# A PRESENTATION OF TWO FAMILIES OF UNIFORMLY BOUNDED REPRESENTATIONS OF CAT(0)-CUBICAL GROUPS AND AN EXAMPLE FROM HYPERBOLIC GEOMETRY

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# Abstract

Geometric group theory is a branch of mathematics in which we explore the characteristics of finitely-generated groups by letting the group act on a particular space and by analyzing the connections between the group's algebraic properties and the geometric and topological properties of the spaces being acted upon.

In the last half of the 20th century, harmonic analysis on a free group was extensively studied and Hilbert space representations of the free group were an integral tool in this research. In 1986, T. Pytlik and R. Szwarc [15] constructed a particularly useful family of uniformly bounded representations of the free group $\mathbb{F}$ acting (by translation) on $\ell^2(\mathbb{F})$. In this dissertation we will extend Pytlik and Szwarc's construction of a holomorphic family of uniformly bounded Hilbert space representations for the free group $\mathbb{F}$ acting on $\ell^2(\mathbb{F})$ to the more general case of a discrete group acting on $\ell^2(X)$, where $X$ is the set of vertices of a CAT(0)-cube complex. We will then show that these representations are identical to another holomorphic family of uniformly bounded Hilbert space representations constructed by E. Guentner and N. Higson using cocycles.

We also examine an example of a discrete group acting on a non-positively curved cube-complex which yields the result that, for every 3-manifold group, there exists a non-positively curved space on which it acts freely.

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# Introduction

## 1.1 Synopsis

Harmonic analysis of the free group has been extensively studied over the last forty to fifty years. In particular, unitary representations are completely understood and several different constructions of holomorphic families of uniformly bounded Hilbert space representations have been explored. In 1986, Pytlik and Szwarc constructed a holomorphic family of uniformly bounded Hilbert space representations for the free group $\mathbb{F}$ acting on $\ell^2(\mathbb{F})$ [15]. Later, Pimsner [14] and Valette [17] constructed a similar holomorphic family of uniformly bounded Hilbert space representations of a discrete group $G$ acting on $\ell^2(X)$, where $X$ is the set of vertices of a simplicial tree. In the first chapter of this dissertation, we will present these constructions, extend the construction of Pytlik and Szwarc to the more general tree case and reprove that the construction of Pytlik and Szwarc and that of Pimsner and Vallette are identical in this slightly generalized setting. We will also prove this for discrete groups that may not be transitive.

Erik Guenter and Nigel Higson extended the construction of Pimsner [14] and Valette [18] for discrete groups acting on trees to construct a holomorphic family of uniformly bounded Hilbert space representations of discrete groups acting on a finite-dimensional CAT(0)-cube complex [9]. In Chapter 2, we will define a CAT(0)-cube complex and in Chapter 3, we will present the construction of Guentner and Higson for CAT(0)-cubical groups and add some new results about this construction. We will then extend the construction of Pytlik and Swarc [15] for free groups to discrete groups acting on a finite-dimensional CAT(0)-cube complex and prove that this new family of representations is identical to that constructed by Guentner and Higson. Finally, in Chapter 4, we will present an interesting example that establishes a new result, that every 3-manifold group acts freely on a non-positively curved space.

## 1.2 History

Geometric group theory is a relatively new area of study in Mathematics. It grew out of combinatorial group theory and began to be recognized as a distinct topic in the 1980's, particularly with the publication of Gromov's "Hyperbolic Groups" [8] in 1987. In general, geometric group theory is a method used to explore the characteristics of finitely-generated groups by letting the group act on a particular space and by analyzing the connections between the group's algebraic properties and the geometric and topological properties of the spaces being acted upon. Often the space being acted upon is the Cayley graph of the group itself, with the word metric giving the metric space structure. In this dissertation, we will extend this to examine a discrete group $G$ acting on the vertex set of a finite-dimensional CAT(0)-cube complex.

In the last half of the 20th century, harmonic analysis on a free group was extensively studied and Hilbert space representations of the free group were an integral tool in this research. In 1986, Pytlik and Szwarc [15] constructed a particularly useful family of uniformly bounded representations of the free group $\mathbb{F}$ acting (by translation) on $\ell^2(\mathbb{F})$. In 1990, Valette [17] constructed another family of representations, using an entirely different (and arguably simpler) method of Pimsner [14], in the more general setting of a discrete group acting on the vertices of a simplicial tree. Valette [17] then showed that this second family is identical to the family constructed earlier by Pytlik and Szwarc in the case of the free group acting on its Cayley graph by translation.

Trees are simple examples of CAT(0)-cube complexes and it might seem natural to wonder if we might be able to generalize these constructions in this direction. In 2007, Guentner and Higson [9] succeeded in generalizing the work of Pimnser and Valette, using analogous methods, to construct a holomorphic family of uniformly bounded representations of a discrete group that admits an action on a finite-dimensional CAT(0)-cube complex. Moreover, when applied to a tree, these representations are exactly those of Pimsner and Valette. In this dissertation, we will generalize the construction of Pytlik and Szwarc to CAT(0)-cubical groups and prove that, as in the simpler tree case, the family of representations constructed is identical to the family constructed using cocycles by Guentner and Higson. We will also demonstrate an interesting example of a CAT(0)-cubical group and use it to establish a new result concerning 3-manifold groups.

## 1.3 The constructions for the free group

It is much easier to understand the constructions for a CAT(0)-cubical group if one first understands the constructions for the free group, particularly as the former are generalizations of the latter. There is a slight difference in the literature that must be addressed (however, this difference is easily rectifiable at a later stage). In the Pytlik-Szwarc construction [15], the free group $\mathbb{F}$ is considered to be acting on its Cayley graph, essentially by translation. In the cocycle construction of Pimsner and Valette [14,18], the authors initially consider a more general setting, that of a topological group acting on a tree. In the later works comparing these constructions, the topological group is restricted to the free group and the tree is restricted to its Cayley graph. We will generalize these comparisons in this chapter to a discrete group acting on a tree.

### 1.3.1 The cocycle construction

As mentioned, the cocycle construction of Pimsner and Valette [14,18] is more general, examining groups acting on simplicial trees. Let $G$ be a discrete group and let $X$ be the set of vertices of a simplicial tree. The natural distance $d(x, y)$ on $X$ is the length of the shortest edge-path between $x$ and $y$, where each edge is of length 1. An action of $G$ on $X$ will be an action $G \times X \to X : (g, x) \mapsto gx$, that preserves the distance $d$ on $X$ by isometries. We may say equivalently that $G$ acts by tree automorphisms.

Suppose that $\pi : G \to \mathcal{B}(\mathcal{H})$ is some unitary representation of the group $G$ on the Hilbert space $\mathcal{H}$. A function $c : X \times X \to \mathcal{B}(\mathcal{H})$ is a *cocycle on $X$ for $\pi$* if it satisfies the following conditions for all $x, y, z \in X$ and all $g \in G$:

(1) $c(x, x) = 1$;

(2) $c(x, y)c(y, z) = c(x, z)$;

(3) $c(gx, gy) = \pi(g)c(x, y)\pi(g)^{-1}$.

In our case, $\mathcal{H}$ will be $\ell^2(X)$ where $X$ is the set of vertices of a simplicial tree on which a discrete group $G$ acts and the representation $\pi$ will be the natural permutation representation of $G$. Cocycles are useful in this setting in that if $c$ is a cocycle for $\pi$ and we fix $x \in X$, then $\pi_c(g) = c(x, gx)\pi(g)$ defines a representation of $G$ into $\mathcal{B}(\ell^2(X))$. Moreover, if $c$ is *uniformly bounded*, that is, if

$$\sup_{x,y \in X} ||c(x, y)|| < \infty,$$

then $\pi_c$ as constructed is uniformly bounded, which is to say that there is a constant $C > 0$ such that $||\pi_c(g)|| < C$ for all $g \in G$. Additionally, if $c(x, y) = c(y, x)^*$ for all $x$ and $y$, that is, if $c(x, y)$ is *unitary*, then the representation $\pi_c$ is unitary.

Let $z \in \mathbb{D} = \{z : |z| < 1\}$ and define $w = \sqrt{1 - z^2}$, where we use the principal branch of the square root (and note that this will be standard throughout this dissertation). Let $\pi(g)$ be the natural permutation representation of $G$ on $\ell^2(X)$. For all $v \in X$, let $\delta_v$ denote the characteristic function of the one element set $\{v\}$. Let $x, y \in X$ be the vertices of an edge. Define $c_z(x,y) \in \mathcal{B}(\ell^2(X))$ by

$$c_z(x,y)\delta_v = \begin{cases} w\delta_x - z\delta_y, & \text{if } v = x, \\ w\delta_y + z\delta_x, & \text{if } v = y, \\ \delta_v, & \text{otherwise.} \end{cases}$$

It is perhaps easier to consider $c_z(x,y)$ as the matrix

$$\begin{pmatrix} w & z \\ -z & w \end{pmatrix},$$

on the two dimensional subspace spanned by the ordered basis $\{\delta_x, \delta_y\}$, and as the identity on the orthogonal complement of this subspace.

**Lemma 1.3.1.** *Let $x$, $y \in X$ be the vertices of an edge. Then $c_z(y,x) = c_z(x,y)^{-1}$.*

*Proof.* Let $x$, $y \in X$ be the vertices of an edge. Then we have

$$c_z(y,x)\delta_v = \begin{cases} w\delta_y - z\delta_x, & \text{if } v = y, \\ w\delta_x + z\delta_y, & \text{if } v = x, \\ \delta_v, & \text{otherwise,} \end{cases}$$

and on the ordered basis $\{\delta_x, \delta_y\}$, we may consider $c_z(y,x)$ as the matrix

$$\begin{pmatrix} w & -z \\ z & w \end{pmatrix},$$

and the identity on the orthogonal complement of this subspace. As each operator is the identity on this orthogonal complement, and as on the subspace itself

$$c_z(x,y)c_z(y,x) = \begin{pmatrix} w & z \\ -z & w \end{pmatrix}\begin{pmatrix} w & -z \\ z & w \end{pmatrix} = \begin{pmatrix} w^2+z^2 & -zw+zw \\ -zw+zw & z^2+w^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

we have that $c_z(y,x) = c_z(x,y)^{-1}$. $\square$

Note also that if $z$ is a real number, $c_z(x,y) = c_z(y,x)^*$, that is, $c_z(x,y)$ is unitary. It is now possible to define the cocycle $c_z(x,y) \in \mathcal{B}(\ell^2(X))$ for arbitrary $x$, $y \in X$. If $x$ and $y$ are arbitrary vertices in the tree, define

$$c_z(x,y) = c_z(v_0,v_1)c_z(v_1,v_2)\cdots c_z(v_{n-1},v_n),$$

where $x = v_0, v_1, \ldots, v_n = y$ are the vertices along *any* edge-path from $x$ to $y$. Any path from $x$ to $y$ can be reduced to the geodesic from $x$ to $y$ by edge cancellation as trees have no cycles and $c_z(u,v) = c_z(v,u)^{-1}$ for every $u$ and $v$ joined by an edge. Thus this definition of $c_z(x,y)$ is independent of path and $c_z(x,y)$ is well-defined.

**Lemma 1.3.2.** *The operator $c_z(x,y)$, as defined above, is a cocycle for $\pi$.*

*Proof.* The first two cocycle properties are clear. We will show that $c_z(gx,gy) = \pi(g)c_z(x,y)\pi(g)^{-1}$ for $x,y \in X$ and $g \in G$. If $x = y$, then $c_z(x,y) = 1 = c_z(gx,gy)$, and the result is obvious. If $x$ and $y$ are the vertices of an edge then, by the distance preserving properties of the group action, $gx$ and $gy$ are also the vertices of an edge and we must consider three possibilities.

Let $b \in X$. If $b \neq x,y$, then $gb$ is neither $gx$ nor $gy$, and $c_z(gx,gy)\pi(g)\delta_b = c_z(gx,gy)\delta_{gb} = \delta_{gb}$ and $\pi(g)c_z(x,y)\delta_b = \pi(g)\delta_b = \delta_{gb}$. If $b = y$, then $\pi(g)c_z(x,y)\delta_y = \pi(g)(w\delta_y + z\delta_x) = w\delta_{gy} + z\delta_{gx}$ and $c_z(gx,gy)\pi(g)\delta_y = c_z(gx,gy)\delta_{gy} = w\delta_{gy} + z\delta_{gx}$. If $b = x$, the calculations are similarly simple. For adjacent vertices $x$, $y \in X$, we then have $\pi(g)^{-1}c_z(gx,gy)\pi(g) = c_z(x,y)$.

Now let $x$, $y \in X$ be arbitrary vertices. Label the vertices of any path from $x$ to $y$

by $x = v_0, v_1, \ldots, v_n = y$. As the group action of $G$ preserves distance, we have that $gx = gv_0, gv_1, \ldots, gv_n = gy$ is a path from $gx$ to $gy$ and

$$\begin{aligned}
\pi(g)^{-1}c_z(gx, gy)\pi(g) &= \pi(g)^{-1}\left(\prod_{i=1}^{n} c_z(gv_{i-1}, gv_i)\right)\pi(g) \\
&= \prod_{i=1}^{n} \pi(g)^{-1}c_z(gv_{i-1}, gv_i)\pi(g) \\
&= \prod_{i=1}^{n} c_z(v_{i-1}, v_i) \\
&= c_z(x, y).
\end{aligned}$$

□

In order to define the representation, we must fix a vertex $x \in X$. The family of representations is then $\{\pi_z : z \in \mathbb{D}\}$ where $\pi_z(g) = c_z(x, gx)\pi(g)$. Valette [17] then shows that the representation just constructed is uniformly bounded by reference to the uniformly bounded representation constructed by Pytlik and Szwarc [15] (outlined below). In Guentner and Higson's paper on CAT(0)-cubical groups [9], they show by direct methods that the cocycle construction for trees is uniformly bounded. We will need the CAT(0)-cubical group analogue of these direct methods later in Chapter 3.

### 1.3.2 The Pytlik-Szwarc construction

In their construction [15], Pytlik and Szwarc do not initially require the free group to have finitely many generators. However, we will restrict ourselves to this case. Let $\mathbb{F}$ be a free group with a fixed finite set of free generators $E$. The association between the free group and the set of reduced words consisting of elements of $E \cup E^{-1}$, with no adjacent factors $aa^{-1}$ or $a^{-1}a$ ($a \in E$), is well known and carefully presented in [5]. Define the *length* of $x$ to be the number of letters in the reduced word associated with $x$, with the provision that $|e| = 0$, where $e$ is the identity element of $\mathbb{F}$. Define $\overline{x}$ to be the element of $\mathbb{F}$ obtained from

$x$ by deleting the last letter and $\delta_x$ to be the characteristic function of the one point set $\{x\}$. Finally define $c_c(\mathbb{F})$ to be the space of all complex functions on $\mathbb{F}$ with finite support, that is, $c_c(\mathbb{F})$ is the space that consists of all linear combinations of $\delta_x$, $x \in \mathbb{F}$. These will later serve as a basis for $\ell^2(\mathbb{F})$.

As the vertices of the Cayley graph of $\mathbb{F}$ are the group elements, we must use some unusual notation. If the group element we are considering is to be used to make the translation operator $\pi(g)$, we will use the standard $g$ and $h$ for the group elements. However, if we are considering the group element as a vertex of the Cayley graph we will use $x$, $y$, $u$ and $v$. This will cease to be an issue after Chapter 1.

For $g \in \mathbb{F}$, let $\pi(g)$ be the translation operator defined by $\pi(g)(\alpha(x)) = \alpha(g^{-1}x)$ where $\alpha$ is a complex function on $\mathbb{F}$. Let $P : c_c(\mathbb{F}) \to c_c(\mathbb{F})$ be the linear operator defined by $P\delta_x = \delta_{\overline{x}}$ for $x \neq e$ and $P\delta_e = 0$.

**Lemma 1.3.3.** *The linear operator $I - zP$ is invertible with inverse $I + zP + z^2P^2 + \dots$*

*Proof.* Let $f \in c_c(\mathbb{F})$. Then $f = \sum_{i=1}^{k} z_i \delta_{x_i}$ for some $x_i \in \mathbb{F}$, $1 \leq i \leq k$. Let $n = \max\{|x_i| : 1 \leq i \leq k\}$. Then $P^m f = 0$ for all integers $m > n$ so that

$$\begin{aligned} f &= (I - z^{n+1}P^{n+1})f \\ &= (I - zP)(I + zP + z^2P^2 + \cdots + z^nP^n)f \\ &= (I - zP)(I + zP + z^2P^2 + \dots)f. \end{aligned}$$

$\square$

Note that the infinite sum $I + zP + z^2P^2 + \dots$ is finite when applied to any element of $c_c(\mathbb{F})$. Define the representation $\pi_z^o$ of $\mathbb{F}$ on $c_c(\mathbb{F})$ by

$$\pi_z^o(g) = (I - zP)^{-1}\pi(g)(I - zP),$$

that is, the conjugation of the left regular representation by the linear operator $I - zP$. Pytlik and Szwarc then show that $\pi_z^o$ extends uniquely to a uniformly bounded representation of $\mathbb{F}$ on $\ell^2(\mathbb{F})$ and that the family of representations is holomorphic on $\{z : |z| < 1\}$. The construction is not yet complete, however. Although $\{\pi_z^o : |z| < 1\}$ is now a holomorphic family of uniformly bounded representations of $\mathbb{F}$ on $\ell^2(\mathbb{F})$, the authors improve these representations to get a new class of representations with some useful properties as follows.

Let $T : \ell^2(\mathbb{F}) \to \ell^2(\mathbb{F})$ be the orthogonal projection onto the one-dimensional subspace $\{z\delta_e : z \in \mathbb{C}\}$. For $|z| < 1$, define the linear operator $T_z : \ell^2(\mathbb{F}) \to \ell^2(\mathbb{F})$ by $T_z = I - T + wT$ where $w = \sqrt{1 - z^2}$ and we again use the principal branch of the square root. In particular

$$T_z \delta_x = \begin{cases} \delta_x, & \text{if } x \neq e, \\ w\delta_x, & \text{if } x = e. \end{cases}$$

It is clear that $T_z$ is bounded and a simple calculation yields that $T_z^{-1} = I - T + \frac{1}{w}T$. Define the representation $\pi_z$ of $\mathbb{F}$ on $\ell^2(\mathbb{F})$ by

$$\pi_z = T_z^{-1}(I - zP)^{-1}\pi(g)(I - zP)T_z.$$

The improved holomorphic family of uniformly bounded representations is then given by $\{\pi_z : |z| < 1\}$.

### 1.3.3 The first two constructions are identical for discrete groups acting transitively on trees

It has been shown that the first two constructions are identical on the free group [17]. However, it is more interesting to extend the Pytlik/Szwarc construction for the free group [15] to discrete groups acting transitively on a simplicial tree and then show that this construction is identical to the cocycle construction. Let $X$ be the set of vertices of a tree on which a discrete group $G$ acts transitively. As in the cocycle construction, fix a vertex $x$

($x$ being $e$ in the Cayley graph of the free group case above) and define $d(x, y)$ to be the length of the shortest edge-path between $x$ and $y$ where each edge is of length 1. For a vertex $v$, with $v \neq x$, define $\overline{v}$ to be the unique vertex on the geodesic from $v$ to $x$ such that $d(x, \overline{v}) = d(x, v) - 1$ and $d(\overline{v}, v) = 1$, that is, $\overline{v}$ is on the geodesic from $v$ to $x$ but "one closer" to $x$.

Define $P$ to be the operator defined by $P\delta_v = \delta_{\overline{v}}$ if $v \neq x$ and $P\delta_x = 0$. Pytlik and Szwarc's construction and their proof that the representations constructed are uniformly bounded rely only on the geometry of the Cayley graph of the free group as a tree and do not involve its regularity. Let $\{\pi_z^P\}_{z\in\mathbb{D}}$ be the adapted family constructed using the operator $P$ [15] but with the fixed point $x$ instead of the group element $e$.

Let $\{\pi_z^C\}_{z\in\mathbb{D}}$ be the family of uniformly bounded representations constructed using the cocycle method [17] above. The following lemma is adapted from Pytlik and Szwarc [15] and the second is from Guentner and Higson [9].

**Lemma 1.3.4.** *For every $z \in \mathbb{D}$, if $x$ is the fixed vertex used to define the representation $\pi_z^P$, then $\pi_z^P(g)\delta_x = z^{d(x,gx)}\delta_x + \sum_{k=0}^{d(x,gx)-1} z^k w P^k \delta_{gx}$.*

*Proof.* Let $z \in \mathbb{D}$ and $g \in G$.

$$\begin{aligned}
\pi_z^P(g)\delta_x &= T_z^{-1}(I - zP)^{-1}\pi(g)(I - zP)T_z\delta_x \\
&= T_z^{-1}(I - zP)^{-1}\pi(g)(I - zP)w\delta_x \\
&= T_z^{-1}(I - zP)^{-1}\pi(g)w\delta_x \\
&= T_z^{-1}(I - zP)^{-1}w\delta_{gx} \\
&= T_z^{-1}(I + zP + z^2P^2 + \dots)w\delta_{gx} \\
&= T_z^{-1}(w\delta_{gx} + wzP\delta_{gx} + wz^2P^2\delta_{gx} + \dots + wz^{d(x,gx)-1}P^{d(x,gx)-1}\delta_{gx} + wz^{d(x,gx)}\delta_x) \\
&= z^{d(x,gx)}\delta_x + \sum_{k=0}^{d(x,gx)-1} z^k w P^k \delta_{gx}.
\end{aligned}$$

$\square$

**Lemma 1.3.5.** *For every $z \in \mathbb{D}$, if $x$ is the fixed vertex used to define the representation, then $\pi_z^C(g)\delta_x = \pi_z^P(g)\delta_x$.*

*Proof.* Let $z \in \mathbb{D}$ and $g \in G$. Let $x = v_0, v_1, \ldots, v_n = gx$ be the vertices on the geodesic path from $x$ to $gx$.

$$
\begin{aligned}
\pi_z^C(g)\delta_x &= c_z(x, gx)\pi(g)\delta_x \\
&= c_z(v_0, v_1)\cdots c_z(v_{n-1}, v_n)\delta_{gx} \\
&= c_z(v_0, v_1)\cdots c_z(v_{n-2}, v_{n-1})(w\delta_{gx} + z\delta_{v_{n-1}}) \\
&= c_z(v_0, v_1)\cdots c_z(v_{n-3}, v_{n-2})(w\delta_{gx} + wz\delta_{v_{n-1}} + z^2\delta_{v_{n-2}}) \\
&= \ldots \\
&= w\delta_{gx} + wz\delta_{v_{n-1}} + wz^2\delta_{v_{n-2}} + \cdots + wz^{n-1}\delta_{v_1} + z^n\delta_{v_0} \\
&= w\delta_{gx} + wzP\delta_{gx} + wz^2P^2\delta_{gx} + \cdots + wz^{d(x,gx)-1}P^{d(x,gx)-1}\delta_{gx} + z^n\delta_x \\
&= \pi_z^P(g)\delta_x.
\end{aligned}
$$

□

This leads to the following proposition involving $\pi_z^P$, and therefore $\pi_z^C$.

