THREE-DIMENSIONAL ZONE MODEL LOG INTERPRETATION

by

Louis J. Shamey

William M. Adams

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and

REMOTE SENSING OF WATER WITH ELECTROMAGNETIC RADIATION
FROM 10^{-1} TO 10^{+15} HERTZ
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ABSTRACT

To facilitate the interpretation of geophysical electrical well-logging data, the apparent resistivity has been calculated in a model in which the experimental situation is approximated by three coaxial cylindrical zones containing the drilling mud, flushed rock, and surrounding rock matrix, respectively. This three-zone model is shown to yield results that differ by as much as 25 percent from the earlier two-zone model calculations.

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INTRODUCTION

A previous report (Shamey and Adams, 1971) dealt with the interpretation of electrical resistivity logs in a cylindrically symmetric geometry in the two-zone coaxial case, *i.e.*, a model calculation in which the physical prototype was approximated by the measuring sonde situated on the axis of an infinite cylindrical well filled with drilling mud and surrounded by a homogeneous, isotropic rock matrix. The present work extends and builds on the previous report in an attempt to make resistivity calculations based on a more realistic model. A cylindrical zone is introduced between the previous two zones to represent the rock matrix flushed with drilling mud. The resistivity of this intermediate zone ranges between those of the inner and outer zones. The diameter of this flushed rock zone can be more than an order of magnitude larger than that of the borehole, so it is appropriate to incorporate this intermediate zone in the model.

The previous report should be read for background information about the basic problem and for a discussion of the basic mathematical and numerical processes involved. The work of Dakhnov (1959) is closely followed.

THREE-ZONE COAXIAL MODEL: THEORY

The physical well-logging problem is approximated by the following model. A cylindrical borehole, of radius r_o, contains drilling mud of resistivity ρ_o. Situated between this well and the rock matrix of resistivity ρ_p is an intermediate cylindrical zone of radius r_i and containing flushed rock of resistivity ρ_i. All zones are considered to be homogeneous and isotropic and to be of infinite vertical extent. The measuring sonde is located on the central axis and is assumed to have negligible radius. This geometry is shown in Figure 1.

Cylindrical coordinates (r, z, ϕ) are used, with r and z reduced by r_o, the radius of the inner cylindrical zone. The origin of the coordinate system is located at the current source (electrode A on the sonde), and the z axis is coaxial with the axis of the well. (See Shamey and Adams, 1971, for further discussion of these background details.) The potential

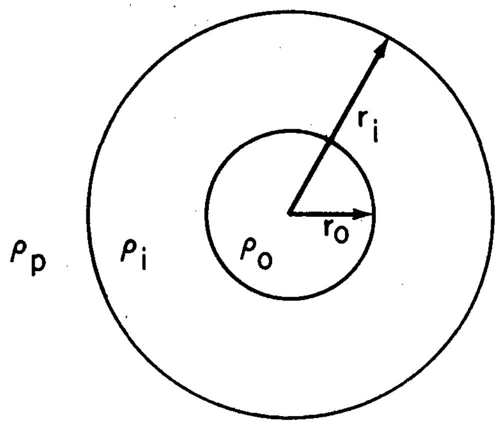

FIGURE 1. GEOMETRY OF THREE-ZONE MODEL (RADIUS r_i HAS BEEN REDUCED BY r_o, AND r_o IS TAKEN TO BE ONE IN THE CALCULATION.

satisfies

$$\nabla^2 U(r,z) = \frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial^2 U}{\partial z^2} = 0 \tag{1}$$

subject to the boundary conditions that U and $\overline{j} = \overline{E}/\rho = -\overline{\nabla}U/\rho$ be continuous at the cylindrical interfaces located at $r = 1$ and $r = r_i$. Using subscripts o, i, and p to denote quantities in the central, intermediate, and outer zones, respectively, the boundary conditions are

$$U_o \Big|_{r=1} = U_i \Big|_{r=1} \tag{2a}$$

$$U_i \Big|_{r=r_i} = U_p \Big|_{r=r_i} \tag{2b}$$

$$\frac{1}{\rho_o}\frac{\partial U_o}{\partial r}\Bigg|_{r=1} = \frac{1}{\rho_i}\frac{\partial U}{\partial r}\Bigg|_{r=1} \tag{2c}$$

$$\frac{1}{\rho_i}\frac{\partial U_i}{\partial r}\Bigg|_{r=r_i} = \frac{1}{\rho_p}\frac{\partial U_p}{\partial r}\Bigg|_{r=r_i} \tag{2d}$$

Also, near the source electrode, the potential must approach its limiting form for a homogeneous, isotropic medium.

