CORE AND NO-TREAT EQUILIBRIUM IN TOURNAMENT GAMES WITH EXTERNALITIES

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ABSTRACT. We consider a situation where coalitions are formed to divide a resource. As in real life, the value of a payoff to a given agent is allowed to depend on the payoff to other agents with whom he shares a common interest. There are various notions of equilibrium for this type of game, including the core and no-treat equilibrium. These stabilities may exist or not, depending on the power structure and the rule for allocating the resource. It is shown that under certain conditions, the no-treat equilibrium can exist even though the core is empty.

1. Introduction

Today, game theory is increasingly being used to model interactions in social science, political science, psychology, and especially economics. But it is actually a field of applied mathematics, one that attempts to mathematically capture behavior in strategic situations in which an individual's success in making choices depends on the choices of others.

The reasons I chose game theory for my master's project were that I got interested in it when I studied game theory in an undergraduate political science class, and that it is closely related to our everyday experience.

Although game theory is used in a lot of disciplines, it requires many advanced mathematical techniques, such as analysis, linear algebra, abstract algebra, and so on. However, game theory is often applied in these disciplines without using those advanced techniques. So I would like to extend my mathematical skills to consider one of the models in economics which is coalition-formation.

Also, I would like to thank Professor Juarez and Professor Nation who have been helping me a lot for this project. Without their help, I cannot complete this project. Also, I would like to thank everyone who helped for this project to complete.

Date: May 15, 2010.

2. Definitions and concepts

We are going to analyze agents who are looking to maximize their share of a divisible resource by being a singleton or forming coalitions.

Let M be a divisible resource, say money. Let N be a set of agents $N = \{1, 2, ..., n\}$, and each agent has an additive preference on his share of money and other agents' shares. Each agent has a power described as $\pi_1, \pi_2, ..., \pi_n$ with $\pi_i \geq 0$ and $\sum_{i=1}^{n} \pi_i = 1$. Also, without loss of generality, we assume that $\sum_{i \in S} \pi_i \neq \sum_{j \in T} \pi_j$ for all $S \neq T$, so we do not have any ties.

A *partition* is a collection of disjoint subsets $S_1, S_2, .., S_k$ of N where $\bigcup_{i=1}^{k} S_i = N$. Each subset S_j in the partition is called a *coalition*. The power of a coalition S is given by $\pi(S) = \sum_{i \in S} \pi_i$. The winning coalition for a partition $\Pi = (S_1, ..., S_k)$ is the subset S_j with $\pi(S_j)$ maximum.

The idea, of course, is that a group of agents might want to form a coalition in order to win the game, or more generally, in order to increase their net utilities (see below).

Let ζ be a function that specifies the allocations of the resource across the winning agents. That is, for any agent $i \in S \subseteq N$, $\zeta_i(S)$ is the allocation of the money to agent i with $\sum_{i=1}^{n} \zeta_i(S) = M$ when coalition S is winning. We assume that ζ is *cross-monotonic* on the size of the coalition, that is $\zeta_i(S) > \zeta_i(T)$ for $i \in S \subset T$.

We are going to consider two rules for dividing money to agents in the winning coalition, which are equal sharing and proportional sharing. Let me introduce these two sharing rules. Let S be the winning coalition.

(1) Equal sharing is given by

$$\zeta_i(S) = \begin{cases} \dfrac{M}{|S|} & \text{if } i \in S, \\ 0 & \text{otherwise.} \end{cases}$$

So under equal sharing, all agents in the winning coalitions share the same amount of the resource.

(2) Proportional sharing is given by

$$\zeta_i(S) = \begin{cases} \dfrac{\pi_i}{\pi(S)} M & \text{if } i \in S, \\ 0 & \text{otherwise.} \end{cases}$$

where $\pi(S) = \sum_{j \in S} \pi_j$ for notational convenience. So under proportional sharing, each agent's share depends on his power and the total power in the winning coalition.

There are other sharing rules besides above two rules, but we only consider these two in this project.

Definition 1. *Externality* is basically a situation in which each agent cares not only about himself, but also possibly cares about the other agents. Those relationships are represented by an $n \times n$ matrix for n agents, with entries m_{ij} representing the externality that agent j imposes on agent i. The *payoff function* is a vector $U = mx$ defined by $U_i(x_1, x_2, ..., x_n) = (mx)_i = \sum_j m_{ij} x_j$. We often assume that $m_{ii} = 1$ as a kind of normalization. In fact, whenever $m_{ii} > 0$ then you can divide that row of m by m_{ii} and not change the results of the game.

Definition 2. For a partition Π, the *net utility* to agent i is $v_i(\Pi) = (m\zeta)_i = \sum_{j \in S^*} m_{ij} \zeta_j(S^*)$ where S^* is the coalition in Π with the largest power. The object of the game is to maximize your net utility. That is, agent i tries to find the partition that maximizes $v_i(\Pi)$. With externalities, this is not the same as maximizing $\zeta_i(\Pi)$. Moreover, with externalities, it is definitely not the same as winning in the sense of being in the coalition with the most power.

Definition 3. The *core* is the set of all partitions with the property that no subset $S \subseteq N$ can improve their net utilities by forming a coalition. That is, a partition Π is in the core (or Π is core-stable) if there does not exist $S \subseteq N$ such that $v_i(\Pi - S, S) > v_i(\Pi)$ for all $i \in S$. Thus there is no set of agents S that would be all be better off by forming a coalition. (If $\Pi = (T_1, ..., T_n)$, then $(\Pi - S, S)$ denotes the partition $(T_1 - S, ...T_n - S, S)$.)

Definition 4. Similarly to the core, we define an alternative notion, *no-treat equilibrium*, call it NTE, under which agents can react to a deviation in a way that harms the agents who originally deviated. So Π is NTE (or Π is *NTE-stable*) if whenever $S \subseteq N$ is such that $v_i(\Pi - S, S) > v_i(\Pi)$ for all $i \in S$, then there exists $T \subseteq N - S$ such that $v_i(\Pi - (S \cup T), T, S) > v_i(\Pi - S, S)$ for all $i \in T$ and $v_i(\Pi - (S \cup T), T, S) < v_i(\Pi)$ for some $i \in S$. This means that at least one member of S will in the long run not profit by deviating to form the coalition S.

