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ABSTRACT

In the dissertation, which mainly deals with commutative semi-prime algebras and representations thereof, we first examined the class of the so-called regular ideals.

Definition 1. *Let A be a commutative semi-prime algebra. For any subset S of A, we define the ideal S^c by $S^c = \{a \in A,\ ax = 0 \text{ for all } x \in S\}$. An ideal I in A is called regular if $I = I^{cc}$.*

In $\underline{B}$, the class of the regular ideals, we introduced the following operations:

$$\wedge I_\nu = \cap I_\nu, \quad \vee I_\nu = (\cap I_\nu^c)^c, \quad I' = I^c,$$

where (I_ν) is any subset of $\underline{B}$ and I any element in $\underline{B}$. We showed that, under these operations, $\underline{B}$ is a Boolean algebra; since for any subset (I_ν) of $\underline{B}$ the intersection $\cap I_\nu$ belongs to $\underline{B}$, it is *complete*; since $I \wedge (\vee I_\nu) = \vee (I \wedge I_\nu)$, where (I_ν) is any subset of $\underline{B}$ and I is any element of $\underline{B}$, it is *distributive*.

Definition 2. *The algebra A satisfies the countable chain condition (c.c.c.), if one of the assertions in the following theorem holds in A.*

Theorem 1. *The following assertions are equivalent.*

1. *Every disjoint family* (b_ν) (*i.e.* $\mu \neq \nu$ implies $b_\mu b_\nu = 0$) *of non-zero elements is countable;*
2. *Every intersection $\cap I_\nu$ of regular ideals I_ν is countably accessible* (*i.e.* there exists a countable subfamily (I_{ν_n}) such that $\cap I_\nu = \cap I_{\nu_n}$).

In the special case where $A = C(X)$, the algebra of the continuous complex-valued functions on the compact Hausdorff space X, we were interested whether or not there exists a *strictly positive measure*, *i.e.* a probability measure with the property that every non-void open

subset of X has positive measure. In this connection we got the following result.

Theorem 2. *The following assertions are equivalent.*

1. *There exists a strictly positive probability measure on* X;
2. *For every non-void regular open set* O_0 *in* X *there exists a bounded regular positive measure* μ_0 *on* $\overline{O}_0$, *such that* $\mu_0(O)$ *is positive for every open set* O, *which is dense in* O_0; *moreover,* X *satisfies the* c.c.c. (*i.e.* every disjoint family of non-void open sets is countable).

In order to conclude 1. from 2. a slightly weaker condition is sufficient: X satisfies the c.c.c. and for every non-void regular open set O_0 together with any subcollection $\underline{C}$ of $\{O,\ O$ open and dense in $O_0\}$ with the property that the interior of any countable intersection $\bigcap O_n$, O_n in $\underline{C}$, again belongs to $\underline{C}$, there exists a measure μ_0 on $\overline{O}_0$, such that $\mu_0(O)$ is positive for every member O of $\underline{C}$.

In terms of Boolean algebras we proved the following.

Theorem 3. *Let* B *be a complete distributive Boolean algebra. The following assertions are equivalent.*

1. *There exists a bounded strictly positive measure* μ *on* B (*i.e.* $p \neq 0$, $p \in B$, implies $\mu(p) > 0$);
2. *For every element* p_0 *in* B *there exists a bounded positive measure* μ *on* B, *such that* $\Sigma\mu(p_i) \neq 0$ *for every disjoint sequence* (p_i) *for which* $p_0 = \vee p_i$; *moreover,* B *satisfies the* c.c.c..

In what follows we assume that the representation

$$U: A \to L(F),$$

where F is any locally convex vector space, has the property that for every ideal I in A the projection

$$f + g \to f, \quad f \in U(I)F,\ g \in U(I^c)F,$$

exists and is continuous.

Among others we proved the following results.

Theorem 4. Let $A = C(X)$, where X is a locally compact Hausdorff space, which has a countable base for its topology. Let F be a normed vector space and let the representation $U: A \to L(F)$ be faithful. Then there exists an element $\phi \in F'$, such that the ideal $\{x \in A,\ \langle U(x)f, \phi\rangle = 0$ for all f in F$\}$ reduces to $\{0\}$, provided that for every $f \in F$ and $\phi \in F'$, the mapping $x \to \langle U(x)f, \phi\rangle$, $x \in A$, is continuous.

Theorem 5. Assume that the vector space F can be written as the topological direct sum of complete metrizable vector spaces H_ν. Let the spaces H_ν be minimal in the sense that there do not exist proper closed invariant subspaces H of H_ν for which the representation U is faithful. Assume that the semi-prime algebra A satisfies the c.c.c.. Then for each ν there exists a vector f_ν, such that $U(A)f_\nu$ is dense in H_ν. Assume, in addition, that the spaces H_ν are Banach spaces and let $U(I)f = \{0\}$, where $f \in F$ and I an ideal in A for which $I^{cc} = A$, imply $f = 0$. Then for each ν there exists an element ϕ_ν in $(\sum_{\mu \neq \nu} H_\mu)^\perp$, such that $U(A)'\phi_\nu$ is w*-dense in $(\sum_{\mu \neq \nu} H_\mu)^\perp$.

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CHAPTER I

INTRODUCTION

In the present work we shall generalize the so-called Gelfand triples for Hilbert spaces to arbitrary locally convex vector spaces. In terms of a given operator (or a family of operators) defined on a Hilbert space H, one often arranges for a triple

$$H_0 \hookrightarrow H \hookrightarrow H_0',$$

where H_0 is a certain locally convex vector space (which may be a nuclear space, a Banach space, etc.), which is dense in H and for which certain invariance conditions hold. The space H_0' is the topological dual of H_0. For concrete examples see e.g. Ju. Berezanskii [2], I. Gelfand and others in [10] and [11], R.A. Hirschfeld [14], K. Maurin [18].

We shall consider a representation

$$U: A \to L(F),$$

where A is a commutative semi-prime algebra (i.e. $a^2 = 0$ implies $a = 0$) over the complexes, F any locally convex vector space and L(F) the algebra of all continuous endomorphisms of F. In A we consider a certain Boolean algebra of ideals. Many of the properties present in case A is generated by a Boolean algebra of idempotents remain valid or can be formulated in terms of this Boolean algebra of ideals.

Our ultimate aim is to arrange for

$$F_0 \hookrightarrow F, \quad F' \hookrightarrow F_0' \quad \text{and} \quad F_0 \hookrightarrow F',$$

where F_0 is U-invariant (i.e. $U(x)F_0 \subset F_0$ for all $x \in A$) and the imbedding $J: F_0 \to F'$ has the property that $U(x)'Jf = JU(x)f$ for all $f \in F_0$ and all $x \in A$; see Theorem 4.2.2 and Theorem 4.3.1.

In Chapter II we shall investigate the properties of semi-prime rings.

More specifically we are interested in the class of the so-called regular ideals; see Definition 2.1.2. In Chapter III we closely examine the "simple" situation, where $A = F = C(X)$, the algebra of all continuous functions on a compact space X, and $U: A \rightarrow L(F)$ is defined by $U(f)g = fg$ for all $f, g \in A$. In order to obtain the injection $F \rightarrow F'$, we need the notion of a strictly positive measure; see Theorem 3.2.3. Finally, in Chapter IV we shall consider the general situation.

The measure-theoretical tools we need are taken from [9] and [13]. For the theory of locally convex vector spaces we use [16] and [23], where a great many results on (partially) ordered vector spaces can be found too. We employ the standard properties of Banach algebras as set forth in [21] and [22]. A treatment of locally convex algebras can be found in [20] and [26]. For a survey of the properties of Boolean algebras see [24]. For properties of (generalized) spectral and/or scalar operators we mention [4], [5] and [17] and the references given there.

CHAPTER II

BOOLEAN ALGEBRAS AND IDEALS

1. A Boolean algebra of a certain class of ideals in a ring.

Throughout the sequel A stands for a commutative ring (with or without identity). The present section is devoted to the construction of a "canonical" Boolean algebra $\underline{B}$ of ideals I in A. No topology on A will be needed for the time being.

Given any set $S \subset A$, we will write $S^c = \{a \varepsilon A,\ aS = \{0\}\}$ for the annihilator of S in A. (The superscript c is reminiscent of set-theoretical complementation.) It is clear that S^c will be an ideal in A (possibly improper) and that $S \subset S^{cc}$. We now impose the following <u>standing hypothesis</u> on A: For every ideal I in A we have $I \cap I^c = \{0\}$. Recall that a commutative ring is semi-prime if it has no nilpotents $\neq 0$.

<u>Proposition</u> 2.1.1. <u>The following properties are equivalent</u>:

(i) <u>For every ideal</u> I <u>in</u> A <u>we have</u> $I \cap I^c = \{0\}$;

(ii) <u>For every ideal</u> I <u>in</u> A, $I^2 = \{0\}$ <u>implies</u> $I = \{0\}$;

(iii) <u>For every element</u> b <u>in</u> A, $b^2 = 0$ <u>implies</u> $b = 0$.

<u>Proof</u>.

(i) => (ii). If $I^2 = \{0\}$, then $I \subset I \cap I^c = \{0\}$.

(ii) => (i). For any ideal I in A, we have $(I \cap I^c)^2 = \{0\}$, so $I \cap I^c = \{0\}$.

(i) => (iii). Suppose $b^2 = 0$. Consider $I = bA$. Then ba belongs to $I \cap I^c = \{0\}$ for all $a \varepsilon A$ and so $bA = \{0\}$. Hence $b \varepsilon A \cap A^c = \{0\}$.

(iii) => (i). If $b \varepsilon I \cap I^c$, then $b^2 = 0$, so $b = 0$.

For more information on semi-prime rings, see <u>e.g</u>. [19].

We now adopt the following definitions.

<u>Definition</u> 2.1.2. <u>An ideal</u> I <u>in</u> A <u>for which</u> $I = I^{cc}$ <u>is called regular</u>.

Definition 2.1.3. *$\underline{B}$ will be the family of all ideals* I (*proper or not*) *in* A, *which are regular.*

We show that all annihilators belong to $\underline{B}$.

Proposition 2.1.4. *For any subset* $S \subset A$ *we have* $S^c \in B$; $\{0\}$ *and* A *belong to* $\underline{B}$.

Proof. We must prove that $S^c = S^{ccc}$. The obvious inclusion $S \subset S^{cc}$, implies $S^c \subset S^{ccc}$. Conversely for any $b \in S^{ccc}$ we have $bS^{cc} = \{0\}$, whence $bS = \{0\}$, so that $b \in S^c$. The remaining statements are obvious.

Before we introduce operations in $\underline{B}$ we agree upon some notation. If $\{I_\nu, \nu \in \Gamma\}$ is a family of ideals in A, then ΣI_ν stands for the ideal of all finite combinations $\Sigma_{i=1}^n a_i$, where $a_i \in \bigcup I_\nu$ for $i = 1,\dots,n$. If I_1 and I_2 are ideals in A, $I_1 I_2$ is the ideal of all finite combinations of the form $\Sigma_{i=1}^n a_i b_i$, where $a_i \in I_1$, $b_i \in I_2$ for $i = 1,\dots,n$. A similar notation is used for the "product" $I_1 I_2 \dots I_m$ of ideals $I_1,\dots,I_m$.

In $\underline{B}$ we introduce two operations:

For an arbitrary family (I_ν), where ν wanders over some index set (which will not be mentioned) and where all I_ν belong to $\underline{B}$, we define

$$\wedge I_\nu = I_\nu,$$
$$\vee I_\nu = (\cap I_\nu^c)^c = (\Sigma I_\nu)^{cc}.$$

It will be seen that for these operations $\underline{B}$ becomes a complete Boolean algebra. Moreover we will notice a striking similarity with ordinary set theory. In order to show these properties we will need the following lemma.

Lemma 2.1.5. *Let* $I_1,\dots,I_n$ *be a finite number of ideals in* A. *Then*

$$(I_1 I_2 \dots I_n)^{cc} = (I_1 \cap \dots \cap I_n)^{cc} = I_1^{cc} \cap \dots \cap I_n^{cc}.$$

The second equality need not hold for infinitely many ideals.

<u>Proof</u>. It will be sufficient to prove the statement for n = 2. Since always, $I_1 I_2 \subset I_1 \cap I_2 \subset I_1^{cc} \cap I_2^{cc}$, we have $(I_1 I_2)^{cc} \subset (I_1 \cap I_2)^{cc} \subset (I_1^{cc} \cap I_2^{cc})^{cc}$. By the equality $I_1^{cc} \cap I_2^{cc} = (I_1^c + I_2^c)^c$ and by Proposition 2.1.4. we get $(I_1^{cc} \cap I_2^{cc})^{cc} = (I_1^c + I_2^c)^{ccc} = (I_1^c + I_2^c)^c = I_1^{cc} \cap I_2^{cc}$. So there remains to show that $I_1^{cc} \cap I_2^{cc} \subset (I_1 I_2)^{cc}$, or equivalently $(I_1 I_2)^c \subset (I_1^{cc} \cap I_2^{cc})^c$. Let $a \in (I_1 I_2)^c$. Then $aI_1 \subset I_2^c$ and so $(aI_1) \cap I_2^{cc} = \{0\}$. For b any element of $I_1 I_2^{cc}$ we have ab is an element of $(aI_1) \cap I_2^{cc} = \{0\}$. So $a \in (I_1 I_2^{cc})^c$, whence $aI_2^{cc} \subset I_1^c$. Thus $(aI_2^{cc}) \cap I_1^{cc} = \{0\}$. Next let b belong to $I_1^{cc} \cap I_2^{cc}$. Then $ab \in (aI_2^{cc}) \cap I_1^{cc} = \{0\}$. From this we finally infer $a \in (I_1^{cc} \cap I_2^{cc})^c$. Next we will give an example in which we will see that the assertion

$$(\cap I_n)^{cc} = \cap I_n^{cc},$$

does not hold for countably many ideals I_n in A. Let A = C[0,1], the ring of all continuous complex valued functions on [0,1]. To each rational number r, 0 < r < 1, we assign the ideal

$$I_r = \{f \in A,\ f(r) = 0\}.$$

Then

$$\cap I_r = \{f \in A,\ f(r) = 0 \text{ for all rational numbers } r\} = \{0\}.$$

So

$$(\cap I_r)^{cc} = \{0\}^{cc} = \{0\}.$$

But

$$I_r^c = \{g \in A,\ gf = 0 \text{ for all } f \in I_r\} = \{0\}.$$

Hence, $I_r^{cc} = A$, from which we see $\cap I_r^{cc} = A$.

In the following statement we collect some of the properties of <u>B</u>.

<u>Theorem</u> 2.1.6. <u>The operations</u> $\wedge$ <u>and</u> $\vee$ <u>satisfy the following rules</u>:

(i) ("<u>Law on complements</u>")

<u>For</u> (I_ν) <u>an arbitrary subset of</u> $\underline{B}$ <u>we have</u>

(a) $\wedge I_\nu \in \underline{B}$, $(\wedge I_\nu)^c = \vee I_\nu^c \in \underline{B}$,

(b) $\vee I_\nu \in \underline{B}$, $(\vee I_\nu)^c = \wedge I_\nu^c \in \underline{B}$.

(ii) ("<u>Distributive laws</u>")

<u>For</u> (I_ν) <u>an arbitrary subset of</u> $\underline{B}$ <u>and</u> $I \in \underline{B}$, <u>we have</u>

(a) $I \vee (\wedge I_\nu) = \wedge(I \vee I_\nu)$,

(b) $I \wedge (\vee I_\nu) = \vee(I \wedge I_\nu)$.

<u>The family</u> $\underline{B}$ <u>is a Boolean algebra, for we have</u>

(iii) $\underline{0}$ $(= \{0\})$ <u>and</u> $\underline{1}$ $(= A)$ <u>belong to</u> $\underline{B}$.

(iv) <u>For every element</u> $I \in \underline{B}$ <u>there exists a uniquely determined element</u> $I_0 \in \underline{B}$, <u>namely</u> $I_0 = I^c$, <u>satisfying</u> $I \wedge I_0 = \underline{0}$ <u>and</u> $I \vee I_0 = \underline{1}$.

<u>Proof</u>.

(i)(a). We will prove that $\wedge I_\nu = (\Sigma I_\nu^c)^c$. In virtue of Proposition 2.1.4, this will show $\wedge I_\nu \in \underline{B}$. For $a \in \wedge I_\nu = \cap I_\nu$ we have, since $I_\nu = I_\nu^{cc}$, $aI_\nu^c = \{0\}$ for all ν. So $a(\Sigma I_\nu^c) = \{0\}$, whence $a \in (\Sigma I_\nu^c)^c$. Conversely, let $a \in (\Sigma I_\nu^c)^c$. Then $a(\Sigma I_\nu^c) = \{0\}$ and so $bI_\nu^c = \{0\}$ for all ν. Hence $a \in I_\nu^{cc} = I_\nu$ for all ν, whence $a \in \cap I_\nu = \wedge I_\nu$.

(i)(b). The first property follows from Proposition 2.1.4, the second is an application of (i)(a).

(ii). In virtue of (i)(a), (b) it is sufficient to prove one of the equalities. Let us prove the second one. Let I and (I_ν) belong to $\underline{B}$. Then it is easy to verify that

$$I(\Sigma I_\nu) = \Sigma I I_\nu.$$

Upon taking second annihilators we get:

$$[I(\Sigma I_\nu)]^{cc} = [\Sigma(II_\nu)]^{cc}.$$

An application of Lemma 2.1.5 to the left-hand side yields:

$$I^{cc} \cap (\Sigma I_\nu)^{cc} = (\bigcap_\nu (II_\nu)^c)^c.$$

Applying the same lemma again we see:

$$(II_\nu)^c = (II_\nu)^{ccc} = (I^{cc} \cap I_\nu^{cc})^c = (I \cap I_\nu)^c.$$

Hence

$$I \cap (\Sigma I_\nu)^{cc} = [\bigcap_\nu (I \cap I_\nu)^c]^c,$$

implying (ii)(b).

Assertion (iii) is obvious.

(iv). Given any $I \in \underline{B}$, let $I_0 \in \underline{B}$ be such that $I \wedge I_0 = \underline{0}$, $I \vee I_0 = \underline{1}$. From $I \cap I_0 = I \wedge I_0 = \underline{0}$ we infer $I_0 \subset I^c$. From $(I^c \cap I_0^c)^c = I \vee I_0 = \underline{1}$ we conclude $I^c \cap I_0^c = (I^c \cap I_0^c)^{cc} = A^c = \{0\}$ and thus $I^c \subset I_0^{cc} = I_0$. Hence $I_0 = I^c$.

<u>Remark</u> 1. In terms of lattice theory $\underline{B}$, together with the operations $\wedge$ and $\vee$, is called a Brouwerian lattice. See [3] for this and related topics.

<u>Remark</u> 2. We did not use the fact that elements of A have negatives. For example, we may apply the results of this section to a cone A of positive functions.

2. Regular ideals.

In the present section we first shall associate the regular ideals ($I = I^{cc}$) of an algebra A of functions on a point set X, to a certain Boolean algebra of subsets of X. We next address ourselves to the main topic of the present work, <u>viz.</u> the regular ideals belonging to a Boolean algebra of idempotents. The results are useful in the spectral theory related to a Boolean algebra of projections defined on a vector space.

Our starting point will be the observation that any subset O of a completely regular space is open if and only if for every $x \in O$ there exists a bounded continuous real-valued function f on X such that

$$\text{(i) } f(x) \neq 0, \quad \text{(ii) } f = 0 \text{ outside of } O.$$

Now let X be some point set, K a field, and A a ring of K-valued functions on X. Using this idea, we will define open sets in X. A subset O of X is said to be hk-open if for every $x \in O$, there exists a function $f \in A$ such that

$$\text{(i) } f(x) \neq 0, \quad \text{(ii) } f = 0 \text{ outside of } O.$$

It is easy to verify that an arbitrary union of hk-open sets is again hk-open. Moreover the intersection of finitely many hk-open sets is hk-open. Let $O_1, \ldots, O_n$ be hk-open sets and $x \in \cap O_i$. Since for each i, O_i is hk-open, there exists a function f_i such that $f_i(x) \neq 0$ and f_i vanishes off O_i. Let $f = f_1 \ldots f_n$. Then $f(x) \neq 0$ and $f = 0$ off $\cap O_i$. It is also easy to verify that for each $f \in A$, the set $\{x \in X,\ f(x) \neq 0\}$ is hk-open. Moreover if L is an arbitrary subset of X then the hk-closure of L consists of all points $x \in X$, for which $f(x) = 0$ for all f which vanish on L. In a formula:

$$\overline{L} = \{x \in X,\ f(x) = 0 \text{ for all } f \in A \text{ for which } f|_L = 0\}.$$

The hk-topology is reminiscent of the classical hull-kernel topology on the maximal ideal space of a commutative Banach algebra; see [15]. In fact, if the above function ring A on X happens to be a Banach algebra with X as its maximal ideal space, then the hk-topology introduced above is readily verified to coincide with the hull-kernel topology (whence the notation hk-topology).

