\sigma$, so that in (2.4) the term $ma_\eta = mg_{\eta\sigma}a^\sigma = g_{\eta\sigma}ma^\sigma = g_{\eta\sigma}f^\sigma = f_\eta$, where the $f^\sigma$ are the (contravariant) force components relative to S. Then eq. (2.4) has the form of the equation of the covariant social force, expressed as a function the velocity of the nation and the covariant potential $A_\epsilon = g_{\epsilon\sigma}A^\sigma$.

Let the expression

$$F_{\epsilon\eta} = \partial_\eta A_\epsilon - \partial_\epsilon A_\eta \tag{2.5}$$

be called the "field tensor" or "field" of the space. Then eq. (2.4) can be written as the more compact expression

$$f_\eta = kv^\epsilon F_{\epsilon\eta} \tag{2.6}$$

This expresses the force acting on a nation as a function of the velocity of the nation and of the social field. From eq. (2.4) it is evident that $F_{\epsilon\eta}$ is anti-symmetric, i.e.

$$\begin{aligned} F_{\epsilon\eta} &= \partial_\eta A_\epsilon - \partial_\epsilon A_\eta \\ &= -(\partial_\epsilon A_\eta - \partial_\eta A_\epsilon) \\ &= -F_{\eta\epsilon} \end{aligned} \tag{2.7}$$

This implies[23] the important results that

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[23] See Technical Appendix C.1.

$$v^{\eta} f_{\eta} = 0 \tag{2.8}$$

i.e. force and velocity are orthogonal, and that

$$v^{\eta} a_{\eta} = 0 \tag{2.9}$$

i.e. accelleration and velocity are orthogonal.

We now proceed to define a class of quantities which are like the transforms of tensors in S to a different reference frame, except that the "dimensionality" of this new frame is greater than the dimensionality of the nation space. Let this new "improper"[24] frame be denoted by $\varphi$ and assume that there are $M + 1 \geq N + 1$ many nations, where $N + 1$ is the dimensionality of the space.[25] Then $\varphi$ is defined as the class of quantities

$$\bar{a}_{\delta\sigma \ldots \eta} = v^{\alpha}_{\delta} v^{\beta}_{\sigma} \cdots v^{\epsilon}_{\eta} a_{\alpha\beta \ldots \epsilon},$$

$$\delta, \sigma, \ldots, \eta = 0, 1, \ldots, M \tag{2.10}$$

where the $a_{\alpha\beta \ldots \epsilon}$ are the components of an arbitrary covariant tensor relative to S, and $v^{\alpha}_{\delta}$, etc. denote the $\alpha$th contravariant component of the velocity of the $\delta$th nation, relative to S. In other words $\varphi$ is the "reference frame" into which quantities in S are transformed by the $v^{\alpha}_{\delta}$ and the "dimensionality" of $\varphi$ equals the number of nations in the space. Any quantity like $\bar{a}_{\delta\sigma \ldots \eta}$

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[24] improper because the higher dimensionality of the new space implies that the transformation has no inverse.

[25] i.e. $N + 1$ = the number of components of vectors in S.

as defined in eq. (2.10) shall be referred to as a covariant tensor in $\varphi$. We now make the following assumption about $\varphi$: Let $a^{\delta\sigma\ldots\eta}$ be an arbitrary contravariant tensor relative to S. Then we assume that there exist quantities $\bar{a}^{\alpha\beta\ldots\epsilon}$ such that

$$a^{\delta\sigma\ldots\eta} = v^{\delta}_{\alpha} v^{\sigma}_{\beta} \ldots v^{\eta}_{\epsilon} \bar{a}^{\alpha\beta\ldots\epsilon},$$
$$\alpha,\beta,\ldots,\epsilon = 0,1,\ldots,M . \tag{2.11}$$

Any quantity of the type $\bar{a}^{\alpha\beta\ldots\epsilon}$ shall be referred to as a contravariant tensor relative to $\varphi$.

The following results can be shown[26] for forces and velocities in $\varphi$: First, let $\bar{v}^{\epsilon}_{\sigma}$ denote the $\sigma$th components in $\varphi$ of the velocity of the $\epsilon$th nation. Then

$$\bar{v}^{\epsilon}_{\sigma} = \begin{cases} 1, & \sigma = \epsilon \\ 0, & \sigma \neq \epsilon; \ \sigma, \epsilon = 0,1,\ldots,M \end{cases} \tag{2.12}$$

i.e. all but the $\epsilon$th component vanish and the $\epsilon$th component = 1. Thus, by construction, the $M + 1$ velocities, each of a nation of the space, are parallel, respectively to the $M + 1$ axes of $\varphi$ and constitute the unit basis vectors of $\varphi$ (see Figure 2). Second, the force equation (2.6) derived above in terms of components in S also holds in $\varphi$, i.e.

$$\bar{f}_{\beta} = k\bar{v}^{\alpha}\bar{F}_{\beta\alpha}, \qquad \beta,\alpha = 0,1,\ldots,M \tag{2.13}$$

where $\bar{f}_{\beta}$, $\bar{v}^{\alpha}$ and $\bar{F}_{\beta\alpha}$ are respectively the force acting on, the velocity of, and the condition of the social field surrounding an

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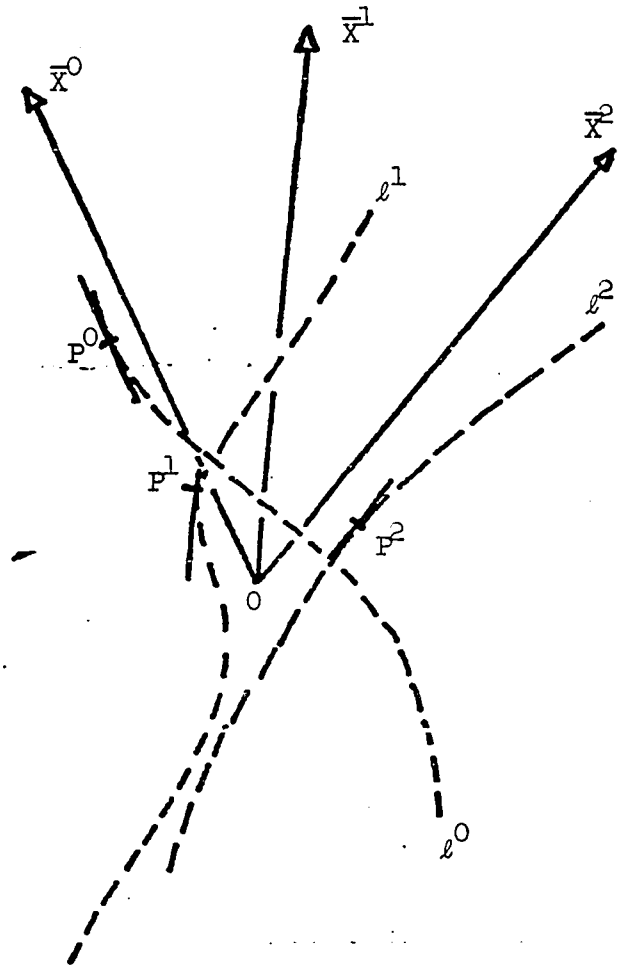
[26] See Technical Appendix C.4.




Legend: Axes $\bar{X}^0$, $\bar{X}^1$ and $\bar{X}^2$ of $\varphi$ are parallel to lines tangent to the paths $\ell^0$, $\ell^1$ and $\ell^2$ of nations 0, 1, and 2 at points $P^0$, $P^1$ and $P^2$, respectively. The origin of $\varphi$ is at 0.

Figure 2

Illustration of the Orientation of the Reference Frame $\varphi$ for the Case of Three Dimensions

arbitrary nation. Third, for an arbitrary nation

$$\overline{f}_{\beta}\overline{v}^{\beta} = \overline{f}^{\sigma}\overline{v}_{\sigma} = 0 \ , \qquad (2.14)$$

$$\beta,\sigma = 0,1,\ldots,M$$

i.e. force and velocity are orthogonal in $\varphi$. Fourth,

$$\overset{\epsilon}{\overline{f}}_{\epsilon} = 0, \quad \epsilon = 0,1,\ldots,M \qquad (2.15)$$

i.e. the $\epsilon$th covariant component of force on the $\epsilon$th (arbitrary) nation vanishes.

## III. CHARACTERISTICS OF NATIONS IMPLIED BY THE TRANSFORMATION LAW FROM S TO $\varphi$

Up to this juncture the discussion has focused solely on the deductive system as an abstraction. No suggestion has yet been offered of the substantive (i.e. empirical) meaning of the tensor quantities which appear in the system. We will now show that certain different, alternate interpretations of this single system have two differing but related predictions of regularities in the behavior of nations. The first alternative interpretation is discussed in the present Chapter.[27]

There are two very inclusive classes of indicators commonly employed to quantitatively represent international relations. Indicators of the first class are said to index the "attributes" of particular social actors (e.g. nations). Indicators of the second class are said to index the "directed behavior" of pairs of actors, in which one of the pair is the initiator and the other is the receiver or "object" of the behavior.[28] We now choose to represent

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[27] It should be emphasized that the possible meanings to be attached to the various tensor quantities are not considered to be exhausted by the present, or any other treatment. At many points in the conception of this paper the writer considered various alternate interpretations which were discarded because they seemed less productive. However, there is no reason in principle to exclude additional interpretations which might be subsequently suggested, and indeed they are desirable as discussed above in Chapter I.

[28] This terminology is also employed by Rummel.

sets of attribute and directed behavior data as follows: 1) $v^{\eta}_{\epsilon}$, $\epsilon = 0,\ldots,M$ shall be regarded as the $\eta$th attribute of the $\epsilon$th nation; 2) $\overset{\epsilon}{f}_{\eta}$, $\epsilon,\eta = 0,\ldots,M$ and $\epsilon \neq \eta$ shall be regarded as the behavior directed from the $\epsilon$th actor to the $\eta$th object. That is, the attributes of a particular nation are to be regarded as the components of its velocity relative to S and the behaviors directed by a particular nation actor to various objects are to be regarded as the components of the force acting on the actor relative to $\varphi$. The restriction $\eta \neq \epsilon$ in 2) reflects the ambiguity concerning what is the empirical meaning of "behavior directed by a nation to itself."

Notice that the above interpretation of velocity components in S does not specify which attributes--demographic, economic, geographic, political, or whatever--are to be associated with components of velocity in S. Notice, also, that only one directed behavior variable is regarded as the force in $\varphi$ on a nation and, again, the specific identity of that variable is not foreclosed. While those specifics clearly are necessary at the point of empirical inquiry, the theory which is being constructed does not at this point distinguish among various attributes and behaviors and that issue shall be momentarily passed by.

By application of eq. (2.10)

$$\overset{\epsilon}{f}_{\kappa} = v^{\eta}_{\kappa}\overset{\epsilon}{f}_{\eta} \tag{3.1}$$

This equality is empirically interpreted for all cases except $\epsilon = \kappa$ as the assertion that behavior from nation $\epsilon$ to nation $\kappa$ is a

linear combination of the $N + 1$ attributes $v_\kappa^0, v_\kappa^1, \ldots, v_\kappa^N$ of $\kappa$.[29] For the case $\epsilon = \kappa$ eq. (2.15) shows that $\overset{\epsilon}{f}_\kappa = 0$ in consequence of the orthogonality between the velocity of, and the force acting on $\epsilon$.

Suppose the nations to be labeled so that $\epsilon = 0$. Then $\overset{o}{f}_o = 0$. From eq. (3.1)

$$\overset{o}{f}_i = v_i^\eta \overset{o}{f}_\eta \tag{3.2}$$

Define $\Delta\overset{\epsilon}{f}_\kappa$ and $\overset{\epsilon}{\Delta v}_\kappa^\eta$ to be the mean deviation transforms obtained by subtracting the arithmetic means $1/M \sum_{\kappa=0}^{M} \overset{\epsilon}{f}_\kappa$ and $1/M \sum_{\kappa=0}^{M} v_\kappa^\eta$, $\kappa \neq \epsilon$ from $\overset{\epsilon}{f}_\kappa$ and the $v_\kappa^\eta$, respectively. Then

$$\left.\begin{aligned} 1/M \sum_{\kappa=0}^{M} \overset{\epsilon}{\Delta v}_\kappa^\eta &= 0 \\ 1/M \sum_{\kappa=0}^{M} \Delta\overset{\epsilon}{f}_\kappa &= 0 \end{aligned}\right\}, \quad \kappa \neq \epsilon \tag{3.3}$$

From (3.2) it follows that

$$\Delta\overset{o}{f}_i = \overset{o}{\Delta v}_i^\eta \overset{o}{f}_\eta \tag{3.4}$$

Define the quantity

$$\overset{\epsilon}{c}_{\sigma\eta} = \overset{\epsilon}{\Delta v}_\kappa^\sigma \overset{\epsilon}{\Delta v}_\kappa^\eta, \quad \kappa \neq \epsilon \tag{3.5}$$

and let $\overset{\epsilon,}{c}_{\lambda\sigma}$ denote the inverse (if it exists) of $\overset{\epsilon}{c}_{\sigma\eta}$, i.e. define $\overset{\epsilon,}{c}_{\lambda\sigma}$ by

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[29] Herein, "$\kappa$" will mean "the $\kappa$th nation."

$$\overset{\epsilon}{c}{}'_{\lambda\sigma}\overset{\epsilon}{c}_{\sigma\eta} = \delta_{\lambda\eta} .$$

Suppose the following assumption holds[30]: that the determinant

$$|\overset{\epsilon}{c}_{\sigma\eta}| \neq 0 \qquad (3.6)$$

Given this assumption the quantity $\overset{\epsilon}{c}{}'_{\lambda\sigma}$ is well defined and eq. (3.4) can be solved[31] for $\overset{o}{f}_{\lambda}$ as

$$\overset{o}{f}_{\lambda} = \overset{o}{c}{}'_{\lambda\sigma}\overset{o}{\Delta}v_i^{\sigma}\overset{o}{\Delta f}_i \qquad (3.7)$$

Since the selection of the oth nation was arbitrary, relabeling of nations shows that for any actor nation $\epsilon$,

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[30] This is equivalent to assuming that the $v_{\kappa}^{\sigma}$ are linearly independent in $\kappa \neq \epsilon$ for all differing pared values of $\sigma$.

[31] Multiplication of both sides of eq. (3.4) by $\overset{o}{\Delta}v_i$ (and, according to the convention, summing on i) shows

$$\overset{o}{\Delta}v_i^{\sigma}\overset{o}{\Delta f}_i = \overset{o}{\Delta}v_i^{\sigma}\overset{o}{\Delta}v_i^{\eta}\overset{o}{f}_{\eta} = \overset{o}{c}_{\sigma\eta}\overset{o}{f}_{\eta} .$$

Multiplication of both sides of the preceeding by the inverse element $\overset{o}{c}{}'_{\lambda\sigma}$ shows

$$\overset{o}{c}{}'_{\lambda\sigma}\overset{o}{\Delta}v_i^{\sigma}\overset{o}{\Delta f}_i = \overset{o}{c}{}'_{\lambda\sigma}\overset{o}{c}_{\sigma\eta}\overset{o}{f}_{\eta} = \delta_{\lambda\eta}\overset{o}{f}_{\eta} = \overset{o}{f}_{\lambda} ;$$

the first and last members of which appear in eq. (3.7).

$$\overset{\epsilon}{f}_{\lambda} = \overset{\epsilon}{c}'_{\lambda\sigma} \overset{\epsilon}{\Delta v}_{\kappa} \sigma \overset{\epsilon}{\Delta f}_{\kappa}, \quad \kappa \neq \epsilon \tag{3.8}$$

This shows that, except for omission of an error term $e_{\kappa}$ on the right, eq. (3.4) is like the customary equation of least squares multiple regression (Johnston, 1963, pp. 106-115) in which, for each value of $\epsilon$, the $\overset{\epsilon}{f}_{\lambda}$ can be regarded as functioning like "$\beta$-weights," selected such as to minimize the error variance

$$1/M \sum_{\kappa=0}^{M} e_{\kappa}^{2}, \quad \kappa \neq \epsilon .$$

The conclusion is thus that, given the linear independence of attributes across nations, eq. (3.1) may be regarded as implying $M + 1$ linear multivariate relationships, one for each actor $\epsilon$, between the behavior directed toward various objects and the respective attributes of the objects.

This conclusion bears a close resemblance to some of the speculations of Rummel. The equality (3.1), given the indicated empirical interpretation, becomes somewhat like what Gleditsch (1969, pp. 13-19) and Rummel (1969, pp. 4, 7-8) have called "attribute theory," viz. the idea that national behavior is a linear function of the attributes of the acting nation. Eq. (3.1) differs, however from this attribute theory (particularly as formulated by Rummel), in the following ways. First, behavior is related to the characteristics of the object rather than those of the actor. Second, the equality asserts that directed behavior $\overset{\epsilon}{f}_{\kappa}$, rather than "total behavior"

$$\sum_{\kappa=0}^{M} \overset{\epsilon}{f}_{\kappa}, \quad \kappa \neq \epsilon$$

as in attribute theory, is connected to nation attributes. Third, the $\beta$-weights of the relationship are specific to the actor $\epsilon$, rather than general to all actors as in attribute theory.

The closest similarity, however, is between eq. (3.4) and Rummel's "field theory Model II" in which he suggested (1969, pp. 3, 16) that the directed behavior of social objects is a linear function of the differences between the attribute values of actor and object, with $\beta$-weights particular to each actor. Using the first of eqs. (3.3), $\overset{\epsilon}{\Delta}v^{\eta}_{\kappa}$ can be written as[32]

$$\overset{\epsilon}{\Delta}v^{\eta}_{\kappa} = (\overset{\epsilon}{\Delta}v^{\eta}_{\kappa} - \overset{\epsilon}{\Delta}v^{\eta}_{\epsilon}) - 1/M \sum_{\kappa=0}^{M} (\overset{\epsilon}{\Delta}v^{\eta}_{\kappa} - \overset{\epsilon}{\Delta}v^{\eta}_{\epsilon}), \quad \kappa \neq \epsilon \qquad (3.9)$$

The quantitative form of Model II is

$$w_{k\rightarrow i} = \Sigma\, \alpha_{kj} d_{ij} \qquad (3.10)$$

where $w_{k\rightarrow i}$ is the behavior directed from a given actor to various objects $i$, $d_{ij}$ is the difference between the $j$th attribute of actor and object and $\alpha_{kj}$ are scalars specific to the given actor. The actual method of testing this relationship involves only the deviations of the $d_{ij}$ from the mean values of each $j$. We can thus make the following correspondences:

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[32] This can be seen as follows: $\overset{\epsilon}{\Delta}v^{\eta}_{\kappa} = \overset{\epsilon}{\Delta}v^{\eta}_{\kappa} - 1/M \sum_{\kappa=0}^{M} \overset{\epsilon}{\Delta}v^{\eta}_{\kappa} = \overset{\epsilon}{\Delta}v^{\eta}_{\kappa} - 1/M \sum_{\kappa=0}^{M} \overset{\epsilon}{\Delta}v^{\eta}_{\kappa} + \overset{\epsilon}{\Delta}v^{\eta}_{\epsilon} - \overset{\epsilon}{\Delta}v^{\eta}_{\epsilon}$ = the right hand side of eq. (3.5), $\kappa \neq \epsilon$.

$$w_{\epsilon\to\kappa} \longleftrightarrow \Delta\bar{f}^{\epsilon}_{\kappa}$$

$$\alpha_{\epsilon\eta} \longleftrightarrow \bar{f}^{\epsilon}_{\eta}$$

$$d_{\kappa\eta} \longleftrightarrow (\Delta v^{\eta}_{\kappa} - \Delta v^{\eta}_{\epsilon}) - 1/M \sum_{\kappa=0}^{M} (\Delta v^{\eta}_{\kappa} - \Delta v^{\eta}_{\epsilon}) \qquad (3.11)$$

Equation (3.9) then shows that eq. (3.4) is algebraically equivalent to the actual method of testing (3.10), except for the self-directed behavior $\bar{f}^{\kappa}_{\kappa}$. The Model II equation still differs from eq. (3.4) because the $d_{ij}$ are the differences of factor scores from a principal components analysis of attributes rather than the raw attributes themselves. In some interpretations of Model II the $w_i$ also differ from the $\bar{f}^{\epsilon}_{\kappa}$ in that $w_i$ is one of the variates of the canonical regression of a sequence of behavior variables on the $d_i$. These differences would be eliminated had we chosen to regard $v^{\eta}_{\kappa}$ and $\bar{f}^{\epsilon}_{\kappa}$ as the above transforms of attribute and behavior variables rather than as the raw variables. Thus we have shown that the form of the Model II equation of Rummel is contained in the eq. (2.10) from which eq. (3.4) was derived.

