COMPLEXITY OF INDEX SETS OF COMPUTABLE LATTICES

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ABSTRACT

We analyze computable algebras in the sense of universal algebra and the index set complexity of properties of such algebras. We look at the difficulty of determining properties of **Con** (**A**), the congruence lattice of an algebra **A**. In particular, we introduce the notion of a class of algebras *witnessing* the complexity of a property of algebras and show that computable lattices witness the Π^0_2-completeness of being simple, as well as witnessing the Σ^0_3-completeness of having finitely many congruences. Finally, in our main result, we show that the property "to be subdirectly irreducible" is Σ^0_3-complete as well, and in the process show that computable lattices witness this.

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CHAPTER 1
INTRODUCTION

When an undergraduate math student takes an algebra course, they may begin with the study of rings, groups, or perhaps monoids. Many computability researchers study computable rings or computable groups. It is interesting to consider the study of computable algebras in general, in the sense of universal algebra. In this paper we study properties of the congruence lattice of computable universal algebras and their complexity. In particular we show computable lattices witness Π^0_2-completeness of being simple, as well witnessing the Σ^0_3-completeness of having finitely many congruences. We also examine the complexity of the property "to be subdirectly irreducible", which we determine to be Σ^0_3-complete.

Complexity of properties of computable algebras have been studied previously and is an area of active research. In the finite case, for instance, it has been shown that for a finite algebra $\mathbf{A}$, calculating $\mathbf{Con}(\mathbf{A})$ and determining whether $\mathbf{A}$ is simple are both **NL**-complete (nondeterministic logarithmic-space) [2, Theorem 2.3,2.4]. Ralph Freese also showed algorithmic complexity results involving Malcev conditions in 2009 [6]. In the infinite case, Bakhadyr Khoussainov and Andrey Morozov showed [9] in 2010 the property "to be simple" to be Π^0_2-complete, witnessable by groups (see Definition 1.2.3). They also showed the property "to have finitely many congruences" to be Σ^0_3-complete, again witnessable by groups, as well as showing that the property "having a congruence lattice with the increasing chain property" is Π^1_1-complete along with the same result for the decreasing chain property..

The study of the complexity of being subdirectly irreducible in particular has also been studied by others, particularly in the finite case. In a 1997 paper, Ralph Freese showed one could decide if a finite computable lattice is subdirectly irreducible in time $O(n^2)$ where n is the cardinality of the lattice in question. In 2002, Clifford Bergman and Giora Slutzki showed that for finite computable algebras the property "to be subdirectly irreducible" is **NL**-complete [2, Theorem 2.4] while also showing that directed graphs witness this complexity. We free ourselves of the finiteness restriction, and examine the complexity of these properties for possibly infinite algebras.

1.1 Definitions and Background

For the purpose of completeness, we include the following definition.

Definition 1.1.1 (computable function)**.** The class of computable functions $\mathcal{C}$ is the smallest class of functions $\mathcal{C}$ in $\bigcup_{k=1}^{\infty} \omega^k$

(i) containing functions

$$\mathcal{O}(x) = 0, \mathcal{S}(x) = x + 1, \text{ and } \pi_{i,k}(x_1, x_2, \ldots, x_k) = x_i$$

for all $i \leq k \in \omega$,

(ii) closed under composition,

(iii) closed under primitive recursion (that is if $g, h \in \mathcal{C}$, the function

$$f(x_1, x_2, \ldots, x_k, n) = \begin{cases} g(x_1, x_2, \ldots, x_k) & \text{if } n = 0 \\ h(x_1, x_2, \ldots, x_k, n-1, f(x_1, x_2, \ldots, x_k, n-1)) & \text{o.w.} \end{cases}$$

is in $\mathcal{C}$), and

(iv) as closed under μ-recursion. That is, given $g \in \mathcal{C}$, where

$$(\forall x_1, x_2, \ldots, x_k)(\exists y)(g(x_1, x_2, \ldots, x_k, y) = 0),$$

the function $f(x_1, x_2, \ldots, x_k) = y_\mu$, where y_μ is the least y where $g(x_1, x_2, \ldots, x_k, y) = 0$, is in $\mathcal{C}$.

From now on however, rather than using the formal definition of computable function, we will freely appeal to Church's Thesis, which asserts that a function is computable if and only if it is computable in the intuitive sense. To the reader to whom "intuitive sense" is inapplicable, it should suffice to know that if a function on ω could be implemented in your favorite programming language, it is computable.

At this point, it is useful to note that any element of ω can be interpreted computably as a finite sequence. For instance given $i = p_1^{a_1+1} \cdot p_2^{a_2+1} \cdot p_n^{a_n+1}$ where $p_1 < p_2 < \ldots < p_n$ are primes, we can interpret i as the sequence $(a_1, a_2, \ldots, a_n)$. Conversely, any sequence of length n can be computably encoded as an i simply by using the first n primes and this encoding scheme. Given $x \in \omega$, we denote its interpretation as a sequence by $x = \langle x_1, x_2, \ldots, x_n \rangle$.

Interpreting elements of ω as sequences is a common trick in computability theory, and we will now do so implicitly throughout the rest of this dissertation.

Definition 1.1.2. A sequence of functions $\{f_i\}$ is *uniformly computable* if there exists a computable function σ where $\sigma(i) = n$ when $f_i : \omega^n \to \omega$ and a computable function f where $f(i, x) = f_i(x_1, x_2, \ldots, x_{\sigma(i)})$ for all $x = \langle x_1, x_2, \ldots, x_{\sigma(i)} \rangle \in \omega$. A *computable set* A is a subset of ω whose characteristic function is computable. A *computable operation* f on a computable set A is a computable function from ω^k to ω for some $k \in \omega$, where for all $a_1, a_2, \ldots, a_k \in A$, we have $f(a_1, a_2, \ldots, a_k) \in A$.

Definition 1.1.3 (algebra)**.** An *algebra* $\mathbf{A}$ is a tuple (A, F) in which A is a nonempty set and $F = \{f_i\}_{i \in I}$ is a sequence of functions where $f_i : A^{n_i} \to A$ and $n_i \in \omega$ for all i for some set I [1, Definition 1.1].

Definition 1.1.4 (subalgebra)**.** If $\mathbf{A} = (A, \{f_i : A^{n_i} \to A\}_{i \in I})$ is an algebra, and $B \subseteq A$ where $B \neq \emptyset$, we call $\mathbf{B} = (B, \{f_i \restriction B^{n_i}_{i \in I}\})$ a *subalgebra* of $\mathbf{A}$ if $f_i \restriction B^{n_i}$ maps into B.

Most structures studied in an upper division undergraduate algebra course, such as groups, monoids, vector spaces, rings, fields, modules, etc., are algebras in this general sense. We define computable versions of these structures.

Definition 1.1.5 (computable algebra)**.** A *computable algebra* $\mathbf{A}$ is a tuple $(A, F = (f_0, f_1, \ldots))$, where

(i) $A \subseteq \omega$ is computable and $A \neq \emptyset$,

(ii) F is a uniformly computable sequence of functions, which are either the empty set or computable operations on A, and

(iii) there exists a computable function $\sigma : \omega \to \omega \cup \{\uparrow\}$ where for any $k \in \omega$, if $\sigma(k) \in \omega$, f_k is a computable operation from $A^{\sigma(k)}$ to A. If $\sigma(k) = \uparrow$, then $f_k = \emptyset$.

We call A and F the *domain* and *operations* of the computable algebra $\mathbf{A}$ respectively. We call σ the *rank function* for $\mathbf{A}$ and the value $\sigma(k)$ the *rank* of f_k. If $\sigma(k) = \uparrow$, we say operation f_k does not exist.

Often, as a notational convention, we will assume implicitly that the domain of $\mathbf{A}$ is A, and the operations of $\mathbf{A}$ are $(f_0, f_1, \ldots)$.

Definition 1.1.6 (computable subalgebra)**.** Given a computable algebra $\mathbf{A} = (A, F)$, if B is a computable subset of A, we say $\mathbf{B} = (B, F)$ is a *computable subalgebra* of $\mathbf{A}$, if $\mathbf{B}$ is a computable algebra.

While it would be tempting to require A to simply be an initial segment of ω, by only requiring A to be a computable subset of ω, we allow for the definition of computable subalgebra to be the intuitive one.

Definition 1.1.7 (uniformly computable algebras)**.** A sequence of computable algebras $\{\mathbf{A}_i = (A_i, F_i = (f_{i,0}, f_{i,1}, \ldots))\}_{i<\omega}$ is uniformly computable if

(1) their characteristic functions $\{A_i\}$ and rank functions $\{\sigma_i\}$ are uniformly computable, and

(2) there exists a computable function $f(i, j, x)$ where for every $x = \langle x_1, x_2, \ldots, x_{\sigma_i(j)} \rangle \in \omega$,

$$f(i, j, x) = f_{i,j}(x_1, x_2, \ldots, x_{\sigma_i(j)}).$$

A computable algebra is an effective version of a general algebra. The study of properties of specific classes of computable algebras is an active area of computability research. We will focus on one particular class of algebras quite a bit, the class of *lattices*.

Definition 1.1.8 (computable lattice)**.** A computable algebra $\mathbf{L} = (L, F)$ is a *computable lattice* if

(i) $\mathbf{L}$ has two binary operations, that is $\sigma(0) = 2 = \sigma(1)$, and $\sigma(n) =\uparrow$ for all $n > 1$, which are idempotent, commutative, and associative, and

(ii) $f_0(n, f_1(n, m)) = n = f_1(n, f_0(n, m))$ for all $n, m \in \omega$.

To align our notation with that of lattice theorists, we will denote one of the operations $\wedge$ and the other $\vee$ and use infix rather than prefix notation. Lattices also define a partial ordering, $\leq_L$, such that $a \leq_L b$ if $a \vee b = b$. In this partial order, $a \wedge b$ is the greatest lower bound of a and b, and $a \vee b$ is the least upper bound of a and b. In fact, any partial order for which every two elements have a glb and a lub induces a lattice. If $a \leq_L b$ and $a \neq b$, we write $a <_L b$. For any partial order, if $a <_L b$, and $c <_L b \Rightarrow a \not<_L c$, we say b *covers* a or a *is covered by* b, and we write this $a \prec b$.

Furthermore for $a, b \in L$, if $a \leq_L b$, we define the *interval* $[a, b]_L$ as the set $\{x \in L : a \leq_L x \leq_L b\}$. We will call subalgebras of a computable lattice *computable sublattices* and note that an interval of a computable lattice is a computable sublattice. If $\mathbf{A}$ is a computable sublattice of $\mathbf{L}$, we say $\mathbf{L}$ is a *computable superlattice* of $\mathbf{A}$.

The *diagram* of a finite order (or finite lattice) is a graphical representation of that order. The elements are represented by circles. The circles representing the elements a and b are connected by an arc if one covers the other. In particular, if $b \prec a$, then the circle representing a is higher than the circle representing b [8, Section 2.1]. We define this formally.

Definition 1.1.9 (diagram of a finite partial order)**.** Let $\mathcal{P}$ be a poset on n elements. A *diagram* of $\mathcal{P}$ is a set of points $\{p_1 = (x_1, y_1), p_2 = (x_2, y_2), \ldots, p_n = (x_n, y_n)\}$ in $\mathbb{R} \times \mathbb{R}$ together with a number of continuous functions called arcs such that:

(1) If $p_i \succ p_j$ then $y_i > y_j$ and there is an arc, a_{ji}, which is the graph of a continuous function of y with domain $[y_j, y_i]$, with $a_{ji}(y_i) = x_i$ and $a_{ji}(y_j) = x_j$ and no other point (x_k, y_k) lying on a_{ji}.

(2) The are no other arcs. A diagram is *planar* if any two arcs only intersect at an endpoint.

Example 1.1.10. *Suppose we have a lattice* $\mathbf{N_5} = (\{0, 1, 2, 3, 4\}, (\wedge, \vee))$*, where* $0 \prec 2 \prec 3 \prec 1$, $0 \prec 4 \prec 1$*, and* 4 *is incomparable to* 2 *and* 3*. Then Figure 1.1 is a diagram of* N_5*.*

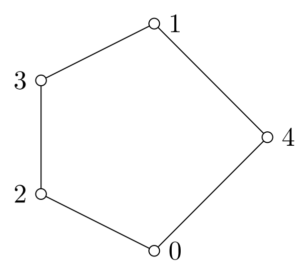

Figure 1.1: The diagram of the lattice N_5

In practice however, in this dissertation we will often informally give “diagrams" for infinite lattices by using ellipses and the reader's imagination. We do so in the introduction of the following well-known family of lattices.

Definition 1.1.11. For $n \in \omega$ or $n = \omega$, define $\mathbf{M}_n = (M_n, (f_1 := \wedge, f_2 := \vee))$ as follows. Let $M_n = \{0, 1, \ldots, n + 1\}$ if $n \in \omega$, and $M_n = \omega$ when $n = \omega$. Let $0 <_L 1$ and for all $m > 1 \in M_n$ let

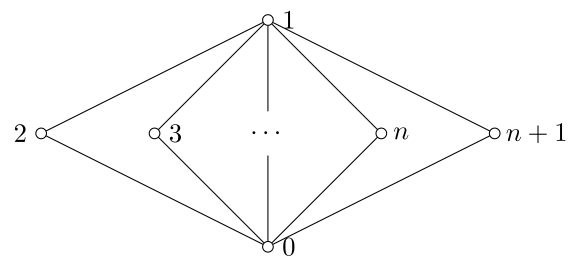

Figure 1.2: The lattice $\mathbf{M}_n$

$x <_L m <_L y$ if and only if $x = 0$ and $y = 1$. This results in the lattice represented in Figure 1.2. This defines a family of lattices, $\{\mathbf{M}_n\}_{n<\omega+1}$.

Definition 1.1.12 (substitution property)**.** Let $\mathbf{A} = (A, (f_0, f_1, \ldots))$ be a computable algebra and θ a binary relation on A. We use the shorthand $a\ \theta\ b$ if $(a, b) \in \theta$.

We say θ has the *substitution property* if for every $n \in \omega$ where $\sigma(n) \neq\uparrow$, we have that

$$a_1\ \theta\ b_1, \quad a_2\ \theta\ b_2, \quad \ldots, \quad a_{\sigma(n)}\ \theta\ b_{\sigma(n)}$$

implies

$$f_n(a_1, a_2, \ldots, a_{\sigma(n)})\ \theta\ f_n(b_1, b_2, \ldots, b_{\sigma(n)}).$$

Intuitively, a relation has the substitution property if it "respects" the operations of an algebra. In the case where the relation is also reflexive, symmetric, and transitive we distinguish it from other relations.