Following standard terminology (cf. P.R. Halmos [12]) a subset $O \subset X$ will be called a regular open set for the hk-topology if O = interior $\overline{O}$. As shown l.c. the family of regular hk-open sets is a Boolean algebra for the operations $O_1 \wedge O_2 = O_1 \cap O_2$, $O_1 \vee O_2 = (O_1 \cup O_2)''$, where O' is the complement of $\overline{O}$ in X. In the next sequence of lemmas and theorems we will establish a one-to-one correspondence between regular ideals in A and regular hk-open sets in X.

Lemma 2.2.1. Let P be a subset of A and form $U = \bigcup_{g \in P^c} \{x \in X,\ g(x) \neq 0\}$.

Then: (i) U is hk-open,

(ii) $f|_U = 0 \iff f \in P^c$,

(iii) $U' = \bigcup_{g \in P^c} \{x \in X,\ g(x) \neq 0\}$.

Proof.

(i) The set $\{x \in X,\ g(x) \neq 0\}$ is hk-open for $g \in A$ and the arbitrary union of such sets is open.

(ii) =>: Let $f|_U = 0$ and $g \in P$. For $x \in U$, we have $f(x) = 0$ and so $f(x)g(x) = 0$. For $x \notin U$, i.e. $g(x) = 0$ for every $g \in P$ and again we have $f(x)g(x) = 0$. Hence $f \in P^c$.

<=: Let $f \in P^c$ and $x \in U$, i.e. $g(x) \neq 0$ for some $g \in P$. Then, since $f \in P^c$, $f(x)g(x) = 0$. So, since $g(x) \neq 0$, $f(x) = 0$.

(iii) By the above remarks $\overline{U} = \bigcap_f \{x \in X,\ f(x) = 0\}$, where the intersection is taken over all f for which $f|_U = 0$. By (ii), $\overline{U} = \bigcap_{f \in P^c} \{x \in X,\ f(x) = 0\}$. Thus $U' = X \setminus \overline{U} = \bigcup_{f \in P^c} \{x \in X,\ f(x) \neq 0\}$.

Lemma 2.2.2. Let P be a subset of A. Then $P = P^{cc} \iff P = \{f \in A,\ f|_O = 0\}$, where O is some hk-open subset of X.

Proof. =>: Consider $O = \bigcup_{g \in P^c} \{x \in X,\ g(x) \neq 0\}$. Then O is a hk-open subset of X and $f|_O = 0$ if and only if $f \in P^{cc} = P$.

<=: Let $P = \{f\epsilon A,\ f|_0 = 0\}$, where 0 is a hk-open subset of X. For every $x \in 0$ there exists a function $f \in A$ such that $f(x) \neq 0$ and $f = 0$ off 0. So if I denotes the ideal $I = \{f\epsilon A,\ f = 0 \text{ off } 0\}$, $0 = \bigcup_{f\epsilon I} \{x\epsilon X,\ f(x) \neq 0\}$. Thus, by (ii) of Lemma 2.2.1, $f|_0 = 0$ if and only if $f \in I^c$. Hence $P = I^c$. Thus, by Proposition 2.1.4, $P = P^{cc}$.

<u>Lemma</u> 2.2.3. <u>A subset</u> 0 <u>of</u> X <u>is a regular hk-open set if and only if there exists a regular ideal</u> I <u>such that</u> $0 = \bigcup_{f\epsilon I} \{x\epsilon X,\ f(x) \neq 0\}$.

<u>Proof.</u> (sufficiency) Let $0 = \bigcup_{f\epsilon I} \{x\epsilon X,\ f(x) \neq 0\}$, where $I = I^{cc}$. Then, by Lemma 2.2.1, $0'' = \bigcup_{f\epsilon I^{cc}} \{x\epsilon X,\ f(x) \neq 0\} = 0$.

(necessity) Let $I = \{f\epsilon A,\ \{x\epsilon X,\ f(x) \neq 0\} \subset 0\}$ where 0 is regular hk-open. Since 0 is hk-open, $0 = \bigcup_{f\epsilon I} \{x\epsilon X,\ f(x) \neq 0\}$. By Lemma 2.2.1, $I^c = \{f\epsilon A,\ f = 0 \text{ on } 0\}$. It follows that $I^c = \{f\epsilon A,\ f = 0 \text{ on } \overline{0}\} =$ $= \{f\epsilon A,\ \{x\epsilon X,\ f(x) \neq 0\} \subset X\setminus\overline{0}\}$. Since $X\setminus\overline{0}$ is hk-open, we get $X\setminus\overline{0} = \bigcup_{f\epsilon I^c} \{x\epsilon X,\ f(x) \neq 0\}$. Again, by Lemma 2.2.1, $f \in I^{cc}$ if and only if f vanishes on $X\setminus\overline{0}$. Hence $I^{cc} = \{f\epsilon A,\ \{x\epsilon X,\ f(x) \neq 0\} \subset \overline{0}\}$. Since for each $f \in A$, $\{x\epsilon X,\ f(x) \neq 0\}$ is hk-open, we infer

$$I^{cc} = \{f\epsilon A,\ \{x\epsilon X,\ f(x) \neq 0\} \subset \mathrm{Int}(\overline{0})\}.$$

Since 0 is regular hk-open, $0 = \mathrm{Int}(\overline{0})$, whence $I = I^{cc}$.

Incidentally we also proved

<u>Lemma</u> 2.2.4. <u>A subset</u> P <u>of</u> A <u>is a regular ideal in</u> A <u>if and only if there exists a regular hk-open subset</u> 0 <u>of</u> X <u>such that</u>

$$P = \{f\epsilon A,\ \{x\epsilon X,\ f(x) \neq 0\} \subset 0\}.$$

As a consequence, we obtain (notation from [12])

Theorem 2.2.5. Let X be some point set, K a field and A a ring of K-valued functions on X. Let X be supplied with the hk-topology. Let $\underline{B}$ be the Boolean algebra of all regular ideals in A and $\tilde{B}$ be the Boolean algebra of all regular hk-open sets in X.

Then, there exists a mapping $u: \underline{B} \to \tilde{B}$ and a mapping $v: \tilde{B} \to \underline{B}$, such that $v \circ u$ = identity on $\underline{B}$ and $u \circ v$ = identity on $\tilde{B}$. Moreover if I, I_1 and I_2 belong to $\underline{B}$, then $u(I_1 \wedge I_2) = u(I_1) \wedge u(I_2)$, $u(I_1 \vee I_2) = u(I_1) \vee u(I_2)$ and $u(I^c) = u(I)'$. Similarly, v has the properties:

$v(O_1 \wedge O_2) = v(O_1) \wedge v(O_2)$, $v(O_1 \vee O_2) = v(O_1) \vee v(O_2)$ and $v(O') = v(O)^c$, where O, O_1 and O_2 are regular hk-open sets in $\tilde{B}$.

Proof. Define $u: \underline{B} \to \tilde{B}$ by $u(I) = \bigcup_{f \in I} \{x \in X,\ f(x) \neq 0\}$, where $I \in \underline{B}$, and define $v: \tilde{B} \to \underline{B}$ by $v(O) = \{f \in A,\ \{x \in X,\ f(x) \neq 0\} \subset O\}$, where $O \in \tilde{B}$.

Then, indeed, by Lemma 2.2.3, u maps $\underline{B}$ onto $\tilde{B}$ and $v \circ u(I) = I$ for all $I \in \underline{B}$. By Lemma 2.2.4, v maps $\tilde{B}$ onto $\underline{B}$ and $u \circ v(O) = O$ for every $O \in \tilde{B}$. By virtue of these facts and since $\underline{B}$ and $\tilde{B}$ are Boolean algebras, it will be sufficient to prove that $u(I_1 \wedge I_2) = u(I_1) \wedge u(I_2)$ for all $I_1, I_2 \in \underline{B}$ and that $u(I^c) = u(I)'$ for all $I \in \underline{B}$.

By definition

$$u(I_1 \wedge I_2) = \bigcup_{f \in I_1 \cap I_2} \{x \in X,\ f(x) \neq 0\}.$$

It is easy to verify that

$$\bigcup_{f \in I_1 \cap I_2} \{x \in X,\ f(x) \neq 0\} = \bigcup_{f \in I_1} \{x \in X,\ f(x) \neq 0\} \cap \bigcup_{f \in I_2} \{x \in X,\ f(x) \neq 0\},$$

whence

$$u(I_1 \wedge I_2) = u(I_1) \cap u(I_2) = u(I_1) \wedge u(I_2).$$

This holds for all I_1 and $I_2 \in \underline{B}$.

If $I \in \underline{B}$, then by definition $u(I) = \bigcup_{f\in I}\{x\in X,\ f(x) \neq 0\}$.
So, by Lemma 2.2.1, $u(I)' = \bigcup_{f\in I^c}\{x\in X,\ f(x) \neq 0\} = u(I^c)$.

Notice that if $A = C(X)$, the algebra of all continuous complex valued functions on the compact Hausdorff space X, then the hk-topology for X coincides with the usual topology.

Next we shall, in a natural way, construct an algebra of "simple functions" belonging to a Boolean algebra. Let B be a Boolean algebra under the operations $\wedge$, $\vee$ and $'$. Its elements will be denoted by $p, q, \ldots$ Let K be a field with members $\lambda, \mu, \ldots$ Let $\mathcal{S}$ be the set of all formal finite combinations of <u>disjoint</u> elements in B, i.e. an element $f \in \mathcal{S}$ is of the form $f = \Sigma_{i=1}^{n}\lambda_i p_i$,
where $\lambda_1,\ldots,\lambda_n \in K$, $p_1,\ldots,p_n \in B$ and $p_i \wedge p_j = 0$ whenever $j \neq i$.
Formally, we define a scalar multiplication, a multiplication and an addition as follows:
If $f_1 = \Sigma_{i=1}^{n}\lambda_i p_i$ and $f_2 = \Sigma_{j=1}^{m}\mu_j q_j$ belong to $\mathcal{S}$,
then $\lambda f_1 = \Sigma_{i=1}^{n}\lambda\lambda_i p_i$, for all $\lambda \in K$,

$$f_1 f_2 = \Sigma_{i=1}^{n}\Sigma_{j=1}^{m}\lambda_i\mu_j p_i \wedge q_j,$$

and $f_1 + f_2 = \Sigma_{i=1}^{n}\Sigma_{j=1}^{m}(\lambda_i + \mu_j)p_i \wedge q_j + \Sigma_{i=1}^{n}\lambda_i p_i \wedge q_1' . \wedge q_m' + \Sigma_{j=1}^{m}\mu_j q_j \wedge p_1' \wedge .. \wedge p_n'$.

We call an element $f = \Sigma_{i=1}^{n}\lambda_i p_i \in \mathcal{S}$ trivial if $\lambda_i \neq 0$ implies $p_i = 0$ and if $p_i \neq 0$ implies $\lambda_i = 0$ for $i = 1,\ldots,n$. An element $f_1 = \Sigma_{i=1}^{n}\lambda_i p_i$ is said to be equivalent to $f_2 = \Sigma_{j=1}^{m}\mu_j q_j$, notation $f_1 \sim f_2$, if the element

$$\Sigma_{i=1}^{n}\Sigma_{j=1}^{m}(\lambda_i - \mu_j)p_i \wedge q_j + \Sigma_{i=1}^{n}\lambda_i p_i \wedge q_1' \wedge .. \wedge q_m' + \Sigma_{j=1}^{m}(-\mu_j)q_j\ p_1' \wedge .. \wedge p_n'$$

is trivial. This relation is an equivalence relation indeed.

If we do not distinguish between elements in $\mathcal{S}$ and their equivalence classes, the (scalar) multiplication and addition, defined above, makes $S = \mathcal{S}/_{\sim}$ into an algebra over K. The class of the trivial elements will become the zero element in S and $(-1)f$ will be the negative of f for each $f \in S$. Upon identifying p and (the class of) $1.p$ for each element $p \in B$, B is in a natural way a subset of S.

Under these identifications we have for instance:

$$pq = p\wedge q, \quad p + q - pq = p\vee q, \quad p + p' = e,$$

where $p, q \in B$ and e is the identity in B.

Example 1. If B is a Boolean algebra of projections defined on a vector space over K. Then S is the algebra of operators spanned by B.

Example 2. If B is a Boolean algebra of subsets of some point set X, then S is (isomorphic to) the algebra of all simple K-valued functions spanned by the characteristic functions of members of B.

Example 3. Let B be the Boolean algebra of the regular open sets of a topological Hausdorff space X. Let $\mathcal{S}$ be the collection of all K-valued functions of the form $\Sigma_{i=1}^{n} \lambda_i \chi_{O_i}$, where for every i, O_i is an open set in X, χ_{O_i} its characteristic function and where all λ_i belong to K. Two functions f_1 and f_2 in $\mathcal{S}$ are said to be equivalent, denoted by $f_1 \sim f_2$, if they coincide on some open set, which is dense in X. Then it is readily verified that $S = \mathcal{S}/_{\sim}$ is isomorphic to the canonical algebra, as constructed above, belonging to B.

A Boolean algebra B is said to be complete if for every decreasing family $(p_\alpha) \subset B$, its meet $\wedge p_\alpha$ exists. It is called distributive if for every decreasing family (p_α) and every element $p \in B$, $p \vee \wedge p_\alpha = \wedge(p \vee p_\alpha)$.

We will prove that, for complete distributive Boolean algebras, an ideal $I \subset S$ is regular (i.e. $I = I^{cc}$) if and only if I is of the form $I = pS$, where p belongs to the underlying Boolean algebra.

We first prove the following lemma.

Lemma 2.2.6. Let B be a complete distributive Boolean algebra, S as above, and let (p_α) be a decreasing family of elements in B. Then $\bigcap p_\alpha S = (\wedge p_\alpha)S$.

Proof. Denote $\wedge p_\alpha$ by p_0. For any $p_\beta \in (p_\alpha)$ we have $p_\beta p_0 = p_\beta \wedge p_\alpha = \underset{\alpha}{\wedge} p_\beta p_\alpha = \underset{\alpha}{\wedge} p_\alpha = p_0$. So, if $f \in S$, then $p_0 f = p_\beta p_0 f$ for all β, whence $p_0 S \subset \bigcap p_\alpha S$. Conversely let $g = \Sigma_{i=1}^{n} \lambda_i r_i \in \bigcap p_\alpha S$. We will show that $p_0 g = g$, whence $\bigcap(p_\alpha S) \subset p_0 S$. We may assume that $\lambda_i \neq 0$ for all i. Then
$p_0 g = \Sigma_{i=1}^{n} \lambda_i p_0 r_i = \Sigma_{i=1}^{n} \lambda_i (\underset{\alpha}{\wedge p}) r_i = \Sigma_{i=1}^{n} \lambda_i \wedge (p_\alpha r_i)$. Since $g = \Sigma_{i=1}^{n} \lambda_i r_i \in \bigcap p_\alpha S$, we certainly have that $g \in p_\alpha S$. Hence, there exist constants μ_j^α, $j = 1,\dots,m$ together with elements q_j^α, $j = 1,\dots,m$ such that $\Sigma_{i=1}^{n} \lambda_i r_i = \Sigma_{j=1}^{m} \mu_j^\alpha p_\alpha q_j^\alpha$ and such that $j \neq k$ implies $q_j^\alpha q_k^\alpha = 0$. Multiplying both sides by r_i and by q_j^α we have $\lambda_i r_i q_j^\alpha = \mu_j^\alpha p_\alpha q_j^\alpha r_i$. Hence, if $p_\alpha q_j^\alpha r_i \neq 0$, we see that $\mu_j^\alpha = \lambda_i$ and so
$\lambda_i r_i = (\Sigma_{i=1}^{n} \lambda_i r_i) r_i = \Sigma_{j=1}^{m} \mu_j^\alpha p_\alpha q_j^\alpha r_i = \Sigma_{j=1}^{m} \lambda_i p_\alpha q_j^\alpha r_i$.
Thus $r_i = \Sigma_{j=1}^{m} p_\alpha q_j^\alpha r_i$, whence $p_\alpha r_i = r_i$.
So we have that $p_0 g = \Sigma_{i=1}^{n} \lambda_i \wedge (p_\alpha r_i) = \Sigma_{i=1}^{n} \lambda_i \wedge r_i = \Sigma_{i=1}^{n} \lambda_i r_i = g$.

Theorem 2.2.7. Let B and S be as in Lemma 2.2.6. An ideal $I \subset S$ is regular if and only if $I = pS$ for some $p \in B$.

Proof. (sufficiency) Let $I = pS$, where $p \in B$. Then $f \in I^c$ if and only if $fp = 0$, or equivalently, $f = f(e - p) = (e - p)f$. Thus $(pS)^c = (e - p)S$, whence $(pS)^{cc} = ((e - p)S)^c = pS$.

(necessity) We will apply Zorn's lemma. Consider a family of increasing idempotents $(p_\alpha) \subset I = I^{cc}$. Since B is a complete Boolean algebra, we have $\bigwedge_\alpha(e - p_\alpha) = e - p_0$, for some $p_0 \in B$. Thus by the previous lemma we have that $\bigcap_\alpha(e - p_\alpha)S = (e - p_0)S$.

We will prove that $p_0 \in I$ and that $p_\alpha p_0 = p_\alpha$ for all α.

We have $\quad I \supset \bigcup\{pS, p \in I, p \in B\}$

and so $\quad I^c \subset \bigcap\{(pS)^c, p \in I, p \in B\}$

$$= \bigcap\{(e - p)S, p \in I, p \in B\} \quad \text{(as above)}$$

$$\subset \bigcap(e - p_\alpha)S = (e - p_0)S,$$

for which we see that $p_0 I^c = \{0\}$ and so $p_0 \in I^{cc} = I$. Moreover it follows that $p_\alpha(e - p_0) = 0$ or $p_\alpha p_0 = p_\alpha$ for all α. Consequently we may apply Zorn's lemma, to the effect that there exists a maximal element $p \in I \cap B$. Suppose there exists an element $f \in I$, $f \notin pS$. Then, by assumption, the element f is of the form $f = \Sigma_{i=1}^n \lambda_i p_i$, where $\lambda_i \neq 0$ and $p_i p_j = 0$ whenever $j \neq i$. Since, for every i, $p_i f = \lambda_i p_i$ it follows that every $p_i \in I$, since $f \notin pA$, at least one $p_j \notin pS$. Consider $q = p_j \vee p = p_j + p - p_j p$. Then $q \in I$ and $q \neq p$ and $pq = p$. Hence p is not maximal, which is a contradiction.

3. The countable chain condition.

In the following chapters we will need a certain countability property of the ring A. We aim to generalize the results on Boolean algebras of projections in locally convex spaces as set forth in [1] and [25]. It will be convenient to give five seemingly different conditions, which turn out to be equivalent.

<u>Lemma</u> 2.3.1. <u>Let</u> A <u>be a semi-prime ring and</u> I <u>an arbitrary ideal in</u> A. <u>Then there exists a family</u> $(b_\nu) \subset I$ <u>such that the following conditions are satisfied</u>;

(i) <u>The family</u> (b_ν) <u>is mutually disjoint</u>: $b_\nu b_\mu = 0$, <u>if</u> $\nu \neq \mu$,

(ii) <u>The family</u> (b_ν) <u>is not trivial</u>: $I^{cc} = (\sum_\nu b_\nu A)^{cc}$.

<u>Proof</u>. Consider the collection of all subsets of I, which satisfy the condition that any two distinct elements have product 0. An application of Zorn's lemma applies to the effect that there exists a maximal subset $\Phi = (b_\nu)$ having this property. We claim that the family Φ also satisfies (ii). Upon letting $I_\nu = b_\nu A$, we have to prove that $I^{cc} = (\Sigma I_\nu)^{cc}$. Since $\Sigma I_\nu \subset I$, we only need to show that $I^{cc} \subset (\Sigma I_\nu)^{cc} = (\bigcap I_\nu^c)^c$ or, equivalently, $I^c \supset \bigcap I_\nu^c$; the latter amounts to $I^{cc} \cap \bigcap I_\nu^c = \{0\}$, which, by Lemma 2.1.5, in turn is equivalent to $I \cap \bigcap I_\nu^c = \{0\}$. Now consider any b_0 in $I \cap \bigcap I_\nu^c$. For every μ we have $b_0 I_\mu \subset I_\mu \cap \bigcap I_\nu^c \subset I_\mu \cap I_\mu^c = \{0\}$. Hence, $b_0 \in I$ annihilates all members of Φ, and so $b_0 = 0$.

For a more concise formulation of the next theorem we shall adopt the following terminology: a family of ideals [ring elements] is said to be <u>disjoint</u> if any two distinct pair has zero intersection [product]. Furthermore, we shall say that an intersection $\bigcap I_\alpha$ of ideals is <u>countably accessible</u> if there is a countable subfamily of indices (α_n) for which $\bigcap_\alpha I_\alpha = \bigcap_n I_{\alpha_n}$.

We now are able to derive the following result.

