DUAL LINEAR SPACES GENERATED BY A NON-DESARGUESIAN CONFIGURATION

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Abstract

A dual linear space is a partial projective plane which contains the intersection of every pair of its lines. Every dual linear space can be extended to a projective plane, usually infinite, by a sequence of one line extensions. Moreover, one may describe necessary conditions for the sequence of one line extensions to terminate after finitely many steps with a finite projective plane. A computer program that attempts to construct a finite projective plane from a given dual linear space by a sequence of one line extension has been written by Dr. Nation. In particular, one would like to extend a dual linear space containing a non-Desarguesian configuration to a finite projectvie plane of non-prime-power order. This dissertation studies the initial dual linear spaces to be used in this algorithm. The main result is that there are 105 non-isomorphic initial dual linear spaces containing the basic non-Desarguesian configuration.

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Chapter 1
Introduction

A dual linear space is a partial projective plane which contains the intersection of every pair of its lines. Every dual linear space can be extended to a projective plane, usually infinite, by a sequence of one line extensions. Moreover, one may describe necessary conditions for the sequence of one line extensions to terminate after finitely many steps with a finite projective plane. A computer program that attempts to construct a finite projective plane from a given dual linear space by a sequence of one line extension has been written by Dr. Nation. In particular, one would like to extend a dual linear space containing a non-Desarguesian configuration to a finite projectvie plane of non-prime-power order. This dissertation studies the initial dual linear spaces to be used in this algorithm. The main result is that there are 105 non-isomorphic initial dual linear spaces containing the basic non-Desarguesian configuration.

We begin by reviewing some elementary properties of finite projective planes in Chapter 2. We will describe how a projective plane may be constructed from a division ring, an affine plane, a ternary ring, or a double loop. We then introduce Desargues' theorem and show that a projective plane constructed from a division ring is a Desarguesian plane. The Bruck-Ryser theorem [4] is the classic result on the order of finite projective planes. By this theorem, there is no projective plane of order n when n is congruent to 1 or 2 mod 4 and n is not a sum of two squares. So there is no projective plane of order 6 or 14. In 1989, C. Lam, L. Thiel and S. Swiercz proved that there is no projective plane of order 10 [10]. Hence 12 and 15 are the first two numbers for which we do not

know whether or not there is a projective plane of that order. Likewise, we do not know whether there is a non-Desarguesian plane of a prime order p when $p \geq 11$.

In Chapter 3, we will state the definition of a linear space. Its dual is sometimes called a semiplane. However, the main focus for that Chapter will be to describe and analyze the algorithm for constructing projective planes by extending finite semiplanes.

1. If a finite semiplane can be extended to a projective plane of order n, then it can be extended through a sequence of one line extensions.

2. We introduce simple and useful necessary conditions for extendibility to a plane of order n.

3. We describe how the computer programs work in general by keeping track of the possible one line extensions that satisfy the conditions.

In Chapter 4, we will prove the main result of this dissertation, that there are 875 different semiplanes generated by a non-Desarguesian configuration, which fall into 105 isomorphism classes. We have checked that all but five of these isomorphism types actually occur in a Hall plane of order 9.

We will summarize the results of our computer search for a projective plane of order 12 in Chapter 5.

Chapter 2
Projective Planes

We begin by reviewing some basic properties of finite projective planes and describing different ways to construct a projective plane in this Chapter. We will use Batten [1], Batten and Beutelspacher [2], Bennett [3], Hall [8], Lam [10], Nation [12], Scherk [13] and Stevenson [14] as the source of the following material.

A *projective plane* is a structure $\Pi = (P, L, \leq)$ where P is a set of points, L is a set of lines, and $p \leq l$ means that the point p is on the line l, satisfying the following axioms:

PP1 Any two distinct points lie on exactly one line.

PP2 Any two distinct lines intersect in exactly one point.

PP3 There exist four points, no three of which are on a line.

If p and q are two distinct points in a projective plane, then we let $p \vee q$ denote the unique line through them. Similarly, $k \wedge l$ denotes the unique point on both lines k and l. If two or more points lie on the same line, the points are *collinear.* If two or more lines pass through the same point, the lines are *concurrent.* If a, b, c, d are four points satisfying PP3, then the quadruple a, b, c, d is called a *complete four-point* or simply a *four-point.*

We would like to note here that axiom PP3 is equivalent in the presence of PP1 and PP2 to the following axiom: Every line has at least 3 points and there are at least 2 lines.

If we add 0 and 1 to the lattice representing the incidences between lines and points, we obtain a complemented modular lattice.

If $\mathcal{T}$ is a statement about points and lines, then the *dual statement* of $\mathcal{T}$ is the statement obtained by interchanging the words 'point' and 'line' with suitable adjustments of the words 'lies on' and 'contains' (or any equivalents). A given statement is true for a projective plane if and only if its dual is true.

Theorem 1. *Let $n \geq 2$ be an integer. In a projective plane Π any one of the following properties implies the rest:*

1. *One line contains exactly $(n+1)$ points.*
2. *One point is on exactly $(n+1)$ lines.*
3. *Every line contains exactly $(n+1)$ points.*
4. *Every point is on exactly $(n+1)$ lines.*
5. *There are exactly (n^2+n+1) points in Π.*
6. *There are exactly (n^2+n+1) lines in Π.*

Proof. Let a, b, c, d be four points, no three on a line, whose existence is given by PP3. For our convenience, we call the condition that no three of the points a, b, c, d are on a line $(\star)$. PP1 and $(\star)$ imply that there are six distinct lines containing a, b, c, d, say, $l_1 = a \vee b$, $l_2 = a \vee c$, $l_3 = a \vee d$, $l_4 = b \vee c$, $l_5 = b \vee d$, and $l_6 = c \vee d$. PP2 and $(\star)$ imply that there are an additional three distinct points x, y, z such that $x = l_1 \wedge l_6$, $y = l_2 \wedge l_5$,

$z = l_3 \wedge l_4$. Moreover, we may check that there are no further incidences between these seven points and these six lines. Before we continue our proof, we would like to prove two claims.

Claim 1: If l is a line with exactly $(n+1)$ points on it, say, $q_1, q_2, ..., q_{n+1}$, and p is a point not on l, then p is on exactly $(n+1)$ lines.

Proof. Since p is not on l, we have $p \vee q_i = l_i$, $i = 1, ..., n+1$, by PP1. If $p \vee q_i = l_i = l_j = p \vee q_j$, then $p \leq q_i \vee q_j = l$, contrary to our assumption. Hence all the l_i's are distinct. Moreover, every line through p intersects l and so must be one of the l_i's. Therefore there are exactly $(n+1)$ lines through p. □

Claim 2: If p is a point on exactly $(n+1)$ lines, say, $k_1, ..., k_{n+1}$, and m is a line not through p , then m has exactly $(n+1)$ points.

Proof. Claim 2 is the dual of Claim 1. □

Now assume property 1. Let l be a line having exactly $(n+1)$ points. Note that at least two of the points a, b, c, d are not on l, say, a and b. By Claim 1, both of them are on exactly $(n+1)$ lines, hence it's clear that property 2 holds. Every line not through a or not through b contains exactly $(n+1)$ points by Claim 2, so every line except possibly l_1 contains exactly $(n+1)$ points. Consider l_2, which does not go through b, and hence has exactly $(n+1)$ points. Since point z is not on l_2, z is on exactly $(n+1)$ lines. And $z \not\leq l_1$ implies that l_1 must also contain exactly $(n+1)$ points. Thus property 3 holds. But now for any point p, we may find a line not through it, and so by Claim 1, p is

on exactly $(n+1)$ lines, giving property 4. Now let p_0 be a particular point and let $l_1, ..., l_{n+1}$ be the lines through p_0. Each of these lines contains exactly n points different from p_0, and since p_0 is joined to any point in the plane by one of the l_i's, they account for all the points of Π, giving a total $1 + (n+1)n = n^2 + n + 1$ points in Π, proving property 5. Similarly, if l_0 is a line containing points $p_1, ..., p_{n+1}$, each of these points is on exactly n additional lines, and since each line in the plane intersects l_0 in one of p_i's, they account for all the lines in Π, giving a total of $n^2 + n + 1$ lines, proving property 6.

We have shown that property 1 implies the remaining five properties. It is clear that property 3 implies the rest. By duality, we have both property 2 and 4 imply the rest. If property 5 holds, then the line $l_1 = a \vee b$ contains $(m+1)$ points for some integer $m \geq 2$, and so by our earlier argument, Π contains $m^2 + m + 1$ points. But since both m and n are positive integers, $m^2 + m + 1 = n^2 + n + 1$ implies $m = n$, whence property 5 implies property 1 and so the remaining ones. Similarly, 6 implies 2 and so the others. Thus all parts of our theorem are proved. □

A finite projective plane is said to be of *order* n if a line contains exactly $(n+1)$ points. By Theorem 1, we know that a projective plane of order n has $n^2 + n + 1$ points and $n^2 + n + 1$ lines, each line contains $n + 1$ points and each point is on $n + 1$ lines.

Figure 1 gives us the projective plane of order 2, which is also referred to as the Fano plane.

Let D be an arbitrary division ring. We would like to construct a plane $\Pi_D = (P, L, \leq)$ as follows. Let $P = \{[x, y, z] \mid x, y, z \in D \text{ and } x = y = z = 0 \text{ is not true } \}$,

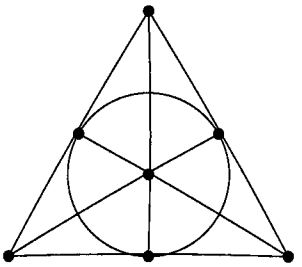

Figure 1: Fano plane.

where $[x, y, z] = [x', y', z']$ iff there exists $r \in D$, $r \neq 0$, such that $x = x'r$, $y = y'r$, $z = z'r$. Let $L = \{\langle a, b, c\rangle \mid a, b, c \in D \text{ and } a = b = c = 0 \text{ is not true }\}$, where $\langle a, b, c\rangle = \langle a', b', c'\rangle$ iff there exists $s \in D$, $s \neq 0$, such that $a = sa'$, $b = sb'$, $c = sc'$. Then define $[x, y, z] \leq \langle a, b, c\rangle$ iff $ax + by + cz = 0$.

Theorem 2. *The plane Π_D is a projective plane.*

Proof. Let $p = [x, y, z]$ and $p' = [x', y', z']$ be distinct points. Since not all of $x, y, z = 0$, we may assume without loss of generality that $x \neq 0$. Furthermore, we may assume that $x = 1$. Since p and p' are distinct, $yx' - y'$ and $zx' - z'$ cannot be both zero; for otherwise, we would have $1 \cdot x' = x'$, $yx' = y'$, $zx' = z'$, which implies that $p = p'$, contrary to our assumption. So we may assume without loss of generality that $yx' - y' \neq 0$.

First, we want to show that there exists a line containing p and p'. Let $L = \langle a, b, c, \rangle$ where $a = -(by + z)$, $b = (zx' - z')(y' - yx')^{-1}$, $c = 1$. Now $ax + by + cz = -(by + z) \cdot 1 + by + 1 \cdot z = 0$. Also $ax' + by' + cz' = -(by + z)x' + by' + 1 \cdot z' = b(y' - yx') - zx' + z' = (zx' - z')(y' - yx')^{-1}(y' - yx') - zx' + z' = 0$. Thus L contains p and p'.

Next, to show uniqueness, suppose that $L' = \langle a', b', c' \rangle$ also contains p and p'. If $c' = 0$, then $a' + b'y = 0$ and $a'x' + b'y' = 0$. Thus $-b'yx' + b'y' = b'(-yx' + y') = 0$. Since $y' - yx' \neq 0$, we have $b' = 0$. Now $a' + b'y = 0$ implies $a' = 0$. Thus $a' = b' = c' = 0$. This contradiction establishes that $c' \neq 0$; we may assume that $c' = 1$. We now have the following equalities:

1. $a + by + z = 0$

2. $ax' + by' + z' = 0$

3. $a' + b'y + z = 0$

4. $a'x' + b'y' + z' = 0$

Equalities (1) and (3) imply that $a + by = a' + b'y$; (2) and (4) imply that $ax' + by' = a'x' + b'y'$. So we have:

5. $a - a' = (b' - b)y$

6. $(a - a')x' = (b' - b)y'$

Now suppose that $b \neq b'$. Then (5) and (6) imply that $(b' - b)yx' = (b' - b)y'$, thus $(b' - b)(yx' - y') = 0$. If $b' - b \neq 0$, then $yx' - y' = 0$, which contradicts our assumption. Hence $b = b'$. Then equality (5) implies that $a = a'$, and so we conclude that $L = L'$. This proves PP1.

The dual property PP2 may be shown in a similar manner reversing the order of products. Finally the points $[0, 0, 1]$, $[0, 1, 0]$, $[1, 0, 0]$ and $[1, 1, 1]$ form a four-point, thus PP3 holds. □

Theorem 2 tells us that projective planes of order k exist for any k a prime power.

Theorem 3. *Let $p = [x, y, z]$, $p' = [x', y', z']$ and $p'' = [x'', y'', z'']$ be distinct points in Π_D. Then p'' is on the line $p \vee p'$ if and only if $x'' = xr + x's$, $y'' = yr + y's$ and $z'' = zr + z's$ for some $r, s \in D$ where $r, s \neq 0$.*

Proof. Let $p \vee p' = \langle a, b, c \rangle$. Note that $ax + by + cz = 0$ and $ax' + by' + cz' = 0$. Thus for $r, s \in D$ and $r, s \neq 0$, $a(xr + x's) + b(yr + y's) + c(zr + z's) = axr + byr + czr + ax's + by's + cz's = (ax + by + cz)r + (ax' + by' + cz')s = 0$. Therefore if $x'' = xr + x's$, $y'' = yr + y's$ and $z'' = zr + z's$, then $p'' \leq p \vee p'$.

Now suppose that $p'' \leq p \vee p'$. Since not all of $x, y, z = 0$, $x, y, z \in D$, we may assume that $x = 1$. Since p and p' are distinct, we know that $yx' - y'$ and $zx' - z'$ cannot be both zero. So without loss of generality, assume $yx' - y' \neq 0$, and thus we may solve the equations for $xr + x's = x''$ and $yr + y's = y''$ for r and s. Since we assumed $x = 1$, we have $r = x'' - x's$ and $s = (y' - yx')^{-1}(y'' - yx'')$. If $c = 0$, then $a + by = 0$ and $ax' + by' = 0$. Thus $-byx' + by' = b(y' - yx') = 0$. Since $y' - yx' \neq 0$, we have $b = 0$. Therefore $a = 0$, and we reach a contradiction. Thus $c \neq 0$, and we may assume $c = 1$. Since p, p' and p'' are collinear, it follows that $a + by + z = 0$, $ax' + by' + z' = 0$ and $ax'' + by'' + z'' = 0$. Thus $z'' = -ax'' - by'' = -a(r + x's) - b(yr + y's) = -ar - ax's - byr - by's = -ar - byr - zr + zr - ax's - by's - z's + z's = (-a - by - z)r - (ax' + by' + z')s + zr + z's = zr + z's$. Clearly, if $r = 0$ or $s = 0$, then p'' would not be distinct from p and p'. Thus $x'' = xr + x's$, $y'' = yr + y's$, $z'' = zr + z's$, and $r, s \neq 0$. □

Corollary 4. *If p, q, r are collinear, there exist coordinates $p_i, q_i, i = 1, 2, 3$, of p, q,*

respectively, such that $r_i = p_i + q_i$ *for given coordinates* r_i *of* r.

G. Desargues proved the following theorem for the real projective plane in the seventeenth century.

Theorem 5 (Desargues' Theorem). *If* x, y, z *and* x', y', z' *are triples of noncollinear points of* $\Pi_{\Re}$ *such that* $x \vee x'$, $y \vee y'$, $z \vee z'$ *intersect in a common point* o, *then corresponding sides meet in three points* u, v, w *which are collinear.*

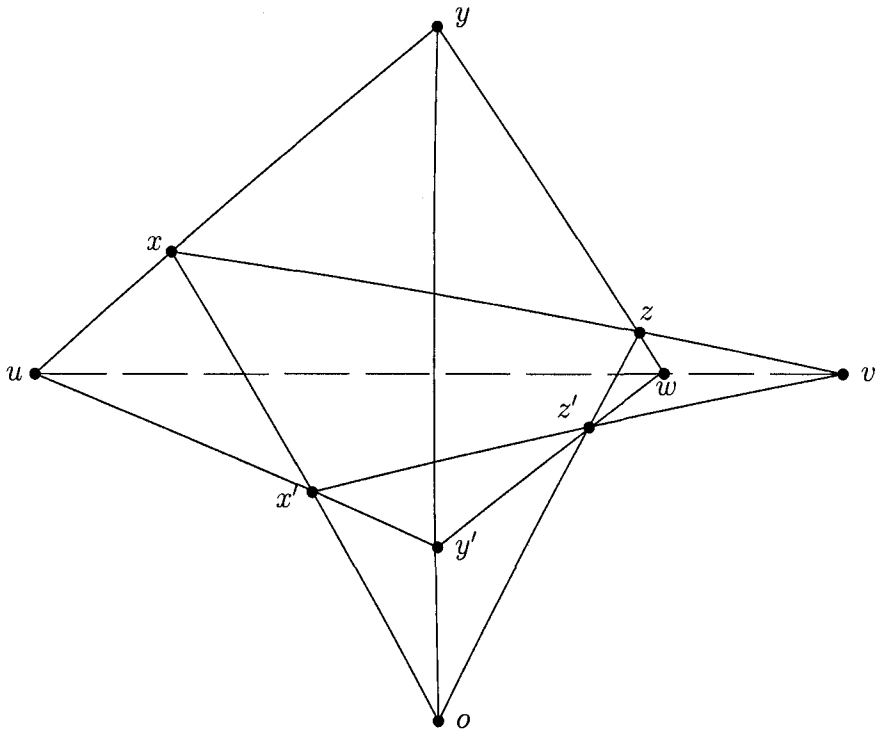

Figure 2: Desarguesian configuration.

The statement of Desargues' Theorem does not hold in every projective plane Π. If Theorem 5 is valid in a projective plane, then it is called a *Desarguesian plane*; otherwise,

it is called a *non-Desarguesian plane.* Figure 2 shows two triangles in a Desarguesian configuration.

Two triangles pqr and $p'q'r'$ are *centrally perspective from point* s if and only if the respective vertices are collinear with s, that is, $p \vee p'$, $q \vee q'$, and $r \vee r'$ pass through s. We say that two triangles are *axially perspective from line* L if and only if the respective sides of the triangles meet on L. Thus in a Desarguesian projective plane, every two centrally perspective triangles are axially perspective.

Theorem 6. *If D is a division ring, then Π_D is a Desarguesian plane.*

Proof. Let p, q, r and p', q', r' be vertices of two triangles centrally perspective from a seventh point s. Let $p = [p_1, p_2, p_3]$, $q = [q_1, q_2, q_3]$, $r = [r_1, r_2, r_3]$, $s = [s_1, s_2, s_3]$, $p' = [p'_1, p'_2, p'_3]$, $q' = [q'_1, q'_2, q'_3]$ and $r' = [r'_1, r'_2, r'_3]$. Since s, p and p' are collinear, by Corollary 4, we may assum that the coordinates of p and p' are such that $p'_i = p_i + s_i, i = 1, 2, 3$. Similarly, we may assume that the coordinates of q and q' are such that $q'_i = q_i + s_i$, and that the coordinates of r and r' are such that $r'_i = r_i + s_i, i = 1, 2, 3$. Let $a = (p \vee q) \wedge (p' \vee q')$, $b = (p \vee r) \wedge (p' \vee r')$, $c = (q \vee r) \wedge (q' \vee r')$. It's then easily checked that $a = [p_1 - q_1, p_2 - q_2, p_3 - q_3]$, $b = [p_1 - r_1, p_2 - r_2, p_3 - r_3]$ and $c = [q_1 - r_1, q_2 - r_2, q_3 - r_3]$. Therefore a, b, c are collinear by Theorem 3. □

The converse of Theorem 6 is also true: if Π is a Desarguesian plane, then Π is isomorphic to Π_D for some division ring D. For the proof, see Stevenson [14].

An *affine* plane is an ordered pair $A = (P, L)$ where P is a nonempty set of elements called *points* and L is a nonempty collection of subsets of P called *lines* that have the

following properties:

A1. Any two distinct points of A are on a unique line.

A2. If p is a point not on the line l, then there is a unique line m through p missing l, i.e., $l \wedge m = \emptyset$. In this case, we say that l is parallel to m, written $l \parallel m$. Note that we consider $l \parallel l$.

A3. There are at least two points on each line; there are at least two lines.

A *pencil* of parallel lines is a maximal set of mutually parallel lines, or equivalently, a set consisting of a line, together with all the lines parallel to it.

Suppose $l \parallel m$, and $m \parallel h$. If there exists $p = l \wedge h$, then l and h are both lines through p parallel to m, which contradicts A2. Hence parallelism is an equivalence relation. Each line belongs to one and only one pencil. Two lines l and m are parallel if and only if they belong to the same pencil.

We may construct a projective plane from an affine plane by adjoining a common new point to each pencil of parallel lines and gathering the new points together to form an additional line, as described below.

Let $A = (P, L)$ be an affine plane. For each pencil Φ of parallel lines, define a symbol P_Φ, called a *point at infinity.* Let $P' = P \cup \{P_\Phi \mid \Phi \text{ a pencil of } A\}$. For each $l \in A$, define $l' = l \cup \{P_\Phi\}$, where Φ is the unique pencil containing l. Let $l_\infty = \{P_\Phi \mid \Phi \text{ a pencil of } A\}$. Finally, let $L' = \{l' \mid l \in A\} \cup \{l_\infty\}$.

Theorem 7. *If $A = (P, L)$ is an affine plane, then $\Pi = (P', L')$ is a projective plane.*

Proof. First, we would like to show that Axiom PP1 holds. Fix $p, q \in \Pi$. If $p, q \in P$, then there exists a unique line $l \in L$ containing p and q. Hence l' is the unique line in Π containing them. If $p \in P$, and $q = P_\Phi$ for some pencil Φ, then p is on a unique $l \in A$ which is in pencil Φ. Thus l' is the unique line in Π containing them. If both p and q are points at infinity, then l_∞ is the unique line in Π containing them.

Now fix $l', m' \in \Pi$. If $l, m \in L$, then they intersect at either some point $p \in P$ or some point at infinity. In either case, we have l' intersects m' at some unique point in Π. If say $m' = l_\infty$, then $l' \wedge m' = P_\Phi$, where P_Φ is the point at infinity on l'. Therefore Axiom PP2 holds.

Since there are at least two points on every $l \in A$, there are at least three points on every $l' \in \Pi$. Fix $l', m' \in \Pi$. Then there are at least two points, say p_1 and p_2, that are on l' but not on m'. Similarly, we can find two points q_1 and q_2 that are on m' but not on l'. Note that no three of the points, p_1, p_2, q_1 and q_2, are on a line. Therefore Axiom PP3 holds, and $\Pi = (P', L')$ is a projective plane. □

Proposition 8. *Let A be a finite affine plane, that is, A has finitely many points. Then there exists an integer $n \geq 2$ such that every line of A has exactly n points.*

Proof. If l is a line of A, we let $r(l)$ denote the number of points on l. Fix two distinct lines l and m of A. If p is a point not on any of those two lines, then A1 and A2 imply that p is on $r(l)+1 = r(m)+1$ lines, so $r(l) = r(m)$. If there is no such point p, then we claim that $l \parallel m$. Suppose not, and let $o = l \wedge m$. A3 implies that there exists a point $a \leq l$, $a \neq o$. By A1, and since $l \neq m$, we have $a \not\leq m$. Hence there is a line k through

a parallel to m. Again, by axioms of affine plane, there exists a point $b \leq k$, $b \neq a$, and $b \not\leq l$. Note that $b \not\leq m$ as well, since $k \parallel m$. So we have a point $b \not\leq l \cup m$, contrary to our assumption. Thus $l \parallel m$. By A3, we may assume there are two points, p_1 and p_2 on l, and two points, q_1 and q_2 on m. Let $k = p_1 \vee q_1$. $q_2 \not\leq l \cup k$. Hence by earlier argument, $r(l) = r(k)$. Similarly, $p_2 \not\leq m \cup k \Longrightarrow r(m) = r(k)$. So we have $r(l) = r(m)$ again. Let $n = r(l)$. Since there are at least two points on each line, $n \geq 2$. □

If each line of a (finite) affine plane contains exactly n points, the plane is said to have *order* n.