Definition 1.1.13 (congruence)**.** If $\mathbf{A}$ is a computable algebra, and θ a binary relation on A, we say θ is a *congruence* if θ is an equivalence relation which has the substitution property. The set of all congruences of $\mathbf{A}$ we call $\mathbf{Con}\,(\mathbf{A})$. For $a \in A$, we define the *congruence class of "a" under* θ as the set $\{b \in A : a\ \theta\ b\}$ and denote this $[\![a]\!]_\theta$. Occasionally, when the context is clear we will suppress the subscript and write only $[\![a]\!]$.

Congruences of algebras correspond to homomorphic images of that algebra, in the sense that every congruence induces a homomorphism, namely the map from the algebra to its congruence classes. Every algebra $\mathbf{A}$ has congruences $0_A = \{(x, x) : x \in A\}$ and $1_A = A \times A$. If these two congruences are the same, then $|A| = 1$ and we call $\mathbf{A}$ *trivial.* If $\mathbf{Con}\,(A) = \{0_A, 1_A\}$ and $\mathbf{A}$ is not trivial, then we call $\mathbf{A}$ *simple.*

It is known for instance that the lattice $\mathbf{M}_n$ is simple when $n \geq 3$. We will use this fact heavily.

Definition 1.1.14 (principal congruence)**.** Given a computable algebra $\mathbf{A}$, and elements $a, b \in A$, we say

$$\mathrm{Cg}_{\mathbf{A}}(a, b) = \bigcap \{\theta \in \mathbf{Con}(\mathbf{A}) : (a, b) \in \theta\}.$$

It is known that $\mathrm{Cg}_{\mathbf{A}}(a, b)$ is a congruence on $\mathbf{A}$ [1, Prop 1.23] and is thus the smallest congruence θ where $a \; \theta \; b$. As before, when the context is clear, we will suppress the subscript and just write $\mathrm{Cg}(a, b)$.

Principal congruences are useful for proving general facts about $\mathbf{Con}(\mathbf{A})$, for instance the following lemma concerning principal congruences in lattices will be extremely useful, and oftentimes used implicitly.

Lemma 1.1.15. *If $\mathbf{L}$ is a lattice and $a \leq b \in L$, then for all $x, y \in [a, b]_L$, $(x, y) \in \mathrm{Cg}(a, b)$.*

Proof. Since $(a, b), (x, x) \in \mathrm{Cg}(a, b)$ we have by the Substitution Property that $(a, x) = (x \wedge a, x \wedge b) \in \mathrm{Cg}_{\mathbf{L}}(a, b)$. Similarly we have $(a, y) \in \mathrm{Cg}_{\mathbf{L}}(a, b)$. By transitivity $(x, y) \in \mathrm{Cg}_{\mathbf{L}}(a, b)$. □

Using this lemma it is possible, for instance, to easily show that $\mathbf{M}_n$ is simple when $n \geq 3$ from the proof that $\mathbf{M}_3$ is simple [1, Lemma 2.9].

We will also use another useful theorem, a consequence of Dilworth [4, Lemma 2.1].

Theorem 1.1.16. *If $\mathbf{L}$ is a lattice, and $a \leq_L b$ then for $\mathrm{Cg}_{\mathbf{L}}(a, b)$ the following are equivalent:*

(i) $[\![a]\!] = [a, b]_L$ and for all $x \notin [a, b]_L$, $[\![x]\!] = \{x\}$;

(ii) for all $x \in \mathbf{L}$, if $x \notin [a, b]_L$ then $(x <_L b \Rightarrow x <_L a)$ and $(a <_L x \Rightarrow b <_L x)$.

1.2 Completeness of Properties of Computable Algebras

Definition 1.2.1 (m-reducibility)**.** A set A is *many-one reducible* to B denoted $(A \leq_m B)$ if there exists a computable function f such that for all a,

$$a \in A \Leftrightarrow f(a) \in B. \qquad \text{([11, 7.1])}$$

We abbreviate many-one reducible as m-reducible. If $A \leq_m B$ and $B \leq_m A$ we say $A \equiv_m B$.

It is known [11, Theorem I] that $\leq_m$ is reflexive and transitive, and that $\equiv_m$ is an equivalence relation. The equivalence classes are called *many-one degrees* or *m-degrees.*

We use the main definition from Khoussainov and Morozov [9, Main Definition].

Definition 1.2.2 (J-complete property of an algebra)**.** Let $\mathcal{J} \subseteq 2^\omega$ (where 2^ω is the power set of ω) be closed downward under m-reducibility. We say that the property $\mathcal{P}$ of algebras in a class $\mathcal{K}$ is $\mathcal{J}$*-complete*, if the following is true:

(1) For all uniformly computable sequences of algebras $\{\mathbf{A}_i\}_{i<\omega}$ in $\mathcal{K}$,

$$\{i : \mathbf{A}_i \text{ satisfies } \mathcal{P}\} \in \mathcal{J}.$$

(2) There exists a uniformly computable sequence of algebras $\{\mathbf{A}_i\}_{i<\omega}$ in $\mathcal{K}$ for which for all $J \in \mathcal{J}$,

$$J \leq_m \{i : \mathbf{A}_i \text{ satisfies } \mathcal{P}\}.$$

If the class $\mathcal{K}$ is the class of all algebras, we suppress mention of it and simply say that the property $\mathcal{P}$ is $\mathcal{J}$-complete. If a property $\mathcal{P}$ satisfies property (1) we say $\mathcal{P}$ is a $\mathcal{J}$*-property* (in $\mathcal{K}$). If it satisfies property (2) we say B is $\mathcal{J}$*-hard* (in $\mathcal{K}$).

We also introduce the notion of a class of algebras $\mathcal{S}$ *witnessing* J-completeness for a class $\mathcal{K}$.

Definition 1.2.3 (witnessing $\mathcal{J}$-completeness)**.** If a property $\mathcal{P}$ is J-complete in $\mathcal{K}$, and if there exists a uniformly computable sequence of algebras in a subclass $\mathcal{S}$ of a class $\mathcal{K}$ witnessing that $\mathcal{P}$ is $\mathcal{J}$*-hard* in $\mathcal{K}$, we say that $\mathcal{S}$ witnesses the J-completeness of B in $\mathcal{K}$.

Intuitively, this means that knowing the solution to determine if any $s \in \mathcal{S}$ has property B is sufficient to determine if any $k \in \mathcal{K}$ has property B, and as such, $\mathcal{S}$ is somehow sufficiently complex in property B to understand the property in general.

Given a sequence of uniformly computable algebras $\{\mathbf{A}_i\}$ and some subset S of that sequence, the set $\{i : \mathbf{A}_i \in S\}$ is called the *index set* of S. We shall examine complexities of index sets of universal algebras. To relate these complexities, we introduce a well known hierarchy of sets closed downward under $\leq_m$ [12, Theorem 1.3(v)].

Definition 1.2.4 (The Arithmetical Hierarchy)**.** The *arithmetical hierarchy* is a method of categorizing some sets according to their expressibility.

(1) A set B is Σ_n^0 if there is a computable function $P(x, y_1, \ldots, y_n)$ such that

$$x \in B \Leftrightarrow (\exists y_1)(\forall y_2)(\exists y_3) \cdots (Qy_n)(P(x, y_1, \ldots, y_n) = 1),$$

where the quantifiers alternate and thus $Q = \exists$ if n is odd and $Q = \forall$ if n even.

(2) A set B is Π_n^0 if there is a computable relation $P(x, y_1, \ldots, y_n)$ such that

$$x \in B \Leftrightarrow (\forall y_1)(\exists y_2)(\forall y_3) \cdots (Qy_n)(P(x, y_1, \ldots, y_n) = 1),$$

where the quantifiers alternate and thus $Q = \forall$ if n is odd and $Q = \exists$ if n even.

In 2010, Khoussainov and Morozov [9] proved the following concerning properties of computable algebras:

(1) The property "to be simple" is Π_2^0-complete,

(2) The property "to have finitely many congruences" is Σ_3^0-complete.

In proving statement (1) and (2), Khoussainov and Morozov proved that computable groups witness Π_2^0-completeness of being simple as well as witnessing Σ_3^0-completeness of having finitely many congruences. We prove that lattices are also sufficient, i.e. groups can be replaced by lattices, and thus also give alternate proofs for (1) and (2) as corollaries. In addition, we examine the complexity of the property "to be subdirectly irreducible" which will be defined below.

In order to prove these results we will need a bit more theory.

Definition 1.2.5 (basic translation)**.** Given an algebra $\mathbf{A} = (A, \{f_i\}_{i<\omega})$, fix $i \in \omega$ and let $n = \sigma(i)$. Furthermore, let $j < n$, and $a_0, \ldots, a_{n-1} \in A$. Then

$$u(x) = f_i(a_0, a_1, \ldots, a_{j-1}, x, a_{j+1}, \ldots, a_{n-1})$$

is a basic translation operation.

Definition 1.2.6 (translation)**.** Given an algebra $\mathbf{A} = (A, \{f_i\}_{i<\omega})$, the translations are a class of functions from A into A, defined as follows:

(1) The identity function $u(x) = x$ and the basic translations are translations;

(2) If u and v are translations so is the composition.

(3) Nothing else is a translations operation.

The next result is one implicitly used and proved by Malcev, (see [7, Theorem 1.10.3] for a treatment of this).

Theorem 1.2.7 (Malcev's Lemma)**.** *For an algebra* $\mathbf{A}, (a,b) \in \mathrm{Cg}_{\mathbf{A}}(c,d)$ *if and only if there exist* $n < \omega$*, a sequence* $a = z_0, z_1, \ldots, z_n = b$ *of elements of* A *and a sequence of translations* $u_0, \ldots, u_{n-1}$ *such that*

$$\{u_i(c), u_i(d)\} = \{z_i, z_{i+1}\}$$

for all $i < n$*.*

Corollary 1.2.8. *If* $\mathbf{B}$ *is a subalgebra of* $\mathbf{A}$*, and* $(a,b) \in \mathrm{Cg}_{\mathbf{B}}(c,d)$*, then* $(a,b) \in \mathrm{Cg}_{\mathbf{A}}(c,d)$*.*

Furthermore, this gives that if $\mathrm{Cg}_{\mathbf{B}}(c,d) = \mathrm{Cg}_{\mathbf{B}}(x,y)$*, then* $\mathrm{Cg}_{\mathbf{A}}(c,d) = \mathrm{Cg}_{\mathbf{A}}(x,y)$

Proof. Since $(a,b) \in \mathrm{Cg}_{\mathbf{B}}(c,d)$, there exists a sequence of elements in B and translations that satisfy Malcev's Lemma. But these will still satisfy Malcev's Lemma in $\mathbf{A}$, which gives $(a,b) \in \mathrm{Cg}_{\mathbf{A}}(c,d)$. □

Using our standard encoding for sequences of numbers, every basic transition $u(x) = f_i(a_0, a_1, \ldots, a_{j-1}, x, a_{j+1}, \ldots, a_{n-1})$ of an algebra $\mathbf{A}$ can be encoded computably as an element of ω, by encoding the sequence

$$(i, j, a_0, a_1, \ldots, a_{n-1}).$$

Furthermore, since any translation is just a sequence of compositions of basic translations and the identity function, we can encode any translation as a finite sequence of encoded basic transitions.

Lemma 1.2.9. *Given a computable algebra* $\mathbf{A}$ *and* $c, d \in A$*, the set* $\mathrm{Cg}_{\mathbf{A}}(c,d)$ *is* Σ^0_1*.*

Proof. We define a computable boolean function $R(a,b,c,d,x)$. Since A is computable, we can check to see if $a,b,c,d \in A$. Given $x \in \omega$, we can computably determine if x can be interpreted as an ordered pair, say $(\vec{z}, \vec{u})$, coded by some fixed convention. Furthermore we can interpret $\vec{z}$ as a finite sequence $(z_0, z_1, \ldots, z_n)$ and check to see if $z_0 = a$ and $z_n = b$. We interpret $\vec{u}$ as a sequence of translations, and check to see if they are valid, as well as the length of $\vec{u}$ is $n-1$. Finally we check if $\{u_i(c), u_i(d)\} = \{z_i, z_{i+1}\}$ for $i < n$. If any of these checks fail we let $R(a,b,c,d,x) = 0$, otherwise $R(a,b,c,d,x) = 1$. Then

$$(a,b) \in \mathrm{Cg}_{\mathbf{A}}(c,d) \Leftrightarrow (\exists x)R(a,b,c,d,x) = 1.$$

So $\{(a,b) : (a,b) \in \mathrm{Cg}_{\mathbf{A}}(c,d)\}$ is Σ_1^0. □

We will also need a technical lemma concerning simple lattices.

Lemma 1.2.10. *Suppose (i) $M = [0_M, 1_M]_L, N = [0_N, 1_N]_L$ are simple sublattices of a lattice $\mathbf{L}$, (ii) $\{1_M, 0_N\} \subseteq M \cap N$, (iii) $1_M \neq 0_N$, and (iv) $[0_M, 1_N]_L = M \cup N$. Then $[0_M, 1_N]_L$ is simple.*

Proof. Let $a \neq b \in M \cup N$. If $a, b \in M$, then since M simple, for all $x, y \in M$, we have $(x,y) \in \mathrm{Cg}_{\mathbf{L}}(a,b)$. But then $(1_M, 0_N) \in \mathrm{Cg}_{\mathbf{L}}(a,b)$, so since N is simple, we have for all $x, y \in N$, $(x,y) \in \mathrm{Cg}_{\mathbf{L}}(1_M, 0_N) \subseteq \mathrm{Cg}_{\mathbf{L}}(a,b)$. So $\mathrm{Cg}_{\mathbf{L}}(a,b) = 1_{[0_M,1_N]_L}\mathbf{L}$. The same argument works with $a, b \in N$. Thus we only need to consider the case where $a \in M \setminus N$ and $b \in N \setminus M$.

In that case $a \vee 1_M = 1_M$, and $b \vee 1_M \in N$, we would have two distinct elements of N related, so the above argument applies. Thus $[0_M, 1_N]_L$ is simple.