$$\lim_{R \to 0} U_o(r,z) = \frac{I\,\rho_a}{4\pi r_o\sqrt{r^2+z^2}} \tag{3}$$

where $R = \sqrt{r^2+z^2}$, I is the current, and ρ_a is the apparent resistivity of the entire heterogeneous medium. The potential must also remain finite as R approaches ∞. Furthermore, there is a basic symmetry that requires that

$$U(r,-z) = U(r,z) . \tag{4}$$

The variables are separated by assuming the solution to be a product of a function of r times a function of z

$$U(r,z) = f(r) \cdot Z(z) . \tag{5}$$

The real separation constant is written m^2 and Laplace's equation (1) becomes

$$\frac{d^2Z(z)}{dz^2} + m^2Z(z) = 0 \tag{6}$$

with solutions sin(mz) and cos(mz), and

$$\frac{d^2f(r)}{dr^2} + \frac{1}{r}\frac{df(r)}{dr} - m^2f(r) = 0 \tag{7}$$

which, upon substitution of x = mr, is the modified Bessel's equation of zero order, with solutions $I_o(mr)$ and $K_o(mr)$.

The general solution to equation (1), with sin(mz) precluded by equation (4), is of the form

$$U(r,z) = \int_0^\infty A(m)\cos(mz)I_o(mr)dm + \int_0^\infty B(m)\cos(mz)K_o(mr)dm \tag{8}$$

where A(m) and B(m) are continuous functions of the variable m to be determined by the boundary conditions.

In the inner zone, $I_o(mr)$ converges to zero as $r \to 0$. However, $K_o(mr)$ diverges as $r \to 0$, in just the proper way to account for the limiting form shown in equation (3). With

$$\int_0^\infty \cos(bt)K_o(at)dt = \frac{\pi/2}{\sqrt{a^2+b^2}} \tag{9}$$

(see Abramowitz and Stegun, 1964, p. 486), the coefficient B(m) is $\rho_o I/(r_o 2\pi^2)$. Therefore, in the inner zone, the potential is

$$U_o(r,z) = \int_0^\infty A_o(m) I_o(mr)\cos(mz)\,dm + \frac{\rho_o I}{r_o 2\pi^2} \int_0^\infty K_o(mr)\cos(mz)\,dm \tag{10}$$

In the intermediate zone, the potential is

$$U_i(r,z) = \int_0^\infty A_i(m) I_o(mr)\cos(mz)\,dm + \int_0^\infty B_i(m) K_o(mr)\cos(mz)\,dm \tag{11}$$

In the outer zone, the coefficient A must be zero because $I_o(mr)$ diverges as $r \to \infty$. The potential is

$$U_p(r,z) = \int_0^\infty B_p(m) K_o(mr)\cos(mz)\,dm \tag{12}$$

In equations (10), (11), and (12) new coefficients are defined by

$$C_o(m) = \frac{2\pi^2 r_o}{\rho_o I} A_o(m) \tag{13a}$$

$$C_i(m) = \frac{2\pi^2 r_o}{\rho_i I} A_i(m) \tag{13b}$$

$$D_i(m) = \frac{2\pi^2 r_o}{\rho_i I} B_i(m) \tag{13c}$$

$$D_p(m) = \frac{2\pi^2 r_o}{\rho_p I} B_p(m) \tag{13d}$$

and the potentials are then written

$$U_o(r,z) = \frac{\rho_o I}{2\pi^2 r_o} \left[\int_0^\infty C_o(m) I_o(mr)\cos(mz)\,dm + \int_0^\infty K_o(mr)\cos(mz)\,dm \right] \tag{14}$$

$$U_i(r,z) = \frac{\rho_i I}{2\pi^2 r_o} \left[\int_0^\infty C_i(m) I_o(mr)\cos(mz)\,dm + \int_0^\infty D_i(m) K_o(mr)\cos(mz)\,dm \right] \tag{15}$$

$$U_p(r,z) = \frac{\rho_p I}{2\pi^2 r_o} \int_0^\infty D_p(m) K_o(mr)\cos(mz)\,dm \tag{16}$$

Boundary conditions (2a) and (2b) are applied. This results in integrations over cos(mz) that are equal to zero; the factors of the integrands that do not depend on z are therefore set equal to zero. These continuity boundary conditions on the potential thus become

$$\rho_o I_o(m) C_o(m) - \rho_i I_o(m) C_i(m) - \rho_i K_o(m) D_i(m) = -\rho_o K_o(m) \tag{17}$$

$$\rho_i I_o(mr_i) C_i(m) + \rho_i K_o(mr_i) D_i(m) - \rho_i K_o(mr_i) D_p(m) = 0 \tag{18}$$