Note that the distinction between the core and NTE is that with a core-stable partition, no group can gain an advantage by forming a new coalition. With an NTE-stable partition, it may be possible for a group to deviate and gain a temporary advantage by forming a new coalition - but if they do so, then yet another coalition can form to punish them.

Definition 5. A *minimally winning coalition* is a winning coalition $S^* \subseteq N$ satisfying $\pi(S^*) > 1/2 > \pi(S^* - \{j\})$ for all $j \in S^*$.

Definition 6. A *minimally winning coalition of minimal size* is a minimally winning coalition $S^* \subseteq N$ satisfying $|S^*| \leq |S|$ for all $S \subseteq N$ with S minimally winning. This is used under the equal sharing.

Definition 7. The *minimally winning coalition of minimal weight* is minimally winning coalition $T^* \subseteq N$ such that $\pi(T^*) \leq \pi(T)$ for all $T \subseteq N$ minimally winning. This is used under the proportional sharing.

With these definitions, what we are going to consider is the following:

- Does there always exist a core-stable partition?
- Can we characterize the set of rules sufficient to have a core-stable partition?
- Does there always exist a NTE-stable partition?
- Can we characterize the set of rules sufficient to have a NTE-stable partition?

3. No externalities

Observe that, if there are no externalities, then as we increase the size of a coalition any winning agent is worse off (because his share decreases), and agents who are not winning would prefer to be winning (because their net utility is zero when losing). This will also be true in the case with externalities under weak conditions.

3.1. **Equally shared case.**

For equally shared case, we cannot guarantee that the core always exists. However, NTE always exists. Here is an example where the core does not exist, but NTE exists.

Example 8. Consider the game with $N = \{1, 2, 3, 4, 5\}$ and $\pi = (.36, .335, .12, .105, .08)$. Then each of the following are partitions with minimally winning coalitions. Now, consider whether these minimally winning coalitions are in the core or NTE.

- Consider the partition $(\{1,2\},\{3,4,5\})$. It is not in the core, because if agent 1 deviates, then he wins: $\pi(1) = .36, \pi(2) = .335, \pi(3,4,5) = .305$. However, it is NTE because if he deviates, then the rest can punish him: in fact, agent 2 can combine with any of 3,4,5 to beat him.
- Consider the partition $(\{1,3,4\},\{2,5\})$. It is not in the core: agent 1 cannot deviate by himself, since $\pi(2,5) = .415$. But he could form a coalition with either agent 3 or 4 that would win. However, if he does so, then the remaining three agents can form a coalition to beat him. So it is NTE.
- $(\{1,4,5\},\{2,3\})$ is not in the core: agent 1 can form a new coalition with agent 2 to get more benefit. He could form a coalition with either agent 4 or 5 that would win. However, if he does so, then the remaining three agents can form a coalition to beat him. So it is NTE.
- Similarly above, $(\{2,3,4\},\{1,5\})$, $(\{2,4,5\},\{1,3\})$, $(\{1,3,5\},\{2,4\})$, and $(\{2,3,5\},\{1,4\})$ is NTE, but not in the core because agent 1 and 2 can be together.

As we have seen above, this game has NTE, for example $(\{1,2\},\{3,4,5\})$. However, there does not exist any core-stable partition because agent(s) in the winning coalition can always deviate. So we found an example that has NTE and empty core.

We are going to characterize below the set of tournaments that have a core-stable partition for equal sharing. Before that, I would like to state a lemma. For next three lemma and propositions, we assume that there is no externality, $m_{ii} > 0$ for all i, and the rule is equally shared.

Lemma 9. *For either equal sharing or proportional sharing,*

1. *if S is a winning coalition and S is contained in T properly, then each member of S is worse off (has a lower net utility) in T than in S.*
2. *Each agent would prefer to be winning rather than losing.*

Proof. Both (1) and (2) are clear in case with no externalities. However this is not very clear in case with externalities. I will show that in lemma 24. □

Proposition 10. Let $N = \{1,2,...,n\}$ and $\pi = (\pi_1, \pi_2, ..., \pi_n)$. If $\pi_i > \frac{1}{2}$ for some i, then $(\{i\}, S_1, S_2, ..., S_m)$ is always in the core (and so in NTE) where $S_j \subseteq N - \{i\}$ and $S_j \cap S_k = \emptyset$.

Proof. This proposition follows from the definition of the core immediately.

□

It is clear that proposition 10 can be applied to the proportionally shared case. Next, we look at a different case.

Proposition 11. Let $N = \{1, 2, 3\}$, $\pi = (\pi_1, \pi_2, \pi_3)$ with $\frac{1}{2} > \pi_1 > \pi_2 > \pi_3$. Then $(\{1\}, \{2, 3\})$ is always in the core, and any partition with one coalition and a singleton is in NTE. This proposition only applies for equal sharing rule.

Proof. This depends only on lemma 9. Since we have $\frac{1}{2} > \pi_1 > \pi_2 > \pi_3$, we know that no agent can win the game being singleton. Now, consider some coalitions. If we have $\Pi = (\{1, 2\}, \{3\})$, then 1 can deviate, so this is not in the core. But if agent 1 deviates, then agent 2 can form a coalition with agent 3 and this leads agent 1 to lose. Therefore, agent 1 does not want to deviate from $\Pi = (\{1, 2\}, \{3\})$, Hence Π is in NTE. The same thing will happen if we have $\Pi = (\{1, 3\}, \{2\})$. However if we have $(\{1\}, \{2, 3\})$, then agent 2 and 3 are better off, so $(\{1\}, \{2, 3\})$ is in the core, so in NTE. □

Proposition 12. For equal sharing with no externality, we have the following results.