Theorem 9. *Let A be an affine plane of order n. Then*

i. A has exactly n^2 points.

ii. Each point is on $n + 1$ lines.

iii. Each pencil contains n lines.

iv. The total number of lines is $n(n + 1)$.

v. There are $n + 1$ pencils of parallel lines.

Proof. (i) Fix a line l, and suppose $p_1, ..., p_n$ are the points on l. Let o be a point not on l. Let $m_1 = p_1 \vee o$, and for $i = 2, ..., n$, let m_i be the unique line through p_i parallel to m_1. Each of the lines $m_1, ..., m_n$ has n points, there are n of these lines, and no pair of lines has a point in common. Therefore there are n^2 many points contained in $m_1 \cup ... \cup m_n$. If p is any point in the plane, then either $p \in m_1$, or there is a unique line k through p

parallel to m_1. In the latter case, k must intersect l at one of the points $p_2, ..., p_n$, for otherwise we would have $m_1 \parallel k \parallel l$, with $p_1 = m_1 \wedge l$, contradicting A2. Hence $k = m_i$ for some $i = 2, ..., n$. Thus every point is in one of the lines $m_1, ..., m_n$. Therefore A has exactly n^2 points.

(ii) Fix a point $p \in A$ and let $l = \{q_1, ..., q_n\}$ be a line not containing p. The lines $l_i = q_i \vee p$ and the unique line l_{n+1} through p parallel to l are all distinct. Moreover, if m is any line through p, either $m \parallel l$, which implies that $m = l_{n+1}$, or $m \wedge l = q_i$ for some i, which imples that $m = l_i$ for some i. Therefore p is on exactly $n + 1$ lines.

(iii) Fix a line $l \in A$. Suppose $p_1 \leq l$. Let p_2 be a point not on l. Then consider the line $p_1 \vee p_2 = m = \{p_1, ..., p_n\}$. For $i = 2, ..., n$, let l_i be the line through p_i parallel to l. There are $n - 1$ such lines, and they are all distinct. If k is any line parallel to l, since l is the only line containing p_1 parallel to k, $k \wedge m = p_i$, for some $i = 2, ..., n$. So $k = l_i$ for some i, and consequently, the lines $l_2, ..., l_n$ are the only lines parallel to l.

(iv) Fix a point $p \in A$, and let $l_1, ..., l_{n+1}$ be the lines through p. Each of this lines is in a pencil consisting of n lines. So the lines through p together with the lines parallel to them account for $n(n + 1)$ distinct lines. Let m be any line in A. If $p \leq m$, then m has been counted as one of the l_i's. If $p \not\leq m$, then there is a unique line k through p parallel to m. In this case, m has also been counted as a line in the same pencil as line k. Therefore there are $n(n + 1)$ lines in A.

(v) Since each of the $n(n + 1)$ lines is in one and only one pencil, and each pencil contains n lines, there are $n + 1$ pencils. □

Note that if we use the construction given just before Theorem 7, we may obtain a projective plane of order n from an affine plane of the same order. This construction may also be reversed: Given a projective plane, an affine plane is obtained by removing any line and all the points on that line. For the details, see Bennett [3].

Now, following Hall [6], [7], we show how to introduce coordinates in an arbitray projective plane.

Let Π be a projective plane and let X, Y, O, I be four points in Π no three on a line. Let $O = (0,0)$ and $I = (1,1)$ and $(X \vee Y) \wedge (O \vee I) = (1)$. Choose a set R which contains $0, 1$ and is in one-to-one correspondence with the points of $O \vee I \setminus (1)$, and let $B = R \setminus \{0,1\}$. For example, if Π has order n, we could take $B = \{2, ..., n-1\}$. Assign the rest of the points on $O \vee I$ distinct labels (b,b) with $b \in B$. For any point P not on $X \vee Y$ or $O \vee I$, assign P the coordinate (a,b) where $(Y \vee P) \wedge (O \vee I) = (a,a)$ and $(X \vee P) \wedge (O \vee I) = (b,b)$. For any point Q on $X \vee Y$ with $Q \neq Y$, the line $O \vee Q$ contains a unique point $(1,m)$ with first coordinate 1; assign Q the coordinate (m). Finally, let $Y = (\infty)$.

Let l_∞ denote the line $X \vee Y$. Note that l_∞ contains (1). If l is any other line through Y except l_∞, then l intersects $O \vee I$ in some point (c,c). In this case, every point except Y on l has coordinates (c,y) for some y, and we let $l = l_c$. We have labelled all the lines through Y. If k is a line not through Y, then $k \wedge l_\infty = (m)$ and $k \wedge l_0 = (0,b)$ for some pair m, b. In this case, let $k = l_{(m,b)}$.

Now note that the lines $l_{(m,b)}$ have the property that they never contain two points with the same x-coordinate, since the line through (c,y) and (c,y') is l_c. On the other

hand, for every $c \in R$, $l_{(m,b)} \wedge l_c$ is some standard point (c, y). Thus we can use these lines to define a ternary operation $T(x, m, b)$ on R by

$$T(x, m, b) = y \iff (x, y) \in l_{(m,b)}$$

The ternary operation T defined above satisfies the following:

T1. $T(0, m, c) = T(a, 0, c) = c$.

T2. $T(1, m, 0) = T(m, 1, 0) = m$.

T3. Given m, b, d with $m \neq 0$ there exists exactly one x such that $T(x, m, b) = d$.

T4. Given a, b, d with $a \neq 0$ there exists exactly one y such that $T(a, y, b) = d$.

T5. Given a, m, d there exists exactly one z such that $T(a, m, z) = d$.

T6. Given m, m', b, b' with $m \neq m'$ there exists exactly one x such that $T(x, m, b) = T(x, m', b')$.

T7. Given a, a', c, c' with $a \neq a'$ there exists exactly one pair m, b such that $T(a, m, b) = c$ and $T(a', m, b) = c'$.

A ternary algebra $R = (R, T, 0, 1)$ satisfying T1 - T7 is called a *ternary ring*. Note that the ternary ring R constructed to coordinatize Π depends on the choice of the quadrangle X, Y, O, I.

Now given a ternary ring $R = (R, T, 0, 1)$, we may construct a plane Π_T as follows. Let $P = \{(x, y) \mid x, y \in R\} \cup \{(x) \mid x \in R\} \cup \{(\infty)\}$. Let $L = l_{(m,b)} \cup l_c \cup l_\infty$, where

$l_{(m,b)} = \{(x,y) \mid y = T(x,m,b), x,y,m,b \in R\} \cup \{(m)\}$, $l_c = \{(x,y) \mid x = c, x,y,c \in R\} \cup \{(\infty)\}$, and $l_\infty = \{(m) \mid m \in R\} \cup \{(\infty)\}$.

Theorem 10. *Π_T is a projective plane.*

Proof. First we show that there exists exactly one line passing through two distinct points p and q.

Case 1. Let $p = (a,b)$ and $q = (c,d)$. If $a = c$, then p and q lie on line l_a. To show that this line is unique, observe that p and q certainly do not occupy l_k for any k, $k \neq a$. Also if $p, q \in l_{(m,k)}$, then $b = T(a,m,k) = T(c,m,k) = d$, and so $p = q$, a contradiction. If $a \neq c$, then by T7, there exists unique one pair m, k such that $T(a,m,k) = b$ and $T(c,m,k) = d$. Thus $l_{(m,k)}$ contains both points. Clearly neither l_∞ nor any line l_s may pass through both points, so the line $l_{(m,k)}$ is unique.

Case 2. Let $p = (a,b)$ and $q = (c)$. By T5, there exists exactly one k such that $T(a,c,k) = b$. Thus $p \leq l_{(c,k)}$ and $q = (c) \leq l_{(c,k)}$. Clearly no other line contains q and q.

Case 3. Let $p = (a,b)$ and $q = (\infty)$. Then it's easy to see that l_a is the only line containing p and q.

Case 4. Let $p = (a)$ and $q = (b)$. Then l_∞ is the only line containing both points.

Case 5. Let $p = (a)$ and $q = (\infty)$. Then again l_∞ is the unique line passing through both points.

So PP1 is satisfied. A similar argument shows that two lines intersect in exactly one

point. As for PP3, it is easy to verify that $(0,0)$, (0), $(1,1)$, and (∞) are four point, no three of which are on a line. □

Theorem 11. *If $(R,+,\cdot)$ is a division ring, then (R,t) is a ternary ring where t is defined by $t(a,b,c) = ab + c$.*

Proof. For any $m, c, a \in R$, we have $t(0,m,c) = 0m + c = c$, and $t(a,0,c) = a0 + c = c$, so T1 holds. For any $m \in R$, $t(1,m,0) = 1m + 0 = m$, and $t(m,1,0) = m1 + 0 = m$, so T2 holds. Given $m, b, d \in R$, with $m \neq 0$, let $x = (d-b)m^{-1}$. Then clearly $t(x,m,b) = d$. Suppose there exists y such that $t(y,m,b) = d$. Then $xm + b = ym + b$ implies that $xm = ym$. Multiplying m^{-1} on both sides, we get $x = y$. So x is unique, and T3 holds. Similar arguments show that both T4 and T5 hold. Given m, m', b, b' with $m \neq m'$. Then $m - m' \neq 0$, so $(m-m')^{-1}$ exists. Let $x = (b'-b)(m-m')^{-1}$. Then $t(x,m,b) - t(x,m',b') = (b'-b)(m-m')^{-1}m + b - (b'-b)(m-m')^{-1}m' - b' = (b'-b)(m-m')^{-1}(m-m') + b - b' = 0$. So $t(x,m,b) = t(x,m',b')$. Suppose there exists y such that $t(y,m,b) = t(y,m',b')$. Then $xm + b - xm' - b' = 0 = ym + b - ym' - b'$, which implies that $x(m-m') = y(m-m')$, hence $(x-y)(m-m') = 0$. Since $m - m' \neq 0$, we get $x - y = 0$. So we have $x = y$, and T6 holds. Given a, a', c, c' with $a \neq a'$. Let $m = (a'-a)^{-1}(c'-c)$ and $b = c - a(a'-a)^{-1}(c'-c)$. Then it is easy to check that $t(a,m,b) = c$ and $t(a',m,b) = c'$. Suppose that u, v is another pair such that $t(a,u,v) = c$ and $t(a',u,v) = c'$. Then we have $au + v = am + b$ and $a'u + v = a'm + b$, which implies that $au + v - am - b = 0 = a'u + v - a'm - b$. So $(a-a')(u-m) = 0$. Since $a - a' \neq 0$, we have $u = m$, and $v = b$ by T5. Therefore T7 holds. □

In view of Theorem 11, we may regard the concept of ternary ring as a generalization of the concept of division ring. Now we want to construct ternary rings that are not derived from division rings, so that we can construct non-Desarguesian projective planes.

A *loop* is an algebra $\mathcal{L} = (L, *, e)$ with the following properties.

1. $a * e = a$ and $e * a = a$ for every $a \in L$.

2. Given $a, b \in L$, there exists unique $x \in L$ such that $a * x = b$.

3. Given $c, d \in L$, there exists unique $y \in L$ such that $y * c = d$.

A *double loop* is an algebra $\mathcal{R} = (R, +, \cdot, 0, 1)$ such that

1. $(R, +, 0)$ is a loop,

2. $(R - \{0\}, \cdot, 1)$ is a loop,

3. $x0 = 0$ and $0x = 0$ for every $x \in R$.

Theorem 12. *A double loop coordinatizes a projective plane if and only if it has the following two properties.*

A. Given m, m', b, b' with $m \neq m'$, there is a unique $x \in R$ such that $xm + b = xm' + b'$.

B. Given a, a', c, c' with $a \neq a'$, there is a unique pair $x, y \in R$ such that $ax + y = c$ and $a'x + y = c'$.

Property A reflects that, in an affine plane, two nonparallel lines should intersect in a unique point. Property B says that two points determine a unique line.

Theorem 13. *In a finite double loop, property A and property B are equivalent.*

For detail of the proof of Theorem 13, see Nation [12].

A *quasifield* is a double loop $\mathcal{Q} = (Q, +, \cdot, 0, 1)$ such that

1. $+$ is associative,

2. $(a + b)c = ac + bc$ for all $a, b, c \in Q$,

3. Given m, m', c with $m \neq m'$, there is a unique $x \in Q$ such that $xm = xm' + c$.

Property 3 in the definition implies that quasifields form a rich class of coordinatizing double loops, and hence a source of non-Desarguesian projective planes. However, the next lemma shows that a projective plane coordinatized by a quasifield will always have prime power order.

Note that property 1 implies that $(Q, +, 0)$ is a group, and property 3 is called the *planar property.*

Lemma 14. *[5] Let $\mathcal{Q} = (Q, +, \cdot, 0, 1)$ be a finite double loop in which addition is associative and the right distributive law holds. Then*

1. $(-a)b = -ab$ for all $a, b \in Q$,

2. $(Q, +, 0)$ is an elementary abelian p-group, i.e., there is a prime p such that $\mid \mathcal{Q} \mid = p^k$ and every element of $\mathcal{Q}$ has additive order p, and

3. the planar property holds, so $\mathcal{Q}$ is a quasifield.

Proof. By the comment before the lemma, we know that $(Q, +, 0)$ is a group. Fix any $a, b \in Q$, $0 = 0b = (-a + a)b = (-a)b + ab$, thus $(-a)b = -ab$.

For any $m \neq 0$, the map $\tau_m : x \mapsto xm$ is an automorphism of $(Q, +, 0)$. Moreover, given any $x, y \in Q$, since $\mathcal{Q}$ is a double loop, there exists unique $g \in Q$ such that $xg = y$. So the automorphism group of $(Q, +, 0)$ is transitive, so each row in the multiplication table of a nonzero element is a permutation, which implies that every nonzero element of Q has the same order. Q is finite, so some element has a prime order p.

To prove that the additive group $(Q, +, 0)$ is commutative, we would like to prove that the planar property holds first. Let $m, m' \in Q$ with $m \neq m'$. Consider the mapping $\sigma_{m,m'} : x \mapsto -xm' + xm$. Suppose there exist $x, y \in Q$ such that $\sigma_{m,m'}(x) = \sigma_{m,m'}(y)$, then $-xm' + xm = \sigma_{m,m'}(x) = \sigma_{m,m'}(y) = -ym' + ym$, which implies that $ym' - xm' = ym - xm$, so $(y - x)m' = (y - x)m$. Transitivity of the automorphism group of $(Q, +, 0)$ and $m \neq m'$ imply that $y - x = 0$, i.e., $x = y$. So the map is one-to-one. Finiteness implies that the map is also onto. Therefore the planar property holds, and hence $\mathcal{Q}$ is a quasifield.

Assume there exist elements $a, b \in Q$ such that $b + a - b = ac$, $c \neq 1$. By the planar property, there exists unique element d such that $-dc + d = b$, which implies that $dc + b = d$, and so $dc + b + a = d + a$. Since $b + a - b = ac$, we have $dc + (ac + b) = d + a$. Since addition is associative, we get $(dc + ac) + b = d + a$, and right distributive law gives $(d + a)c + b = d + a$. Thus, $-(d + a)c + (d + a) = b$. The uniqueness of d then implies $d + a = d$, or $a = 0$. Hence $a \neq 0$ implies $b + a = a + b$ for all $b \in Q$, and this clearly holds if $a = 0$. □

Let F be a finite field, and let $Q = F^d$ be a vector space of dimension d over F. The first projection embeds F into Q, so we can identify $c \in F$ with the vector $(c, 0, ..., 0) \in Q$. Suppose that we can find a set of linear transformations $\rho(m), m \in Q$ such that

1. $\rho(0) = O$, the 0 matrix,

2. $\rho(1) = I$, the identity matrix,

3. the first row of $\rho(u)$ is u,

4. $\rho(m) - \rho(m')$ is nonsingular whenever $m \neq m'$.

If we define multiplication by $u \circ v = u\rho(v)$, then Q is a quasifield.

For example, let $Q = F^2$, where F is a finite field of order greater than 2. Let $f(x) = x^2 - rx - s$ be an irreducible quadratic over F. Define

$$\rho(c, 0) = \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix}$$

and for $d \neq 0$,

$$\rho(c, d) = \begin{pmatrix} c & d \\ -d^{-1}f(c) & -c + r \end{pmatrix}$$

Then Q is called a *Hall quasifield*, which coordinatizes a non-Desarguesian plane.

The classic result on the order of finite projective planes is the Bruck-Ryser Theorem [4].

Theorem 15. *If $n \equiv 1, 2$ mod 4 and n is not the sum of two squares, then there is no projective plane of order n.*

We have shown earlier that there is a Desarguesian projective plane of every prime power. The previous quasifield construction yields a non-Desarguesian projective plane of order q^2 for any prime power $q \geq 3$. Similar construction of non-Desarguesian projective planes may be found in [5]. Thus far, all known finite projective planes have prime power order. There are infinitely many numbers n of the form given in the Bruck-Ryser Theorem, and $6, 14, 21, 22, 30$ are just the first few of them. Lam, Thiel and Swiercz proved that there is no plane of order 10 [10]. Therefore we ask the questions: *Is there a projective plane of non-prime-power order? Is there a non-Desarguesian plane of prime power order?* In our effort to answer the questions, we have done extensive computer search for a non-Desarguesian plane of order 12, and some search for non-Desarguesian planes of order 11 and 15.

Chapter 3
Extending Dual Linear Spaces

We introduced projective planes, some of the properties and different ways to construct Desarguesian or non-Desarguesian planes in the previous chapter. In our effort to answer the questions raised at the end of that chapter, we will introduce the concept of a linear space [1], [2], [3] and its dual, a semiplane. The computer programs written by Dr. Nation implement the one line extension theory [11]. We will explain how these programs work in general and state the results for some of the tests that we have done.

A *linear space* is a triple $S = (P, L, \leq)$ consisting of a set P of elements called *points*, a set L of distinguished subsets of points, called *lines*, and $p \leq l$ means that the point p is on the line l, satisfying the following axioms:

L1 Any two distinct points of S belong to exactly one line of S.

L2 Any line of S contains at least two points of S.

Note that in a linear space, L1 implies that two distinct lines intersect in at most one point. If a linear space S also has the property that there are three points not on a common line, then S is called a *non-trivial linear space.*

We define a *dual linear space*, or *semiplane*, to be a triple $S' = (P', L', \leq')$ statisfying the dual statements of L1 and L2. Thus in a dual linear space, every pair of lines intersect in exactly one point, and every point lies on at least two lines. The dual to the previous observation implies that two points lie on at most one line in a dual linear space.

Again $p \vee q$ denotes the line through points p and q, if there is one. Similarly, $k \wedge l$ denotes the point on both lines k and l, if there is one. As usual, $p \leq l$ means that point p is on line l, or equivalently, that line l passes through point p.

We say that S is a *finite linear space* if S has finitely many points. For any point p in S, we let $r(p)$ denote the number of lines on p. Similarly, we let $r(l)$ denote the number of points on the line l. We shall refer to $r(p)$ and $r(l)$ as the *rank* of the respective point p and line l.

Proposition 16. *Let S be a linear space. If p is a point in S and l is a line not through p, then $r(p) \geq r(l)$.*

Proof. This follows immediately from property L1. □

Proposition 17. *If $S = (P, L, \leq)$ is a finite linear space, then*

$$\sum_{l \in L} r(l) = \sum_{p \in P} r(p)$$

Proof. The sum on the left counts the number of points on each line, line by line. The sum on the right counts the number of lines through each point, point by point. These are just two different ways of counting point-line incidences. Clearly, they are the same. □

Proposition 18. *If $S = (P, L, \leq)$ is a finite linear space, then*

$$\sum_{l \in L} r(l)(r(l) - 1) = |P|(|P| - 1)$$

where $|P|$ is the number of points.

Proof. Since any two distinct points determine a unique line in S, the sum on the left counts the number of ordered pairs of points line by line, which is clearly equal to the right-hand side. □

Proposition 19. *If $S = (P, L, \leq)$ is a finite linear space, then*

$$\sum_{p \in P} r(p)(r(p) - 1) \leq |L|(|L| - 1)$$

where $|L|$ is the number of lines. The equality holds if and only if every pair of lines meet.

Proof. The left-hand side counts the number of intersecting pairs of lines, whereas the right-hand side counts the number of all pairs of lines. □

A linear space with n points, where $n \geq 3$, that has one line containing $n - 1$ points and $n - 1$ lines containing two points, is called a *near pencil.*

The following theorem is the Fundamental Theorem of finite linear spaces. For the proof of the theorem, see [2].

Theorem 20 (Fundamental Theorem). *Let $S = (P, L, \leq)$ be a finite non-trivial linear space. Then $|L| \geq |P|$. Moreover, equality holds if and only if S is a projective plane or a near-pencil.*

Theorem 20 implies that in a finite dual linear space, we have $|P| \geq |L|$.

An *embedding* of a linear space $S = (P, L, \leq)$ into a linear space $S' = (P', L', \leq')$ is a function f mapping P into P' and L into L' which is one-to-one on both points and lines, and such that $p \leq l$ if and only if $f(p) \leq f(l)$.

We note here that any linear space can be embedded into a linear space containing four points such that no three are collinear.

Marshall Hall proved that a linear space S in which one can find four points, no three collinear, is embeddable in a projective plane using the following construction [6]. Consider all pairs of lines which do not meet, and add a new point to any such pair of lines. Now consider all pairs of points not joined by a line, and add a new line through any such pair. Continuing in this way, we obtain a projective plane of usually infinite order. The question remains: *Can a finite linear space be embedded in a finite projective plane? If so, of what order?*

Let $\Sigma = \langle P, L, \leq \rangle$ be a projective plane. Observe that there is a natural correspondence between subsets of L and semiplanes Π contained in Σ. This is given by the map $K \mapsto \Pi_K$, where K is a subset of L and $\Pi_K = \langle Q, K, \leq^* \rangle$ is a semiplane obtained by taking $Q = \{k \wedge \ell : k, \ell \in K \text{ and } k \neq \ell\}$ and $\leq^*$ the restriction of $\leq$ to $Q \times K$.

Let $\Pi = \langle P_0, L_0, \leq_0 \rangle$ be a semiplane, and let $\Sigma = \langle P, L, \leq \rangle$ be a semiplane properly extending Π. Let $k \in L - L_0$, and let $M = \{p \in P_0 : p \leq k\}$. Note that $s \vee t$ is undefined in Π for all pairs $s, t \in M$. Let $D = \{\ell \in L_0 : \ell \wedge k \notin P_0\}$. For each $\ell \in D$, let $q_\ell = \ell \wedge k$; note that $\ell \neq \ell'$ implies $q_\ell \neq q_{\ell'}$. Define an extension $\Pi' = \langle P_1, L_1, \leq_1 \rangle$ of Π as follows.

1. $P_1 = P_0 \cup \{q_\ell : \ell \in D\}$.

2. $L_1 = L_0 \cup \{k\}$.

3. $\leq_1$ is the order induced by Σ, which consists precisely of the relations holding in Π and the new relations $m \leq_1 k$ for all $m \in M$, and $q_\ell \leq_1 \ell$ and $q_\ell \leq_1 k$ for each

$\ell \in D$.

Then Π' is a semiplane with $\Pi \sqsubseteq \Pi' \sqsubseteq \Sigma$. Moreover, $s \vee t$ is defined in Π' for all s, $t \in M$, *viz.*, $s \vee t = k$.

Now let us describe this construction without reference to Σ. Let $\Pi = \langle P_0, L_0, \leq_0 \rangle$ be a semiplane that is not a projective plane, and let $M \subseteq P_0$ be a set of points whose pairwise join is undefined in Π. (We allow $|M| = 0$ or 1, which can happen in the case of a near pencil, but the important case is when $|M| \geq 2$.) Let k be a symbol for a new line. Let $D = \{\ell \in L_0 : p \not\leq_0 \ell \text{ for all } p \in M\}$. For each line $\ell \in D$, introduce a new point q_ℓ. Define $\Pi' = \langle P_1, L_1, \leq_1 \rangle$ by

i. $P_1 = P_0 \cup \{q_\ell : \ell \in D\}$.

ii. $L_1 = L_0 \cup \{k\}$.

iii. $\leq_1$ consists of $\leq_0$, the relations $m \leq_1 k$ for all $m \in M$, and for each $\ell \in D$ the relations $q_\ell \leq_1 \ell$ and $q_\ell \leq_1 k$.