Theorem 2.3.2. *Let A be a commutative semi-prime ring. Then, the following assertions are equivalent.*

(i) *Any disjoint family of arbitrary non-zero ideals is countable;*

(ii) *Any disjoint family of arbitrary non-zero regular ideals is countable;*

(iii) *Any disjoint family of non-zero elements in A is countable;*

(iv) *The intersection of any decreasing family of regular ideals is countably accessible;*

(v) *The intersection of an arbitrary family of regular ideals is countably accessible.*

Proof. We will show (i) => (ii) => (iii) => (iv) => (v) => (i).

(i) => (ii). Trivial.

(ii) => (iii). Let $\Phi = (b_\alpha)$ be a family of non-zero elements in A such that $b_\beta b_\alpha = 0$ for $\beta \neq \alpha$. It is to be shown that Φ is at most countable. Consider the ideals $I_\alpha = b_\alpha A$. By Lemma 2.1.5 we have $I_\alpha^{cc} \cap I_\beta^{cc} = (I_\alpha I_\beta)^{cc} = \{0\}^{cc} = \{0\}$ for $\beta \neq \alpha$. Thus Φ is at most countable, since by (ii), the family (I_α) is so.

(iii) => (iv). Let $\Phi = (I_\alpha)$ be a family of decreasing regular ideals. We have to prove that there exists a countable subfamily (I_{α_n}) such that $\cap I_\alpha = \cap I_{\alpha_n}$. Consider the ideal $I = \Sigma I_\alpha^c$. By the previous lemma there exist elements $b_\nu \in I$ such that

$$(a)\ b_\nu b_\mu = 0 \quad \text{for } \nu \neq \mu ,$$

$$(b)\ I^{cc} = (\Sigma b_\nu A)^{cc}.$$

By (iii) the family (b_ν) is at most countable, say (b_n). Since the family (I_α) is decreasing, the family (I_α^c) is increasing, so we may assume that for every n, $b_n \in I_{\alpha_n}^c$, for some α_n.

Thus $\Sigma b_n A \subset \Sigma I^c_{\alpha_n}$, whence

$$(\cap I_\alpha)^c = (\Sigma I^c_\alpha)^{cc} = I^{cc} = (\Sigma b_n A)^{cc} \subset (\Sigma I^c_{\alpha_n})^{cc} = (\cap I_{\alpha_n})^c.$$

It follows that $\cap I_\alpha \supset \cap I_{\alpha_n}$. We have trivially that $\cap I_{\alpha_n} \supset \cap I_\alpha$. This proves the assertion.

The implication (iv) => (v) follows from the next general result, which has some interest of its own.

Lemma 2.3.3. *Let X be a point set and C a collection of subsets of X which is stable under countable intersections and which has the following property: The intersection of any decreasing family in C is countably accessible.*

Then, every intersection of members of C is countably accessible.

Proof. Given any subcollection $\mathcal{F}$ of C, we must exhibit a countable subset Φ of $\mathcal{F}$, such that $\cap\Phi = \cap\mathcal{F}$. Let $\mathcal{F}_0$ be the collection of all finite intersections of members of $\mathcal{F}$. Consider the family of the countable subsets Φ of $\mathcal{F}_0$. We shall write $\Phi_1 \sim \Phi_2$ whenever $\cap\Phi_1 = \cap\Phi_2$. It is easy to verify that this does define an equivalence relation. Denote the class containing Φ by $\tilde{\Phi}$. We now define a partial order in the set of these equivalence classes: $\tilde{\Phi}_1 > \tilde{\Phi}_2$ if for representations we have $\cap\Phi_1 \subset \cap\Phi_2$. Again it is readily verified, that this relation defines a partial order. Next, let $(\tilde{\Phi}_\nu)$ be a descending family and write $\Phi_\nu = \{L_{\nu,n}, n \in N\}$. Since C is stable under countable intersections, each $\bigcap_n L_{\nu,n}$ belongs to C. Since the family $\tilde{\Phi}_\nu$ is descending, there exists, by assumption, a countable subset $\{\Phi_{\nu_n}\} \subset \{\Phi_\nu\}$, such that

$$\bigcap_\nu \cap\{L, L \in \Phi_\nu\} = \bigcap_n \cap\{L, L \in \Phi_{\nu_n}\}.$$

The right-hand side features an intersection of countably many members of $\mathcal{F}_0$. Let Φ be the set of these elements.

Then, clearly, $\tilde{\Phi}$ is greater than $\tilde{\Phi}_\nu$, for each ν. Hence, by Zorn's lemma, there exists a maximal equivalence class $\tilde{\Phi}_{max}$.

Claim: $$\bigcap\{L,\ L \in \mathcal{F}\} = \bigcap\{L,\ L \in \Phi_{max}\}.$$

Suppose not, then, since $\bigcap\{L,\ L \in \Phi_{max}\}$ certainly contains $\bigcap\{L,\ L \in \mathcal{F}\}$, there exists an element $L_0 \in \mathcal{F}$ such that

$$\bigcap\{L,\ L \in \Phi_{max}\} \cap L_0 \neq \bigcap\{L,\ L \in \Phi_{max}\}.$$

Let $\Phi_0 = \{L \cap L_0,\ L \in \Phi_{max}\}$, then Φ_0 is a countable subset of $\mathcal{F}_0$ for which $\tilde{\Phi}_0 > \tilde{\Phi}_{max}$ and $\tilde{\Phi}_0 \neq \tilde{\Phi}_{max}$. This violates the maximality of $\tilde{\Phi}_{max}$, whence the statement.

In order to show the implication (iv) => (v) we need only to remark that the set of regular ideals is stable under countable intersections. (We even know that it is closed under arbitrary intersections.)

(v) => (i). Let $\Phi = \{I_\alpha\}$ be a family of arbitrary non-zero ideals satisfying $I_\beta \cap I_\alpha = \{0\}$ for $\beta \neq \alpha$. It is to be shown that this family is countable. Consider the family of regular ideals $\{I_\alpha^c\}$. By (v) there exists a countable set $\{I_{\alpha_n}\}$ such that $\bigcap I_{\alpha_n}^c = \bigcap I_\alpha^c$. Claim: $\Phi = \{I_{\alpha_n}\}$. If not, Φ would contain I_0 with $I_0 \neq I_{\alpha_n}$ for all n. Then, $I_0 \cap I_{\alpha_n} = \{0\}$ and so $I_0 \subset I_{\alpha_n}^c$ for all n, whence $I_0 \subset \bigcap I_{\alpha_n}^c = \bigcap I_\alpha^c$. Hence $I_0 \subset I_0^c$ and so $I_0 = \{0\}$ in the semi-prime ring A. This proves the assertion.

<u>Remark</u> 1. As the proof shows, the theorem remains valid if everywhere the the expression "countable" is replaced by "of cardinality $\aleph$", where $\aleph > \aleph_0$.

<u>Remark</u> 2. Condition (iii) enables us to compare our results with results of various authors [1] and [25].

<u>Remark</u> 3. Condition (v) will frequently be used in this sequel.

We are now ready to define the countable chain condition.

Definition 2.3.3. *A commutative semi-prime ring satisfies the countable chain condition* (c.c.c.) *if it satisfies one of the five conditions of Theorem* 2.3.2.

Corollary 2.3.4. *Let* X *be a completely regular topological space and* A *the algebra of all bounded complex-valued functions on* X. *The following assertions are equivalent:*

(i) A *satisfies the countable chain condition;*

(ii) *Every disjoint family of non-empty open sets in* X *is countable;*

(iii) *Every family of open subsets* (O_α) *of* X *contains a countable subfamily* (O_{α_n}) *such that* $\cup O_{\alpha_n}$ *is dense in* $\cup O_\alpha$.

Proof. We will show (i) <=> (ii), (i) <=> (iii).

(i) => (ii). Let (O_α) be a family of mutually disjoint open subsets of X. Then, since X is completely regular, there exists for each α a bounded continuous function f_α such that $f_\alpha \neq 0$ and $f_\alpha = 0$ off O_α. By the countable chain condition for A, the family (f_α) is at most countable and so is the family (O_α).

(ii) => (i). Let (f_α) be a disjoint family of non-zero functions in A. We will show that (f_α) is countable. Consider the family of the open sets $O_\alpha = \{x \in X,\ f_\alpha(x) \neq 0\}$. Then, $\beta \neq \alpha$ implies $O_\alpha \cap O_\beta$ is empty, whence the result.

(i) => (iii). Let (O_α) be a family of open subsets of X.

Consider the set of ideals $\{I_\alpha;\ I_\alpha = \{f\varepsilon A,\ f = 0 \text{ on } O_\alpha\}\}$.

Then, by Lemma 2.2.2, I_α is regular for each α. On account of the previous theorem item (v), there exists a countable family (I_{α_n}) such that $\cap I_{\alpha_n} = \cap I_\alpha$.

Whence, $\{f\varepsilon A,\ f = 0 \text{ on } \cup O_{\alpha_n}\} = \{f\varepsilon A,\ f = 0 \text{ on } \cup O_\alpha\}$.

If $\cup O_{\alpha_n}$ were not dense in $\cup O_\alpha$, there would exist a point $x_0\ \varepsilon\ \cup O_\alpha$ and an open neighbourhood U of x_0 such that $(\underset{n}{\cup} O_{\alpha_n}) \cap U = \phi$ and $x_0\ \varepsilon\ U \cap \cup O_\alpha$.

Since X is completely regular there exists a function $f_0\ \varepsilon$ A such that $f_0(x_0) \neq 0$ and $f_0 = 0$ outside of $U \cap \cup O_\alpha$. Thus $f_0\ \varepsilon \cap I_{\alpha_n}$ and $f_0 \notin \cap I_\alpha$, a contradiction.

(iii) => (i). Let (I_α) be an arbitrary family of regular ideals. We will show that there exists a countable subfamily (I_{α_n}) such that $\cap I_\alpha = \cap I_{\alpha_n}$.

By Lemma 2.2.2, we know that for every α there exists an open subset O_α of X such that $I_\alpha = \{f\varepsilon A,\ f = 0 \text{ on } O_\alpha\}$.

Then $\cap I_\alpha = \underset{\alpha}{\cap}\{f\varepsilon A,\ f = 0 \text{ on } O_\alpha\} = \{f\varepsilon A,\ f = 0 \text{ on } \cup O_\alpha\}$.

But there exists a countable subfamily (O_{α_n}) such that $\cup O_{\alpha_n}$ is dense in $\cup O_\alpha$. Hence $\cap I_\alpha = \{f\varepsilon A,\ f = 0 \text{ on } \cup O_{\alpha_n}\} = \cap I_{\alpha_n}$.

Corollary 2.3.5. *Let X be a completely regular topological space which satisfies the countable chain condition and let O be an open subset of X. Then there exists a countable increasing family of open sets (O_n) such that $\overline{O}_n \subset O$ and $\cup O_n$ is dense in O.*

Proof. Since X is completely regular and O an open subset of X, there exists for each $x \in O$ a bounded non-negative continuous function f such that $f(x) \neq 0$ and $f = 0$ off O. Hence the set O can be written as

$$O = \bigcup_f \{x \in X,\ f(x) \neq 0\},$$

where the union is taken over all bounded non-negative continuous functions f, which vanish outside of O. By the previous corollary there exists a countable subfamily (f_n) such that

$$\bigcup_n \{x \in X,\ f_n(x) \neq 0\} \text{ is dense in } O.$$

Without loss of generality we may assume that $0 \leq f_n(x) \leq 1$ for all x and all n. Define $f_0(x) = \Sigma_{n=1}^{\infty} 2^{-n} f_n(x)$, then f_0 is bounded, continuous and non-negative. Moreover $\bigcup_n \{x \in X,\ f_n(x) \neq 0\} = \{x \in X,\ f_0(x) \neq 0\}$. Finally let $O_n = \{x \in X,\ f_0(x) > n^{-1}\}$.

CHAPTER III

SOME COMMENTS ON STRICTLY POSITIVE FUNCTIONALS

1. Preliminary remarks.

This chapter is entirely devoted to an existence problem on positive measures. Let A be a commutative C*-algebra. A positive functional $\phi \in A' = (A, \|\cdot\|)'$ is said to be <u>strictly positive</u> if $f \in A$, $f \neq 0$ implies $\langle f^*f, \phi\rangle \neq 0$. Does A possess a strictly positive functional? Equivalently, let X be a compact Hausdorff space. Does there exist a strictly positive probability measure, i.e. a regular positive Borel measure μ such that $\mu(X) = 1$ and such that, for every non-void open set O, we have $\mu(O) > 0$?

There are a few well-known cases for which the answer is affirmative.

First, if X is separable, we may take $\phi = \Sigma_{n=1}^{\infty} 2^{-n}\delta_n$, where δ_n is the point evaluation at the n^{th} element of a dense sequence in X.

Second, if X is the closure of an open subset of a compact group, one may take the Haar measure; see e.g. [13], Chapter XI.

Let A be a commutative C*-algebra. Then the existence of a strictly positive functional $\phi \in A'$ implies that A satisfies the <u>countable chain condition</u>. Let $\{f_\gamma, \gamma \in \Gamma\}$ be a family of positive elements in A for which $\|f_\gamma\| = 1$ for all $\gamma \in \Gamma$ and $f_{\gamma_1} f_{\gamma_2} = 0$, whenever $\gamma_1 \neq \gamma_2$. We will show that Γ is countable. Let δ be any positive number and consider the set

$$\Gamma_\delta = \{\gamma \in \Gamma,\ \langle f_\gamma, \phi\rangle \geq \delta\}.$$

Claim: Γ_δ is finite. In fact, if not, then Γ_δ would contain at least countably many distinct elements $\gamma_1, \gamma_2, \ldots$. The sequence $\{g_k\}$, defined by

$$g_k = \Sigma_{i=1}^{k} f_{\gamma_i}, \quad k \in \mathbb{N}$$

would have the properties: $\|g_k\| = 1$ for every k and

$$\langle g_k, \phi\rangle = \Sigma_{i=1}^{k} \langle f_{\gamma_i}, \phi\rangle \geq k\delta.$$

We may suppose that $\|\phi\| = 1$, whence

$$1 = \|g_k\| \geq \langle g_k, \phi\rangle \geq k\delta,$$

for all k, which is impossible.

Hence, Γ_δ is finite and thus $\Gamma = \bigcup\{\Gamma_{\frac{1}{n}}, n = 1, 2, \ldots\}$ is countable, indeed.

We also have the following easy proposition.

Proposition 3.1.1. *Let ϕ be a positive functional on the commutative C^*-algebra A. The following assertions are equivalent:*

(i) *The functional ϕ is strictly positive;*

(ii) *For every non-zero ideal I in A, $\langle IA, \phi\rangle \neq \{0\}$.*

Proof. (i) => (ii). Let $0 \neq f \in I$, then $f^*f \in IA$ and $\langle f^*f, \phi\rangle \neq 0$.

(ii) => (i). Let $f \in A$, $f \neq 0$. Consider the ideal $I = fA$. Then $\langle IA, \phi\rangle \neq \{0\}$, *i.e.* there exists an element $h \in A$ such that $\langle fh, \phi\rangle \neq 0$ and so by the Schwartz inequality:

$$0 \neq |\langle fh, \phi\rangle|^2 \leq \langle f^*f, \phi\rangle\langle h^*h, \phi\rangle,$$

whence $\langle f^*f, \phi\rangle \neq 0$.

It, therefore, seems natural to consider ideals of the form

$$I_\phi = \{f \in A, \langle fg, \phi\rangle = 0 \text{ for all } g \in A\},$$

where ϕ is any element of A'.

Notice that if A has an identity, then I_ϕ is the largest ideal in the kernel of ϕ. Then the task will become to prove the existence of functionals ϕ for which $I_\phi = \{0\}$. As pointed out above, it is necessary to impose the c.c.c. on A. This, however, does not seem to be sufficient. The reason is that the c.c.c. essentially says something about regular ideals: $I_1 \cap I_2 = \{0\}$ implies $I_1^{cc} \cap I_2^{cc} = \{0\}$, or in terms of open sets $O_1 \cap O_2 = \phi$ implies $\mathrm{Int}(\overline{O}_1) \cap \mathrm{Int}(\overline{O}_2) = \phi$, where I_1, I_2 are arbitrary ideals and O_1, O_2 are arbitrary open sets, respectively.

2. Regular functionals and normed algebras.

In this section we will consider a topological algebra which is commutative and semi-prime. Moreover we will assume that for every ideal $I \subset A$ the "projection mapping" $p: IA + I^cA \to IA$, defined by $p(a + b) = a$, $a \in IA$, $b \in I^cA$, is continuous. Remark that a C*-algebra satisfies all these conditions. By A' we will mean the totality of all continuous functionals defined on A. We will say that a functional $\phi \in A'$ is regular if I_ϕ is regular (i.e. $I_\phi^{cc} = I_\phi$).

Example. Let $A = C[0,1]$, equipped with the supremum norm and $g \in A$. Then the functional $f \to \int_0^1 f(t)g(t)dt$ is regular.

One of the problems we face will be whether or not there exist regular functionals. The following lemma gives sufficient conditions in order that the regular functionals separate the points of A.

<u>Lemma</u> 3.2.1. <u>Let the topological commutative semi-prime algebra</u> A <u>satisfy the following conditions:</u>

(i) <u>The topology is locally convex;</u>

(ii) <u>For every ideal</u> I, <u>the mapping</u>

$$p: IA + I^cA \to A,$$

<u>defined by</u>

$$p: a + b \to a, \quad a \in IA,\ b \in I^cA,$$

<u>is continuous;</u>

(iii) <u>For every regular ideal</u> I_0, $I_0 \neq \{0\}$, <u>there exists a functional</u> ϕ_0 <u>such that</u> $\langle IA,\phi_0\rangle \neq \{0\}$, <u>for every closed ideal</u> I <u>for which</u> $I^{cc} = I_0$.

<u>Then</u> $\cap\{I_\phi,\ \phi \text{ regular}\} = \{0\}$.

<u>Proof.</u> Let $I_0 = \cap\{I_\phi,\ \phi \text{ regular}\}$. By Theorem 2.1.6 $I_0 = I_0^{cc}$. We first prove that $\psi \in A'$ implies $I_\psi^{cc} \supset I_0$. Suppose not, <u>i.e.</u> assume $I_\psi^{cc} \cap I_0 \neq I_0$ for some ψ.

Consider the functional

$$\overline{\psi}_0: I_\psi^{cc}A + I_\psi^cA \to \mathbf{C},$$

defined by

$$\overline{\psi}_0: a + b \to \langle b,\psi\rangle \quad , a \in I_\psi^{cc}A,\ b \in I_\psi^cA.$$

Then $\overline{\psi}_0$ is continuous on its domain. Let ψ_0 be a Hahn-Banach extension of $\overline{\psi}_0$ to all of A. Then clearly $I_\psi^{cc} \subset I_{\psi_0}$. For the converse conclusion we have by definition

$$\begin{aligned} I_{\psi_0} &= \{x\in A,\ \langle xy,\psi_0\rangle = 0 \text{ for all } y\in A\} \\ &\subset \{x\in A,\ \langle xb,\psi\rangle = 0 \text{ for all } b\in I_\psi^c\} \\ &\subset \{x\in A,\ \langle xby,\psi\rangle = 0 \text{ for all } b\in I_\psi^c, \text{ all } y\in A\} \\ &= \{x\in A,\ xI_\psi^c \subset I_\psi\} = I_\psi^{cc}. \end{aligned}$$

We conclude that $I_{\psi}^{cc} = I_{\psi_0}$. Our above indirect assumption now becomes

$$I_0 = \{I_\phi,\ I_\phi^{cc} = I_\phi\} = I_0 \cap I_{\psi_0} = I_0 \cap I_\psi^{cc} \neq I_0,$$

which is impossible. Consider now the ideal $I_{\phi_0} \cap I_0$, where ϕ_0 is an element in A' for which $\langle IA, \phi_0 \rangle \neq \{0\}$ holds for every closed ideal $I \subset A$ with $I^{cc} = I_0$. Then, by the definition of I_{ϕ_0}, we have

$$\langle (I_{\phi_0} \cap I_0)A, \phi_0 \rangle = \{0\}.$$

On the other hand, by the property of ϕ_0 and assuming that $I_0 \neq \{0\}$, we have

$$\langle (I_{\phi_0} \cap I_0)A, \phi_0 \rangle \neq \{0\}.$$

Thus, $I_0 = \{0\}$.

The proof of the next lemma is rather technical.

Lemma 3.2.2. *Let the topology for A be defined by a norm. Let again A be semi-prime, commutative and let (ii) of the previous lemma be satisfied. Let (ϕ_n) be a countable family of regular functionals in A'. Then there exists an element $\phi_0 \in A'$ such that $I_{\phi_0} = \bigcap I_{\phi_n}$.*

Proof. We will construct a sequence of regular functionals (ψ_n) such that for all n:

(i) $\|\psi_n\| < 2$,

(ii) $I_{\psi_n} = \bigcap_{m \le n} I_{\phi_n}$,

(iii) $\langle b, \psi_n \rangle = \langle b, \psi_{n+1} \rangle$ for all $b \in I_{\psi_n}^c A$.