(1) If Π is in the core, then $\Pi = (S^*, S_1, ..., S_k)$ with S^* a winning coalition $\pi(S^*) > 1/2$ of minimal size.
(2) If $\Pi = (S^*, S_1, ...S_k)$ is in the core with S^* winning, then $\Pi' = (S^*, N - S^*)$ is also in the core.
(3) $\Pi = (S^*, N - S^*)$ is in the core if and only if there are not subsets $T \subseteq S^*$ and $U \subseteq N - S^*$ such that $\pi(T \cup U) > \pi(S^* - T)$ with $|T \cup U| < |S^*|$ and $\pi(T \cup U) > \pi(N - S - U)$ for some $U \subseteq N - S^*$.

Proof. (1) This is clear since otherwise some new coalition can form to beat S^*.
(2) This is also obvious since otherwise any new coalition that could form to beat S^* in Π', could also form to beat it in Π.
(3) ($\Leftarrow$) Suppose that S^* be the minimally winning coalition of minimal size, and there is no subset $T \subseteq S^*$ such that $\pi(T \cup U) > \pi(S^* - T)$ with $|T \cup U| < |S^*|$ and $\pi(T \cup U) > \pi(N - S - U)$ for some $U \subseteq N - S^*$. This condition implies that no agent in S^* wants to deviate since deviation makes his payoff lower. It follows that all agents in the winning coalition S^* are better off in S^*, that is, there is no subset $T \subseteq N$ such that $v_i(\Pi - T, T) > v_i(\Pi)$ for all $i \in S$, and it follows that $\{S^*, N - S^*\}$ is in the

core by definition.
($\Rightarrow$) Suppose that $\Pi = \{S^*, S_1, S_2, ..., S_m\}$ is a core-stable partition where S^* has the largest power and is of the minimal size. Then each agent in S^* will get $M/|S^*|$ for which they are better off. Now suppose for a contradiction that there is a subset $T \subseteq S^*$ such that $\pi(T \cup U) > \pi(S^* - T)$ with $|T \cup U| < |S^*|$ and $\pi(T \cup U) > \pi(N - S - U)$ for some $U \subseteq N - S^*$ Then agent in the coalition $T \cup U$ gets $M/|T \cup U|$ which is greater than $M/|S^*|$ because we know that $|T \cup U| < |S^*|$. This contradicts that all agents in S^* are better off with $M/|S^*|$. Hence if the core is nonempty, then we have there is no subset $T \subseteq S^*$ such that $\pi(T \cup U) > \pi(S^* - T)$ with $|T \cup U| < |S^*|$ for some $U \subseteq N - S^*$. □

We have found some conditions of tournament games to have a core-stable partition for equal sharing case. In particular, we found that the core is nonempty when $n = 3$, and may be empty when $n = 5$. What happens when $n = 4$? I will leave this question for the future work. But next proposition shows that NTE is always nonempty for this case.

Proposition 13. Under the equal sharing rule, if S^* is minimally winning coalition of minimal size, then any partition with S^* is NTE.

Proof. Let S^* be a minimally winning coalition of minimal size. Since we know that $|S^*| \leq |S|$ for all $S \subseteq N$ with S minimally winning, we get $v_i(S^*) > v_j(S)$ for $i \in S^*$ and $j \in S$. Now, we also know that $\pi(S^*) > 1/2 > \pi(S^* - \{j\})$ for all $j \in S^*$. This implies that $\pi(S^* - \{j\}) < \pi(N - S^* \cup \{j\})$. Hence $(\{S^*\}, \{N - S^*\})$ is in NTE. □

3.2. **Proportionally shared case.** Next, we look at the proportionally shared case. The proportionally shared case is slightly more complicated than equally shared case. It does not only depend on the power structure of the whole set N, but also the power structure of the coalition.

Proposition 14. Let $N = \{1, 2, ..., n\}$ and $\pi = (\pi_1, \pi_2, ..., \pi_n)$. If $\pi_i > \frac{1}{2}$ for some i, then $(\{i\}, S_1, S_2, ..., S_m)$ is always in the core (and so in NTE) where $S_j \subseteq N - \pi_i$ and $S_j \cap S_k = \emptyset$ in proportionally shared case.

Proof. As in proposition 10, this also follows from the definition of the core. □

Proposition 15. Let $N = \{1,2,3\}$, $\pi = (\pi_1, \pi_2, \pi_3)$ with $\frac{1}{2} > \pi_1 > \pi_2 > \pi_3$. then $\Pi_1 = (\{1\}, \{2,3\})$ is always in the core, and NTE coincides with the core in proportionally shared case.

Proof. For the core, it is obvious as in proposition 11. We need to show that $\Pi_2 = (\{1,2\}, \{3\})$ and $\Pi_3 = (\{1,3\}, \{2\})$ are not in NET. Since we have $v_2(\Pi_1) = \pi_2/(\pi_1+\pi_2) > v_2(\Pi_2) = \pi_2/(\pi_2+\pi_3)$, agent 2 prefers to form a coalition with agent 3. Hence Π_2 is not in NTE. Similarly, we have $v_3(\Pi_1) = \pi_3/(\pi_2+\pi_3) > v_3(\Pi_3) = \pi_3/(\pi_1+\pi_3)$, so agent 3 prefers to form a coalition with agent 2. Hence Π_3 is not in NTE. Hence $\Pi_1 = (\{1\}, \{2,3\})$ is always in the core, and NTE coincides with the core in proportionally shared case. □

Proposition 16. For proportional sharing with no externality, we have the following results.