**Proposition 1.3.6.** *Let $G$ be a discrete group that acts transitively on the set $X$ of vertices of a simplicial tree. For every $z \in \mathbb{D}$, the representations $\pi_z^P$ and $\pi_z^C$ of $G$ are cyclic with cyclic vector $\delta_x$.*

*Proof.* Let $z \in \mathbb{D}$ and $v \in X$. If $v = x$, then $\pi_z^P(e)\delta_v = T_z^{-1}(I - zP)^{-1}\pi(e)(I - zP)T_z\delta_x = \delta_x$ and $\pi_z^C(e)\delta_x = c_z(x, x)\delta_x = \delta_x$. Hence $\delta_x$ is in the sets $span\{\pi_z^P(g)\delta_x : g \in G\}$ and $span\{\pi_z^C(g)\delta_x : g \in G\}$.

Now suppose $v \neq x$. As $G$ is transitive, there must exist $g \in G$ such that $gx = v$. as $G$ acts transitively on $X$ there must exist $h \in G$ such that $hx = \overline{gx}$ and as $d(x, \overline{gx}) = d(x, gx) - 1$ and $P\delta_{gx} = \delta_{\overline{gx}}$, then

$$
\begin{aligned}
\pi_z^P(h)\delta_x &= T_z^{-1}(I-zP)^{-1}\pi(h)(I-zP)T_z\delta_x \\
&= T_z^{-1}(I-zP)^{-1}\pi(h)(I-zP)w\delta_x \\
&= T_z^{-1}(I-zP)^{-1}\pi(h)w\delta_x \\
&= T_z^{-1}(I-zP)^{-1}w\delta_{hx} \\
&= T_z^{-1}(I-zP)^{-1}w\delta_{\overline{gx}} \\
&= T_z^{-1}(I+zP+z^2P^2+\dots)w\delta_{\overline{gx}} \\
&= T_z^{-1}(w\delta_{\overline{gx}}+wzP\delta_{\overline{gx}}+wz^2P^2\delta_{\overline{gx}}+\cdots+wz^{d(x,\overline{gx})-1}P^{d(x,\overline{gx})-1}\delta_{\overline{gx}}+wz^{d(x,\overline{gx})}\delta_x) \\
&= z^{d(x,\overline{gx})}\delta_x+\sum_{k=0}^{d(x,\overline{gx})-1} z^k wP^k\delta_{\overline{gx}}
\end{aligned}
$$

which yields

$$
\begin{aligned}
\pi_z^P(g)\delta_x - z\pi_z^P(h)\delta_x &= \left(z^{d(x,gx)}\delta_x+\sum_{k=0}^{d(x,gx)-1} z^k wP^k\delta_{gx}\right) - z\left(z^{d(x,\overline{gx})}\delta_x+\sum_{k=0}^{d(x,\overline{gx})-1} z^k wP^k\delta_{\overline{gx}}\right) \\
&= \sum_{k=0}^{d(x,gx)-1} z^k wP^k\delta_{gx} - \sum_{k=0}^{d(x,gx)-2} z^{k+1} wP^{k+1}\delta_{gx} \\
&= \sum_{k=0}^{d(x,gx)-1} z^k wP^k\delta_{gx} - \sum_{k=1}^{d(x,gx)-1} z^k wP^k\delta_{gx} \\
&= w\delta_{gx}.
\end{aligned}
$$

We now have $\delta_v = \delta_{gx} = \frac{1}{w}\left(\pi_z^P(g)\delta_x - z\pi_z^P(h)\delta_x\right)$ (where $hx = \overline{gx}$) and, as $\pi_z^P(g)\delta_x = \pi_z^C(g)\delta_x$ for all $g \in G$, we also have $\delta_v = \delta_{gx} = \frac{1}{w}\left(\pi_z^C(g)\delta_x - z\pi_z^C(h)\delta_x\right)$. As $G$ acts transitively on $X$, for each $v \in X$, there must exist $k \in G$ such that $v = kx$. This, together with the first result of this proof, shows that $span\{\delta_v : v \in X\} = span\{\delta_{gx} : g \in G\} \subseteq span\{\pi_z^P(g)\delta_x :$

$g \in G\} = span\{\pi_z^C(g)\delta_x : g \in G\}$ and the span of $\{\delta_v : v \in X\}$ is norm-dense in $\ell^2(X)$. Therefore $\pi_z^P$ and $\pi_z^C$ are cyclic representations of $G$ with cyclic vector $\delta_x$. □

As the two representations agree on a cyclic vector, we have the following theorem.

**Theorem 1.3.7.** *Let $G$ be a discrete group that acts transitively on the set $X$ of vertices of a simplicial tree and let $\pi_z^C$ and $\pi_z^P$ be constructed as above. For every $z \in D$, the representations $\pi_z^C$ and $\pi_z^P$ are equal.*

### 1.3.4 The first two constructions are identical for discrete groups acting on trees

We have now seen, by a minor extension of previous work [17], that the two constructions are identical if the discrete group acting on a simplicial tree is acting transitively. However, if the group is not acting transitively, we may not use the previous argument involving cyclic vectors. However, it is possible to demonstrate that the two constructions are identical in the more general case in which the group is discrete but not necessarily acting transitively. This argument is similar to the one we use later in the setting of CAT(0)-cube complexes.

As before, for the rest of this section, let $G$ be a discrete group acting (not necessarily transitively) on a simplicial tree with vertex set $X$ and let $\pi_z^P$ and $\pi_z^C$ be as constructed above (with the cocycle $c_z(x, v)$ as in the cocycle construction of $\pi_z^C$). We will need to generalize our notation as the proofs below will often involve varying the fixed vertex used in the two constructions. For all $z \in \mathbb{D}$ and $v \in X$, let $\pi_{z,v}^P$ and $\pi_{z,v}^C$ be the representations constructed, as above, with fixed vertex $v \in X$.

**Lemma 1.3.8.** *Let $x, v \in X$. For every $z \in \mathbb{D}$, the operator $\pi_{z,x}^C(g)$ is equal to $c_z(x, v)\pi_{z,v}^C(g)c_z(v, x)$.*

*Proof.* Let $z \in \mathbb{D}, g \in G$ and $x,\ v \in X$. We then have $\pi_{z,x}^C(g) = c_z(x, gx)\pi(g)$ and $\pi_{z,v}^C(g) = c_z(v, gv)\pi(g)$. Note also that $c_z(gv, gx)\pi(g) = \pi(g)c_z(v, x)$ as $c_z$ is a cocycle for $\pi$. We then have

$$\begin{aligned}
c_z(x,v)\pi^C_{z,v}(g)c_z(v,x) &= c_z(x,v)c_z(v,gv)\pi(g)c_z(v,x) \\
&= c_z(x,gv)\pi(g)c_z(v,x) \\
&= c_z(x,gv)c_z(gv,gx)\pi(g) \\
&= c_z(x,gx)\pi(g) \\
&= \pi^C_{z,x}(g).
\end{aligned}$$

□

We must now tie the two families of constructions together. We first generalize the linear operators $P$ and $T_z$ from the Pytlik-Szwarc construction [15] to reflect varying base points, essentially by introducing the base points into the notation. For $u$, $v \in X$, $u \neq v$, define $\overline{u}_v$ to be the unique vertex on the geodesic from $u$ to $v$ but one vertex closer to $v$. We may then define $P_{z,v}$ to be the linear operator defined by $P_{z,v}\delta_u = \delta_{\overline{u}_v}$ when $u \neq v$ and $P_{z,v}\delta_v = 0$. Define the linear operator $T_v : \ell^2(X) \to \ell^2(X)$ to be the orthogonal projection onto the one-dimensional subspace $\{z\delta_v : z \in \mathbb{C}\}$ and then define $T_{z,v} = I - T_v + wT_v$ where, as usual, $w = \sqrt{1-z^2}$ using the principal branch of the square root. Then we have

$$\pi^P_{z,v}(g) = T^{-1}_{z,v}(1 - zP_{z,v})^{-1}\pi(g)(1 - zP_{z,v})T_{z,v}.$$

For ease of notation we define $\tilde{c}_z(x,v)$ to be $T^{-1}_{z,x}(1 - zP_{z,x})^{-1}(1 - zP_{z,v})T_{z,v}$.

**Proposition 1.3.9.** *Let $x$, $v \in X$. Then for every $z \in \mathbb{D}$ we have $c_z(x,v) = \tilde{c}_z(x,v)$ on $c_c(X)$.*

*Proof.* If $v = x$, then, $c_z(x,v) = 1$, as it is a cocycle, and $\tilde{c}_z(x,x) = T^{-1}_{z,x}(1 - zP_{z,x})^{-1}(1 - zP_{z,x})T_{z,x} = 1$.

Note also that if $x$, $v$ and $u \in X$,

$$\begin{aligned}\tilde{c}_z(x,u)\tilde{c}_z(u,v) &= T_{z,x}^{-1}(1-zP_{z,x})^{-1}(1-zP_{z,u})T_{z,u}T_{z,u}^{-1}(1-zP_{z,u})^{-1}(1-zP_{z,v})T_{z,v} \\ &= T_{z,x}^{-1}(1-zP_{z,x})^{-1}(1-zP_{z,v})T_{z,v} \\ &= \tilde{c}_z(x,v).\end{aligned}$$

If $d(x,v)=1$, then $x$ and $v$ are adjacent. Let $a \in X$. We have several cases to consider. The first case is that $a$ is neither $x$ nor $v$. Then as $x$ and $v$ are adjacent (and recall that we are working with a simplicial tree), $P_{z,x}\delta_a = P_{z,v}\delta_a$. Moreover

$$\begin{aligned}(1-zP_{z,x})^{-1}(1-zP_{z,v}) &= (1+zP_{z,x}+z^2P_{z,x}^2+\cdots)(1-zP_{z,v}) \\ &= (1+zP_{z,x}+z^2P_{z,x}^2+\dots) - (zP_{z,v}+z^2P_{z,x}P_{z,v}+z^3P_{z,x}^2P_{z,v}+\cdots) \\ &= 1+z(P_{z,x}-P_{z,v})+z^2P_{z,x}(P_{z,x}-P_{z,v})+z^3P_{z,x}^2(P_{z,x}-P_{z,v})+\cdots\end{aligned}$$

so that $(1-zP_{z,x})^{-1}(1-zP_{z,v})\delta_a = \delta_a$. Hence

$$\begin{aligned}\tilde{c}_z(x,v)\delta_a &= T_{z,x}^{-1}(1-zP_{z,x})^{-1}(1-zP_{z,v})T_{z,v}\delta_a \\ &= T_{z,x}^{-1}(1-zP_{z,x})^{-1}(1-zP_{z,v})\delta_a \\ &= T_{z,x}^{-1}\delta_a = \delta_a = c_z(x,v)\delta_a.\end{aligned}$$

The second case is that $a$ is $v$. Then

$$\begin{aligned}
\tilde{c}_z(x,v)\delta_v &= T_{z,x}^{-1}(1 - zP_{z,x})^{-1}(1 - zP_{z,v})T_{z,v}\delta_v \\
&= T_{z,x}^{-1}(1 - zP_{z,x})^{-1}(1 - zP_{z,v})w\delta_v \\
&= T_{z,x}^{-1}(1 - zP_{z,x})^{-1}w\delta_v \\
&= T_{z,x}^{-1}(1 + zP_{z,x} + z^2P_{z,x}^2 + \cdots)w\delta_v \\
&= T_{z,x}^{-1}(w\delta_v + zw\delta_x) \\
&= w\delta_v + z\delta_x \\
&= c_z(x,v)\delta_v.
\end{aligned}$$

The third and final case when $d(x,v) = 1$ is that $a$ is $x$. Then

$$\begin{aligned}
\tilde{c}_z(x,v)\delta_x &= T_{z,x}^{-1}(1 - zP_{z,x})^{-1}(1 - zP_{z,v})T_{z,v}\delta_x \\
&= T_{z,x}^{-1}(1 - zP_{z,x})^{-1}(1 - zP_{z,v})\delta_x \\
&= T_{z,x}^{-1}(1 - zP_{z,x})^{-1}(\delta_x - z\delta_v) \\
&= T_{z,x}^{-1}(1 + zP_{z,x} + z^2P_{z,x}^2 + \cdots)(\delta_x - z\delta_v) \\
&= T_{z,x}^{-1}(\delta_x - z\delta_v - z^2\delta_x) \\
&= T_{z,x}^{-1}((1 - z^2)\delta_x - z\delta_v) \\
&= T_{z,x}^{-1}(w^2\delta_x - z\delta_v) \\
&= w\delta_x - z\delta_v \\
&= c_z(x,v)\delta_x.
\end{aligned}$$

Let $n > 1$ and suppose that $c_z(x,v) = \tilde{c}_z(x,v)$ if $d(x,v) < n$. As $d(x,v) = n > 1$, then $x$ and $v$ are no longer adjacent. However, there must exist a unique geodesic from $x$ to $v$. Label the vertices of this geodesic $x = v_0, v_1, v_2, \ldots, v_n = v$. Then

$$c_z(x, v) = c_z(v_0, v_{n-1})c_z(v_{n-1}, v_n) = \tilde{c}_z(v_0, v_{n-1})\tilde{c}_z(v_{n-1}, v_n) = \tilde{c}_z(x, v).$$

$\square$

**Corollary 1.3.10.** *Let $x$, $v \in X$. For every $z \in \mathbb{D}$, the operator $\pi^P_{z,x}(g)$ is equal to $c_z(x, v)\pi^P_{z,v}(g)c_z(v, x)$.*

*Proof.* Let $z \in \mathbb{D}$, $g \in G$ and $x$, $v \in X$. Then

$$\begin{aligned}
c_z(x, v)\pi^P_{z,v}(g)c_z(v, x) &= \tilde{c}_z(x, v)\pi^P_{z,v}(g)\tilde{c}_z(v, x) \\
&= \tilde{c}_z(x, v)T^{-1}_{z,v}(I - zP_{z,v})^{-1}\pi(g)(I - zP_{z,v})T_{z,v}T^{-1}_{z,v}(I - zP_{z,v})^{-1}(I - zP_{z,x})T_{z,x} \\
&= \tilde{c}_z(x, v)T^{-1}_{z,v}(I - zP_{z,v})^{-1}\pi(g)(I - zP_{z,x})T_{z,x} \\
&= T^{-1}_{z,x}(1 - zP_{z,x})^{-1}(1 - zP_{z,v})T_{z,v}T^{-1}_{z,v}(I - zP_{z,v})^{-1}\pi(g)(I - zP_{z,x})T_{z,x} \\
&= T^{-1}_{z,x}(1 - zP_{z,x})^{-1}\pi(g)(I - zP_{z,x})T_{z,x} \\
&= \pi^P_{z,x}(g).
\end{aligned}$$

$\square$

We now reach the conclusion of this chapter.

**Theorem 1.3.11.** *For every $z \in \mathbb{D}$ and $x \in X$, the representations $\pi^P_{z,x}$ and $\pi^C_{z,x}$ are equal.*

*Proof.* Let $z \in \mathbb{D}$, $g \in G$ and $x$, $v \in X$. We will prove by induction that $\pi^P_{z,x}(g)\delta_v = \pi^C_{z,x}(g)\delta_v$ for all $x$, $v \in X$. The case that $d(x, v) = 0$ is Lemma 1.3.5.

Let $d(x, v) = 1$. As $\pi^C_{z,x}(g) = c_z(x, v)\pi^C_{z,v}(g)c_z(v, x)$ and $\pi^C_{z,x}(g) = c_z(x, v)\pi^C_{z,v}(g)c_z(v, x)$, we have

$$\begin{aligned}
\pi^C_{z,x}(g)\delta_v - \pi^P_{z,x}(g)\delta_v &= c_z(v,x)\pi^C_{z,x}(g)w\delta_v - c_z(v,x)\pi^P_{z,x}(g)w\delta_v \\
&= c_z(v,x)\pi^C_{z,x}(g)w\delta_v - c_z(v,x)\pi^C_{z,x}(g)z\delta_x - c_z(v,x)\pi^P_{z,x}(g)w\delta_v + c_z(v,x)\pi^P_{z,x}(g)z\delta_x \\
&= c_z(v,x)\pi^C_{z,x}(g)c_z(x,v)\delta_v - c_z(v,x)\pi^P_{z,x}(g)c_z(x,v)\delta_v \\
&= \pi^C_{z,v}(g)\delta_v - \pi^P_{z,v}(g)\delta_v \\
&= 0.
\end{aligned}$$

Given $n \geq 0$, assume that $\pi^P_{z,x}\delta_v = \pi^C_{z,x}\delta_v$ for all $x$, $v \in X$ with $d(v,x) < n$. Let $u \in X$ such that $d(x,u) = n \geq 1$. Then there exists $v$ on the geodesic from $x$ to $u$ such that $d(x,v) \geq 1$. Note that $v$ may be $u$. By Proposition 1.3.10, if $v \neq u$ then

$$\begin{aligned}
c_z(v,x)\pi^P_{z,x}(g)\delta_u &= \pi^P_{z,v}(g)c_z(v,x)\delta_u \\
&= \pi^P_{z,v}(g)\delta_u \\
&= \pi^C_{z,v}(g)\delta_u \text{ (by the induction hypothesis)} \\
&= \pi^C_{z,v}(g)c_z(v,x)\delta_u \\
&= c_z(v,x)\pi^C_{z,x}(g)\delta_u \text{ (by Lemma 1.3.8)},
\end{aligned}$$

hence $\pi^P_{z,x}(g)\delta_u = \pi^C_{z,x}(g)\delta_u$. If $v = u$, let $v_x$ be the vertex on the geodesic from $u$ to $x$ such that $d(u,v_x) = 1$. Note that $d(x,v_x) = n-1$. Then

$$\begin{aligned}
c_z(v,x)\pi^P_{z,x}(g)\delta_u &= \pi^P_{z,v}(g)c_z(v,x)\delta_v \\
&= \pi^P_{z,v}(g)(w\delta_v - z\delta_{v_x}) \\
&= \pi^C_{z,v}(g)(w\delta_v - z\delta_{v_x}) \text{ (by the induction hypothesis and Lemma 1.3.5)} \\
&= \pi^C_{z,v}(g)c_z(v,x)\delta_v \\
&= c_z(v,x)\pi^C_{z,x}(g)\delta_u \text{ (by Lemma 1.3.8)},
\end{aligned}$$

hence $\pi^P_{z,x}(g)\delta_u = \pi^C_{z,x}(g)\delta_u$. This concludes the proof. $\square$

# Chapter 2

# CAT(0)-Cube Complexes

## 2.1 Motivations

In Chapter 3 we will generalize the results of Chapter 1 to discrete groups acting on finite dimensional CAT(0)-cube complexes. As geometric group theory is such a young area of mathematical research, some definitions may vary between authors so this chapter will serve to clarify what we mean by a finite dimensional CAT(0)-cube complex. Most of the following has been covered more generally by Martin Bridson in his dissertation [4] and in his book with Andre Haefliger [5], amongst other authors, however we will adapt these results to our particular needs.

In this chapter we will define CAT(0)-cube complexes geometrically. However, it may aid the reader's understanding to be introduced to a combinatorial approach. In this approach, a *cube complex* is a set $X$, called the set of *vertices*, together with a collection of finite subsets of $X$ , called the *cubes* of $X$, with the following properties:

(1) every vertex is a cube;

(2) the intersection of every two cubes is either empty or a cube;

(3) for every cube $C$, there is an integer $n \geq 0$ and a bijection from $C$ to the vertices of a Euclidean cube of side length 1 and dimension $n$ such that the

cubes in $X$ that are subsets of $C$ correspond precisely to the sets of vertices of the faces (of all dimensions) of the Euclidean cube.

It is difficult to define what is meant by the CAT(0) condtion from this combinatorial description. Therefore we will construct a cube complex geometrically below, closely following the definitions and methods of Bridson and Haefliger [5] but narrowed to our particular needs. We will then define a CAT(0) space and establish the characteristics of a finite dimensional CAT(0)-cube complex.

## 2.2 CAT(0) Spaces

We begin with some definitions. Let $(X, d)$ be a metric space. A *geodesic segment* is a topological arc which is isometric to a closed interval of of the real line such that the length of a geodesic segment from $x$ to $y$ in $X$ is equal to $d(x, y)$. If every pair of points in $X$ can be joined by a geodesic segment, we say that $(X, d)$ is a *geodesic metric space*. A *geodesic triangle* $\Delta$ in a metric space $X$ is a triple of points $x, y$ and $z$ in $X$ together with paths joining each pairwise (called the sides) that are geodesic segments in the metric space $X$. A *comparison triangle* $\overline{\Delta}$ for $\Delta$ is a triple of points $\overline{x}, \overline{y}$ and $\overline{z}$ in Euclidean space with geodesic segments joining each pairwise such that $d(x, y) = d(\overline{x}, \overline{y})$, $d(y, z) = d(\overline{y}, \overline{z})$ and $d(x, z) = d(\overline{x}, \overline{z})$. Such a triangle must exist for all $x, y$ and $z$ in $X$ [5]. Note that this triangle is unique up to isometry.

With this we may establish the *thin triangles property* of a CAT(0) space. Given a point $p \in [x, y]$, point $\overline{p}$ in $[\overline{x}, \overline{y}]$ is called a *comparison point* for $p$ if $d(x, p) = d(\overline{x}, \overline{p})$. We extend this definition to comparison points in $[\overline{x}, \overline{z}]$ and $[\overline{y}, \overline{z}]$ similarly. If every point $x \in X$ has a neighborhood such that for all geodesic triangles $\Delta$ in that neighborhood and for all points $p$ and $q$ in $\Delta$ with comparison points $\overline{p}$ and $\overline{q} \in \overline{\Delta}$, we have that $d(p, q) \leq d(\overline{p}, \overline{q})$, then $X$ locally satisfies the CAT(0) inequality and we will say that $X$ is *non-positively curved* (which is sometimes called *locally* CAT(0)). If all geodesic triangles in $X$ satisfy this inequality we will say that $X$ is CAT(0).

## 2.3 Cube complexes

Let the unit $n$-cube $I^n$ be the $n$-fold product $[0,1]^n$ which is isomorphic to a cube in Euclidean $n$-space. For a particular cube $I^n = [0,1]^n$, define a *face* to be a subset of the form

$$S = \prod_{i=1}^{n} S_i \text{ where } S_i \in \{\{0\},\{1\},[0,1]\} \text{ for all } 1 \le i \le n,$$

and a *proper face* to be any face $S$ such that $S \neq I^n$. We may then construct a cube complex by "glueing" cubes together in a particular way.