Boundary conditions (2c) and (2d) are applied. In taking derivatives of the potential, the following relations are used: $I_o(x) = I_1(x)$ and $K_o(x) = -K_1(x)$. Also, in the integrals over m that equal zero, for arbitrary values of z, the factors in the integrand, that do not depend on m, are set equal to zero. These continuity boundary conditions on the derivative of the potential thus become

$$I_1(m) C_o(m) - I_1(m) C_i(m) + K_1(m) D_i(m) = K_1(m) \tag{19}$$

$$I_1(mr_i) C_i(m) - K_1(mr_i) D_i(m) + K_1(mr_i) D_p(m) = 0 \tag{20}$$

Equations (17), (18), (19), and (20) are four simultaneous equations for the four unknown functions $C_o(m)$, $C_i(m)$, $D_i(m)$, and $D_p(m)$. For the interpretation of the electrical resistivity logs, only $C_o(m)$ is of interest because the measuring sonde is located on the axis of the inner zone and hence only the potential $U_o(r,z)$ is required. Determinantal methods were used to determine $C_o(m)$ from these four equations.

$$C_o(m) = X/Y \tag{21}$$

where X is the determinant

$$X = \begin{vmatrix} -\rho_o K_o(m) & -\rho_i I_o(m) & -\rho_i K_o(m) & 0 \\ 0 & \rho_i I_o(mr_i) & \rho_i K_o(mr_i) & -\rho_p K_o(mr_i) \\ K_1(m) & -I_1(m) & K_1(m) & 0 \\ 0 & I_1(mr_i) & -K_1(mr_i) & K_1(mr_i) \end{vmatrix} \tag{22}$$

and where Y is the determinant

$$Y = \begin{vmatrix} \rho_o I_o(m) & -\rho_i I_o(m) & -\rho_i K_o(m) & 0 \\ 0 & \rho_i I_o(mr_i) & \rho_i K_o(mr_i) & -\rho_p K_o(mr_i) \\ I_1(m) & -I_1(m) & K_1(m) & 0 \\ 0 & I_1(mr_i) & -K_1(m_i) & K_1(mr_i) \end{vmatrix} \qquad (23)$$

Further analytical reduction of the function $C_o(m)$ was not necessary because the digital computer performed the determinantal algebra in equations (22) and (23) and evaluated the function from equation (21) in the actual numerical stage of the calculation.

THREE-ZONE COAXIAL MODEL: NUMERICAL CALCULATIONS

The calculation of apparent resistivities is analogous to that of the two-zone coaxial case discussed in Shamey and Adams (1971). The presence of the intermediate zone merely adds more complexity (and two new parameters ρ_i and r_i) to the expression for $C_o(m)$; with this one difference the subsequent development is directly comparable to the previous work. For a normal array,

$$\rho_a = \rho_o \left[1 + \frac{2L'}{\pi} \int_0^\infty C_o(m)\cos(mL')dm \right] \qquad (24)$$

and for the lateral array

$$\rho_a = \rho_o \left[1 + \frac{2L'^2}{\pi} \int_0^\infty C_o(m)\sin(mL')mdm \right] \qquad (25)$$

where $L' = L/r_o$, the reduced electrode spacing parameter. The methods of numerically evaluating equations (24) and (25), as well as a discussion of error bounds, are contained in the previous report.

The apparent resistivities for the three-zone coaxial problem depend on six parameters: ρ_p, ρ_i, ρ_o, L, D_i, and D_o. Resistivities are reduced by ρ_o and lengths are reduced by r_o; these two parameters can therefore be set equal to unity ($\rho_o = 1$, $D_o = 2r_o = 2$) without loss of generality. There remain four independently variable parameters that enter into the calculation of ρ_a.

Preliminary calculations, with limited variance of parameters, were performed to gain insight into the dependence of ρ_a upon D_i, ρ_i, and L. Table 1 shows ρ_a for a normal array, with ρ_p fixed at 10. D_i ranges over values of 5, 10, and 20; ρ_i ranges over values of $\rho_p/3$, $2\rho_p/3$, and ρ_p; and L/D_o ranges over values of 1, 3, 10, and 30. Note that when $\rho_i = \rho_p$ this problem is equivalent to a two-zone model and that ρ_a is therefore independent of D_i. Table 2 shows ρ_a for a more limited range of parameters for both normal and lateral arrays, with ρ_p fixed at 1000.

TABLE 1. APPARENT RESISTIVITY ρ_a FOR NORMAL ARRAY IN THREE-ZONE COAXIAL CASE FOR $\rho_p = 10$.

ZONE	ρ_i \ L/D_o	1.0	3.0	10.0	30.0
	$\rho_p/3$	6.203	10.30	11.20	10.28
$D_i = 5.$	$2\rho_p/3$	7.446	11.24	11.00	10.19
	ρ_p	8.435	12.27	11.09	10.17
	$\rho_p/3$	4.865	8.175	10.99	10.55
$D_i = 10.$	$2\rho_p/3$	6.826	10.22	10.87	10.25
	ρ_p	8.435	12.27	11.09	10.17
	$\rho_p/3$	4.044	6.244	9.634	10.89
$D_i = 20.$	$2\rho_p/3$	6.419	9.257	10.31	10.35
	ρ_p	8.435	12.27	11.09	10.17

TABLE 2. APPARENT RESISTIVITY ρ_a FOR NORMAL ARRAY (LATERAL ARRAY) IN THREE-ZONE COAXIAL CASE FOR ρ_p = 1000.