(1) If Π is in the core, then $\Pi = (S^*, S_1, ..., S_k)$ with S^* a winning coalition $\pi(S^*) > 1/2$ of minimal weight.
(2) If $\Pi = (S^*, S_1, ...S_k)$ is in the core with S^* winning, then $\Pi' = (S^*, N - S^*)$ is also in the core.
(3) $\Pi = \{S^*, N - S^*\}$ is in the core if and only if there is no subset $T \subseteq S^*$ such that $\pi(T \cup U) > \pi(S^* - T)$ with $\pi(T \cup U) < \pi(S^*)$ and $\pi(T \cup U) > \pi(N - S - U)$ for some $U \subseteq N - S^*$.

Proof. (1) This is clear since otherwise some new coalition can form to beat S^*.
(2) This is also obvious since otherwise any new coalition that could form to beat S^* in Π', could also form to beat it in Π.
(3) ($\Leftarrow$) Suppose that S^* be the minimally winning coalition of minimal weight, and there is no subset $T \subseteq S^*$ such that $\pi(T \cup U) > \pi(S^* - T)$ with $\pi(T \cup U) < \pi(S^*)$ and $\pi(T \cup U) > \pi(N - S - U)$ for some $U \subseteq N - S^*$. This condition implies that no agent in S^* wants to deviate since deviation makes his payoff lower. It follows that all agents in the winning coalition S^* are better off in S^*, that is, there is no subset $T \subseteq N$ such that $v_i(\Pi - T, T) > v_i(\Pi)$ for all $i \in S$, and it follows that $\{S^*, N - S^*\}$ is in the core by definition.
($\Rightarrow$) Suppose that $\Pi = \{S^*, S_1, S_2, ..., S_m\}$ is a core-stable partition where S^* has the largest power and is of the minimal weight. Then each agent j in S^* will get $\pi_j M/|\pi(S^*)|$ for which they are better off. Now suppose for a contradiction that there is a subset $T \subseteq S^*$ such that $\pi(T \cup U) > \pi(S^* - T)$ with $\pi(T \cup U) < \pi(S^*)$ and $\pi(T \cup U) > \pi(N - S - U)$ for

some $U \subseteq N - S^*$ Then agent j in the coalition $T \cup U$ gets $\pi_j M/\pi(T \cup U)$ which is greater than $\pi_j M/|\pi(S^*)|$ because we know that $\pi(T \cup U) < \pi(S^*)$. This contradicts that all agents in S^* are better off with $\pi_j M/|\pi(S^*)|$. Hence if the core is nonempty, then we have there is no subset $T \subseteq S^*$ such that $\pi(T \cup U) > \pi(S^* - T)$ with $\pi(T \cup U) < \pi(S^*)$ and $\pi(T \cup U) > \pi(N - S - U)$ for some $U \subseteq N - S^*$.

□

Proposition 17. For proportionally shared case, the minimally winning coalition of minimal weight is always in NTE.

Proof. Let S^* be the minimally winning coalition of minimal weight. Since we know that $\pi(S^*) \leq \pi(S)$ for all $S \subseteq N$ minimally winning, we get $v_i(S^*) > v_j(S)$ for $i \in S^*$ and $j \in S$. Now, we also know that $\pi(S^*) > 1/2 > \pi(S^* - \{j\})$ for all $j \in S^*$. It follows that $\pi(S^* - \{j\}) < \pi(N - S^* \cup \{j\})$, that is, deviation makes the coalition S^* losing. Hence the partition $(\{S^*\}, \{N - S^*\})$ is NTE. □

We have found some conditions of tournament games to have a core-stable and NTE-stable partitions for proportional sharing case. The question is what is the smallest N such that the core can be empty in the proportionally shared case? I will leave this question for the future work.

4. With externality

Now, we consider the cases with externality. That means each agent cares not only about his own share, but also possibly cares about the other agents' shares. This relationship is represented by an $n \times n$ matrix for n agents. Let's look at some examples of the externality matrix to see what kind of power structures makes the core empty, but NTE exists.

Example 18. Consider $N = \{1, 2, 3, 4\}$ where agents 1 and 2 are Muslims, and agents 3 and 4 are Catholic. Agents 1 and 2 care about each other, but do not care about 3 and 4. Similarly, agents 3 and 4 care about each other, but do not care about 1 and 2. In this case, the externality matrix could be following:

$$m = \begin{bmatrix} 1 & \alpha & 0 & 0 \\ \alpha & 1 & 0 & 0 \\ 0 & 0 & 1 & \beta \\ 0 & 0 & \beta & 1 \end{bmatrix}$$

with $1 > \alpha > 0$ and $1 > \beta > 0$.

4.1. **Equally shared case.** Observe that for what π the core is nonempty. Since agents 1 and 2 prefer to form a coalition and so do agents 3 and 4, then for equal sharing, the final benefit will be the following.

If {1,2} are the coalition with the largest power, then we have

$$U_1(\frac{1}{2},\frac{1}{2},0,0)=\frac{1}{2}(1+\alpha)$$

$$U_2(\frac{1}{2},\frac{1}{2},0,0)=\frac{1}{2}(1+\alpha)$$

If {3,4} are the coalition with the largest power, then we have

$$U_3(0,0,\frac{1}{2},\frac{1}{2})=\frac{1}{2}(1+\beta)$$

$$U_4(0,0,\frac{1}{2},\frac{1}{2})=\frac{1}{2}(1+\beta)$$

Let $\pi=(.40,.22,.28,.10)$. Observe some partitions. It is not hard to see that the only possible partition to have core-stable partition and NTE is $(\{1,2\},\{3,4\})$. For this coalition, agent 1 prefers to be a singleton, but cannot deviate because $\pi_1<\pi_2+\pi_3+\pi_4$. So for this π this game has NTE, but the core is empty.