Recalling the earlier observation that the precise identity of the attributes and behavior data was not clear, the above remarks suggest that one criterion for identifying that data is simply to find by trial and error, those sets of data which most closely fit

the equality (3.1).[33]

We conclude this section by stating the essential points which have emerged. First, the Model II equation of Rummel is contained in a system with the following axioms

1) the covariant transformation from a reference frame $S$ to the "improper" reference frame $\varphi$ is as given by eq. (2.10);
2) national attributes, or suitable linear transformations thereof, form the contravariant components in $S$ of the velocities of nations;
3) directed behaviors, or suitable linear transformations thereof, form the covariant components in $\varphi$ of the forces exerted on actor nations;
4) the determinants $|\overset{\epsilon}{c}_{\sigma\eta}|$ are non-vanishing, as in eq. (3.6);
5) for an arbitrary tensor $a^{\delta\sigma\ldots\eta}$ in $S$ there exists a quantity $\bar{a}^{\alpha\beta\ldots\epsilon}$ such that

$$a^{\delta\sigma\ldots\eta} = v^{\delta}_{\alpha} v^{\sigma}_{\beta} \ldots v^{\eta}_{\epsilon} \bar{a}^{\alpha\beta\ldots\epsilon} ,$$

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[33] This is the criterion suggested by Wright (1955, p. 543). Again, that criterion clearly does not relieve the need for substantive precision. A complete theory of international relations would stipulate which variables are the ones which satisfy such a fit. The criterion does suggest an empirical method of resolving the question, however, which is the one actually employed so far by field theoretic researchers. Let the reader be momentarily contented, then, by the remark that whichever variables have been found in prior work to best satisfy Rummel's eq. (3.10) shall also be regarded as the variables of the present eq. (3.1).

where $v_{\alpha}^{\delta}$ is the $\delta$th velocity component in S of the $\alpha$th nation; the $\bar{a}^{\alpha\beta\ldots\epsilon}$ are called the "contravariant" components[34] in $\varphi$ of the tensor $a^{\delta\sigma\ldots\eta}$.

6) The social force acts such that the path of a nation in the space is constrained by the condition $\int L \, d\tau$ = an extremum, where L is given by eq. (2.2), and $\tau$ is a real valued parameter;

7) The metric $g_{\eta\epsilon}$ of S is symmetric.

In the above, axioms 5) and 6) serve only to imply that self-directed behavior is identically zero. The "field potential" vector $A_{\eta}$ which occurs in eq. (2.5) is not explicitly related to the spatial position of nations, nor is $A_{\eta}$ itself given empirical meaning. The importance of axiom 6) is not, then, that Rummel's Model II equation is in some sense a test of the force equation (2.2); the latter is not empirically tested at all. The importance of axiom 6) is rather that the version of field theory represented by the Model II equation is completely consistent with the "physical" idea of social force which eq. (2.2) represents. That is the second essential point which emerges from this analysis.

A third point is that in this analysis the attributes of nations are represented as velocity components rather than, as in Model II, position components of nations. This point is significant in that the heuristic value of calling attributes "velocities" may be greater than the heuristic value of calling them "positions." Once scholars have

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[34] As before, the quotation marks are because $\varphi$ does not completely satisfy the definitional requirements of a reference frame for the space.

developed an operational theory of international relations which agrees with empirical observation and which connects many relevant phenomena together, it will not matter what names the various constructs are given. The results will be the same. At the present juncture, however, it seems to this writer that names are useful for suggesting additional properties to be alert for. The above analysis shows that it may be useful to think of attributes as being like velocity components rather than position components. Similarly, the analysis shows that it may be useful to treat directed behavior like force components relative to the "improper" reference frame $\varphi$.

The fourth conclusion which is drawn is that the Model II equation holds independently of whether the social space is Euclidean. This can be seen by rewriting eq. (3.1) in the equivalent form

$$\overset{\epsilon}{f}_{\kappa} = g_{\eta\sigma} v_{\kappa}^{\eta} \overset{\epsilon}{f}^{\sigma} \qquad (3.1a)$$

from which eq. (3.1) was implicitly obtained by lowering the suffix[35] of $\overset{\epsilon}{f}$. In the above equality the metric $g_{\eta\sigma}$ defines the space in the sense that its symmetry[36] implies the existence of a reference frame $S'$ the metric $g'_{\eta\sigma}$ of which is in diagonal form (Bradbury, 1968, pp. 105-107). If $g'_{\eta\sigma} = \delta_{\eta\sigma}$ then, for a displacement vector $\Delta x^{\eta}$,

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[35] See Technical Appendix B.9.

[36] See Technical Appendix B.4, eq. (2a).

$$g_{\eta\sigma}\Delta x^{\eta}\Delta x^{\sigma} = g'_{\kappa\lambda}\Delta x'^{\kappa}\Delta x'^{\lambda}$$

$$= \delta_{\kappa\lambda}\Delta x'^{\kappa}\Delta x'^{\lambda}$$

$$= (\Delta x'^{0})^2 + (\Delta x'^{1})^2 + \cdots + (\Delta x'^{N})^2 \qquad (3.12)$$

which shows that $S'$ corresponds to a Cartesian reference frame in which the length of the displacement is given by Pathagoras' theorem, i.e. that the space is Euclidean. In general, the diagonal elements of $g'_{\eta\sigma}$ do not equal unity, but may each assume any positive or negative (or zero) value. Thus the admissible type of space is more general than Euclidean[37] and eq. (3.1) does not depend on the Euclidicity of the space. As mentioned in Chapter I, it has been customary in discussions of social field theory to regard the social space as Euclidean. This view was conceptually simple and, in addition, it was the conception of space which usually informs the various statistical methods employed in social inquiry. What has just been shown, however, is that it is unnecessary, at least to the particular behavior-attributes model discussed above.

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[37] Bergmann (1942, p. 164) calls this type "flat space."

## IV. DEVELOPMENT OF A MINKOWSKI METRIC FOR THE SPACE OF NATIONS

It has just been shown that one is free, within wide latitude, to impose at will a metric of one's own choosing on the social space, without contradicting the particular behavior law, eq. (3.1), which was derived. One such possibility is the "Minkowski metric"[38] given by

$$g_{\eta\sigma} = \begin{cases} 1, & \eta = \sigma = 0 \\ -\delta_{\eta\sigma}, & \eta > 0 \quad \text{or} \quad \sigma > 0 \end{cases} \tag{4.1}$$

One question which can be posed is, What would the world of social objects and relations look like, in order that some particular type of space should be its appropriate geometric representation? We choose now to address that question concerning a space having the Minkowski metric of the preceeding equation. In this paper we have undertaken to investigate an interpretation of social, particularly international, relations which would incorporate some of the elements of Wright's social field program but would also give those elements a more exact logical status as components analogous to those of a

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[38] Eq. (4.1) is an extension to $N + 1$ dimensions of the four dimensional metric given by Bradbury (1968, p. 576). The name of Minkowski is associated with an equivalent formulation of the metric, in four dimensions, by Bergmann (1942, pp. 76-81). This metric should not be confused with the distinctly different metric given the same name in discussions of smallest space analysis. See Guttman (1968, p. 475).

mechanical system. The reason for choosing the Minkowski metric is that it also applies in the mechanical analogy, and it is desired to thereby strengthen that analogy.

A consequence of choosing the metric of eq. (4.1) is that some doubt is created as to whether the $N + 1$ components $v_{\kappa}^{\eta}$ for each $\kappa$ of eq. (3.1) should be regarded as the attributes of $\kappa$. To begin with, it is evident from eq. (4.1) that the oth component is set off from the others by the different sign of its corresponding metric component. If the left hand side of eq. (3.12) is written with the aid of eq. (4.1) it becomes, for a displacement vector $\Delta x^{\eta}$

$$\Delta s^2 = g_{\eta\sigma}\Delta x^{\eta}\Delta x^{\sigma} = (\Delta x^0)^2 - (\Delta x^1)^2 - \cdots - (\Delta x^N)^2 \qquad (4.2)$$

which shows the effect of this difference in forming the scalar invariant. Imagine an object to be undergoing motion in the space where $\Delta x^{\eta}$ denote the components of displacement while the object moves over a distance of length $\Delta s$. Then the above equality describes the relationship between the distance and displacement components of that object. It can be shown[39] that, for the path of an object moving according to the Lagrangian of eq. (2.2), $s$ and $\tau$, the parameter of which the position components are a function in eq. (2.1), are related by

$$s = c\tau ,$$

where $c$ is a real valued constant. Since $\tau$ is assumed to be real valued $s$, and therefore an increment $\Delta s$ due to displacement, must

[39] See Technical Appendix C.3.

also be real valued. Therefore

$$(\Delta x^0)^2 - (\Delta x^1)^2 - \cdots - (\Delta x^N)^2 = \Delta s^2 \geq 0$$

which shows

$$(\Delta x^0)^2 \geq \sum_{i=1}^{N} (\Delta x^i)^2 \qquad (4.2a)$$

Therefore

$$(v^0)^2 = (dx^0/d\tau)^2 \geq \sum_{i=1}^{N} (dx^i/d\tau)^2 = \sum_{i=1}^{N} (v^i)^2 \qquad (4.3)$$

In words, the assumption of the Minkowski metric leads to the requirement that one of the velocity components play the special role that its squared value equal or exceed the sum of the squares of the other components (see Figure 3). If the earlier interpretation of velocity as nation attributes was to be retained, then one particular attribute would need to be singled out as having this role[40]. An alternative, which we choose to follow, is to reinterpret "velocity" in such a way that there is a fundamental distinction between one component and the others corresponding to the relationship of eq. (4.3). It would be desirable that this also be done in a way which

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[40]This is a problem in the reasonableness of the interpretation of velocity, not a logical problem. As pointed out in Chapter III, it would be entirely consistent to assume that the metric of eq. (4.1) applies to velocity components regarded as attributes, but to do so would be to assume an empirical relationship among national attributes which seems, to this writer, unfounded.

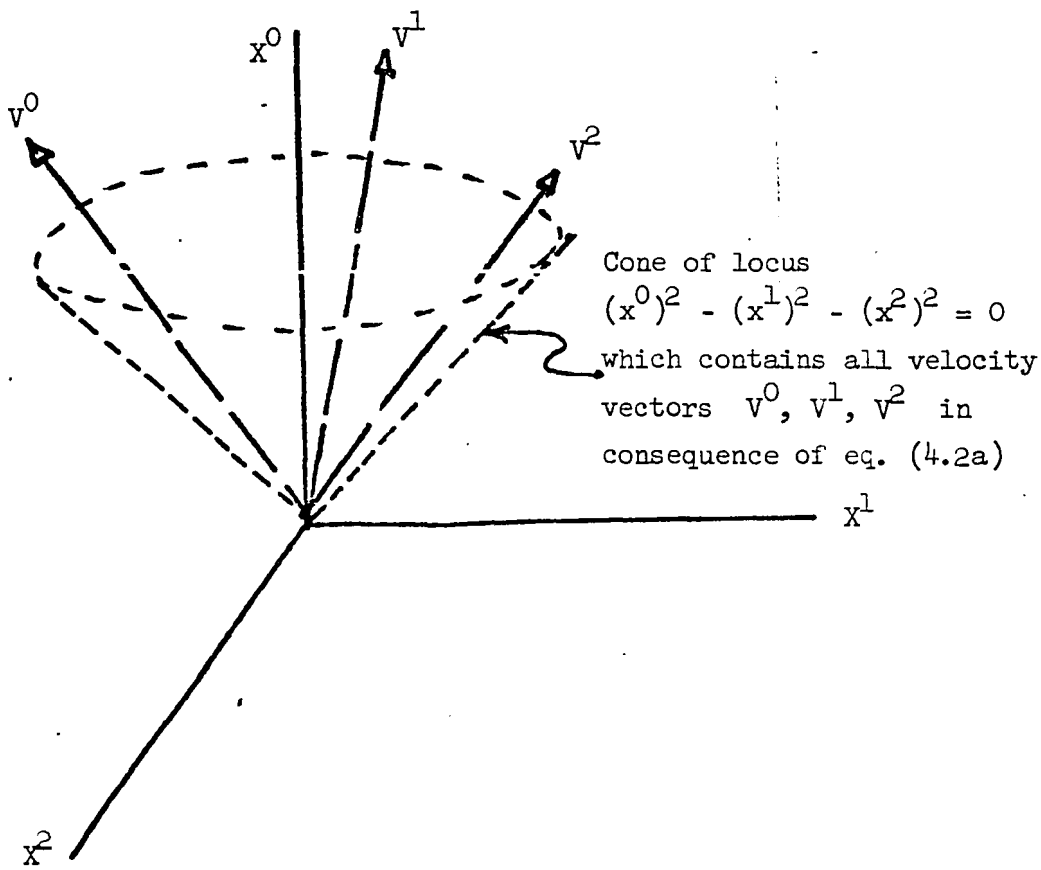


Figure 3

Illustration of the Constraint on Nation Velocities, Due to the Minkowski Metric, for the Case of Three Dimensions with Orthogonal Reference Axes $x^0$, $x^1$ and $x^2$

contains as a special case, given the proper empirical relationship, the identification of the $v_i$ with national attributes. This criterion is partially but not completely satisfied in the following development.

In the physical application of the Minkowski space the distinction between $\Delta x^0$ and the $\Delta x^i$ is the distinction of the time between the occurrence of two events and the separation between them in various spatial directions, provided that $\Delta x^\eta$ is a displacement vector in the space. The special character of $\Delta x^0$ is that it measures how much time has elapsed during the motion of an object, as opposed to how far the object has moved during that time. What, then, is time? Suppose that a material object is observed to repeatedly exhibit some particular discrete condition or behavior and that, insofar as observed, this repetition continues throughout the history of the object. Such a phenomenon involving a particular modal behavior may be referred to as a periodic process, and the object in question is then said to be characterized by periodic behavior of the indicated type. Suppose a series of observations are performed on an object and the presence or absence of specific instances of the indicated behavior noted. For any given observation, the number of occasions upon which the modal behavior has already been observed may serve as an index of that observation, and all such observations may be ordered according to the ascending magnitude of their index values. The existence of what has here been called modal periodic behavior is the operational embodiment

of the idea of time.[41] The presence of a periodic process in social relations is what is meant by a "behavior variable." The reader can perhaps be convinced of this by considering a particular variable, e.g. "number of domestic riots." What is presumed by use of this variable is that at some particular moment, a discrete act occurs that is differentiable from other like acts. One could reword this as 1) prior to this act, a set of background conditions is present, 2) the riot constitutes a disturbance of these prior conditions, 3) subsequent to the riot, the background conditions are restored. Consider another variable, "diplomatic messages to a foreign nation." 1) Prior to the sending of a message, one can imagine that the foreign office of the given nation is engaged in some kind of routine activity, 2) associated with the event of sending the message, there is a change from routine activity corresponding to the effort to formulate and transmit the message, 3) subsequently, the foreign office returns to the routine activity. In the abstract, it can be said that in both examples a single instance of behavior is equivalent to a particular disturbance of a set of background conditions present in the nation. A second presumption in the use of a particular behavior variable is that the behavior occurs, or at least could, in some sense occur, repeatedly for each of the objects (i.e. nations) to which it is applied. Thus, in a particular cross-nation study, the number of instances of domestic riots or of foreign diplomatic messages sent during the period of

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[41] The present conception of time was suggested by a discussion by Weyl (1923, p. 7).

interest is counted for each nation in the sample. Thus social behavior is a periodic process which can be regarded as defining the passage of time in a social sense just as a physical periodic process such as a clock defines the passage of time in a physical sense.

Let there be adopted, then, as the meaning of "social time," the convention that for each nation an index system is to be formed from some periodic social behavior (not yet specified) that is intrinsically interesting to international relations and is specific to the particular nation.[42] Let $w_t$ denote the amount of modal behavior in the tth (calendar) time period. Then $\Delta x^0$, the elapsed social time from $t = 0$ to $t = T$ shall be defined as

$$\Delta x^0 = \left( T \sum_{t=1}^{T} w_t^2 \right)^{1/2} \tag{4.4}$$

where the positive root is taken. The perceptive reader will note that this departs from the custom in physical time measurement. If a single instance of the behavior which $w_t$ measures were taken as analogous to a single tick of the clock, then the conventional procedure would be to regard total elapsed social time = $\sum_{t=1}^{T} w_t$. Instead we have regarded $\Delta x^0$ to be proportional to the root of what Blaylock (1960, p. 244) calls the "variation" of $w_t = 1/T \sum_{t=1}^{T} w_t^2$. In effect, the $\Delta x^0$ as defined by eq. (4.4) puts greater weight on

[42] The definition of "social time" about to be offered differs from the one suggested by Rummel (1970), in part because his definition does not in general satisfy the inequality (4.2a).

behavior which occurs in an unequal distribution over time, i.e. irregularly, than on the same amount of behavior when it occurs in equal distribution over time. This can be seen by the identity[43]

$$(\Delta x^0)^2 = T \sum_{t=1}^{T} w_t^2 = T \sum_{t=1}^{T} (w_t - \bar{w})^2 + \left( \sum_{t=1}^{T} w_t \right)^2 \qquad (4.5)$$

where $\bar{w} = 1/T \Sigma w_t$. Any deviation on $w_t$ from the mean value $\bar{w}$ will contribute positively to the first term on the right. Thus, the sequence which minimizes $(\Delta x_0)^2$, for fixed $\Sigma w_t$, is the one in which $w_t = \bar{w}$, $t = 1,\ldots,T$. In that case the time measure simplifies to the customary $x^0 = \Sigma w_t$. There is also a somewhat different way of viewing $\Delta x^0$. Eq. (4.5) can be written as

$$(\Delta x^0)^2 = T^2 \left[ 1/T \sum_{t=1}^{T} (w_t - \bar{w})^2 + \left( 1/T \sum_{t=1}^{T} w_t \right)^2 \right] \qquad (4.6)$$

The first term within brackets on the right is the customary variance of $w_t$ while the second term = $\bar{w}^2$. Hence $(\Delta x^0)^2$ is also proportional, for fixed T, to the sum of the variance and the arithmetic mean squared of the sequence $w_t$. In summary, it may be noted that $\Delta x^0$ is a measure of elapsed "social time" of an actor which reflects the irregularity of its behavior across calendar time and which simplifies

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[43] The identity can be derived as follows: $T \Sigma (w_t - \bar{w})^2 = T \Sigma (w_t^2 - 2\bar{w}w_t + \bar{w}^2) = T \Sigma w_t^2 - 2T\bar{w} \Sigma w_t + T^2\bar{w}^2 = T \Sigma w_t^2 - 2T(1/T \Sigma w_t) \Sigma w_t + (\Sigma w_t)^2 = T \Sigma w_t^2 - (\Sigma w_t)^2$.

to the sum of behavior in the case of a constant rate of behavior.