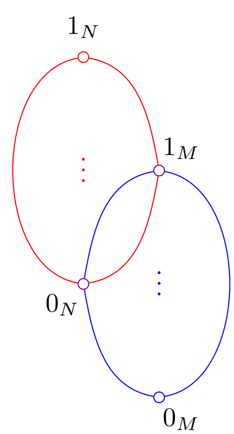

Figure 1.3: An illustration of $[0_M, 1_N]_L$

□

CHAPTER 2
WITNESSING COMPLEXITY VIA COMPUTABLE LATTICES

In the next two chapters, in order to show the hardness of various properties, we will construct various sequences of computable lattices, $\{\mathbf{L}_b\}_{b<\omega}$. In doing so, for convenience, we will define each in terms of the partial order determined by $\wedge$ and $\vee$, being careful to ensure that calculating the meet and join of any two elements is computable. In particular, we construct each $\mathbf{L}_b$ in stages, by defining a series of sublattices $\mathbf{L}_{b,s} = (L_{b,s}, (\wedge_{b,s}, \vee_{b,s}))$, where

$$L_{b,s} \subseteq L_{b,s+1}, \wedge_{b,s} \subseteq \wedge_{b,s+1}, \text{ and } \vee_{b,s} \subseteq \vee_{b,s+1}$$

for all $s \in \omega$. Furthermore $\{L_{b,s}\}$ will be uniformly computable in s, and

$$\mathbf{L}_b = \left(\bigcup_{s<\omega} L_{b,s} = \omega, \left(\wedge_b = \bigcup_{s<\omega} \wedge_{b,s}, \vee_b = \bigcup_{s<\omega} \vee_{b,s} \right) \right).$$

The benefit in defining our lattices this way, is to ensure that they are 1) computable and 2) lattices. We define all the relations on $L_{b,s}$ in terms of a partial order, in such a way that the meets and joins will be computable. In fact, in $\mathbf{L}_b$, for any $x, y \in \omega$, since the meet and join operations are nested, the value of $x \vee y$ does not change once it is encountered. So we need only go to an s large enough that $x, y \in L_{b,s}$, and we can find $x \wedge y$ and $x \vee y$ there. This ensures our operations are computable. In order to ensure our algebras are indeed lattices we use a result found in Quackenbush [10]. The theorem, due to Lasker [10], is the following one:

Theorem 2.0.1. *Let $\mathcal{P}$ be a finite partial order with a planar diagram. If there is at most one element of $\mathcal{P}$ which has no cover and at most one element which covers no point then $\mathcal{P}$ is a lattice.*

This theorem allows us to ensure that our algebras $\mathbf{L}_{b,s}$ are lattices by making sure they have planar diagrams. Since this will give every pair of elements in ω a greatest lower bound and least upper bound, provided that $\mathbf{L}_b$ is a poset, we have that $\mathbf{L}_b$ is a lattice.

When defining these lattices, we will often want to add an element between two other elements in the intuitive way. We introduce a notion, we call "*just between*" to allow us to do that.

Definition 2.0.2 (just between)**.** Given a lattice $\mathbf{L} = (L, (\wedge, \vee))$, and $a, b \in L$. We say c is *just* *between* a and b if

$$a \prec c \prec b \text{ and } (\forall d \in L)(c <_L d \Leftrightarrow d \geq_L b) \text{ and } (\forall d \in L)(c >_L d \Leftrightarrow d \leq_L a).$$

We will denote this

$$a \prec\!\!\prec c \prec\!\!\prec d \text{ in } L.$$

2.1 Simple Computable Lattices

Theorem 2.1.1. *Computable lattices witness the Π^0_2-completeness of being simple.*

Proof. We first show that, given a uniformly computable sequence of algebras $\{\mathbf{A}_i\}$,

$$\{i : \mathbf{A}_i \text{ is simple }\} \in \Pi^0_2.$$

This is shown by noting that in a simple algebra $\mathbf{A}_i$, if $a \neq b$, then $\mathrm{Cg}\,(a, b) = 1_{A_i}$. Thus

$$\begin{aligned}\{i : \mathbf{A}_i \text{ is simple }\} &= \{i : (\forall a, b, c, d)(a, b, c, d \in A_i, c \neq d \to (a, b) \in \mathrm{Cg}\,(c, d))\} \\ &= \{i : (\forall a, b, c, d)(\exists x)(\neg(a, b, c \neq d \in A_i) \text{ or } R(a, b, c, d, x) = 1)\},\end{aligned}$$

which is Π^0_2, where R is as in the proof of Lemma 1.2.9.

Now suppose B is Π^0_2. Then it has been shown [11, Lemma before 14.XV] that there exists a uniformly computable sequence of boolean functions $\{P_b(t)\}_{b<\omega}$ such that

$$b \in B \Leftrightarrow (\overset{\infty}{\exists} t)(P_b(t) = 1),$$

where $(\overset{\infty}{\exists} t)$ means "there exist infinitely many t". We construct a sequence of lattices $\{\mathbf{L}_b\}_{b<\omega}$ with the property that $\mathbf{L}_b$ is simple if and only if $b \in B$. This will give $B \equiv_m \{b : \mathbf{L}_b \text{ is simple}\}$ via the identity function. We define each in stages, and will also define a helper function $T(s)$ which will, informally, be the "top" of the lattice as currently defined by stage s.

Stage 0: We start every $\mathbf{L}_b$ with the lattice operations in Figure 2.1. We also let $T(0) = 3$, in keeping with our desire to have $T(0)$ point at the top after stage s. For technical reasons which will be clear later, we also let $T(-1) = 2$. We note that $L_{b,0} = \{0, 1, \ldots, 5\}$.

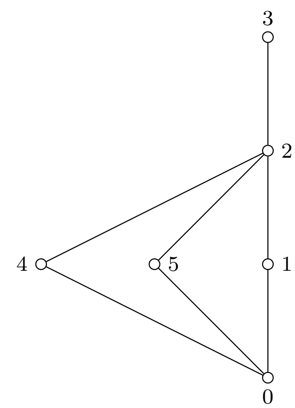

Figure 2.1: The initial lattice

Stage s: We will define lattice operations for all elements less than (in the standard sense) and including $8+3s$. In particular, we just define our lattice operations for additional elements $6+3s, 7+3s$ and $8+3s$. Let $L_{b,s} = L_{b,s-1} \cup \{6+3s, 7+3s, 8+3s\}$.

If $P_b(s) = 0$, then for $6 \leq i \leq 8$ and $n \leq 8+3s$, let

$$0 \prec\!\!\prec (i+3s) \prec\!\!\prec 2$$

in $L_{b,s}$.

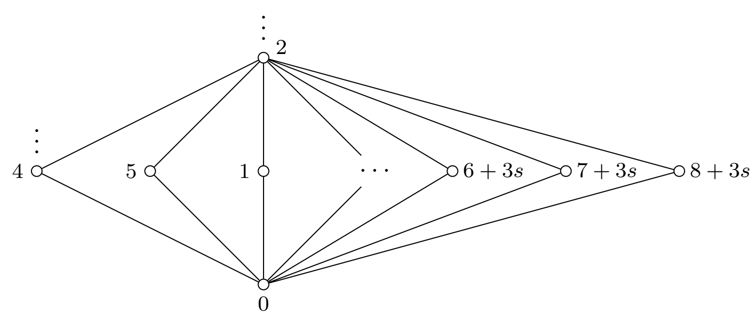

Figure 2.2: Placement of nodes when $P_b(s) = 0$

Since we have not put anything above $T(s)$, we let $T(s) = T(s-1)$.

Otherwise, if $P_b(s) = 1$, we place $6+3s$ at the top, covering $T(s-1)$, i.e. for $n \in L_{b,s}, n \leq_L 6+3s$. Furthermore let

$$T(s-1)+1 \prec\!\!\prec 7+3s, 8+3s \prec\!\!\prec T(s-1).$$

Since we changed the top, we also let $T(t) = 6+3t$ (see Figure 2.3).

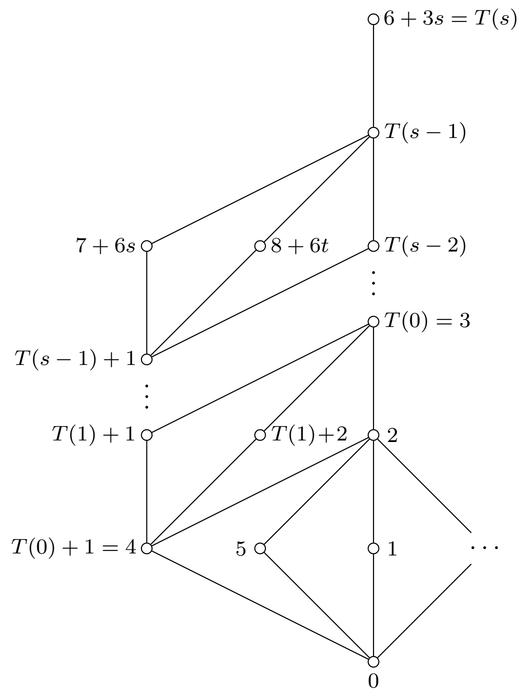

Figure 2.3: Placement of nodes when $P_b(s) = 1$

We claim under this construction,

$$B = \{b : \mathbf{L}_b \text{ is simple}\}.$$

Suppose $b \notin B$. Then there exists s_0 such that for all $s \geq s_0, P_b(s) = 0$. Thus, for all $s \geq s_0$, $T(s) = T(s_0)$. Also, there exists some maximal $s_1 < s_0$ where $T(s_0) \succ T(s_1)^*$. By Theorem 1.1.16 we have that $0_{L_b} \neq \mathrm{Cg}\,(T(s_0), T(s_1)) \neq 1_{L_b}$. Thus $\mathbf{L}_b$ is not simple.

On the other hand, assume $b \in B$. Let $s_k =$ the k^{th} index s such that $T(s) > T(s-1)$. Since $b \in B$, this sequence is defined for all $k \in \omega$. Also let $s_0 = -1$. Then $T(s_k) \prec T(s_{k+1})$ for $k \in \omega$. Note $[0, T(s_0)]_L$ is isomorphic to $\mathbf{M}_n$ for some $n < \omega + 1$, which is simple. Since for every $k > 0$, we have $[T(s_k) + 1, T(s_k)]_L$ isomorphic to $\mathbf{M}_3$ by construction, also simple, then by Lemma 1.2.10, we have that for any k, $[0, T(s_k)]_L$ is a simple sublattice. We claim this shows $\mathbf{L}_b$ is simple, since for any $a \neq b$, $c \neq d$, there exists some s_k above all of them in the lattice theoretical sense, which gives $(c, d) \in \mathrm{Cg}\,(a, b)$ inside of $[0, T(s_k)]_L$. But this gives a sequence required by Theorem 1.2.7

*This was the reason for letting $T(-1) = 2$.

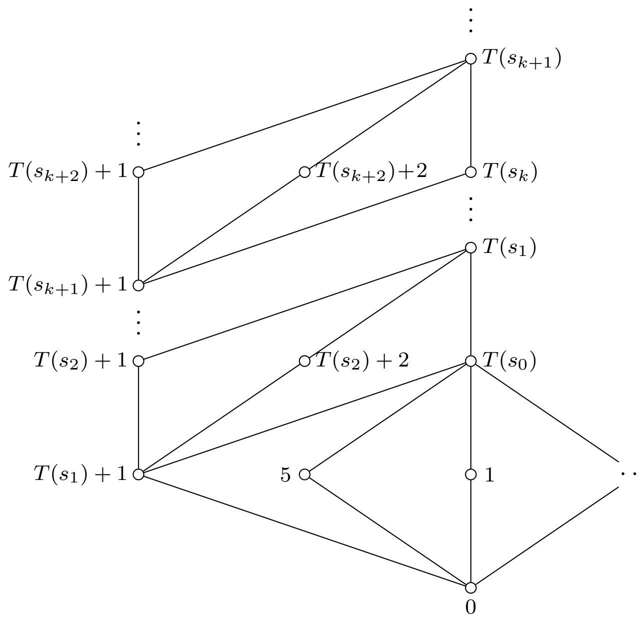

Figure 2.4: $\mathbf{L}_b$ when $b \in B$

which will exist in the superlattice $\mathbf{L}_b$. Thus $a \neq b$ forces $(c,d) \in \operatorname{Cg}(a,b)$, which gives that $\mathbf{L}_b$ is simple. □

2.2 Computable Lattices with Finitely Many Congruences

Theorem 2.2.1. *The computable lattices witness Σ_3^0-completeness of having finitely many congruences.*

Proof. We first show property (1) of Definition 1.2.2. Suppose an algebra **A** has infinitely many congruences. Note that for any $\theta \in \mathbf{Con}(\mathbf{A})$, $\theta = \bigcup_{(a,b)\in\theta} \operatorname{Cg}(a,b)$. Thus if there were only finitely many principal congruences, say n, then there would be at most 2^n many congruences. Thus we must have infinitely many principal congruences. On the other hand, if an algebra has finitely many congruences, then surely it has finitely many principal ones. Thus an algebra has finitely many congruences if and only if it has finitely many principal ones. Thus an algebra **A** has finitely many congruences if and only if

$$(\exists x = \langle a_0, b_0, a_1, b_1, \ldots, a_k, b_k \rangle)(\forall a, b)(\exists i \leq k)\left(\operatorname{Cg}(a,b) = \operatorname{Cg}(a_i, b_i)\right),$$

so "having finitely many congruences" is a Σ_3^0 property.

Now we show property (2) of Definition 1.2.2. Since B is Σ^0_3, there is a uniformly computable sequence of relations $\{P_b(i,j)\}_{b<\omega}$ where

$$b \in B \Leftrightarrow (\exists i)(\overset{\infty}{\exists} j)(P_b(i,j) = 1)$$

[11, Theorem 14.XVII and 14.XV]. We construct a sequence of lattices where

$$B = \{b : \mathbf{L}_b \text{ has finitely many congruences}\},$$

which will show property (2) under the identity function. As before, we shall build $\mathbf{L}_b$ in stages. At stage -1, let $L_{b,-1} = \{2n : n \in \omega\}$ for each b. For the purposes of notation, let $i_k = 2(i + 4k)$ for $i < 5$. Note then that $1_k = 0_{k+1}$ for all k. For uniformity we will write $1_{-1} = 0_0$. We let $0_k \lessdot 2_k, 3_k, 4_k \lessdot 1_k$ in $0, 2, 4, \ldots, 4_k$ for all $k \in \omega$. Note that since $1_k = 0_{k+1} <_L 1_{k+1}$ for all k, we have $1_m <_L 1_n$ if and only if $m < n$. This process yields the lattice $\mathbf{L}_{b,-1} = (2\omega, (\wedge_0, \vee_0))$ shown in Figure 2.5.

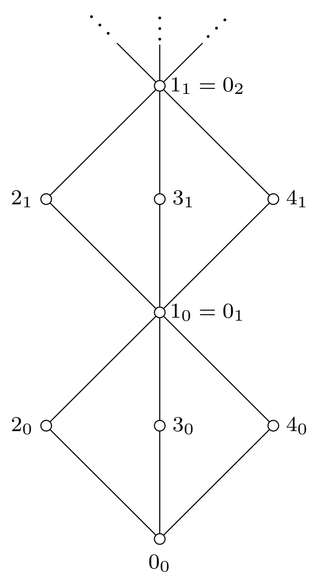

Figure 2.5: The initial lattice $\mathbf{L}_{b,-1}$.

For notation, let $\omega_{-1} = \omega \cup \{-1\}$. We define a family of helper functions $c_s(n)$ on ω_{-1}. Let $c_{-1}(n) = n$ for all $n \in \omega_{-1}$. Given $s \in \omega$, let $\bar{i}(s)$ and $\bar{j}(s)$ be the numbers where $\bar{i}(s) \leq \bar{j}(s)$ and $s = \frac{\bar{j}(s)(\bar{j}(s)+1)}{2} + \bar{i}(s)$. This is the inverse of the enumeration of the lower triangle of an $\omega \times \omega$ matrix shown in Table 2.2.1. Note that $\bar{j}$ is increasing in s.