ZONE	ρ_i \ L/D_o	10.0		30.0	
D_i = 5.	$\rho_p/3$	1225.	(466.3)	1836.	(1732.)
	$2\rho_p/3$	1292.	(482.5)	1952.	(1839.)
D_i = 10.	$\rho_p/3$	1114.	(435.0)	1662.	(1554.)
	$2\rho_p/3$	1242.	(471.2)	1863.	(1759.)
D_i = 20.	$\rho_p/3$	990.9	(395.6)	1489.	(1345.)
	$2\rho_p/3$	1188.	(459.3)	1774.	(1672.)

The presence of the intermediate zone, with resistivity between ρ_o and ρ_p, causes ρ_a to have relative variations by as much as 25 percent as D_i and/or ρ_i vary over the ranges indicated by Tables 1 and 2. These three-zone apparent resistivities may be compared with the corresponding results of a two-zone calculation for the same values of ρ_a and L/D_o; this comparison may be performed either by taking the limiting case $\rho_i \rightarrow \rho_p$ in the present work or by taking the results of Shamey and Adams (1971). The three-zone resistivity ρ_a is usually lower, typically by 5-10 percent but sometimes by as much as 25 percent, than the two-zone ρ_a. The errors caused by the two-zone model's neglect of an intermediate zone are therefore usually small but can be significant for some sets of parameters.

If realistic estimates of ρ_i and D_i could be obtained by some geophysical method, then a unique value of ρ_p could be determined by inverse interpolation of a table of ρ_a values, by numerical procedures similar to those performed in the two-zone model calculations.

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APPENDIX:

COMPUTER PROGRAM

Program THREEZN, which is listed below, calculates the reduced apparent resistivities ρ_a/ρ_o as a function of the reduced resistivities ρ_p/ρ_o and ρ_i/ρ_o and the reduced lengths L/r_o and r_i/r_o. The program and its flow chart are essentially the same as those of the two-zone calculations; the main exception is the function subprogram for determining $C_o(m)$, which in this case depends also upon the diameter and resistivity of the intermediate zone.

```
      PROGRAM THREEZN (INPUT,OUTPUT,FILMPL,PUNCH)
C  WELL-LOGGING RESISTIVITY CALCULATION WITH 3 CONCENTRIC,CYLINDRICAL ZONES.
   SONDE ON AXIS
     0DIMENSION CO(180)         ,XL(8),YIL(8),Y2L(8),Y3L(8),TL(7),FN(180)
     1,GN(180),COSTAR(180),XG( 6),WG( 6),RMS(180),RMC(180),AC(31),AS(31)
     2,SUM(30),XPLOT(180),SUM2(30)
      REAL  L,I0,I1,K0,K1,LP
     0DATA WORD1,WORD2,WORD3,WORD4/10HNORMAL ARR,10HAY
     210HLATERAL AR,10HRAY          /
      DATA WORD5/10H    3 ZONE   /
      PI = 3.141592653
      LPRINT = 1
      NPOINTS = 180
      NNODES = 30
      READ 4,(XG(I),WG(I),I=1, 6).
    4 FORMAT (2F20.10)
      READ 5,XL
      READ 5,Y1L
      READ 5,Y2L
      READ 5,Y3L
    5 FORMAT (8A10)
      READ 20,NCASES
   20 FORMAT (I10)
      RHOO = 1.0   $   RO = 1.0
      DO 500 NCA=1,NCASES
      READ 5,TL
      READ 21,RHOP,L
   21 FORMAT (E10.3,10X,E10.3 )
      DO 500 IDD =1,3
      DO 500 IRHOPP =2,3
      IF( IDD .EQ. 1)   D = 5.0
      IF( IDD .EQ. 2)   D = 10.0
      IF( IDD .EQ. 3)   D = 20.0
      IF( IRHOPP .EQ. 1 ) RHOPP = 1.0
      IF( IRHOPP .EQ. 2 ) RHOPP = (1./3.)*RHOP
      IF( IRHOPP .EQ. 3 ) RHOPP = (2./3.)*RHOP
      IF( IRHOPP .EQ. 4 ) RHOPP = 1.0 * RHOP
      D = D/RC
      RP = D/2.
```

```
C   ***    ***    ***    ***    ***    ***    ***    ***    ***    ***    ***    ***
C  D IS IN REDUCED UNITS (WITH RESPECT TO RO)
C  RP IS RADIUS OF INTERMEDIATE ZONE, IN REDUCED UNITS.
C  RHOP IS RESISTIVITY OF OUTER ZONE, RHOPP IS RESISTIVITY OF INTERMEDIATE
C  ZONE, RHOO IS RESISTIVITY OF INNER ZONE. D IS DIAMETER OF INTERMEDIATE
C  ZONE, DO=2*RO = 2 IS DIAMETER OF INNER ZONE. L IS ELECTRODE SPACING.
C  D  CANNOT BE LESS THAN 2 IN REDUCED UNITS.
C   ***    ***    ***    ***    ***    ***    ***    ***    ***    ***    ***    ***
      PRINT 22,NCA,RHOP,RHOPP,D,L
   220FORMAT (*1CASE NUMBER * I2/* RESISTIVITY OF OUTER ZONE = *F12.1,10
     1X,*RESISTIVITY OF INTERMEDIATE ZONE = *F12.2/ * DIAMETER OF INTERME-
     2DIATE ZONE = *F10.3,20X,*ELECTRODE SPACING L = *F10.1//)
      RMU = RHOP/RHOO $ RMUP = RHOPP/RHOO
      LP = L/RO
      T = 2.*PI/LP
      AC(1)=0.0 $ AC(2)=T/4. $ AS(1)=0.0 $ AS(2)=T/2.
      NNODES1 = NNODES + 1
      DO 10 I=3,NNODES1
      K = I-1
      AC(I) = AC(K) + T/2.
   10 AS(I) = AS(K) + T/2.
C  CONSTRUCT  6 POINT MESH FOR GAUSSIAN INTEGRATION BETWEEN NODES.
      NSTART = -6
      DO 11 I=1,NNODES
      K = I+1
      NSTART = NSTART + 6
      DO 12 J=1,6
      JI = J + NSTART
      RMS(JI)=(AS(K)-AS(I))*XG(J)/2,0 +(AS(K)+AS(I))/2.0
   12 RMC(JI)=(AC(K)-AC(I))*XG(J)/2.0 +(AC(K)+AC(I))/2.0
   11 CONTINUE
C
      XO = 1.00
      DO 30 I=1,NPOINTS
      X = RMC(I) $ Y = RMS(I)
      CO(I) = CF(RHOP,RHOPP,RHOO,X,RP)
      COSTAR (I) = CO(I) + PHIO(RMU,X,XO)
      GN(I) = CF(RHOP,RHOPP,RHOO,Y,RP)*Y*SIN(Y*LP)
   30 FN(I) = COSTAR(I) * COS(X*LP)
      IF( LPRINT .LT. 2) GO TO 39
      PRINT 31