Having adopted a convention for the meaning of the spatial displacement component with positive sign in eq. (4.2), there remains to stipulate the meaning of the displacement components $\Delta x^i$ such that 1) eq. (4.2a) is satisfied and 2) eq. (3.1) can simplify, under certain conditions, to a relation involving national attributes as the $v^{\epsilon}_{\eta}$. (As mentioned above, this second criterion is only partly met. See p. 50.) Let $\Delta a_{tk}$, $k = 1,\ldots,N \leq T$ denote the change in the $k$th attribute during the calendar time period $t - 1$ to $t$, $t = 1,\ldots,T$. We shall stipulate that $\Delta a_{tk}$ corresponds to some attribute of the international system; however, we shall not stipulate which nation (if any), it specifically describes, and if it does, it will be entirely possible that some other quantity $\Delta a_{tm}$, $m \neq k$ corresponds to the same attribute for a different nation. For example, $\Delta a_{t1}$ and $\Delta a_{t2}$ might represent changes in energy consumption for Iran and Poland, respectively. We now define several quantities as follows:

$$\left.\begin{aligned} &A_{jk} \equiv \Delta a_{tj}\Delta a_{tk}, \\ &\beta_{\ell j} \quad \text{the } \ell\text{th eigenvector of } A_{jk},\ \ell = 1,\ldots,N, \\ &q_{t\ell} \quad \text{chosen such that } \Delta a_{tj} \equiv \beta_{\ell j} q_{t\ell} \\ &\lambda^k \quad \text{the } k\text{th eigenvalue of } A_{jk} \end{aligned}\right\} \quad (4.7)$$

and make the following assumptions:

1. The $\Delta a_{tk}$ are linearly independent in the $k$; i.e. for any

k $\Delta a_{tk}$ is linearly independent of any combination of $\Delta a_{tm}$, $m \neq k$.

2. The $\overset{k}{\lambda}$ are all distinct; i.e. $\overset{k}{\lambda} \neq \overset{m}{\lambda}$, $k \neq m$.

3. There exist quantities $\alpha^m$ such that for the modal behavior index $w_t$,

$$w_t = q_{tm}\alpha^m + e_t \tag{4.7a}$$

and

$$q_{tm}e_t = 0 \, . \tag{4.8}$$

Given the above definitions and assumptions it can be shown[44] that the quantities

$$\Delta x^k = (T\overset{k}{\lambda})^{1/2} \alpha^k \tag{4.9}$$

are well defined and that

$$(\Delta x^0)^2 \geq \sum_{k=1}^{N} (\Delta x^k)^2 \, ,$$

i.e. that the inequality of (4.2a) is satisfied. Therefore $\Delta x^\eta$, the displacement vector of a nation, is consistent in its definition with the metric of eq. (4.1) if eqs. (4.4) and (4.9) are taken as its empirical definition, which we choose to do. Notice that the $\Delta x^k$ have been defined as proportional to the weights $\alpha_m$ of a multiple regression of $w_t$ on the factor scores $q_{tm}$ of a principle components analysis (Anderson, 1958, Chpt. 11) of a matrix consisting of the $A_{jk}$,

[44] See Technical Appendix C.5.

except that the data of the $w_t$, the $A_{jk}$ and the $q_{tm}$ are expressed in original rather than mean deviation form. If the terminology of Blaylock is followed the variation (rather than variance) of $w_t$ is partitioned into statistically independent parts $q_{t1}\alpha_1, q_{t2}\alpha_2, \ldots, q_{tN}\alpha_N$ and an error part $e_t$. The following can also be shown[45]: First, the square of the distance traveled by a nation in the social space from $t = 0$ to $t = T$ is

$$\Delta s^2 = T \sum_{t=1}^{T} e_t^2 \qquad (4.10)$$

i.e. it is proportional to the variation of the error term in eq. (4.7a). Second, the squared displacement in any direction, $(\Delta x^k)^2$ is proportional to the sum of the covariance of $q_{tk}$ with $w_t$ and the product of their arithmetic means. Third, the displacement $\Delta x^1$ of a given nation will be proportional to the change in $a_1$ from $t = 0$ to $t = T$, where $a_1$ is one of its attributes, provided 1) its behavior from $t = 0$ to $t = 1$ is constant, 2) $\Delta a_{1t}$ is proportional to the factor $q_{t1}$. If the initial value of $a_1$ at $t = 0$ is small compared with the final value at $t = T$ then $x^1$ will be approximately proportional to the value of $a_1$ at $t = T$. If, in addition, each of the $q_{tk}$, is proportional to the changes $\Delta a_{kt}$ of some attribute $a_k$ of the given nation, then each of the displacements $\Delta x^k$ of that nation will be proportional to the change from $t = 0$ to $t = T$ in the corresponding $a_k$.

From the original definition (Chapter II, above) the components

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[45] See Technical Appendix C.5.

of velocity are the derivatives of position with respect to the parameter $\tau$. Since $s = c\tau$ for some constant $c$ we also have

$$v^{\eta} = dx^{\eta}/d\tau = c\,dx^{\eta}/ds \ .$$

$dx^{\eta}/ds$ may be approximated by $\Delta x^{\eta}/\Delta s$, the ratios of displacements to distance.[46] Thus

$$v^{\eta} \cong c\Delta x^{\eta}/\Delta s \qquad (4.11)$$

In the following this approximation shall be assumed to be exact. Let $\Delta x^{\eta}_{\kappa}$ denote the $\eta$th component of displacement and, $\Delta s^{\kappa}$ the distance corresponding to the displacement of the $\kappa$th nation. Eq. (3.1)

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[46] $\Delta x^{\epsilon}/\Delta s = dx^{\epsilon}/ds$ only if $dx^{\epsilon}/ds = v^{\epsilon}$ is constant for the path segment over which $\Delta x^{\epsilon}$ is evaluated. (For then $\Delta x^{\epsilon} = \int_0^T dx^{\epsilon} = \int_0^T (dx^{\epsilon}/ds)ds = \int_0^T k\ ds = k\ \Delta s$, and $\Delta x^{\epsilon}/\Delta s = k = dx^{\epsilon}/ds$.) Since we have imagined that each nation is subjected to a force $f^{\epsilon}$, the velocity $v^{\epsilon}$ will actually change by the amount $\Delta v^{\epsilon} = \int_0^T dv^{\epsilon} = \int_0^T (dv^{\epsilon}/ds)ds = \int_0^T a^{\epsilon}ds = 1/m \int_0^T f^{\epsilon}ds$, m the "mass" of the nation, during the calendar period 0 to T. $\Delta v^{\epsilon}$ will thus be small for sufficiently large m. To regard the approximation $\Delta x^{\epsilon}/\Delta s$ as valid is thus implicitly to assume that m is sufficiently large for all nations. Since an empirical meaning has not been assigned to the quantity m there is no way of directly checking this assumption.

can be written, with the help of eq. (4.1), as

$$\begin{aligned}\overset{\epsilon}{f}_{\kappa} &= (c\Delta x^{\eta}_{\kappa}/\Delta s^{\kappa})\overset{\epsilon}{f}_{\eta} \\ &= (c\Delta x^{\eta}_{\kappa}/\Delta s^{\kappa})g_{\eta\sigma}\overset{\epsilon}{f}{}^{\sigma} \\ &= c\overset{\epsilon}{f}{}^{o}\Delta x^{o}_{\kappa}/\Delta s^{\kappa} - c\overset{\epsilon}{f}{}^{i}\Delta x^{i}_{\kappa}/\Delta s^{\kappa} \qquad (4.12)\end{aligned}$$

The above expression can be simplified by resort to the orthogonality of the force acting on an object and the velocity of the object. Applying eq. (2.8) to the $\epsilon$th actor

$$g_{\eta\sigma}v^{\eta}_{\epsilon}\overset{\epsilon}{f}{}^{\sigma} = v^{\eta}_{\epsilon}\overset{\epsilon}{f}_{\eta} = 0 .$$

Substitution of $v^{\eta}_{\epsilon}$ from eq. (4.11) (treated as an equality), shows

$$g_{\eta\sigma}(c\Delta x^{\eta}_{\epsilon}/\Delta s^{\epsilon})\overset{\epsilon}{f}{}^{\sigma} = 0 .$$

Expanding the left hand side of the above

$$c\Delta x^{o}_{\epsilon}/\Delta s\overset{\epsilon}{f}{}^{o} - c\Delta x^{i}_{\epsilon}/\Delta s\overset{\epsilon}{f}{}^{i} = 0 ,$$

from which

$$\overset{\epsilon}{f}{}^{o} = (\Delta x^{i}_{\epsilon}/\Delta x^{o}_{\epsilon})\overset{\epsilon}{f}{}^{i} \qquad (4.13)$$

Substitution of the above expression for $\overset{\epsilon}{f}{}^{o}$ into eq. (4.12) yields

$$\begin{aligned}\overset{\epsilon}{f}_{\kappa} &= c(\Delta x^{i}_{\epsilon}/\Delta x^{o}_{\epsilon})\overset{\epsilon}{f}{}^{i}\Delta x^{o}_{\kappa}/\Delta s^{\kappa} - c\overset{\epsilon}{f}{}^{i}\Delta x^{i}_{\kappa}/\Delta s^{\kappa} \\ &= c[(\Delta x^{i}_{\epsilon}/\Delta x^{o}_{\epsilon})\Delta x^{o}_{\kappa}/\Delta s^{\kappa} - \Delta x^{i}_{\kappa}/\Delta s^{\kappa}]\overset{\epsilon}{f}{}^{i} \\ &= c/\Delta s^{\kappa}[\Delta x^{i}_{\epsilon}(\Delta x^{o}_{\kappa}/\Delta x^{o}_{\epsilon}) - \Delta x^{i}_{\kappa}]\overset{\epsilon}{f}{}^{i} . \qquad (4.14)\end{aligned}$$

The above expression can be given one additional simplification due to the fact that $\overset{\epsilon}{f}_{\epsilon} = 0$ (as it should, both by eq. (2.14) and by common usage), that the behavior from an actor to itself is zero. Suppose that the nations of the space are labeled such that $\epsilon = 0$ in eq. (4.14); i.e. the actor of the behavior $\overset{\epsilon}{f}_{\kappa}$ is the oth actor in the enumeration of nations. Then $\overset{o}{f}_{o} = 0$ and from eq. (4.14),

$$\begin{aligned}\overset{o}{f}_{k} &= c/\Delta s^{k}[\Delta x^{i}_{o}(\Delta x^{o}_{k}/\Delta x^{o}_{o}) - \Delta x^{i}_{k}]f^{oi} \\ &= \overset{oi}{\xi}_{k}(cf^{oi}) \ , \end{aligned} \tag{4.15}$$

where

$$\overset{oi}{\xi}_{k} = [\Delta x^{i}_{o}(\Delta x^{o}_{k}/\Delta x^{o}_{o}) - \Delta x^{i}_{k}]/\Delta s^{k} \ .$$

Since the choice of actor was arbitrary, the above can be rewritten with $\epsilon$ and $\kappa$ replacing actor and object labels o and k, respectively, together with the condition $\kappa \neq \epsilon$. Define

$$\overset{\epsilon}{c}_{ij} = \overset{\epsilon_i}{\xi}_{\eta}\overset{\epsilon_j}{\xi}_{\eta}, \quad \eta \neq \epsilon$$

and suppose the following: that the determinants

$$|\overset{\epsilon}{c}_{ij}| \neq 0 \ . \tag{4.15a}$$

This assumption corresponds to the assumption of eq. (3.6), above. The expression (4.15) is in the same form as eq. (3.1) and we can proceed in like manner to treat the solution of the $cf^{oi}$ as a

multiple regression problem of the $\overset{o}{f}_k$ on the $\overset{o_i}{\xi}_k$, where $cf^{o_i}$ is solved analogously to the $\overset{\epsilon}{f}_\lambda$ in eq. (3.8). Equation (4.15) can be applied, in like manner to the solution of the entire ensemble of elements $cf^{\epsilon_i}$.[47]

Some special cases emerge from eq. (4.15) as a consequence of the manner in which the $\Delta x^\eta$ were defined. Suppose, first, that the actor $\epsilon = 0$ in the above is at "rest" relative to the frame S. The physical analogy to which this situation corresponds is one in which the $\Delta x^i_o$ vanish. In the statistical representation which we have adopted it corresponds to the vanishing of the $\alpha$-weights of eq. (4.7a) relating $w_t$, the modal behavior of $\epsilon = 0$ to the attribute change "factors" $q_{ti}$. It can be shown[48] that the non-trivial cases of a weight $\alpha^i = 0$ are that either $q_{ti}$ and $w_t$ are uncorrelated over time or else $q_{ti}$ and $w_t$ have mutual covariance $= -\bar{q}_i\bar{w}$, where $\bar{q}_i$ and $\bar{w}$ are the arithmetic means across $t$ of $q_{ti}$ and $w_t$, respectively. In the case of $\Delta x^i_o = 0$, for all $i$, eq. (4.15) simplifies to

$$\overset{o}{f}_k = -(\Delta x^i_k/\Delta s^k)cf^i_o \; . \qquad (4.16)$$

It was pointed out above that $\Delta x^i_k$ would be proportional to a change

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[47] It is interesting that the quantity within the brackets of eq. (4.15) is similar to Rummel's quantity $d_{ij}$ of eq. (3.10), in that both reflect a difference between quantities indexing actor and object, respectively. Of course, the meanings are not the same because eq. (4.15) contemplates the altered interpretation of nation velocity.

[48] See Technical Appendix C.5.

in an attribute $a_i$ from $t = 0$ to $t = T$ if the modal behavior of k was constant and if $\Delta a_{it}$ was proportional to the factor $q_{ti}$; if also $a_i$ were zero at $t = 0$ then $\Delta x^i_k$ would be proportional to the value of $a_i$ at $t = T$. Given these added conditions, eq. (4.16) shows that the criterion that our definition of the velocity components $v^i$ should be capable of simplifying to the national attributes, with which they were identified in Chapter III, can be partly met. This satisfaction is only partial, however, because once the elements $q_{ti}$ are identified with the attributes of a particular nation they cannot then be identified with the attributes of another, unless by chance the two nations have identical attributes. The simplification cannot, then, apply to the $\Delta x^i$ of more than one nation, except for this exogenous circumstance.[49]

In summary, it has just been shown that a result of requiring that the social space have a reference frame with the Minkowski metric of eq. (4.1) is that there is one nation displacement component $\Delta x^o$ which is set apart from the others, eq. (4.2a), by the relation

[49]Note that if the $q_{tm}$ of eq. (4.7) were assumed to be in mean-deviation form, so that $\bar{q}_m = 0$, then the above interpretation of the $\Delta x^i$ as attributes would disappear entirely. This would also imply that the vanishing of $\Delta x^i$ was always associated with a vanishing over time product moment correlation between $w_t$ and $q_{ti}$, which in turn would imply that any nation exhibiting constant modal behavior $w_t$ = constant over time was at rest in the social space.

$$(\Delta x^o)^2 \geq \sum_i^N (\Delta x^i)^2 \ .$$

An empirical interpretation to the $\Delta x^\eta$ has been construed such that this inequality is tautologically satisfied. The role of the particular metric in the preceeding development is that of an implication, to which the method of constructing $\Delta s$ and the $\Delta x^\eta$ is logically anticedent; i.e. we assume 1) the given definition of $\Delta s$ and the $\Delta x^\eta$ and 2) that the reference frame S is one in which $g_{\eta\epsilon} = 0$, $\eta \neq \epsilon$, which together imply the diagonal elements of $g_{\eta\epsilon}$ given by eq. (4.1).

# V. APPLICATION OF BEHAVIOR-DISPLACEMENT LAW TO THE BEHAVIOR OF NATIONS

In the preceeding chapter the original behavior-attribute relationship given by the transformation law of eq. (3.1) was modified to a relationship, eq. (4.12), between behavior and a construct which was called "displacement," given by

$$\overset{\epsilon}{f}_{\kappa} = cf^{\epsilon_o} \Delta x^{o}_{\kappa}/\Delta s^{\kappa} - cf^{\epsilon_i} \Delta x^{i}_{\kappa}/\Delta s^{\kappa} . \qquad (5.1)$$

Equation (4.15) showed that the quantities $cf^{\epsilon_i}$ can be solved by fixing the value of $\epsilon$ and regarding the $cf^{\epsilon_i}$ as regression weights obtained by regression of $\overset{\epsilon}{f}_{\kappa}$, $\kappa \neq \epsilon$ on quantities $\xi^{\epsilon_i}_{\kappa}$, themselves functions of the $\Delta s^{\kappa}$ and the displacement components. Equation (4.13) showed that

$$f^{\epsilon_o} = (\Delta x^{i}_{\epsilon}/\Delta x^{o}_{\epsilon}) f^{\epsilon_i} .$$

Thus eq. (5.1) states that for fixed $\epsilon$, the behavior $\overset{\epsilon}{f}_{\kappa}$ from a given actor $\epsilon$ to various objects $\kappa$ is the sum of a quantity (the second term on the right), which is linear in the object displacement $\Delta x^{i}_{\kappa}$, plus a second quantity (the first term on the right), which is proportional to the covariation of the $f^{\epsilon_i}$ with the actor displacement $\Delta x^{i}_{\epsilon}$. Thus, the behavior of actor to object depends on the displacements of both in the social space.

Now, let us introduce the definition of the displacements. Substitution of $\Delta x^{i}$ as given by eq. (4.9) into the first line of eq. (4.15) shows

$$\overset{o}{f}_k = c/\Delta s^k\left[\alpha_o^i \sqrt{(T\overset{i}{\lambda})}\,\Delta x_k^o/\Delta x_o^o - \alpha_k^i \sqrt{(T\overset{i}{\lambda})}\right]\overset{oi}{f}$$

$$= T^{1/2}c/\Delta s^k[\alpha_o^i \Delta x_k^o/\Delta x_o^o - \alpha_k^i](\overset{i}{\lambda})^{1/2}\overset{oi}{f} ,$$

where $\alpha_o^i$ and $\alpha_k^i$ denote the regression weights of eq. (4.7) for actor and object, respectively. It can be shown[50] that $\alpha_\eta^i$ is given by

$$\alpha_\eta^i = (\overset{i}{\lambda})^{-1} q_{ti} \overset{\eta}{w}_t \tag{5.2}$$

where $\overset{\eta}{w}_t$ denotes the modal behavior of the $\eta$th nation in eq. (4.7). Substitution of $\alpha_k^i$ and $\alpha_o^i$ as given by eq. (5.2) into the preceeding equation shows

$$\overset{o}{f}_k = T^{1/2}c/\Delta s^k[(\overset{i}{\lambda})^{-1} q_{ti} \overset{o}{w}_t \Delta x_k^o/\Delta x_o^o - (\overset{i}{\lambda})^{-1} q_{ti} \overset{k}{w}_t](\overset{i}{\lambda})^{1/2}\overset{i}{f}_o$$

$$= T^{1/2}c/\Delta s^k(\overset{o}{w}_t \Delta x_k^o/\Delta x_o^o - \overset{k}{w}_t) q_{ti} f_o^i \overset{i}{\lambda}^{-1/2} \tag{5.3}$$

Let

$$\overset{o}{\gamma}_t = -T^{1/2} c q_{ti} f_o^i \overset{i}{\lambda}^{-1/2} . \tag{5.4}$$

Then the above can be expressed as

$$\overset{o}{f}_k = \overset{o}{\gamma}_t(\overset{k}{w}_t - \overset{o}{w}_t \Delta x_k^o/\Delta x_o^o)/\Delta s^k .$$

Since the labeling of the actors is arbitrary, the above equality can be written for arbitrary actor $\epsilon$ and object $\eta$ as

$$\overset{\epsilon}{f}_\eta = \overset{\epsilon}{\gamma}_t(\overset{\eta}{w}_t - \overset{\epsilon}{w}_t \Delta x_\eta^o/\Delta x_\epsilon^o)/\Delta s^\eta . \tag{5.5}$$

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[50] See Technical Appendix C.5, eq. (47).