$\bar{j}(s)$ \ $\bar{i}(s)$	0	1	2	3	4	$\cdots$
0	0					
1	1	2				
2	3	4	5			
3	6	7	8	9		
4	10	11	12	13	14	
$\vdots$			$\vdots$			

Table 2.1: The values of s and its relation to $\bar{i}(s)$ and $\bar{j}(s)$

For $s \in \omega$ let

$$c_s(n) = \begin{cases} c_{s-1}(n) & \text{if } P_b(\bar{i}(s), \bar{j}(s)) = 0 \text{ or } n < \bar{i}(s), \\ c_{s-1}(n+1) & \text{otherwise.} \end{cases}$$

We show some important properties of this family of functions.

Lemma 2.2.2. *$c_{s-1}(n)$ is strictly increasing in n for all $s \in \omega$.*

Proof. We proceed by induction on s. If $s = 0$, then $c_{s-1}(n) = c_{-1} = n$ which is strictly increasing.

Suppose c_{s-1} is strictly increasing. If $P_b(\bar{i}(s), \bar{j}(s)) = 0$, then $c_s = c_{s-1}$ so c_s is strictly increasing. On the other hand, if $P_b(\bar{i}(s), \bar{j}(s)) = 1$, then $c_s(n) = c_{s-1}(n)$ for all $n < \bar{i}(s)$, so c_s is strictly increasing on $-1, 0, 1, \ldots, \bar{i}(s) - 1$. For $n \geq \bar{i}(s)$, $c_s(n) = c_{s-1}(n+1)$, so c_s is strictly increasing on $\bar{i}(s), \bar{i}(s) + 1, \bar{i}(s) + 2, \ldots$. Furthermore,

$$c_s(\bar{i}(s) - 1) = c_{s-1}(\bar{i}(s) - 1) < c_{s-1}(\bar{i}(s) + 1) = c_s(\bar{i}(s)).$$

Thus c_s is strictly increasing on all of ω_{-1}. □

Corollary 2.2.3. *Fix $n \in \omega_{-1}$. $c_{s-1}(n)$ is increasing in s for $s \in \omega$.*

Proof. This follows immediately from Lemma 2.2.2 and the fact that either $c_s(n) = c_{s-1}(n)$ or $c_s(n) = c_{s-1}(n+1) > c_{s-1}(n)$. Thus $c_s(n) \geq c_{s-1}(n)$. □

Lemma 2.2.4. *Let $s_0 < s_1$, and fix $n \in \omega_{-1}$. Then $c_{s_0}(n) < c_{s_1}(n)$ if and only if there exists some s where $s_0 < s \leq s_1$, $P_b(\bar{i}(s), \bar{j}(s)) = 1$ and $\bar{i}(s) \leq n$.*

Proof. (⇐) Suppose there exists some s where $s_0 < s \leq s_1$, $P_b(\bar{i}(s), \bar{j}(s)) = 1$, and $\bar{i}(s) \leq n$. Then

$$\begin{aligned}
c_{s_0}(n) &\leq c_{s-1}(n) && (s_0 < s \text{ and Corollary 2.2.3}) \\
&< c_{s-1}(n+1) && (\text{Lemma 2.2.2}) \\
&= c_s(n) && (\neg\left(P_b(\bar{i}(s), \bar{j}(s)) = 0 \text{ or } n < \bar{i}(s)\right) \text{ and definition of } c_s) \\
&\leq c_{s_1}(n). && (s \leq s_1 \text{ and Corollary 2.2.3})
\end{aligned}$$

(⇒). Suppose for all s where $s_0 < s \leq s_1$, we have$P_b(\bar{i}(s), \bar{j}(s)) = 0$ or $\bar{i}(s) > n$. Then for all m where $0 < m \leq s_1 - s_0$, we have $s_0 < s_0 + m \leq s_1$. So $P_b(\bar{i}(s_0 + m), \bar{j}(s_0 + m)) = 0$ or $\bar{i}(s_0 + m) > n$. Thus by definition of c_s, $c_{s_0+m}(n) = c_{s_0+m-1}(n)$ for all m where $0 < m \leq s_1 - s_0$, and hence $c_{s_0+m}(n) = c_{s_0}(n)$ for all valid $0 < m \leq s_1 - s_0$. Thus $c_{s_0}(n) = c_{s_1}(n)$. □

Lemma 2.2.5. *If $b \notin B$, for any $n \in \omega_{-1}$, there exists an $s_n \in \omega$ where for all $s \geq s_n$, we have $c_s(n) = c_{s_n}(n)$. That is, $c_s(n)$ is eventually constant in s.*

Proof. Fix $n \in \omega_{-1}$. If $b \notin B$, then since

$$b \in B \Leftrightarrow (\exists i)(\overset{\infty}{\exists} j)(P_b(i, j) = 1),$$

for all i, there exists a j_i where for all $j \geq j_i$, we have $P_b(i, j) = 0$. Let $J = \max(j_0, j_1, \ldots, j_n, (n+1))$. Let $s_n = \frac{J(J+1)}{2}$. Note that this is just choosing s_n so that $\bar{j}(s_n) = J$ and $\bar{i}(s_n) = 0$. Then for any $s \geq s_n$, since $\bar{j}$ is increasing, we have $\bar{j}(s) \geq J$. Thus whenever $\bar{i}(s) \leq n$, we have $P_b(\bar{i}(s), \bar{j}(s)) = 0$. Thus for all $s \geq s_n$ we have $P_b(\bar{i}(s), \bar{j}(s)) = 0$ or $n < \bar{i}(s)$, which gives $c_s(n) = c_{s-1}(n)$ for all $s > s_n$. Thus $c_s(n) = c_{s_n}(n)$ for all $s > s_n$. □

Finally, our last lemma regarding c_s.

Lemma 2.2.6. *If $b \in B$, then there exists n where $c_s(m)$ is eventually constant in s for all $m \leq n$, but $c_s(n+1)$ is unbounded in s.*

Proof. Since $b \in B$, we know $(\exists i)(\overset{\infty}{\exists} j)(P_b(i, j) = 1)$. Let $n + 1$ be the least such i. If $n = -1$, since $c_s(-1) = -1$ for all s, we have $c_s(-1)$ constant in s. Otherwise, since for all $i \leq n$, we have there exists a j_i where $j \geq j_i$ implies $P_b(i, j) = 0$. An argument exactly the same as in the proof of Lemma 2.2.5 gives that for each $m \leq n$, we have $c_s(n)$ eventually constant. However, since for $n+1$, there are infinitely many j where $P_b(n+1, j) = 1$, there are infinitely many $j > n+1$ where $P_b(n+1, j) = 1$. Let $\{j_m\}_m \in \omega$ be the increasing sequence of those js. Then for $s_m = \frac{j_m(j_m+1)}{2} + (n+1)$, we have $\bar{i}(s_m) = n+1$ and $\bar{j}(s_m) = j_m$, and thus $P_b(\bar{i}(s_m), \bar{j}(s_m)) = 1$ and $\bar{i}(s_m) = n+1$. Thus $c_{s_m}(n+1) = c_{s_m-1}(n+1+1) \geq n+1+m$. Thus $c_s(n+1)$ is unbounded. $\square$

Keeping these facts in mind, we now define our lattices.

At stage s, let $L_{b,s} = L_{b,s-1} \cup \{4s+1, 4s+3\}$. If $P_b(\bar{i}(s), \bar{j}(s)) = 0$, let

$$0_0 \ll 4s+1, 4s+3 \ll 1_0 \text{ in } L_{b,s}. \qquad \text{(see Figure 2.6)}$$

If $P_b(\bar{i}(s), \bar{j}(s)) = 1$, we let $2_{c_{s-1}(\bar{i}(s))} \ll 4s+1, 4s+3 \ll 2_{c_{s-1}(\bar{i}(s))+1}$ in $L_{b,s}$ (see Figure 2.7). We

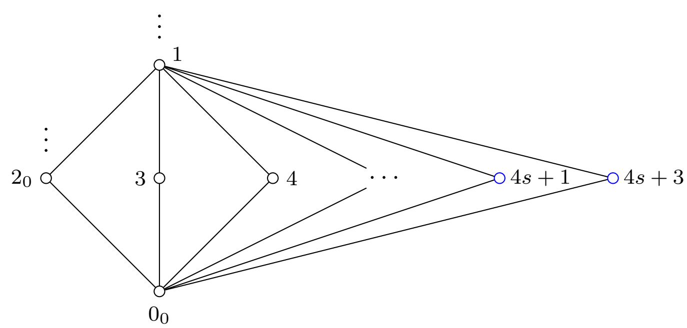

Figure 2.6: Placement of nodes when $P_b(\bar{i}(s), \bar{j}(s)) = 0$.

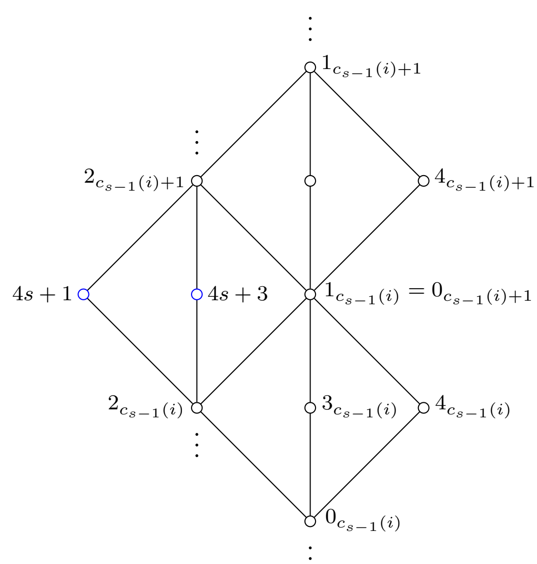

Figure 2.7: Placement of nodes when $P_b(\bar{i}(s), \bar{j}(s)) = 1$, and $i = \bar{i}(s)$.

claim $\mathbf{L}_b$ has finitely many congruences if and only if $b \in B$. For this we prove a couple lemmas.

Lemma 2.2.7. *At any stage s, $1_{c_s(n-1)}$ and $1_{c_s(n)}$ satisfy the conditions of Theorem 1.1.16 for all $n \in \omega$ in $\mathbf{L}_{b,s}$. Also, for any $x \in \mathbf{L}_{b,s}$, we have $x \in [1_{c_s(k-1)}, 1_{c_s(k)}]_{L_{b,s}}$ for some $m \in \omega$.*

Proof. For ease of notation, define $I_{s,n} := [1_{c_s(n-1)}, 1_{c_s(n)}]_{L_{b,s}}$ for $n \in \omega$. Since each c_s is strictly increasing in n, we have $1_{c_s(n-1)} <_L 1_{c_s(n)}$. We need only to show $1_{c_s(n-1)}$ and $1_{c_s(n)}$ satisfy property (ii) of Theorem 1.1.16 and $x \in \mathbf{L}_{b,s} \Rightarrow (\exists k)(x \in I_s, k)$. We proceed by induction on s.

At stage -1, property (ii) follows immediately from the definition of $\mathbf{L}_{b,-1}$ and $\lll$. Also, any $x \in \mathbf{L}_{b,-1}$ is of the form i_k for some $i < 5$ where $k \in \omega$. So $x \in [0_k, 1_k]_{L_{b,s}} = I_{-1,k}$.

Suppose that at stage $s-1$, we have $1_{c_{s-1}(n-1)}$ and $1_{c_{s-1}(n)}$ satisfy Theorem 1.1.16 for all $n \in \omega$ and any $x \in \mathbf{L}_{b,s-1}$ is in some $I_{s-1,k}$. Suppose further $4s+1, 4s+3 \in I_{s,k}$ for some k. Then for any $y \in \{4s+1, 4s+3\}$ where $y \notin I_{s,n}$, $k \neq n$, so either $k \leq n-1$ or $n+1 \leq k$. Thus

$$\begin{aligned} x <_L 1_{c_s(n)} &\Rightarrow 1_{c_s(k-1)} \leq_L y <_L 1_{c_s(n)} && (y \in I_{s,k}) \\ &\Rightarrow k-1 < n && (1_j <_L 1_k \Leftrightarrow j < k) \\ &\Rightarrow k \leq n-1 && (k \leq n-1 \text{ or } n+1 \leq k) \\ &\Rightarrow y \leq_L 1_{c_s(k)} \leq_L 1_{c_s(n-1)} && (y \in I_{s,k}) \\ &\Rightarrow y <_L 1_{c_s(n-1)} && (x \notin I_{s,n}) \end{aligned}$$

and similarly

$$\begin{aligned} 1_{c_s(n-1)} <_L y &\Rightarrow 1_{c_s(n-1)} <_L y \leq_L 1_{c_s(k)} \\ &\Rightarrow n-1 < k \Rightarrow n+1 \leq k \Rightarrow n < k \\ &\Rightarrow 1_{c_s(n)} \leq_L 1_{c_s(k-1)} \leq_L y \\ &\Rightarrow 1_{c_s(n)} <_L y. \end{aligned}$$

Thus to show property (ii), it suffices to show that at every stage s, we have $4s+1, 4s+3 \in I_{s+1,k}$ for some k.

If $P_b(\bar{i}(s), \bar{j}(s)) = 0$, then since $c_s(-1) = 0$, we have $4s+1, 4s+3 \in I_{s,0}$. Furthermore, since $c_s(n) = c_{s-1}(n)$ for all n, this also gives for all $x \in \mathbf{L}_{b,s}$, we have $x \in I_{s,k}$ for some k.

If $P_b(\bar{i}(s), \bar{j}(s)) = 1$, we have if $y \in \{4s+1, 4s+3\}$, then

$$1_{c_{s-1}(\bar{i}(s)-1)} \leq_L 1_{c_{s-1}(\bar{i}(s))-1} \prec 2_{c_{s-1}(\bar{i}(s))} \prec y$$

and

$$y \prec 2_{c_{s-1}(\bar{i}(s))+1} \prec 1_{c_{s-1}(\bar{i}(s))+1} \leq_L 1_{c_{s-1}(\bar{i}(s)+1)}.$$

But $c_s(\bar{i}(s)-1) = c_{s-1}(\bar{i}(s)-1)$ and $c_s(\bar{i}(s)) = c_{s-1}(\bar{i}(s)+1)$, so this and the above give $y \in I_{s,\bar{i}(s)}$.