   31 FORMAT (20X, *VALUES OF FUNCTIONS*/)
      DO 32 I=1,180
   32 PRINT 33,I,RMC(I),CO(I),COSTAR(I),FN(I), RMS(I),GN(I)
   33 FORMAT (5X,I5,5X,4E12.3,10X,2E12.3)
   39 CONTINUE
C  GAUSSIAN INTEGRATION.
      DO 14 I=1,NNODES
      SUM2(I) = 0.0
   14 SUM(I) = 0.0
      SUMTOT = 0.0
      SUMTOT2 = 0.0
      NSTART = -6
```

```
      DO 15 I=1,NNODES
      K = I+1
      NSTART = NSTART + 6
      DO 16 J=1,6
      JI=J + NSTART
      SUM2(I) = SUM2(I) + WG(J)*GN(JI)
   16 SUM(I) = SUM(I) + WG(J)*FN(JI)
      SUM2(I) = SUM2(I) * (AS(K)-AS(I))/2.0
      SUM(I) = SUM(I) * ( AC(K)-AC(I) )/2.0
      SUMTOT = SUMTOT + SUM(I)
      SUMTOT2=SUMTOT2 + SUM2(I)
   15 CONTINUE
  795 CONTINUE
      TEST1 = ABS( SUM(30)-SUM(29))
      TEST2 = ABS( SUM(29)-SUM(28))
      TEST = TEST1
      IF( TEST1 .LE. TEST2) TEST = TEST2
      RATIO = ABS( TEST/SUMTOT)
      IF( RATIO .LE. 5.0E-04) GO TO 800
      PRINT 799,RATIO
  7990FORMAT (* POOR CONVERGENCE, RATIO = *E10.3,4X,*INTEGRATE FOR ANOTHER
     1SET OF 30 HALF-PERIODS.*)
      AC(1)=AC(31)    $  AS(1) = AS(31)
      NNODES1 = NNODES + 1
      DO 710 I=2,NNODES1
      K = I-1
      AC(I) = AC(K) + T/2.
  710 AS(I) = AS(K) + T/2.
      NSTART = - 6
      DO 711 I=1,NNODES
      K = I+1
      NSTART = NSTART + 6
      DO 712 J=1,6
      JI = J + NSTART
      RMS(JI)=(AS(K)-AS(I))*XG(J)/2.0 +(AS(K)+AS(I))/2.0
  712 RMC(JI)=(AC(K)-AC(I))*XG(J)/2.0 +(AC(K)+AC(I))/2.0
  711 CONTINUE
      DO 730 I=1,NPOINTS
      X = RMC(I)    $  Y=RMS(I)
      CO(I) = CF(RHOP,RHOPP,RHOO,X,RP)
      COSTAR(I) = CO(I) + PHIO(RMU,X,XO)
      GN(I) = CF(RHOP,RHOPP,RHOO,Y,RP)*Y*SIN(Y*LP)
  730 FN(I) = COSTAR(I)*COS(X*LP)
C  GAUSSIAN INTEGRATION FOR SECOND SET OF NODES
      DO 714 I=1,NNODES
      SUM2(I) = 0.0
  714 SUM(I) = 0.0
      NSTART = -6
      DO 715 I=1,NNODES
      K = I + 1
      NSTART = NSTART + 6
      DO 716 J=1,6
      JI=J  + NSTART
```

```
      SUM2(I) = SUM2(I) + WG(J)*GN(JI)
  716 SUM(I) = SUM(I) + WG(J)*FN(JI)
      SUM2(I) = SUM2(I) * (AS(K)-AS(I))/2.0
      SUM(I) = SUM(I) * ( AC(K)-AC(I) )/2.0
      SUMTOT = SUMTOT + SUM(I)
      SUMTOT2=SUMTOT2 + SUM2(I)
  715 CONTINUE
      GO TO 795
  800 CONTINUE
      XOLP = X0 * LP
      SUMTOT = SUMTOT + (RMU-1.0)*(SI(XOLP))/LP
      RHOA = RHOO*(1. + 2.*(LP)   *SUMTOT/PI)
      RHOA2=RHOO*(1.0+2.*(LP**2.)*SUMTOT2/PI)
      PRINT 24,RHOA,SUMTOT
   240FORMAT (*OPOTENTIAL SONDE, OR NORMAL ARRAY  -  RHOA = *E12.3,5X,
     1* VALUE OF INTEGRAL = * E12.3/)
      PRINT 25,RHOA2,SUMTOT2
   250FORMAT (*OGRADIENT SONDE, OR LATERAL ARRAY  -  RHOA = *E12.3,5X,
     1* VALUE OF INTEGRAL + * E12.3//)
      PRINT 26
   26 FORMAT (*OCONTRIBUTIONS TO INTEGRAL FROM EACH NODE*/)
      U = 0.0  $ V = 0.0
      DO 27 I=1,NNODES
      U = U + SUM(I) $  V = V + SUM2(I)
      PRINT 28,I,SUM(I),U,SUM2(I),V
   28 FORMAT (2X,I5,5X,2E15.5,15X,2E15.5 )
   27 CONTINUE
      PUNCH 101,RHOA,RHOP,RHOPP,L,D,WORD5,WORD1,WORD2
      PUNCH 101,RHOA2,RHOP,RHOPP,L,D,WORD5,WORD3,WORD4
  101 FORMAT (5E10.3,3A10)
  500 CONTINUE
      END

      FUNCTION CF(X,Y,Z,XM,RP)
C  CALCULATION OF CO(M) FUNCTION FOR THREE CONCENTRIC CYLINDERICAL ZONE CASE.
C  X IS RHOP, Y IS RHOPP, Z IS RHOO, XM IS M, AND RP IS RADIUS OF
C  INTERMEDIATE ZONE (REDUCED WITH RESPECT TO RO).
      REAL K0,K1,I0,I1
      DIMENSION A(4,5),B(4,4)
      DO 5 I=1,4
      DO 5 J=1,5
    5 A(I,J) = 0.0