We will assume that $\|\phi_n\| < 1$ for all n. The construction employs by induction. First, let $\psi_1 = \phi_1$. Now let the functionals $\psi_1, \ldots, \psi_n$ be constructed in such a way that

(a) $\|\psi_k\| \le 2 - \varepsilon$, $k = 1, \ldots, n$ $1 > \varepsilon > 0$,

(b) $I_{\psi_k} = \bigcap_{1 \le k} I_{\phi_1}$, all $k \le n$,

(c) $\langle b, \psi_k \rangle = \langle b, \psi_1 \rangle$, $n \ge k \ge 1$, $b \in I_{\psi_1}^c A$.

We will construct a regular functional ψ_{n+1} such that $\|\psi_{n+1}\| \le 2 - 2^{-1}\varepsilon$ and the family $\psi_1, \ldots, \psi_{n+1}$ satisfies (b) and (c) with n replaced by n+1. By (ii), there exists a constant c_n such that, for all $a \in I_{\psi_n} A$ and $b \in I^c_{\psi_n} A$, the inequality $\|a\| \le c_n \|a + b\|$ is valid.

Define

$$\overline{\psi}_{n+1} : I_{\psi_n} A + I^c_{\psi_n} A \to \mathbf{C}$$

by

$$\overline{\psi}_{n+1} : a + b \to \frac{\varepsilon}{2c_n} \langle a, \phi_{n+1} \rangle + \langle b, \psi_n \rangle,$$

where $a \in I_{\psi_n} A$, $b \in I^c_{\psi_n} A$.

Then
$$\begin{aligned} |\langle a + b, \overline{\psi}_{n+1} \rangle| &= \left| \frac{\varepsilon}{2c_n} \langle a, \phi_{n+1} \rangle + \langle a + b, \psi_n \rangle \right| \\ &\le \frac{\varepsilon}{2c_n} \|a\| + (2 - \varepsilon) \|a + b\| \\ &\le \frac{\varepsilon c_n}{2c_n} \|a + b\| + (2 - \varepsilon) \|a + b\| \\ &= (2 - 2^{-1}\varepsilon) \|a + b\|. \end{aligned}$$

Let ψ_{n+1} be a Hahn-Banach extension of $\overline{\psi}_{n+1}$ to all of A, so that

$$|\langle x, \psi_{n+1} \rangle| \le (2 - 2^{-1}\varepsilon) \|x\|$$

for all $x \in A$.

Then, the family $\psi_1, \ldots, \psi_{n+1}$ satisfies (c). Let us prove (b); then ψ_{n+1} is automatically regular.

By definition

$$\begin{aligned} I_{\psi_{n+1}} &= \{a \in A, \langle ax, \psi_{n+1} \rangle = 0 \text{ for all } x \in A\} \\ &\subset \{a \in A, \langle aby, \psi_n \rangle = 0 \text{ for all } b \in I^c_{\psi_n}, \text{ all } y \in A\} \\ &= \{a \in A, aI^c_{\psi_n} \subset I_{\psi_n}\} \\ &= \{a \in A, aI^c_{\psi_n} \subset I_{\psi_n} \cap I^c_{\psi_n} = \{0\}\} \\ &= I^{cc}_{\psi_n} = I_{\psi_n}. \end{aligned}$$

Hence $\qquad I_{\psi_{n+1}} = I_{\psi_n} \cap I_{\psi_{n+1}}$

(by definition) $\qquad = \{a \varepsilon I_{\psi_n},\ \langle ax,\psi_{n+1}\rangle = 0 \text{ for all } x \varepsilon A\}$

(definition of ψ_{n+1}) $\qquad = \{a \varepsilon I_{\psi_n},\ \langle ax,\phi_{n+1}\rangle = 0 \text{ for all } x \varepsilon A\}$

$\qquad = I_{\psi_n} \cap I_{\phi_{n+1}}$

(induction hypothesis) $\qquad = I_{\phi_1} \cap \cdots \cap I_{\phi_n} \cap I_{\phi_{n+1}}$.

The sequence (ψ_n), obtained in this way, clearly satisfies the following conditions

(i) $\|\psi_n\| < 2$ for all n,

(ii) $I_{\psi_m} \subset I_{\psi_n}$ for $m \geq n$,

(iii) $\langle b,\psi_m\rangle = \langle b,\psi_n\rangle$ for $m > n$ and $b \varepsilon I^c_{\psi_n} A$,

(iv) $\cap I_{\psi_n} = \cap I_{\phi_n}$.

Finally, let $\phi_0 = \Sigma_{n=1}^{\infty} 2^{-n}\psi_n$. We claim that $I_{\phi_0} = \cap I_{\psi_n}$. We shall again use induction. First, we prove that $I_{\phi_0} \subset I_{\psi_1}$. If $a \varepsilon I_{\phi_0}$, then $\langle ax,\phi_0\rangle = 0$, for all $x \varepsilon A$, so certainly $\langle aby,\phi_0\rangle = 0$ for all $b \varepsilon I^c_{\psi_1}$, all $y \varepsilon A$. But, by the properties of the sequence (ψ_n), we have

$$\langle aby,\phi_0\rangle = \Sigma_{n=1}^{\infty} 2^{-n}\langle aby,\psi_n\rangle = \Sigma_{n=1}^{\infty} 2^{-n}\langle aby,\psi_1\rangle,$$

and so

$$aI^c_{\psi_1} \subset I_{\psi_1} \cap I^c_{\psi_1} = \{0\},$$

whence

$$a \varepsilon I^{cc}_{\psi_1} = I_{\psi_1}.$$

We next show that $I_{\phi_0} \subset I_{\psi_n}$ implies $I_{\phi_0} \subset I_{\psi_{n+1}}$.

By definition, we have

$$I_{\phi_0} = \{a\varepsilon I_{\psi_n},\ \ <ax,\phi_0> = 0 \text{ for all } x\varepsilon A\}$$
$$= \{a\varepsilon I_{\psi_n},\ \ \Sigma_{k=1}^{\infty}2^{-k}<ax,\psi_k> = 0 \text{ for all } x\varepsilon A\}$$

(by the fact that $I_{\psi_n} \subset I_{\psi_{n-1}} \subset \quad \subset I_{\psi_1}$)

$$= \{a\varepsilon I_{\psi_n},\ \ \Sigma_{k=n+1}^{\infty}2^{-k}<ax,\psi_k> = 0 \text{ for all } x\varepsilon A\}$$
$$\{a\varepsilon I_{\psi_n},\ \ \Sigma_{k=n+1}^{\infty}2^{-k}<abx,\psi_k>=0 \text{ for all } b\varepsilon I^c_{\psi_{n+1}}, \text{ all } x\varepsilon A\}$$

(by definition of ψ_n)

$$= \{a\varepsilon I_{\psi_n},\ \ \Sigma_{k=n+1}^{\infty}2^{-k}<abx,\psi_{n+1}>=0 \text{ for all } b\varepsilon I^c_{\psi_{n+1}}, \text{ all } x\varepsilon A\}$$
$$= \{a\varepsilon I_{\psi_n},\ \ aI^c_{\psi_{n+1}} \subset I_{\psi_{n+1}}\}$$
$$= \{a\varepsilon I_{\psi_n},\ \ aI^c_{\psi_{n+1}} \subset I_{\psi_{n+1}} \cap I^c_{\psi_{n+1}} = \{0\}\}$$
$$= I_{\psi_n} \cap I^{cc}_{\psi_{n+1}} = I_{\psi_{n+1}}.$$

It follows that $I_{\phi_0} \subset \cap I_{\psi_n}$. The reverse inclusion $\cap I_{\psi_n} \subset I_{\phi_0}$ follows directly from the definitions.

Theorem 3.2.3. *Let the topological commutative semi-prime algebra A satisfy the following conditions:*

(i) *The topology is defined by a norm;*

(ii) *For every ideal I, the mapping* $p\colon IA + I^cA \to A$ *defined by* $p\colon a + b \to a$, *where* $a \varepsilon IA$, $b \varepsilon I^cA$, *is continuous;*

(iii) *The algebra A satisfies the countable chain condition.*

Then the following assertions are equivalent:

(a) *For every regular ideal* $I_0 \neq \{0\}$ *there exists a functional* $\phi_0 \varepsilon A'$ *such that* $<IA,\phi_0> \neq \{0\}$ *for every closed ideal I for which* $I^{cc} = I_0$.

(b) *There exists a functional* ψ_0 *such that* $I_{\psi_0} = \{0\}$.

Remark. The word topological may be omitted if in assertion (a) "every closed ideal I for which $I^{cc} = I_0$" is strengthened to "every ideal I for which $I^{cc} = I_0$".

Proof. (b) => (a). Let I_0 be a non-zero regular ideal (that is, $I_0^{cc} = I_0 \neq \{0\}$) and let $I^{cc} = I_0$. Then $\langle IA, \psi_0 \rangle \neq \{0\}$. In fact, if $\langle IA, \psi_0 \rangle = \{0\}$, then $I \subset I_{\psi_0} = \{0\}$ and so $\{0\} = I^{cc} = I_0$.

(a) => (b). By Lemma 3.2.1 we have $\{0\} = \cap\{I_\phi,\ I_\phi^{cc} = I_\phi\}$. From the c.c.c., we infer that the intersection is countably accessible and so there exists a countable family (ϕ_n) of regular functionals such that $\{0\} = \cap I_{\phi_n}$. But, by Lemma 3.2.2, we know that there exists a functional $\psi_0 \in A'$ such that $I_{\psi_0} = \cap I_{\phi_n}$.

3. Strictly positive functionals.

In this section we shall apply the preceding results to a commutative C*-algebra. We follow standard terminology in calling an element $\psi \in A'$ hermitian if the functional $\psi^*: x \to \overline{\langle x^*, \psi \rangle}$ coincides with ψ or, what is equivalent, ψ takes real values on the hermitian elements of A. A functional $\psi \in A'$ is called positive if it takes non-negative values on the positive elements in A. It is well-known that a positive functional is hermitian. Every ϕ in A' can be written in the form $\phi = \psi_1 + i\psi_2$, where ψ_1 and ψ_2 are hermitian: simply let $\psi_1 = (\phi + \phi^*)/2$ and $\psi_2 = (\phi - \phi^*)/2i$. We also know that every hermitian $\psi \in A'$ admits of a Jordan decomposition $\psi = \psi_1 - \psi_2$, where ψ_1 and ψ_2 are positive functionals in A' and $\|\psi\| = \|\psi_1\| + \|\psi_2\|$ (See [7](2.6.4)); according to Grothendieck, this decomposition is even unique ([7](12.3.4), whether or not A is commutative).

It follows that any $\phi \in A'$ can be uniquely represented in the form

$$\phi = \Sigma_{n=1}^{4} i^n \phi_n,$$

with ϕ_1, ϕ_2, ϕ_3 and ϕ_4 all positive.

The contents of the next lemma is that for suitable chosen positive functionals ϕ_1, ϕ_2, ϕ_3 and ϕ_4 for which $\phi = \Sigma_{n=1}^{4} i^n \phi_n$, we have $I_\phi = \bigcap I_{\phi_n}$.

It then easily follows that for

$$\phi_0 = \Sigma_{n=1}^{4} \phi_n$$

we have

$$I_{\phi_0} = I_\phi.$$

We also need the fact that, for any two positive elements a_1 and a_2 in A, we have

$\{h \epsilon A,\ 0 \le h \le a_1 + a_2\} = \{h \epsilon A,\ 0 \le h \le a_1\} + \{h \epsilon A,\ 0 \le h \le a_2\}$.

It then follows that for ψ any hermitian functional the mapping

$$a \to \sup\{\langle h, \psi\rangle,\ 0 \le h \le a\}$$

is linear on the cone of the positive elements in A.

For more details on vector lattices see e.g. [23].

Lemma 3.3.1. *Let A be a commutative C*-algebra and let $\phi \in A'$. Then there exists a positive functional $\phi_0 \in A'$ such that $I_\phi = I_{\phi_0}$.*

Proof. We first prove that, if $a \in A$ and $0 \le h \le aa^*$, then h belongs to the closure of aA. Since A is a commutative C*-algebra, we know that a closed ideal I is the intersection of the maximal ideals containing I. It follows that, if Δ denotes the maximal ideal space of A, the ideal aA is dense in

$$\bigcap\{\mathrm{Ker}\,\delta,\ \delta \epsilon \Delta,\ \langle a, \delta\rangle = 0\}.$$

So, if $0 \le h \le aa^*$, then $\langle a, \delta\rangle = 0$ implies $\langle h, \delta\rangle = 0$. Hence h belongs to the closure of aA.

Next we write $\phi = \psi_1 + i\psi_2$, where ψ_1 and ψ_2 are hermitian functionals. By definition we have

$$I_\phi = \{a\varepsilon A,\ \langle ax,\phi\rangle = 0 \text{ for all } x\varepsilon A\}$$
$$= \{a\varepsilon A,\ \langle ax,\psi_1 + i\psi_2\rangle = 0 \text{ for all } x\varepsilon A\}$$
$$\subset \{a\varepsilon A,\ \langle aa^*x,\psi_1 + i\psi_2\rangle = 0 \text{ for all } x\varepsilon A,\ x = x^*\}$$
$$= \{a\varepsilon A,\ \langle aa^*x,\psi_1\rangle = 0 \text{ for all } x\varepsilon A,\ x = x^*\}$$
$$\cap\{a\varepsilon A,\ \langle aa^*x,\psi_2\rangle = 0 \text{ for all } x\varepsilon A,\ x = x^*\}$$

(since the hermitian elements span A)

$$= \{a\varepsilon A,\ \langle aa^*x,\psi_1\rangle = 0 \text{ for all } x\varepsilon A\}$$
$$\cap\{a\varepsilon A,\ \langle aa^*x,\psi_2\rangle = 0 \text{ for all } x\varepsilon A\}$$
$$= \{a\varepsilon A,\ aa^* \ \varepsilon\ I_{\psi_1}\cap I_{\psi_2}\}$$

(since $I_{\psi_1}\cap I_{\psi_2}$ is closed)

$$= I_{\psi_1}\cap I_{\psi_2}.$$

The reverse inclusion $I_{\psi_1}\cap I_{\psi_2} \subset I_\phi$ is trivial, whence $I_\phi = I_{\psi_1}\cap I_{\psi_2}$.

Now let ψ be a hermitian functional. Define its "positive variation" ψ_1. First for positive elements in A:

$$\langle a,\tilde{\psi}_1\rangle = \sup\{\langle h,\psi\rangle,\ \ 0 \le h \le a\},\ \ a \ge 0.$$

(e.g. see [23], p.211)

Since A is a vector lattice, $\tilde{\psi}_1$ is linear on A^+, the cone of the positive elements. For arbitrary a ε A, write $a = \Sigma_{n=1}^{4} i^n a_n$, where a_i is positive for i = 1, 2, 3, 4 and $a_1a_3 = a_2a_4 = 0$. Define $\langle a,\psi_1\rangle$ by linear extension. Then ψ_1 is a positive continuous functional on A. Let $\psi_2 = \psi_1 - \psi$. Then for every element a ε A^+ we have

$$\langle a,\psi_2\rangle = \sup\{\langle -h,\psi\rangle,\ \ 0 \le h \le a\},$$

and so ψ_2 is positive.

Next we prove that $I_\psi = I_{\psi_1} \cap I_{\psi_2}$. Let $a \in I_\psi$; that is $\langle aA, \psi\rangle = \{0\}$. If $0 \le h \le aa^*$, then, by the reasoning at the beginning of the proof, h belongs to the closure of the ideal aA. It follows, by continuity, that $\langle h, \psi\rangle = 0$. Hence $\langle aa^*, \psi_1\rangle = \sup\{\langle h, \psi\rangle, 0 \le h \le aa^*\} = 0$. Since ψ_1 is positive, we conclude $a \in I_{\psi_1}$. Similarly we may show that $a \in I_{\psi_2}$. Hence $I_\psi = I_{\psi_1} \cap I_{\psi_2}$, the reverse inclusion, $I_{\psi_2} \cap I_{\psi_1} \subset I_\psi$, being trivial. This method can be employed for the hermitian functionals ψ_1 and ψ_2 in $\phi = \psi_1 + i\psi_2$, providing us with four positive functionals ϕ_1, ϕ_2, ϕ_3 and ϕ_4, so that $I_\phi = \cap I_{\phi_n}$.

We now write down a result which is similar to Theorem 3.2.3.

Theorem 3.3.2. *Let* A *be a commutative* C^*-*algebra. The following assertions are equivalent:*

(i) *There exists a strictly positive functional in* A';

(ii) *There exists a mapping* $T\colon A \to A'$, *which is one-to-one, for which* $\langle ab, Tc\rangle = \langle b, Tca\rangle$ *for all* a, b, c *in* A; *moreover (if* A *does not possess an identity)* A *satisfies the* c.c.c.;

(iii) *For every regular ideal* I_0, $I_0 \ne \{0\}$, *there exists a functional* ϕ_0 *in* A' *such that* $\langle I, \phi_0\rangle \ne \{0\}$, *for all closed ideals* I *for which* $I^{cc} = I_0$; *moreover* A *satisfies the* c.c.c.

Proof. (i) => (ii). Let $\phi \in A'$ be strictly positive. Define $T\colon A \to A'$ as follows: if $a \in A$, then Ta is the functional which assigns to x the number $\langle ax, \phi\rangle$. Thus $\langle x, Ta\rangle = \langle ax, \phi\rangle$ for all a, $x \in A$. It is readily verified that $\langle ab, Tc\rangle = \langle b, T(ca)\rangle$ for all a, b, c in A. We show that T is one-to-one. If a is an element of A for which Ta = 0, then $\langle x, Ta\rangle = 0$ or $\langle ax, \phi\rangle = 0$ for all $x \in A$. In particular, $\langle aa^*, \phi\rangle = 0$ and so $aa^* = 0$, or a = 0.

(ii) => (i). Let $T: A \to A'$ be as in (ii). Define, for every $a \in A$, the functional ϕ_a on A by $\langle x,\phi_a\rangle = \langle x,Ta\rangle$. It is a matter of routine to verify that the ideal I_{ϕ_a} is equal to

$$I_{\phi_a} = \{x\varepsilon A, \ \ ax = 0\}.$$

Hence, by Proposition 2.1.4, $I_{\phi_a}^{cc} = I_{\phi_a}$.

If A has no identity, we know, by the c.c.c., that the intersection $\cap\{I_{\phi_a}, \ \ a\varepsilon A\}$ is countably accessible and hence there exists by Lemma 3.2.2 a functional ϕ_0 such that $I_{\phi_0} = \cap\{I_{\phi_a}, \ \ a\varepsilon A\}$. From the fact that $I_{\phi_a} = \{x\varepsilon A, \ \ ax = 0\}$, we see that $I_{\phi_0} = \{0\}$. By the previous lemma we may assume without loss of generality that ϕ_0 is positive and so

$$I_{\phi_0} = \{x\varepsilon A, \ \langle x^*x,\phi_0\rangle = 0\} = \{0\},$$

showing that ϕ_0 is strictly positive.

If A does have an identity e, the functional $\phi_e: x \to \langle x,Te\rangle$ has the property $I_{\phi_e} = \{0\}$. Again we may assume that ϕ_e is positive and it follows that ϕ_e is strictly positive.

(i) => (iii). Let ϕ be a strictly positive functional on A. Then, $\langle I,\phi\rangle \neq \{0\}$ for every non-zero ideal I. Hence, if $I^{cc} = I_0$, where $I_0^{cc} = I_0 \neq \{0\}$, then $\langle I,\phi\rangle \neq \{0\}$.

That A satisfies the c.c.c. has already been proved above.

(iii) => (i). This is a straightforward application of Theorem 3.2.3 and the previous lemma.

<u>Remark</u> 1. If $\phi_0 \in A'$ is a strictly positive functional, then the mapping: $\{a,b\} \to \langle ab^*,\phi_0\rangle$, defined on $A \times A$, is an inner product which makes A into a Hilbert algebra; see [6], p.330.

Remark 2. A somewhat weaker form of (iii) is sufficient to conclude (i). For every non-zero regular ideal $I_0 \subset A$ together with any collection $\underline{C} \subset \{I,\ I^{cc} = I_0\}$ of closed ideals with the property that every countable (and every finite) intersection

$$\bigcap\{I_n,\ n = 1, 2, \ldots\},\quad I_n \in \underline{C},$$

belongs to $\underline{C}$, there exists a functional $\phi_0 \in A'$ such that $\langle I, \phi_0\rangle \neq \{0\}$ for all ideals $I \in \underline{C}$; moreover A satisfies the c.c.c.

Closely related to this remark is the problem at the end of this chapter.

In the light of Theorem 3.3.2(iii), the existence problem for strictly positive functionals can be reduced to the following one. Let I_0 be a non-zero regular ideal i.e. $I_0^{cc} = I_0 \neq \{0\}$. As in (iii), we consider the collection of those closed ideals I for which $I^{cc} = I_0$. Now select in every such ideal I a non-zero positive element x_I and look at the family $\mathcal{F} = \{x_I\}$.