Finally, for any other $x \in \mathbf{L}_{b,s}$, we have $x \in \mathbf{L}_{b,s-1}$. Thus there exists some k where $x \in I_{s-1,k}$. If $k < \bar{i}(s)$, then $I_{s-1,k} = I_{s,k}$, so we are done. If $\bar{i}(s) < k-1$, then

$$I_{s-1,k} = [1_{c_{s-1}(k-1)}, 1_{c_{s-1}(k)}]_{L_{b,s}} = [1_{c_s(k-2)}, 1_{c_s(k-1)}]_{L_{b,s}} = I_{s,k-1}.$$

If $\bar{i}(s) = k$, then

$$I_{s-1,k} = [1_{c_{s-1}(k-1)}, 1_{c_{s-1}(k)}]_{L_{b,s}} \subseteq [1_{c_{s-1}(k-1)}, 1_{c_{s-1}(k+1)}]_L = [1_{c_s(k-1)}, 1_{c_s(k)}]_{L_{b,s}} =\subseteq I_{s,k}.$$

Finally, if $\bar{i}(s) = k-1$, then

$$I_{s-1,k} = [1_{c_{s-1}(k-1)}, 1_{c_{s-1}(k)}]_{L_{b,s}} \subseteq [1_{c_{s-1}(k-2)}, 1_{c_{s-1}(k)}]_L = [1_{c_s(k-2)}, 1_{c_s(k-1)}]_{L_{b,s}} = I_{s,k-1}.$$

□

Lemma 2.2.8. *For all $s \in \omega$, $[I_{c_{s-1}(n-1)}, 1_{c_{s-1}}(n)]_L$ is a simple sublattice of $\mathbf{L}_{b,s-1}$.*

Proof. We proceed by induction on s.

At $s = 0$, $I_{s-1,n} \cong \mathbf{M}_3$ for all $n \in \omega$, and so each $I_{s-1,n}$ is simple.

Now suppose $I_{s-1,n}$ simple for all $n \in \omega$. If $P_b(\bar{i}(s), \bar{i}(s)) = 0$, we have that every $I_{s-1,n} = I_{s,n}$ is simple by the induction hypothesis, so suppose not.

Then $I_{s,n} = I_{s-1,n}$ for $n < \bar{i}(s)$. For $n > \bar{i}(s)$, we have $I_{s,n} = I_{s-1,n-1}$. Either way, these are simple by our induction hypothesis. So we need really only worry when $\bar{i}(s) = n$. In that case, $I_{s,n} = [1_{c_{s-1}(n-1)}, 1_{c_{s-1}(n+1)}]_L$. We know $I_{s-1,n}$ is simple and $I_{s-1,n+1}$ is simple by assumption. But the only new elements, $4s+1$ and $4s+3$, are in $[2_{c_{s-1}(n)}, 2_{c_{s-1}(n)+1}]_L \cong \mathbf{M}_3$ which is simple, and so Lemma 1.2.10 applies and we get $[1_{c_{s-1}(n-1)}, 2_{c_{s-1}(n)+1}]_L$ is simple. We apply Lemma 1.2.10

once more and this gives $[1_{c_{s-1}(n-1)}, 1_{c_{s-1}(n+1)}]_L$ is simple. Thus $I_{s,n}$ is simple. $\square$

Suppose $b \in B$. By Lemma 2.2.6 there exists an i_0 where $c_s(i_0 - 1)$ is eventually constant but $c_s(i_0)$ is unbounded. Let s_0 be a stage where if $s > s_0$, $c_s(i_0 - 1) = c_{s_0}(i_0 - 1)$. We claim that $\mathbf{X} = \{x : x \geq_L 1_{c_{s_0}(i_0-1)}\}$ is a simple sublattice.

Let $a \neq b, c, d \geq_L 1_{c_{s_0}(i_0-1)}$. Since $c_s(i_0)$ is unbounded in s, there exists some $s > s_0$ where $a, b, c, d \leq_L 1_{c_s(i)}$. But by Lemma 2.2.8, $[1_{c_{s_0}}(i_0 - 1), 1_{c_s}(i_0)]_L$ is simple, so $c, d \in \mathrm{Cg}\,(a, b)$. This gives $\mathbf{X}$ simple. If $i_0 = 1$, $\mathbf{X} = \mathbf{L}_b$ so we would be finished. So suppose $i_0 > 1$. Then $I_{s,0}$ is simple for all $s > s_0$, by Lemma 2.2.7. But $[1_{s,0}, 1_{s,i_0-1}]_L$ is finite and unchanging for all $s > s_0$. Thus can only be finitely many principal congruences with elements in $[1_{s,0}, 1_{s,i_0-1}]_L$. Fix an element $x_0 \in X$ where $x_0 \succ 1_{c_{s_0}(i_0-1)}$. Then for any $x \neq 1_{c_{s_0}(i_0-1)}$ where $x \in X$ and any $y \notin X$, $y <_L 1_{c_{s_0}(i_0-1)}$, so $(x, 1_{c_{s_0}(i_0-1)}) \in \mathrm{Cg}\,(x, y)$ by Lemma 1.1.15. But $\mathbf{X}$ is simple so this gives $(x, x_0) \in \mathrm{Cg}\,(x, y)$. But the same argument gives $(x, x_0) \in \mathrm{Cg}\,(x_0, y)$. Thus there are only finitely many principal congruences with an element in X and one outside. A similar argument can be given for $I_{s,0}$. Thus there are only finitely many principal congruences.

If $b \notin B$, then suppose for the purposes of contradiction that there were only N many congruences. By Lemma 2.2.5, $c_s(n)$ is eventually constant. Take s_0 large enough that for all $s \geq s_0$, you have $c_s(n)$ is constant for all $n \leq N + 2$. By Lemma 2.2.7 and Theorem 1.1.16, $\mathrm{Cg}\left(1_{c_{s_0}(m)}, 1_{c_{s_0}(m+1)}\right) \neq \mathrm{Cg}\left(1_{c_{s_0}(n)}, 1_{c_{s_0}(n+1)}\right)$ if $m \neq n$ and $m, n \leq N + 2$. Thus there are at least $N + 1$ principal congruences, a contradiction. Therefore there are infinitely many congruences. $\square$

CHAPTER 3
SUBDIRECT IRREDUCIBILITY OF COMPUTABLE ALGEBRAS

3.1 Construction of a non-SDI Lattice

Definition 3.1.1 (subdirectly irreducible)**.** A computable algebra $\mathbf{A}$ is *subdirectly irreducible* if there exist two elements $a \neq b \in A$, such that for any $0_{\mathbf{A}} \neq \theta \in \mathbf{Con}(\mathbf{A})$, we have $(a, b) \in \theta$. The congruence $\mathrm{Cg}(a, b)$ is called the *monolith* of $\mathbf{A}$.

This definition is equivalent to the one more commonly seen, that a universal algebra $\mathbf{A}$ is subdirectly irreducible when $|A| > 1$ and whenever $\mathbf{A}$ is isomorphic to a subalgebra of a direct product of algebras, then $\mathbf{A}$ is isomorphic to a subalgebra of one of the factors. In 1944, Birkhoff proved the subdirect representation theorem [3, Theorem 2] of universal algebra which states that every algebra is subdirectly representable by its subdirectly irreducible quotients.

Our goal is to show that the property "to be subdirectly irreducible" is Σ^0_3-complete. To show this property to be Σ^0_3-complete, for any Σ^0_3 set B we will construct a sequence of lattices $\{\mathbf{L}_b\}$ such that $\{\mathbf{L}_b\}$ is subdirectly irreducible when $b \in B$. Each of these will have as a sublattice the sublattice $\mathbf{L} = (L = \{2n : n \in \omega\}, (\wedge\restriction_L, \vee\restriction_L))$ we define below.

For the purposes of notation, let $i_k = 2(i + 4k)$ for $i < 4$. This gives $i_k \in L$. For $k \leq j \in \omega$, define $\wedge$ and $\vee$ on 2ω with the following relations. For $k \in \omega$, let $x <_L 0_k$ if and only if $x = 0_j$ where $j < k$. Similarly let $1_k <_L x$ if and only if $x = 1_j$ where $j < x$. Furthermore let $0_k \prec\!\!\prec 2_k, 3_k \prec\!\!\prec 1_k$ in $\{0, 2, 4, \ldots, 3_k\} \cup \{0_j, 1_j\}_{j \in \omega}$. This yields the sublattice in Figure 3.1.

We first show a few properties of this sublattice.

Lemma 3.1.2. *Suppose* $i_k \neq x \in \{0_k, 1_k, 2_k, 3_k\}$. *Then* $\mathrm{Cg}(i_k, x) = \mathrm{Cg}(0_k, 1_k)$ *and for all* $c, d \notin [0_k, 1_k]_L$ *where* $c \neq d, (c, d) \notin \mathrm{Cg}(0_k, 1_k)$.

That is, the congruence classes of $\mathrm{Cg}(i_k, j_k)$ *are*

$$\{0\}, \{1\}, \ldots \{0_{k-1}\}, \{n : n \geq 0_k\} = [0_k, 1_k]_L.$$

Proof. The sublattice on $\{0_k, 1_k, 2_k, 3_k, 0_{k+1}\}$ is isomorphic to $\mathbf{M}_3$ under the map φ defined by $i_k \mapsto i, 0_{k+1} \mapsto 4$. Since $\mathbf{M}_3$ is simple, Malcev's Lemma (Theorem 1.2.7) gives us a sequence

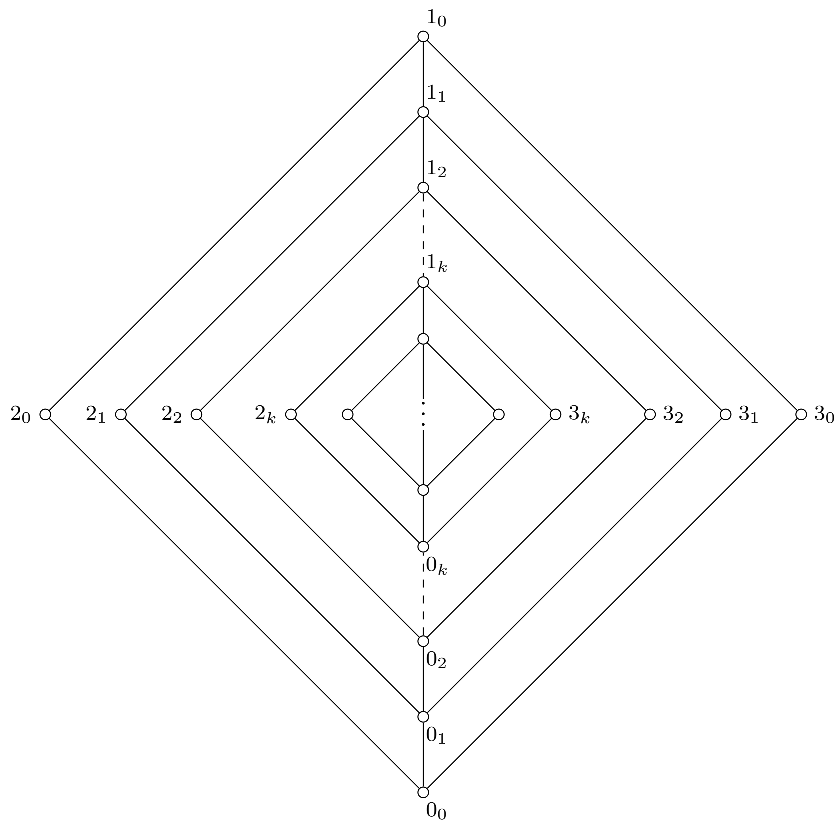

Figure 3.1: The sublattice **L**

of translations of elements of M_3 witnessing $(\varphi(0_k), \varphi(1_k)) \in \mathrm{Cg}\,(\varphi(i_k), \varphi(x))$. The same unary operations and sequence will work in **L** in the obvious way, thus witnessing $(0_k, 1_k) \in \mathrm{Cg}\,(i_k, x)$. Thus $\mathrm{Cg}\,(i_k, x) = \mathrm{Cg}\,(0_k, 1_k)$ in **L**.

The second part follows directly by noting that $[0_k, 1_k]_L$ satisfies the conditions of Lemma 1.1.16.

□

Corollary 3.1.3. *Every non-trivial $\theta \in \mathbf{Con}\,(\mathbf{L})$ is principal. In particular each such θ is of the form $\mathrm{Cg}\,(0_k, 1_k)$ for some $k \in \omega$.*

Proof. Let $\theta \in \mathbf{Con}\,(\mathbf{L})$ be non-trivial. Then there exists $i_k < j_m$ where $i_k \;\theta\; j_m$ and $k, m < 4$. Let k be the smallest such k. We claim $\theta = \mathrm{Cg}\,(0_k, 1_k)$.

By Lemma 3.1.2, we have $\mathrm{Cg}\,(i_k, j_m) = \mathrm{Cg}\,(0_k, 1_k)$, so $\mathrm{Cg}\,(0_k, 1_k) \subseteq \theta$. But for any $(x, y) \in \theta$ where $x, y \geq i_k$ we have $x, y \geq 0_k$, and so $x, y \in [0_k, 1_k]_L$, and thus by Lemma 1.1.15 we have $(x, y) \in \mathrm{Cg}\,(0_k, 1_k)$. This gives $\theta \subseteq \mathrm{Cg}\,(0_k, 1_k)$. □

Lemma 3.1.4. *The congruence lattice of* $\mathbf{Con}\,(\mathbf{L})$ *is a descending chain with a bottom. That is* $\mathbf{Con}\,(\mathbf{L}) \cong (\omega+1, (\wedge = \max(x,y), \vee = \min(x,y))) =: (\omega + \mathbf{1})^d$.

Proof. By Corollary 3.1.3 we have that every non trivial element of $\mathbf{Con}\,(\mathbf{L})$ is principal and of the form $\mathrm{Cg}\,(0_k, 1_k)$. Let $\varphi : \mathbf{Con}\,(\mathbf{L}) \to (\omega+\mathbf{1})^d$ be such that $\mathrm{Cg}\,(0_k,1_k) \mapsto k$, and $0_{\mathbf{L}} \mapsto \omega$. Since if $k \geq j$, $\mathrm{Cg}\,(0_k,1_k) \subseteq \mathrm{Cg}\,(0_j,1_j)$ we have $\mathrm{Cg}\,(0_k,1_k) \vee \mathrm{Cg}\,(0_j,1_j) = \mathrm{Cg}\left(0_{\min(j,k)}, 1_{\min(j,k)}\right)$ and similarly for $\wedge$. $\square$

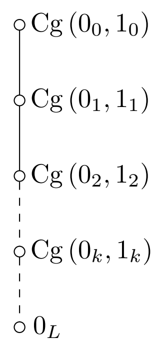

Figure 3.2: The lattice $\mathbf{Con}\,(\mathbf{L})$

Corollary 3.1.5. *L is not subdirectly irreducible.*

Proof. For any possible monolith $\theta = \mathrm{Cg}\,(0_k,1_k)$, we have $(0_k,1_k) \notin \mathrm{Cg}\,(0_{k+1},1_{k+1})$ by the proof of Lemma 3.1.4. Thus there can be no monolith. In fact, $0_L = \bigcap_{k\in\omega} \mathrm{Cg}\,(0_k,1_k)$. $\square$

The idea in our next section is to take this lattice and add elements to it in such away that the congruences collapse.