We will denote the extension Π' obtained thusly by $\alpha_M(\Pi)$. We refer to the construction as a *one line extension*, or the one-line extension of Π determined by M.

Theorem 21. *If Π is a semiplane and M is a set of points whose pairwise join is undefined in Π, then $\alpha_M(\Pi)$ is a semiplane with $\Pi \sqsubseteq \alpha_M(\Pi)$. Moreover, let Σ be a semiplane with $\Pi \sqsubseteq \Sigma$, where $\Pi = \langle P_0, L_0, \leq_0 \rangle$ and $\Sigma = \langle P, L, \leq \rangle$. Let $k \in L - L_0$, and let $M = \{p \in P_0 : p \leq k\}$. Then the subsemiplane $\Pi_{(L_0 \cup \{k\})}$ of Σ is isomorphic to $\alpha_M(\Pi)$.*

Let Π and Σ be two semiplanes. We say that Σ *properly extends* Π if $\Pi \sqsubseteq \Sigma$ and Σ has at least one line not in Π.

Corollary 22. *Let Π be a semiplane. Every semiplane properly extending Π can be obtained as the union of a (possibly infinite) sequence of one line extensions starting with Π. Every projective plane that is a minimal extension of Π (i.e., does not have a proper subplane containing Π) can be obtained as the union of a (possibly infinite) sequence of one line extensions with $|M| \geq 2$.*

The construction of $\alpha_M(\Pi)$ depends, of course, on the choice of M. For example, by always taking $|M| = 2$ we can obtain a sequence whose union is the projective plane freely generated by Π, as constructed by M. Hall.

Now let us look at some changes in parameters affected by a one line extension $\Pi' = \alpha_M(\Pi)$, with the notation as above.

1. $|P_1| = |P_0| + |D|$.

2. $|L_1| = |L_0| + 1$.

Define the *rank* $r_\Pi(p)$ of a point $p \in P_0$ by $r_\Pi(p) = |\{\ell \in L_0 : p \leq_0 \ell\}|$, and define $r_{\Pi'}(p)$, $r_\Pi(\ell)$, $r_{\Pi'}(\ell)$ similarly.

3. For $p \in P_0$,

$$r_{\Pi'}(p) = \begin{cases} r_\Pi(p) + 1 & \text{if } p \in M \\ r_\Pi(p) & \text{if } p \notin M \end{cases}$$

and $r_{\Pi'}(q_\ell) = 2$ for $\ell \in D$.

4. For $\ell \in L_0$,

$$r_{\Pi'}(\ell) = \begin{cases} r_\Pi(\ell) + 1 & \text{if } \ell \in D \\ r_\Pi(\ell) & \text{if } \ell \notin D \end{cases}$$

and $r_{\Pi'}(k) = |M| + |D|$.

5. The new joins that are defined in Π' are

$$s \vee t = k \quad \text{for distinct } s, t \in M,$$

$$q_\ell \vee t = k \quad \text{for } \ell \in D \text{ and } t \in M,$$

$$q_\ell \vee q_m = k \quad \text{for distinct } \ell, m \in D,$$

$$q_\ell \vee u = \ell \quad \text{for } u \leq \ell \in D.$$

Finally, given $\Pi \sqsubseteq \Sigma$, we would like to know when there is a sequence of one line extensions from Π to Σ with $|M| \geq 2$ for every step. This cannot be done, for example, if Π is a subplane of the projective plane Σ. We need that at each step the new line should intersect the existing lines in at least two existing points, i.e., there are at least two existing points whose join is undefined. The following result gives a sufficient condition for this to occur.

Theorem 23. *Let $\Pi \sqsubseteq \Sigma$, where Π is a semiplane and Σ is a projective plane of order n. If Π is not a near pencil or a (sub)plane, then every sequence of one line extensions from Π to Σ has $|M| \geq 2$ for every step.*

We can use Theorem 21 to find a necessary condition for a semiplane $\Pi = \langle P_0, L_0, \leq_0 \rangle$ to be extendible to a projective plane of order n. First, we note the obvious necessary

conditions: $|P_0| \leq n^2+n+1$, $|L_0| \leq n^2+n+1$, $r_\Pi(p) \leq n+1$ for all $p \in P_0$, and $r_\Pi(\ell) \leq n+1$ for all $\ell \in L_0$.

Notice that, by observation 4 above, adding a new point q_ℓ raises the rank of one of the existing lines by exactly one. If Π is to be extended to a plane of order n by a sequence of one line extensions, then all these line ranks must eventually be raised to $n+1$. Thus we must have

$$\sum_{\ell \in L_0} r_\Pi(\ell) + n^2 + n + 1 - |P_0| \geq |L_0|(n+1).$$

It is useful to introduce the function $\rho_n(\Pi) = \sum_{\ell \in L_0} r_\Pi(\ell) + n^2 + n + 1 - |P_0| - |L_0|(n+1)$. The result can then be stated as follows.

Theorem 24. *If Π is a semiplane that can be extended to a projective plane of order n, then $\rho_n(\Pi) \geq 0$.*

After a one line extension $\Pi' = \alpha_M(\Pi)$, adding a new line k, we have

$$\begin{aligned}\Delta\rho_n = \rho_n(\Pi') - \rho_n(\Pi) &= r_{\Pi'}(k) + \left(\sum_{\ell \in L_0} r_{\Pi'}(\ell) - r_\Pi(\ell)\right) - |D| - (n+1) \\ &= r_{\Pi'}(k) + |D| - |D| - (n+1) \\ &= r_{\Pi'}(k) - (n+1).\end{aligned}$$

Thus $\Delta\rho_n$ is nonpositive, i.e., ρ_n is always decreasing, so long as we keep $r_{\Pi'}(k) = |M| + |D|$ at most $n+1$. For emphasis, we record this.

Lemma 25. *If Π and Ψ are semiplanes with $\Pi \sqsubseteq \Psi$, and $r_\Psi(\ell) \leq n+1$ for all lines of Ψ, then $\rho_n(\Pi) \geq \rho_n(\Psi)$, and hence $\Delta\rho_n \leq 0$.*

In practice, we will be trying to extend a relatively small initial semiplane Π to a projective plane Σ of order n by a sequence of one line extensions. Since we need to keep $\rho_n(\Psi) \geq 0$ for each semiplane Ψ in the sequence of extensions, we want to have $\Delta\rho_n$ small, i.e., heuristically we want $r_{\Pi'}(k) = |M| + |D|$ close to $n+1$. If $\Pi = \langle P_0, L_0, \leq_0 \rangle$, we may start with $\rho_n(\Pi)$ relatively big, but soon enough it will decrease to a very small number which often is 0, so we can normally expect that $\rho_n(\Pi) \ll n^2 + n + 1 - |L_0|$, which is the number of one line extensions required to go from Π to Σ. Hence *most* of the one line extensions must have $\Delta\rho_n = 0$. Moreover, since any one line extension raises the rank of an existing point by at most 1, and each new point q_ℓ has rank 2, which means it will take at least $n-1$ extensions to raise the rank of q_ℓ to $n+1$, so no new point q_ℓ can be introduced in the last $n-1$ extensions. So from the conclusion of the nth-to-last extension onwards, each line must contain $n+1$ points. In particular, we must have $r_{\Pi'}(k) = n+1$, and hence $\Delta\rho_n = 0$, for the last n extensions.

Note. The function ρ_n in the condition $\rho_n(\Pi) \geq 0$ depends on n, and for small values it may be decreasing in n. Thus a semiplane Π may be extendible to a plane of order $m < n$ even though $\rho_n(\Pi) < 0$, and indeed this does occur. Let A_4 be the affine plane of order 4. Clearly A_4 can be extended to the projective plane of order 4, but $\rho_5(A_4) = -25$.

Let $\Sigma = \langle P, L, \leq \rangle$ be a semiplane. The relation $\not\leq$ contained in $P \times L$ induces a Galois connection between the points and lines of Σ. The corresponding maps are, for

$R \subseteq P$ and $K \subseteq L$,

$$\lambda_\Sigma(R) = \{\ell \in L : r \not\leq \ell \text{ for all } r \in R\}$$

$$\pi_\Sigma(K) = \{p \in P : p \not\leq k \text{ for all } k \in K\}.$$

The usual properties of a Galois connection hold, e.g., $R \subseteq \pi_\Sigma\lambda_\Sigma(R)$.

For each $R \subseteq P$, define the semiplane $\gamma_\Sigma(R) = \Pi_{(\lambda_\Sigma(R))}$. Note that if $\gamma_\Sigma(R) = \langle P^*, \lambda_\Sigma(R), \leq^* \rangle$, then $p \in P^*$ if and only if p is on at least two lines containing no point of R. Since $p \in P - \pi_\Sigma\lambda_\Sigma(R)$ if and only if p is on at least one line containing no point of R, we have $P^* \subseteq P - \pi_\Sigma\lambda_\Sigma(R) \subseteq P - R$.

Theorem 26. *If $\Sigma = \langle P, L, \leq \rangle$ is a semiplane and $R \subseteq P$, then $\gamma_\Sigma(R)$ is a semiplane. Moreover, let $\Pi = \langle P_0, L_0, \leq_0 \rangle$ be a semiplane with $\Pi \sqsubseteq \Sigma$. If $R \subseteq \pi_\Sigma(L_0)$, i.e., no point of R is on a line of L_0, then $\Pi \sqsubseteq \gamma_\Sigma(R)$, and there is a sequence of one line extensions refining the sequence $\Pi \sqsubseteq \gamma_\Sigma(R) \sqsubseteq \Sigma$.*

Proof. If $R \subseteq \pi_\Sigma(L_0)$, then $\lambda_\Sigma(R) \supseteq \lambda_\Sigma\pi_\Sigma(L_0) \supseteq L_0$. Hence $\Pi_{L_0} \sqsubseteq \Pi_{(\lambda_\Sigma(R))} \sqsubseteq \Pi_L$, so that $\Pi \sqsubseteq \gamma_\Sigma(R) \sqsubseteq \Sigma$. By Corollary 22, there is a sequence of one line extensions going from Π to $\gamma_\Sigma(R)$, which can be extended by a sequence going from $\gamma_\Sigma(R)$ to Σ. □

Recall that for a semiplane $\Pi = \langle P_0, L_0, \leq_0 \rangle$ we have defined

$$\rho_n(\Pi) = \sum_{\ell \in L_0} r_\Pi(\ell) + n^2 + n + 1 - |P_0| - |L_0|(n+1).$$

Theorem 27. *Let $\Pi = \langle P_0, L_0, \leq_0 \rangle$ be a semiplane that can be extended to a projective plane $\Sigma = \langle P, L, \leq \rangle$ of order n. Then $\rho_n(\Pi) = |\pi_\Sigma(L_0)|$.*

Proof. The lines of L_0 contain $|P_0| + \sum_{\ell \in L_0}(n + 1 - r_\Pi(\ell))$ points in Σ. So there are $n^2 + n + 1 + \sum_{\ell \in L_0} r_\Pi(\ell) - |P_0| - |L_0|(n+1) = \rho_n(\Pi)$ points in P that are on no line of L_0. □

Corollary 28. *Let $\Sigma = \langle P, L, \leq \rangle$ be a projective plane of order n, and let $R \subseteq P$. If Π is a semiplane with $\Pi \sqsubseteq \gamma_\Sigma(R)$, then $\rho_n(\Pi) \geq |R|$.*

Proof. If $\Pi = \langle P_0, L_0, \leq_0 \rangle \sqsubseteq \gamma_\Sigma(R)$, then $L_0 \subseteq \lambda_\Sigma(R)$, whence $\pi_\Sigma(L_0) \supseteq \pi_\Sigma\lambda_\Sigma(R) \supseteq R$. □

Given a projective plane $\Sigma = \langle P, L, \leq \rangle$ and a set of points $R \subseteq P$, we would like to know when R is closed, i.e., when $R = \pi_\Sigma\lambda_\Sigma(R)$. Likewise, we would like to know when a set of lines $K \subseteq L$ is closed, i.e., when $K = \lambda_\Sigma\pi_\Sigma(K)$.

If a set R of points is not too large, then it will be closed.

Theorem 29. *Let $\Sigma = \langle P, L, \leq \rangle$ be a projective plane of order n. If $R \subseteq P$ with $|R| \leq n$, then $\pi_\Sigma\lambda_\Sigma(R) = R$, and hence $\rho_n(\gamma_\Sigma(R)) = |R|$.*

Proof. A point $p \in P$ is in $\pi_\Sigma\lambda_\Sigma(R)$ if and only if every line of L containing p also contains some point $r \in R$. Given a point $p \notin R$, consider (in Σ) the set of lines $p \vee R = \{p \vee r : r \in R\}$. If $|p \vee R| = n + 1$ then $p \in \pi_\Sigma\lambda_\Sigma(R)$, while $p \notin \pi_\Sigma\lambda_\Sigma(R)$ whenever $|p \vee R| \leq n$. So if $|R| \leq n$, then $\pi_\Sigma\lambda_\Sigma(R) = R$. □

We have been trying to extend semiplanes containing a non-Desarguesian configuration to projective planes of some finite order using computer programs written by Dr. Nation. All the programs work essentially the same, except possibly accepting

different inputs to suit our testing purposes. The original one basically works in the following way.

1. Read the input of the order of the projective plane and the initial semiplane.

2. Find all choices of M that keep

   (a) $\rho_n \geq 0$,

   (b) $r(k) \leq n+1$, where k is the new line,

   (c) $r(l) \leq n+1$ for all line l in the semiplane,

   (d) $r(p) \leq n+1$ for all point p in the semiplane, and

   (e) the total number of points is no more than n^2+n+1.

   If the number of choices is zero, then go to step (3); otherwise, pick a choice and go to step (4).

3. Remove the last line added, adjust all the parameters. If possible, pick the next choice in line and go to step (4). If there are no more choices, then we have the following two cases:

   (a) If we are not back to the original configuration, repeat step (3).

   (b) Otherwise, all the choices have been tested, and there is no projective plane. Output the conclusion that the semiplane cannot be extended to a projective plane of the given order, and stop the program.

4. Add the line, and adjust all the parameters. If the number of lines is $n^2 + n + 1$, then we have obtained a projective plane. Output the plane and stop the program; otherwise go to step (2).

In principle, this is a finite algorithm. However, even for relatively small orders, the number of choices for M at each stage can be large. Hence, it is not practical to run the program straight through as described.

With a Desargues configuration as the initial input, this program does produce the Desarguesian projective planes of order $3, 4, 5$ and 7. When we tried to extend a semi-plane containing a non-Desarguesian configuration to a plane of order 9, there are simply too many choices for M at each stage. But with a little help from us, which means that we directed the computer to make certain choices for a few steps, then the program was able to produce a plane of that order. For order 12, there are a lot more choices for M and we cannot provide any special help, so we decided to cut the choices for M at 500 and randomly pick choices for the first 22 lines added. Unless we have made the right choices, the program could still run a significant amount of time. So we also control the number of times that we are willing to try for each sequence, once the count has reached that number and if the number of lines is greater or equal to 48, we print all the choices that we have picked for each stage, and start a new sequence. The number 48 was not chosen at random, but rather we realized, based on all the tests that we have done so far, that the sequence of extensions would be a good candidate if the number of lines is able to reach at least 48. We also did tests for order 11 and 15, but we spent most of our time on order 12.

We also applied the idea of genetic algorithms into our testing. We take a good candidate, change the sequence at one or a few places, or we replace a string of numbers by a string from another good candidate, and then we run the program using the new sequence to see whether or not we are able to obtain another good candidate. This method often produces other semiplanes with more than 48 lines.

Chapter 4

Classification of Dual Linear Spaces Generated by a Non-Desarguesian Configuration

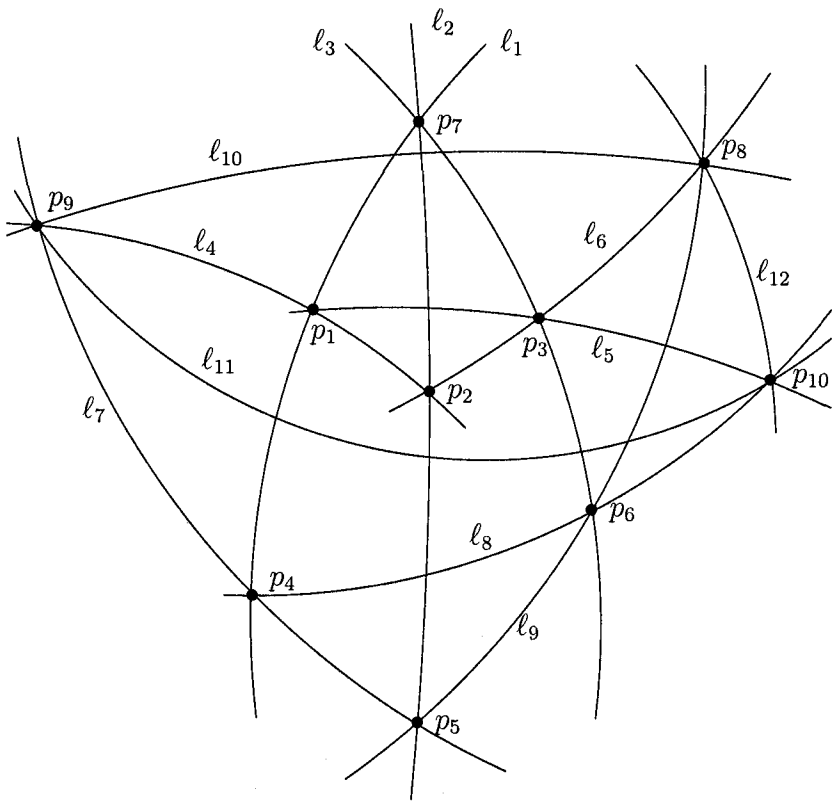

Figure 3: Non-Desarguesian configuration.

Given a non-Desarguesian configuration, there are two basic ways to extend it to a semiplane:

1. make lines go through existing points; or

2. make pairwise nonconcurrent lines meet at new points.

Table 1: Intersection of lines.

$\wedge$	l_1	l_2	l_3	l_4	l_5	l_6	l_7	l_8	l_9	l_{10}	l_{11}	l_{12}
l_1	\	p_7	p_7	p_1	p_1	?	p_4	p_4	?	?	?	?
l_2	p_7	\	p_7	p_2	?	p_2	p_5	?	p_5	?	?	?
l_3	p_7	p_7	\	?	p_3	p_3	?	p_6	p_6	?	?	?
l_4	p_1	p_2	?	\	p_1	p_2	p_9	?	?	p_9	p_9	?
l_5	p_1	?	p_3	p_1	\	p_3	?	p_{10}	?	?	p_{10}	p_{10}
l_6	?	p_2	p_3	p_2	p_3	\	?	?	p_8	p_8	?	p_8
l_7	p_4	p_5	?	p_9	?	?	\	p_4	p_5	p_9	p_9	?
l_8	p_4	?	p_6	?	p_{10}	?	p_4	\	p_6	?	p_{10}	p_{10}
l_9	?	p_5	p_6	?	?	p_8	p_5	p_6	\	p_8	?	p_8
l_{10}	?	?	?	p_9	?	p_8	p_9	?	p_8	\	p_9	p_8
l_{11}	?	?	?	p_9	p_{10}	?	p_9	p_{10}	?	p_9	\	p_{10}
l_{12}	?	?	?	?	p_{10}	p_8	?	p_{10}	p_8	p_8	p_{10}	\

Table 2: Lines through each point.

point	lines
p_1	$l_1\ l_4\ l_5$
p_2	$l_2\ l_4\ l_6$
p_3	$l_3\ l_5\ l_6$
p_4	$l_1\ l_7\ l_8$
p_5	$l_2\ l_7\ l_9$
p_6	$l_3\ l_8\ l_9$
p_7	$l_1\ l_2\ l_3$
p_8	$l_6\ l_9\ l_{10}\ l_{12}$
p_9	$l_4\ l_7\ l_{10}\ l_{11}$
p_{10}	$l_5\ l_8\ l_{11}\ l_{12}$

Figure 3 illustrates the basic non-Desarguesian configuration, that is, two triangles are centrally perspective from a point but not axially perspective from a line, and establishes our notation.

Based on the non-Desarguesian configuration of Figure 3, Table 1 lists the meets of all the lines. A question mark indicates that the meet is undefined. Table 2 gives the detail of which lines pass through which point. By using these two tables, we can easily

check whether method one or two for our extension would work for certain lines. For instance, from Table 2 we know that p_1 is on l_1, l_4 and l_5. So we can only make a line, which does not intersect any of those lines, pass through p_1. In other words, we need to see question marks under those lines on some row of Table 1. Now if we check Table 1 carefully, we know that l_9 would be our only choice. We would like to note here that no more than three lines can meet at a new point because any set of four lines contains at least one intersecting pair. The following is a complete list of all possible ways to extend the non-Desarguesian configuration of Figure 3. When method one is applied, we group them according to the rank of the points, those with rank 4 and those with rank 3. We further separate the latter ones into three smaller groups depending on whether they are on the first triangle p_1, p_2, p_3 or the second triangle p_4, p_5, p_6 or neither.

I. Make a line go through a point of rank 4.

1. $l_1 \longrightarrow p_8$

2. $l_2 \longrightarrow p_{10}$

3. $l_3 \longrightarrow p_9$

II. Make a line go through a point of rank 3.

1. Points on the second triangle.

a. $l_4 \longrightarrow p_6$

b. $l_5 \longrightarrow p_5$

c. $l_6 \longrightarrow p_4$

2. Points on the first triangle.

   a. $l_7 \longrightarrow p_3$

   b. $l_8 \longrightarrow p_2$

   c. $l_9 \longrightarrow p_1$

3. Point on neither of the triangles.

   a. $l_{10} \longrightarrow p_7$

   b. $l_{11} \longrightarrow p_7$

   c. $l_{12} \longrightarrow p_7$

III. Make three pairwise nonconcurrent lines meet at a new point.

   a. l_1, l_6, l_{11}

   b. l_1, l_9, l_{11}

   c. l_2, l_5, l_{10}

   d. l_2, l_8, l_{10}

   e. l_3, l_4, l_{12}

   f. l_3, l_7, l_{12}

IV. Make two nonconcurrent lines meet at a new point.

Since in a projective plane any two points lie on exactly one line, and any two lines intersect in exactly one point, some methods from group I, II and III may not be used simultaneously for our extension. For instance, method I1 cannot go with method III1c,

for otherwise $p_8 = l_1 \wedge l_6 = p_4$, a contradiction. Note that method IV can be used at any time of the extension, and it is the only method which can be used repeatedly. The following list shows how meets will be defined if a method is applied.

I1 $l_1 \wedge l_6 = l_1 \wedge l_9 = l_1 \wedge l_{10} = l_1 \wedge l_{12} = p_8$

I2 $l_2 \wedge l_5 = l_2 \wedge l_8 = l_2 \wedge l_{11} = l_2 \wedge l_{12} = p_{10}$

I3 $l_3 \wedge l_4 = l_3 \wedge l_7 = l_3 \wedge l_{10} = l_3 \wedge l_{11} = p_9$

II1a $l_3 \wedge l_4 = l_4 \wedge l_8 = l_4 \wedge l_9 = p_6$

II1b $l_2 \wedge l_5 = l_5 \wedge l_7 = l_5 \wedge l_9 = p_5$

II1c $l_1 \wedge l_6 = l_6 \wedge l_7 = l_6 \wedge l_8 = p_4$

II2a $l_3 \wedge l_7 = l_5 \wedge l_7 = l_6 \wedge l_7 = p_3$

II2b $l_2 \wedge l_8 = l_4 \wedge l_8 = l_6 \wedge l_8 = p_2$

II2c $l_1 \wedge l_9 = l_4 \wedge l_9 = l_5 \wedge l_9 = p_1$

II3a $l_1 \wedge l_{10} = l_2 \wedge l_{10} = l_3 \wedge l_{10} = p_7$

II3b $l_1 \wedge l_{11} = l_2 \wedge l_{11} = l_3 \wedge l_{11} = p_7$

II3c $l_1 \wedge l_{12} = l_2 \wedge l_{12} = l_3 \wedge l_{12} = p_7$

IIIa $l_1 \wedge l_6 \wedge l_{11} =$ some new point

IIIb $l_1 \wedge l_9 \wedge l_{11} =$ some new point

Table 3: Table of inconsistency.