A significant feature of quantitatively oriented international relations studies is that they have usually conceived of social behavior regularities as either 1) connecting up the variation over time of two sets of data or 2) connecting up the variation across nations at a given time of two sets of data.[51] There seems to have been little or no inquiry whether there might be a relationship between the hypothesized regularities of one type and the hypothesized regularities of the other type nor, do there appear to be any hypotheses connecting over-time data to cross-national data.[52] The hypothesis stated by eq. (5.5) is thus somewhat of a novel type, because the relationship between directed behavior and modal behavior connects the over-time and cross-national variation of behavior. The model which has been developed here implies that directed behavior is a function of the covariation between some actor specific linear combination $\gamma_t$ of attribute changes and the weighted combination of the modal behavior of actor and object which occurs within the parentheses of eq. (5.5).

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[51] For example, the conflict studies of Richardson (1960), McClelland (1968), Holsti, et. al. (1968), Zinnes (1968), Voevodsky (1969), and Singer (1972) are concerned exclusively with variation over time, while the studies of Rummel cited throughout this paper have mostly concerned cross-national variation at a given time.

[52] Rummel (1965, p. 203) suggested an axiom whereby the change in directed behavior over time was related to the relative change of attributes of the actor and object. In a still later study (1971) he suggested a design which combines the data of different nations at different time periods. However, these studies do not explicitly consider the problem of over time versus cross-national behavior.

We can see how that relationship might correspond to a realistic situation by returning to the special case in which the actor nation is imagined to be at rest relative to S. As mentioned in the discussion leading to eq. (4.16), p. 49 above, this corresponds to the restriction $\alpha^i_\eta = 0$, $\eta = \epsilon$, where $\epsilon$ labels the actor and $\alpha^i_\eta$ is the ith weight of the $\eta$th nation in eq. (4.7a). Equation (5.2) shows that this corresponds to

$$(\overset{i}{\lambda})^{-1} q_{ti} \overset{\epsilon}{w}_t = 0 .$$

Substitution of the above value into eq. (5.3) and use of eq. (5.4), (with the indices $\epsilon$ and $\eta$ replacing o and $\kappa$) shows

$$\begin{aligned} \overset{\epsilon}{f}_\eta &= \overset{\epsilon}{\gamma}_t \overset{\eta}{w}_t / \Delta s^\eta \\ &= (\overset{\epsilon}{\gamma}_t - \overset{\epsilon}{\gamma})(\overset{\eta}{w}_t - \overset{\eta}{w})/\Delta s^\eta + T\overset{\epsilon\eta}{\gamma w}/\Delta s^\eta , \end{aligned} \qquad (5.6)$$

where $\overset{\epsilon}{\gamma}$ and $\overset{\eta}{w}$ denote the arithmetic means of the respective quantities. The first term of the second line is the covariance of $\overset{\epsilon}{\gamma}_t$ with $\overset{\eta}{w}_t/\Delta s^\eta$. Thus, eq. (5.6) implies that there is associated with the actor some characteristic pattern $\overset{\epsilon}{\gamma}_t$ of variation over time such that for objects $\eta$ which exhibit total behavior having high covariance with that pattern, the directed behavior from $\epsilon$ to $\eta$ is high. Similarly, the second term, which is proportional to $\overset{\eta}{w}$, shows that the actor also directs behavior to objects having high average amount of total behavior.

The same result as the above can be approximated if, in eq. (5.5),

$\Delta x^{o}_{\epsilon}$ is assumed to be much larger than $\Delta x^{o}_{\eta}$. In that case the right hand term within the parentheses becomes small independently of the covariation between $\overset{\epsilon}{\gamma}_t$ and $\overset{\epsilon}{w}_t$ and the value of $\overset{\epsilon}{\bar{f}}_{\eta}$ approaches eq. (5.6). From eq. (4.4)

$$\Delta x^{o}_{\eta}/\Delta x^{o}_{\epsilon} = \sum_{t=1}^{T} \overset{\eta}{w}_t^2 / \sum_{t=1}^{T} \overset{\epsilon}{w}_t^2 \qquad (5.7)$$

which shows that $\Delta x^{o}_{\eta}/\Delta x^{o}_{\epsilon} \to 0$ if the variation of modal behavior by the actor nation is large compared with the object nation.

V.I Application to the Feudal Interaction Model of Galtung

The discussion of the above two cases can be combined in the assertion that eq. (5.6) is approximated when

$$(\overset{i}{\lambda})^{-1} q_{ti} \overset{\epsilon}{w}_t \Delta x^{o}_{\eta}/\Delta x^{o}_{\epsilon} = \overset{\epsilon_i}{\alpha} \Delta x^{o}_{\eta}/\Delta x^{o}_{\epsilon} \to 0 \; . \qquad (5.8)$$

Given that eq. (5.6) holds in the limit of the condition of eq. (5.8) it would be desirable to select data such that the conditions are actually satisfied for certain actors and that eq. (5.6) is likely to be true for those actors. Then the behavior-displacement transformation law would be accurate for at least a sub-set of the possible cases.

If one observes the pattern of behavior among nations one sees that, for many types of behavior, the pattern approximates an idealization which Johan Galtung (1971, p. 89) has referred to as the "feudal interaction structure," in which the nations of the world are classified into disjoint groups, the group of "center nations" and the group of "periphery nations." The characteristics of interaction are: 1) each

periphery nation interacts with one and only one center nation; 2) no two periphery nations interact; 3) the periphery nations are more numerous than the center nations. This pattern is approximated in the instance of foreign deployments of military personnel, in that many nations host the troop deployments of only one other nation (see Table 2) and, as observed by Galtung (1971, p. 81), in the instance of foreign trade, in that many nations send the preponderant share of their foreign exports to exactly one other nation. (This is illustrated in Table 3, where "preponderance" of exports is defined as sending twenty-five percent or more of total exports to one nation.) The above pattern can be recognized as part of the broader phenomenon which Galtung calls "structural imperialism" (1971, p. 81) and which Organski calls the "international order" (1968, Chpt. 14). For some types of variables it may also be true that each center nation interacts highly with each other center nation. With this modification, the feudal interaction pattern may also be characteristic, to some extent, of foreign state visits by heads of state and other key members of the ruling group.[53]

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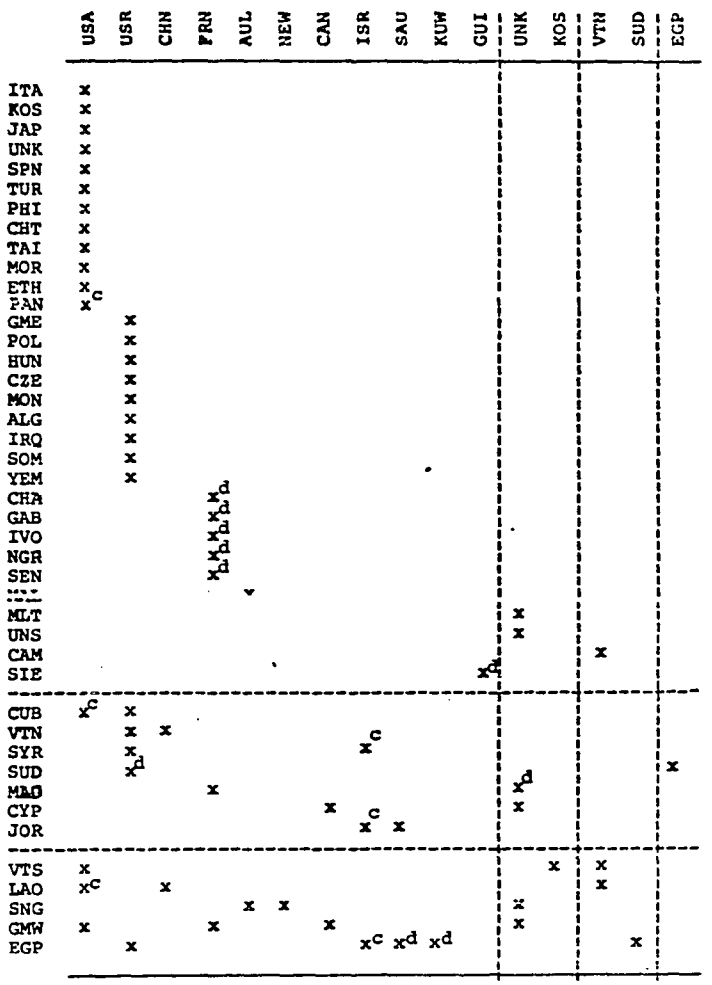
[53]Brams (1969, pp. 584-587) has collected data on international visits between high-level government officials for all nations in the years 1964 and 1965. The use of this data as a representation of feudal structure is suggested by the central role which the developed nations were found to play in such interactions (pp. 590-594). It was not possible, from the article consulted, to check Galtung's conjecture directly against Bram's data since the latter is not displayed in its unprocessed form.

Table 2

Pattern of Military Personnel Deployments of Nations in 1971

Deploying Nations[a]

Receiving (Host) Nations[b]




[a] An "x" in the column of a "deploying nation" and the row of a "recipient nation" indicates that military personnel of the former are located on the national territory of the latter. Deploying nations to the left of the three vertical dashed lines were not the host to deployments of other nations. Deploying nations immediately to the right of the vertical lines were host to deployments by 1, 2 and 3 or more other nations, respectively. See Table 4 for key to nation abbreviations. Source: International Institute for Strategic Studies (1972), except where otherwise indicated.

[b] Recipient nations above the two horizontal dashed lines were each host to the deployment of exactly one nation. Recipient nations below the lines were host to the deployments of 2 and 3 or more nations, respectively.

[c] Deployments not listed in sources, but assumed from materials in the public record.

[d] Source: International Institute for Strategic Studies (1971, pp.65-66).

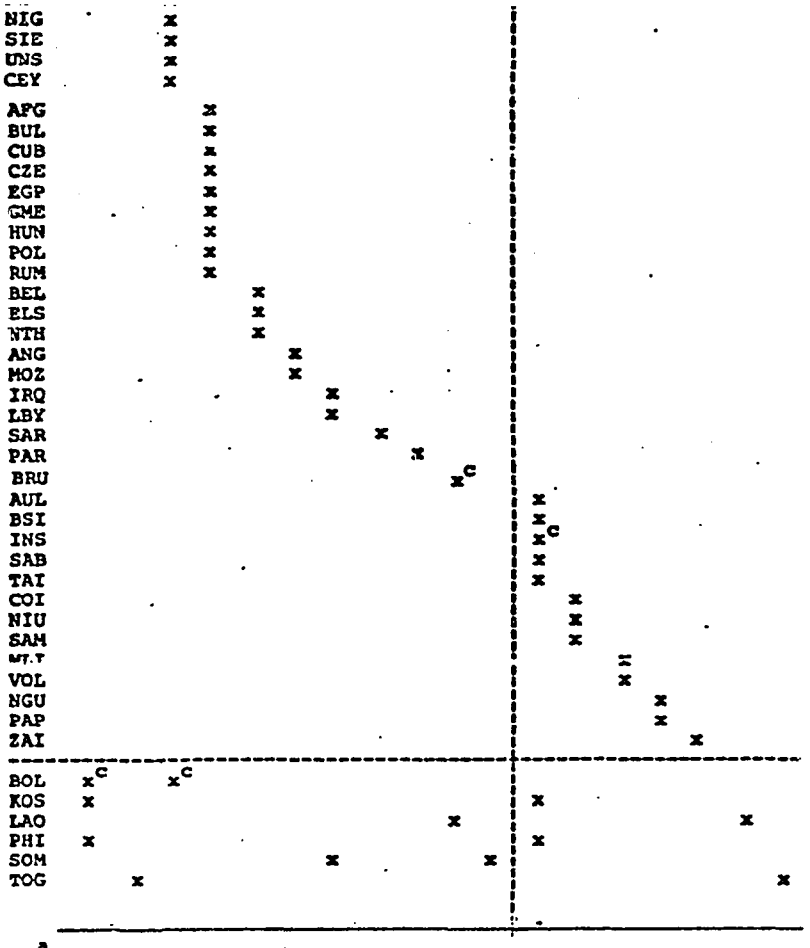
Table 3

Pattern of Preferences of Exporting Nations Toward Importing Nations in 1970

Importing Nations[a]

Exporting Nations[b]

| | USA | FRN | UNK | USR | GMW | POR | ITA | SNG | ARG | MAL | SAU | JAP | NEW | IVO | AUL | BEL | TAI | NTH |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| BRA | x | | | | | | | | | | | | | | | | | |
| BRH | x | | | | | | | | | | | | | | | | | |
| CAN | x | | | | | | | | | | | | | | | | | |
| COL | x | | | | | | | | | | | | | | | | | |
| COS | x | | | | | | | | | | | | | | | | | |
| DOM | x | | | | | | | | | | | | | | | | | |
| ECU | x[c] | | | | | | | | | | | | | | | | | |
| ETH | x | | | | | | | | | | | | | | | | | |
| FRG | x | | | | | | | | | | | | | | | | | |
| GUA | x | | | | | | | | | | | | | | | | | |
| GUY | x | | | | | | | | | | | | | | | | | |
| HON | x | | | | | | | | | | | | | | | | | |
| HGK | x | | | | | | | | | | | | | | | | | |
| ICE | x | | | | | | | | | | | | | | | | | |
| JAM | x | | | | | | | | | | | | | | | | | |
| JAP | x | | | | | | | | | | | | | | | | | |
| MEX | x | | | | | | | | | | | | | | | | | |
| NTA | x | | | | | | | | | | | | | | | | | |
| NIC | x | | | | | | | | | | | | | | | | | |
| PAN | x[d] | | | | | | | | | | | | | | | | | |
| PER | x | | | | | | | | | | | | | | | | | |
| SUR | x | | | | | | | | | | | | | | | | | |
| TRN | x | | | | | | | | | | | | | | | | | |
| VEN | x | | | | | | | | | | | | | | | | | |
| ALG | | x | | | | | | | | | | | | | | | | |
| CAO | | x | | | | | | | | | | | | | | | | |
| CAR | | x | | | | | | | | | | | | | | | | |
| CHA | | x | | | | | | | | | | | | | | | | |
| DAH | | x | | | | | | | | | | | | | | | | |
| GAB | | x | | | | | | | | | | | | | | | | |
| GUD | | x | | | | | | | | | | | | | | | | |
| IVO | | x | | | | | | | | | | | | | | | | |
| MAD | | x | | | | | | | | | | | | | | | | |
| MAR | | x | | | | | | | | | | | | | | | | |
| MOR | | x | | | | | | | | | | | | | | | | |
| NGR | | x | | | | | | | | | | | | | | | | |
| REU | | x | | | | | | | | | | | | | | | | |
| SEN | | x | | | | | | | | | | | | | | | | |
| VTS | | x | | | | | | | | | | | | | | | | |
| BAR | | | x | | | | | | | | | | | | | | | |
| CYP | | | x | | | | | | | | | | | | | | | |
| FIJ | | | x | | | | | | | | | | | | | | | |
| GAM | | | x | | | | | | | | | | | | | | | |
| IRE | | | x | | | | | | | | | | | | | | | |
| MLW | | | x | | | | | | | | | | | | | | | |
| MLT | | | x | | | | | | | | | | | | | | | |
| MRI | | | x | | | | | | | | | | | | | | | |
| NEW | | | x | | | | | | | | | | | | | | | |
| NIG | | | x | | | | | | | | | | | | | | | |
| SIE | | | x | | | | | | | | | | | | | | | |
| UNS | | | x | | | | | | | | | | | | | | | |
| CEY | | | x | | | | | | | | | | | | | | | |
| AFG | | | | x | | | | | | | | | | | | | | |
| BUL | | | | x | | | | | | | | | | | | | | |
| CUB | | | | x | | | | | | | | | | | | | | |
| CZE | | | | x | | | | | | | | | | | | | | |
| EGP | | | | x | | | | | | | | | | | | | | |
| GDE | | | | x | | | | | | | | | | | | | | |
| HUN | | | | x | | | | | | | | | | | | | | |
| POL | | | | x | | | | | | | | | | | | | | |
| RUM | | | | x | | | | | | | | | | | | | | |




[a] An "x" in the column of an "importing nation" and the row of an "exporting nation" indicates that the former was the final known destination of 25% or more of the total 1970 exports (1969 in some cases) of the latter. Enumeration of exporting nations excludes 30 entities not exporting 25% or more of total exports to any one destination. Source indiscriminantly lists both sovereign and dependent entities and no attempt is made here to distinguish the two kinds. Importing nations to the left of the vertical dashed line did not themselves export more than 25% of their own total exports to any other nation. Nations to the right of the line exported above the 25% criterion to one other nation each. See Table 4 for key to nation abbreviations. Source: United Nations (1973).

[b] Exporting nations above the horozontal dashed line are linked to exactly one importing nation by the 25% criterion. Nations below the line are linked to two importing nations by the criterion.

[c] Data for year 1969.

[d] Excluding Canal Zone.