3.2 The Complexity of Subdirect Irreducibility

Theorem 3.2.1. *The property "to be subdirectly irreducible" is Σ^0_3-complete.*

Proof. We first show that given a computable sequence of algebras $\{\mathbf{A_i}\}_{i\in\omega}$ the set $X := \{i :$ $\mathbf{A_i}$ is subdirectly irreducible $\}$ is a Σ^0_3 set. We have that $\mathbf{A_i}$ is subdirectly irreducible if and only if:

$$(\exists a,b)(\forall c,d)\,((a \neq b) \text{ and } (c \neq d \ \ \& \ \ a,b,c,d \in A_i) \Rightarrow (a,b) \in \mathrm{Cg}\,(c,d))\,.$$

Since $\mathbf{A}_i$ is a computable algebra, A_i is computable, and by Lemma 1.2.9 we have that the statement $a \neq b, c \neq d \;\&\; a, b, c, d \in A_i \Rightarrow (a, b) \in \mathrm{Cg}\,(c, d)$ is Σ^0_1. Thus X is Σ^0_3.

It only remains to show that the property "to be subdirectly irreducible" is Σ^0_3-hard. Let B be Σ^0_3. We are now ready to construct our sequence of lattices with the property that $\mathbf{L}_b$ is subdirectly irreducible if and only if $b \in B$.

As in Theorem 2.2.1 we note that there is a uniformly computable sequence of relations $P_b(i, j)$ where

$$b \in B \Leftrightarrow (\exists i)(\overset{\infty}{\exists} j)(P_b(i, j) = 1).$$

We build L_b in stages, letting $L_{b,-1} = \mathbf{L}$ from above. We use the same $c_s(n)$ from Theorem 2.2.1 along with the $\bar{i}, \bar{j}$ notation. Since $c_s(-1) = -1$ for all s, it is convenient to also let $i_0 = i_{-1}$ for $i < 4$.

At stage s, let $L_{b,s} = L_{b,s-1} \cup \{2s + 1\}$.

If $P_b(\bar{i}(s), \bar{j}(s)) = 0$, we let $0_0 \prec\!\!\!\prec 2s + 1 \prec\!\!\!\prec 1_0$ in $L_{b,s}$. (see Figure 3.3).

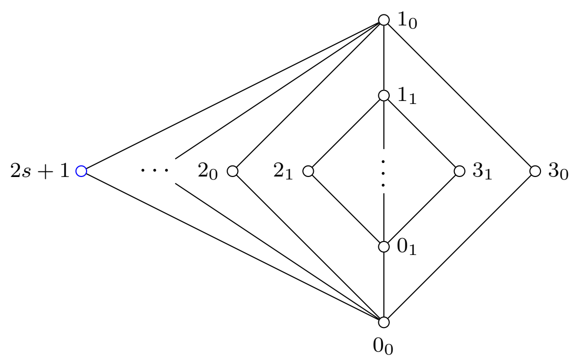

Figure 3.3: Placement of node when $P_b(\bar{i}(s), \bar{j}(s)) = 0$.

If $P_b(\bar{i}(s), \bar{j}(s)) = 1$, we let $2_{c_{s-1}(\bar{i}(s))+1} \prec\!\!\!\prec 2s + 1 \prec\!\!\!\prec 1_{c_{s-1}(\bar{i}(s))}$ in $L_{b,s}$ (see Figure 3.4).

This defines a lattice $\mathbf{L}_b = \bigcup_{s<\omega} \mathbf{L}_{b,s}$. We make some claims concerning this lattice.

Lemma 3.2.2. *Fix b and s. Then $\mathbf{L}_{b,s}$ satisfies the following:*

(1) For all $\theta \in \mathbf{Con}\,(\mathbf{L}_{b,s})$, $\theta = \mathrm{Cg}\left(0_{c_s(n-1)}, 1_{c_s(n-1)}\right)$ for some $n \in \omega$.

(2) Let

$$f(n) = \begin{cases} \mathrm{Cg}\left(0_{c_s(n)}, 1_{c_s(n)}\right) & \text{if } n \in \omega \\ 0_{\mathbf{L}_{b,s}} & \textit{otherwise.} \end{cases}$$

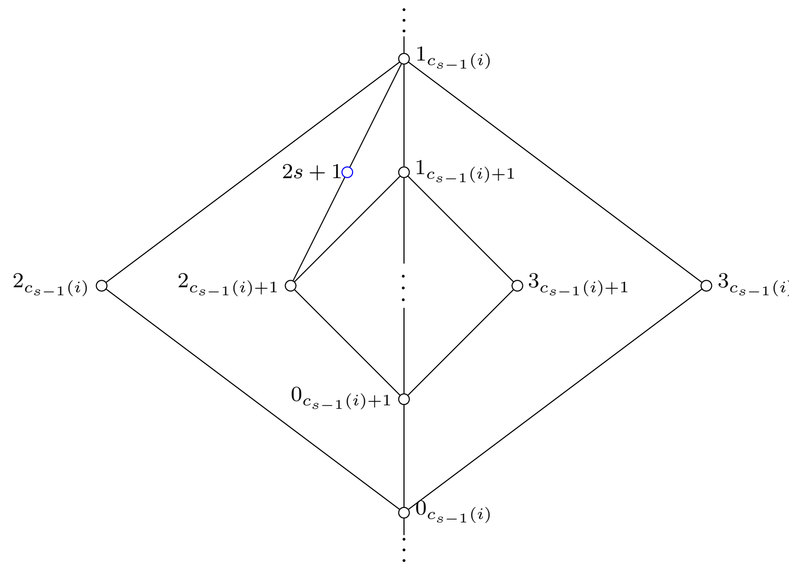

Figure 3.4: Placement of node when $P_b(\bar{i}(s), \bar{j}(s)) = 1$, and $i = \bar{i}(s)$.

Then f is an lattice isomorphism from $(\omega+1)^d$ to $\mathbf{Con}\,(\mathbf{L}_{b,s})$.

(3) For any $n \in \omega$,

$$\mathrm{Cg}\left(0_{c_s(n-1)+1}, 1_{c_s(n-1)+1}\right) = \mathrm{Cg}\left(0_{c_s(n)}, 1_{c_s(n)}\right) = \mathrm{Cg}\left(0_k, 1_k\right)$$

if and only if

$$c_s(n-1)+1 \leq k \leq c_s(n).$$

(4) For all $n \in \omega$, $0_{c_s(n-1)+1}$ and $1_{c_s(n-1)+1}$ satisfy the conditions of Theorem 1.1.16.

That is to say, the set of congruence classes of $\mathrm{Cg}\left(0_{c_s(n-1)+1}, 1_{c_s(n-1)+1}\right)$ are

$$\{[0_{c_s(n-1)+1}, 1_{c_s(n-1)+1}]_L\} \cup \{[x,x]_L : x \notin [0_{c_s(n-1)+1}, 1_{c_s(n-1)+1}]_L\}.$$

Proof. We proceed by induction on s. When $s = -1$, $c_s(n) = n$ for all $n \in \omega$. Since $0_{-1} = 0_0$ and $1_{-1} = 1_0$ by definition, Corollary 3.1.3 handles (1). In addition Lemma 3.1.4 handles (2). As for (3), since $c_{-1}(n) = n$, $c_s(n-1)+1 = c_s(n)$ for all $n \in \omega$. Thus if $c_s(n-1)+1 \leq k \leq c_s(n)$, we have $c_s(n-1)+1 = k = c_s(n)$, which gives

$$\mathrm{Cg}\left(0_{c_s(n-1)+1}, 1_{c_s(n-1)+1}\right) = \mathrm{Cg}\left(0_{c_s(n)}, 1_{c_s(n)}\right) = \mathrm{Cg}\left(0_k, 1_k\right)$$

trivially. On the other hand, suppose $\mathrm{Cg}\left(0_{c_s(n-1)+1}, 1_{c_s(n-1)+1}\right) = \mathrm{Cg}\left(0_{c_s(n)}, 1_{c_s(n)}\right) = \mathrm{Cg}\left(0_k, 1_k\right)$. Recall that Lemma 3.1.2 the congruence classes of $\mathrm{Cg}\left(0_k, 1_k\right)$ as having only one non-trivial class $[0_k, 1_k]_L$. But then $\mathrm{Cg}\left(0_{c_s(n)}, 1_{c_s(n)}\right)$ has only one non-trivial class $[0_{c_s(n)}, 1_{c_s(n)}]_L$. Thus $[0_k, 1_k]_L = [0_{c_s(n)}, 1_{c_s(n)}]_L$ so $k = c_s(n)$. Since $c_s(n-1)+1 = c_s(n)$, this gives $c_s(n-1)+1 = k = c_s(n)$ which satisfies part (3). Lemma 3.1.2 gives exactly part (4). This handles the base case.

Suppose the statement is true for $s-1$. We consider cases depending on the value of $P_b(\bar{i}(s), \bar{j}(s))$. Suppose $P_b(\bar{i}(s), \bar{j}(s)) = 0$. In this case, $c_s(n) = c_{s-1}(n)$ for all $n \in \omega_{-1}$, and $0_0 \prec\!\!\prec 2s+1 \prec\!\!\prec 1_0$.

(1) Let $x \neq y \in L_{b,s}$. We first show for any $y \in L_{b,s}$ where $y \neq 2s+1$, we have $\mathrm{Cg}\left(2s+1, y\right) = \mathrm{Cg}\left(0_0, 0_1\right)$. Since $a, b \in [0_0, 1_0]_L$ for all $a, b \in L_{b,s}$, by Lemma 1.1.15 $\mathrm{Cg}\left(0_0, 1_0\right) = 1_{L_{b,s}}$. So we have $\mathrm{Cg}\left(2s+1, y\right) \subseteq \mathrm{Cg}\left(0_0, 0_1\right)$. We need only show the other inclusion. Suppose $y = 1_0$. Then $((2s+1) \wedge 2_0, 1_0 \wedge 2_0) = (0_0, 2_0) \in \mathrm{Cg}\left((2s+1), y\right)$. The same argument gives $(0_0, 3_0) \in \mathrm{Cg}\left((2s+1), y\right)$. But then $(2_0 \wedge 3_0, 2_0 \vee 3_0) = (0_0, 1_0) \in \mathrm{Cg}\left((2s+1), y\right)$. If $y = 0_0$, the dual argument gives the same. If $y \neq 0_0, 1_0$, then $((2s+1) \wedge y, (2s+1) \vee y = (0_0, 1_0)) \in \mathrm{Cg}\left((2s+1), y\right)$. Thus if either x or y is $2s+1$, we have $\mathrm{Cg}\left(x, y\right) = \mathrm{Cg}\left(0_0, 1_0\right)$. But $0_0 = 0_{-1}$ and $1_0 = 1_{-1}$ by definition, and $c_s(-1) = -1$ for all $s \in \omega_{-1}$. Thus $\mathrm{Cg}\left(2s+1, y\right) = \mathrm{Cg}\left(0_{c_s(0-1)}, 1_{c_s(0-1)}\right)$. If both x and y are not $2s+1$, then in $\mathbf{L}_{b,s-1}$ we have $\mathrm{Cg}_{\mathbf{L}_{b,s-1}}\left(x, y\right) = \mathrm{Cg}_{\mathbf{L}_{b,s-1}}\left(0_{c_{s-1}(n-1)}, 1_{c_{s-1}(n-1)}\right)$ for some $n \in \omega$. But then Malcev's Lemma gives series of elements and sequences of translations which are valid in $\mathbf{L}_{b,s}$ since $\mathbf{L}_{b,s-1}$ is a subalgebra of $\mathbf{L}_{b,s}$. Thus $\mathrm{Cg}_{\mathbf{L}_{b,s}}\left(x, y\right) = \mathrm{Cg}_{\mathbf{L}_{b,s}}\left(0_{c_{s-1}(n-1)}, 1_{c_{s-1}(n-1)}\right) = \mathrm{Cg}_{\mathbf{L}_{b,s}}\left(0_{c_{s-1}(n)}, 1_{c_s(n-1)}\right)$. Thus all principal congruences are of the correct form. We now show all congruences are principal.

Suppose $\theta \in \mathbf{Con}\left(\mathbf{L}_{b,s}\right)$. If $(2s+1, x) \in \theta$ for some $x \neq 2s+1$, then $\theta = \mathrm{Cg}\left(0_0, 1_0\right)$, from the above. If not, then $\theta \setminus \{(x,y) : x = 2s+1 \,\mathrm{or}\, y = 2s+1\} \in \mathbf{Con}\left(\mathbf{L}_{b,s-1}\right)$, so $\theta = \mathrm{Cg}_{\mathbf{L}_{b,s-1}}\left(0_{c_{s-1}(n-1)}, 1_{c_{s-1}(n-1)}\right)$ by the induction hypothesis. But then $\theta = \mathrm{Cg}_{\mathbf{L}_{b,s}}\left(0_{c_{s-1}(n-1)}, 1_{c_{s-1}(n-1)}\right)$ in $\mathbf{L}_{b,s}$ again as a consequence of Malcev's Lemma. So θ is principal in $\mathbf{L}_{b,s}$. This with the above gives (1).

(4) Fix $n \in \omega$. Since $c_s(n-1) = c_{s-1}(n-1)$, $0_{c_{s-1}(n-1)} = 0_{c_s(n-1)}$ and $1_{c_{s-1}(n-1)} = 1_{c_s(n-1)}$. Suppose $x \in \mathbf{L}_{b,s}$. If $x \neq 2s+1$, then condition (ii) is true from the induction hypothesis. So suppose $x = 2s+1$. If $c_s(n-1)+1 = 0$, then $x \in [0_{c_s(n-1)+1}, 1_{c_s(n-1)+1}]_L$, so (ii) of Theorem 1.1.16 is satisfied. If $c_s(n-1)+1 > 0$, we have $x \notin [0_{c_s(n-1)+1}, 1_{c_s(n-1)+1}]_L$. But $x \not\prec\!\!\prec_L 1_{c_s(n-1)+1}$ and $0_{c_s(n-1)+1} \not\prec\!\!\prec_L x$ for all $n \in \omega$. So (ii) is again satisfied.

(2) Since we showed (1) we get that the map f is onto. But part (4) gives that for all $n \neq m > 0$, $\mathrm{Cg}\left(0_{c_s(n)+1}, 1_{c_s(n)+1}\right) \neq \mathrm{Cg}\left(0_{c_s(m)+1}, 1_{c_s(m)+1}\right)$ since $[0_{c_s(n)+1}, 1_{c_s(n)+1}]_L \neq [0_{c_s(m)+1}, 1_{c_s(m)+1}]_L$ if

$n \neq m$. Thus f is one to one. Also since $[0_{c_s(n)+1}, 1_{c_s(n)+1}]_L \subseteq [0_{c_s(m)+1}, 1_{c_s(m)+1}]_L$ if and only if $m < n$, this gives an order isomorphism.