I1	II1c II2c II3a II3c IIIa IIIb
I2	II1b II2b II3b II3c IIIc IIId
I3	II1a II2a II3a II3b IIIe IIIf
II1a	I3 II2b II2c IIIe
II1b	I2 II2a II2c IIIc
II1c	I1 II2a II2b IIIa
II2a	I3 II1b II1c IIIf
II2b	I2 II1a II1c IIId
II2c	I1 II1a II1b IIIb
II3a	I1 I3 II3b II3c IIIc IIId
II3b	I2 I3 II3a II3c IIIa IIIb
II3c	I1 I2 II3a II3b IIIe IIIf
IIIa	I1 II1c II3b IIIb
IIIb	I1 II2c II3b IIIa
IIIc	I2 II1b II3a IIId
IIId	I2 II2b II3a IIIc
IIIe	I3 II1a II3c IIIf
IIIf	I3 II2a II3c IIIe

IIIc $l_2 \wedge l_5 \wedge l_{10} =$ some new point

IIId $l_2 \wedge l_8 \wedge l_{10} =$ some new point

IIIe $l_3 \wedge l_4 \wedge l_{12} =$ some new point

IIIf $l_3 \wedge l_7 \wedge l_{12} =$ some new point

Using the list above, we obtain the table of inconsistency which gives a complete list of all the conflicts among the methods.

Lemma 30. *Altogether at most six applications of methods from groups I, II1, II2, II3 and III can be made.*

Proof. If we are to choose seven or more methods from the five groups, not necessarily

from all five, then we are forced to take at least two methods from at least one of the five groups. Note that no more than one method may be chosen from group II3. Also note that only up to three methods may be taken from group III at a time.

Case I: If we take any two methods from group I, then by Table 3, we know that we may possibly pick the remaining one from the same group, one method from group II1, one method from group II2, and two methods from group III. But the two methods from group III don't go together. Therefore, we may take no more than six methods in this case.

For example, if we take I1 and I2, then Table 3 suggests that I3, II1a, II2a, IIIe and IIIf are the possible choices to add. But IIIe and IIIf are inconsistent, hence we may only take up to six methods for this case.

Case II: If we take any two methods from group II1, then we may possibly take the reaming one from the same group, one method from group I, one method from group II3, and three methods from group III.

a. If we take the one from group I, then we may take up to two from group III with or without taking one from group II3.

b. If we take the remaining one from the same group, then we can either take up to three methods from group III, or take one from group II3 together with no more than two from group III.

c. If we do not take the possible one from group I and the remaining one from the same group, then we can take up to a total of three from group II3 and

III.

In any case, we may only take a maximum of six methods.

For example, suppose we take II1a and II1b. Then I1, II1c, any one from group II3, IIIa, IIIb, IIId, and IIIf are the possible choices from which we may pick. If we choose I1, then we cannot pick II1c, II3a, II3c, IIIa and IIIb. So now we are left with II3b, IIId and IIIf as our possible choices. Table 3 suggests that II3b has no conflicts with IIId or IIIf. Therefore we may pick all three of them, which gives us a total of six methods. If we choose II1c, which is the remaining method from the same group, then I1 and IIIa are eliminated from our list. Since II3a does not go with IIId, II3b does not go with IIIb and II3c does not go with IIIf, we can either take IIIb, IIId and IIIf all at once, or we can take only two of those and one from group II3. Either case gives a total of six methods. Suppose we ignore both I1 and II1c. Since no more than three can be chosen from group III, again we are left with only six choices.

Case III: If we take any two methods from group II2, then a similar argument as in Case II will lead to the same conclusion.

Case IV: If we take any two from group III, then only one method from group I, four methods in total from group II1 and II2, one method from group II3, and two methods from group III could possibly be added. But the two methods from group III are inconsistent, and none of them is consistent with either the one from group I or the one from group II3. We may choose up to three of the four methods from group

II1 and II2 at a time, but if we choose the one method from group I, then only up to two of those methods are possible. Therefore only up to six methods may be taken at a time in this case.

For example, suppose we take IIIa and IIIc, then I3, II1a, II2a, II2b, II2c, II3c, IIIe and IIIf are our possible choices to add. Notice that IIIe and IIIf are inconsistent, and both of them are inconsistent with I3 and II3c. Also notice that we may choose II2a, II2b and II2c out of the four methods from group II1 and II2, but if we choose I3, then we may only take II2b and II2c. Therefore we may take only up to four methods from the list which implies a maximum of six methods for this case.

□

Proposition 31. *There are* 875 *ways to extend the non-Desarguesian configuration of Figure* 3 *to a semiplane without adding a new line .*

Proof. The proof of this proposition is a straight forward case analysis using our table of inconsistency and the above lemma. The cases are listed in the order of the total number of methods which we pick from group I, II1, II2, II3, and III. Note that method IV is the only method which we may use repeatedly, and when we list the case I1 II1a, for example, we mean apply methods I1 and II1a first to our initial configuration, and then apply method IV as many times as necessary until the meets of every pair of lines are defined. Also if we say *pick 2-1* in the proof, we mean *pick two methods from one group, and pick the third method from a group that follows in the list.* We would like to give a special note here for a few inconsistencies which will be encountered often: no

more than one method can be picked from group II3; IIIa IIIb cannot go together, and the same for the pairs IIIc IIId and IIIe IIIf. The symbol, $\longrightarrow\longleftarrow$, in the proof indicates inconsistency.

Case I: Pick none from the five groups.

1	IV

Case II: Pick one at a time.

2	I1
3	I2
4	I3
5	II1a
6	II1b
7	II1c
8	II2a
9	II2b
10	II2c
11	II3a
12	II3b
13	II3c
14	IIIa
15	IIIb
16	IIIc
17	IIId

18 IIIe

19 IIIf

Case III: Pick two at a time.

A. Pick two from the same group.

20 I1 I2

21 I1 I3

22 I2 I3

23 II1a II1b

24 II1a II1c

25 II1b II1c

26 II2a II2b

27 II2a II2c

28 II2b II2c

29 IIIa IIIc

30 IIIa IIId

31 IIIa IIIe

32 IIIa IIIf

33 IIIb IIIc

34 IIIb IIId

35 IIIb IIIe

36 IIIb IIIf

37 IIIc IIIe

38 IIIc IIIf

39 IIId IIIe

40 IIId IIIf

B. Pick 1-1.

I1 →← II1c, II2c, II3a, II3c, IIIa, IIIb.

41 I1 II1a

42 I1 II1b

43 I1 II2a

44 I1 II2b

45 I1 II3b

46 I1 IIIc

47 I1 IIId

48 I1 IIIe

49 I1 IIIf

I2 →← II1b, II2b, II3b, II3c, IIIc, IIId.

50 I2 II1a

51 I2 II1c

52 I2 II2a

53 I2 II2c

54 I2 II3a

55 I2 IIIa

56 I2 IIIb

57 I2 IIIe

58 I2 IIIf

I3 →← II1a, II2a, II3a, II3b, IIIe, IIIf.

59 I3 II1b

60 I3 II1c

61 I3 II2b

62 I3 II2c

63 I3 II3c

64 I3 IIIa

65 I3 IIIb

66 I3 IIIc

67 I3 IIId

II1a →← II2b, II2c, IIIe.

68 II1a II2a

69 II1a II3a

70 II1a II3b

71 II1a II3c

72 II1a IIIa

73 II1a IIIb

74 II1a IIIc

75 II1a IIId

76 II1a IIIf

II1b →← II2a, II2c, IIIc.

77 II1b II2b

78 II1b II3a

79 II1b II3b

80 II1b II3c

81 II1b IIIa

82 II1b IIIb

83 II1b IIId

84 II1b IIIe

85 II1b IIIf

II1c →← II2a, II2b, IIIa.

86 II1c II2c

87 II1c II3a

88 II1c II3b

89 II1c II3c

90 II1c IIIb

91 II1c IIIc

92 II1c IIId

93 II1c IIIe

94 II1c IIIf

II2a →← IIIf.

95 II2a II3a

96 II2a II3b

97 II2a II3c

98 II2a IIIa

99 II2a IIIb

100 II2a IIIc

101 II2a IIId

102 II2a IIIe

II2b $\rightarrow\leftarrow$ IIId.

103 II2b II3a

104 II2b II3b

105 II2b II3c

106 II2b IIIa

107 II2b IIIb

108 II2b IIIc

109 II2b IIIe

110 II2b IIIf

II3c $\rightarrow\leftarrow$ IIIb.

111 II2c II3a

112 II2c II3b

113 II2c II3c

114 II2c IIIa

115 II2c IIIc

116 II2c IIId

117 II2c IIIe

118 II2c IIIf

II3a $\rightarrow\leftarrow$ IIIc, IIId.

119 II3a IIIa

120 II3a IIIb

121 II3a IIIe

122 II3a IIIf

II3b $\rightarrow\leftarrow$ IIIa, IIIb.

123 II3b IIIc

124 II3b IIId

125 II3b IIIe

126 II3b IIIf

II3c $\rightarrow\leftarrow$ IIIe, IIIf.

127 II3c IIIa

128 II3c IIIb

129 II3c IIIc

130 II3c IIId

Case IV: Pick three at a time.

A. Pick three from the same group.

131 I1 I2 I3

132 II1a II1b II1c

133 II2a II2b II2c

134 IIIa IIIc IIIe

135 IIIa IIIc IIIf

136 IIIa IIId IIIe

137 IIIa IIId IIIf

138 IIIb IIIc IIIe

139 IIIb IIIc IIIf

140 IIIb IIId IIIe

141 IIIb IIIc IIIf

B. Pick 2-1.

I1 I2 →← II1b, II1c, II2b, II2c, II3a, II3b, II3c, IIIa, IIIb, IIIc, IIId.

142 I1 I2 II1a

143 I1 I2 II2a

144 I1 I2 IIIe

145 I1 I2 IIIf

I1 I3 →← II1a, II1c, II2a, II2c, II3a, II3b, II3c, IIIa, IIIb, IIIe, IIIf.

146 I1 I3 II1b

147 I1 I3 II2b

148 I1 I3 IIIc

149 I1 I3 IIId

I2 I3 →← II1a, II1b, II2a, II2b, II3a, II3b, II3c, IIIc, IIId, IIIe, IIIf.

150 I2 I3 II1c

151 I2 I3 II2c

152 I2 I3 IIIa

153 I2 I3 IIIb

II1a II1b →← IIIc, IIIe.

154 II1a II1b II3a

155 II1a II1b II3b

156 II1a II1b II3c

157 II1a II1b IIIa

158 II1a II1b IIIb

159 II1a II1b IIId

160 II1a II1b IIIf

II1a II1c →← IIIa, IIIe.

161 II1a II1c II3a

162 II1a II1c II3b

163 II1a II1c II3c

164 II1a II1c IIIb

165 II1a II1c IIIc

166 II1a II1c IIId

167 II1a II1c IIIf

II1b II1c →← IIIa, IIIc.

168 II1b II1c II3a

169 II1b II1c II3b

170 II1b II1c II3c

171 II1b II1c IIIb

172 II1b II1c IIId

173 II1b II1c IIIe

174 II1b II1c IIIf

II2a II2b →← IIId, IIIf.

175 II2a II2b II3a

176 II2a II2b II3b

177 II2a II2b II3c

178 II2a II2b IIIa

179 II2a II2b IIIb

180 II2a II2b IIIc

181 II2a II2b IIIe

II2a II2c →← IIIb, IIIf.

182 II2a II2c II3a

183 II2a II2c II3b

184 II2a II2c II3c

185 II2a II2c IIIa

186 II2a II2c IIIc

187 II2a II2c IIId

188 II2a II2c IIIe

II2b II2c →← IIIb, IIId.

189 II2b II2c II3a

190 II2b II2c II3b

191 II2b II2c II3c

192 II2b II2c IIIa

193 II2b II2c IIIc

194 II2b II2c IIIe

195 II2b II2c IIIf

C. Pick 1-2.

I1 →← II1c, II2c, II3a, II3c, IIIa, IIIb.

196 I1 II1a II1b

197 I1 II2a II2c

198 I1 IIIc IIIe

199 I1 IIIc IIIf

200 I1 IIId IIIe

201 I1 IIId IIIf

I2 →← II1b, II2b, II3b, II3c, IIIc, IIId.

202 I2 II1a II1c

203 I2 II2a II2c

204 I2 IIIa IIIe

205 I2 IIIa IIIf

206 I2 IIIb IIIe

207 I2 IIIb IIIf

I3 →← II1a, II2a, II3a, II3b, IIIe, IIIf.

208 I3 II1b II1c

209 I3 II2b II2c

210 I3 IIIa IIIc

211 I3 IIIa IIId

212 I3 IIIb IIIc

213 I3 IIIb IIId

II1a →← II2b, II2c, IIIe.

214 II1a IIIa IIIc

215 II1a IIIa IIId

216 II1a IIIa IIIf

217 II1a IIIb IIIc

218 II1a IIIb IIId

219 II1a IIIb IIIf

220 II1a IIIc IIIf

221 II1a IIId IIIf

II1b →← II2a, II2c, IIIc.

222 II1b IIIa IIId

223 II1b IIIa IIIe

224 II1b IIIa IIIf

225 II1b IIIb IIId

226 II1b IIIb IIIe

227 II1b IIIb IIIf

228 II1b IIId IIIe

229 II1b IIId IIIf

II1c →← II2a, II2b, IIIa.

230 II1c IIIb IIIc

231 II1c IIIb IIId

232 II1c IIIb IIIe

233 II1c IIIb IIIf

234 II1c IIIc IIIe

235 II1c IIIc IIIf

236 II1c IIId IIIe

237 II1c IIId IIIf

II2a →← IIIf.

238 II2a IIIa IIIc

239 II2a IIIa IIId

240 II2a IIIa IIIe

241 II2a IIIb IIIc

242 II2a IIIb IIId

243 II2a IIIb IIIe

244 II2a IIIc IIIe

245 II2a IIId IIIe

II2b →← IIId.

246 II2b IIIa IIIc

247 II2b IIIa IIIe

248 II2b IIIa IIIf

249 II2b IIIb IIIc

250 II2b IIIb IIIe

251 II2b IIIb IIIf

252 II2b IIIc IIIe

253 II2b IIIc IIIf

II2c →← IIIb.

254 II2c IIIa IIIc

255 II2c IIIa IIId

256 II2c IIIa IIIe

257 II2c IIIa IIIf

258 II2c IIIc IIIe

259 II2c IIIc IIIf

260 II2c IIId IIIe

261 II2c IIId IIIf

II3a →← IIIc, IIId.

262 II3a IIIa IIIe

263 II3a IIIa IIIf

264 II3a IIIb IIIe

265 II3a IIIb IIIf

II3b →← IIIa, IIIb.

266 II3b IIIc IIIe

267 II3b IIIc IIIf

268 II3b IIId IIIe

269 II3b IIId IIIf

II3c →← IIIe, IIIf.

270 II3c IIIa IIIc

271 II3c IIIa IIId

272 II3c IIIb IIIc

273 II3c IIIb IIId

D. Pick 1-1-1.

I1 II1a →← II2b, II2c, II3a, II3c, IIIa, IIIb, IIIe.

274 I1 II1a II2a

275 I1 II1a II3b

276 I1 II1a IIIc

277 I1 II1a IIId

278 I1 II1a IIIf

I1 II1b →← II2a, II2c, II3a, II3c, IIIa, IIIb, IIIc.

279 I1 II1b II2b

280 I1 II1b II3b

281 I1 II1b IIId

282 I1 II1b IIIe

283 I1 II1b IIIf

I1 II2a →← II3a, II3c, IIIa, IIIb, IIIf.

284 I1 II2a II3b

285 I1 II2a IIIc

286 I1 II2a IIId

287 I1 II2a IIIe

I1 II2b →← II3a, II3c, IIIa, IIIb, IIId.

288 I1 II2b II3b

289 I1 II2b IIIc

290 I1 II2b IIIe

291 I1 II2b IIIf

I1 II3b →← IIIa, IIIb.

292 I1 II3b IIIc

293 I1 II3b IIId

294 I1 II3b IIIe

295 I1 II3b IIIf

I2 II1a →← II2b, II2c, II3b, II3c, IIIc, IIId, IIIe.

296 I2 II1a II2a

297 I2 II1a II3a

298 I2 II1a IIIa

299 I2 II1a IIIb

300 I2 II1a IIIf

I2 II1c →← II2a, II2b, II3b, II3c, IIIa, IIIc, IIId.

301 I2 II1c II2c

302 I2 II1c II3a

303 I2 II1c IIIb

304 I2 II1c IIIe

305 I2 II1c IIIf

I2 II2a →← II3b, II3c, IIIc, IIId, IIIf.

306 I2 II2a II3a

307 I2 II2a IIIa

308 I2 II2a IIIb

309 I2 II2a IIIe

I2 II2c →← II3b, II3c, IIIb, IIIc, IIId.

310 I2 II2c II3a

311 I2 II2c IIIa

312 I2 II2c IIIe

313 I2 II2c IIIf

I2 II3a →← IIIc, IIId.

314 I2 II3a IIIa

315 I2 II3a IIIb

316 I2 II3a IIIe

317 I2 II3a IIIf

I3 II1b →← II2a, II2c, II3a, II3b, IIIc, IIIe, IIIf.

318 I3 II1b II2b

319 I3 II1b II3c

320 I3 II1b IIIa

321 I3 II1b IIIb

322 I3 II1b IIId

I3 II1c →← II2a, II2b, II3a, II3b, IIIa, IIIe, IIIf.

323 I3 II1c II2c

324 I3 II1c II3c

325 I3 II1c IIIb

326 I3 II1c IIIc

327 I3 II1c IIId

I3 II2b →← II3a, II3b, IIId, IIIe, IIIf.

328 I3 II2b II3c

329 I3 II2b IIIa

330 I3 II2b IIIb

331 I3 II2b IIIc

I3 II2c →← II3a, II3b, IIIb, IIIe, IIIf.

332 I3 II2c II3c

333 I3 II2c IIIa

334 I3 II2c IIIc

335 I3 II2c IIId

I3 II3c →← IIIe, IIIf.

336 I3 II3c IIIa

337 I3 II3c IIIb

338 I3 II3c IIIc

339 I3 II3c IIId

II1a II2a →← IIIe, IIIf.

340 II1a II2a II3a

341 II1a II2a II3b

342 II1a II2a II3c

343 II1a II2a IIIa

344 II1a II2a IIIb

345 II1a II2a IIIc

346 II1a II2a IIId

II1a II3a →← IIIc, IIId, IIIe.

347 II1a II3a IIIa

348 II1a II3a IIIb

349 II1a II3a IIIf

II1a II3b →← IIIa, IIIb, IIIe.

350 II1a II3b IIIc

351 II1a II3b IIId

352 II1a II3b IIIf

II1a II3c →← IIIe, IIIf.

353 II1a II3c IIIa

354 II1a II3c IIIb

355 II1a II3c IIIc

356 II1a II3c IIId

II1b II2b →← IIIc, IIId.

357 II1b II2b II3a

358 II1b II2b II3b

359 II1b II2b II3c

360 II1b II2b IIIa

361 II1b II2b IIIb

362 II1b II2b IIIe

363 II1b II2b IIIf

II1b II3a →← IIIc, IIId.

364 II1b II3a IIIa

365 II1b II3a IIIb

366 II1b II3a IIIe

367 II1b II3a IIIf

II1b II3b →← IIIa, IIIb, IIIc.

368 II1b II3b IIId

369 II1b II3b IIIe

370 II1b II3b IIIf

II1b II3c →← IIIc, IIIe, IIIf.

371 II1b II3c IIIa

372 II1b II3c IIIb

373 II1b II3c IIId

II1c II2c →← IIIa, IIIb.

374 II1c II2c II3a

375 II1c II2c II3b

376 II1c II2c II3c

377 II1c II2c IIIc

378 II1c II2c IIId

379 II1c II2c IIIe

380 II1c II2c IIIf

II1c II3a →← IIIa, IIIc, IIId.

381 II1c II3a IIIb

382 II1c II3a IIIe

383 II1c II3a IIIf

II1c II3b →← IIIa, IIIb.

384 II1c II3b IIIc

385 II1c II3b IIId

386 II1c II3b IIIe

387 II1c II3b IIIf

II1c II3c →← IIIa, IIIe, IIIf.

388 II1c II3c IIIb

389 II1c II3c IIIc

390 II1c II3c IIId

II2a II3a →← IIIc, IIId, IIIf.

391 II2a II3a IIIa

392 II2a II3a IIIb

393 II2a II3a IIIe

II2a II3b →← IIIa, IIIb, IIIf.

394 II2a II3b IIIc

395 II2a II3b IIId

396 II2a II3b IIIe

II2a II3c →← IIIe, IIIf.

397 II2a II3c IIIa

398 II2a II3c IIIb

399 II2a II3c IIIc

400 II2a II3c IIId

II2b II3a →← IIIc, IIId.

401 II2b II3a IIIa

402 II2b II3a IIIb

403 II2b II3a IIIe

404 II2b II3a IIIf

II2b II3b →← IIIa, IIIb, IIId.

405 II2b II3b IIIc

406 II2b II3b IIIe

407 II2b II3b IIIf

II2b II3c →← IIId, IIIe, IIIf.

408 II2b II3b IIIa

409 II2b II3b IIIb

410 II2b II3b IIIc

II2c II3a →← IIIb, IIIc, IIId.

411 II2c II3a IIIa

412 II2c II3a IIIe

413 II2c II3a IIIf

II2c II3b →← IIIa, IIIb.

414 II2c II3b IIIc

415 II2c II3b IIId

416 II2c II3b IIIe

417 II2c II3b IIIf

II2c II3c →← IIIb, IIIe, IIIf.

418 II2c II3c IIIa

419 II2c II3c IIIc

420 II2c II3c IIId

Case V: Pick four at a time.

A. Pick 3-1 or 1-3.

Note: I1 I2 I3 don't go with anything.

II1a II1b II1c →← I1, I2, I3, II2a, II2b, II2c, IIIa, IIIc, IIIe.

421 II1a II1b II1c II3a

422 II1a II1b II1c II3b

423 II1a II1b II1c II3c

424 II1a II1b II1c IIIb

425 II1a II1b II1c IIId

426 II1a II1b II1c IIIf

II2a II2b II2c →← I1, I2, I3, II1a, II1b, II1c, IIIb, IIId, IIIf.

427 II2a II2b II2c II3a

428 II2a II2b II2c II3b

429 II2a II2b II2c II3c

430 II2a II2b II2c IIIa

431 II2a II2b II2c IIIc

432 II2a II2b II2c IIIe

IIIa IIIc IIIe →← I1, I2, I3, II1a, II1b, II1c II3a, II3b, II3c.

433 II2a IIIa IIIc IIIe

434 II2b IIIa IIIc IIIe

435 II2c IIIa IIIc IIIe

IIIa IIIc IIIf →← I1, I2, I3, II1b, II1c, II2a, II3a, II3b, II3c.

436 II1a IIIa IIIc IIIf

437 II2b IIIa IIIc IIIf

438 II2c IIIa IIIc IIIf

IIIa IIId IIIe →← I1, I2, I3, II1a, II1c, II2b, IIIa, IIIb, IIIc.

439 II1b IIIa IIId IIIe

440 II2a IIIa IIId IIIe

441 II2c IIIa IIId IIIe

IIIa IIId IIIf →← I1, I2, I3, II1c, II2a, II2b, II3a, II3b, II3c.

442 II1a IIIa IIId IIIf

443 II1b IIIa IIId IIIf

444 II2c IIIa IIId IIIf

IIIb IIIc IIIe →← I1, I2, I3, II1a, II1b, II2c, II3a, II3b, II3c.

445 II1c IIIb IIIc IIIe

446 II2a IIIb IIIc IIIe

447 II2b IIIb IIIc IIIe

IIIb IIIc IIIf →← I1, I2, I3, II1b, II2a, II2c, II3a, II3b, II3c.

448 II1a IIIb IIIc IIIf

449 II1c IIIb IIIc IIIf

450 II2b IIIb IIIc IIIf

IIIb IIId IIIe →← I1, I2, I3, II1a, II2b, II2c, II3a, II3b II3c.