Table 4

Key to Nation Abbreviations

| | | | |
|---|---|---|---|
| Afghanistan | AFG | Libyan Arab Republic | LBY |
| Algeria | ALG | Madagascar (Malagasy Republic) | MAD |
| Angola | ANG | Malawi | MLW |
| Argentina | ARG | Malaysia (West Malasia, Table 3) | MAL |
| Australia | AUL | Sabah | SAB |
| Barbados | BAR | Sarawak | SAR |
| Belgium-Luxembourg | BEL | Mali | MLI |
| Bolivia | BOL | Malta | MLT |
| Brazil | BRA | Martinique | MAR |
| British Honduras | BRH | Mauritius | MRI |
| British Solomon Islands | BSI | Mexico | MEX |
| Brunei | BRU | Mongolia | MON |
| Bulgaria | BUL | Morocco | MOR |
| Cameroon | CAO | Mozambique | MOZ |
| Canada | CAN | Netherlands | NTH |
| Central African Republic | CAR | Netherlands Antilies | NTA |
| Chad | CHA | New Guinea | NGU |
| China, Peoples Republic of | CHN | New Hebrides | NHB |
| China, Republic of (Taiwan) | CHT | New Zealand | NEW |
| Colombia | COL | Nicaragua | NIC |
| Cook Islands | COI | Niger | NGR |
| Costa Rica | COS | Nigeria | NIG |
| Cuba | CUB | Niue Island | NIU |
| Cyprus | CYP | Panama | PAN |
| Czechoslovakia | CZE | Papua | PAP |
| Dahomey | DAH | Paraguay | PAR |
| Dominican Republic | DOM | Peru | PER |
| Ecuador | ECU | Philippines | PHI |
| Egypt | EGY | Poland | POL |
| El Salvador | ELS | Portugal | POR |
| Ethiopia | ETH | Reunion | REU |
| Fiji | FIJ | Rumania | RUM |
| France | FRN | Saudi Arabia | SAU |
| French Guiana | FRG | Sengal | SEN |
| Gabon | GAB | Sierra Leone | SIE |
| Gambia | GAM | Singapore | SNG |
| German Democratic Republic | GME | Somalia | SOM |
| Germany, Federal Republic of | GMW | South Africa, Union of | UNS |
| Guadeloupe | GUD | Spain | SPN |
| Guatemala | GUA | Sri Lanka (Ceylon) | CEY |
| Guinea | GUI | Sudan | SUD |
| Guyana | GUY | Surinam | SUR |
| Honduras | HON | Syrian Arab Republic | SYR |
| Hong Kong | HGK | Thailand | TAI |
| Hungary | HUN | Togo | TOG |
| Iceland | ICE | Trinidad and Tobago | TRN |
| Indonesia | INS | Turkey | TUR |
| Iraq | IRQ | Union of Soviet Socialist Republics | USR |
| Ireland | IRE | United Kingdom | UNK |
| Italy | ITA | United States | USA |
| Ivory Coast | IVO | Upper Volta | VOL |
| Jamaica | JAM | Venezuela | VEN |
| Japan | JAP | Viet Nam, North | VTN |
| Jordan | JOR | Viet Nam, South | VTS |
| Khmer Republic (Cambodia) | CAM | Western Samoa | SAM |
| Korea, Republic of | KOS | Yemen | YEM |
| Kuwait | KUW | Zaire | ZAI |
| Laos | LAO | | |

Another feature which seems to characterize the interaction between center and periphery nations is that the behaviors mentioned above are not reciprocated. As illustrated in Tables 2 and 3, the periphery nation does not usually deploy military personnel to the center nation and the center nation does not direct a large share of its exports to the periphery nation. One might conjecture that, also, visits by state officials of the periphery nation to the center are often not reciprocated.

For behavior that is characterized by the feudal interaction structure we shall henceforth regard the center nation as the "actor" and the periphery nation as the "object," even for behavior initiated by the latter.[54] This can be done in a consistent manner for various center-periphery pairs, provided that the direction of non-zero behavior is either always (or usually) from center to periphery or always from periphery to center for various pairs. This amounts to saying that for certain behavior indices the sense of the direction of action is to be reversed from what is customary. For example, non-reciprocated trade dependence (Table 2) is generally from periphery to center, so that "% total exports of A going to B" is to be regarded as an action initiated by B, with importing nation as actor. Military

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[54] It is clear that an element of judgement is necessary to decide when the behavior to or from a periphery nation is sufficiently concentrated on one other nation to regard the behavior as essentially one-directional. Also, it will not be possible in practice to unambiguously classify every nation as belonging either to the center or to the periphery. For example, Galtung lists the U.S.S.R. in both the periphery and the center (1971, pp. 105 and 114, n. 19). This discussion assumes that the number of clear cut cases is sufficient to make the analysis meaningful.

troop deployments, however appear to be usually from center to periphery (Table 3), so that the customary direction of "number of A's troops deployed to B" is to be retained, with the deploying nation counted as actor. This can be argued as a meaningful convention from a different viewpoint. It is customarily assumed that the direction of directed behavior is unambiguous, i.e. that an arbitrary datum can be regarded as clearly reflecting the initiative of one of the pair of nations. In the case of center-periphery pairs this is not necessarily clear. For example, if periphery nation x exports a raw commodity primarily to center nation y, one might be inclined to regard this as just as much at the initiative of y as the ostensible actor x. The convention described above is not necessarily contra-intuitive to the substantive meaning of the interaction.[55]

Another distinction between center and periphery, which might reasonably be conjectured, is that there are a large number of system attributes, to which 1) the modal behavior of a center nation is unrelated, in the sense that the increments from one occasion to the

[55]This procedure has the interesting consequence, within the abstract spatial model, that all of the force components of each periphery nation $\kappa$ vanish: by hypothesis, $\kappa$ initiates no behavior to other periphery nations nor to center nations other than its dominant nation; by the above convention $\kappa$ initiates no behavior to its dominant nation. Therefore $\overset{\kappa}{f}_{\sigma} = 0$ for all $\sigma$. Eq. (3.1) shows that this is satisfied if $\overset{\kappa\eta}{f} = 0$, i.e. if the velocity of $\kappa$ is constant.

next of these attributes does not have a high degree of covariation with its modal behavior, but to which 2) the modal behavior of one or more peripheral nations has high covariance. Attributes such as the world price of a particular raw commodity, or concerning the internal condition of a particular peripheral nation seem likely candidates as variables the over time variations of which would affect the modal behavior, over time, of one or more peripheral nations.[56] If the $q_{ti}$ are composed of linear combinations of such attributes then the quantity $q_{ti}\overset{\epsilon}{w}_t$ of eq. (5.8) would also be small. Conversely, the "error variation" $1/T \Sigma e_t^2$, $e_t$ as defined in eq. (4.7a), would be small for periphery nations and large for center nations. By eq. (4.10), $\Delta s^2$ would be small or large, respectively.

Still a third distinction might well apply between center and periphery nations. McClelland and Hoggard (1969, p. 716) have reported finding that five nations, the U.S.A., U.S.S.R., China, United Kingdom and France, accounted for 40 percent of the total number, initiated

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[56]Since the attribute values of many peripheral nations are small compared with the center nations, it might also be desirable to determine the $\Delta a_{tj}$ from the logarithms of the $a_{tj}$, i.e. define $\Delta a_{tj} = \log a_{tj} - \log a_{t-1,j}$, so as to enhance the role of changes in small values of the $a_{tj}$. This would also have the consequence of giving equal emphasis to attribute changes which were proportionately the same, regardless of their absolute magnitude; e.g. a 10 percent change in $a_{tj}$ over $a_{t-1,j}$ would produce the same value $\Delta a_{tj}$, regardless of the size of $a_{tj}$.

by all nations, of a broad class of acts reported in the daily New York Times in a particular recent year. If these are taken as "center" nations, the above finding suggests that the amount of modal behavior done by center nations may be much greater than the amount done by a periphery nation.[57] In that case the variation $\Delta x^{o}$ in the modal behavior of a center nation would be large, compared with $\Delta x^{o}$ of a periphery nation. Thus, we conjecture that the quantity $q_{ti}{}^{\epsilon}w_{t}\Delta x^{o}_{\eta}/\Delta x^{o}_{\epsilon}$ is small when $\epsilon$ and $\eta$ are center and periphery nations, respectively, and eq. (5.8), hence eq. (5.6), is approximated.

Now it is possible to see how the feudal pattern of interaction between center and periphery might be replicated by eq. (5.5). Given that the actor is a center nation, the above discussion has shown that eq. (5.6) can be expected to approximate the directed behavior of the actor. It was also shown above that it could be expected that for some value of $i$ the covariation between $q_{ti}$ and $w_{t}^{\eta}$ would be large for $\eta$ a periphery nation. Suppose that $q_{t1}$ is the quantity such that $q_{t1}{}^{\eta}w_{t}$ is large, and suppose that $\eta$ interacts primarily with the center nation $\epsilon$. By eq. (5.4) $\gamma_{t}^{\epsilon}$ is proportional to

$$q_{ti}f^{\epsilon_i} = q_{t1}f^{\epsilon_1} + q_{t2}f^{\epsilon_2} + \cdots + q_{tN}f^{\epsilon_N} .$$

For $f^{\epsilon_1}$ sufficiently larger than $f^{\epsilon_2},\ldots,f^{\epsilon_N}$ the equality $\gamma_{t}^{\epsilon} = q_{t1}f^{\epsilon_1}$ holds approximately. Then $\gamma_{t}^{\epsilon}w_{t}^{\eta}$ will be large. As discussed above, we may also conjecture that $\Delta s^{\eta}$ is small. Reference

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[57] Although the distribution of behavior might vary from one datum to another.

to eq. (5.6) then shows that $\overset{\epsilon}{f}_{\eta}$, the behavior from $\epsilon$ to $\eta$ is large. The representation of behavior from $\epsilon$ to periphery nations is complete if it is assumed that the covariation $\overset{\epsilon}{\gamma}_t \overset{\sigma}{\gamma}_t$ is small for any two different center nations $\epsilon$ and $\sigma$ and that the behavior $w_t$ of every periphery nation has high covariation with the $\gamma_t$ of exactly one center nation. Then $\overset{\epsilon}{f}_{\eta}$ is large just for its "own" periphery nations and small for the periphery nations "belonging" to other center nations, which is the desired result. The case of behavior toward objects, themselves center nations, should show a different result, consistent with the above conjecture that center nations interact highly among themselves. Referring to eq. (5.8) $\Delta x^o_{\eta}$ is large, according to our analysis, because of the assumed large variation of the behavior of center nations, so that $\Delta x^o_{\eta}/\Delta x^o_{\epsilon}$ is no longer small if $\eta$ is a center nation. The simplification of eq. (5.8) can then no longer be regarded as applicable and the full directed behavior eq. (5.5) must be used, which shows that the weighted difference $(\overset{\eta}{w}_t - \overset{\epsilon}{w}_t \Delta x^o_{\eta}/\Delta x^o_{\epsilon})/\overset{\eta}{\Delta s}$ must have large covariation with $\overset{\epsilon}{\gamma}_t$. The conjectures which have been presented seem to point to the opposite conclusion, since $\overset{\epsilon}{\gamma}_t \overset{\eta}{w}_t$ and $\overset{\epsilon}{\gamma}_t \overset{\epsilon}{w}_t$ are argued to be small and $\overset{\eta}{\Delta s}$, large in the above discussion, for $\epsilon$ and $\eta$ center nations. Our model of directed behavior may be less realistic of the behavior of center nations and thus its applicability may depend on the accuracy of the earlier conjecture that periphery nations are in the majority in the international system. An alternative conjecture is that $\overset{\epsilon}{\gamma}_t \overset{\epsilon}{w}_t$ is, contrary to the above, large so that the distinction between

$\overline{f}^{\epsilon}_{\eta}$ to $\eta$ periphery versus $\eta$ center depends entirely on the ratio $\Delta x^{o}_{\eta}/\Delta x^{o}_{\epsilon}$ in eq. (5.5). This alternative would imply a large amount of behavior toward $\eta$ center nations while preserving eq. (5.6) as the special case applying to $\eta$ periphery nations and, as in the previous analysis, implying the large $\overline{f}^{\epsilon}_{\eta}$ of those periphery nations linked to $\epsilon$ and the small $\overline{f}^{\epsilon}_{\eta}$ of those linked to some other center nation.

The preceeding discussion is admittedly highly speculative, both as to the accuracy of the "feudal structure" which is assumed of the international system and as to the patterns of covariation over time which are judged to replicate that description. More important is, first, that a possible substantive implication of the behavior law of eq. (5.1) has been explicated, second, that it has been shown how this implication might be a realistic feature of the international system, namely the feudal model suggested by Galtung and, third, that a suggestion, as to what sort of directed behavior, modal behavior and attribute change data should be employed as the interpretation of the behavior law, can be elucidated.

To arrange for the possibility that the feudal model might be replicated, the modal behavior data should be of a type for which each nation that is clearly identified as belonging to the "center" has high over time variation and for which each nation clearly belonging to the periphery has low overtime variation. The datum on "total number of acts initiated," as collected in the McClelland and Hoggard study might qualify in this respect. The attribute change data should include indices of economic activity, particularly pertaining to

international transactions (such as the world prices of the principal commodities of international trade), and might well also include variables particular to groups of nations (such as the military personnel and armaments of principal military and diplomatic bloc members) and to individual nations (such as indices of internal conflict or political activity). A primary criterion of selection would be that the various nations be sharply differentiated in the respective covariation of their $w_t$ with the $\Delta a_k$ change data, so as to enhance the possibility of replicating the differential directed behavior pattern hypothesized above. This should be tried out empirically with various specific indices. Finally, the index of directed behavior should be of the kind which exhibits the center-periphery distinction.[58]

The above is as specific a stipulation of the choice of indices as is appropriate, since what has been conjectured is meant to be applicable to a wide variety of data; i.e. it is offered as a general hypothesis about the structure of directed behavior in the international system.

One more stipulation concerns the time interval over which the variables are to be defined. It seems to be a reasonable working assumption that the interval $t - 1$ to $t$ in eq. (4.7a) should correspond to at least one year, since it might well require up to a year (or more) for attribute changes to produce the imagined effect

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[58] This writer conjectures that such a distinction would be readily identifiable in many directed behavior indices. Such a conjecture is entirely consonant with Galtung's assertion of the ubiquity of structural imperialism (1971, pp. 91-94).

on modal behavior. For the eq. (4.7a) to be substantively meaningful requires, also, that T, the order of $w_t$, exceed N, the order of $\alpha^m$. Perhaps T = 20, N = 10 would be a suitable starting point. (Since N is the number of space-like dimensions, we are talking about an N + 1 = 11 dimensional social space.) The data of the directed behavior $\overline{f}^{\epsilon}_{\eta}$ is evaluated over a finite period of time. This period is conventionally taken to be one year; i.e. directed behavior from A to B usually means the number of acts of a certain kind taking place during a one year period. This convention is entirely arbitrary, however. There does not appear to be any intrinsic reason for the one year interval. In view of the present hypothesis that directed behavior relates to conditions over the entire span t = 0 to t = T it might be more meaningful to similarly interpret $\overline{f}^{\epsilon}_{\eta}$ as the amount of behavior from $\epsilon$ to $\eta$ evaluated from t = 0 to t = T.

If behavior is regarded as pertaining only to sovereign nations then the period of time over which directed and modal behaviors are to be evaluated leads to an ambiguity for nations which gain sovereign status during the period, i.e. after t = 0 and before t = T. The problem is that indices derived from the amount of behavior may be artificially small because they have not been evaluated over the complete period. Also, measures involving the terms $w_t$ will be incomplete for values of t preceeding the moment of sovereignty. For the case of modal behavior this ambiguity can be resolved either by the additional constraint that the indices of $w_t$ must all be capable of evaluation independently of the legal status of the national entity

or, by the convention that, prior to sovereignty, the modal behavior of the parent governing entity is regarded as also the modal behavior of the given dependent nation. For the case of directed behavior the ambiguity can be resolved, again, by observing the constraint that the index be capable of evaluation independently of legal status or, by prorating the amount of directed behavior over the total period $t = 0$ to $t = T$ (e.g. doubling the value observed during sovereignty of the behavior variable for a nation that is sovereign for half the period). For directed behavior, the first alternative of a judicious choice of indices appears to be less arbitrary. This would limit somewhat the choice of indices; however many possible choices would remain. (From the above examples, official diplomatic visits might be eliminated, but not military deployments or trade dependence.) If a composite index of modal behavior, such as the one by McClelland and Hoggard, were used, then a mixture of the first and second alternatives might be required. That is, some of the behaviors making up the index would be defined regardless of sovereign status; others would not and the data on the parent nation would need to be substituted for the period preceeding independence.

It is important to take note that, by admitting the relevance of data concerning non-sovereign entities, the model is saying, in effect, that a sovereign nation which is an object in the social space is no less an object in the time preceeding its independence. This approach contrasts with much current thinking, including the previously cited work of Rummel and Singer, which consciously chooses to focus

on the sovereign entities, sui generis.

One test, then, of the above interpretation of the behavior law, eq. (4.15), is to form the quantities $\xi^{\epsilon_i}_{\eta}$ from data on over time modal behavior and attribute change variables, in the manner indicated above, and to regress various directed behaviors $\overline{f}^{\epsilon}_{\eta}$ upon them. The criterion of this test would be the goodness of fit for various directed behaviors and various actors $\epsilon$. A more elaborate test would involve regression of a sequence of various behaviors

$$\overline{f}^{\epsilon}_{\eta,1}, \overline{f}^{\epsilon}_{\eta,2}, \ldots, \overline{f}^{\epsilon}_{\eta,P}, \quad P \leq N ,$$

$N + 1$ the number of components of $S$, on the $\xi^{\epsilon_i}_{\eta}$. This would test the behavior law with some linear combination

$$a_1 \overline{f}^{\epsilon}_{\eta,1} + a_2 \overline{f}^{\epsilon}_{\eta,2} + \cdots + a_P \overline{f}^{\epsilon}_{\eta,P}$$

in the role of force components.

# VI. SUMMARY AND SOME SUGGESTIONS FOR ADDITIONAL EXPLORATION

We began by showing how the Model II equation of Rummel can be obtained from the interpreted analytic system developed in chapters II and III, in a manner independent of whether the space is Euclidean. In chapter IV this system was modified by stipulating a metric for the reference frame S and by giving a different meaning to the velocity components in S of nations. In chapter V this new system was shown to imply a directed behavior law which might plausibly replicate the feudal model of international interaction suggested by Galtung. The new system has axioms which are the same as axioms 1), 3), 5) and 6) of the old system (p. 31, above). Axioms 2), 4) and 7) are modified in the new system as follows:

2′) the velocities of nations are approximated by the ratio $c\Delta x^{\eta}/\Delta s$, eq. (4.11);

4′) the determinants $|c^{\epsilon'}_{ij}|$ are non-vanishing, as in eq. (4.15a);

7′) the metric $g_{\eta\epsilon}$ of S is diagonal.

There are additional axioms, stemming from the method of constructing the displacement components $\Delta x^{\epsilon}$ and the quantity $\Delta s$, which do not correspond to axioms in the old system. In summary form they are

8′) the invariant length of the displacement of a nation is $\Delta s$, as given by eq. (4.10);

9′) the quantities $\Delta x^{i}$ are well defined given a specific

empirical meaning to the modal behavior index $w_t$ and to the attribute change variables $\Delta a_{tj}$. (The sufficient conditions that they be well defined are the three assumptions stated on pp. 43-44);

10′) for every nation $\epsilon$ there is at least one value of $t = 1,\ldots,T$ for which $w_t > 0$, $w_t$ the modal behavior of the given nation.

As mentioned in chapter IV, the particular values of the metric, eq. (4.1), i.e. that the space is the Minkowski type, can be derived from axioms 7′) and 8′) and the method of constructing the displacements.

There are several directions in which the present exploration might be continued. The result, eq. (4.15) constitutes, as it stands, an option for empirical study. It is the view of this writer, however, that the present empirical implication should be taken as an indication that a useful start has been made in developing the model, rather than primarily as a suggestion that empirical studies commence at once. Additional development of the model may, in time, indicate a sharpening of the criteria for selecting the variables, a further modification or generalization of the directed behavior law, additional, more easily tested social laws or, connections with other propositions already tested.