4.2. **Proportionally shared case.** Let $\pi=(\pi_1,\pi_2,\pi_3,\pi_4)$ and observe the payoff.

$$U_1=x_1+\alpha x_2 \qquad U_2=\alpha x_1+x_2$$

$$U_3=x_3+\beta x_4 \qquad U_4=\beta x_3+x_4$$

In this case, x_i depends on π, α, and β. If $\pi=(.40,.39,.11,.10)$, $\alpha=1/10$, and $\beta=1/10$, then agent 1 might prefer to form a coalition with agent 3 instead of agent 2 since

$$v_1(\{1,2\},\{3,4\})=\frac{40}{40+39}+\frac{1}{10}\frac{39}{40+39}\simeq 0.56,$$

and

$$v_1(\{1,3\},\{2,4\})=\frac{40}{40+11}\simeq 0.78,$$

so

$$v_1(\{1,2\},\{3,4\})<v_1(\{1,3\},\{2,4\}).$$

In particular, if we have $\frac{\pi_1}{\pi_1+\pi_3}>\frac{\pi_1}{\pi_1+\pi_2}(1+\alpha)$, then we can guarantee that agent 1 prefers to form a coalition with agent 3 because $v_1(\{1,2\}\{3,4\})<v_1(\{1,3\}\{2,4\})$.

Example 19. Consider $N = \{1, 2, 3, 4\}$ with agents 1 and 2 are Yankees fans, and agents 3 and 4 are Mets fans. Agents 1 and 2 do not care about each other, but dislike agents 3 and 4. Similarly, agents 3 and 4 do not care about each other, but hate agents 1 and 2. In this case, the externality matrix could be following,

$$m = \begin{bmatrix} 1 & 0 & \beta & \beta \\ 0 & 1 & \beta & \beta \\ \alpha & \alpha & 1 & 0 \\ \alpha & \alpha & 0 & 1 \end{bmatrix}$$

with $\alpha < 0$ and $\beta < 0$.

4.3. **Equally shared case.** Similarly to the previous case, agents 1 and 2 prefer to form a coalition and so do agents 3 and 4. Then for equal sharing, the final benefit will be the following.

If {1,2} is the coalition with the largest power, then we have

$$U_1(\frac{1}{2}, \frac{1}{2}, 0, 0) = \frac{1}{2} \qquad U_2(\frac{1}{2}, \frac{1}{2}, 0, 0) = \frac{1}{2}$$

$$U_3(\frac{1}{2}, \frac{1}{2}, 0, 0) = \alpha < 0 \qquad U_4(\frac{1}{2}, \frac{1}{2}, 0, 0) = \alpha < 0$$

If {3,4} is the coalition with the largest power, then we have

$$U_1(0, 0, \frac{1}{2}, \frac{1}{2}) = \beta < 0 \qquad U_2(0, 0, \frac{1}{2}, \frac{1}{2}) = \beta < 0$$

$$U_3(0, 0, \frac{1}{2}, \frac{1}{2}) = \frac{1}{2} \qquad U_4(0, 0, \frac{1}{2}, \frac{1}{2}) = \frac{1}{2}$$

Let $\pi = (.40, .22, .28, .10)$ as before. Observe some partitions. It is not hard to see that the only possible coalition to have core and NTE is $(\{1, 2\}, \{3, 4\})$. For this coalition, agent 1 prefers to be a singleton, but cannot deviate because $\pi_1 < \pi_2 + \pi_3 + \pi_4$. So for this π this game has NTE, but the core is empty.

4.4. **Proportionally shared case.** Let $\pi = (\pi_1, \pi_2, \pi_3, \pi_4)$ and observe the payoff.

$$U_1 = x_1 + \beta x_3 + \beta x_4 \qquad U_2 = x_2 + \beta x_3 + \beta x_4$$

$$U_3 = \alpha x_1 + \alpha x_2 + x_3 \qquad U_4 = \alpha x_1 + \alpha x_2 + x_4$$

As in the previous example, x_i depends on π, α, and β. But since $\beta < 0$, agent 1 can hardly form a coalition with agent 3. Agent 1's payoff does not depend on α, we only consider π and β. Let $\pi = (.40, .39, .11, .10)$. Then if we have $\frac{\pi_1}{\pi_1 + \pi_3}(1 + \beta) > \frac{\pi_1}{\pi_1 + \pi_2}$, then we can guarantee that agent 1 prefers to form a coalition with agent 3 because $v_1(\{1,2\}\{3,4\}) < v_1(\{1,3\}\{2,4\})$.

Example 20. Let's look at one more example, which is $N = \{1,2,3,4\}$ with agents 1 and 2 are Muslim, and agents 3 and 4 are Jewish. Agents 1 and 2 care about each other, but dislike agents 3 and 4. Similarly, agents 3 and 4 care about each other, but dislike agents 1 and 2. In this case, the externality matrix could be following,

$$m = \begin{bmatrix} 1 & \alpha & \beta & \beta \\ \alpha & 1 & \beta & \beta \\ \gamma & \gamma & 1 & \delta \\ \gamma & \gamma & \delta & 1 \end{bmatrix}$$

with $1 > \alpha > 0$, $\beta < 0$, $\gamma < 0$, and $1 > \delta > 0$.

4.5. **Equally shared case.** Observe for what π is core nonempty. In this case agents 1 and 2 strongly prefer to form a coalition and so do agents 3 and 4, the final benefit will be the following.

If {1,2} are the coalition with the largest power, then we have

$$U_1(\frac{1}{2},\frac{1}{2},0,0) = \frac{1}{2}(1+\alpha) \qquad U_2(\frac{1}{2},\frac{1}{2},0,0) = \frac{1}{2}(1+\alpha)$$

$$U_3(\frac{1}{2},\frac{1}{2},0,0) = \gamma < 0 \qquad U_4(\frac{1}{2},\frac{1}{2},0,0) = \gamma < 0$$

If {3,4} is the coalition with the largest power, then we have

$$U_1(0,0,\frac{1}{2},\frac{1}{2}) = \beta < 0 \qquad U_2(0,0,\frac{1}{2},\frac{1}{2}) = \beta < 0$$

$$U_3(0,0,\frac{1}{2},\frac{1}{2}) = \frac{1}{2}(1+\delta) \qquad U_4(0,0,\frac{1}{2},\frac{1}{2}) = \frac{1}{2}(1+\delta)$$