451 II1b IIIb IIId IIIe

452 II1c IIIb IIId IIIe

453 II2a IIIb IIId IIIe

IIIb IIId IIIf →← I1, I2, I3, II2a, II2b, II2c, II3a, II3b, II3c.

454 II1a IIIb IIId IIIf

455 II1b IIIb IIId IIIf

456 II1c IIIb IIId IIIf

B. Pick 2-2.

Note: Any two from group I cannot go with any two from any other group.

II1a II1b →← II2a, II2b, II2c, IIIc, IIIe.

457 II1a II1b IIIa IIId

458 II1a II1b IIIa IIIf

459 II1a II1b IIIb IIId

460 II1a II1b IIIb IIIf

461 II1a II1b IIId IIIf

II1a II1c →← II2a, II2b II2c, IIIa, IIIe.

462 II1a II1c IIIb IIIc

463 II1a II1c IIIb IIId

464 II1a II1c IIIb IIIf

465 II1a II1c IIIc IIIf

466 II1a II1c IIId IIIf

II1b II1c →← II2a, II2b, II2c, IIIa, IIIc.

467 II1b II1c IIIb IIId

468 II1b II1c IIIb IIIe

469 II1b II1c IIIb IIIf

470 II1b II1c IIId IIIe

471 II1b II1c IIId IIIf

II2a II2b →← IIId, IIIf.

472 II2a II2b IIIa IIIc

473 II2a II2b IIIa IIIe

474 II2a II2b IIIb IIIc

475 II2a II2b IIIb IIIe

476 II2a II2b IIIc IIIe

II2a II2c →← IIIb, IIIf.

477 II2a II2c IIIa IIIc

478 II2a II2c IIIa IIId

479 II2a II2c IIIa IIIe

480 II2a II2c IIIc IIIe

481 II2a II2c IIId IIIe

II2b II2c →← IIIb, IIId.

482 II2b II2c IIIa IIIc

483 II2b II2c IIIa IIIe

484 II2b II2c IIIa IIIf

485 II2b II2c IIIc IIIe

486 II2b II2c IIIc IIIf

C. Pick 2-1-1.

I1 I2 →← II1b, II1c, II2b, II2c, II3a, II3b, II3c, IIIa, IIIb, IIIc, IIId.

II1a →← II2b, II2c, IIIe.

487 I1 I2 II1a II2a

488 I1 I2 II1a IIIf

II2a →← IIIf.

489 I1 I2 II2a IIIe

I1 I3 →← II1a, II1c, II2a, II2c, II3a, II3b, II3c, IIIa, IIIb, IIIe, IIIf.

II1b →← II2a, II2c, IIIc.

490 I1 I3 II1b II2b

491 I1 I3 II1b IIId

II2b →← IIId.

492 I1 I3 II2b IIIc

I2 I3 →← II1a, II1b, II2a, II2b, II3a, II3b, II3c, IIIc, IIId, IIIe, IIIf.

II1c →← II2a, II2b, IIIa.

493 I2 I3 II1c II2c

494 I2 I3 II1c IIIb

II2c →← IIIb.

495 I2 I3 II2c IIIa

II1a II1b →← II2a, II2b, II2c, IIIc, IIIe.

II3a →← IIIc, IIId.

496 II1a II1b II3a IIIa

497 II1a II1b II3a IIIb

498 II1a II1b II3a IIIf

II3b →← IIIa, IIIb.

499 II1a II1b II3b IIId

500 II1a II1b II3b IIIf

II3c →← IIIe, IIIf.

501 II1a II1b II3c IIIa

502 II1a II1b II3c IIIb

503 II1a II1b II3c IIId

II1a II1c →← II2a, II2b, II2c, IIIa, IIIe.

504 II1a II1c II3a IIIb

505 II1a II1c II3a IIIf

506 II1a II1c II3b IIIc

507 II1a II1c II3b IIId

508 II1a II1c II3b IIIf

509 II1a II1c II3c IIIb

510 II1a II1c II3c IIIc

511 II1a II1c II3c IIId

II1b II1c →← II2a, II2b, II2c, IIIa, IIIc.

512 II1b II1c II3a IIIb

513 II1b II1c II3a IIIe

514 II1b II1c II3a IIIf

515 II1b II1c II3b IIId

516 II1b II1c II3b IIIe

517 II1b II1c II3b IIIf

518 II1b II1c II3c IIIb

519 II1b II1c II3c IIId

II2a II2b →← IIId, IIIf.

520 II2a II2b II3a IIIa

521 II2a II2b II3a IIIb

522 II2a II2b II3a IIIe

523 II2a II2b II3b IIIc

524 II2a II2b II3b IIIe

525 II2a II2b II3c IIIa

526 II2a II2b II3c IIIb

527 II2a II2b II3c IIIc

II2a II2c →← IIIb, IIIf.

528 II2a II2c II3a IIIa

529 II2a II2c II3a IIIe

530 II2a II2c II3b IIIc

531 II2a II2c II3b IIId

532 II2a II2c II3b IIIe

533 II2a II2c II3c IIIa

534 II2a II2c II3c IIIc

535 II2a II2c II3c IIId

II2b II2c →← IIIb, IIId.

536 II2b II2c II3a IIIa

537 II2b II2c II3a IIIe

538 II2b II2c II3a IIIf

539 II2b II2c II3b IIIc

540 II2b II2c II3b IIIe

541 II2b II2c II3b IIIf

542 II2b II2c II3c IIIa

543 II2b II2c II3c IIIc

D. Pick 1-2-1.

I1 →← II1c, II2c, II3a, II3c, IIIa, IIIb.

544 I1 II1a II1b II3b

545 I1 II1a II1b IIId

546 I1 II1a II1b IIIf

547 I1 II2a II2b II3b

548 I1 II2a II2b IIIc

549 I1 II2a II2b IIIe

I2 →← II1b, II2b, II3b, II3c, IIIc, IIId.

550 I2 II1a II1c II3a

551 I2 II1a II1c IIIb

552 I2 II1a II1c IIIf

553 I2 II2a II2c II3a

554 I2 II2a II2c IIIa

555 I2 II2a II2c IIIe

I3 →← II1a, II2a, II3a, II3b, IIIe, IIIf.

556 I3 II1b II1c II3c

557 I3 II1b II1c IIIb

558 I3 II1b II1c IIId

559 I3 II2b II2c II3c

560 I3 II2b II2c IIIa

561 I3 II2b II2c IIIc

E. Pick 1-1-2.

I1 →← II1c, II2c, II3a, II3c, IIIa, IIIb.

II1a →← II2b, II2c, IIIe.

562 I1 II1a IIIc IIIf

563 I1 II1a IIId IIIf

II1b →← II2a, II2c, IIIc.

564 I1 II1b IIId IIIe

565 I1 II1b IIId IIIf

II2a →← IIIf.

566 I1 II2a IIIc IIIe

567 I1 II2a IIId IIIe

II2b →← IIId.

568 I1 II2b IIIc IIIe

569 I1 II2b IIIc IIIf

II3b →← IIIa, IIIb.

570 I1 II3b IIIc IIIe

571 I1 II3b IIIc IIIf

572 I1 II3b IIId IIIe

573 I1 II3b IIId IIIf

I2 →← II1b, II2b, II3b, II3c, IIIc, IIId.

II1a →← II2b, II2c, IIIe.

574 I2 II1a IIIa IIIf

575 I2 II1a IIIb IIIf

II1c →← II2a, II2b, IIIa.

576 I2 II1c IIIb IIIe

577 I2 II1c IIIb IIIf

II2a →← IIIf.

578 I2 II2a IIIa IIIe

579 I2 II2a IIIb IIIe

II2c →← IIIb.

580 I2 II2c IIIa IIIe

581 I2 II2c IIIa IIIf

II3a →← IIIc, IIId.

582 I2 II3a IIIa IIIe

583 I2 II3a IIIa IIIf

584 I2 II3a IIIb IIIe

585 I2 II3a IIIb IIIf

I3 →← II1a, II2a, II3a, II3b, IIIe, IIIf.

II1b →← II2a, II2c, IIIc.

586 I3 II1b IIIa IIId

587 I3 II1b IIIb IIId

II1c →← II2a, II2b, IIIa.

588 I3 II1c IIIb IIIc

589 I3 II1c IIIb IIId

II2b →← IIId.

590 I3 II2b IIIa IIIc

591 I3 II2b IIIb IIIc

II2c →← IIIb.

592 I3 II2c IIIa IIIc

593 I3 II2c IIIa IIId

II3c →← IIIe, IIIf.

594 I3 II3c IIIa IIIc

595 I3 II3c IIIa IIId

596 I3 II3c IIIb IIIc

597 I3 II3c IIIb IIId

II1a →← II2b II2c IIIe.

II2a →← IIIf. 598 II1a II2a IIIa IIIc

599 II1a II2a IIIa IIId

600 II1a II2a IIIb IIIc

601 II1a II2a IIIb IIId

II3a →← IIIc, IIId.

602 II1a II3a IIIa IIIf

603 II1a II3a IIIb IIIf

II3b →← IIIa, IIIb.

604 II1a II3b IIIc IIIf

605 II1a II3b IIId IIIf

II3c →← IIIe, IIIf.

606 II1a II3c IIIa IIIc

607 II1a II3c IIIa IIId

608 II1a II3c IIIb IIIc

609 II1a II3c IIIb IIId

II1b →← II2a, II2c, IIIc.

II2b →← IIId.

610 II1b II2b IIIa IIIe

611 II1b II2b IIIa IIIf

612 II1b II2b IIIb IIIe

613 II1b II2b IIIb IIIf

II3a →← IIIc, IIId.

614 II1b II3a IIIa IIIe

615 II1b II3a IIIa IIIf

616 II1b II3a IIIb IIIe

617 II1b II3a IIIb IIIf

II3b →← IIIa, IIIb.

618 II1b II3b IIId IIIe

619 II1b II3b IIId IIIf

II3c →← IIIe, IIIf.

620 II1b II3c IIIa IIId

621 II1b II3c IIIb IIId

II1c →← II2a, II2b, IIIa.

II2c →← IIIb

622 II1c II2c IIIc IIIe

623 II1c II2c IIIc IIIf

624 II1c II2c IIId IIIe

625 II1c II2c IIId IIIf

II3a →← IIIc, IIId.

626 II1c II3a IIIb IIIe

627 II1c II3a IIIb IIIf

II3b →← IIIa, IIIb.

628 II1c II3b IIIc IIIe

629 II1c II3b IIIc IIIf

630 II1c II3b IIId IIIe

631 II1c II3b IIId IIIf

II3c →← IIIe, IIIf.

632 II1c II3c IIIb IIIc

633 II1c II3c IIIb IIId

II2a →← IIIf.

II3a →← IIIc, IIId.

634 II2a II3a IIIa IIIe

635 II2a II3a IIIb IIIe

II3b →← IIIa, IIIb.

636 II2a II3b IIIc IIIe

637 II2a II3b IIId IIIe

II3c →← IIIe, IIIf.

638 II2a II3c IIIa IIIc

639 II2a II3c IIIa IIId

640 II2a II3c IIIb IIIc

641 II2a II3c IIIb IIId

II2b →← IIId.

II3a →← IIIc, IIId.

642 II2b II3a IIIa IIIe

643 II2b II3a IIIa IIIf

644 II2b II3a IIIb IIIe

645 II2b II3a IIIb IIIf

II3b →← IIIa, IIIb.

646 II2b II3b IIIc IIIe

647 II2b II3b IIIc IIIf

II3c →← IIIe, IIIf.

648 II2b II3c IIIa IIIc

649 II2b II3c IIIb IIIc

II2c →← IIIb.

II3a →← IIIc, IIId.

650 II2c II3a IIIa IIIe

651 II2c II3a IIIa IIIf

II3b →← IIIa, IIIb.

652 II2c II3b IIIc IIIe

653 II2c II3b IIIc IIIf

654 II2c II3b IIId IIIe

655 II2c II3b IIId IIIf

II3c →← IIIe, IIIf.

656 II2c II3c IIIa IIIc

657 II2c II3c IIIa IIId

F. Pick 1-1-1-1.

I1 →← II1c, II2c, II3a, II3c IIIa, IIIb.

II1a →← II2b, II2c IIIe.

II2a →← IIIf.

658 I1 II1a II2a II3b

659 I1 II1a II2a IIIc

660 I1 II1a II2a IIId

661 I1 II1a II3b IIIc

662 I1 II1a II3b IIId

663 I1 II1a II3b IIIf

664 I1 II2a II3b IIIc

665 I1 II2a II3b IIId

666 I1 II2a II3b IIIe

II1b →← II2a, II2c, IIIc.

II2b →← IIId.

667 I1 II1b II2b II3b

668 I1 II1b II2b IIIe

669 I1 II1b II2b IIIf

670 I1 II1b II3b IIId

671 I1 II1b II3b IIIe

672 I1 II1b II3b IIIf

673 I1 II2b II3b IIIc

674 I1 II2b II3b IIIe

675 I1 II2b II3b IIIf

I2 →← II1b, II2b, II3b, II3c, IIIc, IIId.

II1a →← II2b, II2c, IIIe.

II2a →← IIIf.

676 I2 II1a II2a II3a

677 I2 II1a II2a IIIa

678 I2 II1a II2a IIIb

679 I2 II1a II3a IIIa

680 I2 II1a II3a IIIb

681 I2 II1a II3a IIIf

682 I2 II2a II3a IIIa

683 I2 II2a II3a IIIb

684 I2 II2a II3a IIIe

II1c →← II2a, II2b, IIIa.

II2c →← IIIb.

685 I2 II1c II2c II3a

686 I2 II1c II2c IIIe

687 I2 II1c II2c IIIf

688 I2 II1c II3a IIIb

689 I2 II1c II3a IIIe

690 I2 II1c II3a IIIf

691 I2 II2c II3a IIIa

692 I2 II2c II3a IIIe

693 I2 II2c II3a IIIf

I3 →← II1a, II2a, II3a, II3b, IIIe, IIIf.

II1b →← II2a, II2c, IIIc.

II2b →← IIId.

694 I3 II1b II2b II3c

695 I3 II1b II2b IIIa

696 I3 II1b II2b IIIb

697 I3 II1b II3c IIIa

698 I3 II1b II3c IIIb

699 I3 II1b II3c IIId

700 I3 II2b II3c IIIa

701 I3 II2b II3c IIIb

702 I3 II2b II3c IIIc

II1c →← II2a, II2b, IIIa.

II2c →← IIIb.

703 I3 II1c II2c II3c

704 I3 II1c II2c IIIc

705 I3 II1c II2c IIId

706 I3 II1c II3c IIIb

707 I3 II1c II3c IIIc

708 I3 II1c II3c IIId

709 I3 II2c II3c IIIa

710 I3 II2c II3c IIIc

711 I3 II2c II3c IIId

II1a →← II2b, II2c IIIe.

II2a →← IIIf.

712 II1a II2a II3a IIIa

713 II1a II2a II3a IIIb

714 II1a II2a II3b IIIc

715 II1a II2a II3b IIId

716 II1a II2a II3c IIIa

717 II1a II2a II3c IIIb

718 II1a II2a II3c IIIc

719 II1a II2a II3c IIId

II1b →← II2a, II2c, IIIc.

II2b →← IIId.

720 II1b II2b II3a IIIa

721 II1b II2b II3a IIIb

722 II1b II2b II3a IIIe

723 II1b II2b II3a IIIf

724 II1b II2b II3b IIIe

725 II1b II2b II3b IIIf

726 II1b II2b II3c IIIa

727 II1b II2b II3c IIIb

II1c →← II2a, II2b, IIIa.

II2c →← IIIb.

728 II1c II2c II3a IIIe

729 II1c II2c II3a IIIf

730 II1c II2c II3b IIIc

731 II1c II2c II3b IIId

732 II1c II2c II3b IIIe

733 II1c II2c II3b IIIf

734 II1c II2c II3c IIIc

735 II1c II2c II3c IIId

Case VI: Pick five at a time.

A. Pick 3-2 or 2-3.

Note: I1 I2 I3 don't go with anything and any two from group I don't go with three from another group.

II1a II1b II1c →← II2a, II2b, II2c, IIIa, IIIc, IIIe.

736 II1a II1b II1c IIIb IIId

737 II1a II1b II1c IIIb IIIf

738 II1a II1b II1c IIId IIIf

II2a II2b II2c →← II1a, II1b, II1c, IIIb, IIId, IIIf.

739 II2a II2b II2c IIIa IIIc

740 II2a II2b II2c IIIa IIIe

741 II2a II2b II2c IIIc IIIe

742 II1a II1b IIIa IIId IIIf

743 II1a II1b IIIb IIId IIIf

744 II1a II1c IIIb IIIc IIIf

745 II1a II1c IIIb IIId IIIf

746 II1b II1c IIIb IIId IIIe

747 II1b II1c IIIb IIId IIIf

748 II2a II2b IIIa IIIc IIIe

749 II2a II2b IIIb IIIc IIIe

750 II2a II2c IIIa IIIc IIIe

751 II2a II2c IIIa IIId IIIe

752 II2b II2c IIIa IIIc IIIe

753 II2b II2c IIIa IIIc IIIf

B. Pick 3-1-1 or 1-3-1 or 1-1-3.

Note: I1 I2 I3 don't go with anything.

II1a II1b II1c →← I1, I2, I3, II2a, II2b, II2c, IIIa, IIIc, IIIe.

754 II1a II1b II1c II3a IIIb

755 II1a II1b II1c II3a IIIf

756 II1a II1b II1c II3b IIId

757 II1a II1b II1c II3b IIIf

758 II1a II1b II1c II3c IIIb

759 II1a II1b II1c II3c IIId

II2a II2b II2c →← I1, I2, I3, II1a, II1b, II1c, IIIb, IIId, IIIf.

760 II2a II2b II2c II3a IIIa

761 II2a II2b II2c II3a IIIe

762 II2a II2b II2c II3b IIIc

763 II2a II2b II2c II3b IIIe

764 II2a II2b II2c II3c IIIa

765 II2a II2b II2c II3c IIIc

Note: Pick one from group I or group II3, then no three can be picked from group III.

II1a II2a don't go with IIIe, IIIf, so no three can be picked from group III. Similarly for II1b II2b and II1c II2c.

C. Pick 2-2-1.

Note: Any two from group I don't go with any two from another group. Any two from II1 don't go with any two from II2.

None for this case.

D. Pick 2-1-2.

Note: Any two from group I don't go with any two from another group.

II1a II1b $\rightarrow\leftarrow$ II2a, II2b, II2c, IIIc, IIIe.

766 II1a II1b II3a IIIa IIIf

767 II1a II1b II3a IIIb IIIf

768 II1a II1b II3b IIId IIIf

769 II1a II1b II3c IIIa IIId

770 II1a II1b II3c IIIb IIId

II1a II1c $\rightarrow\leftarrow$ II2a, II2b, II2c, IIIa, IIIe.

771 II1a II1c II3a IIIb IIIf

772 II1a II1c II3b IIIc IIIf

773 II1a II1c II3b IIId IIIf

774 II1a II1c II3c IIIb IIIc

775 II1a II1c II3c IIIb IIId

II1b II1c →← II2a, II2b, II2c, IIIa, IIIc.

776 II1b II1c II3a IIIb IIIe

777 II1b II1c II3a IIIb IIIf

778 II1b II1c II3b IIId IIIe

779 II1b II1c II3b IIId IIIf

780 II1b II1c II3c IIIb IIId

II2a II2b →← IIId, IIIf.

781 II2a II2b II3a IIIa IIIe

782 II2a II2b II3a IIIb IIIe

783 II2a II2b II3b IIIc IIIe

784 II2a II2b II3c IIIa IIIc

785 II2a II2b II3c IIIb IIIc

II2a II2c →← IIIb, IIIf.

786 II2a II2c II3a IIIa IIIe

787 II2a II2c II3b IIIc IIIe

788 II2a II2c II3b IIId IIIe

789 II2a II2c II3c IIIa IIIc

790 II2a II2c II3c IIIa IIId

II2b II2c →← IIIb, IIId.

791 II2b II2c II3a IIIa IIIe

792 II2b II2c II3a IIIa IIIf

793 II2b II2c II3b IIIc IIIe

794 II2b II2c II3b IIIc IIIf

795 II2b II2c II3c IIIa IIIc

E. Pick 1-2-2.

I1 →← II1c, II2c, II3a, II3c, IIIa, IIIb.

II1a II1b →← II2a, II2b, II2c IIIc, IIIe.

796 I1 II1a II1b IIId IIIf

II2a II2b →← IIId, IIIf.

797 I1 II2a II2b IIIc IIIe

I2 →← II1b, II2b, II3b, II3c, IIIc, IIId.

II1a II1c →← II2a, II2b, II2c, IIIa, IIIe.

798 I2 II1a II1c IIIb IIIf

II2a II2c →← IIIb, IIIf.

799 I2 II2a II2c IIIa IIIe

I3 →← II1a, II2a, II3a, II3b, IIIe, IIIf.

II1b II1c →← II2a, II2b, II2c, IIIa, IIIc.

800 I3 II1b II1c IIIb IIId

II2b II2c →← IIIb, IIId.

801 I3 II2b II2c IIIa IIIc

F. Pick 2-1-1-1.

I1 I2 →← II1b, II1c, II2b, II2c, II3a, II3b, II3c, IIIa, IIIb, IIIc, IIId.

II1a →← II2b, II2c, IIIe.

II2a →← IIIf.

So nothing works with I1 I2. Similarly, nothing works with I1 I3 and I2 I3.

Note: Any two from group II1 don't go with anything from group II2.

None for this case.

G. Pick 1-2-1-1.

I1 →← II1c, II2c, II3a, II3c, IIIa, IIIb.

II1a II1b →← II2a, II2b, II2c, IIIc, IIIe.

802 I1 II1a II1b II3b IIId

803 I1 II1a II1b II3b IIIf

II2a II2b →← IIId, IIIf.

804 I1 II2a II2b II3b IIIc

805 I1 II2a II2b II3b IIIe

I2 →← II1b, II2b, II3b, II3c, IIIc, IIId.

II1a II1c →← II2a, II2b, II2c, IIIa, IIIe.

806 I2 II1a II1c II3a IIIb

807 I2 II1a II1c II3a IIIf

II2a II2c →← IIIb, IIIf.

808 I2 II2a II2c II3a IIIa

809 I2 II2a II2c II3a IIIe

I3 →← II1a, II2a, II3a, II3b, IIIe, IIIf.

II1b II1c →← II2a, II2b, II2c, IIIa, IIIc.

810 I3 II1b II1c II3c IIIb

811 I3 II1b II1c II3c IIId

II2b II2c →← IIIb, IIId.

812 I3 II2b II2c II3c IIIa

813 I3 II2b II2c II3c IIIc

Note: Any one from group II1 doesn't go with any two from group II2.

H. Pick 1-1-2-1.

Note: Any two from group II2 don't go with anything from II1.

None for this case.

I. Pick 1-1-1-2.

I1 →← II1c, II2c, II3a, II3c, IIIa, IIIb.

II1a →← II2b, II2c, IIIe.

II2a →← IIIf.

II3b →← IIIa, IIIb.

814 I1 II1a II3b IIIc IIIf

815 I1 II1a II3b IIId IIIf

II1b →← II2a, II2c, IIIc.

II2b →← IIId.

816 I1 II1b II3b IIId IIIe

817 I1 II1b II3b IIId IIIf

818 I1 II2a II3b IIIc IIIe

819 I1 II2a II3b IIId IIIe

820 I1 II2b II3b IIIc IIIe

821 I1 II2b II3b IIIc IIIf

I2 →← II1b, II2b, II3b, II3c IIIc, IIId.

II1a →← II2b, II2c, IIIe.

II2a →← IIIf.

II3a →← IIIc, IIId.

822 I2 II1a II3a IIIa IIIf

823 I2 II1a II3a IIIb IIIf

II1c →← II2a, II2b, IIIa.

II2c →← IIIb.

824 I2 II1c II3a IIIb IIIe

825 I2 II1c II3a IIIb IIIf

826 I2 II2a II3a IIIa IIIe

827 I2 II2a II3a IIIb IIIe

828 I2 II2c II3a IIIa IIIe

829 I2 II2c II3a IIIa IIIf

I3 →← II1a, II2a, II3a, II3b, IIIe, IIIf.

II1b →← II2a, II2c, IIIc.

II2b →← IIId.