In eq. (4.15) the value of the displacements $\Delta x^o$ compared with the $\Delta x^i$ are constrained by the inequality of (4.2a), that

$$(\Delta x^o)^2 \geq \sum_{i=1}^{N} (\Delta x^i)^2 ,$$

from which one concludes that the nations are in a Minkowski type space. Expression (4.2a) itself has no empirical significance, however, because it is satisfied tautologically by the construction of $\Delta x^o$ and the $\Delta x^i$. Although the meaning of eq. (4.15) would then be different, one might equally well have assumed that all the displacements were of the form $\Delta x^i$, for which an Euclidean metric $g_{\eta\epsilon} = \delta_{\eta\epsilon}$ would then be the appropriate representation of the space. This would constitute a different directed behavior law, so it might be supposed that a difference in the goodness of fit of one law from the other is a test of the appropriate geometric assumption. Such a test, however, would actually be comparing two alternative analytic systems, involving different definitions of displacement. This would not settle the question whether one alternative is better because its displacements, given a fixed empirical meaning, do or do not satisfy expression (4.2a). A better approach would seek a redefinition of displacement such that 1) expression (4.2a) becomes empirically falsifiable and 2) the redefined displacements approach the present definition just in case (4.2a) is satisfied.[59] The question of the appropriateness of the Minkowski assumption would then be directly testable and the resulting system would be more comprehensive than at present.

A related problem is to stipulate a meaning for the position of

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[59] In the physical application the Minkowski metric has the testable consequence that the classical velocity $[\sum_{i=1}^{N} (\Delta x^i)^2/(\Delta x^o)^2]^{1/2}$ of a ray of light is the same in all reference frames.

nations in the space. It is important to note that an element of the original extension of field theory, proposed in the verbal exposition of chapter II (pp. 12-13, above) does not receive operational expression in the above. That extension reflects the idea that force is a function of both the position of nations in the space and their motion. If, following Wright and Rummel, one were to regard the spatial coordinates of nations to be given by their attribute values, then changes in position (i.e. motion) would correspond to changes in those same attributes. Then the extended field principle would imply that changes in attributes as well as the attributes themselves are related to behavior. This has not been done. Instead, the attributes were first reassigned to the role of velocity components, in order to arrive at the Model II eq. (3.10), and then modified entirely to the role of providing the attribute change variables $\Delta a$, in order to obtain the behavior law eq. (4.15) and its applications.

The original idea of spatial position as itself a determinant of behavior remains an abstraction, as the independent variable of the second of eqs. (2.2), determining the values of the potentials $A^{\sigma}$ which, in turn, determine the forces $f_{\eta}$ in eq. (2.6). Only the forces are directly observed. If an empirical interpretation of spatial position was known, that interpretation would directly relate to the above question of the relationship among the displacement components. If $x^{\eta}$ and $\bar{x}^{\eta}$ denote the position of a given actor at $t = 0$ and $t = T$, respectively, the displacement from $t = 0$ to $t = T$ is given

by

$$\Delta x^{\eta} = \bar{x}^{\eta} - x^{\eta} .$$

Hence, the empirical stipulation of position would imply another means of checking the inequality (4.2a). Such a development would also permit a direct consideration of the above conjecture that behavior is a function of both position and velocity of nations in the space.

The technique of smallest space analysis (Guttman, 1968) may possibly lead to a means for accomplishing the definition of spatial position. This technique positions objects in a coordinate system such that the distance between them is a function of the degree of their similarity according to some function such as the Pearson product moment correlation. The interest in this method lies in its asserted independence of applicability from any particular assumptions about the metric of the space (Guttman, 1968, pp. 474-6). Until smallest space analysis is carefully considered, especially from the standpoint of a Minkowski metric as defined here, the above line of thought remains entirely speculative.

In this paper a start has been made in exploring the use of a mechanistic conception of international relations, by showing how the components of force given by eq. (2.6) can be interpreted as the amount of directed behavior from a given actor, the subject of the force, to various objects. Since that force is derived exactly in analogy to the method of deriving the physical electrodynamic force, other properties known in the physical application become feasible subjects

of speculation concerning the social force. Two of the most important physical laws are the conservation of momentum and the conservation of electric charge in a closed system. Using the notation introduced in chapter II, the momentum of a single object can be defined by

$$p^{\eta} = mv^{\eta} ,$$

where $m$ and $v^{\eta}$ are the mass and velocity of the object. If $\epsilon$ indexes the objects of a closed system, the former conservation law can be expressed by

$$\sum_{\epsilon} {}^{\epsilon}p^{\eta} = \text{constant} ,$$

from which

$$\begin{aligned} \sum_{\epsilon} {}^{\epsilon}f^{\eta} &= \sum_{\epsilon} {}^{\epsilon}ma^{\eta} \\ &= \sum_{\epsilon} m d{}^{\epsilon}v^{\eta}/d\tau \\ &= \sum_{\epsilon} d{}^{\epsilon}p^{\eta}/d\tau \\ &= d/d\tau \sum_{\epsilon} {}^{\epsilon}p^{\eta} \\ &= 0 . \end{aligned}$$

Since the quantities ${}^{\epsilon}cf^{\eta}$ are determined in the solutions to eq. (4.15), the above equality could be directly checked. The result would be meaningless, however, until it was determined what is to be meant by a "closed" international system. Hence, an effort to define the idea of closure is warranted.

Similarly, the idea of conservation of charge can be related to the present considerations. Equation (2.2) is, in its physical interpretation, the Lagrangian of an electrically charged particle. The constant $k$ in that expression is proportional to the amount of charge on the particle. The principle of conservation of charge is thus

$$\sum_{\epsilon} {}^{\epsilon}k = \text{constant} , \tag{6.1}$$

where $\epsilon$ indexes the objects of a closed system. Again the meaning of closure of the international system would need to be determined for this principle to be meaningful.

The above possibility is related to another lacuna of the present model, the indeterminateness of the potential function $A^{\sigma}$. From eqs. (2.5) and (2.6), the force vector can be expressed as

$$f_{\eta} = kv^{\epsilon}(\partial_{\eta}A_{\epsilon} - \partial_{\epsilon}A_{\eta}) .$$

Since the force and velocity of a nation are known in the manner outlined in the preceeding pages, an independent stipulation of the values of $A_{\epsilon}$ would imply the value of $k$, thus permitting a determination of eq. (6.1). It should be noted that, since the $A^{\sigma}$ are functions of position, the resolution of this question would depend on progress with the above problem of resolving the position of nations in the space.

Finally it is entirely possible that the most productive further use of the mechanical analogy does not involve a direct elucidation of

the empirical content of the above assertions. The alternative is that the most readily observable aspects of a mechanistic model may come from the derived implications of the above laws, rather than from the laws themselves. Indeed, the appropriate analytic system may be a modification of the physical analogy in which one or more of these laws appear as tautologies which serve merely to connect other quantities in the system. An example from the above is the velocity of a nation in the reference frame $\varphi$, which is identically given by eq. (2.12). The importance of the suggestions offered in this chapter is that they indicate some of the points of departure for additional thinking about the question of a mechanistic aspect to social behavior.

## Technical Appendix[60]

A. Notational conventions

A.1. Kronecker delta, $\delta_{\eta\epsilon}$

$\delta_{\eta\epsilon}$ is defined by

$$\delta_{\eta\epsilon} = \begin{cases} 1, & \eta = \epsilon \\ 0, & \eta \neq \epsilon \end{cases} .$$

A.2. Gradient, $\partial_\eta A_\epsilon$

$\partial_\eta A_\epsilon$ is defined as the ensemble of partial derivatives of $A_\epsilon$ with respect to the position components (i.e. spatial coordinates) $x^\eta$:

$$\partial_\eta A_\epsilon = \partial A_\epsilon / \partial x^\eta .$$

A.3. $\eta$th order quantity, $a^{\sigma\eta \ldots \epsilon}_{\alpha\beta \ldots \gamma}$

We shall define an $\eta$th order quantity to be any ordered sequence of real-valued numbers which is indexed by $\eta$ different littorial suffixes. For example, the second order entity $x^{ij}$ can be written as the square array

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[60] Helpful discussions of the notations and concepts employed in this paper, especially of the idea of tensors, were found in Hagedorn (1964, pp. 116-123); Bergmann (1942, pp. 47-67), and Bradbury (1968, pp. 92-112). The latter source compares the treatments of linear transformations in tensor and matrix notations.

$$\begin{matrix} x^{11} & x^{12} & x^{13} & \dots & x^{1N} \\ x^{21} & x^{22} & x^{23} & \dots & x^{2N} \\ x^{31} & x^{32} & x^{33} & \dots & x^{3N} \\ \vdots & \vdots & \vdots & & \vdots \\ x^{N1} & x^{N2} & x^{N3} & \dots & x^{NN} \end{matrix} .$$

The first order entity $a_k$ is the linear array

$$a_1, a_2, a_3, \dots, a_N .$$

Numbers which appear without suffixes (including numbers having an index only above them--not a suffix--e.g. $\Delta^k s$) are regarded as zero-order quantities. For example the symbol "s" denotes a zero order quantity.

A.4. Summation and indexing conventions

Note that the following conventions are extremely important to following the discussions in this paper. Five conventions are observed in regard to indexed quantities, unless otherwise stated in the text:

1. In expressions where a particular lower case littorial index occurs exactly once in any one term the expression is regarded as an abbreviation for the whole sequence of numerical indices. E.g. $x_k^{ij}$ means (for expansion in the index i)

$$x_k^{1j}, x_k^{2j}, x_k^{3j}, \dots, x_k^{Nj}$$

and similarly for expansion in the indices j and k.

2. In expressions where a particular lower case littorial suffix occurs exactly twice[61] in any one term the expression is regarded as stipulating summation[62] on that suffix. E.g. $x_i y^i$ means

$$\sum_{i=1}^{N} x_i y^i = x_1 y^1 + x_2 y^2 + x_3 y^3 + \cdots + x_N y^N .$$

3. The above two conventions do not apply to numerical indices, e.g. to "$v^0$."

4. Latin indices assume the range of integers from 1 to N. Greek indices assume the range of integers from 0 to N. N has a fixed value throughout the discussion. E.g. $x^\upsilon$ means

$$x^\upsilon, \quad \upsilon = 0,1,2,3,\ldots,N .$$

5. Indices occurring in parentheses are not regarded as defining an ordered entiry; e.g. $\lambda(k)$, $v^{(\upsilon)}$ are not 1st order entities; the

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[61] Suffixes occurring three or more times are mistakes. (This remark does not include other indices, above the quantity, e.g. the index $\epsilon$ in $\overset{\epsilon}{f}$.)

[62] The main advantage of the sequence and summation conventions is compactness of exposition. For example, the expression

$$\sum_{\eta=0}^{N} \sum_{\epsilon=0}^{N} \sum_{\sigma=0}^{N} g_{\eta\epsilon} v^\eta c_\sigma^{\epsilon\kappa} \overset{\sigma}{f}, \quad \kappa = 0,1,2,3,\ldots N$$

would be written instead as

$$g_{\eta\epsilon} v^\eta c_\sigma^{\epsilon\kappa} \overset{\sigma}{f} .$$

summation convention does not apply to such indices.

The primary exception to the above is that for some expressions the range of some indices and the range of summation on some paired suffices is up to some integer greater than N. This is indicated in the text.

A.5. Transformation

If two quantities $\bar{a}_{\delta\sigma \ldots \eta}$ and $a_{\alpha\beta \ldots \epsilon}$ are related by

$$\bar{a}_{\delta\sigma \ldots \eta} = c_{\alpha\delta} c_{\beta\sigma} \ldots c_{\epsilon\eta} a_{\alpha\beta \ldots \epsilon}$$

then $c_{\alpha\delta}$ is said to define a transformation from $a_{\alpha\beta \ldots \epsilon}$ to $\bar{a}_{\delta\sigma \ldots \eta}$. E.g., if $x^i$ and $y^j$ are related by

$$x^i = \sum_{j=1}^{N} c_{ij} y^j$$

$$= c_{ij} y^j \quad \text{(by the summation convention)}$$

then $c_{ij}$ defines a transformation from $y^j$ to $x^i$. Note that in this paper the above definition will be applied even if the order of the two connected quantities are different. E.g., if the suffix i in the above example had a range of $i = 1, \ldots, M \neq N$, $c_{ij}$ would still be regarded as defining, what we shall call, an "improper" transformation. Except for those results which involve the inverse transformation (defined below), the following results are indifferent to the respective orders. (The scalar invariance of inner products, eq. (7) below, is the only result which assumes the existence of an inverse.) The concept

of improper transformation relates to discussions of the "improper reference frame $\varphi$," beginning with eq. (24), below.

A.6. Inverse transformation.

If the range of the two indices $\alpha$ and $\eta$ of a transformation $c_{\alpha\eta}$ are equal, and if there exists a quantity $c'_{\lambda\alpha}$ such that

$$\sum_{\alpha} c'_{\lambda\alpha} c_{\alpha\delta} = c'_{\lambda\alpha} c_{\alpha\eta} \quad \text{(using the summation convention)}$$

$$= \delta_{\lambda\eta}$$

then $c'_{\lambda\alpha}$ shall be called the inverse transformation of $c_{\alpha\eta}$.

B. Definitions and Elementary Properties of Various Tensor Quantities

B.1. Contravariant tensor, $a^{\alpha\beta\epsilon \cdots \gamma}$

Imagine that there is a definite collection of objects and that for this collection there are two alternative systems of measurement $S$ and $S'$ each of which implies the values of a particular first order quantity associated with each object. Let $x^{\epsilon}$ denote the numerical values of this quantity as implied by the system $S$ and, $x'^{\eta}$, the numerical values of the same quantity as implied by $S'$, for a particular object. Suppose also that there exists a transformation $c_{\eta\epsilon}$ such that

$$x'^{\eta} = c_{\eta\epsilon} x^{\epsilon} \tag{1}$$

Then the $c_{\eta\epsilon}$ are referred to as the <u>transformation from</u> $S$ <u>to</u> $S'$ and $S$ and $S'$ are called <u>reference frames</u> for the $x$ quantities.

A contravariant tensor is defined as any arbitrary nth order quantity

1. the values of which as implied by S are $a^{\alpha\beta\epsilon \cdots \gamma}$ and, as implied by $S'$ are $a'^{\delta\sigma\eta \cdots \lambda}$,
2. for which $a'^{\delta\sigma\eta \cdots \lambda} = c_{\delta\alpha} c_{\sigma\beta} c_{\eta\epsilon} \cdots c_{\lambda\gamma} a^{\alpha\beta\epsilon \cdots \gamma}$.

The $a^{\alpha\beta\epsilon \cdots \gamma}$ and the $a'^{\delta\sigma\eta \cdots \lambda}$ are then referred to as the original and transformed components of the same tensor in S and $S'$, respectively. If there exists another system of measurement $S''$ implying values $x''^{\sigma}$ related to $x'^{\eta}$ by a set of quantities $b_{\sigma\eta}$ in the manner of eq. (1), then $S''$ is also a reference frame and any quantity

$$a''^{\delta\sigma\eta \cdots \lambda} = b_{\delta\alpha} b_{\sigma\beta} b_{\eta\epsilon} \cdots b_{\lambda\gamma} a'^{\alpha\beta\epsilon \cdots \gamma}$$

forms the components in $S''$ of a tensor if $a'^{\alpha\beta\epsilon \cdots \lambda}$ are the components in $S'$ of a tensor.

B.2. Contravariant vector, $a^{\alpha}$

A contravariant vector is defined as a first order contravariant tensor.[62] Unless otherwise stated, the term "vector" shall mean contravariant vector.

B.3. Scalar invariant

A scalar s is regarded as a zero-order tensor if 1) s is either a constant or is a function of one or more other tensors, 2) the value of s remains unchanged upon substitution of the values

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[62] Note that $x^{\epsilon}$ and $x'^{\eta}$ above are thereby the components in S and $S'$, respectively, of a contravariant vector.

of the transformed components in place of the original values for each tensor in its argument. In this case $s$ is also called a "scalar invariant" or just "invariant" of the transformation.

B.4. Metric tensor, $g_{\eta\epsilon}$ of a reference frame

Let $S$ and $S'$ be arbitrary reference frames and let $a^{\eta}$ and $a'^{\sigma}$ denote the components in $S$ and $S'$, respectively, of a vector representing the coordinate differences, as determined by $S$ and $S'$, between two fixed points in a space. Suppose that with $S$ there is associated a second order quantity $g_{\eta\epsilon}$ and with $S'$, a second order quantity $g'_{\sigma\kappa}$ such that

$$g'_{\sigma\kappa}a'^{\sigma}a'^{\kappa} = g_{\eta\epsilon}a^{\eta}a^{\epsilon} \tag{2}$$

Then $g_{\eta\epsilon}$ is called the <u>metric</u>[63] of $S$. Throughout the discussion it shall be assumed that $g_{\epsilon\eta}$ is symmetric, i.e. that

$$g_{\epsilon\eta} = g_{\eta\epsilon} \, . \tag{2a}$$

B.5. Covariant tensor, $a_{\alpha\beta\epsilon \ldots \gamma}$

Substitution into the left hand side of eq. (2) of the value of $a'^{\sigma}$ as given by eq. (1) shows that

$$\begin{aligned} g'_{\sigma\kappa}a'^{\sigma}a'^{\kappa} &= g'_{\sigma\kappa}(c_{\sigma\eta}a^{\eta})(c_{\kappa\epsilon}a^{\epsilon}) \\ &= g'_{\sigma\kappa}c_{\sigma\eta}c_{\kappa\epsilon}a^{\eta}a^{\epsilon} \, . \end{aligned} \tag{3}$$

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[63] By symmetry $g'_{\sigma\kappa}$ is the metric of $S'$.

Comparing eqs. (2) and (3) we obtain

$$g_{\eta\epsilon}a^{\eta}a^{\epsilon} = g'_{\sigma\kappa}c_{\sigma\eta}c_{\kappa\epsilon}a^{\eta}a^{\epsilon} \tag{4}$$

Because the components $a^{\eta}$ have (as yet) completely arbitrary (real) numerical values, eq. (4) is satisfied if and only if

$$g_{\eta\epsilon} = g'_{\sigma\kappa}c_{\sigma\eta}c_{\kappa\epsilon} \, . \tag{5}$$

This shows that the transformation $c_{\sigma\eta}$ relating the values of a contravariant tensor in S and S′ also relate the values of $g_{\eta\epsilon}$ and $g'_{\sigma\kappa}$, but in the opposite direction. That is, if $c_{\sigma\eta}$ transforms the components of a contravariant tensor from S to S′ then $c_{\sigma\eta}$ transforms the components of $g_{\sigma\kappa}$ to $g_{\eta\epsilon}$. A <u>covariant tensor</u> is defined as any arbitrary nth order entity

1. the values of which as implied by S are $a_{\alpha\beta\epsilon \ldots \gamma}$ and, as implied by S′ are $a'_{\delta\sigma\eta \ldots \lambda}$;
2. for which $a_{\alpha\beta\epsilon \ldots \gamma} = c_{\delta\alpha}c_{\sigma\beta}c_{\eta\epsilon} \cdots c_{\lambda\gamma}a'_{\delta\sigma\eta \ldots \lambda}$, where $c_{\eta\epsilon}$ is the transformation of eq. (1).

The following table summarizes the distinction between contravariant and covariant tensors.

$$a^{\alpha\beta\epsilon \ldots \gamma} \xrightarrow{c_{\eta\epsilon}} a'^{\delta\sigma\eta \ldots \lambda}$$

$$a'_{\delta\sigma\eta \ldots \lambda} \xrightarrow{c_{\eta\epsilon}} a_{\alpha\beta\epsilon \ldots \gamma}$$

Table 5. Distinction Between Contravariant and Covariant Tensors

From eq. (5) it is evident that the metric $g_{\eta\epsilon}$ is a second order covariant tensor. A covariant vector is defined as a first order covariant tensor.