We will define a *cube complex* $K$ as in Bridson-Haefliger [5] to be the quotient of a disjoint union of cubes $X = \amalg_{\lambda \in \Lambda} I_\lambda^{n_\lambda}$ by an equivalence relation $\sim$ with the requirement that the restrictions $p_\lambda : I_\lambda^{n_\lambda} \to K$ of the natural projection $p : X \to K = X/\sim$ satisfy:

(1) for every $\lambda \in \Lambda$ the map $p_\lambda$ is injective;

(2) for $\lambda, \mu \in \Lambda$, if $p_\lambda(I_\lambda^{n_\lambda}) \cap p_\mu(I_\mu^{n_\mu}) \neq \emptyset$ then there is an isometry $h_{\lambda,\mu}$ from a face $T_\lambda \subset S_\lambda$ onto a face $T_\mu \subset S_\mu$ such that $p_\lambda(x) = p_\mu(x')$ if and only if $x' = h_{\lambda,\mu}(x)$.

In effect, the equivalence relation constructs the cube complex by "glueing" cubes together and identifying matched vertices with no cube being glued to itself and at most one glueing isometry between any two cubes. Note also that our definition of a cube complex yields the property that the intersection of every two cubes is either empty or a cube and also the other combinatorial properties mentioned above.

If $n = \max\{n_\lambda | \lambda \in \Lambda\}$ exists, we say that the cube complex is *finite dimensional.* In this dissertation, we will only work with finite dimensional cube complexes. Also, for the rest of this dissertation, we will use the term "cubes" for images of the cubes from $X$ in $K$ and also for the images of all their proper faces in $K$. As in the case of trees, we will call a

0-cube a *vertex* and a 1-cube an *edge*. We define an *interior point* of the cube complex $K$ to be a point that does not lie in any proper face of $K$ and the *interior* of $K$ to be the set of all its interior points. Finally, we note for the interested reader that Bridson and Haefliger [5] refer to the cube complexes constructed above as *cubical complexes* and refer to more general complexes of cubes, without the same restrictions, as *cubed complexes*. We now have the following lemma.

**Lemma 2.3.1.** *For every point $x \in K$, there is a unique cube in $K$ containing $x$ in its interior. It is a face of every cube in $K$ containing $x$.*

*Proof.* Let $x$ be a point in $K$. Suppose two cubes in $K$ contain $x$. Then the intersection of those two cubes also contains $x$ and this intersection is also a cube in $K$. This cube has dimension at most the minimum dimension of the two original cubes and is contained in both of the original cubes. Therefore there must be a cube of minimum dimension containing $x$. □

We will call this unique cube the *support* of $x \in K$, denoted *supp*$(x)$.

## 2.4 Defining a metric on a cube complex

Much of Chapter 3 will depend on CAT(0)-cube complexes being complete geodesic metric spaces. As such, we must define a metric and show this to be the case. Note that the metric defined in this chapter will not in fact be either of the two metrics we use in Chapter 3, but instead allows for their existence. Moreover, the $K = X/\sim$ that we have defined above includes all points in the disjoint union of cubes, either on faces or interior to a cube, whereas in Chapter 3 we will use $X$ to refer only to the set of vertices of the cube complex and the metrics we establish will be defined only for these vertices.

Let $K$ be a cube complex, as defined above, and assume that it is connected, that is, that every pair of points can be connected by an $n$-string as follows. For $x, y \in K$, define an

$n$-*string* $\Sigma$ from $x$ to $y$ to be a sequence $\Sigma = (v_0, v_1, v_2, \ldots, v_n)$ such that $v_0 = x$ and $v_n = y$ with each $v_i \in X$ and such that, for $1 \leq i \leq n$, there exists a cube $C_i$ containing $v_{i-1}, v_i$. As each pair $v_{i-1}, v_i$ is in $C_i$, which is isomorphic to a Euclidean cube and the gluing maps are isometries, we can adopt the usual Euclidean metric for each pair. Call each such submetric $d_{C_i}$. We may then define the *length* of $\Sigma$ to be

$$l(\Sigma) = \sum_{i=1}^{n} d_{C_i}(v_{i-1}, v_i),$$

and the *intrinsic pseudometric* on $X$ to be $d(x,y) = \inf\{l(\Sigma) : \Sigma \text{ a string from } x \text{ to } y\}$. It is clear from the construction that $d$ is non-negative, symmetric and that $d$ also satisfies the triangle inequality. In order to prove that the intrinsic psuedometric defined above is indeed a metric, we need to show that $d(x,y) = 0$ if and only if $x = y$. For this we will need the following definition. For a cube $C \in K$ containing $x$, define $\epsilon(x, C) = \inf\{d_C(x, F) : F \text{ a face of } C \text{ and } x \notin F\}$, with the added condition that if $C$ is the 0-cube consisting solely of $x$, then $\epsilon(x, C) = 1$. Further define $\epsilon(x) = \inf\{\epsilon(x, C) : C \subset K \text{ a cube containing } x\}$. Note that this implies that $\epsilon(x) \leq 1$ as we are working with a connected non-trivial cube complex.

**Proposition 2.4.1.** *The intrinsic pseudometric, defined above, is a metric.*

*Proof.* Let $x \in K$. If $x \in C \subset D$ where $C$ and $D$ are cubes of $K$, then $\epsilon(x, C) = \epsilon(x, D)$. Therefore for any cube $C \in K$ with $x \in C$, $\epsilon(x, C) = \epsilon(x, supp(x))$. If $x$ is a vertex, that is, if $supp(x)$ is the zero-dimensional cube containing $x$, then $\epsilon(x) = 1$ as we are in a cube complex. Otherwise $\epsilon(x) \geq \epsilon(x, supp(x)) > 0$ by Euclidean geometry. Therefore $\epsilon(x) > 0$ for all $x \in K$.

Let $y \in K$ and $d(x,y) < \epsilon(x)$. We can then see that any cube $C$ containing $y$ also contains $x$. To show this we will instead show that if $\Sigma = (x_0, x_1, \ldots, x_m)$ is an $m$-string of length $\ell(\Sigma) < \epsilon(x)$ from $x = x_0$ to $y = x_m$, with $m \geq 2$, then $\Sigma' = (x_0, x_2, \ldots, x_m)$ is an $(m-1)$-string with $\ell(\Sigma') \leq \ell(\Sigma)$, which is sufficient. Moreover, this will imply that $d(x,y) = d_C(x,y)$.

Let the $m$-string from $x$ to $y$ be defined as above. There must be a cube $C_2$ containing $x_1$ and $x_2$. Since $\ell(\Sigma) < \epsilon(x)$ we have that $x_0 = x$ also belongs to $C_2$. Then $d_{C_2}(x_0, x_2) \leq d_{C_2}(x_0, x_1) + d_{C_2}(x_1, x_2)$, hence $\ell(\Sigma') \leq \ell(\Sigma)$.

We then have that for any two vertices $x, y \in X$, if $d(x, y) = 0 \leq \epsilon(x)$, then any cube $C$ containing $y$ contains $x$ and $d(x, y) = d_C(x, y) = 0$, hence $x = y$. □

We define a *geodesic metric space* to be a metric space in which every pair of points in $K$ can be connected by a geodesic segment. We conclude this section with the following result from Bridson's dissertation.

**Theorem 2.4.2** ([4]). *If the connected cube complex $K$ is finite-dimensional, then $(K, d)$ is a complete geodesic space.*

## 2.5 CAT(0)-cube complexes

We may now define CAT(0)-cube complexes and examine some of their properties. We will first need to define the *link* of a vertex, which we will denote $Lk(v)$. We will follow the methods of Sageev [3]. Let $C$ be a cube in $X$. We define a *local edge* of $C$ to be a subinterval of length 1/3 on an edge of $C$ with a vertex of $C$ at one end of this subinterval. A local edge in $K$ is then the image of a local edge in $C$. For each vertex $v \in K$ we can then define $Lk(v)$ to be the simpicial complex with a vertex for every local edge in $K$ containing $v$ and in which a set of vertices in span a simplex if and only if they all came from the same cube $C$ in $X$. We will also need the following definition. The abstract simplicial complex $L$ is called a *flag complex* if every finite subset of its vertices that are pairwise joined by edges also spans a simplex in $L$.

In their book, Bridson and Haefliger [5] refine a result of Gromov from abstract simplicial complexes to cubed complexes which, as mentioned above, are more general than the cube complexes we have defined. A slight refinement then gives the following theorem.

**Theorem 2.5.1** (Gromov's Link Condition [5, 11])**.** *A finite dimensional cube complex is non-positively curved if and only if the link of each of its vertices is a flag simplicial complex.*

We will use this theorem together with the following theorem to establish our result.

**Theorem 2.5.2** (Cartan-Hadamard [5])**.** *If a complete, non-positively curved metric space is simply connected, then it is a* CAT(0)*-space.*

So we have reduced the problem of establishing that a complete, simply connected metric space is CAT(0) to proving that it is non-positively curved. Moreover, we have established an arguably easier method to determine whether a cube complex is non-positively curved.

**Theorem 2.5.3.** *A cube complex is* CAT(0) *if and only if it is simply connected and satisfies Gromov's link condition.*

As we will not be using the metric defined above in Chapter 3, it will helpful to to be able to consider whether a cube complex $K$ is simply connected without referring to the metric topology. Let $K_m$ be $K$ with the metric topology, and $K_q$ be $K$ with the quotient topology. It is known that $K_q$ is finer than $K_m$ so that if $i$ is the identity map on $K$, then $i : K_q \to K_m$ is continuous [12]. However, it is known that if $K$ is not locally finite, then $i$ is not a homeomorphism [12]. However, we have the following.

**Proposition 2.5.4** ([11, 12])**.** *Let $K$ be a cube complex . Then the identity map $i : K_q \to K_m$ is a homotopy equivalence.*

**Corollary 2.5.5.** *A cube complex is simply connected in the quotient topology if and only if it is simply connected in the metric topology.*

## 2.6 Further definitions and properties of CAT(0)-cube complexes

We must now establish some definitions and properties of CAT(0) cube complexes. Let $K$ be a CAT(0) cube complex as defined above. Recall that $K = X/\sim$ but, for ease of notation, we will use $X$ instead of $K$, as is the convention. For a given cube $C = [0,1]^n \subset X$, define a *midplane* to be an $(n-1)$-dimensional plane with coordinates $\{x_1, x_2, \ldots, x_n\}$ where $x_i = 1/2$ for some $i \in \{1, 2, \ldots, n\}$ and all other $x_j \in [0,1]$. We may form an equivalence relation on the midplanes of our cube complex $K$. Given two midplanes $M$ and $N$, we say that they are *hyperplane equivalent* if there is a sequence of midplanes $M = M_0, M_1, \ldots, M_j = N$ such that $M_i \cap M_{i+1}$ is also a midplane for all $i \in \{1, 2, \ldots, n-1\}$. We define a *hyperplane* to be an equivalence class of midplanes and we will call two hyperplanes parallel if they do not intersect. There are several properties of hyperplanes that we will need.

**Lemma 2.6.1** ([16]). *Every hyperplane in $X$ separates $X$ into exactly two components.*

For two vertices $x$, $y \in X$, define $H(x,y)$ to be the set of all hyperplanes separating $x$ and $y$ and define the *geometric distance* from $x$ to $y$, denoted, $d(x,y)$, to be the minimum length $n$ of an edge-path $x = v_0, v_1, \ldots, v_n = y$ between them. Note that this is not the same metric we defined in Section 2.4 but this is what we will mean by $d(x,y)$ from this point onward.

We now list several properties of hyperplanes in CAT(0)-cube complexes.

**Proposition 2.6.2** ([16]). *An edge-path in a* CAT(0) *cube complex from vertices $x$ to $y$ crosses each hyperplane in $H(x,y)$. An edge-path from $x$ to $y$ is a geodesic if and only if it crosses only the hyperplanes of $H(x,y)$ and crosses each one of these exactly once.*

**Corollary 2.6.3.** *For any two vertices $x$, $y \in X$, the geometric distance $d(x,y)$ is equal to $|H(x,y)|$.*

**Corollary 2.6.4** ([9]). *Let $x$, $y$ and $v$ be vertices in a* CAT(0) *cube complex $X$. The following are equivalent:*

(1) *$v$ lies on a geodesic from $x$ to $y$;*

(2) *$d(x,y) = d(x,v) + d(v,y)$;*

(3) *$H(x,v) \cap H(v,y) = \emptyset$;*

(4) $H(x,v) \cap H(v,y) = \emptyset$ *and* $H(x,v) \cup H(v,y) = H(x,y)$*;*

(5) $H(x,v) \subseteq H(x,y)$*;*

(6) $H(x,v) \subseteq H(x,y)$ *and* $H(v,y) \subseteq H(x,y)$*.*

**Proposition 2.6.5** ([13]). *A hyperplane in a* CAT(0) *cube complex* $X$ *does not self-intersect. In fact, if a vertex* $x \in X$ *is adjacent to a hyperplane* $H$*, then there is a unique edge adjacent to* $x$ *that crosses* $H$*.*

**Proposition 2.6.6** ([13]). *If two hyperplanes in a* CAT(0) *cube complex* $X$ *intersect and are both adjacent to the same vertex, then they intersect in a square containing that vertex.*

**Proposition 2.6.7** ([9]). *Let* $H_1, \ldots, H_n$ *be hyperplanes in a* CAT(0) *cube complex* $X$*, all adjacent to a vertex* $x \in X$*. If there is a vertex* $y \in X$ *such that each hyperplane* $H_i$ *separates* $x$ *from* $y$*, then there is a cube of dimension* $n$ *in which all the hyperplanes* $H_i$ *intersect.*

**Proposition 2.6.8** ([9]). *Two parallel hyperplanes, that is, non-intersecting, separate a* CAT(0) *cube complex into exactly three components.*

We will also need several properties of cube complexes in Chapter 3 that are less directly related to hyperplanes. We first borrow some definitions from Niblo and Reeves [13], who define a *cube-path* as follows. Let $\{C_i\}_0^n$ be a sequence of cubes, each of dimension at least 1, such that for $1 \leq i \leq n$, $C_{i-1} \cap C_i = v_i$ where $v_i$ is a vertex of the cube complex. That is, each cube meets its successor in a single vertex. We call this sequence a *cube-path* if $C_i$ is the (unique) cube of minimal dimension containing $v_i$ and $v_{i+1}$. Note that $v_i$ and $v_{i+1}$ are diagonally opposite vertices of $C_i$. We define $v_0$ to be the vertex of $C_0$ that is diagonally opposite $v_1$, and $v_{n+1}$ to be the vertex of $C_n$ that is diagonally opposite $v_n$. We call the $v_i$, vertices of the cube-path and $v_0$ the initial vertex and $v_{n+1}$ the terminal vertex of the cube-path. We define the *length* of a cube-path to be to be the number of cubes in the sequence. Note that a cube-path defines a family of edge-paths from $v_0$ to $v_{n+1}$ which travel from $v_i$ to $v_{i+1}$ via a geodesic in the 1-skeleton of $C_i$.

We are mainly interested in a particular type of cube-path which functions somewhat like a geodesic. Let $x$ and $y$ be vertices in $K$. Then there are a set of hyperplanes, $H(x, y)$, separating $x$ and $y$. A cube-path from $x = v_0, v_1, \ldots, v_n = y$ is a *normal cube-path* if, for all $1 \leq i \leq n$, the vertex $v_{i-1}$ is separated from $v_i$ by all hyperplanes in $H(v_{i-1}, y)$ that are adjacent to $v_{i-1}$. The length of a normal cube-path will then be the number of vertices in this sequence. Alternatively, we may think of the length as the number of cubes in the sequence.

**Proposition 2.6.9** ([13]). *Given two vertices $x, y \in X$, there is a unique normal cube-path from $x$ to $y$.* (*The order is important here, since in general normal cube-paths are not reversible.*)

**Proposition 2.6.10** ([13]). *A normal cube-path achieves the minimum length among all cube-paths joining its endpoints.*

**Proposition 2.6.11** ([13]). *Given a normal cube-path $\{C_i\}_0^n$ with $C_{i-1} \cap C_i = v_i$ for all $1 \leq i \leq n$ ($y = v_0, x = v_n$), then every edge-path through the set $\{v_i\}_1^n$ such that the edge-path from $v_{i-1}$ to $v_i$ is a geodesic for $1 \leq i \leq n$ is a geodesic from $y$ to $x$.*

We will be using two different metrics in Chapter 3. We have already defined the *geometric distance* between two vertices to be the minimum length of an edge-path between them and we now define the *cubic distance* between two vertices to be the length of a normal cube-path between them. Note that although normal cube-paths are not reversible, this distance is symmetric [13]. The final definition we will need is that an *action* of a group on a cube complex $X$ is an action on the set of vertices of $X$ that maps cubes to cubes.

# Chapter 3

# The Constructions for CAT(0)-Cubical Groups

## 3.1 The cocycle construction of Guentner and Higson

In their 2007 paper "Weak Amenability of CAT(0)-Cubical Groups" [9], Guentner and Higson constructed a holomorphic family of uniformly bounded representations of a discrete group acting on a CAT(0)-cube complex. They accomplished this by extending the cocycle method that Pimsner and Valette [14,18] had used for discrete groups acting on trees. We will present this construction below with some clarifications of our own. In particular, Guentner and Higson [9] did not need to exactly calculate the matrix coefficients of their representation in order to prove that their representation was uniformly bounded. We will need to clarify these as we will be extending the construction of Pytlik and Szwarc [15] and proving that the representations that we have constructed are equal to those constructed by Guentner and Higson for any particular $z \in \mathbb{D}$. Note that we will henceforth use $X$ for the set of a vertices of a CAT(0)-cube complex, as is the convention.

The following is a summary of the work of Guentner and Higson [9]. As before, let $z \in \mathbb{D}$ and let $w = \sqrt{1 - z^2}$. Let $X$ be the set of vertices of a CAT(0)-cube complex and let $x$ and $y$ be adjacent vertices in $X$, that is, $x$ and $y$ are connected by an edge. Let $H$ be the hyperplane that separates $x$ and $y$. We will orient this hyperplane by denoting the half-space containing $x$ by $H^+$ and the half-space containing $y$ by $H^-$. The sets of vertices adjacent to $H$ will be of particular interest to us. Denote by $\partial H^+$ the set of vertices adjacent to $H$ that are in $H^+$ and denote by $\partial H^-$ the set of vertices adjacent to $H$ that are in $H^-$. For a vertex $v \in \partial H^{\pm}$, there is a unique edge adjacent to $v$ crossing $H$ [13]. Define $\overline{v}$ to be the unique vertex opposite $v$ across $H$. As in the case of trees, define a bounded operator $c_z(x, y)$ on $\ell^2(X)$ by

$$c_z(x,y)\delta_v = \begin{cases} w\delta_v - z\delta_{\overline{v}}, & \text{if } v \in \partial H^+ \\ w\delta_v + z\delta_{\overline{v}}, & \text{if } v \in \partial H^- \\ \delta_v, & \text{if } v \notin \partial H^+ \cup \partial H^- \end{cases}$$

Although this operator is analogous to the one constructed in the tree case, there is a significant difference as it is nontrivial on the basis vector $\delta_v$ for every $v$ adjacent to $H$. As a result, it may be non-trivial on basis vectors other than $\delta_x$ and $\delta_y$.

For every pair of vertices $v$ and $\overline{v}$ adjacent across $H$, the two dimensional subspace spanned by the ordered basis $\{\delta_v, \delta_{\overline{v}}\}$ is reducing for $c_z(x, y)$. The subspace spanned by all $\delta_v$ such that $v$ is not adjacent to $H$ is also reducing for $c_z(x, y)$. Further $c_z(x, y)$ is the direct sum of a family of operators on these two-dimensional subspaces and the identity operator on their joint orthogonal complement.

As in the case for trees, if $v \in \partial H^+$, $c_z(x, y)$ is given by the matrix

$$\begin{pmatrix} w & z \\ -z & w \end{pmatrix}$$

on the two-dimensional subspace with ordered basis $\{\delta_v, \delta_{\overline{v}}\}$. We may extend $c_z(x, y)$ to edge-paths as before also. If $x, y \in X$ with $x = v_0, v_1, \ldots, v_n = y$ an edge-path from $x$ to $y$, then define

$$c_z(x,y) = c_z(v_0,v_1)c_z(v_1,v_2)\cdots c_z(v_{n-1},v_n).$$

As in the tree case, it is easy to verify that $c_z(u,v)^{-1} = c_z(v,u)$ for every pair of adjacent vertices $u, v \in X$ and for every $z \in \mathbb{D}$. However, proving that the edge-path definition of $c_z(x,y)$ is independent of path is slightly more involved than in the tree case.

**Proposition 3.1.1** ([9])**.** *The expression $c_z(x,y) = c_z(v_0,v_1)c_z(v_1,v_2)\cdots c_z(v_{n-1},v_n)$ defining $c_z(x,y)$ for general $x, y \in X$ is independent of the edge-path $v_0, v_1, \ldots, v_n$ connecting $x$ to $y$.*

The key difference from the tree case is that even with cancellation of inverse operators, there may be more than one edge-path between two vertices in the case of CAT(0)-cube complexes. We begin with a definition. A *corner move* transforms an edge-path by changing a string in the path from $\{u,v,w\}$ to $\{u,t,w\}$ where $\{u,v,t,w\}$ form a square (2-cube) in the cube complex. Note that a corner move alters neither the endpoints of an edge-path nor its length. The authors then quote a result of Sageev [16] that demonstrates that any two paths in such a complex with the same endpoints are related by a sequence of simple cancellations and corner moves and that such transformations leave $c_z(x,y)$ unaltered.

Guentner and Higson then prove that $c_z : X \times X \to \mathcal{B}(\ell^2(X))$ is a cocycle for the permutation representation of $G$ on $\ell^2(X)$. This allows the authors, once a particular $x \in X$ is fixed, to construct the family of representations $\{\pi_z : z \in \mathbb{D}\}$ of $G$ into the bounded invertible operators on $\ell^2(X)$ defined by

$$\pi_z(g) = c_z(x,gx)\pi(g).$$

As in the case for trees, the operator $\pi_z(g)$ is a polynomial in $z$ and $w$ with coefficients in $\mathcal{B}(\ell^2(X))$, hence the family $\{\pi_z(g) : z \in \mathbb{D}\}$ is holomorphic. The authors then prove that the cocycle $c_z(x,y)$ is uniformly bounded for each $z \in D$, which further implies that the representation $\pi_z$ is uniformly bounded for each $z \in D$. We will refer the reader to Guentner and Higson's paper [9] for the details as this proof is quite involved. For $x, y \in X$, the authors [9] define the matrix coefficient $c_{ab}$ by

$$c_z(x,y)\delta_b = \sum_{a \in X} c_{ab}\delta_a,$$

which yields $c_{ab} = \langle c_z(x,y)\delta_b, \delta_a \rangle$ and the following result. Note that $d(a,b)$ will be the geometric distance between $a$ and $b$. Guentner and Higson conclude their calculations with the following proposition.