830 I3 II1b II3c IIIa IIId

831 I3 II1b II3c IIIb IIId

II1c →← II2a, II2b, IIIa.

II2c →← IIIb.

832 I3 II1c II3c IIIb IIIc

833 I3 II1c II3c IIIb IIId

834 I3 II2b II3c IIIa IIIc

835 I3 II2b II3c IIIb IIIc

836 I3 II2c II3c IIIa IIIc

837 I3 II2c II3c IIIa IIId

II1a →← II2b, II2c, IIIe.

II2a →← IIIf.

II3a →← IIIc, IIId.

II3b →← IIIa, IIIb.

II3c →← IIIe, IIIf.

838 II1a II2a II3c IIIa IIIc

839 II1a II2a II3c IIIa IIId

840 II1a II2a II3c IIIb IIIc

841 II1a II2a II3c IIIb IIId

II1b →← II2a, II2c, IIIc.

II2b →← IIId.

842 II1b II2b II3a IIIa IIIe

843 II1b II2b II3a IIIa IIIf

844 II1b II2b II3a IIIb IIIe

845 II1b II2b II3a IIIb IIIf

II1c →← II2a, II2b, IIIa.

II2c →← IIIb.

846 II1c II2c II3b IIIc IIIe

847 II1c II2c II3b IIIc IIIf

848 II1c II2c II3b IIId IIIe

849 II1c II2c II3b IIId IIIf

J. Pick 1-1-1-1-1.

I1 →← II1c, II2c, II3a, II3c, IIIa, IIIb.

II1a →← II2b, II2c, IIIe.

II2a →← IIIf.

850 I1 II1a II2a II3b IIIc

851 I1 II1a II2a II3b IIId

II1b →← II2a, II2c, IIIc.

II2b →← IIId.

852 I1 II1b II2b II3b IIIe

853 I1 II1b II2b II3b IIIf

I2 →← II1b, II2b, II3b, II3c, IIIc, IIId.

II1a →← II2b, II2c, IIIe.

II2a →← IIIf.

854 I2 II1a II2a II3a IIIa

855 I2 II1a II2a II3a IIIb

II1c →← II2a, II2b, IIIa.

II2c →← IIIb.

856 I2 II1c II2c II3a IIIe

857 I2 II1c II2c II3a IIIf

I3 →← II1a, II2a, II3a, II3b, IIIe, IIIf

II1b →← II2a, II2c, IIIc.

II2b →← IIId.

858 I3 II1b II2b II3c IIIa

859 I3 II1b II2b II3c IIIb

II1c →← II2a, II2b, IIIa.

II2c →← IIIb.

860 I3 II1c II2c II3c IIIc

861 I3 II1c II2c II3c IIId

Case VII: Pick six at a time.

A. Any combination with 3 from one group.

Note: I1 I2 I3 don't go with anything. II1a II1b II1c don't go with anything

from group I and group II2. II2a II2b II2c don't go with anything from group I and group II1.

862 II1a II1b II1c IIIb IIId IIIf

863 II2a II2b II2c IIIa IIIc IIIe

864 II1a II1b II1c II3a IIIb IIIf

865 II1a II1b II1c II3b IIId IIIf

866 II1a II1b II1c II3c IIIb IIId

867 II2a II2b II2c II3a IIIa IIIe

868 II2a II2b II2c II3b IIIc IIIe

869 II2a II2b II2c II3c IIIa IIIc

B. Any combination with 2 from one group.

Note: Any two from group I may possibly go with one from group II1, one from group II2 and one from group III, which will only make five in total. So we cannot pick two from group I.

Note: Any two from group II1 don't go with anything from group II2, and vice versa.

870 I1 II1a II1b II3b IIId IIIf

871 I1 II2a II2b II3b IIIc IIIe

872 I2 II1a II1c II3a IIIb IIIf

873 I2 II2a II2c II3a IIIa IIIe

874 I3 II1b II1c II3c IIIb IIId

875 I3 II2b II2c II3c IIIa IIIc

□

Theorem 32. *Any automorphism of the non-Desarguesian configuration of Figure* 3 *fixes* p_7.

Proof. Since p_7 has rank 3, it could probably be mapped only to one of the points from p_1 to p_7. Note that every point on the lines passing through p_7 has rank 3, but there always exists a point of rank 4 on at least one of the lines passing through any of the points from p_1 to p_6. Therefore, any automorphism of the non-Desarguesian configuration of Figure 3 must map p_7 to p_7. □

Corollary 33. *There are 12 automorphisms of the non-Desarguesian configuration of Figure* 3.

Proof. Since any automorphism will fix p_7, we may only map l_1, l_2, and l_3 into themselves. This implies that we may map p_1 to itself, p_2, p_3, p_4, p_5, or p_6. And under each of the cases mentioned, there are two choices to map p_2 to. Therefore we obtain 12 automorphisms. Table 4 gives all the details. □

Among the 875 extensions, some of them may be isomorphic. Table 5 shows how the methods change when automorphisms are applied.

Theorem 34. *We can extend the non-Desarguesian configuration of Figure* 3 *to 105 semiplanes up to isomorphism without adding a new line.*

When we apply the automorphisms to the semiplanes listed in Theorem 31, we obtain a list of all 105 isomorphism classes, which is shown at the end of this Chapter. In this

Table 4: Automorphisms.

*	1	2	3	4	5	6	7	8	9	10	11	12
p_7	p_7	p_7	p_7	p_7	p_7	p_7	p_7	p_7	p_7	p_7	p_7	p_7
p_1	p_1	p_1	p_2	p_2	p_3	p_3	p_4	p_4	p_5	p_5	p_6	p_6
p_2	p_2	p_3	p_1	p_3	p_1	p_2	p_5	p_6	p_4	p_6	p_4	p_5
p_3	p_3	p_2	p_3	p_1	p_2	p_1	p_6	p_5	p_6	p_4	p_5	p_4
p_4	p_4	p_4	p_5	p_5	p_6	p_6	p_1	p_1	p_2	p_2	p_3	p_3
p_5	p_5	p_6	p_4	p_6	p_4	p_5	p_2	p_3	p_1	p_3	p_1	p_2
p_6	p_6	p_5	p_6	p_4	p_5	p_4	p_3	p_2	p_3	p_1	p_2	p_1
p_8	p_8	p_8	p_{10}	p_{10}	p_9	p_9	p_8	p_8	p_{10}	p_{10}	p_9	p_9
p_9	p_9	p_{10}	p_9	p_8	p_{10}	p_8	p_9	p_{10}	p_9	p_8	p_{10}	p_8
p_{10}	p_{10}	p_9	p_8	p_9	p_8	p_{10}	p_{10}	p_9	p_8	p_9	p_8	p_{10}
l_1	l_1	l_1	l_2	l_2	l_3	l_3	l_1	l_1	l_2	l_2	l_3	l_3
l_2	l_2	l_3	l_1	l_3	l_1	l_2	l_2	l_3	l_1	l_3	l_1	l_2
l_3	l_3	l_2	l_3	l_1	l_2	l_1	l_3	l_2	l_3	l_1	l_2	l_1
l_4	l_4	l_5	l_4	l_6	l_5	l_6	l_7	l_8	l_7	l_9	l_8	l_9
l_5	l_5	l_4	l_6	l_4	l_6	l_5	l_8	l_7	l_9	l_7	l_9	l_8
l_6	l_6	l_6	l_5	l_5	l_4	l_4	l_9	l_9	l_8	l_8	l_7	l_7
l_7	l_7	l_8	l_7	l_9	l_8	l_9	l_4	l_5	l_4	l_6	l_5	l_6
l_8	l_8	l_7	l_9	l_7	l_9	l_8	l_5	l_4	l_6	l_4	l_6	l_5
l_9	l_9	l_9	l_8	l_8	l_7	l_7	l_6	l_6	l_5	l_5	l_4	l_4
l_{10}	l_{10}	l_{12}	l_{11}	l_{12}	l_{11}	l_{10}	l_{10}	l_{12}	l_{11}	l_{12}	l_{11}	l_{10}
l_{11}	l_{11}	l_{11}	l_{10}	l_{10}	l_{12}	l_{12}	l_{11}	l_{11}	l_{10}	l_{10}	l_{12}	l_{12}
l_{12}	l_{12}	l_{10}	l_{12}	l_{11}	l_{10}	l_{11}	l_{12}	l_{10}	l_{12}	l_{11}	l_{10}	l_{11}

Table 5: Changes in methods corresponding to automorphisms.

method	1	2	3	4	5	6	7	8	9	10	11	12
I1	I1	I1	I2	I2	I3	I3	I1	I1	I2	I2	I3	I3
I2	I2	I3	I1	I3	I1	I2	I2	I3	I1	I3	I1	I2
I3	I3	I2	I3	I1	I2	I1	I3	I2	I3	I1	I2	I1
II1a	II1a	II1b	II1a	II1c	II1b	II1c	II2a	II2b	II2a	II2c	II2b	II2c
II1b	II1b	II1a	II1c	II1a	II1c	II1b	II2b	II2a	II2c	II2a	II2c	II2b
II1c	II1c	II1c	II1b	II1b	II1a	II1a	II2c	II2c	II2b	II2b	II2a	II2a
II2a	II2a	II2b	II2a	II2c	II2b	II2c	II1a	II1b	II1a	II1c	II1b	II1c
II2b	II2b	II2a	II2c	II2a	II2c	II2b	II1b	II1a	II1c	II1a	II1c	II1b
II2c	II2c	II2c	II2b	II2b	II2a	II2a	II1c	II1c	II1b	II1b	II1a	II1a
II3a	II3a	II3c	II3b	II3c	II3b	II3a	II3a	II3c	II3b	II3c	II3b	II3a
II3b	II3b	II3b	II3a	II3a	II3c	II3c	II3b	II3b	II3a	II3a	II3c	II3c
II3c	II3c	II3a	II3c	II3b	II3a	II3b	II3c	II3a	II3c	II3b	II3a	II3b
IIIa	IIIa	IIIa	IIIc	IIIc	IIIe	IIIe	IIIb	IIIb	IIId	IIId	IIIf	IIIf
IIIb	IIIb	IIIb	IIId	IIId	IIIf	IIIf	IIIa	IIIa	IIIc	IIIc	IIIe	IIIe
IIIc	IIIc	IIIe	IIIa	IIIe	IIIa	IIIc	IIId	IIIf	IIIb	IIIf	IIIb	IIId
IIId	IIId	IIIf	IIIb	IIIf	IIIb	IIId	IIIc	IIIe	IIIa	IIIe	IIIa	IIIc
IIIe	IIIe	IIIc	IIIe	IIIa	IIIc	IIIa	IIIf	IIId	IIIf	IIIb	IIId	IIIb
IIIf	IIIf	IIId	IIIf	IIIb	IIId	IIIb	IIIe	IIIc	IIIe	IIIa	IIIc	IIIa

list, we provide two additional properties about the semiplanes in each isomorphism class, which are number of points in the semiplanes and the ρ value which is computed with $n = 12$. In summary, one semiplane, with 37 points, forms one class by itself, 6 semiplanes with 35 points form one class, 9 semiplanes with 34 points form 2 classes, 15 semiplanes with 33 points form 3 classes, 42 semiplanes with 32 points form 4 classes, 47 semiplanes with 31 points form 7 classes, 75 semiplanes with 30 points form 9 classes, 111 semiplanes with 29 points form 12 classes, 95 semiplanes with 28 points form 11 classes, 126 semiplanes with 27 points form 15 classes, 120 semiplanes with 26 points form 12 classes, 79 semiplanes with 25 points form 9 classes, 66 semiplanes with 24 points form 9 classes, 45 semiplanes with 23 points form 5 classes, 26 semiplanes with 22 points form 3 classes, 6 semiplanes with 21 points form one class and 6 semiplanes with 20 points

form one class. In the list, the parenthetical number(s) following a semiplane indicate(s) the automorphism(s) which will map the first semiplane in the class to that semiplane. Note that semiplanes with the same number of points may not have the same ρ value. Also note that a semiplane with fewer points does not necessarily have a smaller ρ value than a semiplane with more points.

Class 1. Semiplane with 37 points and $\rho_{12} = 51$.

(a) IV

Class 2. Semiplanes with 33 points and $\rho_{12} = 48$.

(a) I1,IV

(b) I2,IV (3,4,9,10)

(c) I3,IV (5,6,11,12)

Class 3. Semiplanes with 34 points and $\rho_{12} = 49$.

(a) II1a,IV

(b) II1b,IV (2,5)

(c) II1c,IV (4,6)

(d) II2a,IV (7,9)

(e) II2b,IV (8,11)

(f) II2c,IV (10,12)

Class 4. Semiplanes with 34 points and $\rho_{12} = 49$.

(a) II3a,IV

(b) II3b,IV (3,5,9,11)

(c) II3c,IV (2,4,8,10)

Class 5. Semiplanes with 35 points and $\rho_{12} = 50$.

(a) IIIa,IV

(b) IIIb,IV (7,8)

(c) IIIc,IV (3,4)

(d) IIId,IV (9,10)

(e) IIIe,IV (5,6)

(f) IIIf,IV (11,12)

Class 6. Semiplanes with 29 points and $\rho_{12} = 45$.

(a) I1,I2,IV

(b) I1,I3,IV (2,5,8,11)

(c) I2,I3,IV (4,6,10,12)

Class 7. Semiplanes with 31 points and $\rho_{12} = 47$.

(a) II1a,II1b,IV

(b) II1a,II1c,IV (3,4)

(c) II1b,II1c,IV (5,6)

(d) II2a,II2b,IV (7,8)

(e) II2a,II2c,IV (9,10)

(f) II2b,II2c,IV (11,12)

Class 8. Semiplanes with 33 points and $\rho_{12} = 49$.

(a) IIIa,IIIc,IV

(b) IIIa,IIIe,IV (2,5)

(c) IIIb,IIId,IV (7,9)

(d) IIIb,IIIf,IV (8,11)

(e) IIIc,IIIe,IV (4,6)

(f) IIId,IIIf,IV (10,12)

Class 9. Semiplanes with 33 points and $\rho_{12} = 49$.

(a) IIIa,IIId,IV

(b) IIIa,IIIf,IV (2,11)

(c) IIIb,IIIc,IV (3,7)

(d) IIIb,IIIe,IV (5,8)

(e) IIIc,IIIf,IV (4,12)

(f) IIId,IIIe,IV (6,10)

Class 10. Semiplanes with 30 points and $\rho_{12} = 46$.

(a) I1,II1a,IV

(b) I1,II1b,IV (2)

(c) I1,II2a,IV (7)

(d) I1,II2b,IV (8)

(e) I2,II1a,IV (3)

(f) I2,II1c,IV (4)

(g) I2,II2a,IV (9)

(h) I2,II2c,IV (10)

(i) I3,II1b,IV (5)

(j) I3,II1c,IV (6)

(k) I3,II2b,IV (11)

(l) I3,II2c,Iv (12)

Class 11. Semiplanes with 30 points and $\rho_{12} = 46$.

(a) I1,II3b,IV

(b) I2,II3a,IV (3,4,9,10)

(c) I3,II3c,IV (5,6,11,12)

Class 12. Semiplanes with 31 points and $\rho_{12} = 47$.

(a) I1,IIIc,IV

(b) I1,IIId,IV (7)

(c) I1,IIIe,IV (2)

(d) I1,IIIf,IV (8)

(e) I2,IIIa,IV (3)

(f) I2,IIIb,IV (9)

(g) I2,IIIe,IV (4)

(h) I2,IIIf,IV (10)

(i) I3,IIIa,IV (5)

(j) I3,IIIb,IV (11)

(k) I3,IIIc,IV (6)

(l) I3,IIId,IV (12)

Class 13. Semiplanes with 31 points and $\rho_{12} = 47$.

(a) II1a,II2a,IV

(b) II1b,II2b,IV (2,5,8,11)

(c) II1c,II2c,IV (4,6,10,12)

Class 14. Semiplanes with 31 points and $\rho_{12} = 47$.

(a) II1a,II3a,IV

(b) II1a,II3b,IV (3)

(c) II1b,II3b,IV (5)

(d) II1b,II3c,IV (2)

(e) II1c,II3a,IV (6)

(f) II1c,II3c,IV (4)

(g) II2a,II3a,IV (7)

(h) II2a,II3b,IV (9)

(i) II2b,II3b,IV (11)

(j) II2b,II3c,IV (8)

(k) II2c,II3a,IV (12)

(l) II2c,II3c,IV (10)

Class 15. Semiplanes with 31 points and $\rho_{12} = 47$.

(a) II1a,II3c,IV

(b) II1b,II3a,IV (2,5)

(c) II1c,II3b,IV (4,6)

(d) II2a,II3c,IV (7,9)

(e) II2b,II3a,IV (8,11)

(f) II2c,II3b,IV (10,12)

Class 16. Semiplanes with 32 points and $\rho_{12} = 48$.

(a) II1a,IIIa,IV

(b) II1a,IIIc,IV (3)

(c) II1b,IIIa,IV (2)

(d) II1b,IIIe,IV (5)

(e) II1c,IIIc,IV (4)

(f) II1c,IIIe,IV (6)

(g) II2a,IIIb,IV (7)

(h) II2a,IIId,IV (9)

(i) II2b,IIIb,IV (8)

(j) II2c,IIId,IV (10)

(k) II2c,IIIf,IV (12)

Class 17. Semiplanes with 32 points and $\rho_{12} = 48$.

(a) II1a,IIIb,IV

(b) II1a,IIId,IV (3)

(c) II1b,IIIb,IV (2)

(d) II1b,IIIf,IV (5)

(e) II1c,IIId,IV (4)

(f) II1c,IIIf,IV (6)

(g) II2a,IIIa,IV (7)

(h) II2a,IIIc,IV (9)

(i) II2b,IIIa,IV (8)

(j) II2b,IIIe,IV (11)

(k) II2c,IIIc,IV (10)

(l) II2c,IIIe,IV (12)

Class 18. Semiplanes with 32 points and $\rho_{12} = 48$.

(a) II1a,IIIf,IV

(b) II1b,IIId,IV (2,5)

(c) II1c,IIIb,IV (4,6)

(d) II2a,IIIe,IV (7,9)

(e) II2b,IIIc,IV (8,11)

(f) II2c,IIIa,IV (10,12)

Class 19. Semiplanes with 32 points and $\rho_{12} = 48$.

(a) II3a,IIIa,IV

(b) II3a,IIIb,IV (7)

(c) II3a,IIIe,IV (6)

(d) II3a,IIIf,IV (12)

(e) II3b,IIIc,IV (3)

(f) II3b,IIId,IV (9)

(g) II3b,IIIe,IV (5)

(h) II3b,IIIf,IV (11)

(i) II3c,IIIa,IV (2)

(j) II3c,IIIb,IV (8)

(k) II3c,IIIc,IV (4)

(l) II3c,IIId,IV (10)

Class 20. Semiplane with 25 points and $\rho_{12} = 42$.

(a) I1,I2,I3,IV

Class 21. Semiplanes with 28 points and $\rho_{12} = 45$.

(a) II1a,II1b,II1c,IV

(b) II2a,II2b,II2c,IV (7,8,9,10,11,12)

Class 22. Semiplanes with 31 points and $\rho_{12} = 48$.

(a) IIIa,IIIc,IIIe,IV

(b) IIIb,IIId,IIIf,IV (7,8,9,10,11,12)

Class 23. Semiplanes with 31 points and $\rho_{12} = 48$.

(a) IIIa,IIIc,IIIf,IV

(b) IIIa,IIId,IIIe,IV (2,5)

(c) IIIa,IIId,IIIf,IV (10,12)

(d) IIIb,IIIc,IIIe,IV (4,6)

(e) IIIb,IIIc,IIIf,IV (8,11)

(f) IIIb,IIId,IIIe,IV (7,9)

Class 24. Semiplanes with 26 points and $\rho_{12} = 43$.

(a) I1,I2,II1a,IV

(b) I1,I2,II2a,IV (7,9)

(c) I1,I3,II1b,IV (2,5)

(d) I1,I3,II2b,IV (8,11)

(e) I2,I3,II1c,IV (4,6)

(f) I2,I3,II2c,IV (10,12)

Class 25. Semiplanes with 27 points and $\rho_{12} = 44$.

(a) I1,I2,IIIe,IV

(b) I1,I2,IIIf,IV (7,9)

(c) I1,I3,IIIc,IV (2,5)

(d) I1,I3,IIId,IV (8,11)

(e) I2,I3,IIIa,IV (4,6)

(f) I2,I3,IIIb,IV (10,12)

Class 26. Semiplanes with 28 points and $\rho_{12} = 45$.

(a) II1a,II1b,II3a,IV

(b) II1a,II1b,II3c,IV (2)

(c) II1a,II1c,II3b,IV (3)

(d) II1a,II1c,II3c,IV (4)

(e) II1b,II1c,II3a,IV (6)

(f) II1b,II1c,II3b,IV (5)

(g) II2a,II2b,II3a,IV (7)

(h) II2a,II2b,II3c,IV (8)

(i) II2a,II2c,II3b,IV (9)

(j) II2a,II2c,II3c,IV (10)

(k) II2b,II2c,II3a,IV (12)

(l) II2b,II2c,II3b,IV (11)

Class 27. Semiplanes with 28 points and $\rho_{12} = 45$.

(a) II1a,II1b,II3b,IV

(b) II1a,II1c,II3a,IV (3,4)

(c) II1b,II1c,II3c,IV (5,6)

(d) II2a,II2b,II3b,IV (7,8)

(e) II2a,II2c,II3a,IV (9,10)

(f) II2b,II2c,II3c,IV (11,12)

Class 28. Semiplanes with 29 points and $\rho_{12} = 46$.

(a) II1a,II1b,IIIa,IV

(b) II1a,II1c,IIIc,IV (3,4)

(c) II1b,II1c,IIIe,IV (5,6)

(d) II2a,II2b,IIIb,IV (7,8)

(e) II2a,II2c,IIId,IV (9,10)

(f) II2b,II2c,IIIf,IV (11,12)

Class 29. Semiplanes with 29 points and $\rho_{12} = 46$.

(a) II1a,II1b,IIIb,IV

(b) II1a,II1c,IIId,IV (3,4)

(c) II1b,II1c,IIIf,IV (5,6)

(d) II2a,II2b,IIIa,IV (7,8)

(e) II2a,II2c,IIIc,IV (9,10)

(f) II2b,II2c,IIIe,IV (11,12)

Class 30. Semiplanes with 29 points and $\rho_{12} = 46$.

(a) II1a,II1b,IIId,IV

(b) II1a,II1b,IIIf,IV (2)

(c) II1a,II1c,IIIb,IV (3)

(d) II1a,II1c,IIIf,IV (4)

(e) II1b,II1c,IIIb,IV (5)

(f) II1b,II1c,IIId,IV (6)

(g) II2a,II2b,IIIc,IV (7)

(h) II2a,II2b,IIIe,IV (8)

(i) II2a,II2c,IIIa,IV (9)

(j) II2a,II2c,IIIe,IV (10)

(k) II2b,II2c,IIIa,IV (11)

(l) II2b,II2c,IIIc,IV (12)

Class 31. Semiplanes with 27 points and $\rho_{12} = 44$.

(a) I1,II1a,II1b,IV

(b) I1,II2a,II2b,IV (7,8)

(c) I2,II1a,II1c,IV (3,4)

(d) I2,II2a,II2c,IV (9,10)

(e) I3,II1b,II1c,IV (5,6)

(f) I3,II2b,II2c,IV (11,12)

Class 32. Semiplanes with 29 points and $\rho_{12} = 46$.

(a) I1,IIIc,IIIe,IV

(b) I1,IIId,IIIf,IV (7,8)

(c) I2,IIIa,IIIe,IV (3,4)

(d) I2,IIIb,IIIf,IV (9,10)

(e) I3,IIIa,IIIc,IV (5,6)

(f) I3,IIIb,IIId,IV (11,12)

Class 33. Semiplanes with 29 and $\rho_{12} = 46$.

(a) I1,IIIc,IIIf,IV

(b) I1,IIIe,IIId,IV (2,7)

(c) I2,IIIa,IIIf,IV (3,10)

(d) I2,IIIb,IIIe,IV (4,9)

(e) I3,IIIa,IIId,IV (5,12)

(f) I3,IIIb,IIIc,IV (6,11)

Class 34. Semiplanes with 30 points and $\rho_{12} - 45$.