      DO 6 I=1,4
      DO 6 J=1,4
    6 B(I,J)   0.0
      XMRP = XM*RP
      A(1,1) =  Z*I0(XM)
      A(1,2) = -Y*I0(XM)
      A(1,3) = -Y*K0(X,M)
      A(1,5) = -Z*K0(XM)
      A(2,2) =  Y*I0(XMRP)
      A(2,3) =  Y*K0(XMRP)
      A(2,4) = -X*K0(XMRP)
```

```
      A(3,1) =  I1(XM)
      A(3,2) = -A(3,1)
      A(3,3) =  K1(XM)
      A(3,5) =  A(3,3)
      A(4,2) =  I1(XMRP)
      A(4,3) = -K1(XMRP)
      A(4,4) =  K1(XMRP)
      DO 10 I=1,4
      DO 10 J=1,4
   10 B(I,J) = A(I,J)
      XD = DET4(B)
      DO I1 I=1,4
   11 B(I,1) = A(I,5)
      CF = XN/XD
      RETURN $ END
      FUNCTION DET3(A)
C   CALCULATES THE DETERMINANT OF 3 X 3 MATRIX.
      DIMENSION A(3,3)
      X1 = A(1,1)*(A(2,2)*A(3,3)-A(2,3)*A(3,2))
      X2 = -A(2,1)*(A*1,2)*A(3,3)-A(1,3)*A(3,2))
      X3 = A(3,1)*(A(1,2)*A(2,3)-A(1,3)*A(2,2))
      DET3 = X1 + X2 + X3
      RETURN $ END
      FUNCTION DET4(B)
C   CALCULATES DETERMINANT OF 4 X 4 MATRIX.
      DIMENSION B(4,4),X(4),A(3,3)
      DO 20 I=1,4
      DO 15 J=1,3
      DO 15 K=1,3
      KK = K + 1
      JJ = J
      IF(J .GE. I) JJ = J+1
   15 A(J,K) = B(JJ,KK)
   20 X(I) = B(I,1) * DET3(A)
      DET4 = X(1) - X(2) +X(3) -X(4)
      RETURN $ END
      FUNCTION PHIO(RMU,X,X0)
      PHIO = (RMU - 1.0) * ALOG(X/X0)
      IF( X .GE. X0)   PHIO = 0.0
      RETURN $ END
      SUBROUTINE MAXMIN (F,N,XMAX,XMIN)
      DIMENSION F(N)
      XMAX = F(1) $ XMIN = F(1)
      IF( N .EQ. 1 ) RETURN
      DO 10 I=2,N
      IF( F(I) .GE. XMAX) XMAX = F(I)
   10 IF( F(I) .LE. XMIN) XMIN = F(I)
      RETURN
      END
      REAL FUNCTION K0(X)
      REAL I0
      K0 = 0.
      IF(X.LT.2.) GO TO 120
```

```
      X2 = 2./X
      K0 = 1./SQRT(X)*EXP(-X)*(1.25331414-.07832358*X2 +.02189568*X2**2
     2 -.01062446*X2**3+.00587872*X2**4-.0025154*X2**5+.00053208*X2**6)
      RETURN
  120 IF(X.LT.0.) RETURN
      T = X/3.75
      X2 = X/2.
      I0=1.+3.5156229*T**2+3.0899424*T**4+1.2067492*T**6+.2659732*T**8
     2  +.0360768*T**10 +.0045813*T**12
      K0=-ALOG(X2)*I0-.57721566+.42278420*X2**2+.2306976*X2**4+
     2 .0348859*X2**6+.00262698*X2**8+.0001075*X2**10+.0000074*X2**12
      RETURN
      END

      REAL FUNCTION K1(X)
      REAL I1
      K1 = 0.
      IF(X.LT.2.) GO TO 130
      X2 = 2./X
      K1 = 1./SQRT(X)*EXP(-X)*(1.25331414+.2349861*X2-.0365562*X2**2 +
     2 .01504268*X2**3-.00780353*X2**4+.00325614*X2**5-.00068245*X2**6)
      RETURN
  130 IF (X.LT.0.) RETURN
      T = X/3.75
      X2 = X/2.
      I1 = X*(.5+.87890594*T**2+.51498869*T**4+.15084934*T**6 +
     2 .02658733*T**8+.00301532*T**10+.00032411*T**12)
      K1 = I1*ALOG(X2)+1./X*(1.+.15443144*X2**2-.67278579*X2**4-
     2 .18156897*X2**6-.0191940*X2**8-.00110404*X2))10-.00004686*X2**12)
      RETURN
      END

      FUNCTION I0(X)
      REAL I0
      DIMENSION EI(3)
      CALL BESSI(X,EI)
      I0 = EI(1)
      RETURN $ END
      FUNCTION I1(X)
      REAL I1
      DIMENSION EI(3)
      CALL BESSI(X,EI)
      I1 = EI(2)
      RETURN $ END
      SUBROUTINE BESSI (X,CKE,EI)
      DIMENSION FIRST (4),EI(3),COEF(4),CKE(3),A(10,4)
      DATA (A =  0.42278420,   .23069756,   .03488590,
     1            .00262698,   .00010750,   .00000740, 3(0.0), 6.0,
     2            .15443144,  -.67278579,  -.18156897,
     3           -.01919402,  -.00110404,  -.00004686, 3(0.0), 6.0,
     4           -.07832358,   .02189568,  -.01062446,