**Proposition 3.1.2** ([9]). *Let $x$ and $y$ be vertices of $X$. For $a, b \in X$, if $c_{ab}$ is nonzero for some $z \in \mathbb{D}$, then*

$$c_{ab} = \pm z^{d(a,b)} w^{\ell},$$

*for some non-negative integer $\ell$ not exceeding twice the dimension of $X$.*

Our contribution will be to determine the sign of such $c_{ab}$ and to clarify the value of $\ell$. We will need several propositions and lemmas of Guentner and Higson [9] which we will state without proof. We will then state and prove our new assertions but it may be helpful for the reader to have a brief preview of the methods. The central idea is to construct a geodesic from $x$ to $y$ that allows us to more easily calculate the exponent $\ell$ of $w$ and to determine the sign of $c_{ab}$. Although this geodesic allows us to easily calculate the matrix coefficients, its construction is fairly elaborate.

We now need several definitions. For two vertices $u$, $v \in X$, recall that $H(u,v)$ is defined to be the set of all hyperplanes separating $u$ and $v$. We then define a *geodesic order* on $H(u,v)$ to be linear order on $H(u,v)$ for which there exists a geodesic edge-path from $u$ to $v$ such that $H < H'$ if and only if the path crosses $H$ before it crosses $H'$. Finally, for a set $S$ of vertices in $X$, the *convex hull* of $S$ is defined to be the intersection of all half-spaces containing the vertices of $S$.

We will need the following results, some of which we will state without proof.

**Lemma 3.1.3** ([9]). *Let $x$ and $y$ be two vertices in $X$. If $c_{ab}$ is nonzero for some $z \in \mathbb{D}$, $H(a,b) \subseteq H(x,y)$.*

**Proposition 3.1.4** ([9]). *Let $x$, $y$ and $b$ be vertices of $X$. If $c_{ab}$ is nonzero for some $z \in \mathbb{D}$, then $a$ lies in the convex hull of $\{x, y, b\}$.*

**Lemma 3.1.5** ([9]). *A linear ordering $\{H_1, \dots H_n\}$ on $H(x,y)$ is a geodesic ordering if and only if the vertex $v_0 = x$ is adjacent to $H_1$ and for each $i = 1, \dots, n$ the vertex $v_i$ obtained by successively reflecting $v_0$ across $H_1, \dots, H_{i-1}$ is adjacent to $H_i$. In this case the sequence of vertices $v_0, \dots, v_n$ is a geodesic edge-path from $x$ to $y$.*

**Lemma 3.1.6** ([9]). *If $c_{ab}$ is nonzero for some $z \in \mathbb{D}$, then every geodesic order on $H(x,y)$ induces a geodesic order on $H(a,b)$.*

Our contributions now follow.

**Lemma 3.1.7.** *Let $H$ and $K$ be two non-intersecting hyperplanes in $H(x,y)$. If there exists a geodesic order on $H(x,y)$ with $H < K$, then $H < K$ in every geodesic order on $H(x,y)$.*

*Proof.* Let $H$ and $K$ be two non-intersecting hyperplanes in $H(x,y)$ and suppose there exists a geodesic order $H(x,y)$ with $H < K$. Let $H^+$ be the half-space corresponding to the hyperplane $H$ that contains $x$ and $H^-$ the opposite half-space. Define $K^+$ similarly. Then $H^+ \cap K^- = \emptyset$, hence $H^+$ is properly contained in $K^+$. Therefore any geodesic from $x$ to $y$ must cross $H$ before it crosses $K$. $\square$

**Corollary 3.1.8.** *For any two vertices $u$, $v \in X$, if $H, K \in H(u,v)$ do not intersect, then neither $u$ nor $v$ can be adjacent to both hyperplanes.*

**Lemma 3.1.9.** *For any two vertices $u$, $v \in X$, if $H \in H(u,v)$ and $H$ is not adjacent to $v$ then there must exist $K \in H(u,v)$ such that $H$ and $K$ do not intersect. Moreover, there exists one such $K$ with $v$ adjacent to $K$.*

*Proof.* Let $H \in H(u,v)$ with $H$ not adjacent to $v$. Construct a geodesic $u = u_0, u_1, \ldots, u_n = v$ with corresponding geodesic order $\{H_1, H_2, \ldots, H_n\} = H(u,v)$. Then $H = H_i$ for some $1 \leq i \leq n-1$. If $H_{i+1}$ does not intersect $H$ we are done. Suppose otherwise. As $H_{i+1}$ intersects $H$, by Proposition 2.6.6, $u_{i+1}$ must be adjacent to $H = H_i$. The same argument then holds for $u_{i+2}, \ldots, u_n = v$ which is a contradiction. Hence there must exist $K \in H(u_{i+1}, v)$ that does not intersect $H_i$. If $K$ is not adjacent to $v$, we may repeat this process with until we find a $K'$ that is adjacent to $v$. As $K'$ is parallel to $K$, it is also parallel to $H$. $\square$

**Lemma 3.1.10.** *With the above notation, if $c_{ab}$ is nonzero, then there do not exist two non-intersecting hyperplanes in $H(a,b)$ with $a$ on the same side of each hyperplane as $y$.*

*Proof.* Let $c_{ab}$ be nonzero and suppose that $H$ and $K$ are two non-intersecting hyperplanes in $H(a,b)$ with $a$ on the same side of each hyperplane as $y$. By a previous lemma, any geodesic from $x$ to $y$ must cross one of these first. Without loss of generality, let $H$ occur first in every geodesic from $x$ to $y$. Let $H^+$ and $K^+$ be the half-spaces formed by $H$ and $K$ respectively with $x$ in that half-space and let $H^-$ and $K^-$ be the half-spaces formed by $H$ and $K$ respectively with $y$ in that half-space.

Let $v$ be a vertex on a geodesic from $x$ to $y$ that is after $H$ on the geodesic but before $k$. Then as $b$ is in $H^+$, the support of $c_z(v,y)\delta_b$ is contained in $H^+$ because $H$ is not used in $c_z(v,y)$. Further the support of $c_z(x,y)$ is contained in $K^+$. Moreover, the support of $c_z(x,y)\delta_b = c_z(x,v)c_z(v,y)\delta_b$ is contained in $K^+$ as $K$ is not used in $c_z(x,v)$. This implies the contradiction that $c_{ab}$ is zero. $\square$

For two vertices $x$, $y \in X$, we define the *interval* $[x,y]$ to be the set of all vertices on a geodesic edge-path from $x$ to $y$. One may also describe this set as the vertices in the intersection of all half-spaces containing $x$ and $y$. Note that if $v \in [x,y]$, no hyperplanes separate $v$ from both $x$ and $y$.

We must also establish some new notation. Define $t(a,b)$ to be the number of hyperplanes in $H(a,b)$ that separate $a$ and $x$. For three vertices $u, v, c \in X$, define $H(u,v;c)$ to be the set of hyperplanes that separate $u$ from $v$ and also separate $u$ from $c$ with the added condition that they also be adjacent to $c$. The sets $H(x, y; a)$ and $H(y, x; b)$ will be of particular interest to us. We will denote by $H(x, y; a)\Delta H(y, x; b)$ the symmetric difference of these two sets and define $\ell(a, b)$ to be the cardinality of this set.

**Lemma 3.1.11.** *With the above notation, if $c_{ab}$ is nonzero, then*

$$H(y, x; b)\backslash H(a, b) \ \cup \ H(x, y; a)\backslash H(a, b) = H(y, x; b)\Delta H(x, y; a).$$

*Proof.* Let $H \in H(y, x; b)\backslash H(a, b)$. Then $H$ must separate $x$ from $y$ and $b$ from $y$. Hence $b$ must be on the same side of $H$ as $x$. As $H$ does not separate $a$ from $b$, we must also have that $a$ is on the same side of $H$ as $x$, which further implies that $H \notin H(x, y; a)$. Similarly, if $H \in H(x, y; a)\backslash H(a, b)$, then $H$ must separate $x$ from $y$ and $x$ from $a$ but not $a$ from $b$. Hence $b$ must be on the same side of $H$ as $y$ which implies that $H \notin H(y, x; b)$. Therefore $H(y, x; b)\backslash H(a, b) \ \cup \ H(x, y; a)\backslash H(a, b) \subseteq H(y, x; b)\Delta H(x, y; a)$.

Let $H$ be a hyperplane in $H(y, x; b)\Delta H(x, y; a)$ and assume for a contradiction that $H \in H(a, b)$. Either $H \in H(y, x; b)\backslash H(x, y; a)$ or $H \in H(x, y; a)\backslash H(y, x; b)$, so first consider the case that $H \in H(y, x; b)\backslash H(x, y; a)$. As $H \notin H(x, y; a)$, we must have that $H$ does not separate $x$ from $a$ or $H$ is not adjacent to $a$.

Suppose that the former is true, that is, suppose that $H$ does not separate $x$ from $a$. As $H \in H(y, x; b)$, $H$ separates $y$ from $b$ and $y$ from $x$, so $b$ and $x$ must be on the same side of $H$. Hence $H$ does not separate $a$ and $b$ which is a contradiction.

Now suppose that the latter is true, that is, that $H$ is not adjacent to $a$. Recall that $b$ is on the same side of $H$ as $x$, so as $H \in H(a, b)$, $a$ is on the same side of $H$ as $y$. Hence $H$ must be oriented in reverse in any geodesic order of $H(x, y)$ and its induced geodesic order of $H(a, b)$. Furthermore, as $H$ is not adjacent to $a$, there must exist one hyperplane $H'$ that separates $a$ from $b$ that does not intersect $H$. As $H'$ does not intersect $H$ we must have that $H' \in H(a, b)$ also. This hyperplane $H'$ will also be oriented in reverse in the geodesic orders of $H(x, y)$ and $H(a, b)$ which contradicts our assumption that $c_{ab}$ is nonzero by the previous lemma.

The case that $H \in H(x, y; a) \backslash H(y, x; b)$ is exactly analogous to the case for $H \in H(y, x; b) \backslash H(x, y; a)$. □

For the following lemmas we will need several new vertices. Define $a_0$ to be the vertex that is separated from $a$ by all of the hyperplanes in $H(x, y; a)$ and $a_1$ to be the vertex separated from $a_0$ by all of the hyperplanes in $H(x, y; a) \backslash H(a, b)$. Note that we may think of $a_1$ as the vertex separated from $a$ by all of the hyperplanes in $H(x, y; a) \cap H(a, b)$. Simlarly, define $b_0$ to be the vertex that is separated from $b$ by all of the hyperplanes in $H(y, x; b)$ and $b_1$ to be the vertex separated from $b_0$ by all of the hyperplanes in $H(y, x; b) \backslash H(a, b)$. We may then think of $b_1$ as the vertex separated from $b$ by all the hyperplanes in $H(y, x; b) \cap H(a, b)$.

**Lemma 3.1.12.** *With the vertices constructed above, if $c_{ab}$ is nonzero then $H(a, a_1) = H(b, b_1)$.*

*Proof.* Let $H \in H(a, a_1) = H(x, y; a) \cap H(a, b)$. Then $a$ and $y$ are in one half-space with respect to $H$ and $b, x$ and $a_1$ are in the other and $a$ is adjacent to $H$. Suppose $H$ is not in $H(b, b_1) = H(y, x; b) \cap H(a, b)$. Then $b$ must not be adjacent to $H$. By Lemma 3.1.9, there must exist $K \in H(a, b) \subseteq H(x, y)$ such that $H$ and $K$ do not intersect with $a$ on the same side of each hyperplane as $y$, contradicting Lemma 3.1.10.

Let $H \in H(b, b_1) = H(y, x; b) \cap H(a, b)$.Then $b$ and $x$ are in one half-space with respect to $H$ and $a, y$ and $b_1$ are in the other and $b$ is adjacent to $H$. Suppose $H$ is not in $H(a, a_1) = H(x, y; a) \cap H(a, b)$.Then $a$ must not be adjacent to $H$. Again by Lemma 3.1.9, there must exist $K \in H(a, b) \subseteq H(x, y)$ such that $H$ and $K$ do not intersect with $a$ on the same side of each hyperplane as $y$, contradicting Lemma 3.1.10. □

**Lemma 3.1.13.** *With the vertices constructed above, if $c_{ab}$ nonzero then $H(a_1, b_1) = H(a, b)$.*

*Proof.* As noted above, $H(a, a_1) = H(x, y; a) \cap H(a, b)$ and $H(b, b_1) = H(y, x; b) \cap H(a, b)$. Let $H \in H(a_1, b_1)$. If $H$ separates $a$ and $a_1$ then we must have $H \in H(a, b)$. Suppose $H$ does not separate $a$ and $a_1$ and assume $H$ is not in $H(a,b)$. Then $H$ separates $b$ and $b_1$ which is a contradiction as $H(b,b_1) \subseteq H(a,b)$.

Let $H \in H(a,b)$. Suppose $H \notin H(a_1,b_1)$. Then $H$ separates $a$ from $a_1$ but does not separate $b$ from $b_1$, or vice versa, either of which is a contradiction as $H(a,a_1) = H(b,b_1)$ by Lemma 3.1.12. $\square$

**Lemma 3.1.14.** *With the above notation, if $c_{ab}$ is nonzero and $a$ and $b$ are in the interval $[x,y]$, then any geodesic from $y$ to $x$ that passes through $a_1$ and $b_1$ must pass through $b_1$ before $a_1$.*

*Proof.* Assume otherwise, then there must be a hyperplane $H$ in $H(a_1,b_1) = H(a,b)$ with $a_1$ on the same side of $H$ as $y$ and $b_1$ on the same side of $H$ as $x$. However $a$ may not be in the same half-space with respect to $H$ as $x$ and $b_1$ as then $H$ separates $a$ and $a_1$ which is a contradiction as $H(a,a_1) = H(x,y;a) \cap H(a,b)$ so $H$ must separate $x$ and $a$. Hence $a$ is in the same half-space as $y$ and $a_1$ and, as $H \in H(a,b)$, $b$ must be in the same half-space as $x$.

As $b$ and $b_1$ are then in the same half-space with respect to $H$, we have that $H \notin H(b,b_1) = H(y,x;b) \cap H(a,b)$. However $H \in H(a,b)$ and $H$ separates $y$ from $b$, so we must have that $b$ is not adjacent to $H$. By Lemma 3.1.9, there must exist $K \in H(a,b) \subseteq H(x,y)$ such that $H$ and $K$ do not intersect with $a$ on the same side of each hyperplane as $y$, contradicting Lemma 3.1.10. $\square$

**Lemma 3.1.15.** *With the above notation, if $c_{ab}$ is nonzero and $a$ and $b$ are in the interval $[x,y]$ there is a geodesic order on $H(x,y)$ such that $H(y,x;b)\backslash H(a,b) < H(a,b) < H(x,y;a)\backslash H(a,b)$.*

*Proof.* By the previous lemma, any geodesic through $a_1$ and $b_1$ from $y$ to $x$ passes through $b_1$ first. This geodesic order will be induced by the following geodesic: first follow any geodesic from $y$ to $b_0$; then follow any geodesic from $b_0$ to $b_1$; then follow any geodesic from $b_1$ to $a_1$; then follow any geodesic from $a_1$ to $a_0$; and, finally, follow any geodesic from $a_0$ to $x$.

We first note that $H(b_0, y) = H(b, y) \backslash H(y, x; b)$ and $H(b_1, b_0) = H(y, x; b) \backslash H(a, b)$, hence both sets are subsets of $H(b, y)$. As $b_0$ is separated from $b_1$ by $H(y, x; b) \backslash H(a, b) \subseteq H(y, b)$ we have that this geodesic passes through $b_0$ before $b_1$ and that $H(b_1, b_0)$, and $H(b_0, y)$ are disjoint.

As $H(b_1, b_0) = H(y, x; b) \backslash H(a, b)$, we can see that $H(b_1, b_0)$ and $H(a, b)$ are disjoint. Now suppose that there exists $H \in H(a, b)$ that is also in $H(b_0, y) = H(b, y) \backslash H(y, x; b)$. Then $a$ and $y$ are in the same half-space with respect to $H$ and $b$ and $x$ are in the other half-space. As $H \notin H(y, x; b)$, $H$ must not be adjacent to $b$. Then, as before, by Lemma 3.1.9, there must exist $K \in H(a, b) \subseteq H(x, y)$ such that $H$ and $K$ do not intersect with $a$ on the same side of each hyperplane as $y$, contradicting Lemma 3.1.10.

An analogous result to this argument gives us that the sets $H(x, a_0)$, $H(a_0, a_1)$ and$H(a, b)$ are pairwise disjoint which, together with the lemmas above, establishes the result. □

Recall that our goal is to determine the sign and the exponent of $w$ of the matrix coefficients $c_{ab}$. We will accomplish this by following the geodesic that we have constructed above. We will see that every hyperplane in $H(a, b)$ will contribute a factor of $z$ or $-z$ and that every hyperplane in $H(y, x; b) \Delta H(x, y; a)$ will contribute a factor of $w$. To accomplish this we will need some new notation. For a fixed geodesic order on $H(x, y)$, let $x = v_0, \ldots, v_n = y$ be the corresponding geodesic edge-path from $x$ to $y$ with corresponding hyperplanes $H_1, \ldots, H_n$. For a given $a$ and $b$, we will also have a sequence $\{c_0, \ldots, c_n\}$ that will be induced by this geodesic order in a manner outlined below. In particular, let $c_0 = b$. We will now define a second sequence $\Sigma_\alpha = (k_\alpha, \ell_\alpha, t_\alpha)$ recursively for $\alpha = 0, 1, \ldots, n$. We define $\Sigma_0 = (0, 0, 0)$. Assume that $\Sigma_0, \ldots, \Sigma_\alpha$ and $c_0, \ldots, c_\alpha$ have been defined. Consider the hyperplane $H_{n-\alpha}$. We will define $\Sigma_{\alpha+1}$ and $c_{\alpha+1}$ by considering the relationship between this hyperplane and the vertices $c_\alpha$, $a$ and $x$.

(1) If $H_{n-\alpha}$ is not adjacent to $c_\alpha$, then $c_{\alpha+1} = c_\alpha$ and $\Sigma_{\alpha+1} = \Sigma_\alpha$;

(2) If $H_{n-\alpha}$ is adjacent to $c_\alpha$ but does not separate $a$ from $c_\alpha$, then $c_{\alpha+1} = c_\alpha$

and $\Sigma_{\alpha+1} = \Sigma_\alpha + (0,1,0)$, that is, $\ell_{\alpha+1} = \ell_\alpha + 1$ and the remaining $k$ and $t$ are unchanged;

(3) If $H_{n-\alpha}$ is adjacent to $c_\alpha$ and separates $a$ from $c_\alpha$, then $c_{\alpha+1}$ is the vertex $\overline{c_\alpha}$ adjacent to $c_\alpha$ across $H_{n-\alpha}$ and:

   (a) if $H_{n-\alpha}$ *does not* separate $a$ from $x$ then $\Sigma_{\alpha+1} = \Sigma_\alpha + (1,0,0)$, that is, $k$ is incremented by 1 and $\ell$ and $t$ are unchanged;

   (b) if $H_{n-\alpha}$ *does* separate $a$ from $x$ then $\Sigma_{\alpha+1} = \Sigma_\alpha + (1,0,1)$, that is, $k$ and $t$ are incremented by 1 and $\ell$ is unchanged.

We will need the following lemma.

**Lemma 3.1.16.** *Let $H$ be a hyperplane and $f \in \ell^2(X)$. If the support of $f$ is contained in one half-space with respect to $H$ and $H \notin H(u,v)$, then the support of $c_z(u,v)f$ is contained in the same half-space with respect to $H$.*

*Proof.* Let $H$ be a hyperplane and let $u$ and $v$ be adjacent vertices in $X$. Let $K$ be the hyperplane separating $u$ and $v$. Let $y \in X$. If $y$ is not adjacent to $K$, then $c_z(u,v)\delta_y = \delta_y$ and the result holds. If $y$ is adjacent to $K$ then $c_z(u,v)\delta_y = w\delta_y \pm z\delta_{\overline{y}}$ where $\overline{y}$ is the vertex adjacent to $y$ across $K$. In either case, both $y$ and $\overline{y}$ are in the same half-space with respect to $H$ as they are adjacent across $K$. This naturally extends to cocycles where $u$ and $v$ are not adjacent and, as $span\{\delta_x | x \in X\}$ is a basis for $\ell^2(X)$, to $\ell^2(X)$. $\square$

**Proposition 3.1.17.** *With the above notation, if $c_n \neq a$ then $c_{ab}$ is zero, and if $c_n = a$, then $c_{ab} = (-1)^{t_n} z^{k_n} w^{\ell_n}$.*

*Proof.* We will now prove by induction that

$$c_{ab} = (-1)^{t_\alpha} z^{k_\alpha} w^{\ell_\alpha} \left[ c_z(v_0, v_1) \ldots c_z(v_{n-\alpha-1} v_{n-\alpha}) \delta_{c_\alpha}(a) \right]. \tag{3.1}$$

If $\alpha = 0$, we must show that $c_{ab} = [c_z(v_0, v_1) \dots c_z(v_{n-1}v_n)\delta_b(a)]$, which is the definition of $c_{ab}$. Now assume for some fixed $0 \leq \alpha < n-1$ that (3.1) holds. We then wish to show that

$$c_{ab} = (-1)^{t_{\alpha+1}} z^{k_{\alpha+1}} w^{\ell_{\alpha+1}} \left[c_z(v_0, v_1) \dots c_z(v_{n-\alpha-2}v_{n-\alpha-1})\delta_{c_{\alpha+1}}(a)\right].$$

However, this is equivalent to showing that

$$\begin{aligned} &(-1)^{t_\alpha} z^{k_\alpha} w^{\ell_\alpha} \left[c_z(v_0, v_1) \dots c_z(v_{n-\alpha-1}v_{n-\alpha})\delta_{c_\alpha}(a)\right] = \\ &(-1)^{t_{\alpha+1}} z^{k_{\alpha+1}} w^{\ell_{\alpha+1}} \left[c_z(v_0, v_1) \dots c_z(v_{n-\alpha-2}v_{n-\alpha-1})\delta_{c_{\alpha+1}}(a)\right]. \end{aligned} \tag{3.2}$$

<u>Case 1: Suppose $H_{n-\alpha}$ is not adjacent to $c_\alpha$.</u> By definition, $c_{\alpha+1} = c_\alpha$ and $\Sigma_{\alpha+1} = \Sigma_\alpha$ and, since $c_\alpha$ is not adjacent to $H_{n-\alpha}$, we have that $c_z(v_{n-\alpha-1}, v_{n-\alpha})\delta_{c_\alpha} = \delta_{c_\alpha} = \delta_{c_{\alpha+1}}$. Hence (3.2) follows.