B.6. Mixed tensor, $a^{\sigma\eta \ldots \epsilon}_{\alpha\beta \ldots \gamma}$

A mixed tensor is an nth order quantity which transforms like a contravariant tensor in some of its suffixes and like a covariant tensor in the others. For example, if $a'^{\sigma}_{\eta} = c_{\sigma\alpha} c'_{\beta\eta} a^{\alpha}_{\beta}$, where $c'_{\beta\eta}$ is the inverse of $c_{\eta\beta}$, then $a^{\alpha}_{\beta}$ is a mixed tensor.

B.7. Inner Product, $g_{\eta\epsilon} a^{\eta} b^{\epsilon}$ of two vectors

Let $c^{\eta\epsilon}$ again define the transformation of contravariant vectors from S to S′. Given arbitrary vectors $a^{\eta}$ and $b^{\epsilon}$, the inner product is defined as the scalar formed by

$$g_{\eta\epsilon} a^{\eta} b^{\epsilon} .$$

Let the components in the above expression relative to S′ be $g'_{\rho\lambda}$, $a'^{\rho}$ and $b'^{\lambda}$, respectively. Since, by eq. (5)

$$g_{\eta\epsilon} = c_{\sigma\eta} c_{\kappa\epsilon} g'_{\sigma\kappa} ,$$

we have

$$\begin{aligned} c'_{\eta\rho} c'_{\epsilon\lambda} g_{\eta\epsilon} &= c'_{\eta\rho} c_{\sigma\eta} c'_{\epsilon\lambda} c_{\kappa\epsilon} g'_{\sigma\kappa} \\ &= \delta_{\rho\sigma} \delta_{\lambda\kappa} g'_{\sigma\kappa} \\ &= g'_{\rho\lambda} , \end{aligned} \tag{6}$$

where $c'_{\eta\rho}$ is the inverse of $c_{\rho\eta}$ (defined such that $c'_{\eta\rho}c_{\sigma\eta} = \delta_{\rho\sigma}$). By supposition, the $a'^{\rho}$ and $b'^{\lambda}$ are transformed from the contravariant components $a^{\eta}$ and $b^{\epsilon}$ in S, so that

$$\begin{aligned} g'_{\rho\lambda}a'^{\rho}b'^{\lambda} &= (c'_{\eta\rho}c'_{\epsilon\lambda}g_{\eta\epsilon})(c_{\rho\alpha}a^{\alpha})(c_{\lambda\beta}b^{\beta}) \\ &= (c'_{\eta\rho}c_{\rho\alpha})(c'_{\epsilon\lambda}c_{\lambda\beta})g_{\eta\epsilon}a^{\alpha}b^{\beta} \\ &= \delta_{\eta\alpha}\delta_{\epsilon\beta}g_{\eta\epsilon}a^{\alpha}b^{\beta} \\ &= g_{\eta\epsilon}a^{\eta}b^{\epsilon} . \end{aligned} \tag{7}$$

This shows that inner product is a scalar invariant of the space.

B.8. Mutual orthogonality of two vectors, $g_{\eta\epsilon}a^{\eta}a^{\epsilon} = 0$

If, for two vectors $a^{\eta}$ and $b^{\epsilon}$ it is true that their inner product vanishes, i.e.

$$g_{\eta\epsilon}a^{\eta}b^{\epsilon} = 0 ,$$

then they will be described as <u>mutually orthogonal</u> in the space.

B.9. Lowering of vector suffixes

Suppose that $a^{\epsilon}$ and $a'^{\eta}$ denote the components in S and S′ respectively of an arbitrary vector. Now define first order quantities $a_{\sigma}$ and $a_{\lambda}$ by

$$\begin{aligned} a_{\eta} &= g_{\eta\epsilon}a^{\epsilon} \\ a'_{\sigma} &= g'_{\sigma\kappa}a'^{\kappa} \end{aligned} \tag{8}$$

where the g's are the respective components of the metrics of S and $S'$. From eq. (5)

$$g_{\eta\epsilon}a^{\epsilon} = g'_{\sigma\kappa}c_{\sigma\eta}c_{\kappa\epsilon}a^{\epsilon} .$$

Substituting the $S'$ value of $c_{\kappa\epsilon}a^{\epsilon}$ shows that

$$g_{\eta\epsilon}a^{\epsilon} = g'_{\sigma\kappa}c_{\sigma\eta}a'^{\kappa}$$

$$= c_{\sigma\eta}(g'_{\sigma\kappa}a'^{\kappa}) .$$

From the identities of eqs. (6), this is equivalent to

$$a_{\eta} = c_{\sigma\eta}a'_{\sigma} \tag{9}$$

which shows that $a_{\eta}$ and $a'_{\sigma}$ are the components of a covariant vector in S and $S'$, respectively. In view of this result, it is possible to express any contravariant vector to which the metric tensor has been applied, in the manner of eqs. (6), as a covariant vector. In this case, $a_{\sigma}$ is referred to as the covariant form of the vector $a^{\epsilon}$ in S and similarly, in $S'$. The application of the metric to a vector in this manner is called "lowering the index" of the vector. In particular, the invariant of eq. (7) can be expressed as

$$g_{\eta\epsilon}a^{\eta}b^{\epsilon} = a_{\epsilon}b^{\epsilon} . \tag{10}$$

C. Miscellaneous derived results

C.1. Orthogonality of force and velocity

The following relation holds between the velocity of an object and the force acting upon it. Consider the scalar

$$mg_{\eta\sigma}v^{\eta}dv^{\sigma}/d\tau = v^{\eta}f_{\eta} \ . \tag{11}$$

By eq. (2.4)

$$v^{\eta}f_{\eta} = kv^{\eta}v^{\epsilon}F_{\epsilon\eta}$$

from which

$$(1/k)v^{\eta}f_{\eta} = v^{\eta}v^{\epsilon}F_{\epsilon\eta} \tag{12}$$

Now the terms of the right hand side of eq. (12) can be written in a matrix in which the $\epsilon$th, $\eta$th component occupies the $\eta$th row and $\epsilon$th column, as follows:

$$\begin{matrix} v^{o}v^{o}F_{oo}, & v^{o}v^{1}f_{1o}, & \cdots & v^{o}v^{N}F_{No} \\ v^{1}v^{o}F_{o1}, & v^{1}v^{1}F_{11}, & \cdots & v^{1}v^{N}F_{N1} \\ \vdots & \vdots & & \vdots \\ v^{N}v^{o}F_{oN}, & v^{N}v^{1}F_{1N}, & \cdots & v^{N}v^{N}F_{NN} \ . \end{matrix}$$

The summation convention states that $v^{\eta}v^{\epsilon}F_{\epsilon\eta}$ is to be formed by adding all of these matrix elements together. By eq. (2.5) the diagonal elements of this matrix vanish.[64] Each below-diagonal element $v^{(\eta)}v^{(\epsilon)}F_{(\epsilon)(\eta)}$, $\eta > \epsilon$ can be paired with exactly one above-diagonal element $v^{(\epsilon)}v^{(\eta)}F_{(\eta)(\epsilon)}$ which, by eq. (2.5), is its negative. Upon summation the off-diagonal terms eq. (12) will therefore cancel pairwise

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[64] i.e. they all become = 0.

(because of the anti-symmetry), i.e.

$$v^{\eta}v^{\epsilon}F_{\epsilon\eta} = 0$$

which shows that

$$v^{\eta}f_{\eta} = 0 \tag{13}$$

i.e. force and velocity are orthogonal. Comparison with eq. (12) shows that also

$$v^{\eta}dv_{\eta}/d\tau = g_{\eta\sigma}v^{\eta}dv^{\sigma}/d\tau = 0 \tag{14}$$

C.2. Transformation law of velocity and accelleration

Since $x^{\eta}$ is a vector it transforms to a different reference frame according to

$$x'^{\kappa} = c_{\kappa\eta}x^{\eta} .$$

Therefore

$$dx^{\kappa}/d\tau = d/d\tau(c_{\kappa\eta}x^{\eta})$$

$$= c_{\kappa\eta}dx^{\eta}/d\tau$$

or

$$v'^{\kappa} = c_{\kappa\eta}v^{\eta} , \tag{15}$$

which shows that $v^{\eta}$ is also a vector. This result in turn implies that

$$dv'^{\kappa}/d\tau = d/d\tau(c_{\kappa\eta}v^{\eta})$$

$$= c_{\kappa\eta}dv^{\eta}/d\tau \qquad (16)$$

i.e. $dv^{\eta}/d\tau$ is also a vector. Therefore, by the result of eq. (7) the expression

$$v^{\eta}dv_{\eta}/d\tau$$

in eq. (14) is a scalar invariant. Now define $a^{\sigma} = dv^{\sigma}/d\tau$. By the definition of $v^{\sigma} = dx^{\sigma}/d\tau$, $a^{\sigma}$ is the second derivative of $x^{\sigma}$ with respect to $\tau$, i.e. $a^{\sigma} = d^2x^{\sigma}/d\tau^2$. $a^{\sigma}$ shall be referred to as the accelleration with respect to $\tau$ of the object with position $x^{\sigma}$. From eq. (16), accelleration is also a vector.

C.3. Absolute invariance of velocity

Consider the derivative $d/d\tau(v_{\eta}v^{\eta})$. Evaluation of this derivative yields

$$d/d\tau(v_{\eta}v^{\eta}) = d/d\tau(g_{\sigma\eta}v^{\sigma}v^{\eta})$$

$$= g_{\sigma\eta}d/d\tau(v^{\sigma}v^{\eta})$$

$$= g_{\sigma\eta}(v^{\sigma}dv^{\eta}/d\tau + v^{\eta}dv^{\sigma}/d\tau)$$

$$= g_{\sigma\eta}v^{\sigma}dv^{\eta}/d\tau + g_{\sigma\eta}v^{\eta}dv^{\sigma}/d\tau \ .$$

By symmetry of the metric the last line can be rewritten as

$$g_{\sigma\eta}v^{\sigma}dv^{\eta}/d\tau + g_{\eta\sigma}v^{\eta}dv^{\sigma}/d\tau \ ,$$

which, by the summation convention can also be written as

$$g_{\eta\sigma}v^{\eta}dv^{\sigma}/d\tau + g_{\eta\sigma}v^{\eta}dv^{\sigma}/d\tau$$

$$= 2g_{\eta\sigma}v^{\eta}dv^{\sigma}/d\tau \ .$$

By eq. (14), the last line vanishes, which shows that

$$d/d\tau(v_{\eta}v^{\eta}) = 0 \ . \tag{17}$$

Integrating both sides of eq. (17) with respect to $\tau$ yields

$$\int d/d\tau(v_{\eta}v^{\eta})d\tau = \int d(v_{\eta}v^{\eta}) = v_{\eta}v^{\eta} = \text{constant} \tag{18}$$

which, by the result of eq. (7) is a scalar invariant. Define the quantity $s$ by

$$s^2 = g_{\epsilon\eta}x^{\epsilon}x^{\eta}$$

$$= x_{\eta}x^{\eta} \ .$$

Then

$$ds^2/d\tau^2 = d/d\tau^2(x_{\eta}x^{\eta})$$

$$= (dx_{\eta}/d\tau)(dx^{\eta}/d\tau)$$

$$= v_{\eta}v^{\eta}$$

$$= \text{constant},$$

or

$$ds/d\tau = (\text{constant})^{1/2} = c,$$

where $c$ is defined as the value of the root of the constant. Thus

$$s = \int ds = \int (\text{constant})^{1/2}\, d\tau = (\text{constant})^{1/2}\, \tau \qquad (19)$$

This shows that $s$ and $\tau$ are related to each other by a scalar constant which is an invariant. Let this constant be denoted by $c$. Then eqs. (18) and (19) can be written as

$$\left.\begin{aligned} v_\eta v^\eta &= c^2 \\ s &= c\tau \end{aligned}\right\}. \qquad (20)$$

From its definition $c$ is evidently real. Since $\tau$ is assumed to be real $s$ must also be real.

C.4. Characteristics of the improper reference frame $\varphi$

Let $v_\delta^\alpha$ denote the $\alpha$th contravariant component of the velocity of the $\delta$th nation, relative to S. Assume there are $M + 1 \geq N + 1$ many nations. Let the new "improper" frame $\varphi$ be defined as the class of quantities

$$\bar{a}_{\delta\sigma \ldots \eta} = v_\delta^\alpha v_\sigma^\beta \ldots v_\eta^\epsilon a_{\alpha\beta \ldots \epsilon};\ \delta,\sigma,\ldots,\eta = 0,1,\ldots,M \qquad (21)$$

where the $a_{\alpha\beta \ldots \epsilon}$ are the components of an arbitrary covariant tensor relative to S; i.e. $\varphi$ is the "frame" into which quantities in S are transformed by the $v_\epsilon^\eta$. In particular is

$$\bar{g}_{\kappa\sigma} = v^{\eta}_{\kappa} v^{\epsilon}_{\sigma} g_{\eta\epsilon}, \qquad \kappa,\sigma = 0,1,\ldots,M . \tag{22}$$

If $a^{\delta\sigma\ldots\eta}$ is an arbitrary contravariant tensor relative to S, then we make the following assumption: that there exist quantities $\bar{a}^{\alpha\beta\ldots\epsilon}$ such that

$$a^{\delta\sigma\ldots\eta} = v^{\delta}_{\alpha} v^{\sigma}_{\beta} \ldots v^{\eta}_{\epsilon} \bar{a}^{\alpha\beta\ldots\epsilon} \tag{23}$$

This, in effect, is an assumption of the existence of contravariant tensors in $\varphi$, complimentary to the definition of covariant tensors in $\varphi$ given by eq. (21). Any quantity of the type $\bar{a}^{\alpha}$ shall be referred to as a contravariant vector relative to $\varphi$. Applying eq. (23) to the metric tensor $g^{\eta\mu}$ we see that there exists a $\bar{g}^{\sigma\epsilon}$ such that

$$g^{\eta\mu} = v^{\eta}_{\sigma} v^{\mu}_{\epsilon} \bar{g}^{\sigma\epsilon} .$$

Multiplying the above equation by $a_{\eta} a_{\mu}$ on both sides and applying eq. (21) shows that

$$a^{\eta} a^{\mu} g_{\eta\mu} = a_{\eta} a_{\mu} g^{\eta\mu} = a_{\eta} a_{\mu} v^{\eta}_{\sigma} v^{\mu}_{\epsilon} \bar{g}^{\sigma\epsilon}$$

$$= \bar{a}_{\sigma} \bar{a}_{\epsilon} \bar{g}^{\sigma\epsilon} \tag{23a}$$

In words, $\bar{g}^{\sigma\epsilon}$ preserves, in $\varphi$, the quantity $x^{\eta} x^{\mu} g_{\eta\mu}$ (i.e. the "magnitude" of any vector $x^{\eta}$ is preserved in transformations from S to $\varphi$). $\bar{g}^{\sigma\epsilon}$ will be accordingly referred to as the "metric" of $\varphi$.[65]

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[65] In Appendix B.5, above, it was shown how the assumption that vector magnitude is preserved between two reference frames implies the covariant transformation law, Table 5 and eq. (21), for metric tensors. Here somewhat the contrary procedure has been employed: the contravariant transformation law, eq. (23) is assumed, from which the preservation of vector magnitude, eq. (23a) is derived.

Now let $\bar{v}_\kappa$ denote the velocity of an arbitrary nation relative to $\varphi$, as defined in eq. (21). Suppose the nations in the space to be enumerated such that this is the "oth" nation (i.e. it is the first one enumerated). Let $v_\eta$ denote the covariant velocity of this nation relative to S. Then by lowering of indices[66]

$$v_\eta = g_{\eta\epsilon} v_o^\epsilon \tag{24}$$

and, by eq. (21)

$$\bar{v}_\kappa = v_\kappa^\eta v_\eta, \quad \kappa = 0,1,\ldots,M . \tag{25}$$

Substitution of eq. (24) into eq. (25) shows that

$$\bar{v}_\kappa = v_\kappa^\eta g_{\eta\epsilon} v_o^\epsilon, \quad \kappa = 0,1,\ldots,M . \tag{26}$$

Now let $\bar{x}^\sigma$ be the components in $\varphi$ of any vector which is parallel to the oth axis of $\varphi$. Then $\bar{x}^\sigma = 0$ for $\sigma \neq 0$ and by lowering of indices

$$\begin{aligned} \bar{x}_\kappa/\bar{x}^o &= (\bar{g}_{\kappa\sigma}\bar{x}^\sigma)/\bar{x}^o \\ &= \bar{g}_{\kappa o}\bar{x}^o/\bar{x}^o \\ &= \bar{g}_{\kappa o} , \quad \kappa = 0,1,\ldots,M . \end{aligned} \tag{27}$$

By eq. (22)

$$\bar{g}_{\kappa o} = v_\kappa^\eta v_o^\epsilon g_{\eta\epsilon} , \quad \kappa = 0,1,\ldots,M$$

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[66] See Technical Appendix B.9, above.

which, by rearrangement of the right hand terms and, by reference to eq. (26) shows

$$\bar{g}_{\kappa o} = v^{\eta}_{\kappa} g_{\eta\epsilon} v^{\epsilon}_{o}$$

$$= \bar{v}_{\kappa} , \quad \kappa = 0,1,\ldots,M . \tag{28}$$

Comparing eqs. (28) and (27), we see that

$$\bar{v}_{\kappa} = \bar{x}_{\kappa}/\bar{x}^{o} , \quad \kappa = 0,1,\ldots,M , \tag{29}$$

i.e. that the velocity of the oth (arbitrary) nation is parallel to the oth axis of $\varphi$.