Claim. There exists a strictly positive functional on A if and only if the family $\mathcal{F}$ can be chosen in such a manner that there is a positive functional $\phi_0 \in A'$ which does not vanish at any point of $\mathcal{F}$.

In fact, suppose, indirectly, that for each choice of $\mathcal{F}$ every positive $\phi \in A'$ the set $\mathcal{F}_\phi = \{x \in \mathcal{F},\ \langle x,\phi\rangle = 0\}$ is non-empty. Since, for every sequence $(\phi_n) \subset A'$, $\phi_n \geq 0$, $\|\phi_n\| \leq 1$, the functional $\phi = \Sigma_{n=1}^{\infty} 2^{-n}\phi_n$ has again these properties, it follows that for any countable collection $(\mathcal{F}_{\phi_n})$, the intersection $\bigcap \mathcal{F}_{\phi_n}$ $(= \mathcal{F}_\phi)$ is non-void. This is impossible if $A = C(X)$, where X is compact and separable. Neither is it possible in case $\mathcal{F}$ is weakly compact (= weakly countably compact according to Eberlein, see e.g. [23], p.185).

However, we were not able to construct such a weakly compact family $\mathcal{F}$.

As another consequence of the theorem we have

Theorem 3.3.3. *Let X be a compact Hausdorff space. The following assertions are equivalent:*

(i) *There exists a strictly positive finite Borel measure μ on X;*

(ii) *The space X satisfies the c.c.c. and, for any non-void regular open set O_0, there exists a bounded regular positive measure μ_0 on $\overline{O}_0$ such that $\mu_0(O) > 0$ for every open set O which is dense in O_0.*

Proof. We consider the algebra $A = C(X)$ of all continuous complex-valued functions on X. Recall the one-to-one correspondence between regular ideals in A and regular open sets in X; see Theorem 2.2.5.

The mapping

$$I \rightarrow \bigcup_{f \in I} \{x \in X,\ f(x) \neq 0\}$$

is a bijection between the collection of closed ideals I A and the topology of X: the collection of the open subsets.

Its inverse is given by

$$O \rightarrow \{f \in A,\ \{x \in X,\ f(x) \neq 0\} \subset O\},$$

where O is any open subset of X.

The restriction of these mappings to the regular ideals and regular open sets respectively establishes a one-to-one correspondence between the regular ideals and the regular open sets.

Consider a pair (I_0, O_0), where O_0 is a regular open set belonging to the regular ideal I_0. Then under the above mappings the collection of ideals $\{I \subset A,\ I \text{ closed},\ I^{cc} = I_0\}$ is in one-to-one correspondence with the collection of open sets $\{O \subset X,\ O \text{ open and dense in } O_0\}$.

After these preparatory remarks we now proceed with the proof of the theorem.

(i) => (ii). Clear.

(ii) => (i). By virtue of Theorem 3.3.2 it is sufficient to exhibit a functional satisfying condition (iii) in that theorem. Let I_0 be any regular ideal in $A = C(X)$ and O_0 its corresponding regular open set. By (ii) there exists a measure μ_0 on $\overline{O}_0$ such that $\mu_0(O) > 0$ for every dense open subset O of O_0. Given any closed ideal I for which $I^{cc} = I_0$, the set

$$O = \bigcup_{f\varepsilon I} \{x\varepsilon X,\ f(x) \neq 0\}$$

is open and dense in O_0. Since μ_0 is regular there exists a compact subset $K \subset O$ such that $\mu_0(K) > 0$. Let f_0 be any function satisfying the following conditions: $f_0(x) \geq 0$ for all $x\varepsilon X$, $f_0(x) = 1$ for all $x\varepsilon K$ and $f_0(x) = 0$ for all x off O. Such a function exists, since X is compact (and so normal). The function f_0 belongs to I and we have $\mu_0(f_0) \neq 0$. Since μ_0 may be viewed as a continuous functional on $C(\overline{O}_0)$ and I_0 is in a natural way a subspace of $C(\overline{O}_0)$, the measure μ_0 on $\overline{O}_0$ defines a continuous functional on I_0. Let ϕ_0 be any Hahn-Banach extension of μ_0 to all of A, then ϕ_0 does satisfy condition (iii) in Theorem 3.3.2.

Again, let X be a compact Hausdorff space. Take a non-empty regular open set O_0 in X and consider the following hypothesis on O_0.

Hypothesis (*). There exists a family $\{u_\gamma,\ \gamma\varepsilon\Gamma\}$ in $A = C(X)$, together with a family of points $\{x_\gamma,\ \gamma\varepsilon\Gamma\} \subset X$, such that the following conditions are satisfied:

(i) $u_\gamma(x_\gamma) \neq 0$ for every $\gamma \varepsilon \Gamma$;

(ii) For every open set O dense in O_0, there exists an element $\gamma \varepsilon \Gamma$ such that $\{x\varepsilon X,\ u_\gamma(x) \neq 0\} \subset O$;

(iii) The functional

$$\phi_0 : \Sigma_{i=1}^{n}\lambda_i u_{\gamma_i} \to \Sigma_{i=1}^{n}\lambda_i u_{\gamma_i}(x_{\gamma_i})$$

is well-defined and continuous on the vector space spanned by the family $\{u_\gamma, \gamma\varepsilon\Gamma\}$.

Remark 1. The closure of $\{x_\gamma, \gamma\varepsilon\Gamma\}$ has non-void interior.

Remark 2. Whereas the collections $\{u_\gamma\}$ and $\{x_\gamma\}$ can always be chosen in such a way that (i) and (ii) are satisfied, (iii) is the crucial condition.

Remark 3. A motive for looking at this type of conditions is furnished by the fact that if instead of the family $\{u_\gamma, \gamma\varepsilon\Gamma\}$ we would have taken the collection of characteristic functions

$$\{\chi_O, O \text{ open and dense in } O_0\},$$

then the functional

$$\Sigma_{i=1}^{n}\lambda_i \chi_{O_i} \to \Sigma_{i=1}^{n}\lambda_i$$

has property (iii), if we take the supremum norm for defining the topology.

Theorem 3.3.4. Let X be a compact Hausdorff space. A sufficient condition for the existence of a strictly positive measure is that every non-void regular open set O_0 satisfies hypothesis (*) and that X satisfies the countable chain condition. Moreover, if X is connected then these conditions are also necessary.

Proof. (sufficiency) We will check assertion (iii) in Theorem 3.3.2. Let I_0 be any regular ideal and O_0 be the corresponding regular open set. If $\{u_\gamma, \gamma\varepsilon\Gamma\}$ and $\{x_\gamma, \gamma\varepsilon\Gamma\}$ are as in hypothesis (*) for the set O_0, then the functional ϕ_0, which is defined on the vector space spanned by the family $\{u_\gamma, \gamma\varepsilon\Gamma\}$ admits a Hahn-Banach extension to all of $C(X)$. This extension satisfies the conditions put forth in Theorem 3.3.2 item (iii).

(necessity) Assume X to be connected. Let I_0 be any regular ideal and O_0 the corresponding regular open set. Select, for every open dense subset O of O_0, a function u_O such that $\|u_O\| = 1$, $1 \geq u_O(x) \geq 0$ for all x in X and $\{x \in X,\ u_O(x) \neq 0\} \subset O$. Let M be the subspace of C(X) spanned by the family $\{u_O,\ O$ open and dense in $O_0,\ O \neq O_0\}$. Let $\phi\colon C(X) \to \mathbf{C}$ be a strictly positive functional on C(X) originating from the strictly positive measure μ on X. We may assume that $\|\phi\| = 1$.

Thus, for every such O, we have

$$0 < \langle u_O, \phi\rangle \leq \|u_O\| = 1.$$

Since $O \neq O_0$, $O \subset O_0$, there exists a point $x_1 \in X$ such that $u_O(x_1) = 0$. In addition, since $\|u_O\| = 1$, there exists a point $x_2 \in X$ such that $u_O(x_2) = 1$. By the assumption that X is connected, there exists a point x_O in O such that $\langle u_O, \phi\rangle = u_O(x_O)$. Hence the functional $\phi_0\colon M \to \mathbf{C}$, defined by

$$\phi_0(\Sigma_{i=1}^n \lambda_i u_{O_i}) = \Sigma_{i=1}^n \lambda_i u_{O_i}(x_{O_i})$$

is continuous on M. And so the family $\{u_O\}$ together with the family $\{x_O\}$ and ϕ_0 does satisfy the conditions (i), (ii) and (iii) in hypothesis (*).

4. Boolean algebras and strictly positive measures.

In this section we shall consider a Boolean algebra B together with a "canonical" algebra S of "simple functions". As in Chapter II section 2, S consists of all formal linear combinations

$$f = \Sigma_{i=1}^n \lambda_i p_i,\quad \lambda_1,\dots,\lambda_n \in C,\ p_1,\dots,p_n \in B,\ p_i \wedge p_j = 0 \text{ whenever } j \neq i,$$

modulo the set of all "trivial sequences" i.e. all formal combinations of the form $f = \Sigma_{i=1}^n \lambda_i p_i$, where $\lambda_i \neq 0$ implies $p_i = 0$ and $p_i \neq 0$ implies $\lambda_i = 0$.

We made l.c. S into an algebra by defining (scalar) multiplication and addition in the following manner:

If $f = \Sigma_{i=1}^{n}\lambda_i p_i$, $g = \Sigma_{j=1}^{m}\mu_j q_j$ and $\lambda \in \mathbb{C}$,

then $\lambda f = \Sigma_{i=1}^{n}\lambda\lambda_i p_i$, $\quad fg = \Sigma_{i=1}^{n}\Sigma_{j=1}^{m}\lambda_i\mu_j p_i \wedge q_j$ and

$f + g = \Sigma_{i=1}^{n}\Sigma_{j=1}^{m}(\lambda_i + \mu_j)p_i \wedge q_j + \Sigma_{i=1}^{n}\lambda_i p_i \wedge q_1' \wedge \ldots \wedge q_m' + \Sigma_{j=1}^{m}\mu_j q_j \wedge p_1' \wedge \ldots \wedge p_n'$;

here $p \vee p' = e$ and $p \wedge p' = 0$ for all $p \in B$.

These definitions coincide with the usual Boolean operations:

$p \wedge q = pq$, $p' = e - p$, $p \vee q = p + q - pq$, for all p and q in B.

In the algebra S we define a norm

$$\|\Sigma_{i=1}^{n}\lambda_i p_i\| = \max\{|\lambda_i|,\ 1 \le i \le n\},$$

and an involution

$$(\Sigma_{i=1}^{n}\lambda_i p_i)^* = \Sigma_{i=1}^{n}\bar{\lambda}_i p_i.$$

Except for completeness, $(S, \| \|)$ has the usual properties of a C^*-algebra. There is a natural way to introduce a partial order in S: an element $f = \Sigma_{i=1}^{n}\lambda_i p_i$ is said to be positive ($f \ge 0$) if $p_i \ne 0$ implies $\lambda_i \ge 0$. Consequently $f \ge g$, if $f - g \ge 0$. The cone $S^+ = \{f \in S,\ f \ge 0\}$ is generating in the sense that every element $f \in S$ can be written in the form

$$f = f_1 - f_2 + i(f_3 - f_4),$$

where $f_1, f_2, f_3, f_4 \in S^+$ and $f_1 f_2 = f_3 f_4 = 0$.

A measure on B (or a functional on S) is defined as an element of $(S, \| \|)'$. A measure is said to be positive if it is positive-valued (or 0) on B. As in the general case, I_ϕ is the largest ideal in the kernel of ϕ and ϕ is said to be regular if $I_\phi^{cc} = I_\phi$.

We need two technical lemmas.

Lemma 3.4.1. *Let* S, B *be as above and let* $\phi \in S'$.

Then $\|\phi\| = \sup\{\Sigma_{i=1}^{\infty}|<p_i,\phi>|\}$, *where the supremum is taken over all mutually disjoint sequences* $(p_i) \subset B$.

Proof. Let $f = \Sigma_{i=1}^{n}\lambda_i p_i \in S$. Then we have

$$|<f,\phi>| = |\Sigma_{i=1}^{n}\lambda_i<p_i,\phi>| \leq \Sigma_{i=1}^{n}|\lambda_i||<p_i,\phi>| \leq \|f\| \Sigma_{i=1}^{n}|<p_i,\phi>|.$$

Hence $\|\phi\| = \sup\{|<f,\phi>|,\ f\in S,\ \|f\| = 1\} \leq \sup\{\Sigma_{i=1}^{\infty}|<p_i,\phi>|\}$.

Conversely let $(p_i) \subset B$ be a mutually disjoint sequence.

Let $f_n = \Sigma_{i=1}^{n}\lambda_i p_i$, where

$$\lambda_i = \begin{cases} 0 & \text{if } <p_i,\phi> = 0 \\ \dfrac{|<p_i,\phi>|}{<p_i,\phi>} & \text{if } <p_i,\phi> \neq 0. \end{cases}$$

Then $\|f_n\| \leq 1$, and $<f_n,\phi> = \Sigma_{i=1}^{n}|<p_i,\phi>|$.

This holds for all n and for all mutually disjoint sequences $(p_i) \subset B$, whence $\|\phi\| \geq \sup\{\Sigma_{i=1}^{\infty}|<p_i,\phi>|\}$.

Analogous to the above C*-algebra situation we have

Lemma 3.4.2. *Let* S, B *be as above and let* $\phi \in S'$. *Then there exists a positive functional* $\phi_0 \in S'$ *such that* $I_\phi = I_{\phi_0}$.

Proof. Basically the proof is the same as for Lemma 3.3.1. The only problem is that S is not complete. We shall outline the proof. First define hermitian functionals ψ_1 resp. ψ_2 on S as follows:

$<f,\psi_1> = (<f,\phi> + \overline{<f^*,\phi>})/2$ and $<f,\psi_2> = (<f,\phi> - \overline{<f^*,\phi>})/2i$

for all $f \in S$. Then $\|\psi_1\| \leq \|\phi\|$, $\|\psi_2\| \leq \|\phi\|$ and $\phi = \psi_1 + i\psi_2$.

We claim that an element $p \in B$ belongs to I_ϕ if and only if $p \in I_{\psi_1} \cap I_{\psi_2}$.

By definition p belongs to I_ϕ if and only if $<pq,\psi_1 + i\psi_2> = 0$ or, since ψ_1 and ψ_2 are hermitian, $<pq,\psi_1> = <pq,\psi_2> = 0$ for all q in B.

Hence $p \in I_\phi$ if and only if $p \in I_{\psi_1} \cap I_{\psi_2}$. Now let $f = \Sigma_{i=1}^{n} \lambda_i p_i$ belong to I_ϕ, then $\lambda_i p_i = f p_i$ and so if $\lambda_i \neq 0$, we have $p_i \in I_\phi$ and thus $p_i \in I_{\psi_1} \cap I_{\psi_2}$. It follows $I_\phi \subset I_{\psi_1} \cap I_{\psi_2}$. The converse inclusion is trivial.

Next, let ψ be a hermitian functional. Define $\psi_1 : S \to \mathbf{C}$ as follows; for p any element of B

$$\langle p, \psi_1 \rangle = \sup\{\langle pq, \psi\rangle,\ q \in B\}.$$

Using the fact that for disjoint elements p_1 and p_2 in B the equality $p_1 B + p_2 B = (p_1 + p_2)B$ is valid, we easily infer that for such elements $\langle p_1 + p_2, \psi_1 \rangle = \langle p_1, \psi_1\rangle + \langle p_2, \psi_1 \rangle$. The latter enables us to define $\psi_1 : S \to \mathbf{C}$ by linear extension.

Elementary estimations show that

$$\sup\{\Sigma_{i=1}^{\infty} \langle p_i, \psi_1 \rangle\} \leq \sup\{\Sigma_{i=1}^{\infty} |\langle p_i, \psi\rangle|\},$$

from which we conclude that $\|\psi_1\| \leq \|\psi\|$.

Similarly we define $\psi_2 : S \to \mathbf{C}$. If p belongs to B , then $\langle p, \psi_2 \rangle = \sup\{-\langle pq, \psi\rangle,\ q \in B\}$. By linear extension we define ψ_2 on all of S. It is readily verified that $\psi = \psi_1 - \psi_2$ and that $I_\psi = I_{\psi_1} \cap I_{\psi_2} = I_{\psi_1 + \psi_2}$. As a consequence of Theorem 3.2.3 we obtain the result.

<u>Theorem</u> 3.4.3. <u>Let</u> B, S <u>be as above</u>. <u>The following assertions are equivalent</u>:

(i) <u>There exists a functional which is positive for every</u> $p \in B$;

(ii) <u>The Boolean algebra</u> B <u>satisfies the</u> c.c.c. <u>and for every regular ideal</u> $I_0 \subset S$ <u>there exists a functional</u> $\phi_0 \in S'$ <u>such that for every mutually disjoint sequence</u> $(p_i) \subset B$, <u>for which</u> $(\Sigma p_i S)^{cc} = I_0$, <u>we have</u> $\Sigma_{i=1}^{\infty} |\langle p_i, \phi\rangle| \neq 0$.

Proof. (i) => (ii). Obvious.

(ii) => (i). We first prove that for any regular ideal I_0 there exists a functional ϕ_0, such that $\langle I,\phi_0\rangle \neq 0$ for every ideal I for which $I^{cc} = I_0$. By Lemma 2.3.1 there exists for every ideal I, for which $I^{cc} = I_0$, a mutually disjoint family $(p_i) \subset B$, such that $(\Sigma p_i S)^{cc} = I_0$, $(p_i) \subset I$. Assuming that B satisfies the c.c.c., this family is necessarily countable. Since there exists a functional $\phi_0 \in S'$, such that for every sequence (p_i) for which $(\Sigma p_i S)^{cc} = I_0$, we have $\Sigma_{i=1}^{\infty}|\langle p_i,\phi_0\rangle| \neq 0$, it follows $\langle I,\phi_0\rangle \neq \{0\}$ for any ideal for which $I^{cc} = I_0$. Theorem 3.2.3 applies to the effect that there exists a functional $\phi_0 \in S'$ so that $I_{\phi_0} = \{0\}$. By the previous lemma we may assume that ϕ_0 is positive and so, if $p \in B$ and $\langle p,\phi_0\rangle = 0$, then $p \in I_{\phi_0}$, whence $p = 0$.

Remark. If B is complete and distributive condition (ii) may, by virtue of Theorem 2.2.7, be replaced by:

(ii') The Boolean algebra satisfies the c.c.c. and for every element $p_0 \in B$ there exists a measure $\phi_0 \in S'$ such that for every mutually disjoint sequence $(p_i) \subset B$ for which $\vee p_i = p_0$, we have $\Sigma_{i=1}^{\infty}|\langle p_i,\phi_0\rangle| \neq 0$.

We conclude this chapter by mentioning the following open problem.

Problem. Let $\underline{C}$ be a collection of dense open subsets of the compact Hausdorff space X, for which $\cap\{O,\ O\in\underline{C}\}$ is void and which is closed under countable intersections in the sense that for every countable subcollection (O_n) the open set $\mathrm{Int}(\cap O_n)$ is again a member of $\underline{C}$. Does there exist such a collection? If so, does there exist a regular positive Borel measure μ on X such that $\mu(O) \neq 0$ for every O in $\underline{C}$?

If there exists a regular positive Borel measure μ with the latter property, then there exists a regular positive Borel measure μ_0 on X such that $\mu_0(O) = 1$ for every $O \in \underline{C}$. To see this, consider the equality

$$\inf\{\mu(O),\ O\epsilon\underline{C}\} = \inf\{\mu(O_i),\ i = 1, 2, .. \}$$

for a suitable countable subcollection $(O_i) \subset \underline{C}$. Let $O_\mu = \mathrm{Int}(\bigcap O_i)$. Then the open set O_μ belongs to $\underline{C}$ and $\mu(O\cap O_\mu) = \mu(O_\mu)$ for every O in $\underline{C}$.
Finally define

$$\mu_0(B) = \frac{\mu(B\cap O_\mu)}{\mu(O_\mu)}$$

for every Borel set B. Then μ_0 is a regular Borel measure on X with the property that $\mu_0(O) = 1$ for every $O \in \underline{C}$.
Also notice that, by the assumption $\bigcap\{O,\ O\epsilon\underline{C}\}$ is void, the space X cannot possess isolated points. See [8], Lemme 8, for a situation reminiscent to the above one.

CHAPTER IV

GENERALIZED GELFAND TRIPLES

1. Representations of semi-prime algebras.

In this chapter we shall consider a commutative semi-prime algebra A, a locally convex topological vector space F and a faithful representation U of A into L(F), the algebra of all continuous linear operators in F. In section 2 we investigate the general situation. Moreover we specify to the case where A satisfies certain strong countability conditions; see Lemma 4.2.3. In section 3 we consider the situation where U(I)F is dense in $U(I^{cc})F$ for each I belonging to a certain class of ideals. Our purpose is to arrange for the situation

$$F_0 \to F, \quad F' \to F_0',$$

in such a way that

(i) F_0 is an invariant dense subspace of F;

(ii) there exists a mapping

$$T: F_0 \to F' \quad (\text{or } F_0')$$

such that for every $a \in A$, $U(a)'Tf = T(U(a)f)$, for all $f \in F_0$.