<u>Case 2: $H_{n-\alpha}$ is adjacent to $c_\alpha$ but does not separate $a$ from $c_\alpha$.</u> Then $c_{\alpha+1} = c_\alpha$ and $\Sigma_{\alpha+1} = \Sigma_\alpha + (0,1,0)$ and $c_z(v_{n-\alpha-1}, v_{n-\alpha})\delta_{c_\alpha} = w\delta_{c_\alpha} - z\delta_{\overline{c_\alpha}}$. However, by Lemma 3.1.16, we have that $c_z(v_0, v_1) \dots c_z(v_{n-\alpha-2}, v_{n-\alpha-1})\delta_{\overline{c_\alpha}}(a) = 0$, hence

$$\begin{aligned} \text{LHS of (3.2)} &= (-1)^{t_\alpha} z^{k_\alpha} w^{\ell_\alpha} \left[c_z(v_0, v_1) \dots c_z(v_{n-\alpha-2}, v_{n-\alpha-1}) \left(w\delta_{c_\alpha}\right)\right](a) \\ &= (-1)^{t_\alpha} z^{k_\alpha} w^{\ell_\alpha+1} \left[c_z(v_0, v_1) \dots c_z(v_{n-\alpha-2}, v_{n-\alpha-1})\delta_{c_{\alpha+1}}\right](a) \\ &= \text{RHS of (3.2)}. \end{aligned}$$

<u>Case 3: Suppose $H_{n-\alpha}$ is adjacent to $c_\alpha$and separates $a$ from $c_\alpha$.</u> Then $c_{\alpha+1}$ is the vertex $\overline{c_\alpha}$ adjacent to $c_\alpha$ across $H_{n-\alpha}$.

<u>Subcase (i):</u> If the hyperplane *does not* separate $a$ from $x$, then $\Sigma_{\alpha+1} = \Sigma_\alpha + (1,0,0)$ and

$$c_z(v_0, v_1) \dots c_z(v_{n-\alpha-1}, v_{n-\alpha})\delta_{c_\alpha} = c_z(v_0, v_1) \dots c_z(v_{n-\alpha-2}, v_{n-\alpha-1}) \left(w\delta_{c_\alpha} + z\delta_{c_{\alpha+1}}\right).$$

However, by Lemma 3.1.16, we have that $c_z(v_0,v_1)\dots c_z(v_{n-\alpha-2},v_{n-\alpha-1})\delta_{c_\alpha}(a)=0$, hence

$$\begin{aligned}
\text{LHS of (3.2)} &= (-1)^{t_\alpha} z^{k_\alpha} w^{\ell_\alpha}\left[c_z(v_0,v_1)\dots c_z(v_{n-\alpha-2},v_{n-\alpha-1})\left(z\delta_{c_{\alpha+1}}\right)\right](a)\\
&= (-1)^{t_\alpha} z^{k_\alpha+1} w^{\ell_\alpha}\left[c_z(v_0,v_1)\dots c_z(v_{n-\alpha-2},v_{n-\alpha-1})\delta_{c_{\alpha+1}}\right](a)\\
&= \text{RHS of (3.2)}.
\end{aligned}$$

Subcase (ii):If the hyperplane *does* separate $a$ from $x$, then $\Sigma_{\alpha+1}=\Sigma_\alpha+(1,0,1)$ and

$$c_z(v_0,v_1)\dots c_z(v_{n-\alpha-1},v_{n-\alpha})\delta_{c_\alpha} = c_z(v_0,v_1)\dots c_z(v_{n-\alpha-2},v_{n-\alpha-1})\left(w\delta_{c_\alpha}-z\delta_{c_{\alpha+1}}\right).$$

However, by Lemma 3.1.16, we have that $c_z(v_0,v_1)\dots c_z(v_{n-\alpha-2},v_{n-\alpha-1})\delta_{c_\alpha}(a)=0$, hence

$$\begin{aligned}
\text{LHS of (3.2)} &= (-1)^{t_\alpha} z^{k_\alpha} w^{\ell_\alpha}\left[c_z(v_0,v_1)\dots c_z(v_{n-\alpha-2},v_{n-\alpha-1})\left(-z\delta_{c_{\alpha+1}}\right)\right](a)\\
&= (-1)^{t_\alpha+1} z^{k_\alpha+1} w^{\ell_\alpha}\left[c_z(v_0,v_1)\dots c_z(v_{n-\alpha-2},v_{n-\alpha-1})\delta_{c_{\alpha+1}}\right](a)\\
&= \text{RHS of (3.2)}.
\end{aligned}$$

We now have that $c_{ab}=(-1)^{t_n}z^{k_n}w^{\ell_n}\delta_{c_n}(a)$ which is only nonzero if $c_n=a$ and, in that case, $c_{ab}=(-1)^{t_n}z^{k_n}w^{\ell_n}$. □

We may now conclude with the following result.

**Proposition 3.1.18.** *With the above notation, if $c_{ab}$ is nonzero and $a$ and $b$ are in the interval $[x,y]$, then $c_{ab}=(-1)^{t(a,b)}z^{d(a,b)}w^{\ell(a,b)}$.*

*Proof.* By Proposition 3.1.17, $c_{ab}=(-1)^{t_n}z^{k_n}w^{\ell_n}$ and this calculation is independent of the geodesic chosen from $x$ to $y$. We will follow the geodesic from Lemma 3.1.15, which divides $H(x,y)$ into the geodesic order $H(b_0,y)<H(b_1,b_0)<H(a_1,b_1)<H(a_0,a_1)<H(x,a_0)$ where $H(a_1,b_1)=H(a,b)$, $H(a_0,a_1)=H(x,y;a)\backslash H(a,b)$ and $H(b_0,b_1)=H(y,x;b)\backslash H(a,b)$ by the construction of $a_0,a_1,b_0$ and $b_1$.

As we follow this geodesic from $y$ to $x$, all of the hyperplanes in $H(y, b_0)$ are not adjacent to $c_0 = b$ by the construction of $b_0$. Thus they fall into the first case of Proposition 3.1.17, that is, $\Sigma_{\alpha+1} = \Sigma_\alpha$. Hence these hyperplanes to not increment $k$, $t$ or $\ell$. Note that $c_\alpha = b$ for these hyperplanes.

We next follow the geodesic through all of the hyperplanes of $H(b_0, b_1) = H(y, x; b)\backslash H(a, b)$. Each of these fall into the second case of Proposition 3.1.17, that is, each of these is adjacent to $c_0 = b$ but does not separate $a$ from $c_0$. Hence $\Sigma_{\alpha+1} = \Sigma_\alpha + (0, 1, 0)$, that is, for each of these, $\ell_{\alpha+1} = \ell_\alpha + 1$. Again note that $c_\alpha = b$ for these hyperplanes.

We next follow the geodesic through all of the hyperplanes of $H(a_1, b_1) = H(a, b)$. Each of these fall into the third case of Proposition 3.1.17, that is, each of these is adjacent to $c_\alpha$, but does not separate $a$ from $c_\alpha$. If the hyperplane does not separate $a$ from $x$, then $\Sigma_{\alpha+1} = \Sigma_\alpha + (1, 0, 0)$. If the hyperplane does separate $a$ from $x$, then then $\Sigma_{\alpha+1} = \Sigma_\alpha + (1, 0, 1)$. At the end of this process, $c_\alpha = a$.

We next follow the geodesic through all of the hyperplanes of $H(a_1, a_0) = H(x, y; a)\backslash H(a, b)$. Each of these fall into the second case of Proposition 3.1.17, that is, each of these is adjacent to $c_\alpha = a$ but does not separate $a$ from $c_\alpha = a$. Hence $\Sigma_{\alpha+1} = \Sigma_\alpha + (0, 1, 0)$, that is, for each of these, $\ell_{\alpha+1} = \ell_\alpha + 1$.

Finally we follow the geodesic through $H(a_0, x)$. Each hyperplane in this set is not adjacent to $c_\alpha = a$, so falls into the first case of Proposition 3.1.17, that is, $\Sigma_{\alpha+1} = \Sigma_\alpha$. Hence these hyperplanes to not increment $k$, $t$ or $\ell$.

From this geodesic we can see that $t_n = t(a, b)$, $k_n = d(a, b)$, and $\ell_n = \ell(a, b)$ as these were defined above. □

We will see in the conclusion of Section 3.3, that this result is enough for our needs, even though we have restricted ourselves to the case that $a$ and $b$ are in the interval $[x, y]$. However, we can calculate these matrix coefficients in full generality. Define the *median* of three vertices $x$, $y$ and $v$ to be the unique vertex in $[x, y] \cap [y, v] \cap [v, x]$, and we denote this vertex $m(x, y, v)$ [10]. This yields the following three lemmas.

**Lemma 3.1.19.** *Let $x$, $y$, $v \in X$. For any hyperplane $H$, if two of the three vertices are in the same half-space with respect to $H$, then the median, $m(x, y, v)$ is in that half-space.*

*Proof.* Let $x$, $y$, $v \in X$. Let $H$ be a hyperplane with $x$ and $y$ in the same half-space with respect to $H$. The interval $[x, y]$ lies entirely in the same half-space as $x$ and $y$ and $m(x, y, v) \in [x, y]$, hence is also in this half-space. $\square$

**Lemma 3.1.20.** *Let $x$, $y$ and $v$ be vertices in $X$ and $H \in H(x, y)$. Then $H$ separates $y$ from $v$ if and only if $H$ separates $y$ from $m(x, y, v)$.*

*Proof.* Suppose $H$ separates $y$ from $v$, then $x$ and $v$ are in the same half-space and then $m(x, y, v)$ must be in this half-space by the previous lemma. Now let $H$ separate $y$ from $m(x, y, v)$ and assume $y$ does not separate $y$ from $v$. Then we have a contradiction of the previous lemma as $m(x, y, v)$ must be in the same half-space as $y$ and $v$. $\square$

Note that we may reverse the order of $x$ and $y$ in the above lemma to obtain the same result for the vertex $x$. Note also that this implies that for any hyperplane $H$ in $H(x, y)$, the median $m(x, y, v)$ must be in the same half-space with respect to $H$ as $v$. For ease of notation, for a vertex $u \in X$, define $u'$ to be $m(x, y, u)$ and and note that this vertex is in the interval $[x, y]$. We will now examine some properties of medians.

**Lemma 3.1.21.** *For vertices $x$, $y$, $a$ and $b$ in $X$, if $H(a, b) \subset H(x, y)$, the set of hyperplanes $H(a', b')$ is contained in the set of hyperplanes $H(a, b)$.*

*Proof.* Let $H \in H(a', b')$. Then $H$ must separate $x$ and $y$ as, if not, then $x$ and $y$ are on the same side of $H$ and hence both medians are also on that side. Suppose then that $H$ does not separate $a$ and $b$. Then $a$ and $b$ are on the same side of $H$. If $x$ is on this same side, then so are $a'$ and $b'$ and again $H$ fails to separate these medians. If $x$ is on the other side, then as $H$ separates $x$ from $y$, we have that $a$, $b$ and $y$ are on the same side of $H$ and, as before, $H$ fails to separate the medians. Hence $H \in H(a, b)$ and we have that $H(a', b') \subseteq H(a, b)$. $\square$

**Lemma 3.1.22.** *For vertices $x$, $y$, $a$ and $b$ in $X$, if $H(a,b) \subset H(x,y)$ then $H(a,b) = H(a',b')$.*

*Proof.* By the previous lemma, we have that $H(a',b') \subseteq H(a,b)$. Let $H \in H(a,b)$. Then $H$ separates $a$ fom $b$ and also $x$ from $y$. Suppose $a$ is in the same half-space as $x$. Then by Lemma 3.1.19, $a'$ is in the same half-space as $x$. As $H$ is in $H(a,b)$ and $H(x,y)$, $b$ is in the same half-space as $y$ and, by Lemma 3.1.19, $b'$ is in this half-space also. Hence $H \in H(a',b')$. If $a$ is in the same half-space as $y$ a similar argument shows that $H \in H(a',b')$. Therefore $H(a,b) \subseteq H(a',b')$, hence $H(a,b) = H(a',b')$. □

**Corollary 3.1.23.** *If $H \in H(x,y)$ and $u$ is adjacent to $H$ then $u'$ is adjacent to $H$. Moreover, in this case, if $\overline{u}$ is the vertex adjacent to $u$ across $H$, then $u'$ is adjacent to $(\overline{u})'$ across $H$.*

*Proof.* Let $H$, $u$ and $\overline{u}$ be as above. Then by Lemma 3.1.22, $H(u',(\overline{u})') = H(u,\overline{u}) = \{H\}$. Hence $u'$ is adjacent to $(\overline{u})'$ across $H$. □

**Lemma 3.1.24.** *For vertices $x$, $y$, $a$ and $b$ in $X$,if $H(a,b) \subseteq H(x,y)$, then $H(y,x;b) \subseteq H(y,x;b')$ and $H(x,y;a) \subseteq H(x,y;a')$. Furthermore $H(y,x;b) \backslash H(a,b) \subseteq H(y,x;b') \backslash H(a,b)$ and $H(x,y;a) \backslash H(a,b) \subseteq H(x,y;a') \backslash H(a,b)$.*

*Proof.* The first assertion is clear from Lemmas 3.1.20 and 3.1.23. The second assertion is clear from the first. □

In order to calculate the coefficient $c_{ab}$ in general, we will be comparing the coefficients $c_{ab}$ and $c_{a'b'}$. These are not exactly equal as the following examples in the cube complex $\mathbb{Z} \oplus \mathbb{Z}$ will show. Note that $c_{ab} = z^3w$ and $c_{a'b'} = z^3w^2$.

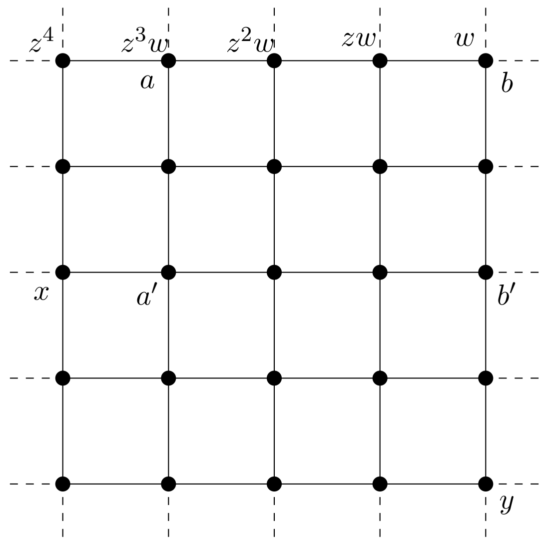


Figure 3.1: $c_z(x,y)\delta_b$

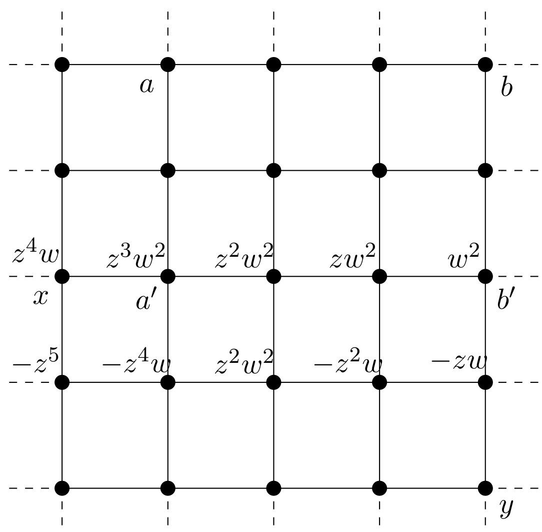


Figure 3.2: $c_z(x,y)\delta'_b$

We will need the following lemmas to connect the related values of $c_{ab}$ and $c_{a'b'}$. In the following, let $a$, $b \in X$ and let $c_\alpha$ and $\Sigma_\alpha = (k_\alpha, \ell_\alpha, t_\alpha)$ be as constructed in the process above for these vertices and let $c_n^*$ and $\Sigma_n^* = (k_\alpha^*, \ell_\alpha^*, t_\alpha^*)$ be the corresponding values when following the process for $a'$ and $b'$.

**Lemma 3.1.25.** *With the above notation, if $H(a,b) \subset H(x,y)$, then for all $0 \leq \alpha \leq n$, we have $H(a,c_\alpha) \subset H(a,b)$.*

*Proof.* We prove this by induction. When $\alpha = 0$, $c_0 = b$ so $H(a,c_\alpha) = H(a,b)$. Suppose for some $0 \leq \alpha < n$, we have $H(a,c_\alpha) \subset H(a,b)$. By the process used to create the $c_\alpha$, we have that $c_{\alpha+1} = c_\alpha$ except in the case that the hyperplane being crossed is adjacent to $c_\alpha$ and also separates $c_\alpha$ from $a$.

In this case, $c_{\alpha+1} = \overline{c_\alpha}$, the vertex across $H$ from $c_\alpha$, so $H(a,c_{\alpha+1}) = H(a,c_\alpha)\backslash\{H\} \subset H(a,b)$. □

**Corollary 3.1.26.** *With the above notation, if $H(a,b) \subset H(x,y)$, then for all $0 \leq \alpha \leq n$, we have $H(a,c_\alpha) = H(a',c'_\alpha)$.*

*Proof.* This is an immediate consequence of the previous lemma and Lemma 3.1.22. □

**Lemma 3.1.27.** *With the above notation, if $H(a,b) \subset H(x,y)$, then $c_\alpha^* = c'_\alpha$ for all $0 \leq \alpha \leq n$.*

*Proof.* We will prove by induction, for all $0 \leq \alpha \leq n$, that $c_\alpha^* = c'_\alpha$. When $n = 0$, we have $c_0 = b$ and $c_0^* = b'$, hence $c_0^* = c'_0$. Now suppose $c_\alpha^* = c'_\alpha$ for some $0 \leq \alpha < n$. Recall that the next hyperplane we cross is $H_{n-\alpha}$.

<u>Case 1: Suppose $H_{n-\alpha}$ is adjacent to $c_\alpha$ and $H \in H(a,c_\alpha)$.</u> By Corollary 3.1.23, we have that $H_{n-\alpha}$ is also adjacent to $c'_\alpha = c_\alpha^*$ and by Corollary 3.1.26, $H \in H(a',c'_\alpha) = H(a',c_\alpha^*)$. Then $c_{\alpha+1}$ is the vertex across $H_{n-\alpha}$ from $c_\alpha$ and $c_{\alpha+1}^*$ is the vertex across $H_{n-\alpha}$ from $c_\alpha^*$, so, again by Corollary 3.1.23, $c_{\alpha+1}^* = c'_{\alpha+1}$.

Case 2: Suppose $H_{n-\alpha}$ is adjacent to $c_\alpha$ and $H \notin H(a, c_\alpha)$. By Corollary 3.1.23, we have that $H_{n-\alpha}$ is also adjacent to $c'_\alpha = c^*_\alpha$ and by Corollary 3.1.26, $H \notin H(a', c'_\alpha) = H(a', c^*_\alpha)$. Then $c_{\alpha+1} = c_\alpha$ and $c^*_{\alpha+1} = c^*_\alpha$, so $c^*_{\alpha+1} = c'_{\alpha+1}$.

Case 3: Suppose $H_{n-\alpha}$ is not adjacent to $c_\alpha$. Then $c_{\alpha+1} = c_\alpha$ and we have two possibilities. If $H_{n-\alpha}$ is not adjacent to $c^*_\alpha = c'_\alpha$, then $c^*_{\alpha+1} = c^*_\alpha$, so again $c^*_{\alpha+1} = c'_{\alpha+1}$. If $H_{n-\alpha}$ is adjacent to $c'_\alpha$, we still have that $c^*_{\alpha+1} = c^*_\alpha = c'_\alpha = c'_{\alpha+1}$ unless $H_{n-\alpha}$ is in $H(a', c^*_\alpha)$. We will see that this is not possible.

Recall that by Corollary 3.1.26, we have $H(a, c_\alpha) = H(a', c'_\alpha) = H(a', c^*_\alpha)$. Suppose $H_{n-\alpha}$ is adjacent to $c^*_\alpha = c'_\alpha$ and $H_{n-\alpha}$ is in $H(a', c^*_\alpha) = H(a, c_\alpha)$. As $c_\alpha$ is not adjacent to $H_{n-\alpha}$, by Lemma 3.1.9 there must exist a hyperplane $K$ in $H(a, c_\alpha)$ that does not intersect $H$ with $c_\alpha$ adjacent to $K$.

Then by Corollary 3.1.23, $K$ is adjacent to $c'_\alpha$. However, we then have $H, K \in H(a', c'_\alpha)$ with both adjacent to $c'_\alpha$, in which case it has been shown that $H$ and $K$ must intersect [13], which is a contradiction. This concludes the proof. □

**Corollary 3.1.28.** *With the above notation, if $c_{ab}$ is nonzero, then $c_{a'b'}$ is nonzero.*

*Proof.* By Proposition 3.1.17, if $c_{ab}$ is non-zero, then $c_n = a$. Then, by Lemma 3.1.27, we have that $c^*_n = a'$. Finally, again by Proposition 3.1.17, we have that $c_{a'b'}$ is nonzero. □

We wish now to establish a relationship between $c_{ab}$ and $c_{a'b'}$ as we have previously shown that $c_{a'b'}$ can be exactly calculated. We will see that $c_{ab}$ and $c_{a'b'}$ differ only by a factor of $w$. In order to do so, we will again refer to the geodesic of Lemma 3.1.15, but constructed for $a'$ and $b'$ in this case.

We will also need a new definition. Consider the sets $(H(y, x; b')\backslash H(a', b'))\backslash(H(y, x; b)\backslash H(a, b))$ and $(H(x, y; a')\backslash H(a', b'))\backslash(H(x, y; a)\backslash H(a, b))$. Note that by Lemmas 3.1.11 and 3.1.22 and Corollary 3.1.28, if $c_{ab}$ is nonzero, these sets are disjoint. Define $r(a, b)$ to be the cardinality of this disjoint union.

**Lemma 3.1.29.** *With the above notation, if $c_{ab}$ is nonzero, then $c_{ab} = w^{-r(a,b)}c_{a'b'}$.*

*Proof.* By Proposition 3.1.17, if $c_{ab}$ is nonzero, then $c_{ab} = (-1)^{t_n}z^{k_n}w^{\ell_n}$. By Corollary 3.1.28, we then have that $c_{a'b'}$ is nonzero and, again by Proposition 3.1.17, we then have that $c_{a'b'} = (-1)^{t_n^*}z^{k_n^*}w^{\ell_n^*}$. Moreover, by Proposition 3.1.1, we also have that these calculations are independent of the geodesic chosen from $x$ to $y$. We will follow the geodesic from Lemma 3.1.15, calculated using $a'$ and $b'$, which divides $H(x,y)$ into the geodesic order $H((b')_0,y) < H((b')_1,(b')_0) < H((a')_1,(b')_1) < H((a')_0,(a')_1) < H(x,(a')_0)$ where $H((a')_1,(b')_1) = H(a',b') = H(a,b)$, $H((a')_0(a')a_1) = H(x,y;(a'))\backslash H(a',b') \supset H(x,y;a)\backslash H(a,b)$ and $H((b')_0,(b')_1) = H(y,x;b')\backslash H(a',b') \supset H(y,x;b)\backslash H(a,b)$ by the construction of $(a')_0, (a')_1, (b')_0$ and $(b')_1$.