     5            .00587872,  -.00251540,   .00053208, 3(0.0), 6.0,
     6            .23498619,  -.03655620,   .01504268,
     7           -.00780353,   .00325614,  -.00068245, 3(0.0), 6.0  )
```

```
      CALL BESSI (X,EI)
      IF(X .LT. 2.0) 10,20
   10 T = X /2.0
      XP = ALOG(T)
      FIRST(1) = -XP * EI(1) - 0.57721566
      FIRST(2) =  X * XP * EI(2) + 1.0
      FACTOR = T * T
      COEF(1) = 1.0
      COEF(2) = 1.0 / X
      JJ = 1
      GO TO 50
   20 T = 2.0 / X
      FIRST(3) = FIRST(4) = 1.25331414
      JJ = 3
      COEF(3) = COEF(4) = 1.0 / (SQT (X) * EXP (X) )
      FACTOR = T
   50 JEND = JJ + 1
      I = 0
      DO 200 J = JJ,JEND
      I = I + 1
      PROD = 1.0
      SUM = 0.0
      KEND = A(10,J)
      DO 100 K = 1,KEND
      PROD = PROD * FACTOR
      SUM = SUM + PROD * A(K,J)
  100 CONTINUE
      CKE(I) = COEF(J) * (FIRST(J) + SUM)
  200 CONTINUE
      CKE(3) = (2.0/X) * CKE(2) + CKE(1)
      RETURN
      END

      SUBROUTINE BESSI (X,EI)
      DIMENSION A(10,4),FIRST(4),COEF(4),EI(3)
      DATA(FIRST = 1.0,0.5,2(0.39894228)),
     1    (A =  3.5156229, 3.0899424, 1.2067492,
     2           .2659732,  .0360768,  .0045813, 3(0.0), 6.0,
     3           .87890594,  .51498869,  .15084934,
     4           .02658733,  .00301532,  .00032411, 3(0.0), 6.0,
     5           .01328592,  .00225319, -.00157565,
     6           .00916281, -.02057706,  .02635537,
     7          -.01647633,  .00392377,                0.0,  8.0,
     8          -.03988024, -.00362018,  .00163801,
     9          -.01031555,  .02282967, -.02895312,
     1           .01787654, -.00420059,                0.0 , 8.0  )
      T = X / 3.75
      COEF(1) = 1.0
      COEF(2) = X
      COEF(3) = COEF(4) = EXP(X) / SQRT (X)
      IF(X .LT. 3.75) 10,20
   10 FACTOR = T * T
      JJ = 1
```

```
      GO TO 50
   20 FACTOR = 1.0 / T
      JJ = 3
   50 JEND = JJ + 1
      I = 0
      DO 200 J = JJ,JEND
      I = I + 1
      PROD = 1.0
      SUM = 0.0
      KEND = A(10,J)
      DO 100 K = 1,KEND
      PROD = PROD * FACTOR
      SUM = SUM + PROD * A(K,J)
  100 CONTINUE
      EI(1) = COEF(J) * (FIRST(J) + SUM)
  200 CONTINUE
      EI(3) = (-2.0/X) * EI(2) + EI(1)
      RETURN
      END
      FUNCTION SI(X)
C  CALCULATION OF THE SINE INTEGRAL FUNCTION.
      DIMENSION XG(10),WG(10),Z(10),F(10)
C  USE 10 POINT GAUSSIAN INTEGRATION BETWEEN NODES OF (SIN(Z))/Z.
      XG(6) = 0.148874338981631  $ WG(6) = 0.295524224714753
      XG(7) = 0.433395394129247  $ WG(7) = 0.269266719309996
      XG(8) = 0.679409568299024  $ WG(8) = 0.219086362515982
      XG(9) = 0.865063366688985  $ WG(9) = 0.149451349150581
      XG(10)= 0.973906528517172  $ WG(10)= 0.066671344308688
      XG(5) = -XG(6)      $  WG(5) = WG(6)
      XG(4) = -XG(7)      $  WG(4) = WG(7)
      XG(3) = -XG(8)      $  WG(3) = WG(8)
      XG(2) = -XG(9)      $  WG(2) = WG(9)
      XG(1) = -XG(10)     $  WG(1) = WG(10)
      PI = 3.141592653589
      T = PI
      A = -T  $  B = 0.0
      SI = 0.0
      ITEST = 0
   10 A = A + T  $  B = B + T
      IF( X .LE. B) ITEST = 1
      BP = B
      IF( ITEST .EQ. 1 ) BP = X
      DO 15 I=1,10
      Z(I) = (BP-A)*XG(I)/2.0 + (BP+A)/2.0
   15 F(I) = ( SIN(Z(I)))/Z(I)
      SUM = 0.0
      DO 16 I=1,10
   16 SUM = SUM + F(I) * WG(I)
      SUM = SUM * (BP-A)/2.0
      SI = SI + SUM
      IF ( ITEST .EQ. 0 ) GO TO 10
      RETURN $ END
```