4.6. **Proportionally shared case.** Let $\pi = (\pi_1, \pi_2, \pi_3, \pi_4)$ and observe the payoff.

$$U_1 = x_1 + \alpha x_2 + \beta x_3 + \beta x_4 \qquad U_2 = \alpha x_1 + x_2 + \beta x_3 + \beta x_4$$

$$U_3 = \gamma x_1 + \gamma x_2 + x_3 + \delta x_4 \qquad U_4 = \gamma x_1 + \gamma x_2 + \delta x_3 + x_4$$

In this case it is very hard for agent 1 to form a coalition with agent 3 and 4 because they are going to harm agent 1. But if we let $\pi = (.40, .39, .05, .16)$, $\alpha = 1/10$, and $\beta = -1/10$, then agent 1 might prefer to form a coalition with agent 3 instead of agent 2 since

$$v_1(\{1,2\},\{3,4\}) = \frac{40}{40+39} + \frac{1}{10}\frac{39}{40+39} \simeq 0.56,$$

and

$$v_1(\{1,3\},\{2,4\}) = \frac{40}{40+5} - \frac{1}{10}\frac{5}{40+5} \simeq 0.8,$$

so

$$v_1(\{1,2\},\{3,4\}) < v_1(\{1,3\},\{2,4\}).$$

In particular, if we have $\frac{\pi_1}{\pi_1+\pi_3}(1+\beta) > \frac{\pi_1}{\pi_1+\pi_2}(1+\alpha)$ then we can guarantee that agent 1 prefers to form a coalition with agent 3 because $v_1(\{1,2\}\{3,4\}) < v_1(\{1,3\}\{2,4\})$.

We have seen several examples to see how externality works in tournament games. Now, let's look at a specific example which has no NTE and empty core.

Example 21. We consider the model with following externality matrix for equally shared case;

$$m = \begin{bmatrix} 1 & \frac{1}{3} & 0 \\ 0 & 1 & \frac{1}{3} \\ \frac{1}{3} & 0 & 1 \end{bmatrix}$$

and $\pi = (1/3, 1/3, 1/3)$.
Suppose agents 1 and 2 form a coalition. Then the net utility is going to be

$$U_1(\frac{1}{2},\frac{1}{2},0) = 1\times\frac{1}{2} + \frac{1}{3}\times\frac{1}{2} + 0\times 0 = \frac{2}{3}$$
$$U_2(\frac{1}{2},\frac{1}{2},0) = 0\times\frac{1}{2} + 1\times\frac{1}{2} + \frac{1}{3}\times 0 = \frac{1}{2}$$
$$U_3(\frac{1}{2},\frac{1}{2},0) = \frac{1}{3}\times\frac{1}{2} + 0\times\frac{1}{2} + 1\times 0 = \frac{1}{6}$$

In this case, agent 2 can deviate and form a new coalition with agent 3 to get more benefit.

$$U_1(0,\frac{1}{2},\frac{1}{2}) = 1\times 0 + \frac{1}{3}\times\frac{1}{2} + 0\times\frac{1}{2} = \frac{1}{6}$$
$$U_2(0,\frac{1}{2},\frac{1}{2}) = 0\times 0 + 1\times\frac{1}{2} + \frac{1}{3}\times\frac{1}{2} = \frac{2}{3}$$
$$U_3(0,\frac{1}{2},\frac{1}{2}) = \frac{1}{3}\times 0 + 0\times\frac{1}{2} + 1\times\frac{1}{2} = \frac{1}{2}$$

In this case, agent 3 can deviate and form a new coalition with 1 to get more benefit.

$$U_1(\frac{1}{2},0,\frac{1}{2}) = 1\times\frac{1}{2} + \frac{1}{3}\times 0 + 0\times\frac{1}{2} = \frac{1}{2}$$
$$U_2(\frac{1}{2},0,\frac{1}{2}) = 0\times\frac{1}{2} + 1\times 0 + \frac{1}{3}\times\frac{1}{2} = \frac{1}{6}$$

$$U_3(\frac{1}{2}, 0, \frac{1}{2}) = \frac{1}{3} \times \frac{1}{2} + 0 \times 0 + 1 \times \frac{1}{2} = \frac{2}{3}$$

In this case, agent 1 can deviate and form a new coalition with agent 2 to get more benefit. Therefore, this particular example has no NTE and empty core.

Note that this situation is called a *cycle*. The cycle is a property of the externality matrix, not the game. Although it occurs in more than three agents case, I will state the condition for $n = 3$.

Let

$$m = \begin{bmatrix} 1 & m_{12} & m_{13} \\ m_{21} & 1 & m_{23} \\ m_{31} & m_{32} & 1 \end{bmatrix}$$

be externality matrix where $\sum_{i \neq j} |m_{ij}| < 1$. There are two possibilities to have cycle, the one is $m_{12} > m_{13}, m_{23} > m_{21}, m_{31} > m_{32}$, and the other is $m_{12} < m_{13}, m_{23} < m_{21}, m_{31} < m_{32}$.

Lemma 22. *If $n = 3$ and there is a cycle in the externality matrix, and if $\pi_i < 1/2$ for all i, then there is no partition in NTE.*

Proof. If $\pi_i < 1/2$, then a coalition has to be formed to be in NTE. However in no coalition, agents are better off since we know that $v_1(\{1,2\},\{3\}) > v_1(\{1,3\},\{2\})$, $v_2(\{2,3\},\{1\}) > v_2(\{1,2\},\{3\})$, $v_3(\{1,3\},\{2\}) > v_3(\{2,3\},\{1\})$ if $m_{12} > m_{13}, m_{23} > m_{21}, m_{31} > m_{32}$. Also, $v_1(\{1,2\},\{3\}) < v_1(\{1,3\},\{2\})$, $v_2(\{2,3\},\{1\}) < v_2(\{1,2\},\{3\})$, $v_3(\{1,3\},\{2\}) < v_3(\{2,3\},\{1\})$ if $m_{12} < m_{13}, m_{23} < m_{21}, m_{31} < m_{32}$ Hence there is no partition in NTE. □

Above, we looked at simple externality matrices. Next I would like to observe a general case of a two-person game.