As we follow this geodesic from $y$ to $x$, all of the hyperplanes in $H(y,(b')_0)$ are not adjacent to $c_0^* = b'$ by the construction of $(b')_0$. By Corollary 3.1.23, these hyperplanes are not adjacent to $c_0 = b$ either. Thus $\Sigma_{\alpha+1}^* = \Sigma_\alpha^*$ and $\Sigma_{\alpha+1} = \Sigma_\alpha$. Note that $c_\alpha^* = c_0^* = b'$ and $c_\alpha = c_0 = b$ for these hyperplanes.

We next follow the geodesic through all of the hyperplanes of $H((b')_0,(b')_1) = H(y,x;b')\backslash H(a',b')$. We have seen in Proposition 3.1.18 that each of these is adjacent to $c_0^* = b'$ but does not separate $a'$ from $c_0^*$. Hence $\Sigma_{\alpha+1}^* = \Sigma_\alpha^* + (0,1,0)$. By Corollary 3.1.26, each of these hyperplanes separates $a$ from $c_0$. However a hyperplane in this set may or may not be adjacent to $c_0$. If the hyperplane is in $H(y,x;b)\backslash H(a,b)$, then $\Sigma_{\alpha+1} = \Sigma_\alpha + (0,1,0) = \Sigma_\alpha^* + (0,1,0) = \Sigma_{\alpha+1}^*$. Otherwise, that is, if the hyperplane is in $(H(y,x;b')\backslash H(a,b))\backslash H(y,x;b)\backslash H(a,b))$, then $\Sigma_{\alpha+1} = \Sigma_\alpha = \Sigma_\alpha^* = \Sigma_{\alpha+1}^* - (0,1,0)$. Again note that $c_\alpha^* = c_0^* = b'$ and $c_\alpha = c_0 = b$ for these hyperplanes.

We next follow the geodesic through all of the hyperplanes of $H((a)_1',(b)_1') = H(a',b') = H(a,b)$. We have seen in Proposition 3.1.18 that each of these is adjacent to $c_\alpha^*$, but does not separate $a'$ from $c_\alpha^*$. By Lemma 3.1.26, each such hyperplane is adjacent to $c_\alpha$ but does not separate $a$ from $c_\alpha$. If the hyperplane does not separate $a'$ from $x$, then $\Sigma_{\alpha+1}^* = \Sigma_\alpha^* + (1,0,0)$. By Lemma 3.1.20, each such hyperplane does not separate $a$ from $x$, so

$\Sigma_{\alpha+1} = \Sigma_\alpha + (1,0,0) = \Sigma^*_\alpha + (1,0,0) = \Sigma^*_{\alpha+1}$. If the hyperplane does separate $a$ from $x$, then then $\Sigma^*_{\alpha+1} = \Sigma^*_\alpha + (1,0,1)$. By Lemma 3.1.20, each such hyperplane does separate $a$ from $x$, so $\Sigma_{\alpha+1} = \Sigma_\alpha + (1,0,1) = \Sigma^*_\alpha + (1,0,1) = \Sigma^*_{\alpha+1}$. Note at the end of this process, $c^*_\alpha = a'$ and $c_\alpha = a$.

We next follow the geodesic through all of the hyperplanes of $H((a')_1,(a')_0) = H(x,y;(a'))\backslash H(a',b')$. We have seen in Proposition 3.1.18 that each of these is adjacent to $c^*_\alpha = a'$ but does not separate $a'$ from $c^*_\alpha = a'$. Hence $\Sigma^*_{\alpha+1} = \Sigma_\alpha + (0,1,0)^*$. By Corollary 3.1.26, no such hyperplane separates $a$ from $c_\alpha$. However a hyperplane in this set may or may not be adjacent to $c_\alpha = a$. If the hyperplane is in $H(x,y;a)\backslash H(a,b)$, then $\Sigma_{\alpha+1} = \Sigma_\alpha + (0,1,0) = \Sigma_\alpha + (0,1,0)^* = \Sigma^*_{\alpha+1}$. Otherwise, that is, if the hyperplane is in $(H(x,y;a')\backslash H(a,b))\backslash H(x,y;a)\backslash H(a,b))$, then $\Sigma_{\alpha+1} = \Sigma_\alpha = \Sigma^*_\alpha = \Sigma^*_{\alpha+1} - (0,1,0)$. Again note that $c^*_\alpha = a'$ and $c_\alpha = a$ for these hyperplanes.

Finally we follow the geodesic through $H((a')_0,x)$. Each hyperplane in this set is not adjacent to $c^*_\alpha = a'$, so $\Sigma^*_{\alpha+1} = \Sigma^*_\alpha$. By Corollary 3.1.23, each of these is also not adjacent to $c_\alpha = a$, so $\Sigma_{\alpha+1} = \Sigma_\alpha$.

By Proposition 3.1.17, we have that these processes correctly calculate the values of $c_{a'b'}$ and $c_{ab}$ and, by the above, we can see that the only difference between these values occurs in the union of the disjoint sets $(H(y,x;b')\backslash H(a,b))\backslash H(y,x;b)\backslash H(a,b))$ and $(H(x,y;a')\backslash H(a,b))\backslash H(x,y;a)\backslash H(a,b))$. That is, $t_n = t^*_n$,$k_n = k^*_n$ and $\ell_n$ differs from $\ell^*_n$ by a factor of $w$ for every hyperplane in the above mentioned disjoint union. As the geodesic used traverses all of $H(x,y)$, this suffices for the result. □

**Lemma 3.1.30.** *With the above notation, if $H(a,b) \subseteq H(x,y)$, then:*

(1) $t(a',b') = t(a,b)$;

(2) $d(a',b') = d(a,b)$.

*If, moreover, $c_{ab}$ is nonzero, then:*

(3) $\ell(a',b') = \ell(a,b) + r(a,b)$.

*Proof.* Recall that we defined $t(a,b)$ to be the number of hyperplanes in $H(a,b)$ that separate $a$ and $x$. As $H(a',b') = H(a,b)$, by Lemma 3.1.20, this is exactly equal to $t(a',b')$. This proves the first assertion. The second assertion is a consequence of Lemma 3.1.22. The third assertion is by the definition of $\ell(a,b)$ and $r(a,b)$. □

**Theorem 3.1.31.** *With the above notation, if $c_{ab}$ is nonzero, then $c_{ab} = (-1)^{t(a,b)} z^{d(a,b)} w^{\ell(a,b)}$.*

*Proof.* By the three previous lemmas,

$$\begin{aligned} c_{ab} &= w^{-r(a,b)} c_{a'b'} \\ &= w^{-r(a,b)} (-1)^{t(a',b')} z^{d(a',b')} w^{\ell(a',b')} \\ &= (-1)^{t(a',b')} z^{d(a',b')} w^{\ell(a',b')-r(a,b)} \\ &= (-1)^{t(a,b)} z^{d(a,b)} w^{\ell(a,b)}. \end{aligned}$$

□

Our refinement of the result of Guentner and Higson, in the case that $a$ and $b$ are in the interval $[x,y]$, will be needed in Section 3.3. However, Guentner and Higson [9] used their less refined result to show that the cocycle $c_z(x,y)$ defined above is uniformly bounded for all $z \in \mathbb{D}$ and therefore that the associated representation $\pi_z = c_z(x,y)\pi(g)$ is also uniformly bounded for all $z \in \mathbb{D}$. As in the case of trees, when discussing more than one family of representations, we will denote by $\{\pi_z^C : |z| < 1\}$ the family of representations constructed by Guentner and Higson using cocycles.

## 3.2 The Extension of the Construction of Pytlik and Szwarc

As before, let $G$ be a discrete group acting on a CAT(0)-cube complex with vertex set $X$ and, for $x \in X$, let $\delta_x$ be the characteristic function of the one point set $\{x\}$. We will follow the methods used by Pytlik and Szwarc for the free group to construct a holomorphic family of representations of $G$ on $\ell^2(X)$. Define $c_c(X) \subset \ell^2(X)$ to be the space of all finitely supported complex functions on $X$. This space consists of all (complex) linear combinations of the $\delta_x$ with $x \in X$. As before, for all $x$, $y \in X$, let $d(x, y)$ be the geometric distance between $x$ and $y$.

Now fix a vertex $x$ in $X$. For the remainder of this section, consider this same vertex to be fixed . We define an *admissable cube-path* from $y$ to $a$ (in the direction of $x$), say $\mathcal{C} = (C_0, C_1 \dots C_n)$, to be one such that every hyperplane in every cube $C_i$ should separate $x$ and $y$ with $y \in C_0$ and $a \in C_n$ and also that each such hyperplane must separate $a$ from $y$. Given an admissable cube-path $\mathcal{C} = (C_0, C_1, \dots, C_n)$ from $y$ to $a$, the *weight* of $\mathcal{C}$, denoted $w(\mathcal{C})$, is defined to be $(-1)^n$ where $n$ is the number of even dimensional cubes in $\mathcal{C}$. Alternatively, $w(\mathcal{C}) = \prod_{i=0}^{n} (-1)^{\dim C_i - 1}$.

For every vertex $y \in X$, let $C_x(y)$ be the set of all vertices that are on a geodesic from $y$ to $x$ and in a cube with $y$. Define $P_j : c_c(X) \to c_c(X)$ to be the linear operator such that for all $y$ and $a \in X$ ,

$$P_j \delta_y(a) = \begin{cases} 1, & \text{if } a \in C_x(y),\ d(y, a) = j \geq 0 \\ 0, & \text{otherwise.} \end{cases}$$

where $d(y, a)$ is the geometric distance from $y$ to $a$. Note that $P_0$ is actually the identity operator, $P_j \delta_y = 0$ for every $j$ greater than the dimension of the cube complex and $P_j \delta_y(a) \neq 0$ implies that $d(a, x) \leq d(y, x)$. We then define the linear operator $Q_z = zP_1 - z^2 P_2 + z^3 P_3 - \dots + (-1)^{n-1} z^n P_n \pm \dots$. Note that $Q_z \delta_y$ will have only $d - 1$ nonzero terms, where $d$ is the dimension of the cube $C_x(y)$. We then also define the related operator $P_z = 1 - Q_z$ for later ease of use. The following diagram shows a simple example of $Q_z \delta_y$ for a three dimensional cube. In this cube $Q_z \delta_y = z\delta_{v_3} + z\delta_{v_5} + z\delta_{v_6} - z^2 \delta_{v_1} - z^2 \delta_{v_2} - z^2 \delta_{v_4} + z^3 \delta_{v_0}$.

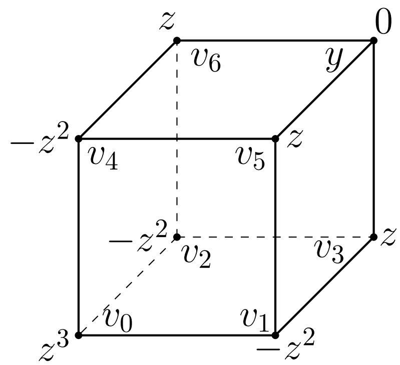


Figure 3.3: $Q_z\delta_y$

**Lemma 3.2.1.** *For every complex number $z$, the linear operator $P_z$ is invertible and*

$$P_z^{-1} = 1 + Q_z + Q_z^2 + \dots.$$

*Proof.* Let $f \in c_c(X)$. Then $f$ is a linear combination of $\delta_v, v \in X$, that is, $f = \sum_{v\in V} c_v\delta_v$ for some finite subset $V$ of $X$ and complex numbers $c_v \in \mathbb{C}, v \in V$. Let $n = \max\{d(v,x) : v \in V\}$ where $x$ is the previously chosen fixed vertex. Then for $m > n$, $Q_z^m\delta_v = 0$ for all $v \in V$ and we have $Q_z^m f$=0. Therefore

$$f = (1 - Q_z^m)f = (1 - Q_z)\left(1 + Q_z + Q_z^2 + \dots Q_z^{m-1}\right) f = (1 - Q_z)\left(1 + Q_z + Q_z^2 + \dots\right) f.$$

We conclude that $P_z^{-1} = 1 + Q_z + Q_z^2 + \dots$, which is a well-defined linear operator as $(1 + Q_z + Q_z^2 + \dots)f$ has only finitely many nonzero terms for every $f \in c_c(X)$. $\square$

**Lemma 3.2.2.** *Let $a \in X$ and $f \in c_c(X)$. If $(Q_zf)(a) \neq 0$ then there exists $b \in X$ such that $f(b) \neq 0$ and a cube $C$ such that $a$ and $b$ are vertices of $C$ and all hyperplanes in $C$ separate $b$ and $x$.*

*Proof.* Let $B \subseteq X$ be the collection of all $b$ such that $a$ lies on an admissible cube-path from

$b$ to $x$ and that $b$ is in a cube common with $a$. Then

$$((Q_z)f)(a) = Q_z \sum_{b \in B} f(b)\delta_b(a) = \sum_{b \in B} f(b) Q_z \delta_b(a).$$

Thus if $(Q_z f)(a) \neq 0$, there must exist some $b$ such that $f(b) \neq 0$ and $Q_z\delta_b(a) \neq 0$, the latter implying that $a$ and $b$ are in the same cube. $\square$

We will now need some new notation. Let $\mathcal{C}(y,a)$ be the set of all admissible cube-paths from $a$ to $y$ and let $\mathcal{C}_n(y,a)$ be the subset of these consisting of all admissible cube-paths from $y$ to $a$ of length $n$.

**Proposition 3.2.3.** *Let $y$ and $a$ be vertices in $X$ with $d(y,a) \geq 1$ and $(Q_z^n \delta_y)(a) \neq 0$. Then*

$$(Q_z^n \delta_y)(a) = z^{d(y,a)} \sum_{\mathcal{C} \in \mathcal{C}_n(y,a)} w(\mathcal{C}). \tag{3.3}$$

*Proof.* We will first prove this for $n = 1$. In this case $y$ and $a$ are in the same cube and there is only one admissible cube-path of length 1 and the minimum dimension sub-cube connecting $a$ and $y$ will have dimension $d(y,a)$. Let $C_1$ be this minimum dimension cube. Then

$$(Q_z\delta_y)(a) = (-1)^{d(y,a)-1} z^{d(y,a)} = z^{d(y,a)}(-1)^{dim\ C_1 - 1} = z^{d(y,a)} w(C_1) = z^{d(y,a)} \sum_{\mathcal{C} \in \mathcal{C}_n(y,a)} w(\mathcal{C})$$

Now for an arbitrary $n$, an admissible cube-path of cubic length $n$ from $y$ to $a$ can be given by a concatenation of an admissible cube-path of length 1 from $y$ to a vertex one closer in cube distance to $a$ and an admissible cube-path of length $n-1$ from that vertex to $a$. Let this set of vertices be denoted by $\{v_i\}_{i \in I}$ and note that these form a cube with $y$. Further note that for each $1 \leq i \leq n$, $\mathcal{C}_1(y, v_i)$ is a single element set. Then we may think of $\mathcal{C}_n(y,a)$ as the disjoint union of the sets $S_i = \{(D_i, C_1, \ldots, C_{n-1}) | D_i \in \mathcal{C}_1(y, v_i), (C_1, \ldots, C_{n-1}) \in \mathcal{C}_{n-1}(v_i, a)\}$. Then

$$\begin{aligned} Q_z^n \delta_y &= Q_z^{n-1} \left(Q_z \delta_y\right) \\ &= \sum_{i \in I} Q_z^{n-1} \left((-1)^{d(y,v_i)-1} z^{d(y,v_i)}\right) \delta_{v_i}. \end{aligned}$$

Now assume that (3.3) holds for all $1 \leq k < n$. Then

$$\begin{aligned} Q_z^n \delta_y(a) &= \sum_{i \in I} (-1)^{d(y,v_i)-1} z^{d(y,v_i)} Q_z^{n-1}(\delta_{v_i})(a) \\ &= \sum_{i \in I} (-1)^{d(y,v_i)} z^{d(y,v_i)-1} z^{d(v_i,a)} \sum_{\mathcal{C} \in \mathcal{C}_{n-1}(v_i,a)} w(\mathcal{C}) \\ &= z^{d(y,a)} \sum_{i \in I} (-1)^{d(y,v_i)-1} \sum_{\mathcal{C} \in \mathcal{C}_{n-1}(v_i,a)} w(\mathcal{C}) \\ &= z^{d(y,a)} \sum_{i \in I} w(D_i) \sum_{\mathcal{C} \in \mathcal{C}_{n-1}(v_i,a)} w(\mathcal{C}) \\ &= z^{d(y,a)} \sum_{\mathcal{E} \in \mathcal{C}_n(y,a)} w(\mathcal{E}) \end{aligned}$$

as $w(D_i) = (-1)^{d(y,v_i)-1}$ and $d(y, v_i) + d(v_i, a) = d(y, a)$ since $v_i$ is on a geodesic from $y$ to $a$. Therefore (3.3) holds. □

**Proposition 3.2.4.** *Let $y$ and $a$ be vertices in $X$. Then*

$$\sum_{\mathcal{C} \in \mathcal{C}(y,a)} w(\mathcal{C}) = 1.$$

*Proof.* We will prove this by induction on the geometric distance from $y$ to $a$. Suppose $d(a, y) = 1$. Then there is only one admissible cube-path from $y$ to $a$. This cube is 1-dimensional, hence has weight 1.

Suppose $d(a, y) = n$. The admissible cube-paths from $y$ to $a$ are determined by the vertices one closer in cubic distance to $y$ than $a$. As these vertices are all cubic distance one from $a$, they are all in a cube with $a$. Moreover, they must be in a cube together as they are all on an admissible cube-path from $y$ to $a$.

Hence these vertices, together with $a$, form a cube, say of dimension $k$. Note that this cube contains $2^k-1$ vertices other than $a$ which we will describe as $V=\{v_i : 1 \leq i \leq 2^k-1\}$. Let $\mathcal{C}(y, v_i)C_i$ be the set of all admissable cube-paths from $y$ to $v_i$ concatenated with the cube-path $C_i$ of cube-length 1 from $v_i$ to $a$. We may then describe the admissible cube-paths from $y$ to $a$ as the disjoint union

$$\mathcal{C}(y, a) = \bigcup_{i=1}^{2k-1} \mathcal{C}(y, v_i)C_i$$

where $C_i$ is the single cube containing $y$ and $v_i$.

As this is a disjoint union, we may then calculate the sum of the weights counting the vertices in increasing order of their geometric distance from $y$ and noting that for a cube of dimension $k$ containing $a$, the number of vertices of geometric distance $m$ from $a$ is $\binom{k}{m}$. Similar to above, we note that every admissible cube-path $\mathcal{C} \in \mathcal{C}(y, a)$ can be formed as a concatenation of an admissible cube-path from $y$ to some vertex $v_i \in V$ followed by the cube-path of length 1 from $v_i$ to $a$. Let $C_i$ be the cube of minimum dimension containing $v_i$ and $a$ and $\mathcal{C}_i$ the cube-path of cubic length 1 from $v_i$ to $a$. This gives

$$
\begin{aligned}
\sum_{\mathcal{C}\in\mathcal{C}(y,a)} w(\mathcal{C}) &= \sum_{i=1}^{2^k-1} \sum_{\mathcal{D}\in\mathcal{C}(y,v_i)} w(\mathcal{D})w(\mathcal{C}_i) \\
&= \sum_{i=1}^{2^k-1} \sum_{\mathcal{D}\in\mathcal{C}(y,v_i)} w(\mathcal{D}) \cdot (-1)^{dim\ C_i-1} \\
&= \sum_{i=1}^{2^k-1} (-1)^{dim\ C_i-1} \\
&= \sum_{v_i\in V} (-1)^{dim\ C_i-1} \\
&= \binom{k}{1}\cdot(-1)^0 + \binom{k}{2}\cdot(-1)^1 + \cdots + \binom{k}{k}\cdot(-1)^{k-1} \\
&= 1,
\end{aligned}
$$

where in the end, rather than sum over all vertices $v_i$, we sum over the sets of vertices of distance $m$ from $a$, $1 \leq m \leq k$. The final equality is from the following well known result about alternating binomial numbers:

$$\sum_{j=0}^{k} \binom{k}{j}(-1)^j(1)^{k-j} = 0.$$

□

Note that if $a = y$ or if $a$ is not on a geodesic from $y$ to $x$, $Q_z^n\delta_y(a) = 0$ for all $n$ by the definition of $Q_z$, hence $P_z^{-1}\delta_y(a) = 1$.

**Proposition 3.2.5.** *Let $a$ be on a geodesic from $y$ to $x$ with $a \neq y$. Then $P_z^{-1}\delta_y(a) = z^{d(y,a)}$.*

*Proof.* Let $a$ be on a geodesic from $y$ to $x$, $a \neq y$. Then

$$
\begin{aligned}
P_z^{-1}\delta_y(a) &= \left(1 + \sum_{n=1}^{\infty} Q_z\right)\delta_y(a) \\
&= \delta_y(a) + \sum_{n=1}^{\infty}\left(z^{d(y,a)} \sum_{\mathcal{C}\in\mathcal{C}_n(y,a)} w(\mathcal{C})\right) \\
&= 0 + z^{d(y,a)} \sum_{\mathcal{C}\in\mathcal{C}(y,a)} w(\mathcal{C}) \\
&= z^{d(y,a)}
\end{aligned}
$$

$\square$

For $z \in \mathbb{C}, |z| < 1$, let $w = \sqrt{1-z^2}$ where $\sqrt{1-z^2}$ denotes the principal branch of the square root. Let $p$ be the maximum dimension cube in the cube complex. For each vertex $v$ let $p_v$ be the number of directions geodesics may start from $v$ to $x$, that is, the number of hyperplanes separating $v$ from $x$ and adjacent to $v$. Then define a linear operator $T_z : c_c(X) \to c_c(X)$ by $T_z\delta_v = w^{p-p_v}\delta_v$. It is easy to see that $T_z^{-1}\delta_v = w^{p_v-p}\delta_v$.

We may now define a representation $\pi_z^P(g)$ of the group $G$ into the invertible linear transformations on $c_c(X)$. Let $\pi(g) : c_c(X) \to c_c(X)$ be the permutation representation of $G$. Note that for $f \in c_c(X)$, we have $\pi(g)f(x) = f(g^{-1}x)$ and, in particular, $\pi(g)(\delta_x) = \delta_{gx}$. We define, for $|z| < 1$,

$$\pi_z^P(g) = T_z^{-1}P_z^{-1}\pi(g)P_zT_z$$

and note that $\pi_z^P(g)$ is a conjugation of the permutation representation by $P_zT_z$.