(a) II1a,IIIa,IIIc,IV

(b) II1b,IIIa,IIIe,IV (2,5)

(c) II1c,IIIc,IIIe,IV (4,6)

(d) II2a,IIIb,IIId,IV (7,9)

(e) II2b,IIIb,IIIf,IV (8,11)

(f) II2c,IIId,IIIf,IV (10,12)

Class 35. Semiplanes with 30 points and $\rho_{12} = 47$.

(a) II1a,IIIa,IIId,IV

(b) II1a,IIIb,IIIc,IV (3)

(c) II1b,IIIa,IIIf,IV (2)

(d) II1b,IIIb,IIIe,IV (5)

(e) II1c,IIIc,IIIf,IV (4)

(f) II1c,IIId,IIIe,IV (6)

(g) II2a,IIIa,IIId,IV (9)

(h) II2a,IIIb,IIIc,IV (7)

(i) II2b,IIIa,IIIf,IV (11)

(j) II2b,IIIb,IIIe,IV (8)

(k) II2c,IIIc,IIIf,IV (12)

(l) II2c,IIId,IIIe,IV (10)

Class 36. Semiplanes with 30 points and $\rho_{12} = 47$.

(a) II1a,IIIa,IIIf,IV

(b) II1a,IIIc,IIIf,IV (3)

(c) II1b,IIIa,IIId,IV (2)

(d) II1b,IIId,IIIe,IV (5)

(e) II1c,IIIb,IIIc,IV (4)

(f) II1c,IIIb,IIIe,IV (6)

(g) II2a,IIIb,IIIe,IV (7)

(h) II2a,IIId,IIIe,IV (9)

(i) II2b,IIIb,IIIc,IV (8)

(j) II2b,IIIc,IIIf,IV (11)

(k) II2c,IIIa,IIId,IV (10)

(l) II2c,IIIa,IIIf,IV (12)

Class 37. Semiplanes with 30 points and $\rho_{12} = 47$.

(a) II1a,IIIb,IIId,IV

(b) II1b,IIIb,IIIf,IV (2,5)

(c) II1c,IIId,IIIf,IV (4,6)

(d) II2a,IIIa,IIIc,IV (7,9)

(e) II2b,IIIa,IIIe,IV (8,11)

(f) II2c,IIIc,IIIe,IV (10,12)

Class 38. Semiplanes with 30 points and $\rho_{12} = 47$.

(a) II1a,IIIb,IIIf,IV

(b) II1a,IIId,IIIf,IV (3)

(c) II1b,IIIb,IIId,IV (2)

(d) II1b,IIId,IIIf,IV (5)

(e) II1c,IIIb,IIId,IV (4)

(f) II1c,IIIb,IIIf,IV (6)

(g) II2a,IIIa,IIIe,IV (7)

(h) II2a,IIIc,IIIe,IV (9)

(i) II2b,IIIa,IIIc,IV (8)

(j) II2b,IIIc,IIIe,IV (11)

(k) II2c,IIIa,IIIc,IV (10)

(l) II2c,IIIa,IIIe,IV (12)

Class 39. Semiplanes with 30 points and $\rho_{12} = 47$.

(a) II3a,IIIa,IIIe,IV

(b) II3a,IIIb,IIIf,IV (7,12)

(c) II3b,IIIc,IIIe,IV (3,5)

(d) II3b,IIId,IIIf,IV (9,11)

(e) II3c,IIIa,IIIc,IV (2,4)

(f) II3c,IIIb,IIId,IV (8,10)

Class 40. Semiplanes with 30 points and $\rho_{12} = 47$.

(a) II3a,IIIa,IIIf,IV

(b) II3a,IIIb,IIIe,IV (6,7)

(c) II3b,IIIc,IIIf,IV (3,11)

(d) II3b,IIId,IIIe,IV (5,9)

(e) II3c,IIIa,IIId,IV (2,10)

(f) II3c,IIIb,IIIc,IV (4,8)

Class 41. Semiplanes with 27 points and $\rho_{12} = 44$.

(a) I1,II1a,II2a,IV

(b) I1,II1b,II2b,IV (2,8)

(c) I2,II1a,II2a,IV (3,9)

(d) I2,II1c,II2c,IV (4,10)

(e) I3,II1b,II2b,IV (5,11)

(f) I3,II1c,II2c,IV (6,12)

Class 42. Semiplanes with 27 points and $\rho_{12} = 44$.

(a) I1,II1a,II3b,IV

(b) I1,II1b,II3b,IV (2)

(c) I1,II2a,II3b,IV (7)

(d) I1,II2b,II3b,IV (8)

(e) I2,II1a,II3a,IV (3)

(f) I2,II1c,II3a,IV (4)

(g) I2,II2a,II3a,IV (9)

(h) I2,II2c,II3a,IV (10)

(i) I3,II1b,II3c,IV (5)

(j) I3,II1c,II3c,IV (6)

(k) I3,II2b,II3c,IV (11)

(l) I3,II2c,II3c,IV (12)

Class 43. Semiplanes with 28 points and $\rho_{12} = 45$.

(a) I1,II1a,IIIc,IV

(b) I1,II1b,IIIe,IV (2)

(c) I1,II2a,IIId,IV (7)

(d) I1,II2b,IIIf,IV (8)

(e) I2,II1a,IIIa,IV (3)

(f) I2,II1c,IIIe,IV (4)

(g) I2,II2a,IIIb,IV (9)

(h) I2,II2c,IIIf,IV (10)

(i) I3,II1b,IIIa,IV (5)

(j) I3,II1c,IIIc,IV (6)

(k) I3,II2b,IIIb,IV (11)

(l) I3,II2c,IIId,IV (12)

Class 44. Semiplanes with 28 points and $\rho_{12} = 45$.

(a) I1,II1a,IIId,IV

(b) I1,II1b,IIIf,IV (2)

(c) I1,II2a,IIIc,IV (7)

(d) I1,II2b,IIIe,IV (8)

(e) I2,II1a,IIIb,IV (3)

(f) I2,II1c,IIIf,IV (4)

(g) I2,II2a,IIIa,IV (9)

(h) I2,II2c,IIIe,IV (10)

(i) I3,II1b,IIIb,IV (5)

(j) I3,II1c,IIId,IV (6)

(k) I3,II2b,IIIa,IV (11)

(l) I3,II2c,IIIc,IV (12)

Class 45. Semiplanes with 28 points and $\rho_{12} = 45$.

(a) I1,II1a,IIIf,IV

(b) I1,II1b,IIId,IV (2)

(c) I1,II2a,IIIe,IV (7)

(d) I1,II2b,IIIc,IV (8)

(e) I2,II1a,IIIf,IV (3)

(f) I2,II1c,IIIb,IV (4)

(g) I2,II2a,IIIe,IV (9)

(h) I2,II2c,IIIa,IV (10)

(i) I3,II1b,IIId,IV (5)

(j) I3,II1c,IIIb,IV (6)

(k) I3,II2b,IIIc,IV (11)

(l) I3,II2c,IIIa,IV (12)

Class 46. Semiplanes with 28 points and $\rho_{12} = 45$.

(a) I1,II3b,IIIc,IV

(b) I1,II3b,IIId,IV (7)

(c) I1,II3b,IIIe,IV (2)

(d) I1,II3b,IIIf,IV (8)

(e) I2,II3a,IIIa,IV (3)

(f) I2,II3a,IIIb,IV (9)

(g) I2,II3a,IIIe,IV (4)

(h) I2,II3a,IIIf,IV (10)

(i) I3,II3c,IIIa,IV (5)

(j) I3,II3c,IIIb,IV (11)

(k) I3,II3c,IIIc,IV (6)

(l) I3,II3c,IIId,IV (12)

Class 47. Semiplanes with 28 points and $\rho_{12} = 45$.

(a) II1a,II2a,II3a,IV

(b) II1a,II2a,II3b,IV (3,9)

(c) II1b,II2b,II3b,IV (5,11)

(d) II1b,II2b,II3c,IV (2,8)

(e) II1c,II2c,II3a,IV (6,12)

(f) II1c,II2c,II3c,IV (4,10)

Class 48. Semiplanes with 28 points and $\rho_{12} = 45$.

(a) II1a,II2a,II3c,IV

(b) II1b,II2b,II3a,IV (2,5,8,11)

(c) II1c,II2c,II3b,IV (4,6,10,12)

Class 49. Semiplanes with 29 points and $\rho_{12} = 46$.

(a) II1a,II2a,IIIa,IV

(b) II1a,II2a,IIIb,IV (7)

(c) II1a,II2a,IIIc,IV (3)

(d) II1a,II2a,IIId,IV (9)

(e) II1b,II2b,IIIa,IV (2)

(f) II1b,II2b,IIIb,IV (8)

(g) II1b,II2b,IIIe,IV (5)

(h) II1b,II2b,IIIf,IV (11)

(i) II1c,II2c,IIIc,IV (4)

(j) II1c,II2c,IIId,IV (10)

(k) II1c,II2c,IIIe,IV (6)

(l) II1c,II2c,IIIf,IV (12)

Class 50. Semiplanes with 29 points and $\rho_{12} = 46$.

(a) II1a,II3a,IIIa,IV

(b) II1a,II3b,IIIc,IV (3)

(c) II1b,II3b,IIIe,IV (5)

(d) II1b,II3c,IIIa,IV (2)

(e) II1c,II3a,IIIe,IV (6)

(f) II1c,II3c,IIIc,IV (4)

(g) II2a,II3a,IIIb,IV (7)

(h) II2a,II3b,IIId,IV (9)

(i) II2b,II3b,IIIf,IV (11)

(j) II2b,II3c,IIIb,IV (8)

(k) II2c,II3a,IIIf,IV (12)

(l) II2c,II3c,IIId,IV (10)

Class 51. Semiplanes with 29 points and $\rho_{12} = 46$.

(a) II1a,II3a,IIIb,IV

(b) II1a,II3b,IIId,IV (3)

(c) II1b,II3b,IIIf,IV (5)

(d) II1b,II3c,IIIb,IV (2)

(e) II1c,II3a,IIIf,IV (6)

(f) II1c,II3c,IIId,IV (4)

(g) II2a,II3a,IIIa,IV (7)

(h) II2a,II3b,IIIc,IV (9)

(i) II2b,II3b,IIIe,IV (11)

(j) II2b,II3c,IIIa,IV (8)

(k) II2c,II3a,IIIe,IV (12)

(l) II2c,II3c,IIIc,IV (10)

Class 52. Semiplanes with 29 points and $\rho_{12} = 46$.

(a) II1a,II3a,IIIf,IV

(b) II1a,II3b,IIIf,IV (3)

(c) II1b,II3b,IIId,IV (5)

(d) II1b,II3c,IIId,IV (2)

(e) II1c,II3a,IIIb,IV (6)

(f) II1c,II3c,IIIb,IV (4)

(g) II2a,II3a,IIIe,IV (7)

(h) II2a,II3b,IIIe,IV (9)

(i) II2b,II3b,IIIc,IV (11)

(j) II2b,II3c,IIIc,IV (8)

(k) II2c,II3a,IIIa,IV (12)

(l) II2c,II3c,IIIa,IV (10)

Class 53. Semiplanes with 29 points and $\rho_{12} = 46$.

(a) II1a,II3c,IIIa,IV

(b) II1a,II3c,IIIc,IV (3)

(c) II1b,II3a,IIIa,IV (2)

(d) II1b,II3a,IIIe,IV (5)

(e) II1c,II3b,IIIc,IV (4)

(f) II1c,II3b,IIIe,IV (6)

(g) II2a,II3c,IIIb,IV (7)

(h) II2a,II3c,IIId,IV (9)

(i) II2b,II3a,IIIb,IV (8)

(j) II2b,II3a,IIIf,IV (11)

(k) II2c,II3b,IIId,IV (10)

(l) II2c,II3b,IIIf,IV (12)

Class 54. Semiplanes with 29 points and $\rho_{12} = 46$.

(a) II1a,II3c,IIIb,IV

(b) II1a,II3c,IIId,IV (3)

(c) II1b,II3a,IIIb,IV (2)

(d) II1b,II3a,IIIf,IV (5)

(e) II1c,II3b,IIId,IV (4)

(f) II1c,II3b,IIIf,IV (6)

(g) II2a,II3c,IIIa,IV (7)

(h) II2a,II3c,IIIc,IV (9)

(i) II2b,II3a,IIIa,IV (8)

(j) II2b,II3a,IIIe,IV (11)

(k) II2c,II3b,IIIc,IV (10)

(l) II2c,II3b,IIIe,IV (12)

Class 55. Semiplanes with 25 points and $\rho_{12} = 47$.

(a) II1a,II1b,II1c,II3a,IV

(b) II1a,II1b,II1c,II3b,IV (3,5)

(c) II1a,II1b,II1c,II3c,IV (2,4)

(d) II2a,II2b,II2c,II3a,IV (7,12)

(e) II2a,II2b,II2c,II3b,IV (9,11)

(f) II2a,II2b,II2c,II3c,IV (8,10)

Class 56. Semiplanes with 26 points and $\rho_{12} = 44$.

(a) II1a,II1b,II1c,IIIb,IV

(b) II1a,II1b,II1c,IIId,IV (3,4)

(c) II1a,II1b,II1c,IIIf,IV (5,6)

(d) II2a,II2b,II2c,IIIa,IV (7,8)

(e) II2a,II2b,II2c,IIIc,IV (9,10)

(f) II2a,II2b,II2c,IIIe,IV (11,12)

Class 57. Semiplanes with 28 points and $\rho_{12} = 46$.

(a) II1a,IIIa,IIIc,IIIf,IV

(b) II1b,IIIa,IIId,IIIe,IV (2,5)

(c) II1c,IIIb,IIIc,IIIe,IV (4,6)

(d) II2a,IIIb,IIId,IIIe,IV (7,9)

(e) II2b,IIIb,IIIc,IIIf,IV (8,11)

(f) II2c,IIIa,IIId,IIIf,IV (10,12)

Class 58. Semiplanes with 28 points and $\rho_{12} = 46$.

(a) II1a,IIIa,IIId,IIIf,IV

(b) II1a,IIIb,IIIc,IIIf,IV (3)

(c) II1b,IIIa,IIId,IIIf,IV (2)

(d) II1b,IIIb,IIId,IIIe,IV (5)

(e) II1c,IIIb,IIIc,IIIf,IV (4)

(f) II1c,IIIb,IIId,IIIe,IV (6)

(g) II2a,IIIa,IIId,IIIe,IV (9)

(h) II2a,IIIb,IIIc,IIIe,IV (7)

(i) II2b,IIIa,IIIc,IIIf,IV (11)

(j) II2b,IIIb,IIIc,IIIe,IV (8)

(k) II2c,IIIa,IIIc,IIIf,IV (12)

(l) II2c,IIIa,IIId,IIIe,IV (10)

Class 59. Semiplanes with 28 points and $\rho_{12} = 46$.

(a) II1a,IIIb,IIId,IIIf,IV

(b) II1b,IIIb,IIId,IIIf,IV (2,5)

(c) II1c,IIIb,IIId,IIIf,IV (4,6)

(d) II2a,IIIa,IIIc,IIIe,IV (7,9)

(e) II2b,IIIa,IIIc,IIIe,IV (8,11)

(f) II2c,IIIa,IIIc,IIIe,IV (10,12)

Class 60. Semiplanes with 27 points and $\rho_{12} = 45$.

(a) II1a,II1b,IIIa,IIId,IV

(b) II1a,II1b,IIIa,IIIf,IV (2)

(c) II1a,II1c,IIIb,IIIc,IV (3)

(d) II1a,II1c,IIIc,IIIf,IV (4)

(e) II1b,II1c,IIIb,IIIe,IV (5)

(f) II1b,II1c,IIId,IIIe,IV (6)

(g) II2a,II2b,IIIb,IIIc,IV (7)

(h) II2a,II2b,IIIb,IIIe,IV (8)

(i) II2a,II2c,IIIa,IIId,IV (9)

(j) II2a,II2c,IIId,IIIe,IV (10)

(k) II2b,II2c,IIIa,IIIf,IV (11)

(l) II2b,II2c,IIIc,IIIf,IV (12)

Class 61. Semiplanes with 27 points and $\rho_{12} = 45$.

(a) II1a,II1b,IIIb,IIId,IV

(b) II1a,II1b,IIIb,IIIf,IV (2)

(c) II1a,II1c,IIIb,IIId,IV (3)

(d) II1a,II1c,IIId,IIIf,IV (4)

(e) II1b,II1c,IIIb,IIIf,IV (5)

(f) II1b,II1c,IIId,IIIf,IV (6)

(g) II2a,II2b,IIIa,IIIc,IV (7)

(h) II2a,II2b,IIIa,IIIe,IV (8)

(i) II2a,II2c,IIIa,IIIc,IV (9)

(j) II2a,II2c,IIIc,IIIe,IV (10)

(k) II2b,II2c,IIIa,IIIe,IV (11)

(l) II2b,II2c,IIIc,IIIe,IV (12)

Class 62. Semiplanes with 27 points and $\rho_{12} = 45$.

(a) II1a,II1b,IIId,IIIf,IV

(b) II1a,II1c,IIIb,IIIf,IV (3,4)

(c) II1b,II1c,IIIb,IIId,IV (5,6)

(d) II2a,II2b,IIIc,IIIe,IV (7,8)

(e) II2a,II2c,IIIa,IIIe,IV (9,10)

(f) II2b,II2c,IIIa,IIIc,IV (11,12)

Class 63. Semiplanes with 23 points and $\rho_{12} = 41$.

(a) I1,I2,II1a,II2a,IV

(b) I1,I3,II1b,II2b,IV (2,5,8,11)

(c) I2,I3,II1c,II2c,IV (4,6,10,12)

Class 64. Semiplanes with 24 points and $\rho_{12} = 42$.

(a) I1,I2,II1a,IIIf,IV

(b) I1,I3,II1b,IIId,IV (2,5)

(c) I2,I3,II1c,IIIb,IV (4,6)

(d) I1,I2,II2a,IIIe,IV (7,9)

(e) I1,I3,II2b,IIIc,IV (8,11)

(f) I2,I3,II2c,IIIa,IV (10,12)

Class 65. Semiplanes with 26 points and $\rho_{12} = 44$.

(a) II1a,II1b,II3a,IIIa,IV

(b) II1a,II1b,II3c,IIIa,IV (2)

(c) II1a,II1c,II3b,IIIc,IV (3)

(d) II1a,II1c,II3c,IIIc,IV (4)

(e) II1b,II1c,II3a,IIIe,IV (6)

(f) II1b,II1c,II3b,IIIe,IV (5)

(g) II2a,II2b,II3a,IIIb,IV (7)

(h) II2a,II2b,II3c,IIIb,IV (8)

(i) II2a,II2c,II3b,IIId,IV (9)

(j) II2a,II2c,II3c,IIId,IV (10)

(k) II2b,II2c,II3a,IIIf,IV (12)

(l) II2b,II2c,II3b,IIIf,IV (11)

Class 66. Semiplanes with 26 points and $\rho_{12} = 44$.

(a) II1a,II1b,II3a,IIIb,IV

(b) II1a,II1b,II3c,IIIb,IV (2)

(c) II1a,II1c,II3b,IIId,IV (3)

(d) II1a,II1c,II3c,IIId,IV (4)

(e) II1b,II1c,II3a,IIIf,IV (6)

(f) II1b,II1c,II3b,IIIf,IV (5)

(g) II2a,II2b,II3a,IIIa,IV (7)

(h) II2a,II2b,II3c,IIIa,IV (8)

(i) II2a,II2c,II3b,IIIc,IV (9)

(j) II2a,II2c,II3c,IIIc,IV (10)

(k) II2b,II2c,II3a,IIIe,IV (12)

(l) II2b,II2c,II3b,IIIe,IV (11)

Class 67. Semiplanes with 26 points and $\rho_{12} = 44$.

(a) II1a,II1b,II3a,IIIf,IV

(b) II1a,II1b,II3c,IIId,IV (2)

(c) II1a,II1c,II3b,IIIf,IV (3)

(d) II1a,II1c,II3c,IIIb,IV (4)

(e) II1b,II1c,II3a,IIIb,IV (6)

(f) II1b,II1c,II3b,IIId,IV (5)

(g) II2a,II2b,II3a,IIIe,IV (7)

(h) II2a,II2b,II3c,IIIc,IV (8)

(i) II2a,II2c,II3b,IIIe,IV (9)

(j) II2a,II2c,II3c,IIIa,IV (10)

(k) II2b,II2c,II3a,IIIa,IV (12)

(l) II2b,II2c,II3b,IIIc,IV (11)

Class 68. Semiplanes with 26 points and $\rho_{12} = 44$.

(a) II1a,II1b,II3b,IIId,IV

(b) II1a,II1b,II3b,IIIf,IV (2)

(c) II1a,II1c,II3a,IIIb,IV (3)

(d) II1a,II1c,II3a,IIIf,IV (4)

(e) II1b,II1c,II3c,IIIb,IV (5)

(f) II1b,II1c,II3c,IIId,IV (6)

(g) II2a,II2b,II3b,IIIc,IV (7)

(h) II2a,II2b,II3b,IIIe,IV (8)

(i) II2a,II2c,II3a,IIIa,IV (9)

(j) II2a,II2c,II3a,IIIe,IV (10)

(k) II2b,II2c,II3c,IIIa,IV (11)

(l) II2b,II2c,II3c,IIIc,IV (12)

Class 69. Semiplanes with 24 points and $\rho_{12} = 42$.

(a) I1,II1a,II1b,II3b,IV

(b) I2,II1a,II1c,II3a,IV (3,4)

(c) I3,II1b,II1c,II3c,IV (5,6)

(d) I1,II2a,II2b,II3b,IV (7,8)

(e) I2,II2a,II2c,II3a,IV (9,10)

(f) I3,II2b,II2c,II3c,IV (11,12)

Class 70. Semiplanes with 25 points and $\rho_{12} = 43$.

(a) I1,II1a,II1b,IIId,IV

(b) I1,II1a,II1b,IIIf,IV (2)

(c) I2,II1a,II1c,IIIb,IV (3)

(d) I2,II1a,II1c,IIIf,IV (4)

(e) I3,II1b,II1c,IIIb,IV (5)

(f) I3,II1b,II1c,IIId,IV (6)

(g) I1,II2a,II2b,IIIc,IV (7)

(h) I1,II2a,II2b,IIIe,IV (8)

(i) I2,II2a,II2c,IIIa,IV (9)

(j) I2,II2a,II2c,IIIe,IV (10)

(k) I3,II2b,II2c,IIIa,IV (11)

(l) I3,II2b,II2c,IIIc,IV (12)

Class 71. Semiplanes with 26 points and $\rho_{12} = 44$.

(a) I1,II1a,IIIc,IIIf,IV

(b) I1,II1b,IIId,IIId,IV (2)

(c) I2,II1a,IIIa,IIIf,IV (3)

(d) I2,II1c,IIIb,IIIe,IV (4)

(e) I3,II1b,IIIa,IIId,IV (5)

(f) I3,II1c,IIIb,IIIc,IV (6)

(g) I1,II2a,IIId,IIIe,IV (7)

(h) I1,II2b,IIIc,IIIf,IV (8)

(i) I2,II2a,IIIb,IIIe,IV (9)

(j) I2,II2c,IIIa,IIIf,IV (10)

(k) I3,II2b,IIIb,IIIc,IV (11)

(l) I3,II2c,IIIa,IIId,IV (12)

Class 72. Semiplanes with 26 points and $\rho_{12} = 44$.

(a) I1,II1a,IIId,IIIf,IV

(b) I1,II1b,IIId,IIIf,IV (2)

(c) I2,II1a,IIIb,IIIf,IV (3)

(d) I2,II1c,IIIb,IIIf,IV (4)

(e) I3,II1b,IIIb,IIId,IV (5)

(f) I3,II1c,IIIb,IIId,IV (6)

(g) I1,II2a,IIIc,IIIe,IV (7)

(h) I1,II2b,IIIc,IIIe,IV (8)

(i) I2,II2a,IIIa,IIIe,IV (9)

(j) I2,II2c,IIIa,IIIe,IV (10)

(k) I3,II2b,IIIa,IIIc,IV (11)

(l) I3,II2c,IIIa,IIIc,IV (12)

Class 73. Semiplanes with 26 points and $\rho_{12} = 44$.