Example 23. Let $\pi = (\pi_1, \pi_2)$ and assume without loss of generality $\pi_1 > \pi_2$. Consider the following externality matrix,

$$m = \begin{bmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{bmatrix}$$

Then their final utilities are the followings.

$$U_1(x_1, x_2) = \alpha_1 \times x_1 + \alpha_2 \times x_2$$

$$U_2(x_1, x_2) = \beta_1 \times x_1 + \beta_2 \times x_2$$

(1) Equally shared case

Let $\Pi_1 = (\{1\},\{2\})$ and $\Pi_2 = (\{1,2\})$. Then $v_1(\Pi_1) = \alpha_1$, $v_2(\Pi_1) = \beta_1$, $v_1(\Pi_2) = \alpha_1/2 + \alpha_2/2$, and $v_2(\Pi_2) = \beta_1/2 + \beta_2/2$. If $\alpha_1 > \alpha_2$, then agent 1 wants Π_1. Hence Π_1 is in the core. If

$\alpha_1 < \alpha_2$, then agent 1 prefers Π_2. If also $\beta_2 > \beta_1$, then both prefer Π_2 so that is in the core. If $\alpha_1 < \alpha_2$ but $\beta_2 < \beta_1$, then agent 1 wants Π_2 and agent 2 wants Π_1. Hence Π_1 is in the core. And Π_2 is not, since agent 2 could improve his share by deviating.

(2) Proportionally shared case

Let $\Pi_1 = (\{1\},\{2\})$ and $\Pi_2 = (\{1,2\})$. Then $v_1(\Pi_1) = \alpha_1$, $v_2(\Pi_1) = \beta_1$, $v_1(\Pi_2) = \alpha_1\pi_1 + \alpha_2\pi_2$, and $v_2(\Pi_2) = \beta_1\pi_1 + \beta_2\pi_2$. If $\alpha_1 > \alpha_2\pi_2/(1-\pi_1)$, then agent 1 wants Π_1. Hence Π_1 is in the core. If $\alpha_1 < \alpha_2\pi_2/(1-\pi_1)$, then agent 1 prefers Π_2. If also $\beta_1 < \beta_2\pi_2/(1-\pi_1)$, then both prefer Π_2 so that is in the core. If $\alpha_1 < \alpha_2\pi_2/(1-\pi_1)$ but $\beta_1 > \beta_2\pi_2/(1-\pi_1)$, then agent 1 wants Π_2 and agent 2 wants Π_1. Hence Π_2 is in the core. And Π_1 is not, since agent 1 could improve his share by forming a coalition.

We have seen more general case of externality in example 23. Next, I would like to state a lemma.

Lemma 24. *Let $S \subseteq N$. Assume that whenever $i \in S \subseteq N$ and $j \notin S$, we have $m_{ij} < \sum_{k\in S} \frac{m_{ik}}{|S|}$ for equally shared case and $m_{ij} < \sum_{k\in S} \frac{m_{ik}}{|\pi(S)|}$ for proportionally shared case. Then as we increase the size of coalition, any winning agent is worse off (cross-monotonicity) for both equal sharing and proportional sharing cases.*

Proof. First, prove for equally shared case. For an agent $i \in S$, the net utility in the coalition S is $U_i = \sum_{k\in S} m_{ik}\frac{M}{|S|}$. For the same agent, in the coalition $S \cup \{j\}$, the net utility is $U_i' = \sum_{k\in S\cup j} m_{ik}\frac{M}{|S|+1}$. Subtracting and simplifying, $U_i - U_i' = \frac{M}{|S|+1}(\sum_{k\in S}\frac{m_{ik}}{|S|} - m_{ij})$. So $U_i - U_i' > 0$ whenever $m_{ij} < \sum_{k\in S}\frac{m_{ik}}{|S|}$.

Next, we prove for proportionally shared case. For an agent $i \in S$, the net utility in the coalition S is $U_i = \sum_{k\in S} m_{ik}\frac{\pi_i M}{|\pi(S)|}$. For the same agent, in the coalition $S \cup \{j\}$, the net utility is $U_i' = \sum_{k\in S\cup j} m_{ik}\frac{\pi_i M}{|\pi(S\cup\{j\})|}$. Subtracting and simplifying,
$U_i - U_i' = \frac{M}{|\pi(S\cup\{j\})|}(\sum_{k\in S}\frac{m_{ik}}{|\pi(S)|} - m_{ij})$. So $U_i - U_i' > 0$ whenever $m_{ij} < \sum_{k\in S}\frac{m_{ik}}{|\pi(S)|}$. □

This lemma can be applied to the case with $m_{ii} = 1$ and $\sum_{i \neq j} |m_{ij}| < 1$. Now we are interested in finding a set of externality matrices for which we can guarantee that the core is empty or nonempty for any π.

Theorem 25. *In the case with externality, we have the following results.*

(1) *If $n = 2, 3$, then any matrix of the form*

$$m = \begin{bmatrix} 1 & \varepsilon & \varepsilon \\ \varepsilon & 1 & \varepsilon \\ \varepsilon & \varepsilon & 1 \end{bmatrix}$$

always has a nonempty core for any π in both equal sharing and proportional sharing.

(2) *Let $n = 3$ and*

$$m = \begin{bmatrix} 1 & m_{12} & m_{13} \\ m_{21} & 1 & m_{23} \\ m_{31} & m_{32} & 1 \end{bmatrix}$$

where $m_{12} \neq m_{13}$ and $m_{21} \neq m_{23}$ and $m_{31} \neq m_{32}$ and $\sum_{i \neq j} |m_{ij}| < 1$. Then we can find π such that (π, M) has empty core in both sharing rules.