As in Chapter 1, in the next section we will show that this representation is identical to the cocycle representation of Guentner and Higson on $c_c(X)$. As such, our representation will inherit the property of being uniformly bounded for all $z \in \mathbb{D}$ and may then be extended by continuity to $\ell^2(X)$.

## 3.3 The two constructions are identical for discrete groups acting on the cube complex

As in the case of trees, it is possible to prove that $\pi_z^C$ and $\pi_z^P$ are in fact identical if the group $G$ acts transitively on the set of vertices $X$ of a CAT(0)-cube complex using cyclic vectors. However, we wish to consider the more general case where the group is not necessarily acting transitively on the cube complex.

Let $z \in \mathbb{D}$, let $G$ be a discrete group acting on the CAT(0)-cube complex $X$. Let the two families of representations $\pi_z^C$ and $\pi_z^P$ be constructed as above. As in the case for trees, we will generalize our notation so that for all $z \in D$ and $v \in X$, $\pi_{z,v}^C$ and $\pi_{z,v}^P$ will be the families of representations constructed as above with initial fixed point $v$. Recall Propositon 3.1.28 which we restate.

**Proposition 3.3.1.** *Fix a vertex $x$. Then for $y \in X$ and $c_{ab} = \langle c_z(x,y)\delta_b, \delta_a\rangle$, if $c_{ab}$ is nonzero, then $c_{ab} = (-1)^{t(a,b)} z^{d(a,b)} w^{\ell(a,b)}$.*

Recall here that $H(y,x;b)$ was defined to be the subset of those hyperplanes in $H(x,y)$ that are adjacent to $b$ and separate $b$ from $y$; $H(x,y;a)$ was defined similarly for $a$ with respect to $x$; $\ell(a,b)$ was defined to be $|H(y,x;b)\Delta H(x,y;a)|$; and $t(a,b)$ was defined to be the number of hyperplanes in $H(a,b)$ with $a$ on the same side as $y$.

**Corollary 3.3.2.** *For all $a \in X$ on a geodesic from $x$ to $gx$, $\langle \pi_{z,x}^C(g)\delta_x, \delta_a\rangle = z^{d(a,gx)} w^{\ell(a,gx)}$ and $\ell(a,gx) = |H(x,gx;a)|$.*

*Proof.* Recall that $\pi_z^C(g)\delta_x = c_z(x,gx)\delta_{gx}$. We have that $H(gx,x;gx) = \emptyset$ hence $\ell(a,gx) = |H(x,gx;a)|$. Moreover, $a$ is on a geodesic from $x$ to $gx$, so $t(a,gx)$ is the number of hyperplanes in $H(a, gx)$ with $a$ on the same side as $gx$, hence $t(a, gx) = 0$.

$$\begin{aligned}\langle \pi_z^C(g)\delta_x, \delta_a\rangle &= \langle c_z(x, gx)\pi(g)\delta_x, \delta_a\rangle \\ &= \langle c_z(x, gx)\delta_{gx}, \delta_a\rangle \\ &= (-1)^{t(a,gx)} z^{d(a,gx)} w^{\ell(a,gx)} \\ &= z^{d(a,gx)} w^{\ell(a,gx)}.\end{aligned}$$

□

As in the tree case, we must now tie the two families of constructions together. As before, we first generalize the linear operators $P_z$ and $T_z$ from the Pytlik-Szwarc construction. Each was constructed using a fixed point $x \in X$. We now wish to vary this fixed point so for all $v \in X$, define $P_{z,v}$, $Q_{z,v}$ and $T_{z,v}$ to be the linear operators constructed using $v \in X$ as the fixed point. In the new notation

$$\pi_{z,v}^P(g) = T_{z,v}^{-1} P_{z,v}^{-1} \pi(g) P_{z,v} T_{z,v}.$$

**Lemma 3.3.3.** *Let $z \in D, g \in G$. Then for all $x \in X$ we have $\pi_{z,x}^C(g)\delta_x = \pi_{z,x}^P(g)\delta_x$.*

*Proof.* Let $v \in X$ be on a geodesic from $x$ to $gx$. Recall from Section 3.2 that for a particular vertex $v$, we defined $p_v$ to be the number of hyperplanes adjacent to $v$ that separate $v$ from $x$ and $p$ to be the dimension of the maximum dimension cube in the cube complex. Note that in this case $p_v = |H(x, gx; v)| = \ell(v, gx)$ by the previous corollary. By Proposition 3.2.5, for all vertices $v$ on a geodesic from $gx$ to $x$, $P_{z,x}^{-1}\delta_{gx}(v) = z^{d(v,gx)}$ which further implies that

$\langle T_z^{-1} P_{z,x}^{-1} \delta_{gx}, \delta_v \rangle = T_z^{-1} P_{z,x}^{-1} \delta_{gx}(v) = w^{p_v - p} z^{d(v,gx)}$. Then

$$\begin{aligned}
\langle \pi_{z,x}^P(g)\delta_x, \delta_v \rangle &= \langle T_{z,x}^{-1} P_{z,x}^{-1} \pi(g) P_{z,x} T_{z,x} \delta_x, \delta_v \rangle \\
&= \langle T_{z,x}^{-1} P_{z,x}^{-1} \pi(g) P_{z,x} w^p \delta_x, \delta_v \rangle \\
&= \langle T_{z,x}^{-1} P_{z,x}^{-1} \pi(g) w^p \delta_x, \delta_v \rangle \\
&= \langle T_{z,x}^{-1} P_{z,x}^{-1} w^p \delta_{gx}, \delta_v \rangle \\
&= w^{p_v} z^{d(v,gx)} \\
&= w^{\ell(v,gx)} z^{d(v,gx)} \\
&= \langle \pi_{z,x}^C(g)\delta_x, \delta_v \rangle.
\end{aligned}$$

If $v \in X$ is not on a geodesic from $x$ to $gx$, then $\langle \pi_{z,x}^C(g)\delta_x, \delta_v \rangle = \langle c_z(x, gx)\delta_{gx}, \delta_v \rangle = 0$ as $v$ is not in the interval $[x, gx]$. Furthermore

$$\begin{aligned}
\langle \pi_{z,x}^P(g)\delta_x, \delta_v \rangle &= \langle T_{z,x}^{-1} P_{z,x}^{-1} \pi(g) P_{z,x} T_{z,x} \delta_x, \delta_v \rangle \\
&= \langle T_{z,x}^{-1} P_{z,x}^{-1} \pi(g) P_{z,x} w^p \delta_x, \delta_v \rangle \\
&= \langle w^p T_{z,x}^{-1} P_{z,x}^{-1} \pi(g)(1 - Q_{z,x}) \delta_x, \delta_v \rangle \\
&= \langle w^p T_{z,x}^{-1} P_{z,x}^{-1} \pi(g) \delta_x, \delta_v \rangle \\
&= \langle w^p T_{z,x}^{-1} P_{z,x}^{-1} \delta_{gx}, \delta_v \rangle.
\end{aligned}$$

By the construction of $T_{z,x}^{-1}$ and $P_{z,x}^{-1}$, we can see that the support of $T_{z,x}^{-1} P_{z,x}^{-1} \delta_{gx}$ is the interval $[x, gx]$, hence $\langle \pi_{z,x}^P(g)\delta_x, \delta_v \rangle = 0$, which completes the proof. $\square$

For the following lemma, recall that $c_z(gx, gv)\pi(g) = \pi(g)c_z(x, v)$ as $c_z$ is a cocycle for $\pi$.

**Lemma 3.3.4.** *Let $x$, $v \in X$. If $z \in \mathbb{D}$ then $\pi_{z,x}^C(g) = c_z(x, v)\pi_{z,v}^C(g)c_z(v, x)$.*

*Proof.* Let $z \in \mathbb{D}$, $g \in G$ and $x$, $v \in X$. Then $\pi_{z,x}^C(g) = c_z(x, gx)\pi(g)$ and $\pi_{z,v}^C(g) =$

$c_z(v, gv)\pi(g)$. Hence,

$$
\begin{aligned}
c_z(x,v)\pi^C_{z,v}(g)c_z(v,x) &= c_z(x,v)c_z(v,gv)\pi(g)c_z(v,x) \\
&= c_z(x,gv)\pi(g)c_z(v,x) \\
&= c_z(x,gv)c_z(gv,gx)c_z(gx,gv)\pi(g)c_z(v,x) \\
&= c_z(x,gx)c_z(gx,gv)\pi(g)c_z(v,x) \\
&= c_z(x,gx)\pi(g)c_z(x,v)c_z(v,x) \\
&= c_z(x,gx)\pi(g) \\
&= \pi^C_{z,x}(g)
\end{aligned}
$$

□

**Proposition 3.3.5.** *Let $x$, $v \in X$. Then for every $z \in \mathbb{D}$ we have $c_z(x,v) = T^{-1}_{z,x}P^{-1}_{z,x}P_{z,v}T_{z,v}$.*

*Proof.* If $v = x$, then clearly $c_z(x,v) = 1 = T^{-1}_{z,x}P^{-1}_{z,x}P_{z,v}T_{z,v}$. If $d(x,v) = 1$, then $x$ and $v$ are adjacent. Let $a \in X$ and let $H$ be the hyperplane separating $x$ and $v$.

Case 1: $a = v$. Recall that for all $y, a \in X$, we have $T_{z,y}\delta_a = w^{p-p_a}\delta_a$ and $T^{-1}_{z,y}\delta_a = w^{p_a-p}\delta_a$ where $p$ is the maximum dimension cube in the cube complex and $p_a$ is the number of hyperplanes adjacent to $a$ that separate $a$ and $x$. Hence,

$$
\begin{aligned}
T^{-1}_{z,x}P^{-1}_{z,x}P_{z,v}T_{z,v}\delta_v &= T^{-1}_{z,x}P^{-1}_{z,x}P_{z,v}w^p\delta_v \\
&= w^pT^{-1}_{z,x}P^{-1}_{z,x}\delta_v \\
&= w^pT^{-1}_{z,x}(1 + Q_{z,x} + Q^2_{z,x} + \cdots)\delta_v \\
&= w^pT^{-1}_{z,x}(\delta_v + z\delta_x) \\
&= w^p(w^{1-p}\delta_v + zw^{-p}\delta_x) \\
&= w\delta_v + z\delta_x \\
&= c_z(x,v)\delta_v.
\end{aligned}
$$

Case 2: $a = x$. In this case, there is one less hyperplane adjacent to $a$ that separates $a$ and $x$. Hence

$$\begin{aligned}
T_{z,x}^{-1}P_{z,x}^{-1}P_{z,v}T_{z,v}\delta_x &= T_{z,x}^{-1}P_{z,x}^{-1}P_{z,v}w^{p-1}\delta_x \\
&= w^{p-1}T_{z,x}^{-1}P_{z,x}^{-1}(\delta_x - z\delta_v) \\
&= w^{p-1}T_{z,x}^{-1}(1 + Q_{z,x} + Q_{z,x}^2 + \cdots)(\delta_x - z\delta_v) \\
&= w^{p-1}T_{z,x}^{-1}(\delta_x - z\delta_v - z^2\delta_x) \\
&= w^{p-1}T_{z,x}^{-1}(w^2\delta_x - z\delta_v) \\
&= w^{p-1}(w^{2-p}\delta_x - zw^{1-p}\delta_v) \\
&= w\delta_x - z\delta_v \\
&= c_z(x,v)\delta_x.
\end{aligned}$$

Case 3: $a \neq x, v$. This case must be split into three subcases.

Subcase $i$: Suppose that $a$ is not adjacent to the hyperplane separating $x$ and $v$.

Then $Q_{z,x}\delta_a = Q_{z,v}\delta_a$, $T_{z,x}\delta_a = T_{z,v}\delta_a = w^r$ and $T_{z,x}^{-1}\delta_a = T_{z,v}^{-1}\delta_a = w^{-r}$ for some positive integer $r$. Moreover

$$\begin{aligned}
P_{z,x}^{-1}P_{z,v} &= (1 + Q_{z,x} + Q_{z,x}^2 + \cdots)(1 - Q_{z,v}) \\
&= (1 + Q_{z,x} + Q_{z,x}^2 + \cdots) - (Q_{z,v} + Q_{z,x}Q_{z,v} + Q_{z,x}^2Q_{z,v} + \cdots) \\
&= 1 + (Q_{z,x} - Q_{z,v}) + Q_{z,x}(Q_{z,x} - Q_{z,v}) + Q_{z,x}^2(Q_{z,x} - Q_{z,v}) + \cdots \\
&= 1 + P_{z,x}^{-1}(Q_{z,x} - Q_{z,v}).
\end{aligned}$$

This, together with the fact that $Q_{z,x}\delta_a = Q_{z,v}\delta_a$, gives

$$\begin{aligned}
T_{z,x}^{-1}P_{z,x}^{-1}P_{z,v}T_{z,v}\delta_a &= T_{z,x}^{-1}(1 + P_{z,x}^{-1}(Q_{z,x} - Q_{z,v}))w^r\delta_a \\
&= w^r T_{z,x}^{-1}(\delta_a + P_{z,x}^{-1}(Q_{z,x} - Q_{z,v})\delta_a) \\
&= w^r T_{z,x}^{-1}\delta_a \\
&= \delta_a \\
&= c_z(x,v)\delta_a.
\end{aligned}$$

<u>Subcase $ii$: Suppose $a$ is adjacent to $H$ and in the same half-space as $v$.</u>

In this subcase, and in the following subcase, we will prove the equivalent result $P_{z,x}T_{z,x}c_z(x,v) = P_{z,v}T_{z,v}$. Let $\overline{a}$ be the vertex adjacent to $a$ across $H$, that is, in the same half-space as $x$. Then $T_{z,v}\delta_a = T_{z,x}\delta_{\overline{a}} = wT_{z,x}\delta_a = w^r$ for some positive integer $r$. Let $C$ be the first cube in the normal cube-path from $a$ to $x$, $C_x$ be the first cube in the normal cube-path from $\overline{a}$ to $x$ and $C_v$ the first cube in the normal cube-path from $a$ to $v$. Note that the set of vertices of $C$ is the disjoint union of the sets of vertices of $C_x$ and $C_v$. We will examine the vertices of $C_x$ and $C_v$ separately. Note also that if $b \in X$ is not in $C$, $\langle Q_{z,v}T_{z,v}\delta_a, \delta_b\rangle = 0 = \langle Q_{z,x}T_{z,x}c_z(x,v)\delta_a, \delta_b\rangle$.

If $b \in C_v$, let $d(b,a) = n$. Then $d(b,\overline{a}) = n+1$, $\langle P_{z,x}\delta_{\overline{a}}, \delta_b\rangle = 0$ and $\langle P_{z,v}T_{z,v}\delta_a, \delta_b\rangle = \langle P_{z,v}w^r\delta_a, \delta_b\rangle = (-1)^n z^n w^r$. We also have that

$$\begin{aligned}
\langle P_{z,x}T_{z,x}c_z(x,v)\delta_a, \delta_b\rangle &= \langle P_{z,x}T_{z,x}(w\delta_a + z\delta_{\overline{a}}), \delta_b\rangle \\
&= \langle wP_{z,x}T_{z,x}\delta_a, \delta_b\rangle + \langle zP_{z,x}T_{z,x}\delta_{\overline{a}}, \delta_b\rangle \\
&= \langle wP_{z,x}w^{r-1}\delta_a, \delta_b\rangle + \langle zP_{z,x}w^r\delta_{\overline{a}}, \delta_b\rangle \\
&= \langle w^rP_{z,x}\delta_a, \delta_b\rangle + \langle w^rzP_{z,x}\delta_{\overline{a}}, \delta_b\rangle \\
&= (-1)^n z^n w^r + 0 \\
&= \langle P_{z,v}T_{z,v}\delta_a, \delta_b\rangle.
\end{aligned}$$

If $b \in C_x$, let $d(b, \overline{a}) = n$. Then $d(b,a) = n+1$ and $\langle P_{z,v}T_{z,v}\delta_a, \delta_b\rangle = \langle P_{z,v}w^r\delta_a, \delta_b\rangle = 0$. We also have that

$$\begin{aligned}
\langle P_{z,x}T_{z,x}c_z(x,v)\delta_a, \delta_b\rangle &= \langle P_{z,x}T_{z,x}(w\delta_a + z\delta_{\overline{a}}), \delta_b\rangle \\
&= \langle wP_{z,x}T_{z,x}\delta_a, \delta_b\rangle + \langle zP_{z,x}T_{z,x}\delta_{\overline{a}}, \delta_b\rangle \\
&= \langle wP_{z,x}w^{r-1}\delta_a, \delta_b\rangle + \langle zP_{z,x}w^r\delta_{\overline{a}}, \delta_b\rangle \\
&= (-1)^{n+1} z^{n+1} w^r + (-1)^n z^{n+1} w^r \\
&= 0 \\
&= \langle P_{z,v}T_{z,v}\delta_a, \delta_b\rangle.
\end{aligned}$$

Subcase $iii$: Suppose $a$ is adjacent to $H$ and in the same half-space as $x$.

As before, let $\overline{a}$ be the vertex adjacent to $a$ across $H$ which is now in the same half-space as $v$. Then$T_{z,x}\delta_a = wT_{z,v}\delta_a = wT_{z,x}\delta_{\overline{a}} = w^r$ for some positive integer $r$. As before, let $C$ be the first cube in the normal cube-path from $\overline{a}$ to $x$, $C_x$ the first cube in the normal path from $a$ to $x$ and $C_v$ the first cube in the normal path from $\overline{a}$ to $v$. Note that we again have that the set of vertices of $C$ is the disjoint union of the set of vertices of $C_x$ and $C_v$ and that if $b \in X$ is not in $C$, $\langle P_{z,v}T_{z,v}\delta_a, \delta_b\rangle = 0 = \langle P_{z,x}T_{z,x}c_z(x,v)\delta_a, \delta_b\rangle$.

If $b \in C_v$, let $d(b,a) = n$. Then $d(b,\overline{a}) = n-1$, $\langle wP_{z,x}w^r\delta_a, \delta_b\rangle = 0$ and $\langle P_{z,v}T_{z,v}\delta_a, \delta_b\rangle = \langle P_{z,v}w^{r-1}\delta_a, \delta_b\rangle = (-1)^n z^n w^{r-1}$ and

$$\begin{aligned}
\langle P_{z,x}T_{z,x}c_z(x,v)\delta_a, \delta_b\rangle &= \langle P_{z,x}T_{z,x}(w\delta_a - z\delta_{\overline{a}}), \delta_b\rangle \\
&= \langle wP_{z,x}T_{z,x}\delta_a, \delta_b\rangle - \langle zP_{z,x}T_{z,x}\delta_{\overline{a}}, \delta_b\rangle \\
&= \langle wP_{z,x}w^r\delta_a, \delta_b\rangle - \langle zP_{z,x}w^{r-1}\delta_{\overline{a}}, \delta_b\rangle \\
&= 0 - (-1)^{n-1}z^n w^{r-1} \\
&= \langle P_{z,v}T_{z,v}\delta_a, \delta_b\rangle.
\end{aligned}$$

If $b \in C_x$, let $d(b,\overline{a}) = n$. Then $d(b,a) = n-1$ and $\langle P_{z,v}T_{z,v}\delta_a, \delta_b\rangle = \langle P_{z,v}w^{r-1}\delta_a, \delta_b\rangle = (-1)^{n-1}z^{n-1}w^{r-1}$ and

$$\begin{aligned}
\langle P_{z,x}T_{z,x}c_z(x,v)\delta_a, \delta_b\rangle &= \langle P_{z,x}T_{z,x}(w\delta_a - z\delta_{\overline{a}}), \delta_b\rangle \\
&= \langle wP_{z,x}T_{z,x}\delta_a, \delta_b\rangle - \langle zP_{z,x}T_{z,x}\delta_{\overline{a}}, \delta_b\rangle \\
&= \langle wP_{z,x}w^r\delta_a, \delta_b\rangle - \langle zP_{z,x}w^{r-1}\delta_{\overline{a}}, \delta_b\rangle \\
&= (-1)^{n-1}z^{n-1}w^{r+1} - (-1)^n z^{n+1}w^{r-1} \\
&= (-1)^{n-1}z^{n-1}w^{r-1}(w^2+z^2) \\
&= (-1)^{n-1}z^{n-1}w^{r-1} \\
&= \langle P_{z,v}T_{z,v}\delta_a, \delta_b\rangle.
\end{aligned}$$

If $d(x,v) > 1$, then $x$ and $v$ are no longer adjacent. However, Guentner and Higson [9] have previously shown that $c_z(x,v)$ is not dependent on the geodesic (or even path) chosen from $x$ to $v$. Label the vertices of any particular geodesic from $x$ to $v$ by $x = v_0, v_1, v_2, \ldots, v_n = v$. Note that for $0 \leq i < n-1$,

$$T_{z,v_i}^{-1}P_{z,v_i}^{-1}P_{z,v_{i+1}}T_{z,v_{i+1}}T_{z,v_{i+1}}^{-1}P_{z,v_{i+1}}^{-1}P_{z,v_{i+2}}T_{z,v_{i+2}} = T_{z,v_i}^{-1}P_{z,v_i}^{-1}P_{z,v_{i+2}}T_{z,v_{i+2}}.$$

Therefore

$$\begin{aligned}
c_z(x,v) &= c_z(v_0,v_1)c_z(v_1,v_2)c_z(v_2,v_3)\cdots c_z(v_{n-1},v_n) \\
&= T_{z,v_0}^{-1}P_{z,v_0}^{-1}P_{z,v_n}T_{z,v_n} \\
&= T_{z,x}^{-1}P_{z,x}^{-1}P_{z,v}T_{z,v}.
\end{aligned}$$

□

**Corollary 3.3.6.** *Let $x, v \in X$. For every $z \in \mathbb{D}$, the representation $\pi_{z,x}^P(g)$ is equal to $c_z(x,v)\pi_{z,v}^P(g)c_z(v,x)$.*

*Proof.* Let $z \in \mathbb{D}$, $g \in G$ and $x$, $v \in X$. Then

$$\begin{aligned}
c_z(x,v)\pi_{z,v}^P(g)c_z(v,x) &= T_{z,x}^{-1}P_{z,x}^{-1}P_{z,v}T_{z,v}T_{z,v}^{-1}P_{z,v}^{-1}\pi(g)P_{z,v}T_{z,v}T_{z,v}^{-1}P_{z,v}^{-1}P_{z,x}T_{z,x} \\
&= T_{z,x}^{-1}P_{z,x}^{-1}\pi(g)P_{z,x}T_{z,x} \\
&= \pi_{z,x}^P(g).
\end{aligned}$$

□

**Theorem 3.3.7.** *For every $z \in \mathbb{D}$ and $x \in X$, the representations $\pi_{z,x}^P$ and $\pi_{z,x}^C$ are equal.*

*Proof.* Let $z \in \mathbb{D}$, $g \in G$. We have already seen that for all $x \in X$, $\pi_{z,x}^P(g)\delta_x = \pi_{z,x}^C(g)\delta_x$. Fix $x \in X$ and let $v \in X$. We will prove the assertion by induction on the geometric distance from $x$ to $v$. Let $d(x,v) = 1$. Then the only admissible cube-path from $x$ to $v$ is the edge joining $x$ and $v$. We then have that

$$
\begin{aligned}
&c_z(v,x)\pi^C_{z,x}(g)w\delta_v - c_z(v,x)\pi^P_{z,x}(g)w\delta_v\\
&= c_z(v,x)\pi^C_{z,x}(g)z\delta_x + c_z(v,x)\pi^C_{z,x}(g)w\delta_v - c_z(v,x)\pi^P_{z,x}(g)z\delta_x - c_z(v,x)\pi^P_{z,x}(g)w\delta_v\\
&= c_z(v,x)\pi^C_{z,x}(g)(z\delta_x + w\delta_v) - c_z(v,x)\pi^P_{z,x}(g)(z\delta_x + w\delta_v)\\
&= c_z(v,x)\pi^C_{z,x}(g)c_z(x,v)\delta_v - c_z(v,x)\pi^P_{z,x}(g)c_z(x,v)\delta_v\\
&= \pi^C_{z,v}(g)\delta_v - \pi^P_{z,v}(g)\delta_v \text{ (by Corollary 3.3.6)}\\
&= 0.
\end{aligned}
$$

As $w$ is a nonzero complex number and $c_z(x,v)$ is an invertible operator, we then have $\pi^C_{z,x}(g)\delta_v = \pi^P_{z,x}(g)\delta_v$.