(3)($\Rightarrow$) We first show a small claim.

Claim. $\mathrm{Cg}_{\mathbf{L}_{b,s}}(0_k, 1_k) = \mathrm{Cg}_{\mathbf{L}_{b,s}}(0_j, 1_j) \Rightarrow \mathrm{Cg}_{\mathbf{L}_{b,s-1}}(0_k, 1_k) = \mathrm{Cg}_{\mathbf{L}_{b,s-1}}(0_j, 1_j)$.

Reason: Since $(0_k, 1_k) \in \mathrm{Cg}_{\mathbf{L}_{b,s}}(0_j, 1_j)$, there exist sequences $\{z_i\}_{i \leq n}$, in $\mathbf{L}_{b,s}$ and $\{u_i\}_{i<n}$ witnessing Malcev's Lemma. Since $0_0 \prec\!\!\prec 2s+1 \prec\!\!\prec 1_0$, and $0_0 \prec\!\!\prec 3_0 \prec\!\!\prec 1_0$, for any translation $u(x)$, $u(2s+1) = u(3_0)$. Thus we can replace any instance of $2s+1$ in either $\{z_i\}$ or $\{u_i\}$ with 3_0. This yields a witnessing sequence for Malcev's Lemma entirely in $\mathbf{L}_{b,s-1}$. Thus $(0_k, 1_k) \in \mathrm{Cg}_{\mathbf{L}_{b,s-1}}(0_j, 1_j)$. The same argument gives the $(0_j, 1_j) \in \mathrm{Cg}_{\mathbf{L}_{b,s-1}}(0_j, 1_j)$. Thus $\mathrm{Cg}_{\mathbf{L}_{b,s-1}}(0_k, 1_k) = \mathrm{Cg}_{\mathbf{L}_{b,s-1}}(0_j, 1_j)$.

End of Claim.

Suppose $\mathrm{Cg}_{\mathbf{L}_{b,s}}\left(0_{c_s(n-1)+1}, 1_{c_s(n-1)+1}\right) = \mathrm{Cg}_{\mathbf{L}_{b,s}}\left(0_{c_s(n)}, 1_{c_s(n)}\right) = \mathrm{Cg}_{\mathbf{L}_{b,s}}(0_k, 1_k)$, then by the above claim, we have $\mathrm{Cg}_{\mathbf{L}_{b,s-1}}\left(0_{c_s(n-1)+1}, 1_{c_s(n-1)+1}\right) = \mathrm{Cg}_{\mathbf{L}_{b,s-1}}\left(0_{c_s(n)}, 1_{c_s(n)}\right) = \mathrm{Cg}_{\mathbf{L}_{b,s-1}}(0_k, 1_k)$. But then by the induction hypothesis, $c_{s-1}(n-1)+1 \leq k \leq c_s(n)$. Since $c_s = c_{s-1}$, this proves (3).

(3)($\Leftarrow$) Suppose $c_s(n-1)+1 \leq k \leq c_s(n)$. Since $c_s = c_{s-1}$, by our induction hypothesis we have $\mathrm{Cg}_{\mathbf{L}_{b,s-1}}\left(0_{c_s(n-1)+1}, 1_{c_s(n-1)+1}\right) = \mathrm{Cg}_{\mathbf{L}_{b,s-1}}\left(0_{c_s(n)}, 1_{c_s(n)}\right) = \mathrm{Cg}_{\mathbf{L}_{b,s-1}}(0_k, 1_k)$. Since $\mathbf{L}_{b,s-1}$ is a subalgebra of $\mathbf{L}_{b,s}$, by repeated applications of Corollary 1.2.8, we have $\mathrm{Cg}_{\mathbf{L}_{b,s}}\left(0_{c_s(n-1)+1}, 1_{c_s(n-1)+1}\right) = \mathrm{Cg}_{\mathbf{L}_{b,s}}\left(0_{c_s(n)}, 1_{c_s(n)}\right) = \mathrm{Cg}_{\mathbf{L}_{b,s}}(0_k, 1_k)$.

That handles the case where $P_b(\bar{i}(s), \bar{j}(s)) = 0$. Suppose then that $P_b(\bar{i}(s), \bar{j}(s)) = 1$. Then for all $n < \bar{i}(s)$, $c_s(n) = c_{s-1}(n)$, and for all $n \geq \bar{i}(s)$, $c_s(n) = c_{s-1}(n+1)$, and $2_{c_{s-1}(\bar{i}(s))+1} \prec\!\!\prec 2s+1 \prec\!\!\prec 1_{c_{s-1}(\bar{i}(s))}$ in $\mathbf{L}_{b,s}$. Note that this gives that $c_s(\omega_{-1}) = c_{s-1}(\omega_{-1}) \setminus \{c_{s-1}(\bar{i}(s))\}$.

(1) We first show that $\mathrm{Cg}_{\mathbf{L}_{b,s}}\left(0_{c_{s-1}(\bar{i}(s))}, 1_{c_{s-1}(\bar{i}(s))}\right) = \mathrm{Cg}_{\mathbf{L}_{b,s}}\left(0_{c_{s-1}(\bar{i}(s)+1)}, 1_{c_{s-1}(\bar{i}(s)+1)}\right)$. Note that since $0_{c_{s-1}(\bar{i}(s))+1}, 1_{c_{s-1}(\bar{i}(s))+1} \in [0_{c_{s-1}(\bar{i}(s))}, 1_{c_{s-1}(\bar{i}(s))}]_L$, by Lemma 1.1.15 we have $(0_{c_{s-1}(\bar{i}(s))+1}, 1_{c_{s-1}(\bar{i}(s))+1}) \in \mathrm{Cg}_{\mathbf{L}_{b,s}}\left(0_{c_{s-1}(\bar{i}(s))}, 1_{c_{s-1}(\bar{i}(s))}\right)$. We need only show the other inclusion.

Now $((2s+1) \vee 0_{c_{s-1}(\bar{i}(s))+1}, (2s+1) \vee 1_{c_{s-1}(\bar{i}(s))+1}) = (2s+1, 1_{c_{s-1}(\bar{i}(s))}) \in \mathrm{Cg}_{\mathbf{L}_{b,s}}\left(0_{c_{s-1}(\bar{i}(s))+1}, 1_{c_{s-1}(\bar{i}(s))+1}\right)$. But then $(2_{c_{s-1}(\bar{i}(s))} \wedge (2s+1), 2_{c_{s-1}(\bar{i}(s))} \wedge 1_{c_{s-1}(\bar{i}(s))}) = (0_{c_{s-1}(\bar{i}(s))}, 2_{c_{s-1}(\bar{i}(s))}) \in \mathrm{Cg}_{\mathbf{L}_{b,s}}\left(0_{c_{s-1}(\bar{i}(s))+1}, 1_{c_{s-1}(\bar{i}(s))+1}\right)$. The same argument with $3_{c_{s-1}(\bar{i}(s))}$ gives $(0_{c_{s-1}(\bar{i}(s))}, 3_{c_{s-1}(\bar{i}(s))}) \in \mathrm{Cg}_{\mathbf{L}_{b,s}}\left(0_{c_{s-1}(\bar{i}(s))+1}, 1_{c_{s-1}(\bar{i}(s))+1}\right)$. But then $(2_{c_{s-1}(\bar{i}(s))} \vee 3_{c_{s-1}(\bar{i}(s))}, 2_{c_{s-1}(\bar{i}(s))} \wedge 3_{c_{s-1}(\bar{i}(s))}) = (0_{c_{s-1}(\bar{i}(s))}, 1_{c_{s-1}(\bar{i}(s))}) \in \mathrm{Cg}_{\mathbf{L}_{b,s}}\left(0_{c_{s-1}(\bar{i}(s))+1}, 1_{c_{s-1}(\bar{i}(s))+1}\right)$.

Now suppose $x \neq y \in L_{b,s-1}$. By induction hypothesis part (1) there exists an $n \in \omega$ where $\mathrm{Cg}_{\mathbf{L}_{b,s-1}}(x, y) = \mathrm{Cg}_{\mathbf{L}_{b,s-1}}\left(0_{c_{s-1}(n-1)}, 1_{c_{s-1}(n-1)}\right)$. Then by Lemma 1.2.8, we know $\mathrm{Cg}_{\mathbf{L}_{b,s}}(x, y) =$

$\mathrm{Cg}_{\mathbf{L}_{b,s}}\left(0_{c_{s-1}(n-1)}, 1_{c_{s-1}(n-1)}\right)$. If $n-1<\bar{i}(s)$, then $c_s(n-1)=c_{s-1}(n-1)$, so $\mathrm{Cg}_{\mathbf{L}_{b,s}}(x,y)=\mathrm{Cg}_{\mathbf{L}_{b,s}}\left(0_{c_s(n-1)}, 1_{c_s(n-1)}\right)$. Now if $n-1=\bar{i}(s)$, the above gives $\mathrm{Cg}_{\mathbf{L}_{b,s}}\left(0_{c_{s-1}(n-1)}, 1_{c_{s-1}(n-1)}\right)=\mathrm{Cg}_{\mathbf{L}_{b,s}}\left(0_{c_{s-1}(n)}, 1_{c_{s-1}(n)}\right)=\mathrm{Cg}_{\mathbf{L}_{b,s}}\left(0_{c_{s-1}(n+1-1)}, 1_{c_{s-1}(n+1-1)}\right)$. If $n-1>\bar{i}(s)$, then $n-2\geq\bar{i}(s)$, so $c_s(n-2)=c_{s-1}(n-1)$. This gives $\mathrm{Cg}_{\mathbf{L}_{b,s}}(x,y)=\mathrm{Cg}_{\mathbf{L}_{b,s}}\left(0_{c_s(n-1-1)}, 1_{c_s(n-1-1)}\right)$. Since $n-2\geq\bar{i}(s)$, we know $n-1\in\omega$. Thus we have shown that for any $x\neq y\in L_{b,s-1}$, there exists an $n\in\omega$ where $\mathrm{Cg}_{\mathbf{L}_{b,s}}(x,y)=\mathrm{Cg}_{\mathbf{L}_{b,s}}\left(0_{c_s(n-1)}, 1_{c_s(n-1)}\right)$. Now suppose $y=2s+1$, and $x\in L_{b,s-1}$. Since $\mathrm{Cg}_{\mathbf{L}_{b,s}}(2s+1,x)=\mathrm{Cg}_{\mathbf{L}_{b,s}}((2s+1)\vee x,(2s+1)\wedge x)$ and $2_{c_{s-1}(\bar{i}(s))+1}\lessdot 2s+1\lessdot 1_{c_{s-1}(\bar{i}(s))}$, either $(2s+1, 2_{c_{s-1}(\bar{i}(s))+1})\in\mathrm{Cg}_{\mathbf{L}_{b,s}}(2s+1,x)$ or $(2s+1, 1_{c_{s-1}(\bar{i}(s))})\in\mathrm{Cg}_{\mathbf{L}_{b,s}}(2s+1,x)$ as a consequence of Lemma 1.1.15.

If $(2s+1, 1_{c_{s-1}(\bar{i}(s))})\in\mathrm{Cg}_{\mathbf{L}_{b,s}}(2s+1,x)$, the exact argument 3 paragraphs ago gives $(0_{c_{s-1}(\bar{i}(s))}, 1_{c_{s-1}(\bar{i}(s))})\in\mathrm{Cg}_{\mathbf{L}_{b,s}}(2s+1,x)$.

On the other hand, suppose $(2s+1, 2_{c_{s-1}(\bar{i}(s))+1})\in\mathrm{Cg}_{\mathbf{L}_{b,s}}(2s+1,x)$. Then $(2s+1\vee 1_{c_{s-1}(\bar{i}(s))+1}, 2_{c_{s-1}(\bar{i}(s))+1}\vee 1_{c_{s-1}(\bar{i}(s))+1})=(1_{c_{s-1}(\bar{i}(s))}, 1_{c_{s-1}(\bar{i}(s))+1})\in\mathrm{Cg}_{\mathbf{L}_{b,s}}(2s+1,x)$. But then $(2_{c_{s-1}(\bar{i}(s))}\wedge 1_{c_{s-1}(\bar{i}(s))}, 2_{c_{s-1}(\bar{i}(s))}\wedge 1_{c_{s-1}(\bar{i}(s))+1})=(2_{c_{s-1}(\bar{i}(s))}, 0_{c_{s-1}(\bar{i}(s))})\in\mathrm{Cg}_{\mathbf{L}_{b,s}}(2s+1,x)$. The same argument with $3_{c_{s-1}(\bar{i}(s))}$ gives $(3_{c_{s-1}(\bar{i}(s))}, 0_{c_{s-1}(\bar{i}(s))})\in\mathrm{Cg}_{\mathbf{L}_{b,s}}(2s+1,x)$. But from above this gives $(0_{c_{s-1}(\bar{i}(s))}, 1_{c_{s-1}(\bar{i}(s))})\in\mathrm{Cg}_{\mathbf{L}_{b,s}}(2s+1,x)$.

Now if $x\in[0_{c_{s-1}(\bar{i}(s))}, 1_{c_{s-1}(\bar{i}(s))}]_L$, then $\mathrm{Cg}_{\mathbf{L}_{b,s}}\left(0_{c_{s-1}(\bar{i}(s))}, 1_{c_{s-1}(\bar{i}(s))}\right)=\mathrm{Cg}_{\mathbf{L}_{b,s}}(2s+1,x)$. But above we showed that $\mathrm{Cg}_{\mathbf{L}_{b,s}}\left(0_{c_{s-1}(\bar{i}(s))}, 1_{c_{s-1}(\bar{i}(s))}\right)=\mathrm{Cg}_{\mathbf{L}_{b,s}}\left(0_{c_s(\bar{i}(s))}, 1_{c_s(\bar{i}(s))}\right)$. So $\mathrm{Cg}_{\mathbf{L}_{b,s}}\left(0_{c_s(\bar{i}(s)+1-1)}, 1_{c_s(\bar{i}(s)+1-1)}\right)$. If $x\notin[0_{c_{s-1}(\bar{i}(s))}, 1_{c_{s-1}(\bar{i}(s))}]_L$, then by the previous part there exists some $n\in\omega$ where $\mathrm{Cg}_{\mathbf{L}_{b,s}}\left(1_{c_{s-1}(\bar{i}(s))}, x\right)=\mathrm{Cg}_{\mathbf{L}_{b,s}}\left(0_{c_s(n-1)}, 1_{c_s(n-1)}\right)$. But either $0_{c_s(n-1)}, 1_{c_s(n-1)}\in[0_{c_s(\bar{i}(s)+1-1)}, 1_{c_s(\bar{i}(s)+1-1)}]_L$ or $0_{c_s(\bar{i}(s)+1-1)}, 1_{c_s(\bar{i}(s)+1-1)}\in[0_{c_s(n-1)}, 1_{c_s(n-1)}]_L$. Let θ be the bigger one. Then $(2s+1,x)\in\theta$, since $(1_{c_{s-1}(\bar{i}(s))}, x)\in\theta$ and $(1_{c_{s-1}(\bar{i}(s))}, 2s+1)\in\theta$. Thus $\mathrm{Cg}_{\mathbf{L}_{b,s}}(2s+1,x)=\theta$, which is of the proper form.