2. The general situation.

First let us agree upon the notation. The vector space F is equipped with a locally convex Hausdorff topology, defined by a family of semi-norms Γ; L(F) denotes the algebra of all continuous linear operators in F. The topological dual of F is designated by F' and if S belongs to L(F), then S' is its dual. We shall deal with a faithful representation

$$U: A \to L(F)$$

of a given semi-prime algebra A.

A subspace $H \subset F$ is said to be invariant if $U(x)H \subset H$ for all $x \in A$, similarly a subspace $H' \subset F'$ is called invariant if $U(x)'H' \subset H'$ for all $x \in A$. If I is an ideal in A and H a subspace of F, $U(I)H$ will denote the vector span of all elements of the form $U(a)h$, $a \in I$, $h \in H$. A definition of the same type is used for subspaces of the dual space F'. We adopt the following definitions.

Definition 4.2.1. *The topology on F is said to be U-compatible if for every regular ideal I (equivalently for every ideal) the mapping*

$$P: U(I)F + U(I^c)F \to F,$$

defined by

$$P: f_1 + f_2 \to f_1,$$

$f_1 \in U(I)F$, $f_2 \in U(I^c)F$, *is well-defined and continuous.*

Similarly, a semi-norm $p \in \Gamma$ *is said to be U-compatible if for every ideal* I *in* A *there exists a constant* $c = c_I$ *such that*

$$p(f_1) \le cp(f_1 + f_2),$$

for $f_1 \in U(I)F$ *and* $f_2 \in U(I^c)F$.

Example. Let A be the algebra generated by a complete distributive Boolean algebra B of projections in L(F). Then by Theorem 2.2.7, an ideal I is regular if and only if $I = uA$ for some projection $u \in A$. If $U(a)f = a(f)$ for $f \in F$ and if $I = uA$, then $U(I)F = uF$ and $U(I^c)F = (e - u)F$ so that the projection $u: U(I)F + U(I^c)F \to F$ is continuous, indeed. If, moreover, B is equicontinuous, then by [25] we may assume that every semi-norm in the calibration Γ of F is U-compatible. As a matter of fact B. Walsh proves a much stronger compatibility in this case; see [25], Proposition 2.3, 2.4.

Unless stated otherwise, A will be equipped with the following weak operator topology. A subbasis at 0 is given by open neighbourhoods of the form $\{a\epsilon A,\ |\langle U(a)f,\phi\rangle| < 1\}$, where $f \in F$, $\phi \in F'$. This locally convex topology in A is the inverse image under U of the weak operator topology in L(F); since U is faithful, it is a Hausdorff topology. We are interested in the following types of closed ideals:

those of the form $I_f = \{a\epsilon A,\ U(a)f = 0\}$, where $f \in F$,

those of the form $I_\phi = \{a\epsilon A,\ U(a)'\phi = 0\}$, where $\phi \in F'$ and

those of the form $I_p = \{a\epsilon A,\ U(a)F \subset N(p)\}$, where p is a semi-norm in Γ and $N(p) = \{f\epsilon F,\ p(f) = 0\}$. An element $f \in F$ ($\phi \in F'$, $p \in \Gamma$ resp.) is said to be regular if I_f (I_ϕ, I_p resp.) is a regular ideal in A.

<u>Example</u>. Let $A = F = C(R)$, the algebra of the complex-valued continuous functions on $\mathbb{R}$, equipped with the topology defined by the family of semi-norms $\Gamma = \{p_K,\ p_K(f) = \sup\{|f(x)|,\ x\epsilon K\},\ K \subset \mathbb{R} \text{ compact}\}$. Then, every $f \in F$ is regular; every semi-norm p_K is U-compatible; and, if O is a bounded open subset of $\mathbb{R}$, the functional $f \to \int_O f(x)dx$ is regular. Finally, a semi-norm p_K is regular if and only if there exists an open subset $O \subset \mathbb{R}$ such that O is dense in K.

The theorem we want to prove reads as follows.

<u>Theorem</u> 4.2.2. <u>Let</u> A <u>be a semi-prime algebra, which satisfies the</u> c.c.c. <u>Let</u> $(F, \mathcal{T}_F)$ <u>be a locally convex vector space</u>, $U: A \to L(F)$ <u>an algebra homomorphism. Assume there exists a family of closed invariant subspaces</u> $\{H_\nu,\ \nu \in \Lambda\}$ <u>for which the following conditions are satisfied.</u>

(i) <u>The spaces</u> H_ν <u>are disjoint</u>: $H_{\nu_0} \cap \mathrm{cl}(\sum_{\nu\neq\nu_0} H_\nu) = \{0\}$, $(\nu_0 \in \Lambda)$

(ii) <u>They span</u> F: $H_{\nu_0} + \mathrm{cl}(\sum_{\nu\neq\nu_0} H_\nu) = F$, $(\nu_0 \in \Lambda)$

(iii) The mapping

$$P_{\nu_0} : F \to F,$$

defined by

$$P_{\nu_0} : h_0 + h \to h_0, \quad h_0 \in H_{\nu_0}, \; h \in \mathrm{cl}(\sum_{\nu \neq \nu_0} H_\nu)$$

is continuous for all $\nu_0 \in \Lambda$.

(iv) The topology $\mathcal{T}_F$, restricted to H_ν is U-compatible for all $\nu \in \Lambda$.

(v) Every H_ν is a "copy of A" in the sense that there exists an ideal S_ν (proper or not) in A together with a mapping U_ν: $S_\nu \to H_\nu$, such that (1) $U_\nu(S_\nu)$ is dense in H_ν,

(2) $U(a)U_\nu(b) = U_\nu(ab)$,

for all $a \in A$, $b \in S_\nu$.

(vi) The representation U restricted to H_ν is faithful: $U(x)H_\nu = \{0\}$ implies $x = 0$.

(vii) For every $\nu_0 \in \Lambda$, the family of regular functionals

$$\mathcal{J}_{\nu_0} = \{\phi \in (\sum_{\nu \neq \nu_0} H_\nu)^{\perp}, \; I_\phi = I_\phi^{cc}\}$$

has the property

$$\cap\{I_\phi, \; \phi \in \mathcal{J}_{\nu_0}\} = \{0\}.$$

(viii) For every $\nu \in \Lambda$ and every countable increasing family (I_n) of regular ideals for which $(\cup I_n)^{cc} = A$, the subspace $U_\nu(S_\nu \cap \cup I_n)$ is dense in H_ν.

Then there exists a family of invariant locally convex vector spaces $(F_\nu; \mathcal{T}_\nu) \subset F$ such that

(a) For each ν, F_ν is densely imbeddable into $(H_\nu, \mathcal{T}_F)$.

(b) The locally convex direct sum $(F_0, \mathcal{T}_0) = \oplus (F_\nu, \mathcal{T}_\nu)$ is densely imbeddable into $(F, \mathcal{T})$.

(c) The space $F' = (F, \mathcal{T}_F)'$ may be considered as a subspace of $F_0' = (F_0, \mathcal{T}_0)'$.

(d) There exists a mapping

$$T\colon F_0 \to F',$$

such that for every $\nu \neq \nu_0$,

$$\langle F_\nu, T(F_{\nu_0})\rangle = \{0\}.$$

(e) For every $\nu \in \Lambda$, $T(F_\nu)$ is invariant under $U(x)'$ for all $x \in A$. In fact we have

$$U(x)'T(f) = T(U(x)f)$$

for all $f \in F_0$.

Before we prove this, we like to make a few comments. This type of theorem is proved by W. Bade [1] in the special case that F is a Banach space and A an algebra of measurable functions on a Stone space X; for every simple function $a \in A$, $a = \Sigma_{i=1}^{n}\lambda_i\chi_{B_i}$, $U(a) = \Sigma_{i=1}^{n}\lambda_i U(\chi_{B_i})$, where $\{U(\chi_{B_i})\}$ is a Boolean algebra of commuting projections in F. In this case the spaces H_ν are cyclic in the sense that $H_\nu = U(A)f_\nu$, for some $f_\nu \in F$. The family (f_ν) can be chosen in such a way that $U(a)f_\nu = 0$, $a \in A$, implies $a = 0$. Under these circumstances one can show that, if H_ν admits a topological complement H_1 in F, then there exists a functional $\phi_\nu \in H_1^\perp \subset F'$ such that $U(A)'\phi_\nu$ is w*-dense in $H_1^\perp$ and the mapping

$$T\colon H_\nu \to H_1^\perp,$$

defined by

$$T\colon U(a)f_\nu \to U(a)'\phi_\nu, \quad a \in A$$

is one-to-one, linear and satisfies

$$TU(a)g = U(a)Tg,$$

for all $g \in H_\nu$.

We try to exhibit a similar construction in our general case.

Upon imposing more conditions on A (*e.g.* C*-algebra, vector lattice) and/or more conditions on F one can strengthen considerably the above result. Condition (v) is readily verified if H_ν is of the form $cl(U(A)f_\nu)$, where $f_\nu \in F$. Simply let $U_\nu(a) = U(a)f_\nu$ for $a \in A$.
The following lemma gives sufficient conditions in order that condition (vii) of the theorem is fulfilled.

Lemma 4.2.3. *Let* H_0, H *be subspaces of the locally convex vector space* F *such that* $H_0 \;\; H = \{0\}$ *and the projection mapping*

$$h_0 + h \to h_0 , \quad h_0 \in H_0 , \; h \in H$$

is continuous. Moreover, let H_0 *be invariant under* U.
In order that the ideal $J_0 = \{x \in A,\ U(x)H_0 = \{0\}\}$ *is equal to*

$$\bigcap \{x \in A,\ \langle U(x)H_0 , \phi\rangle = \{0\}\},$$

where the intersection is taken over all elements $\phi \in H^\perp$, *for which the ideal* $\{x \in A,\ \langle U(x)H_0 , \phi_0\rangle = \{0\}\}$ *is regular, either one of the following conditions is sufficient:*

(i) *For every closed ideal* $I \subset A$, $U(I)H_0$ *is dense in* $U(I^{cc})H_0$;

(ii) *For every regular ideal* $I_0 \subset A$, *for which* $I_0 \supset J_0$, $I_0 \neq J_0$, *there exists a continuous semi-norm* p *on* F *and a countable family of elements* $(a_n) \subset A$ *such that both*

(a) $p(U(a_n)H_0) \neq \{0\}$ *for all* n,

(b) *every closed ideal* I *for which* $\{x \in A,\ p(U(x)H_0) = \{0\}\} \cap I_0 \subset I \subset I_0$, $I^{cc} = I_0$ *contains at least one element* a_n.

Moreover the topology of F, *restricted to* H_0, *must be* U-*compatible.*

Proof. Sufficiency of (i). By definition,

$$\begin{aligned} J_0 &= \{x\varepsilon A,\ U(x)H_0 = \{0\}\} \\ &= \{x\varepsilon A,\ \langle U(x)H_0,\phi\rangle = \{0\} \text{ for all } \phi\varepsilon F'\} \\ &\subset \{x\varepsilon A,\ \langle U(x)H_0,\phi\rangle = \{0\} \text{ for all } \phi\varepsilon H^{\perp}\}. \end{aligned}$$

We claim that the converse inclusion is also true. Let $\langle U(x)H_0,\phi\rangle = \{0\}$ for all $\phi \;\varepsilon\; H^{\perp}$ and take $\phi_0 \;\varepsilon\; F'$. We are going to show that $\langle U(x)H_0,\phi_0\rangle = \{0\}$.

First, define the functional

$$\overline{\phi}_1 : H_0 + H \to \mathbb{C},$$

by

$$\overline{\phi}_1 : h_0 + h \to \langle h_0,\phi_0\rangle.$$

By the Hahn-Banach theorem there exists a continuous functional ϕ_1 defined on all of F, which is an extension of $\overline{\phi}_1$. So, if x is an element in A such that $\langle U(x)H_0,\phi\rangle = \{0\}$, for all $\phi \;\varepsilon\; H^{\perp}$, then

$$\begin{aligned} \langle U(x)h_0,\phi_0\rangle &= \langle U(x)h_0,\phi_1\rangle + \langle U(x)h_0,\phi_0\rangle - \langle U(x)h_0,\phi_1\rangle \\ &= 0 + \langle U(x)h_0,\phi_0\rangle - \langle U(x)h_0,\phi_0\rangle \\ &= 0. \end{aligned}$$

This shows that $J_0 = \bigcap_{\phi}\{x\varepsilon A,\ \langle U(x)H_0,\phi\rangle = \{0\},\ \phi\varepsilon H^{\perp}\}$.

To complete the proof it suffices to show that for every $\phi \;\varepsilon\; H^{\perp}$, the ideal I_ϕ defined by $I_\phi = \{x\varepsilon A,\ \langle U(x)H_0,\phi\rangle = \{0\}\}$ is regular. From the definition of I_ϕ it follows that $\langle U(I_\phi)H_0,\phi\rangle = \{0\}$. Since $U(I_\phi)H_0$ is dense in $U(I_\phi^{cc})H_0$, we have $\langle U(I_\phi^{cc})H_0,\phi\rangle = \{0\}$. Hence, since I_ϕ is the largest ideal I for which $\langle U(I)H_0,\phi\rangle = \{0\}$, it follows $I_\phi \supset I_\phi^{cc}$, whence $I_\phi = I_\phi^{cc}$.

Sufficiency of (ii). Define again, for any $\phi \in H^{\perp} = \{\phi \in F', \langle H,\phi\rangle = \{0\}\}$, the ideal I_ϕ by $I_\phi = \{x \in A, \langle U(x)H_0,\phi\rangle = \{0\}\}$ and let $I_0 = \cap\{I_\phi, \phi \in H^{\perp}, I_\phi = I_\phi^{cc}\}$. We first prove that for every $\psi \in H^{\perp}$, $I_\psi^{cc} \cap I_0 = (I_\psi \cap I_0)^{cc} = I_0$. Since we always have that $(I_\psi \cap I_0)^{cc} = I_\psi^{cc} \cap I_0^{cc}$, it is sufficient to prove that both $I_0 = I_0^{cc}$ and $I_\psi^{cc} \cap I_0 = I_0$. The fact that $I_0 = I_0^{cc}$ is a consequence of Theorem 2.1.6. Next, let there exist an element $\psi \in H^{\perp}$ for which $I_\psi^{cc} \cap I_0 \neq I_0$.

Define the functional

$$\overline{\psi}_0 : U(I_\psi^{cc})H_0 + U(I_\psi^{c})H_0 + H \to \mathbb{C}$$

by

$$\overline{\psi}_0 : h_0 + h_1 + h \to \langle h_1,\psi\rangle,$$

where $h_0 \in U(I_\psi^{cc})H_0$, $h_1 \in U(I_\psi^{c})H_0$, $h \in H$.

Then, since the topology on F restricted to H_0 is U-compatible, it follows that $\overline{\psi}_0$ is continuous on its domain and hence, by the Hahn-Banach theorem, admits of a continuous extension ψ_0 to all of F. Notice that $\psi_0 \in H^{\perp}$ and that $I_\psi^{cc} \subset I_{\psi_0}$. We must prove that the converse inclusion is valid, too. By definition, we have

$$\begin{aligned} I_{\psi_0} &= \{x \in A, \langle U(x)H_0,\psi_0\rangle = \{0\}\} \\ &\subset \{x \in A, \langle U(x)U(I_\psi^{c})H_0,\psi_0\rangle = \{0\}\} \\ &= \{x \in A, \langle U(xI_\psi^{c})H_0,\psi\rangle = \{0\}\} \\ &= \{x \in A, xI_\psi^{c} \subset I_\psi \cap I_\psi^{c} = \{0\}\} \\ &= I_\psi^{cc}, \end{aligned}$$

from which the desired conclusion $I_{\psi_0} = I_\psi^{cc}$ follows. Hence it follows $I_{\psi_0} \cap I_0 \neq I_0$. However this is impossible, since, by definition, I_{ψ_0} contains I_0.

Next, consider the space $G = H_0 + H$, equipped with the topology of F. Let G' denote the dual space of G and let, for each semi-norm p in the calibration Γ of F, H'_p be the subspace of G' defined by

$$H'_p = \{\phi\varepsilon G', \text{ there exists a constant } c = c_\phi, \text{ such that } |\langle h_0 + h,\phi\rangle| \leq cp(h_0) \text{ for all } h_0\varepsilon H_0,\ h\varepsilon H\}.$$

Endowing H'_p with the norm

$$\|\phi\|_p = \sup\{|\langle h_0 + h,\phi\rangle|,\ p(h_0) < 1,\ h\varepsilon H\},$$

it becomes a Banach space.

Again, let for $\phi \varepsilon G'$, I_ϕ denote the ideal

$$\{x\varepsilon A,\ \langle U(x)H_0,\phi\rangle = \{0\}\}.$$

Notice that every functional $\phi \varepsilon G'$ can be extended to a continuous functional defined on all of F. Moreover, if $\phi \varepsilon G'$ and $\langle H,\phi\rangle = \{0\}$, then every extension $\tilde{\phi}$ of ϕ to F has the property that $I_{\tilde{\phi}} = I_\phi$. In other words, if $\phi \varepsilon G'$, then I_ϕ is closed in the weak operator topology. By the above remark we even have that for every $\phi \varepsilon G'$, $\phi \varepsilon H^\perp$, $I_\phi^{cc} \cap I_0 = I_0$.

We shall prove that the space H'_p can be written as

$$H'_p = \bigcup_I\{\phi\varepsilon H'_p,\ \langle H,\phi\rangle = \{0\},\ I_\phi \supset I\},$$

where the union is taken over all closed ideals $I \subset A$ for which both $\{x\varepsilon A,\ p(U(x)H_0) = \{0\}\}\cap I_0 \subset I \subset I_0$, and $I^{cc} = I_0$.

Recall that I_0 is the ideal defined by $I_0 = \bigcap\{I_\phi,\ I_\phi = I_\phi^{cc},\ \phi\varepsilon H^\perp\}$, where $H^\perp = \{\phi\varepsilon F',\ \langle H,\phi\rangle = \{0\}\}$.

If ϕ is an element in H'_p, then $|\langle h_0 + h,\phi\rangle| \leq cp(h_0)$ for all $h_0 \varepsilon H_0$, $h \varepsilon H$, and so $\langle H,\phi\rangle = \{0\}$. Also if $a \varepsilon A$ belongs to $I_p = \{x\varepsilon A,\ p(U(x)H_0) = \{0\}\}$, then $|\langle U(a)h_0,\phi\rangle| \leq cp(U(a)h_0) = 0$ for all $h_0 \varepsilon H_0$, whence $I_p \subset I_\phi$.

By the previous comments it follows that the ideal $I = I_\phi \cap I_0$ is closed and has the properties $I_p \cap I_0 \subset I \subset I_0$ and $I^{cc} = I_0$. We have to prove that $I_0 = J_0$, where $J_0 = \{a \in A,\ U(a)H_0 = \{0\}\}$. We clearly have $I_0 \supset J_0$ and I_0 is regular. Suppose, indirectly, that $I_0 \neq J_0$. By condition (ii) there exists a semi-norm p and a countable family (a_n) of elements in A for which

(a) $p(U(a_n)H_0) \neq \{0\}$ for every n, and

(b) every closed ideal I for which $I_p \cap I_0 \subset I \subset I_0$, $I^{cc} = I_0$, contains at least one element a_n.

Let H_p' be the subspace belonging to this semi-norm. Then, by what has been proved above, the Banach space H_p' can be written as the countable union

$$H_p' = \bigcup \{\phi \in H_p',\ \langle H,\phi\rangle = \{0\},\ a_n \in I_\phi\}.$$

It is readily verified that for each n, the sets

$$\{\phi \in H_p',\ \langle H,\phi\rangle = \{0\},\ a_n \in I_\phi\},$$

are closed subspaces of H_p'. A Baire category argument applies to the effect that at least one of the subspaces

$$\{\phi \in H_p',\ \langle H,\phi\rangle = \{0\},\ a_n \in I_\phi\}$$

contains an open neighbourhood and, therefore, coincides with H_p'. It follows that there exists an element $a_n \in A$ for which $p(U(a_n)H_0) \neq \{0\}$ and $\langle U(a_n)H_0, H_p'\rangle = \{0\}$. However, this is impossible as the next argument shows. Let $p(U(a_n)h_0) = 1$ and define $\bar{\psi}$: $\mathbf{C}U(a_n)h_0 \rightarrow \mathbf{C}$, by $\bar{\psi}$: $\lambda U(a_n)h_0 \rightarrow \lambda$. By the Hahn-Banach theorem $\bar{\psi}$ can be extended to all of H_0 in such a way that $|\langle h_1, \bar{\psi}\rangle| \leq p(h_1)$ for all $h_1 \in H_0$. Finally define ψ: $G \rightarrow \mathbf{C}$ as follows $\psi(h_1 + h) = \bar{\psi}(h_1)$ for all $h_1 \in H_0$, $h \in H$.

Then $\psi \in H_p'$ and $\langle U(a_n)h_0, \psi\rangle = 1$.