Now assume that the assertion holds for all vertices $y \in X$ such that $d(x,y) \leq n-1$. Let $v \in X$ be such that $d(x,v) = n$. Then by the cocycle construction of Guentner and Higson, $c_z(x,v)\delta_v$ is a linear combination of elements of the set $\{\delta_y | d(y,x) \leq n, y \text{ on a geodesic from } v \text{ to } x\}$ and $\langle c_z(x,v)\delta_v, \delta_v\rangle = w^k$ for some positive integer $k$. We may then let $c_z(x,v)\delta_v = w^k\delta_v + \xi$ where $\xi$ is a linear combination of $\delta_y$ with all $y$ such that $d(x,y) < n$. By the induction hypothesis, $\pi^C_{z,x}(g)\xi = \pi^P_{z,x}(g)\xi$. Recalling again that $c_z(v,x)$ is invertible and that $w$ is nonzero, we have that

$$
\begin{aligned}
&c_z(v,x)\pi^C_{z,x}(g)w^k\delta_v - c_z(v,x)\pi^P_{z,x}(g)w^k\delta_v\\
&= c_z(v,x)\pi^C_{z,x}(g)w^k\delta_v + c_z(v,x)\pi^C_{z,x}(g)\xi - c_z(v,x)\pi^P_{z,x}(g)w^k\delta_v - c_z(v,x)\pi^P_{z,x}(g)\xi\\
&= c_z(v,x)\pi^C_{z,x}(g)(w^k\delta_v + \xi) - c_z(v,x)\pi^P_{z,x}(g)(w^k\delta_v + \xi)\\
&= c_z(v,x)\pi^C_{z,x}(g)c_z(x,v)\delta_v - c_z(v,x)\pi^P_{z,x}(g)c_z(x,v)\delta_v\\
&= \pi^C_{z,v}(g)\delta_v - \pi^P_{z,v}(g)\delta_v \text{ (by Corollary 3.3.6)}\\
&= 0
\end{aligned}
$$

As before, we then have $\pi^C_{z,x}(g)\delta_v = \pi^P_{z,x}(g)\delta_v$. $\square$

Our family of representations inherits the properties of being holomorphic and uniformly bounded from the work of Guentner and Higson.

# Chapter 4

# An Example

## 4.1 Constructing a Cube Complex in $H^3$

As the representations that we have studied and created above are of groups acting on a CAT(0)-cube complex, in this chapter we will examine an example of a group acting on a non-positively curved (or locally CAT(0)) cube complex which yields an interesting result. We will investigate how three manifold groups can act freely on a non-positively curved cube complex. By a three manifold group we will mean the fundamental group of a compact orientable three manifold without boundary. The group we will examine will be a finite index subgroup of the orbifold group $U$ of the Borromean rings and it will be acting on a quotient space of $H^3$. [6]

In order to construct the cube complex we wish to examine, we must first examine a tessellation of $H^3$ by dodecahedra, hence we will need the following geometry. A regular Euclidean dodecahedron can be cubulated into 8 cubes in a natural way, as indicated below in Figure 4.2, where we have depicted a cube-like polyhedron $C_0$ contained in the unit cube in the positive octant. The coordinates are the usual $(x, y, z)$ coordinates and $P, Q$ and $R$ are faces of $C_0$.

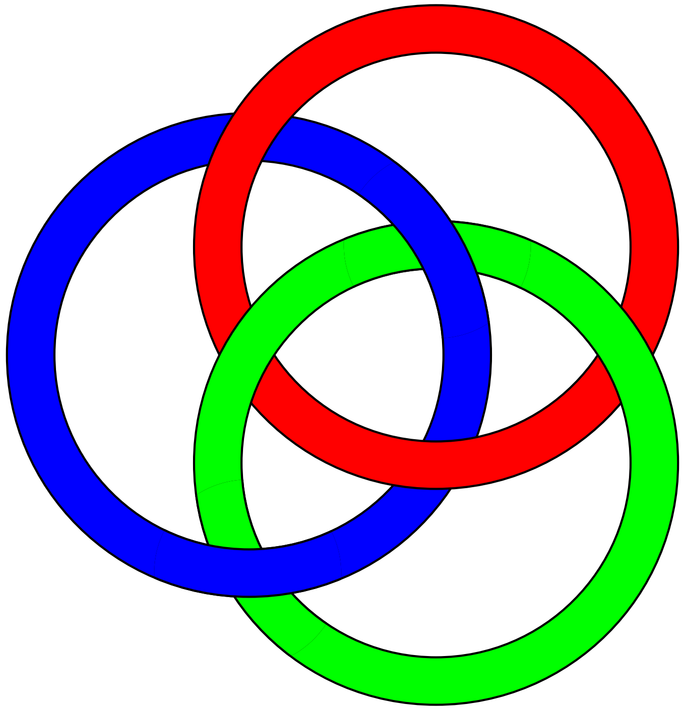

Figure 4.1: The Borromean Rings

Let $H$ be the order 8 group generated by reflections in the $xy, xz$ and $yz$ planes so that $H$ is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Then for any choice of $a, 0 < a < 1$, we define $D(a)$ to be the union of $h(C_0)$ for all $h \in H$. In particular, if we let $a = \frac{1}{2}(3 - \sqrt{5})$, the three planes $P, Q$ and $R$ depicted in Figure 4.2 intersect at the point $(t, t, t)$ with $t = \frac{1}{2}(\sqrt{5} - 1)$. Note that $t = 1 - a$.

**Lemma 4.1.1.** *The polyhedron $D(a)$ is a regular Euclidean dodecahedron.*

*Proof.* We can see that the faces of $C_0$ that are subsets of the axis planes will be internal in $D(a)$, hence need not be considered. As $|H| = 8$, the resulting polyhedron will have 24 quadrangular faces. However, pairs of these are coplanar and form 12 regular pentagons.

As an example, from the vertices of $P$ we can calculate a normal vector to $P$ to be $(1, 0, 1 - a)$. Let $P'$ be the face of the polygon resulting in reflecting $P$ in the $xz$ plane. It will have vertices $(1, 0, 0), (1, -a, 0), (t, -t, t)$ and $(a, 0, 1)$. From these vertices we find the

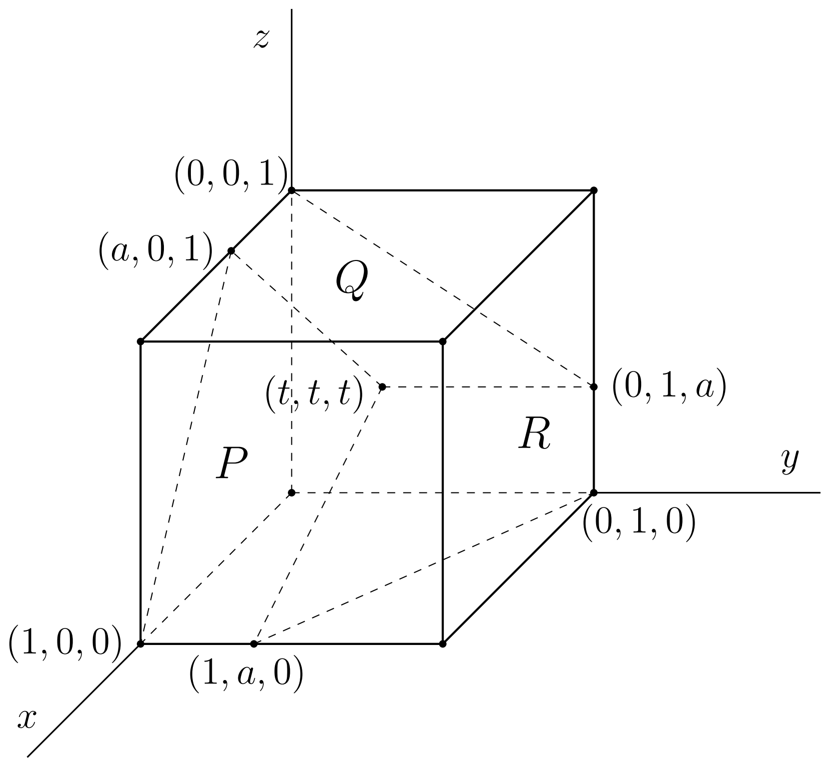


Figure 4.2: Topological Cube

same normal vector hence $P$ and $P'$ are coplanar.

Denote by $P^*$ the pentagon consisting of the union of $P$ and $P'$. It has vertices $(1, -a, 0)$, $(1, a, 0)$, $(t, t, t)$, $(a, 0, 1)$, and $(t, -t, t)$. It is easy to show that these form a regular pentagon and further that the other 22 faces of the polygon form 11 more regular pentagons. □

Thus $D = D(a)$ is naturally decomposed into eight "topological" cubes by the eight octants. We can then choose a sphere at infinity centered at the origin, of minimum radius $t\sqrt{3} = \frac{\sqrt{3}}{2}(\sqrt{5}-1)$, and our regular dodecahedron becomes a regular hyperbolic dodecahedron in the Klein model for $H^3$. In particular, if we let the radius of the sphere at infinity be the square root of the golden mean, we arrive at a useful result.

**Lemma 4.1.2.** *Let the radius of the sphere at infinity be* $\sqrt{\frac{1}{2}(\sqrt{5}+1)}$*, then the dihedral angle of the dodecahedron becomes* $90°$.

*Proof.* Let $S^\infty$ denote the sphere at infinity. Let $P^*$ be the pentagon composed of $P$ from

Figure 4.2 together with its reflection in the $xz$ plane $P'$, as in the proof of Lemma 4.1.1. We have seen that a normal vector to $P^*$ is $(1, 0, 1-a)$ hence $P^*$ sits in the plane $x+(1-a)z = 1$. For ease of notation, we will denote this plane $P^\infty$. Let $Q^*$ be the adjacent pentagon that shares an edge with $P^*$ adjoining the vertices $(1, -a, 0)$ and $(1, a, 0)$. It is easy to show that $Q^*$ sits in the plane $x + (a-1)z = 1$. Denote this plane $Q^\infty$.

Let $R$ denote the radius of the sphere at infinity. The planes $P^\infty$ and $Q^\infty$ intersect $S^\infty$ in circles that intersect at two points, one of which is $(1, \sqrt{R^2-1}, 0)$. As we are working in the Klein model, the dihedral angle between the two planes will be the same as the angle of intersection of these two circles. As $(1, 0, 1-a)$ is normal to $P^\infty$, it is also normal to the circle $P^\infty \cap S^\infty$. We also have that $(1, \sqrt{R^2-1}, 0)$ is normal to $P^\infty \cap S^\infty$, hence their cross product is a tangent vector to this circle. It is slightly simpler to use $t = 1-a$ to calculate this cross product to be $(-t\sqrt{R^2-1}, t, \sqrt{R^2-1})$. Similarly, $(1, 0, a-1)$ and $(1, \sqrt{R^2-1}, 0)$ are normal to $Q^\infty \cap S^\infty$ and their cross product is $(t\sqrt{R^2-1}, -t, \sqrt{R^2-1})$. If we then set the dot product of these two vectors to be zero, it is then simple algebra to show that the solution is $R = \sqrt{\frac{1}{2}(\sqrt{5}+1)}$. □

A presentation of the hyperbolic orbifold group $U$ of the Borromean rings is given by

$$U = \langle a, b, c | a^4, b^4, c^4, ab\bar{c}\bar{b}c = b\bar{c}\bar{b}ca, bc\bar{a}\bar{c}a = c\bar{a}\bar{c}ab, ca\bar{b}\bar{a}b = a\bar{b}\bar{a}bc \rangle$$

where $a$, $b$ and $c$ represent rotations of $90^o$ in the axes drawn in the dodecahedron above and $S^3 = H^3/U$ is defined by pasting matching faces of the dodecahedron, hence has singular set the Borromean rings [6]. For example, in the dodecahedron, any 2 pentagons that share a common edge that lies on an axis of rotation are pasted together by the $90°$ rotation in that axis.

The group $U$ always contains rotations, both $90°$ and $180°$, hence does not act freely on $H^3$. However, s $U$ is a Kleinian group, we are guaranteed that there exists a finite index subgroup of $U$ that does act freely [2]. We will shortly define this subgroup.

From the presentation of $U$ we see that we can define a homomorphism $\alpha$ that maps $a, b$ and $c$ to 1 in $\mathbb{Z}_4$. We simply check the relations to establish that this is a homormorphism. We may then define a subgroup $A$ of $U$ to be the kernel of this homomorphism. $A$ is then a subgroup of index 4. Moreover $a, b$ and $c$ represent rotations and any nontrivial rotation will be conjugate to $a^j, b^j$ or $c^j$ where $j = 1, 2$ or 3 [6]. This follows from the fact that any two oriented meridians corresponding to the same oriented component of a link are conjugate in the fundamental group of the complement of that link and from the fact that the only orientation preserving hyperbolic isometry with a fixed point is a rotation [6]. It follows from this observation that the kernel of $\alpha$, $A$, contains no rotations, hence acts freely on $H^3$.

Thus we have a 4-fold regular branched covering of $S^3$, with branch set the Borromean rings. We shall denote manifold that is the branch cover of $S^3$ by $W$. The cover $W$ is a compact hyperbolic manifold (not just orbifold), tessellated by four dodecahedra [7]. This tessellation of $W$ by four dodecahedra lifts to a tessellation of $H^3$ using the universal covering space map $\rho : H^3 \longrightarrow W$ and pulling back the tessellation.

Note that as the dodecahedra in $H^3$ are regular right dihedral angled dodecahedra, two distinct dodecahedra in $H^3$ intersect in a common pentagonal face of both, in a common edge, in a common vertex or not at all. Our decomposition of each dodecahedra into 8 cube-like polyhedra then lifts to a decomposition of $W$ into 32 cubes. This decomposition of $W$ then lifts to a decomposition of $H^3$ into "topological" cubes.

There are two types of vertices of each of these cubes, vertices that don't belong to axes of rotation and vertices that do. Those that don't belong to axes of rotation have neighborhoods that are embedded as, if we remove the axes of rotation, we have a covering space map from $H^3 - \{\text{axes of rotation}\}$ to $S^3 - \{\text{the Borromean rings}\}$. The link of a vertex is defined by local edges and there can be no identification of local edges. The axes that do belong to axes of rotation have links that look like a double cone on an octagon (see figure 4.3. below). If the group contains a rotation, the link in the quotient space will contain a bigon and the quotient space would not be non-positively curved. However, we will see below that our group contains no rotations, hence the quotient space will be non-positively curved.

Alternatively, we may examine the links of the vertices of $W$. There are two types of vertices. One type has a dihedral angle of $90^o$ and the link of these types of vertices can be seen in the first figure below. The second has a dihedral angle of $45^o$ and the link of these can be seen in the second figure below. In each case, the trianglular simplices of these links are embedded in a topological cube in $H^3$, hence each link is a flag simplicial complex and $W$ is non-positively curved.

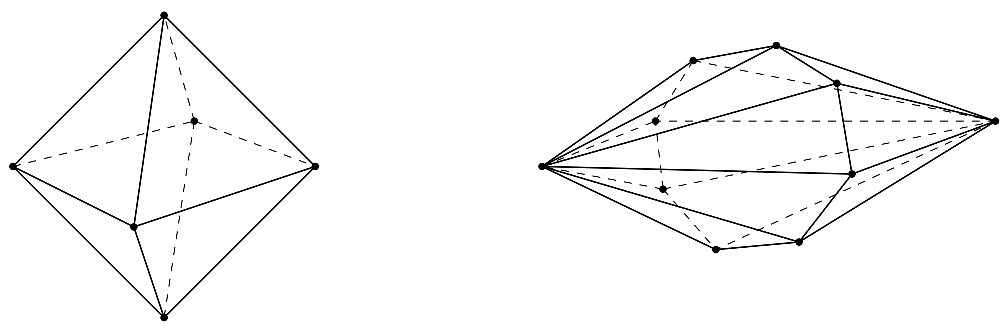

Figure 4.3: Links of vertices in $W$

We can define homeomorphisms from the hyperbolic cube-like polyhedra to Euclidean cubes in such a way that the Bridson-Haefliger definition of a cube complex in Chapter 2 above is satisfied.

## 4.2 Establishing the Group and Conclusions

Now let $M$ be an arbitrary closed oriented 3-manifold with three manifold group $\Pi$, that is, $\Pi$ is the fundamental group of $M$ ($\Pi = \pi_1(M)$). As before, let $U$ be the orbifold group of the Borromean rings and $A$ the kernel of the homomorphism $\alpha$ defined above. Let $G$ be the finite index subgroup of $U$ such that the quotient space of $H^3$ under the action of $G$ is homeomorphic to $M$ which is guaranteed to exist [6]. Let $TOR(G)$ be the subgroup of $G$

generated by the rotations in $G$.

**Lemma 4.2.1.** *With the above definitions, both $A \cap G$ and $TOR(G)$ are normal subgroups of $G$.*

*Proof.* As $A$ is the kernel of a homomorphism from $U$ to $\mathbb{Z}_4$, $A$ is a normal subgroup of $U$. Hence $A \cap G$ is a normal subgroup of $G \subset U$. Let $t$ be a generator in $TOR(G)$ and $g \in G$. As $t$ is a rotation, it has finite order, say $n$. Then $(gtg^{-1})^n = gt^n g^{-1} = e$, hence $gtg^{-1}$ is a rotation, hence a generator in $TOR(G)$. Since the generating set is closed under conjugation, $TOR(G)$ is closed under conjugation. $\square$

**Lemma 4.2.2.** *Any one $90°$ rotation in $TOR(G)$ together with $A \cap G$ generates $G$.*

*Proof.* It is clear from the proof of the universality of $U$ (ref) that the group $G$ that gives rise to $M$ always contains $90°$ rotations. Let $g \in G$. Let $t$ be any $90°$ rotation in $G$. Suppose $\alpha(g) = i$ where $i = 0, 1, 2$ or $3$. Then $\alpha(t^{4-i}g) = 0$, hence $t^{4-i}g = a \in A$. Therefore $g = t^i a \in t^i A$. $\square$

The previous lemma gives a commutative diagram of group inclusions shown below in Figure 4.4. Note that all subgroups are normal subgroups.

$$\begin{tikzcd}
 & G & \\
TOR(G) \arrow[ur] & & A \cap G \arrow[ul] \\
 & A \cap TOR(G) \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

Figure 4.4: Subgroups

We will need a theorem of Armstrong .

**Theorem 4.2.3** ([1])**.** *Let $M$ be a 3-manifold, let $G$ be the subgroup of $U$ such that $H^3/G \cong M$ and let $TOR(G)$ be the subgroup of $G$ generated by rotations. Then $\pi_1(M)$ is isomorphic to $G/TOR(G)$.*

The diagram of group inclusions in the previous section, together with a basic result from group theory, and Armstrong's theorem yield the following corollary.

**Corollary 4.2.4.** *With the above notation,* $\pi_1(M) \cong (A \cap G)/(A \cap TOR(G))$

We now have another diagram of group inclusions $e \lhd A \cap TOR(G) \lhd A \cap G$ which yields a series of covering space maps

$$H^3 \longrightarrow H^3/(A \cap TOR(G)) \longrightarrow H^3/(A \cap G).$$

Note that in this example we have regular coverings as the corresponding subgroups are normal.

Then, by covering space theory, as the group $A$ is acting freely, and as $H^3$ is the simply connected universal cover for both of these spaces, we have $\pi_1(H^3/(A \cap TOR(G))) = A \cap TOR(G)$ and $\pi_1(H^3/(A \cap G)) = A \cap G$. Therefore the group of covering space transformations of $H^3/(A \cap TOR(G))$ as a covering space of $H^3/(A \cap G))$ is isomorphic to

$$(A \cap G)/(A \cap TOR(G)) \cong \pi_1(M).$$

Moreover, all of the covering space maps preserve the cubulation of the space $H^3/(A \cap TOR(G))$ induced by the cubulation of $H^3$. We may summarize the preceding in a theorem. Recall that a non-positively curved space need not be simply connected.

**Theorem 4.2.5.** *Let* $\Pi$ *be a 3-manifold group. Then there is a non-positively curved cube complex,* $H^3/(A \cap TOR(G))$*, on which* $\Pi$ *acts freely.*

*Proof.* As $A$ acts freely, $A \cap G$ and $A \cap TOR(G)$ act freely. As a result, both $H^3/(A \cap G)$ and $H^3/(A \cap TOR(G))$ satisfy the link condition and are therefore non-positively curved. □

As the action preserves dimension, we can then restrict the above covering space maps to the 2-skeleton which yields the following corollary.

**Corollary 4.2.6.** *Let* $\Pi$ *be a* 3*-manifold group. Then there are* 2*-dimensional cube complexes* $K$ *and* $L$ *such that* $\Pi$ *acts freely preserving the cubulation on* $K$ *as a group of deck transformations with base space* $L$*.*

*Proof.* In this case, the cube-complex $K$ is the 2-skeleton of the cube-complex $H^3/(A \cap G)$ and the cube-complex $L$ is the 2-skeleton of the cube complex $H^3/(A \cap TOR(G))$. $\square$

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