Thus we have shown part (1) for principal congruences. Let $\theta\in\mathbf{Con}(L_{b,s})$. Then for every $(a,b)\in\theta$, there exists a $n_{a,b}$ where $\mathrm{Cg}_{\mathbf{L}_{b,s}}(a,b)=\mathrm{Cg}_{\mathbf{L}_{b,s}}\left(0_{c_s(n_{a,b}-1)}, 1_{c_s(n_{a,b}-1)}\right)$. Then $\theta=\bigcup_{(a,b)\in\theta}\mathrm{Cg}_{\mathbf{L}_{b,s}}(a,b)=\bigcup_{(a,b)\in\theta}\mathrm{Cg}_{\mathbf{L}_{b,s}}\left(0_{c_s(n_{a,b}-1)}, 1_{c_s(n_{a,b}-1)}\right)=\mathrm{Cg}_{\mathbf{L}_{b,s}}\left(0_{c_s(\min\{n_{a,b}\}-1)}, 1_{c_s(\min\{n_{a,b}\}-1)}\right)$. This shows (1) in general.

(4) Fix $n\in\omega$, and $x\in\mathbf{L}_{b,s}$. Now if $x\in[0_{c_s(n-1)+1}, 1_{c_s(n-1)+1}]_L$ property (ii) is vacuously true. So suppose $x\notin[0_{c_s(n-1)+1}, 1_{c_s(n-1)+1}]_L$. Suppose $x\not\geq 2s+1$. Since $c_s(\omega_{-1})=c_{s-1}(\omega_{-1})\setminus\{c_{s-1}(\bar{i}(s))\}$, we have that $c_s(n-1)=c_{s-1}(m-1)$ for some $m\in\omega$. If $x<_L 1_{c_s(n-1)+1}$, then $x<_L 1_{c_{s-1}(m-1)+1}$. But then our induction hypothesis gives $x<_L 0_{c_{s-1}(m-1)+1}=0_{c_s(n-1)+1}$. The

case where $0_{c_s(n-1)+1} <_L x$ is similar.

Suppose then that $x = 2s+1$. Since $x <_L 1_{c_s(n-1)+1}$ if and only if $x \in [0_{c_s(n-1)+1}, 1_{c_s(n-1)+1}]_L$, we need only worry about the case where $0_{c_s(n-1)+1} <_L x$. Now $x \in [0_{c_s(n-1)+1}, 1_{c_s(n-1)+1}]_L$ when $c_s(n-1)+1 \leq c_{s-1}\bar{i}(s)$. When $c_s(n-1)+1 > c_{s-1}(\bar{i}(s))+1$, $0_{c_s(n-1)+1} \not<_L x$. Thus we need only worry about the case where $c_s(n-1)+1 = c_{s-1}(\bar{i}(s))+1$. But this is when $c_s(n-1) = c_{s-1}(\bar{i}(s))$ a contradiction.

(2) Again, showing (1) gives us that the map f is onto. The induction hypothesis and Lemma 1.2.8 gives us that for all $n \in \omega$, $\mathrm{Cg}_{\mathbf{L}_{b,s}}\left(0_{c_s(n-1)+1}, 1_{c_s(n-1)+1}\right) = \mathrm{Cg}_{\mathbf{L}_{b,s}}\left(0_{c_s(n)}, 1_{c_s(n)}\right)$. Furthermore, part (4) gives us that each $n \in \omega$, the congruence classes of $\mathrm{Cg}_{\mathbf{L}_{b,s}}\left(0_{c_s(n)}, 1_{c_s(n)}\right)$ are different, so that gives us one to one. And similar to the previous case, since $[0_{c_s(n)+1}, 1_{c_s(n)+1}]_L \subseteq [0_{c_s(m)+1}, 1_{c_s(m)+1}]_L$ if and only if $m < n$, this gives an order isomorphism.

(3)($\Rightarrow$) Suppose $\mathrm{Cg}_{\mathbf{L}_{b,s}}\left(0_{c_s(n-1)+1}, 1_{c_s(n-1)+1}\right) = \mathrm{Cg}_{\mathbf{L}_{b,s}}\left(0_{c_s(n)}, 1_{c_s(n)}\right) = \mathrm{Cg}_{\mathbf{L}_{b,s}}(0_k, 1_k)$. By (4), since $(0_k, 1_k) \in \mathrm{Cg}_{\mathbf{L}_{b,s}}\left(0_{c_s(n-1)+1}, 1_{c_s(n-1)+1}\right)$, we have $0_k, 1_k \in [0_{c_s(n-1)+1}, 1_{c_s(n-1)+1}]_L$. Thus $k \geq c_s(n-1)+1$.

By the induction hypothesis there exists an m where $\mathrm{Cg}_{\mathbf{L}_{b,s-1}}(0_k, 1_k) = \mathrm{Cg}_{\mathbf{L}_{b,s-1}}\left(0_{c_{s-1}(m)}, 1_{c_{s-1}(m)}\right)$ and $c_{s-1}(m-1)+1 \leq k \leq c_{s-1}(m)$. . If there exists an j where $c_s(j) = c_{s-1}(m)$, then $\mathrm{Cg}_{\mathbf{L}_{b,s}}(0_k, 1_k) = \mathrm{Cg}_{\mathbf{L}_{b,s}}\left(0_{c_s(j)}, 1_{c_s(j)}\right)$. But by part (2), f is one to one onto $\mathbf{Con}\,(\mathbf{L}_{b,s})$ so then $j = n$. But that gives $c_s(n) = c_{s-1}(m) \geq k$. If there does not exist such a j, then $m = \bar{i}(s)$ since $c_s(\omega_{-1}) = c_{s-1}(\omega_{-1}) \setminus \{c_{s-1}(\bar{i}(s))\}$. But $\mathrm{Cg}_{\mathbf{L}_{b,s}}(0_k, 1_k) = \mathrm{Cg}_{\mathbf{L}_{b,s}}\left(0_{c_{s-1}(m)}, 1_{c_{s-1}(m)}\right)$ by Lemma 1.2.8. But in proving (1) we showed $\mathrm{Cg}_{\mathbf{L}_{b,s}}\left(0_{c_{s-1}(\bar{i}(s))}, 1_{c_{s-1}(\bar{i}(s))}\right) = \mathrm{Cg}_{\mathbf{L}_{b,s}}\left(0_{c_{s-1}(\bar{i}(s)+1)}, 1_{c_{s-1}(\bar{i}(s)+1)}\right)$. Again since $c_s(\omega_{-1}) = c_{s-1}(\omega_{-1}) \setminus \{c_{s-1}(\bar{i}(s))\}$, there must be a $j \in \omega$ where $c_s(j) = c_{s-1}(\bar{i}(s)+1)$. But then again, $\mathrm{Cg}_{\mathbf{L}_{b,s}}(0_k, 1_k) = \mathrm{Cg}_{\mathbf{L}_{b,s}}\left(0_{c_s(j)}, 1_{c_s(j)}\right)$, so by (2) we have $n = j$. So then $k \leq c_{s-1}(m) < c_{s-1}(m+1) = c_s(j) = c_s(n)$. Thus $c_s(n-1)+1 \leq k \leq c_s(n)$.

(3)($\Leftarrow$) Suppose $c_s(n-1)+1 \leq k \leq c_s(n)$. If there exists some m where $c_{s-1}(m) = c_s(n)$ and $c_{s-1}(m-1) = c_s(n-1)$, then by induction and Lemma 1.2.8 we have $\mathrm{Cg}_{\mathbf{L}_{b,s}}\left(0_{c_s(n-1)+1}, 1_{c_s(n-1)+1}\right) = \mathrm{Cg}_{\mathbf{L}_{b,s}}\left(0_{c_s(n)}, 1_{c_s(n)}\right) = \mathrm{Cg}_{\mathbf{L}_{b,s}}(0_k, 1_k)$. If there does not exist such an m, then $n = \bar{i}(s)$. But since $\mathrm{Cg}_{\mathbf{L}_{b,s}}\left(0_{c_{s-1}(\bar{i}(s))}, 1_{c_{s-1}(\bar{i}(s))}\right) = \mathrm{Cg}_{\mathbf{L}_{b,s}}\left(0_{c_{s-1}(\bar{i}(s)+1)}, 1_{c_{s-1}(\bar{i}(s)+1)}\right) = \mathrm{Cg}_{\mathbf{L}_{b,s}}\left(0_{c_s(\bar{i}(s))}, 1_{c_s(\bar{i}(s))}\right)$, and that $c_s(n-1) = c_{s-1}(n-1)$ because $n = \bar{i}(s)$. We get $\mathrm{Cg}_{\mathbf{L}_{b,s}}\left(0_{c_s(n-1)+1}, 1_{c_s(n-1)+1}\right) = \mathrm{Cg}_{\mathbf{L}_{b,s}}\left(0_{c_s(n)}, 1_{c_s(n)}\right)$. Furthermore, since $c_s(n-1)+1 \leq k \leq c_s(n)$, we have $\mathrm{Cg}_{\mathbf{L}_{b,s}}\left(0_{c_s(n-1)+1}, 1_{c_s(n-1)+1}\right) \subseteq \mathrm{Cg}_{\mathbf{L}_{b,s}}(0_k, 1_k) \subseteq \mathrm{Cg}_{\mathbf{L}_{b,s}}\left(0_{c_s(n)}, 1_{c_s(n)}\right) =$

$\mathrm{Cg}_{\mathbf{L}_{b,s}}\left(0_{c_s(n-1)+1}, 1_{c_s(n-1)+1}\right)$. So $\mathrm{Cg}_{\mathbf{L}_{b,s}}\left(0_k, 1_k\right) = \mathrm{Cg}_{\mathbf{L}_{b,s}}\left(0_{c_s(n)}, 1_{c_s(n)}\right)$.

□

We are now ready to prove our claim. We show $\mathbf{L}_b$ is subdirectly irreducible if and only if $b \in B$.

($\Rightarrow$) : Suppose $b \notin B$. By Lemma 2.2.5, we have for any i, $c_s(i)$ is eventually constant in s. Define $c(i) = \lim_{s\to\infty} c_s(i)$. So for any $x \neq y$, there exists some maximal i where $x, y \in \left[0_{c(i)}, 1_{c(i)}\right]_L$. Then $(x, y) \notin \mathrm{Cg}\left(0_{c(i)+1}, 1_{c(i)+1}\right)$, so thus no x, y can produce a monolith. In other words, $0_{L_b} = \bigcap_{i\in\omega} \mathrm{Cg}\left(0_{c(i)}, 1_{c(i)}\right)$ and no $\mathrm{Cg}\left(0_{c(i)}, 1_{c(i)}\right) = 0_{L_b}$. Thus $\mathbf{L}_b$ is not subdirectly irreducible.

($\Leftarrow$) : Suppose $b \in B$. By Lemma 2.2.6, there exists n where $c_s(m)$ is eventually constant in s for all $m \leq n$, but $c_s(n+1)$ is unbounded in s. Suppose $c_s(m)$ is constant for all $m \leq n$ and $s \geq s_0$. We show $\mathrm{Cg}\left(0_{c_{s_0}(n)+1}, 1_{c_{s_0}(n)} + 1\right)$ is a monolith for $\mathbf{L}_b$.

Suppose $\theta \in \mathbf{Con}\left(\mathbf{L}_b\right)$ is nontrivial. Then there exists $x \neq y \in \omega$ where $(x, y) \in \theta$. Consider a stage s_1 where $x, y \in \mathbf{L}_{b,s_1}$. Take $s = \max(s_0, s_1)$. Then by Lemma 3.2.2 part (1), there exists a $k \in \omega$ where $\mathrm{Cg}_{\mathbf{L}_{b,s}}(x, y) = \mathrm{Cg}_{\mathbf{L}_{b,s}}\left(0_{c_s(k-1)}, 1_{c_s(k-1)}\right)$. By Lemma 1.2.8, $\mathrm{Cg}_{\mathbf{L}_b}(x, y) = \mathrm{Cg}_{\mathbf{L}_b}\left(0_{c_s(k-1)}, 1_{c_s(k-1)}\right)$. If $c_s(k-1) \leq c_{s_0}(n) + 1$, then $0_{c_{s_0}(n)+1}, 1_{c_{s_0}(n)+1} \in \left[0_{c_s(k-1)}, 1_{c_s(k-1)}\right]_L$ so by Lemma 1.1.15, we have $(0_{c_{s_0}(n)+1}, 1_{c_{s_0}(n)+1}) \in \theta$. On the other hand, if $c_s(k-1) > c_{s_0}(n) + 1$, since $c_s(n+1)$ is unbounded, there exists a stage $s' \geq s$ where $c_{s'}(n+1) \leq c_s(k-1)$. Then by Lemma 3.2.2 part (3), $\mathrm{Cg}_{\mathbf{L}_{b,s'}}\left(0_{c_{s'}(n)+1}, 1_{c_{s'}(n)+1}\right) = \mathrm{Cg}_{\mathbf{L}_{b,s'}}\left(0_{c_{s'}(n)}, 1_{c_{s'}(n)}\right)$. But $c_{s'}(n) + 1 = c_{s_0}(n) + 1$, so $(0_{c_{s_0}(n)+1}, 1_{c_{s_0}(n)+1}) \in \theta$. Thus $\mathrm{Cg}\left(0_{c_{s_0}(n)+1}, 1_{c_{s_0}(n)} + 1\right)$ is a monolith for $\mathbf{L}_b$. □

Corollary 3.2.3. *The computable lattices witness the Σ^0_3-completeness of being subdirectly irreducible.*

Proof. Immediate from the previous construction. □

Remark 3.2.4. *We also note that the construction of the $\{\mathbf{L}_b\}$ in Theorem 3.2.1 provides an alternative proof for Theorem 2.2.1, although a more complicated one, as $\mathbf{L}_b$ has finitely many congruences if and only if $b \in B$.*

3.3 Epilogue

The usefulness of these results is mostly in the negative. The main result shows that no characterization of subdirectly irreducible for a general algebra can be expressed in anything below a Σ^0_3 formula. In fact these results show that no characterization of a lattice in particular being simple, having finitely many congruences, or being subdirectly irreducible can be expressible in anything below a Π^0_2 or Σ^0_3 formula respectively.

The first two constructions are done using modular lattices, showing in fact that these properties can be witnessed using modular lattices. It is an open question whether or not the Σ^0_3 complexity of being subdirectly irreducible can bet witnessed using modular lattices. It is our conjecture that it is likely, though more difficult.

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