A Boolean algebra B is said to be complete (σ-complete) if for every (countable) increasing family $(p_\alpha) \subset B$, its supremum $\vee p_\alpha$ exists as an element of B. It is said to be distributive (σ-distributive) if for every (countable) increasing family (p_α) and every element $p \in B$, $p \wedge \vee p_\alpha = \vee(p \wedge p_\alpha)$.

Example 1. Let B be a complete (σ-complete) distributive (σ-distributive) Boolean algebra of continuous projections defined on a topological vector space F. Let, for every (countable) increasing family $(p_\alpha) \subset B$, $\cup p_\alpha F$ be dense in $(\vee p_\alpha)F$ (and let B satisfy the c.c.c.). Let A denote the algebra of all finite complex combinations of projections in B and define $U: A \to L(F)$ by $U(a)f = af$ for $a \in A$ and $f \in F$. Then, $U(I)F$ is dense in $U(I^{cc})F$ for every ideal $I \subset A$. (The proof of this hinges upon the fact that $I^{cc} = (\vee p_\alpha)A$, for a suitable chosen increasing family in $B \cap I$. If A satisfies the c.c.c., then this family can be chosen to be countable, see Lemma 2.3.1.)

Example 2. Let X be a locally compact Hausdorff space, which has a countable base for its topology. Let A denote the algebra of all complex-valued continuous functions on X and assume A to be equipped with the topology of uniform convergence on compact subsets of X. Let $U: A \to L(F)$ be a faithful representation. Then there exists a countable family of functions $(f_n) \subset A$, $f_n \neq 0$, such that every closed ideal in A contains at least one f_n and in particular, if the functionals

$$a \to \langle U(a)f, \phi\rangle$$

are continuous on A for all $f \in F$ and all $\phi \in F'$, property (ii) in Lemma 4.2.3 holds.

The next example shows that (i) and (ii) in the previous lemma need not go together.

Example 3. Let $A = C[0,1]$, the algebra of all complex-valued continuous functions on $[0,1]$, $F = L^2[0,1]$, the Hilbert space of all square integrable functions and $U(f)g = fg$ for $f \in A$, $g \in F$. The triple (A, F, U) has the properties of the previous example; we will show that there exists a closed ideal I in A, for which $I^{cc} = A$ and for which $U(I)F$ is not dense in $U(A)F = F$. Let $U \subset [0,1]$ be an open set, dense in $[0,1]$, of Lebesgue measure less than ε, $1 > \varepsilon > 0$. Let I be the ideal defined by $I = \{f \varepsilon A, \{x \varepsilon [0,1], f(x) \neq 0\} \subset U\}$. Then $I^{cc} = A$, but $U(I)F$ is not dense in F. To see this, consider the function $h = 1 - \chi_U$ in F. If f belongs to $U(I)F$, then $\|f - h\|^2 = \|f\|^2 + \|h\|^2 \geq \|h\|^2 \geq 1 - \varepsilon$.

Such an open set U exists: following G. Helmberg, Math. Zeitschr. 83, 261-266 (1964), we define for r_n the n^{th} rational number in R, V_n by $V_n = (r_n - \varepsilon 2^{-n-1}, r_n + \varepsilon 2^{-n-1})$. Let $O = \cup V_n$ and $U = (0,1) \cap O$. Then U is dense in $[0,1]$ and the Lebesgue measure of U is less than ε.

For the proof of Theorem 4.2.2 we need one more technical lemma. Lemma 4.2.4 is in fact a generalization of Lemma 3.2.2.

Lemma 4.2.4. *Let A be a semi-prime algebra, F a locally convex vector space, $U: A \to L(F)$ a representation. As in the previous lemma, let H_0 and H be two subspaces for which the projection mapping $h_0 + h \to h_0$, $h_0 \in H_0$, $h \in H$, exists and is continuous. Assume that H_0 is invariant and that the topology on F restricted to H_0 is U-compatible. Let $(\phi_n) \subset H^\perp$ be a countable family of functionals for which $\{x \varepsilon A, \langle U(x)H_0, \phi_n \rangle = \{0\}\}$ is a regular ideal for each n.*

Then, there exists a countable family of functionals $(\psi_n) \subset H^{\perp}$ such that the following conditions are satisfied:

(i) $\bigcap_{k\leq n} \{x\varepsilon A,\ \langle U(x)H_0,\phi_k\rangle = \{0\}\} = \{x\varepsilon A,\ \langle U(x)H_0,\psi_n\rangle = \{0\}\}$, for all n;

(ii) $\langle U(b)h_0,\psi_m\rangle = \langle U(b)h_0,\psi_n\rangle$ for all elements

$b \in \{x\varepsilon A,\ \langle U(x)H_0,\psi_n\rangle = \{0\}\}^c$, all $h_0 \in H_0$ and all $m \geq n$.

In particular it follows that, for each n, the ideal

$\{x\varepsilon A,\ \langle U(x)H_0,\psi_n\rangle = \{0\}\}$ is regular and that

$\bigcap_n\{x\varepsilon A,\ \langle U(x)H_0,\phi_n\rangle = \{0\}\} = \bigcap_n\{x\varepsilon A,\ \langle U(x)H_0,\psi_n\rangle = \{0\}\}$.

Assume, in addition, that

(a) there exists a fixed semi-norm p whose restriction to H_0 is U-compatible;

(b) the family (ϕ_n) has the property that, for each n, there exists a constant c_n such that $|\langle h_0,\phi_n\rangle| \leq c_n p(h_0)$, for all $h_0 \in H_0$.

Then, the family (ψ_n) can be chosen in such a way that it has not only properties (i) and (ii), but also satisfies the following:

$|\langle h_0,\psi_n\rangle| \leq p(h_0)$, for all $h_0 \in H_0$ and each integer n. It easily follows that any Hahn-Banach extension ψ_0 of the functional

$$\bar{\psi}_0\colon H_0 + H \to \mathbb{C},$$

defined by

$$\bar{\psi}_0\colon h_0 + h \to \Sigma_{n=1}^{\infty} 2^{-n}\langle h_0,\psi_n\rangle,$$

has the property that

$$\{x\varepsilon A,\ \langle U(x)H_0,\psi_0\rangle = \{0\}\} = \bigcap_n\{x\varepsilon A,\ \langle U(x)H_0,\phi_n\rangle = \{0\}\}.$$

Proof. We first prove the second assertion. So, let A, F, H_0, H, U be as in the lemma and let there exist a countable family (ϕ_n) of functionals in $H^{\perp}$, and a semi-norm p which restricted to H_0 is U-compatible, such that for each n we have $|\langle h_0,\phi_n\rangle| \leq c_n p(h_0)$ for all $h_0 \in H_0$ and a suitable chosen constant c_n. We must prove that there exists a countable family (ψ_n) such that the conclusions of the lemma are valid. Upon dividing by an appropriate constant we may assume that for each n,

$$|\langle h_0,\phi_n\rangle| \leq p(h_0), \quad h_0 \in H_0.$$

The construction proceeds by induction. First, let $\psi_1 = \phi_1$. Now, let the functionals $\psi_1,\ldots,\psi_n$ in $H^{\perp}$ be constructed in such a way that

(a) $|\langle h_0,\psi_k\rangle| \leq (2-\varepsilon)p(h_0)$, all $h_0 \in H_0$, $k = 1,\ldots,n$, $1 > \varepsilon > 0$,

(b) $\{x\varepsilon A,\ \langle U(x)H_0,\psi_k\rangle = \{0\}\} = \bigcap_{1\leq k} \{x\varepsilon A,\ \langle U(x)H_0,\phi_1\rangle = \{0\}\}$, for all $k \leq n$,

(c) $\langle h_1,\psi_k\rangle = \langle h_1,\psi_1\rangle$, $n \geq k \geq 1$, all $h_1 \varepsilon U(\{x\varepsilon A,\ \langle U(x)H_0,\psi_1\rangle = \{0\}\}^c)H_0$.

Upon writing I_k for the ideal $I_k = \{x\varepsilon A,\ \langle U(x)H_0,\psi_k\rangle = \{0\}\}$, (c) may be reformulated as $\langle h_1,\psi_k\rangle = \langle h_1,\psi_1\rangle$, $n \geq k \geq 1$, $h_1 \in U(I_1^c)H_0$.

By the compatibility of p there exists a constant d_n, such that

$$p(h_1) \leq d_n p(h_1 + h_2),$$

for $h_1 \in U(I_n)H_0$ and $h_2 \in U(I_n^c)H_0$.

Under this induction hypothesis we shall construct a functional $\psi_{n+1} \in H^{\perp}$, such that $|\langle h_0,\psi_{n+1}\rangle| \leq (2 - 2^{-1}\varepsilon)p(h_0)$, for all $h_0 \in H_0$, and for which the family $\psi_1,\ldots,\psi_{n+1}$, satisfies conditions (b) and (c) above with n replaced by $n+1$.

Define

$$\overline{\psi}_{n+1}: U(I_n)H_0 + U(I_n^c)H_0 \to \mathbb{C},$$

by

$$\overline{\psi}_{n+1}: h_1 + h_2 \to \frac{\varepsilon}{2d_n} \langle h_1, \phi_{n+1} \rangle + \langle h_2, \psi_n \rangle,$$

where $h_1 \in U(I_n)H_0$ and $h_2 \in U(I_n^c)H_0$.

The functional $\overline{\psi}_{n+1}$ is well-defined on its domain. In fact, if h_1, h_1' are in $U(I_n)H_0$ and h_2, h_2' are in $U(I_n^c)H_0$, then $h_1 - h_1' \in U(I_n)H_0$ and $h_2 - h_2' \in U(I_n^c)H_0$ and so $|\langle h_1, \phi_{n+1}\rangle - \langle h_1', \phi_{n+1}\rangle| = |\langle h_1 - h_1', \phi_{n+1}\rangle|$ $\leq p(h_1 - h_1') \leq d_n p(h_1 - h_1' + h_2 - h_2') = d_n p(0) = 0.$

Hence
$$\begin{aligned}
&\frac{\varepsilon}{2d_n} \langle h_1', \phi_{n+1}\rangle + \langle h_2', \psi_n\rangle \\
&= \frac{\varepsilon}{2d_n} \langle h_1, \phi_{n+1}\rangle + \langle h_1' + h_2', \psi_n\rangle \\
&= \frac{\varepsilon}{2d_n} \langle h_1, \phi_{n+1}\rangle + \langle h_1 + h_2, \psi_n\rangle \\
&= \frac{\varepsilon}{2d_n} \langle h_1, \phi_{n+1}\rangle + \langle h_2, \psi_n\rangle.
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
|\langle h_1 + h_2, \overline{\psi}_{n+1}\rangle| &= \left|\frac{\varepsilon}{2d_n} \langle h_1, \phi_{n+1}\rangle + \langle h_2, \psi_n\rangle\right| \\
&= \left|\frac{\varepsilon}{2d_n} \langle h_1, \phi_{n+1}\rangle + \langle h_1 + h_2, \psi_n\rangle\right| \\
&\leq \frac{\varepsilon}{2d_n} p(h_1) + (2 - \varepsilon)p(h_1 + h_2) \\
&\leq \frac{\varepsilon}{2d_n} d_n p(h_1 + h_2) + (2 - \varepsilon)p(h_1 + h_2) \\
&= (2 - 2^{-1}\varepsilon)p(h_1 + h_2),
\end{aligned}$$

for all $h_1 \in U(I_n)H_0$ and $h_2 \in U(I_n^c)H_0$.

Let $\tilde{\psi}_{n+1}$ be any Hahn-Banach extension of $\overline{\psi}_{n+1}$ to all of H_0, so that

$$|\langle h_0, \tilde{\psi}_{n+1}\rangle| \leq (2 - 2^{-1}\varepsilon)p(h_0), \quad \text{for all } h_0 \in H_0,$$

and define

$$\psi_{n+1}: F \to \mathbf{C},$$

as being any Hahn-Banach extension to all of F of the functional

$$h_0 + h \to \langle h_0, \tilde{\psi}_{n+1}\rangle, \quad h_0 \in H_0,\ h \in H.$$

Then, the functional ψ_{n+1} satisfies (a) and the family $\psi_1, \dots, \psi_{n+1}$ satisfies (c) with n replaced by n+1. Let us check (b).

Consider the ideal $\{x \varepsilon A,\ \langle U(x)H_0, \psi_{n+1}\rangle = \{0\}\}$, denoted by I_{n+1}. We clearly have

$$\begin{aligned} I_{n+1} &\subset \{x \varepsilon A,\ \langle U(x)U(I_n^c)H_0, \psi_{n+1}\rangle = \{0\}\} \\ &= \{x \varepsilon A,\ \langle U(xI_n^c)H_0, \psi_n\rangle = \{0\}\} \\ &= \{x \varepsilon A,\ xI_n^c \subset I_n\} \\ &= I_n^{cc} = I_n. \end{aligned}$$

Hence

$$\begin{aligned} I_{n+1} &= I_n \cap I_{n+1}, \text{ which is by definition,} \\ &= \{x \varepsilon I_n,\ \langle U(x)H_0, \psi_{n+1}\rangle = \{0\}\} \\ &= \{x \varepsilon I_n,\ \langle U(x)H_0, \phi_{n+1}\rangle = \{0\}\} \\ &= \{x \varepsilon I_n,\ x \varepsilon I_{\phi_{n+1}}\} \\ &= I_n \cap I_{\phi_{n+1}}, \end{aligned}$$

from which (b) follows.

The sequence (ψ_n), obtained in this way, has the following properties:

(i) $|\langle h_0, \psi_n\rangle| < 2p(h_0)$, for all $h_0 \in H_0$,

(ii) $I_m \subset I_n$, for $m \geq n$,

(iii) $\langle U(b)h_0, \psi_m\rangle = \langle U(b)h_0, \psi_n\rangle$ for all $b \in I_n^c$, all $h_0 \in H_0$, all $m \geq n$,

(iv) $I_n = \bigcap_{k \leq n} \{x \varepsilon A,\ \langle U(x)H_0, \phi_k\rangle = \{0\}\}$.

Upon dividing by 2, we may assume that for every n, $|\langle h_0, \psi_n \rangle| < p(h_0)$, for all $h_0 \in H_0$. There remains to be proved that, if ψ_0 is any Hahn-Banach extension of the functional

$$h_0 + h \to \Sigma_{n=1}^{\infty} 2^{-n} \langle h_0, \psi_n \rangle,$$

then ψ_0 has the property that

$$\{x \varepsilon A, \langle U(x)H_0, \psi_0 \rangle = \{0\}\} = \bigcap_n \{x \varepsilon A, \langle U(x)H_0, \psi_n \rangle = \{0\}\}.$$

This will be done by induction again. If $\langle U(x)H_0, \psi_0 \rangle = \{0\}$, then $\langle U(x)U(I_1^c)H_0, \psi_0 \rangle = \{0\}$ and so $\Sigma_{n=1}^{\infty} 2^{-n} \langle U(xI_1^c)H_0, \psi_1 \rangle = \{0\}$, or $xI_1^c \subset I_1$ i.e. $x \varepsilon I_1^{cc} = I_1$. The remaining part of the proof is exactly the same as in the proof of Lemma 3.2.2 and may be omitted.

In the general case we, again, proceed by induction. First, define, $\psi_1 = \phi_1$. Suppose that the functionals $\psi_1, .., \psi_n$ are constructed in such a way that

(i) $\psi_k \varepsilon H^{\perp}$, $k = 1, .., n$,

(ii) $\{x \varepsilon A, \langle U(x)H_0, \psi_k \rangle = \{0\}\} = \bigcap_{1 \leq k} \{x \varepsilon A, \langle U(x)H_0, \phi_1 \rangle = \{0\}\}$, $k = 1, .., n$,

(iii) $\langle U(b)h_0, \psi_k \rangle = \langle U(b)h_0, \psi_1 \rangle$, $h_0 \varepsilon H_0$, $b \varepsilon I_1^c$, $n \geq k \geq 1$.

(Again, as above, I_1 denotes the ideal $I_1 = \{x \varepsilon A, \langle U(x)H_0, \psi_1 \rangle = \{0\}\}$.)

Let ψ_{n+1} be any Hahn-Banach extension to all of F of the functional,

$$\overline{\psi}_{n+1} : U(I_n)H_0 + U(I_n^c)H_0 + H \to \mathbb{C},$$

defined by

$$\overline{\psi}_{n+1} : h_1 + h_2 + h \to \langle h_1, \phi_{n+1} \rangle + \langle h_2, \psi_n \rangle,$$

where $h_1 \varepsilon U(I_n)H_0$, $h_2 \varepsilon U(I_n^c)H_0$ and $h \varepsilon H$.

Then by the same argumentation as above the family $\psi_1, .., \psi_{n+1}$ satisfies (ii) and (iii) with n replaced by n+1.

The construction of ψ_{n+1} in this manner is possible by the facts that the topology of F, restricted to H_0, is U-compatible and that the projection $h_0 + h \to h_0$, $h_0 \in H_0$ and $h \in H$, is continuous. Clearly, the family (ψ_n) obtained in this way satisfies the conclusions of the lemma.

Proof of Theorem 4.2.2.

By assumption (vii) we know that for every H_{ν_0}, the family of regular functionals in $(\sum_{\nu\neq\nu_0} H_\nu)^\perp$ has the property

$$\{0\} = \cap\{I_\phi,\ I_\phi = I_\phi^{cc},\ \phi \in (\sum_{\nu\neq\nu_0} H_\nu)^\perp\},$$

where by definition for every $\phi \in F'$, I_ϕ denotes the ideal

$$I_\phi = \{x \in A,\ \langle U(x)F,\phi\rangle = \{0\}\}.$$

For brevity we shall write $H_0 = H_{\nu_0}$, $H = \mathrm{cl} \sum_{\nu\neq\nu_0} H_\nu$.

Then, the pair H_0, H satisfies the conditions of the previous lemma. Since A satisfies the c.c.c. we know that there exists a countable family $(\phi_n) \subset H^\perp$ such that $\{0\} = \cap\{I_{\phi_n},\ I_{\phi_n}^{cc} = I_{\phi_n}\}$. By the previous lemma, we may suppose that the sequence of ideals (I_{ϕ_n}) has the following properties:

(i) $I_{\phi_m} \subset I_{\phi_n}$, for all $m \geq n$,

(ii) $\langle U(b)h_0,\phi_m\rangle = \langle U(b)h_0,\phi_n\rangle$, for all $m \geq n$, all $h_0 \in H_0$, all $b \in I_{\phi_n}^c$,

(iii) $I_{\phi_n}^{cc} = I_{\phi_n}$, for all n.

Denote by K_{ν_0} the ideal $K_{\nu_0} = \cup I_{\phi_n}^c$ and let F_{ν_0} be the subspace $F_{\nu_0} = U_{\nu_0}(S_{\nu_0} \cap K_{\nu_0})$, where U_{ν_0} is the mapping of assumption (v). Then, by (viii), F_{ν_0} is dense in H_0. Define $T_{\nu_0}(f)$, for f of the form $f = U_{\nu_0}(a_{\nu_0})$, where $a_{\nu_0} \in S_{\nu_0} \cap I_{\phi_n}^c$, by $T_{\nu_0}(f) = U(a_{\nu_0})'\phi_n$. Then T_{ν_0} is a well-defined, linear and one-to-one map defined on F_{ν_0} and taking its values in F'. Moreover, it has the following invariance property.

If $x \in A$ and $f \in F_{\nu_0}$ as above, then $U(x)'T_{\nu_0}(f) = U(x)'U(a_{\nu_0})\phi_n$ $= U(xa_{\nu_0})'\phi_n = T_{\nu_0}(U_{\nu_0}(xa_{\nu_0})) = T_{\nu_0}(U(x)U_{\nu_0}(a_{\nu_0})) = T_{\nu_0}(U(x)f)$. We can follow this procedure for every $\nu \in \Lambda$, thereby providing ourselves with a family of invariant subspaces $(F_\nu, \mathcal{T}_\nu)$ as described in the theorem. Let $(F_0, \mathcal{T}_0) = \oplus (F_\nu, \mathcal{T}_\nu)$ and define T_0: $F_0 \to F'$ as follows: a general element in F_0 being of the form $f_0 = \Sigma f_\nu$, where $f_\nu \in F_\nu$ and only finitely many terms are non-zero, $T_0(f_0)$ is by definition $T_0(f_0) = \Sigma T_\nu(f_\nu)$. Then T_0 satisfies (d) and (e) of the theorem.

A simple example featuring the situation of the theorem is the following one.

Example. Let $A = F = C(R)$, the space of the complex-valued continuous functions on R and let $U(f)g = fg$ for all $f \in A$, $g \in F$. In this case there is only one invariant space H_ν involved, namely F itself. If U_0: $A \to F$ is the identity map, then the conditions of Theorem 4.2.2 are readily verified. We may take for F_0 the space of all continuous functions of compact support. For the sequence of regular functionals (ϕ_n) we may take $\langle f,\phi_n\rangle = \int_{-n}^{+n} f(t)dt$, where $f \in F$ and n is a positive integer. For the map J_0: $F_0 \to F'$, we take the mapping defined by

$$\langle f,J_0(f_0)\rangle = \int_{-\infty}^{+\infty} f(t)f_0(t)dt$$

for all $f \in F$, $f_0 \in F_0$.