(a) I1,II3b,IIIc,IIIe,IV

(b) I1,II3b,IIId,IIIf,IV (7,8)

(c) I2,II3a,IIIa,IIIe,IV (3,4)

(d) I2,II3a,IIIb,IIIf,IV (9,10)

(e) I3,II3c,IIIa,IIIc,IV (5,6)

(f) I3,II3c,IIIb,IIId,IV (11,12)

Class 74. Semiplanes with 26 points and $\rho_{12} = 44$.

(a) I1,II3b,IIIc,IIIf,IV

(b) I1,II3b,IIId,IIIe,IV (2,7)

(c) I2,II3a,IIIa,IIIf,IV (3,10)

(d) I2,II3a,IIIb,IIIe,IV (4,9)

(e) I3,II3c,IIIa,IIId,IV (5,12)

(f) I3,II3c,IIIb,IIIc,IV (6,11)

Class 75. Semiplanes with 27 points and $\rho_{12} = 45$.

(a) II1a,II2a,IIIa,IIIc,IV

(b) II1a,II2a,IIIb,IIId,IV (7,9)

(c) II1b,II2b,IIIa,IIIe,IV (2,5)

(d) II1b,II2b,IIIb,IIIf,IV (8,11)

(e) II1c,II2c,IIIc,IIIe,IV (4,6)

(f) II1c,II2c,IIId,IIIf,IV (10,12)

Class 76. Semiplanes with 27 points and $\rho_{12} = 45$.

(a) II1a,II2a,IIIa,IIId,IV

(b) II1a,II2a,IIIb,IIIc,IV (3,7)

(c) II1b,II2b,IIIa,IIIf,IV (2,11)

(d) II1b,II2b,IIIb,IIIe,IV (5,8)

(e) II1c,II2c,IIIc,IIIf,IV (4,12)

(f) II1c,II2c,IIId,IIIe,IV (6,10)

Class 77. Semiplanes with 27 points and $\rho_{12} = 45$.

(a) II1a,II3a,IIIa,IIIf,IV

(b) II1a,II3b,IIIc,IIIf,IV (3)

(c) II1b,II3b,IIId,IIIe,IV (5)

(d) II1b,II3c,IIIa,IIId,IV (2)

(e) II1c,II3a,IIIb,IIIe,IV (6)

(f) II1c,II3c,IIIb,IIIc,IV (4)

(g) II2a,II3a,IIIb,IIIe,IV (7)

(h) II2a,II3b,IIId,IIIe,IV (9)

(i) II2b,II3b,IIIc,IIIf,IV (11)

(j) II2b,II3c,IIIb,IIIc,IV (8)

(k) II2c,II3a,IIIa,IIIf,IV (12)

(l) II2c,II3c,IIIa,IIId,IV (10)

Class 78. Semiplanes with 27 points and $\rho_{12} = 45$.

(a) II1a,II3a,IIIb,IIIf,IV

(b) II1a,II3b,IIId,IIIf,IV (3)

(c) II1b,II3b,IIId,IIIf,IV (5)

(d) II1b,II3c,IIIb,IIId,IV (2)

(e) II1c,II3a,IIIb,IIIf,IV (6)

(f) II1c,II3c,IIIb,IIId,IV (4)

(g) II2a,II3a,IIIa,IIIe,IV (7)

(h) II2a,II3b,IIIc,IIIe,IV (9)

(i) II2b,II3b,IIIc,IIIe,IV (11)

(j) II2b,II3c,IIIa,IIIc,IV (8)

(k) II2c,II3a,IIIa,IIIe,IV (12)

(l) II2c,II3c,IIIa,IIIc,IV (10)

Class 79. Semiplanes with 27 points and $\rho_{12} = 45$.

(a) II1a,II3c,IIIa,IIIc,IV

(b) II1b,II3a,IIIa,IIIe,IV (2,5)

(c) II1c,II3b,IIIc,IIIe,IV (4,6)

(d) II2a,II3c,IIIb,IIId,IV (7,9)

(e) II2b,II3a,IIIb,IIIf,IV (8,11)

(f) II2c,II3b,IIId,IIIf,IV (10,12)

Class 80. Semiplanes with 27 points and $\rho_{12} = 45$.

(a) II1a,II3c,IIIa,IIId,IV

(b) II1a,II3c,IIIb,IIIc,IV (3)

(c) II1b,II3a,IIIa,IIIf,IV (2)

(d) II1b,II3a,IIIb,IIIe,IV (5)

(e) II1c,II3b,IIIc,IIIf,IV (4)

(f) II1c,II3b,IIId,IIIe,IV (6)

(g) II2a,II3c,IIIa,IIId,IV (9)

(h) II2a,II3c,IIIb,IIIc,IV (7)

(i) II2b,II3a,IIIa,IIIf,IV (11)

(j) II2b,II3a,IIIb,IIIe,IV (8)

(k) II2c,II3b,IIIc,IIIf,IV (12)

(l) II2c,II3b,IIId,IIIe,IV (10)

Class 81. Semiplanes with 27 points and $\rho_{12} = 45$.

(a) II1a,II3c,IIIb,IIId,IV

(b) II1b,II3a,IIIb,IIIf,IV (2,5)

(c) II1c,II3b,IIId,IIIf,IV (4,6)

(d) II2a,II3c,IIIa,IIIc,IV (7,9)

(e) II2b,II3a,IIIa,IIIe,IV (8,11)

(f) II2c,II3b,IIIc,IIIe,IV (10,12)

Class 82. Semiplanes with 24 points and $\rho_{12} = 42$.

(a) I1,II1a,II2a,II3b,IV

(b) I1,II1b,II2b,II3b,IV (2,8)

(c) I2,II1a,II2a,II3a,IV (3,9)

(d) I2,II1c,II2c,II3a,IV (4,10)

(e) I3,II1b,II2b,II3c,IV (5,11)

(f) I3,II1c,II2c,II3c,IV (6,12)

Class 83. Semiplanes with 25 points and $\rho_{12} = 43$.

(a) I1,II1a,II2a,IIIc,IV

(b) I1,II1a,II2a,IIId,IV (7)

(c) I1,II1b,II2b,IIIe,IV (2)

(d) I1,II1b,II2b,IIIf,IV (8)

(e) I2,II1a,II2a,IIIa,IV (3)

(f) I2,II1a,II2a,IIIb,IV (9)

(g) I2,II1c,II2c,IIIe,IV (4)

(h) I2,II1c,II2c,IIIf,IV (10)

(i) I3,II1b,II2b,IIIa,IV (5)

(j) I3,II1b,II2b,IIIa,IV (11)

(k) I3,II1c,II2c,IIIc,IV (6)

(l) I3,II1c,II2c,IIId,IV (12)

Class 84. Semiplanes with 25 points and $\rho_{12} = 43$.

(a) I1,II1a,II3b,IIIc,IV

(b) I1,II1b,II3b,IIIe,IV (2)

(c) I1,II2a,II3b,IIId,IV (7)

(d) I1,II2b,II3b,IIIf,IV (8)

(e) I2,II1a,II3a,IIIa,IV (3)

(f) I2,II1c,II3a,IIIe,IV (4)

(g) I2,II2a,II3a,IIIb,IV (9)

(h) I2,II2c,II3a,IIIf,IV (10)

(i) I3,II1b,II3c,IIIa,IV (5)

(j) I3,II1c,II3c,IIIc,IV (6)

(k) I3,II2b,II3c,IIIb,IV (11)

(l) I3,II2c,II3c,IIId,IV (12)

Class 85. Semiplanes with 25 points and $\rho_{12} = 43$.

(a) I1,II1a,II3b,IIId,IV

(b) I1,II1b,II3b,IIIf,IV (2)

(c) I1,II2a,II3b,IIIc,IV (7)

(d) I1,II2b,II3b,IIIe,IV (8)

(e) I2,II1a,II3a,IIIb,IV (3)

(f) I2,II1c,II3a,IIIf,IV (4)

(g) I2,II2a,II3a,IIIa,IV (9)

(h) I2,II2c,II3a,IIIe,IV (10)

(i) I3,II1b,II3c,IIIb,IV (5)

(j) I3,II1c,II3c,IIId,IV (6)

(k) I3,II2b,II3c,IIIa,IV (11)

(l) I3,II2c,II3c,IIIc,IV (12)

Class 86. Semiplanes with 25 points and $\rho_{12} = 43$.

(a) I1,II1a,II3b,IIIf,IV

(b) I1,II1b,II3b,IIId,IV (2)

(c) I1,II2a,II3b,IIIe,IV (7)

(d) I1,II2b,II3b,IIIc,IV (8)

(e) I2,II1a,II3a,IIIf,IV (3)

(f) I2,II1c,II3a,IIIb,IV (4)

(g) I2,II2a,II3a,IIIe,IV (9)

(h) I2,II2c,II3a,IIIa,IV (10)

(i) I3,II1b,II3c,IIId,IV (5)

(j) I3,II1c,II3c,IIIb,IV (6)

(k) I3,II2b,II3c,IIIc,IV (11)

(l) I3,II2c,II3c,IIIa,IV (12)

Class 87. Semiplanes with 26 points and $\rho_{12} = 44$.

(a) II1a,II2a,II3a,IIIa,IV

(b) II1a,II2a,II3a,IIIb,IV (7)

(c) II1a,II2a,II3b,IIIc,IV (3)

(d) II1a,II2a,II3b,IIId,IV (9)

(e) II1b,II2b,II3b,IIIe,IV (5)

(f) II1b,II2b,II3b,IIIf,IV (11)

(g) II1b,II2b,II3c,IIIa,IV (2)

(h) II1b,II2b,II3c,IIIb,IV (8)

(i) II1c,II2c,II3a,IIIe,IV (6)

(j) II1c,II2c,II3a,IIIf,IV (12)

(k) II1c,II2c,II3c,IIIc,IV (4)

(l) II1c,II2c,II3c,IIId,IV (10)

Class 88. Semiplanes with 26 points and $\rho_{12} = 44$.

(a) II1a,II2a,II3c,IIIa,IV

(b) II1a,II2a,II3c,IIIb,IV (7)

(c) II1a,II2a,II3c,IIIc,IV (3)

(d) II1a,II2a,II3c,IIId,IV (9)

(e) II1b,II2b,II3a,IIIa,IV (2)

(f) II1b,II2b,II3a,IIIb,IV (8)

(g) II1b,II2b,II3a,IIIe,IV (5)

(h) II1b,II2b,II3a,IIIf,IV (11)

(i) II1c,II2c,II3b,IIIc,IV (4)

(j) II1c,II2c,II3b,IIId,IV (10)

(k) II1c,II2c,II3b,IIIe,IV (6)

(l) II1c,II2c,II3b,IIIf,IV (12)

Class 89. Semiplanes with 24 points and $\rho_{12} = 43$.

(a) II1a,II1b,II1c,IIIb,IIId,IV

(b) II1a,II1b,II1c,IIIb,IIIf,IV (2,5)

(c) II1a,II1b,II1c,IIId,IIIf,IV (4,6)

(d) II2a,II2b,II2c,IIIa,IIIc,IV (7,9)

(e) II2a,II2b,II2c,IIIa,IIIe,IV (8,11)

(f) II2a,II2b,II2c,IIIc,IIIe,IV (10,12)

Class 90. Semiplanes with 25 points and $\rho_{12} = 44$.

(a) II1a,II1b,IIIa,IIId,IIIf,IV

(b) II1a,II1c,IIIb,IIIc,IIIf,IV (3,4)

(c) II1b,II1c,IIIb,IIId,IIIe,IV (5,6)

(d) II2a,II2b,IIIb,IIIc,IIIe,IV (7,8)

(e) II2a,II2c,IIIa,IIId,IIIe,IV (9,10)

(f) II2b,II2c,IIIa,IIIc,IIIf,IV (11,12)

Class 91. Semiplanes with 25 points and $\rho_{12} = 44$.

(a) II1a,II1b,IIIb,IIId,IIIf,IV

(b) II1a,II1c,IIIb,IIId,IIIf,IV (3,4)

(c) II1b,II1c,IIIb,IIId,IIIf,IV (5,6)

(d) II2a,II2b,IIIa,IIIc,IIIe,IV (7,8)

(e) II2a,II2c,IIIa,IIIc,IIIe,IV (9,10)

(f) II2b,II2c,IIIa,IIIc,IIIe,IV (11,12)

Class 92. Semiplanes with 23 points and $\rho_{12} = 42$.

(a) II1a,II1b,II1c,II3a,IIIb,IV

(b) II1a,II1b,II1c,II3a,IIIf,IV (6)

(c) II1a,II1b,II1c,II3b,IIId,IV (3)

(d) II1a,II1b,II1c,II3b,IIIf,IV (5)

(e) II1a,II1b,II1c,II3c,IIIb,IV (2)

(f) II1a,II1b,II1c,II3c,IIId,IV (4)

(g) II2a,II2b,II2c,II3a,IIIa,IV (7)

(h) II2a,II2b,II2c,II3a,IIIe,IV (12)

(i) II2a,II2b,II2c,II3b,IIIc,IV (9)

(j) II2a,II2b,II2c,II3b,IIIe,IV (11)

(k) II2a,II2b,II2c,II3c,IIIa,IV (8)

(l) II2a,II2b,II2c,II3c,IIIc,IV (10)

Class 93. Semiplanes with 24 points and $\rho_{12} = 43$.

(a) II1a,II1b,II3a,IIIa,IIIf,IV

(b) II1a,II1b,II3c,IIIa,IIId,IV (2)

(c) II1a,II1c,II3b,IIIc,IIIf,IV (3)

(d) II1a,II1c,II3c,IIIb,IIIc,IV (4)

(e) II1b,II1c,II3a,IIIb,IIIe,IV (6)

(f) II1b,II1c,II3b,IIId,IIIe,IV (5)

(g) II2a,II2b,II3a,IIIb,IIIe,IV (7)

(h) II2a,II2b,II3c,IIIb,IIIc,IV (8)

(i) II2a,II2c,II3b,IIId,IIIe,IV (9)

(j) II2a,II2c,II3c,IIIa,IIId,IV (10)

(k) II2b,II2c,II3a,IIIa,IIIf,IV (12)

(l) II2b,II2c,II3b,IIIc,IIIf,IV (11)

Class 94. Semiplanes with 24 points and $\rho_{12} = 43$.

(a) II1a,II1b,II3a,IIIb,IIIf,IV

(b) II1a,II1b,II3c,IIIb,IIId,IV (2)

(c) II1a,II1c,II3b,IIId,IIIf,IV (3)

(d) II1a,II1c,II3c,IIIb,IIId,IV (4)

(e) II1b,II1c,II3a,IIIb,IIIf,IV (6)

(f) II1b,II1c,II3b,IIId,IIIf,IV (5)

(g) II2a,II2b,II3a,IIIa,IIIe,IV (7)

(h) II2a,II2b,II3c,IIIa,IIIc,IV (8)

(i) II2a,II2c,II3b,IIIc,IIIe,IV (9)

(j) II2a,II2c,II3c,IIIa,IIIc,IV (10)

(k) II2b,II2c,II3a,IIIa,IIIe,IV (12)

(l) II2b,II2c,II3b,IIIc,IIIe,IV (11)

Class 95. Semiplanes with 24 points and $\rho_{12} = 43$.

(a) II1a,II1b,II3b,IIId,IIIf,IV

(b) II1a,II1c,II3a,IIIb,IIIf,IV (3,4)

(c) II1b,II1c,II3c,IIIb,IIId,IV (5,6)

(d) II2a,II2b,II3b,IIIc,IIIe,IV (7,8)

(e) II2a,II2c,II3a,IIIa,IIIe,IV (9,10)

(f) II2b,II2c,II3c,IIIa,IIIc,IV (11,12)

Class 96. Semiplanes with 23 points and $\rho_{12} = 42$.

(a) I1,II1a,II1b,IIId,IIIf,IV

(b) I2,II1a,II1c,IIIb,IIIf,IV (3,4)

(c) I3,II1b,II1c,IIIb,IIId,IV (5,6)

(d) I1,II2a,II2b,IIIc,IIIe,IV (7,8)

(e) I2,II2a,II2c,IIIa,IIIe,IV (9,10)

(f) I3,II2b,II2c,IIIa,IIIc,IV (11,12)

Class 97. Semiplanes with 22 points and $\rho_{12} = 41$.

(a) I1,II1a,II1b,II3b,IIId,IV

(b) I1,II1a,II1b,II3b,IIIf,IV (2)

(c) I2,II1a,II1c,II3a,IIIb,IV (3)

(d) I2,II1a,II1c,II3a,IIIf,IV (4)

(e) I3,II1b,II1c,II3c,IIIb,IV (5)

(f) I3,II1b,II1c,II3c,IIId,IV (6)

(g) I1,II2a,II2b,II3b,IIIc,IV (7)

(h) I1,II2a,II2b,II3b,IIIe,IV (8)

(i) I2,II2a,II2c,II3a,IIIa,IV (9)

(j) I2,II2a,II2c,II3a,IIIe,IV (10)

(k) I3,II2b,II2c,II3c,IIIa,IV (11)

(l) I3,II2b,II2c,II3c,IIIc,IV (12)

Class 98. Semiplanes with 23 points and $\rho_{12} = 42$.

(a) I1,II1a,II3b,IIIc,IIIf,IV

(b) I1,II1b,II3b,IIId,IIIe,IV (2)

(c) I1,II2a,II3b,IIId,IIIe,IV (7)

(d) I1,II2b,II3b,IIIc,IIIf,IV (8)

(e) I2,II1a,II3a,IIIa,IIIf,IV (3)

(f) I2,II1c,II3a,IIIb,IIIe,IV (4)

(g) I2,II2a,II3a,IIIb,IIIe,IV (9)

(h) I2,II2c,II3a,IIIa,IIIf,IV (10)

(i) I3,II1b,II3c,IIIa,IIId,IV (5)

(j) I3,II1c,II3c,IIIb,IIIc,IV (6)

(k) I3,II2b,II3c,IIIb,IIIc,IV (11)

(l) I3,II2c,II3c,IIIa,IIId,IV (12)

Class 99. Semiplanes with 23 points and $\rho_{12} = 43$.

(a) I1,II1a,II3b,IIId,IIIf,IV

(b) I1,II1b,II3b,IIId,IIIf,IV (2)

(c) I1,II2a,II3b,IIIc,IIIe,IV (7)

(d) I1,II2b,II3b,IIIc,IIIe,IV (8)

(e) I2,II1a,II3a,IIIb,IIIf,IV (3)

(f) I2,II1c,II3a,IIIb,IIIf,IV (4)

(g) I2,II2a,II3a,IIIa,IIIe,IV (9)

(h) I2,II2c,II3a,IIIa,IIIe,IV (10)

(i) I3,II1b,II3c,IIIb,IIId,IV (5)

(j) I3,II1c,II3c,IIIb,IIId,IV (6)

(k) I3,II2b,II3c,IIIa,IIIc,IV (11)

(l) I3,II2c,II3c,IIIa,IIIc,IV (12)

Class 100. Semiplanes with 24 points and $\rho_{12} = 43$.

(a) II1a,II2a,II3c,IIIa,IIIc,IV

(b) II1a,II2a,II3c,IIIb,IIId,IV (7,9)

(c) II1b,II2b,II3a,IIIa,IIIe,IV (2,5)

(d) II1b,II2b,II3a,IIIb,IIIf,IV (8,11)

(e) II1c,II2c,II3b,IIIc,IIIe,IV (4,6)

(f) II1c,II2c,II3b,IIId,IIIf,IV (10,12)

Class 101. Semiplanes with 24 points and $\rho_{12} = 43$.

(a) II1a,II2a,II3c,IIIa,IIId,IV

(b) II1a,II2a,II3c,IIIb,IIIc,IV (3,7)

(c) II1b,II2b,II3a,IIIa,IIIf,IV (2,11)

(d) II1b,II2b,II3a,IIIb,IIIe,IV (5,8)

(e) II1c,II2c,II3b,IIIc,IIIf,IV (4,12)

(f) II1c,II2c,II3b,IIId,IIIe,IV (6,10)

Class 102. Semiplanes with 22 points and $\rho_{12} = 41$.

(a) I1,II1a,II2a,II3b,IIIc,IV

(b) I1,II1a,II2a,II3b,IIId,IV (7)

(c) I1,II1b,II2b,II3b,IIIe,IV (2)

(d) I1,II1b,II2b,II3b,IIIf,IV (8)

(e) I2,II1a,II2a,II3a,IIIa,IV (3)

(f) I2,II1a,II2a,II3a,IIIb,IV (9)

(g) I2,II1c,II2c,II3a,IIIe,IV (4)

(h) I2,II1c,II2c,II3a,IIIf,IV (10)

(i) I3,II1b,II2b,II3c,IIIa,IV (5)

(j) I3,II1b,II2b,II3c,IIIb,IV (11)

(k) I3,II1c,II2c,II3c,IIIc,IV (6)

(l) I3,II1c,II2c,II3c,IIId,IV (12)

Class 103. Semiplanes with 22 points and $\rho_{12} = 42$.

(a) II1a,II1b,II1c,IIIb,IIId,IIIf,IV

(b) II2a,II2b,II2c,IIIa,IIIc,IIIe,IV (7-12)

Class 104. Semiplanes with 21 points and $\rho_{12} = 41$.

(a) II1a,II1b,II1c,II3a,IIIb,IIIf,IV

(b) II1a,II1b,II1c,II3b,IIId,IIIf,IV (3,5)

(c) II1a,II1b,II1c,II3c,IIIb,IIId,IV (2,4)

(d) II2a,II2b,II2c,II3a,IIIa,IIIe,IV (7,12)

(e) II2a,II2b,II2c,II3b,IIIc,IIIe,IV (9,11)

(f) II2a,II2b,II2c,II3c,IIIa,IIIc,IV (8,10)

Class 105. Semiplanes with 20 points and $\rho_{12} = 40$.

(a) I1,II1a,II1b,II3b,IIId,IIIf,IV

(b) I1,II2a,II2b,II3b,IIIc,IIIe,IV (7,8)

(c) I2,II1a,II1c,II3a,IIIb,IIIf,IV (3,4)

(d) I2,II2a,II2c,II3a,IIIa,IIIe,IV (9,10)

(e) I3,II1b,II1c,II3c,IIIb,IIId,IV (5,6)

(f) I3,II2b,II2c,II3c,IIIa,IIIc,IV (11,12)

Chapter 5
Conclusions

If a finite semiplane can be extended to a projective plane of order n, then it can be extended through a sequence of one line extensions. Dr. Nation observed that it was necessary to keep $\rho_n \geq 0$ and this provides a good criterion for restricting the choices for extensions. The algorithm for one line extensions was implemented to computer programs which can be used to attempt to construct non-Desarguesian projective planes.

The main result of this dissertation is that there are 105 non-isomorphic semiplanes generated by a non-Desarguesian configuration, which serve as the initial configuration to the programs. So far we have done more than a hundred thousand hours of computer search. In all cases, ρ_n decreases to 0 in just a few steps. At the beginning of our computer search, we focused mainly on extending semiplane 105 to a plane of order 12, since it is a minimal case. But after testing all the 105 possibilities, we decided to shift our focus to semiplanes 41, 42 and 86 which seem to produce more good candidates than the others. However, we should properly test our semiplanes more thoroughly in the future. We did work on semiplane 98 for a while, but once we found that it cannot be extended to a Hall plane of order 9 and also its dual, we stopped using that configuration as initial input.

From all the tests that we have done so far on extending non-Desarguesian semiplanes to a plane of order 12, we have more than fourteen hundred cases in which we were able to extend the semiplanes from 12 lines to 49 lines, and more than thirty cases to 50 lines, and one case so far to 51 lines. We did some search on order 11 and 15 as well.

In the construction of all the lower order planes, we observed that once the number of lines exceeded roughly one third of the total required, the extension to a projective plane was completed immediately with only one choice for the extension in all remaining steps. While we do not understand the exact nature of this phenomenon (and hope to work on it in the future), it is clear that we are not necessarily as far from a plane of order 12 as it may appear.

We also checked that there are no embeddings among the 105 isomorphism types. All of them can be extended to a Hall plane of order 9, except five of them, namely semiplanes 89, 90, 95, 96 and 98. These five types cannot be extended to the dual of that Hall plane either, together with two additional types, semiplanes 93 and 99.

For other results on projective plane of order 12, see Hall and Wilkinson [9] and Suetake [15].

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