Proof. (1) By the previous example, we can conclude that the core is always nonempty for any ϵ for two-person game. For three-person game, if $\varepsilon > 1$, then it is clear that $\Pi = (\{1, 2, 3\})$ is in the core because all three agents share the same amount, $1/3 + \varepsilon/3 + \varepsilon/3 > 1/2 + \varepsilon/2 > 1$. If $\varepsilon < 1$ let $\pi = \{\pi_1, \pi_2, \pi_3\}$ with $\pi_1 > \pi_2 > \pi_3$. For this externality matrix, we get $U_1(x_1, x_2, x_3) = x_1 + \varepsilon x_2 + \varepsilon x_3$, $U_2(x_1, x_2, x_3) = \varepsilon x_1 + x_2 + \varepsilon x_3$, $U_3(x_1, x_2, x_3) = \varepsilon x_1 + \varepsilon x_2 + x_3$. Therefore, it does not matter with which agent to form a coalition, their share will be the same. It follows that this game can be now treated as if this was not an externality case except for the final benefit. If we have $\pi_1 > \frac{1}{2}$ and $\varepsilon < 1$, then we are done because agent 1 can be a singleton as in proposition 10 for equal sharing and in proposition 14 for proportional sharing. Next, suppose that we have $\pi_1 < \frac{1}{2}$. Then by example 11 and 15, we know that $(\{2, 3\}, \{1\})$ is in the core. Hence any matrix of this form has a nonempty core if $n = 2, 3$ in both sharing rules.

(2) Let

$$m = \begin{bmatrix} 1 & m_{12} & m_{13} \\ m_{21} & 1 & m_{23} \\ m_{31} & m_{32} & 1 \end{bmatrix}$$

be externality matrix, not all m_{ij}'s are the same. First, consider the equal sharing rule, then we get $U_1(x_1, x_2, x_3) = x_1 + m_{12}x_2 + m_{13}x_3$, $U_2(x_1, x_2, x_3) = m_{21}x_1 + x_2 + m_{23}x_3$, $U_3(x_1, x_2, x_3) = m_{31}x_1 + m_{32}x_2 + x_3$. First of all, we have seen that if we have a cycle in the externality matrix and $\pi_i < 1/2$ for $i = 1, 2, 3$, then clearly we have an empty core. There are two possibilities to have cycle, the one is $m_{12} > m_{13}, m_{23} > m_{21}, m_{31} > m_{32}$, and the other is $m_{12} < m_{13}, m_{23} < m_{21}, m_{31} < m_{32}$. Let's look at m_{ij}'s which do not make cycles.
Case 1: Suppose $m_{12} > m_{13}$. Then if $m_{21} > m_{23}$, then $(\{1, 2\}, \{3\})$ will be the winning coalition and if we let $\pi_1 > \pi_2 > \pi_3$, the game has an empty core. If $m_{21} < m_{23}$, then we have $m_{31} < m_{32}$ in order not to have a cycle. Then $(\{2, 3\}, \{1\})$ will be the winning coalition and if we let $\pi_2 > \pi_3 > \pi_1$, the game has an empty core.
Case 2: Suppose $m_{12} < m_{13}$. Then if $m_{31} > m_{32}$, then $(\{1, 3\}, \{2\})$ will be the winning coalition and if we let $\pi_1 > \pi_3 > \pi_2$, the game has an empty core. If $m_{31} < m_{32}$, then we have $m_{21} < m_{23}$ in order not to have a cycle. Then $(\{2, 3\}, \{1\})$ will be the winning coalition and if we let $\pi_2 > \pi_3 > \pi_1$, the game has an empty core. Hence for any matrix of this form, we can find π such that (π, M) has empty core in equally shared case.
For proportional sharing case, we can apply above results to find π to make the core empty. For example, in case 1 we found that if $m_{12} > m_{13}$ and $m_{21} > m_{23}$, then $\pi_1 > \pi_2 > \pi_3$ makes the core empty. If we let $\pi_1 = \pi_2 + \delta = \pi_3 + 2\delta$ for very small δ, then the result is the same as in the equally shared case. Do the same thing to all the cases that we found in equally shared case, and we conclude that for any matrix of this form, we can find π such that (π, M) has empty core in proportionally shared case. Hence any matrix of this form has a nonempty core if $n = 2, 3$ in both sharing rules.

□

5. Conclusions and future work

We have been considering two sharing rules to analyze agents who are looking to maximize their share of a divisible resource by being a singleton or forming coalitions. We considered two cost sharing rules, and characterize the existence of the core and NTE. This is a good start for analyzing strategic situations. For the possible extensions of this project, we have the alternate equilibrium besides the core and

NTE. We can also consider the different ζ (cost sharing rules). I can think of one possible cost sharing rule, which is inversely proportional sharing. I am going to analyze that for what π and externality the game has nonempty core or empty core. And, of course, I need to prove the theorem 25 for more than three-person game. There should be a lot to be done beyond this project. I am looking to complete these extensions of this project in the near future.

References

[1] F. Bloch, *Sequential formation of coalitions in games with externalities and fixed payoff division*, Games Econ. Behav. **14** (1996), 901–923.

[2] I. Hafalir, *Efficiency in coalition games with externalities*, Games and Economic Behavior (2007) 61242-258

[3] E. Maskin, *Bargaining, coalitions and externalities. Working paper*, Institute for Advanced Study (2003)

[4] D. Ray, *A Game-Theoretic Perspective on Coalition Formation*, Oxford University Press (2007)

[5] G. de Clippel and R. Serrano, *Marginal contributions and externalities in the value*, Forthcoming Econometrica

[6] D. Ray and R. Vohra, *A theory of endogenous coalition structures*, Games Econ. Behav. **26** (1999), 286-336

[7] O. Moeschlin and D. Pallaschke, *Game Theory and Mathematical Economics*, North-Holland Publishing Company. (1981)

[8] E. Rasmusen, *Games and Information*, Blackwell Publishers. (1989)