Corollary 4.2.5. Assume, in addition to (i) - (viii), of Theorem 4.2.2, that the topology of F, restricted to H_ν is given by a norm for each $\nu \in \Lambda$. Then F_0 in the theorem may be taken $(F_0, \mathcal{T}_0) = \oplus (U_\nu(S_\nu), \mathcal{T}_\nu)$, where $\mathcal{T}_\nu$ is the topology on F restricted to $U_\nu(S_\nu)$.

Proof. This result is a consequence of the second assertion in Lemma 4.2.4. In fact, for every $\nu_0 \in \Lambda$, there exists a functional ϕ_0 in $(\sum_{\nu \neq \nu_0} H_\nu)^\perp$ for which $I_{\phi_0} = \{0\}$. Then $T_0: U_{\nu_0}(S_{\nu_0}) \to F'$ may be defined by

$$T_0(f_0) = U(a)'\phi_0, \quad f_0 = U_{\nu_0}(a),\ a \in S_{\nu_0}.$$

3. The situation where $U(I)F$ is dense in $U(I^{cc})F$.

In this section we examine the situation where, for every ideal $I \subset A$, $U(I)F$ is a dense subset of $U(I^{cc})F$. We shall prove the following result.

Theorem 4.3.1. Let $U: A \to L(F)$ be a faithful representation of the semiprime algebra A, which satisfies the c.c.c.. Assume, there exists a family of closed invariant subspaces $\{H_\nu, \nu\in\Lambda\}$ such that (i) - (iv) of Theorem 4.2.2 are satisfied. Moreover, let every subspace H_ν be minimal, in the sense that there does not exist a proper closed invariant subspace $H \subset H_\nu$, for which the representation U, restricted to H, is faithful. Finally, let for every $f \in F$ the ideal $\{x\in A, U(x)f = 0\}$ be regular. Then the same conclusions can be drawn as in Theorem 4.2.2.

For the proof we need the following lemmas. Lemma 4.3.2 justifies the title of this section.

Lemma 4.3.2. Let H_0 be a minimal subspace, for which U is faithful, and let the topology of F, restricted to H_0, be U-compatible. Then

(a) For every ideal $I \subset A$, $H_0 = \mathrm{cl}U(I)H_0 + \mathrm{cl}U(I^c)H_0$;

(b) For every ideal $I \subset A$, $U(I)H_0$ is dense in $U(I^{cc})H_0$; $U(A)H_0$ is dense in H_0.

Proof. Let I be an arbitrary ideal in A. Consider the subspace $H = U(I)H_0 + U(I^c)H_0$. Then $U(x)H = \{0\}$ implies $x \in I^{cc} \cap I^c = \{0\}$.

Thus, by the minimality, $H_0 = cl(U(I)H_0 + U(I^c)H_0)$. By the U-compatibility, it follows that $H_0 = cl(U(I)H_0) + cl(U(I^c)H_0)$. This proves (a).
It follows $U(I^{cc})H_0 = U(I^{cc})cl(U(I)H_0)$. Hence $U(I^{cc} \cap I)H_0$ is dense in H_0. So, certainly, the same holds for $U(I)H_0$. Since $U(x)U(A)H_0 = \{0\}$ implies $x = 0$ and since H_0 is minimal, we conclude $U(A)H_0$ is dense in H_0.

Lemma 4.3.3. *Let H_0 be a closed invariant subspace of F and let h_1 and h_2 be two vectors in H_0. Denote by I the ideal $I = \{x \epsilon A,\ U(x)h_1 = \{0\}\}^{cc}$, let $H_0 = clU(I)H_0 + clU(I^c)H_0$ and let the projection $H_0 \to clU(I)H_0$ exist and be continuous. Then there exist a vector $h \epsilon H_0$ such that $\{x \epsilon A,\ U(x)h = \{0\}\} = I \cap \{x \epsilon A,\ U(x)h_2 = \{0\}\}$. Moreover, h can be chosen in such a way that for all $x \epsilon I^c$, $U(x)h_1 = U(x)h$.*

Proof. Let P_{cc}, P_c denote the projections on $clU(I)H_0$ and $clU(I^c)H_0$ resp. and define $h = P_{cc}h_2 + P_c h_1$. It easily follows that $x \epsilon I$ implies $U(x)h = U(x)h_2$ and that $x \epsilon I^c$ implies $U(x)h = U(x)h_1$. Consider the ideal $\{x \epsilon A,\ U(x)h = \{0\}\}$. This ideal is contained in $\{x \epsilon A,\ U(x)U(I^c)h = \{0\}\}$
$= \{x \epsilon A,\ U(xI^c)h = \{0\}\} = \{x \epsilon A,\ U(xI^c)h_1 = \{0\}\} =$
$= \{x \epsilon A,\ xI^c \subset \{y \epsilon A,\ U(y)h_1 = \{0\}\}\} \subset \{x \epsilon A,\ xI^c \subset I\} = \{x \epsilon A,\ x \epsilon I\} = I$.
Hence, $\{x \epsilon A,\ U(x)h = \{0\}\} = \{x \epsilon I,\ U(x)h = \{0\}\} = \{x \epsilon I,\ U(x)h_2 = \{0\}\}$
$= I \cap \{x \epsilon A,\ U(x)h_2 = \{0\}\}$. This proves the lemma.

Recall that a vector $f \epsilon F$ is called regular if the ideal $\{x \epsilon A,\ U(x)f = 0\}$ is regular.

Lemma 4.3.4. *Let H_0 be a closed invariant subspace of F and let (h_n) be a countable family of regular elements in H_0. Let, for every ideal $I \subset A$, $H_0 = clU(I)H_0 + clU(I^c)H_0$ and the projection $H_0 \to clU(I)H_0$ be continuous.*

Then there exists a sequence (g_n) of elements in H_0, such that the family of ideals (I_n), where $I_n = \{x\epsilon A,\ U(x)g_n = 0\}$, has the properties:

(i) $I_n = I_n^{cc}$, $\cap I_n = \cap\{x\epsilon A,\ U(x)h_n = 0\}$;

(ii) $x \in I_m^c$ implies $U(x)g_n = U(x)g_m$ for all $n \geq m$;

(iii) $I_n = \bigcap_{k\leq n} \{x\epsilon A,\ U(x)h_k = 0\}$.

Proof. By induction, using Lemma 4.3.3.

Lemma 4.3.5. Let H_0 be an invariant subspace. Assume, there does not exist an element $h \in H_0$, $h \neq 0$, such that $U(I)h = \{0\}$, for some ideal $I \subset A$, for which $I^{cc} = A$. Then every $h \in H_0$ is regular.

Proof. Let $I = \{x\epsilon A,\ U(x)h = 0\}$, where h is an arbitrary element in H_0. Then, $U(I + I^c)U(I^{cc})h = \{0\}$ and so, since $(I + I^c)^{cc} = A$ and $U(I^{cc})h \subset H_0$, $U(I^{cc})h = \{0\}$. Since I is the largest ideal J for which $U(J)h = \{0\}$, it follows $I \supset I^{cc}$.

Proof of Theorem 4.3.1.

We shall prove, with the additional knowledge, that (v) - (viii) of Theorem 4.2.2 are satisfied too. Condition (vi) is valid by assumption: U restricted to H_ν is faithful for every $\nu \in \Lambda$. Condition (vii) is an application of Lemma 4.3.2(b) and Lemma 4.2.3(i). So (v) and (viii) remain to be checked. Let H_ν be one of the minimal invariant subspaces.

Then $\{x\epsilon A,\ U(x)H_\nu = \{0\}\} = \cap\{x\epsilon A,\ U(x)h = 0\} = \{0\}$, where the intersection is taken over all elements h in H_ν. By assumption, each of the ideals $\{x\epsilon A,\ U(x)h = 0\}$ is regular. Hence, since A satisfies the c.c.c., we may assume that there exists a countable family $(h_n) \subset H_\nu$, such that $\{0\} = \cap\{x\epsilon A,\ U(x)h_n = 0\}$. By Lemma 4.3.4, we may assume that this family has the properties ($I_n = \{x\epsilon A,\ U(x)h_n = 0\}$): (a) $I_m \subset I_n$, for $m \geq n$, and (b) $U(x)h_n = U(x)h_m$ for $x \in I_n^c$ and $m \geq n$.

Define $S_\nu = \bigcup I_n^c$ and U_ν: $S_\nu \to H_\nu$ by $U_\nu(x) = U(x)h_n$, for $x \in I_n^c$. In order to complete the proof, it is sufficient to show that $U_\nu(S_\nu \cap (\bigcup J_n))$ is dense in H_ν, for every increasing countable family of regular ideals (J_n) for which $(\bigcup J_n)^{cc} = A$. It suffices to prove that the representation U, restricted to $U_\nu(S_\nu \cap (\bigcup J_n))$, is faithful. Since the sequence (I_n^c) is increasing and the same holds for the family (J_n), it easily follows that $S_\nu \cap (\bigcup J_n) = \bigcup_n (I_n^c \cap J_n)$. So, if $U(x)U_\nu(S_\nu \cap (\bigcup J_n)) = \{0\}$, then for every n and every $a \in I_n^c \cap J_n$, $U(x)U_\nu(a) = 0$. By the definition of U_ν, it follows that $U(x)U(a)h_n = 0$ for all $a \in I_n^c \cap J_n$. Recalling the definition of I_n, we get $x(I_n^c \cap J_n) \subset I_n$ for all n, and hence $x(I_n^c \cap J_n) = \{0\}$. From which $x(\bigcup(I_n^c \cap J_n)) = x((\bigcup I_n^c) \cap (\bigcup J_m)) = \{0\}$. Thus $x \in ((\bigcup I_n^c) \cap (\bigcup J_m))^c$. We prove that $((\bigcup I_n^c) \cap (\bigcup J_m))^c = \{0\}$. From Lemma 2.1.5, it follows that $((\bigcup I_n^c) \cap (\bigcup J_m))^c = ((\bigcup I_n^c) \cap (\bigcup J_m))^{ccc} = ((\bigcup I_n^c)^{cc} \cap (\bigcup J_m)^{cc})^c = ((\bigcap I_n^{cc})^c \cap (\bigcup J_m)^{cc})^c$.
Since $I_n^{cc} = I_n$ for all n, $\bigcap I_n = \{0\}$ and $(\bigcup J_m)^{cc} = A$, it follows $((\bigcup I_n^c) \cap (\bigcup J_m))^c = (\{0\}^c \cap A)^c = A^c = \{0\}$.

Corollary 4.3.6. *Let in Theorem 4.3.1, H_ν be complete metrizable. Then there exists an element $f_\nu \in H_\nu$ such that $H_\nu = \mathrm{cl}U(A)f_\nu$. If the topology of F makes H_ν into a Banach space, then there exists an element ϕ_ν in $(\sum_{\mu \neq \nu} H_\mu)^\perp$ such that $U(A)'\phi_\nu$ is w*-dense in $(\sum_{\mu \neq \nu} H_\mu)^\perp$, provided that there does not exist an element $h_0 \in H_\nu$, $h_0 \neq 0$, for which $U(I)h_0 = \{0\}$ for some ideal I in A, $I^{cc} = A$.*

Proof. In order to show the first assertion, it is sufficient to construct, for a countable family $(h_n) \subset H$ with the properties ($I_n = \{x \in A,\ U(x)h_n = 0\}$): (a) $I_n^{cc} = I_n$ for all n, (b) $I_m \subset I_n$ for $m \geq n$, (c) $U(x)h_m = U(x)h_n$ for $m \geq n$, $x \in I_n^c$, an element $h \in H_\nu$, such that $\{x \in A,\ U(x)h = 0\} = \bigcap I_n$.

Let (p_k) be an increasing countable family of semi-norms, which defines the topology of H_ν.

Define h by $h = \Sigma_{n=1}^{\infty} \frac{1}{2^n(1 + p_n(h_n))} h_n$.

Then, since for $n, m > s$, $n, m > 1 - \ln \varepsilon/\ln 2$,

$$p_s(\Sigma_{k=n}^{m} \frac{1}{2^k(1 + p_k(h_k))} h_k) < \varepsilon,$$

h belongs to H_ν.

By induction one may show that $\{x \varepsilon A,\ U(x)h = 0\} = \cap I_n$.

The second assertion is a consequence of Lemma 4.2.4, Lemma 4.3.5, the way Theorem 4.3.1 is proved and the following proposition, which has some interest in its own.

Proposition 4.3.7. Let $U: A \to L(F)$ be a representation of the semi-prime algebra A. Let H_0 and H be closed invariant subspaces for which $H_0 \cap H = \{0\}$, $H_0 + H = F$ and for which the mapping

$$h_0 + h \to h_0,\ h_0 \varepsilon H_0,\ h \varepsilon H,$$

is continuous.

Then the following assertions are equivalent:

(a) The space H_0 is a minimal closed invariant subspace for which U is faithful; moreover, if $U(I)h_0 = \{0\}$, $h_0 \varepsilon H_0$, for some ideal $I \subset A$, for which $I^{cc} = A$, then $h_0 = 0$.

(b) The space $H^{\perp}$ is minimal, in the sense that there does not exist a w*-closed invariant subspace $H_0' \subset H^{\perp}$, for which $U(x)'H_0' = \{0\}$, $x \varepsilon A$, implies $x = 0$; moreover, for every ideal $I \subset A$, for which $I^{cc} = A$, $U(I)H_0$ is dense in H_0.

Proof. (a) $\Rightarrow$ (b). Let I be an arbitrary ideal in A for which $I^{cc} = A$. Then $U(x)U(I)H_0 = \{0\}$, $x \in A$, implies $x = 0$. By the minimality of H_0, it follows that $U(I)H_0$ is dense in H_0. Next, let $H_0' \subset H^{\perp}$ be a w*-closed invariant subspace, for which $U(x)'H_0' = \{0\}$, $x \in A$, implies $x = 0$. We shall show that $H_0' = H^{\perp}$. Let H_1 be the closed invariant subspace defined by $H_1 = \{h \in H_0, \langle h, H_0'\rangle = \{0\}\}$ and let I denote the ideal $I = \{x \in A, U(x)H_1 = \{0\}\}$. Then, since H_0' is w*-closed, $H_0' = (H + H_1)^{\perp} = H^{\perp} \cap H_1^{\perp}$. Consider the subspace $G = U(I)H_0 + U(I^c)H_1$. We first prove that G is dense in H_0. By assumption (a), it suffices to prove that $U(x)G = \{0\}$, $x \in A$, implies $x = 0$. Let $x \in A$, for which $U(x)G = \{0\}$. Then both $U(xI)H_0 = \{0\}$ and $U(xI^c)H_1 = \{0\}$. It easily follows that $xI = \{0\}$ and $xI^c \subset I$, from which $x = 0$. Since H_0 and H are topological complementary subspaces in F, we infer that the space

$$\{\phi \in H^{\perp}, \langle U(I)H_0 + U(I^c)H_1, \phi\rangle = \{0\}\}$$

reduces to $\{0\}$.

Hence,

$$\{\phi \in H^{\perp}, U(I)'\phi = \{0\} \text{ \underline{and} } U(I^c)'\phi \subset H_1^{\perp}\} = \{0\},$$

from which

$$\{\phi \in H^{\perp}, U(I)'\phi = \{0\} \text{ \underline{and} } U(I^c)'\phi \subset H_1^{\perp} \cap H^{\perp}\} = \{0\}.$$

Or, equivalently,

$$\{\phi \in H^{\perp}, U(I)'\phi = \{0\} \text{ \underline{and} } U(I^c)'\phi \subset H_0'\} = \{0\}.$$

Since $U(I^c)'H_0'$ is contained in the left-hand side of the equality, we have $U(I^c)'H_0' = \{0\}$. Hence, since $U(x)'H_0' = \{0\}$, $x \in A$, implies $x = 0$, we get $I^c = \{0\}$. By the definition of the ideal I, we have $U(I)H_1 = \{0\}$; this together with the fact that $I^{cc} = A$ implies, by assumption (a), that $H_1 = \{0\}$. Hence, $H_0' = H^{\perp} \cap H_1^{\perp} = H^{\perp}$.

(b) => (a). First, let I be an ideal in A for which $I^{cc} = A$ and let h_0 be an element in H_0 for which $U(I)h_0 = \{0\}$. Since $U(x)'U(I)'H^{\perp} = \{0\}$ implies $x \in I^c = \{0\}$, it follows, by the minimality of $H^{\perp}$, that $U(I)'H^{\perp}$ is w*-dense in $H^{\perp}$. So, since $U(I)h_0 = \{0\}$ implies $\langle h_0, U(I)'H^{\perp}\rangle = \{0\}$, $\langle h_0, H^{\perp}\rangle = \{0\}$. Since H_0 and H are topological complementary subspaces in F, it follows $\langle h_0, F'\rangle = \{0\}$, and so $h_0 = 0$.

Next, let $H_1 \subset H_0$ be a closed invariant subspace for which $U(x)H_1 = \{0\}$, $x \in A$, implies $x = 0$. We shall prove that $H_1 = H_0$. Consider the w*-closed subspace $H_0' = \{\phi \in H^{\perp}, \langle H_1, \phi\rangle = \{0\}\} = (H_1 + H)^{\perp}$ together with the ideal $I = \{x \in A, U(x)'H_0' = \{0\}\}$. We first prove that the space

$$G' = U(I)'H^{\perp} + U(I^c)'H_0'$$

is a w*-dense subspace of $H^{\perp}$. By the minimality of $H^{\perp}$, it is sufficient to prove that $U(x)'G' = \{0\}$, $x \in A$, implies $x = 0$. So let $x \in A$ be such that $U(x)'G' = \{0\}$. Since $U(xI)'H^{\perp} = \{0\}$, we get $xI = \{0\}$. Since $U(xI^c)'H_0' = \{0\}$, it follows, by the definition of I, that $xI^c \subset I$. From these remarks we easily infer that $x = 0$. Since G' is a w*-dense subspace of $H^{\perp}$, it follows that the space $\{h \in H_0, \langle h, G'\rangle = \{0\}\}$ reduces to $\{0\}$. Equivalently,

$$\{h \in H_0,\ \langle U(I)h, H^{\perp}\rangle = \{0\}\ \underline{\text{and}}\ \langle U(I^c)h, H_0'\rangle = \{0\}\} = \{0\}.$$

Since $H_1 + H$ is a closed subspace of F and $H_0' = (H_1 + H)^{\perp}$, it follows

$$\{h \in H_0,\ U(I)h = \{0\}\ \underline{\text{and}}\ U(I^c)h \subset H_1 + H\} = \{0\}.$$

Thus,

$$\{h \in H_0,\ U(I)h = \{0\}\ \underline{\text{and}}\ U(I^c)h \subset H_1\} = \{0\}.$$

Since the space $U(I^c)H_1$ is contained in the left-hand side of the equality, we have $U(I^c)H_1 = \{0\}$. Since $U(x)H_1 = \{0\}$, $x \in A$, implies $x = 0$, it follows $I^c = \{0\}$ and so $I^{cc} = A$.

By the definition of I we have $\langle U(I)H_0, H_0'\rangle = \{0\}$. Since $I^{cc} = A$, it follows, by assumption, that $U(I)H_0$ is dense in H_0. Hence, $\langle H_0, H_0'\rangle = \{0\}$. Thus, since $H_0' \subset H^{\perp}$ and $F = H_0 + H$, $H_0' = \{0\}$. So, $H_1 + H$ is a dense subspace of F; whence $H_1 = H_0$.

Remark. Proposition 4.3.7 shows the symmetry between the representation U and the "dual" representation, defined by $x \to U(x)'\phi$, $x \in A$, $\phi \in F'$.

The results in the third section show that the theory is nice, if we assume that, roughly speaking, F can be decomposed into a direct sum of closed invariant subspaces, which are minimal in the sense of Theorem 4.3.1. In this case we necessarily have that U(I)F is dense in $U(I^{cc})F$ for every ideal $I \subset A$. It seems to be worthwhile to develop a theory of spectral operators along the lines of this chapter. In particular, it might be useful to assume that the representation U has the above property, viz. U(I)F is dense in $U(I^{cc})F$ for every ideal $I \subset A$. One might call an operator S A-spectral, if it commutes with U(a) for every element a in A; one might say that it is A-scalar, if $S = U(a)$ for some element a in A. We mention the following two open problems.

Problem 1. Let $U: A \to L(F)$ be a faithful representation of the semiprime algebra A. Assume that for every ideal I in A, U(I)F is dense in $U(I^{cc})F$. Do there exist minimal closed invariant subspaces for which U is faithful?

Problem 2. Assume that F can be decomposed as a direct sum of Banach spaces, which are minimal in the sense of Theorem 4.3.1. Assume that A is a vector lattice. Is it possible to choose the "cyclic" vectors f_ν and ϕ_ν of Corollary 4.3.6 in such a way, that the expression $\langle U(a)f_\nu, \phi_\nu\rangle$ is positive for every element a in the positive cone of A?

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