At this juncture we would like to conclude, by raising of indices, that $\bar{v}^{\kappa} = \bar{x}^{\kappa}/\bar{x}^{o}$. This is not possible, however, because the greater dimensionality of $\varphi$ compared with S implies that $\bar{g}_{\nu\epsilon}$ is singular, and raising of indices in a reference frame implicitly involves taking the inverse of the covariant tensor of that frame. A different approach must be employed. Backing up a bit, we see that by its definition

$$\bar{x}_{\nu}/\bar{x}^{o} = \bar{g}_{\nu\epsilon}\bar{x}^{\epsilon}/\bar{x}^{o} .$$

Thus eq. (29) is equivalent to

$$\bar{g}_{\nu\epsilon}\bar{x}^{\epsilon}/\bar{x}^{o} = \bar{v}_{\nu} .$$

Substitution of $\bar{g}_{\nu\epsilon}$ as given by eq. (22) into the above shows that

$$v^{\sigma}_{\nu} v^{\kappa}_{\epsilon} g_{\sigma\kappa}\bar{x}^{\epsilon}/\bar{x}^{o} = \bar{v}_{\nu} .$$

Multiplying both sides by $v^{\nu}_{\alpha}$ yields

$$v^{\nu}_{\alpha} v^{\sigma}_{\nu} v^{\kappa}_{\epsilon} g_{\sigma\kappa} \overline{x}^{\epsilon}/\overline{x}^{o} = v^{\nu}_{\alpha} \overline{v}_{\nu} \, . \tag{29a}$$

Because of the greater dimensionality of $\varphi$, the order of $\nu$ exceeds the order of $\alpha$ and $\sigma$, the determinant

$$|k^{\sigma}_{\alpha}| = |v^{\nu}_{\alpha} v^{\sigma}_{\nu}| \neq 0 \, , \tag{29b}$$

provided that the v's are linearly independent across $\nu$ for differing paired values of $\alpha$ and $\sigma$. But the assumption that the determinants $|\overset{\epsilon}{c}_{\sigma\eta}| \neq 0$, eq. (3.6) implies that the v's are linearly independent in all components except the $\epsilon$th which implies they are linearly independent in all components. Let k' denote the inverse of k. By eq. (29b) k' exists and from eq. (29a)

$$\begin{aligned} k'^{\alpha}_{\beta} v^{\nu}_{\alpha} \overline{v}_{\nu} &= k'^{\alpha}_{\beta} v^{\nu}_{\alpha} v^{\sigma}_{\nu} v^{\kappa}_{\epsilon} g_{\sigma\kappa} \overline{x}^{\epsilon}/\overline{x}^{o} \\ &= \delta^{\sigma}_{\beta} v^{\kappa}_{\epsilon} g_{\sigma\kappa} \overline{x}^{\epsilon}/\overline{x}^{o} \\ &= v^{\kappa}_{\epsilon} g_{\beta\kappa} \overline{x}^{\epsilon}/\overline{x}^{o} \, . \end{aligned}$$

Multiplication of both sides of the above by $g^{\lambda\beta}$ shows that

$$\begin{aligned} g^{\lambda\beta} k'^{\alpha}_{\beta} v^{\nu}_{\alpha} \overline{v}_{\nu} &= \delta^{\lambda}_{\kappa} v^{\kappa}_{\epsilon} \overline{x}^{\epsilon}/\overline{x}^{o} \\ &= v^{\lambda}_{\epsilon} \overline{x}^{\epsilon}/\overline{x}^{o} \, . \end{aligned} \tag{29c}$$

At this point the greater dimensionality of $\varphi$ prevents the further isolation of $\overline{x}^{\epsilon}/\overline{x}^{o}$ on the right of eq. (29c), since the order of $\epsilon$ exceeds the order of $\lambda$. By application of eqs. (23) and (21), and lowering of indices of $v^{\mu}$ in S, however, we have that

$$\overline{v}_\nu = v^\alpha_\nu g_{\alpha\mu} v^\mu_\sigma \overline{x}^\sigma$$

$$= v_{\nu\mu} v^\mu_\sigma \overline{v}^\sigma \ .$$

Substitution of the above into the left hand side of eq. (29c) and replacement of the bound index $\sigma$ by $\epsilon$ shows

$$v^\lambda_\epsilon \overline{x}^\epsilon / \overline{x}^o = g^{\lambda\beta} k' {}^\alpha_\beta v^\nu_\alpha v_{\nu\mu} v^\mu_\sigma \overline{v}^\sigma$$

$$= g^{\lambda\beta} k' {}^\alpha_\beta v^\mu_k v_{\sigma\mu} \overline{v}^\sigma$$

$$= v^\lambda_\sigma \overline{v}^\sigma$$

$$= v^\lambda_\epsilon \overline{v}^\epsilon \ . \qquad (29d)$$

Since the $v^\lambda_\epsilon$ are arbitrary; eq. (29d) shows that

$$\overline{x}^\epsilon / \overline{x}^o = \overline{v}^\epsilon \qquad (29e)$$

which is the desired result.

The above equation and reference to the component values of $\overline{x}^\epsilon$ shows that the contravariant components in $\varphi$ of the velocity of the oth nation must be constrained by

$$\overline{v}^\sigma = 0, \quad \sigma \neq 0, \quad \sigma = 1, \ldots, M \ .$$

Since the oth nation was arbitrarily selected the same result is demonstrated, by appropriate relabeling of indices, for all nations; i.e. if $\overline{v}^\sigma_\epsilon$ is the velocity in $\varphi$ of the $\epsilon$th nation then

$$\bar{v}^{\sigma}_{\epsilon} = 0, \quad \sigma \neq \epsilon, \quad \sigma,\epsilon = 0,1,..,,M . \tag{30}$$

Evidently, $\varphi$ is defined such that its axes are respectively parallel to the velocities of nations.

By eq. (23) it is seen that

$$v^{\sigma} = v^{\sigma}_{\kappa}\bar{v}^{\kappa}, \quad \kappa = 0,1,\ldots,M \tag{31}$$

where $v^{\sigma}$, $\bar{v}^{\kappa}$ are the velocity in S and $\varphi$, respectively, of an arbitrary nation. Suppose again that the nations are labeled such that this arbitrary nation is the oth. Application of eq. (30) shows that $\bar{v}^{\kappa} = 0$, $\kappa \neq 0$ so that eq. (31) becomes

$$v^{\sigma} = v^{\sigma}_{o}\bar{v}^{o} .$$

But $v^{\sigma}$ and $v^{\sigma}_{o}$ both denote the velocity in S of the oth nation, i.e. $v^{\sigma} = v^{\sigma}_{o}$, so that the preceding equality can be written

$$v^{\sigma} = v^{\sigma}\bar{v}^{o} ,$$

This shows that $\bar{v}^{o} = 1$. As before, this result is valid for any nation, so eq. (30) can be elaborated as

$$\bar{v}^{\sigma}_{\epsilon} = \delta_{\sigma\epsilon} , \quad \sigma,\epsilon = 0,1,\ldots,M . \tag{32}$$

We will now show that the force equation (2.6) also applies in

$\varphi$. From the definition of $\varphi$,

$$\bar{f}_\eta = v_\eta^\epsilon f_\epsilon \tag{33}$$

where $\bar{f}_\eta$ are the covariant force components relative to $\varphi$. Substituting the value of $f_\epsilon$ given by eq. (2.6), we obtain

$$\bar{f}_\eta = k v_\eta^\epsilon v^\sigma F_{\epsilon\sigma} ,$$

where $v^\sigma$ is the velocity of an arbitrary nation, $F_{\epsilon\sigma}$ is the social field in the spatial region surrounding it, and $\bar{f}_\eta$ are the components in $\varphi$ of the force acting upon it. By rearrangement of terms and, upon substitution of the value of $v^\sigma$ given by eq. (31), we obtain

$$\begin{aligned} \bar{f}_\eta &= k v^\sigma v_\eta^\epsilon F_{\epsilon\sigma} \\ &= k(v_\kappa^\sigma \bar{v}^\kappa) v_\eta^\epsilon F_{\epsilon\sigma} \\ &= k\bar{v}^\kappa (v_\kappa^\sigma v_\eta^\epsilon F_{\epsilon\sigma}) . \end{aligned}$$

But the quantity in parentheses is the transformation to $\varphi$ of the covariant tensor $F_{\epsilon\sigma}$; i.e. $v_\kappa^\sigma v_\eta^\epsilon F_{\epsilon\sigma} = \bar{F}_{\eta\kappa}$. Thus

$$\bar{f}_\eta = k\bar{v}^\kappa \bar{F}_{\eta\kappa} \tag{34}$$

which is the same force equation in $\varphi$ as eq. (26) in S.[67] Expansion of terms and application of eq. (32) simplifies this equality to

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[67] $k$ is assumed to be a scalar invariant of the transformation.

$$\bar{f}_\eta = k(\bar{v}^{o}_{\epsilon}\bar{F}_{\eta o} + \bar{v}^{1}_{\epsilon}\bar{F}_{\eta 1} + \cdots + \bar{v}^{(\epsilon)}_{\epsilon}\bar{F}_{\eta(\epsilon)} + \cdots + \bar{v}^{N}_{\epsilon}\bar{F}_{\eta N})$$

$$= k\bar{v}^{(\epsilon)}_{\epsilon}\bar{F}_{\eta(\epsilon)}$$

$$= k\bar{F}_{\eta(\epsilon)} \quad . \tag{34a}$$

For the $\epsilon$th nation; i.e. the $N + 1$ components of $\bar{f}_\eta$ are proportional to $N + 1$ of the components of the social field. An additional simplification is implied by the orthogonality of force and velocity, eq. (13). Let $f^{(\sigma)}_{\epsilon}$ denote the force acting on the $\sigma$th nation. Then, by eq. (13),

$$v^{\epsilon}_{\sigma}\, f^{(\sigma)}_{\epsilon} = 0 \quad .$$

By eq. (33) this is the $\sigma$th component of $f^{(\sigma)}_{\eta}$, i.e.

$$f^{(\sigma)}_{\sigma} = v^{\epsilon}_{\sigma}\, f^{(\sigma)}_{\epsilon} \quad .$$

Combining the above two equations shows that eq. (34a) can be amended to

$$\bar{f}^{\sigma}_{\eta} = \begin{cases} 0, & \eta = \sigma \\ k\bar{F}_{\eta\sigma}, & \eta \neq \sigma \end{cases} \quad . \tag{34b}$$

C.5. Characteristics of spatial components of the displacement of nations, $\Delta x^k$

Let $w_t$ denote the amount of behavior occuring during the calendar time period $t - 1$ to $t$, $t = 1,\ldots,T$, and let $\Delta a_{tk}$, $k = 1,\ldots,N \leq T$ denote the change in the $k$th attribute during the same period. We assume that the $\Delta a_{tk}$ are linearly independent in the index $k$; i.e.

that $\Delta a_{tk}$ is independent of any combination of the $\Delta a_{tm}$, $m \neq k$. Define $\beta_{\ell j}$ as the $\ell$th eigenvector[68] of $A_{jk} = \Delta a_{tj}\Delta a_{tk}$. We stipulate that the $\beta_{\ell j}$ are normed such that $\sum_{j=1}^{N} \beta_{\ell j}^2 = 1$. Define $q_{t\ell}$ such that

$$\Delta a_{tj} = \beta_{\ell j} q_{t\ell} \cdot \tag{35}$$

We assume that the eigenvalues of $A_{jk}$ are distinct. Then it can be shown[69] that

$$\beta_{mj}\beta_{\ell j} = \delta_{m\ell} \cdot$$

Therefore

$$\beta_{mj}\Delta a_{tj} = \beta_{mj}\beta_{\ell j} q_{t\ell} = \delta_{m\ell} q_{t\ell} = q_{tm} \tag{36}$$

which gives the explicit solution of the q's. Now assume that there are quantities $\alpha^m$ such that

$$w_t = q_{tm}\alpha^m + e_t \tag{37}$$

where $e_t$ is such that

$$q_{tm}e_t = 0 \tag{38}$$

for arbitrary behavior $w_t$; i.e. we assume that any behavior variable is a linear function across time of the q's, plus an error term $e_t$ which is orthogonal to the q's. Now define

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[68] Bradbury (1968), p. 105

[69] Bradbury (1968), pp. 102, 106-107.

$$c_{km} = q_{tk}q_{tm} .$$

By the assumption of linear independence in $k$ of the $\Delta a_{tk}$ and by eq. (36), the $q_{tm}$ are linearly independent in $m$ and $|c_{km}| \neq 0$. Therefore the inverse $c'_{\ell k}$ of $c_{km}$ is defined and $\alpha_\ell$ can be solved as

$$q_{tk}w_t = q_{tk}q_{tm}\alpha^m = c_{km}\alpha^m$$

from which

$$\begin{aligned} c'_{\ell k}q_{tk}w_t &= c'_{\ell k}c_{km}\alpha^m \\ &= \delta_{\ell m}\alpha^m \\ &= \alpha^\ell , \end{aligned} \tag{39}$$

which shows that $\alpha^\ell$ is well defined. By eq. (36)

$$\begin{aligned} q_{tm}q_{tk} &= (\beta_{mj}\Delta a_{tj})(\beta_{k\ell}\Delta a_{t\ell}) \\ &= \beta_{mj}\beta_{k\ell}A_{j\ell} . \end{aligned}$$

Since $\beta_{mj}$ is the $m$th eigenvector of $A_{j\ell}$, it is by definition that

$$\beta_{mj}\beta_{k\ell}A_{j\ell} = \overset{k}{\lambda}\delta_{mk} ,$$

where $\overset{k}{\lambda}$ is the $k$th eigenvalue of $A_{j\ell}$, which shows that

$$q_{tm}q_{tk} = \overset{k}{\lambda}\delta_{mk} . \tag{40}$$

Then, using eqs. (37), (38) and (40) we can express $\sum_{t=1}^{T} w_t^2$ explicitly as a function of the $\alpha$'s and $\lambda$'s:

$$\sum_{t=1}^{T} w_t^2 = \Sigma\,(q_{tm}\alpha^m + e_t)^2$$

$$= (q_{tm}\alpha^m + e_t)(q_{tk}\alpha^k + e_t)$$

$$= (q_{tm}q_{tk})\alpha^m\alpha^k + 2q_{tm}e_t\alpha^m + \sum_{t=1}^{T} e_t^2$$

$$= {}^{k}\lambda\delta_{mk}\alpha^m\alpha^k + \sum_{t=1}^{T} e_t^2$$

$$= \sum_{k=1}^{N} {}^{k}\lambda(\alpha^k)^2 + \sum_{t=1}^{T} e_t^2 \,. \qquad (41)$$

Suppose that the kth spatial component of displacement is defined by

$$\Delta x^k = T^{1/2}\,{}^{k}\lambda^{1/2}\,\alpha^k \qquad (42)$$

Then multiplication of both sides of equation (41) by T, and substitution of $\Delta x^o$ from eq. (4.4) and the $\Delta x^k$ given by eq. (42) shows that

$$(\Delta x^o)^2 = \sum_{k=1}^{N} (\Delta x^k)^2 + T \sum_{t=1}^{T} e_t^2$$

and, since $T \sum_{t=1}^{T} e_t^2 \geq 0$, we have

$$(\Delta x^o)^2 \geq \sum_{k=1}^{N} (\Delta x^k)^2$$

which is the same as eq. (4.2a). This shows that the displacement components $\Delta x^\eta$ as defined by eqs. (4.4) and (42) are consistent with the metric given by eq. (4.1).

From eqs. (4.2), (37), and (42) it is evident that

$$s^2 = (\Delta x^o)^2 - \sum_{k=1}^{N} (\Delta x^k)^2$$

$$= T \sum_{t=1}^{T} e_t^2 \qquad (43)$$

i.e. that the square of the distance traveled by a nation in the social space during the time $t = 0$ to $t = T$ is proportional to the variation of the error term $e_t$ occuring in the regression eq. (37). From the definitions of $q_{tm}$, $\beta_{mj}$ and $x^k$ it follows that

$$T\alpha^{(k)} q_{tk} w_t = T q_{tk} q_{tm} \alpha^m \alpha^{(k)}$$

$$= T\lambda^k \delta_{mk} \alpha^m \alpha^{(k)}$$

$$= T\lambda^k (\alpha^k)^2$$

$$= (\Delta x^k)^2 \, . \qquad (44)$$

The expression on the left can also be written as

$$T\alpha^{(k)} q_{tk} w_t = T^2 \alpha^{(k)} [1/T \, (q_{tk} - \bar{q}_k)(w_t - \bar{w}) + \bar{q}_k \bar{w}] \qquad (45)$$

where $\bar{q}_k$ and $\bar{w}$ are the respective arithmetic means of $q_{tk}$ and $w_t$ across $t$. Since the first term within the brackets of eq. (45) is the covariance of $q_{tk}$ with $w_t$, comparison of eqs. (44) and (45) shows that $(\Delta x^k)^2$ is proportional to the sum of the covariance of $q_{tk}$ with $w_t$ and the product of their means.

Suppose now that the behavior of some particular nation is $w_t = w$, a constant, $t = 1,\ldots,T$ and that some attribute $a_1$ of

this nation is related to the $q$'s by $\beta_{11} = 1$, $\beta_{\ell 1} = 0$, $\ell > 1$. Then by eq. (35)

$$\Delta a_{t1} = \beta_{\ell 1} q_{t\ell} = q_{t1} .$$

Substituting the above values for $q_{t1}$ and $w_t$ into eq. (44) shows that

$$\begin{aligned} (\Delta x^1)^2 &= T\alpha^1 q_{t1} w_t \\ &= T\alpha^1 \Delta a_{t1} w_t \\ &= T\alpha^1 w \left( \sum_{t=1}^{T} \Delta a_{t1} \right) . \end{aligned} \tag{46}$$

By eq. (40)

$$\begin{aligned} c_{km} &= q_{tk} q_{tm} \\ &= \overset{m}{\lambda} \delta_{mk} . \end{aligned}$$

Therefore

$$c'_k = (\overset{\ell}{\lambda})^{-1} \delta_{\ell k}$$

and eq. (39) becomes

$$\begin{aligned} \alpha^\ell &= c'_{\ell k} q_{tk} w_t \\ &= (\overset{\ell}{\lambda})^{-1} \delta_{\ell k} q_{tk} w_t \\ &= (\overset{\ell}{\lambda})^{-1} q_{t\ell} w_t . \end{aligned} \tag{47}$$

In particular, for the $\alpha^1$ of eq. (46)

$$\alpha^1 = (\overset{1}{\lambda})^{-1} q_{t1} w_t$$

$$= (\overset{1}{\lambda})^{-1} w \sum_{t=1}^{T} \Delta a_{t1} .$$

Substituting the above into eq. (46) shows that

$$(x^1)^2 = Tw^2 (\overset{1}{\lambda})^{-1} \left( \sum_{t=1}^{T} \Delta a_{t1} \right)^2$$

or

$$x^1 = T^{1/2} w (\overset{1}{\lambda})^{-1/2} \sum_{t=1}^{T} \Delta a_{t1} \qquad (48)$$

i.e. the total change in the attribute $a_1$ of the given nation is then proportional to its displacement $x^1$.

The conditions under which a particular displacement component $\Delta x^i$ vanishes can be seen from the following. The solution of the $\alpha^i$ is shown by eq. (47) to be proportional to

$$q_{ti} w_t = (q_{ti} - \bar{q}_i)(w_t - \bar{w}) + T \bar{q}_i \bar{w} .$$

Since the constant of proportionality $= \overset{i}{\lambda}^{-1}$, the reciprocal of the ith eigenvalue, $\overset{i}{\lambda}$ finite proves $\overset{i}{\lambda}^{-1} \neq 0$. Thus, the vanishing of $\Delta x^{oi}$ requires that

$$1/T \; (q_{ti} - \bar{q}_i)(w_t - \bar{w}) = -\bar{q}_i \bar{w} . \qquad (49)$$

The left hand side of eq. (49) is the covariance of $q_{ti}$ with $w_t$ over time. By construction $w_t \geq 0$ which shows $\bar{w} \geq 0$. If $\bar{w} = 0$

then $w_t = 0$, which satisfies eq. (49) for all values of the $q_{ti}$. This case is trivial since the actor nation then exhibits no behavior whatever, obviating the whole analysis. If $\bar{w} > 0$ and if $\bar{q}_i = 0$, then eq. (49) is satisfied if $q_{ti}$ and $w_i$ are uncorrelated.[70] If $\bar{q}_i < 0$ then eq. (49) is satisfied if $q_{ti}$ and $w_i$ have positive covariance of magnitude $-\bar{q}_i\bar{w}$. If $\bar{q}_i > 0$ the eq. (49) is satisfied if $q_{ti}$ and $w_t$ have negative covariance of magnitude $|-\bar{q}_i\bar{w}| = \bar{q}_i\bar{w}$. Thus, for $\Delta x^{o_i}$ to vanish in a non-trivial way either 1) $q_{ti}$ and $w_i$ are uncorrelated over time or 2) they have positive or negative covariance of magnitude $\bar{q}_i\bar{w}$, depending on the sign of $\bar{q}_i$.

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[70] In the sense of the Pearson product moment correlation.

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