OMEGA FUNCTION: A THEORETICAL INTRODUCTION

VU NGOC NGUYEN

Abstract. This paper investigates the theory behind a new universal performance measure (the so called Omega function), which was first introduced by Con Keating and William F. Shadwick in 2002 (see [1]). In the first section, we review some rudimentary probability. We then define the Omega function, introduce some of its properties, and prove these properties without continuity assumptions. We also define the standard dispersion, a new statistic derived from the Omega function. We prove one new theorem about the range of the standard dispersion for a finite sample. The structure of the second section on the Omega function follows closely that of a recent talk given by Ana Cascon and William Shadwick in [4]. In the last section, we demonstrate these properties with real-life data.

Acknowledgments

First and foremost, I would like to give special thanks to Professor Thomas Ramsey for introducing me to the application aspect of mathematics. Without his guidance and support, this thesis would not have been possible. I would also like to thank Professor George Wilkens for all the coaching sessions along with numerous suggestions. I would like to thank my dad for his constant support during the last several years. I would like to thank my fellow graduate students Chee Chen, Quinn Culver, Ni Lu, and Shashi Singh for all the help, and more importantly, for making graduate studies much more enjoyable. Finally, I would like to thank the students, staff and faculty of the Department of Mathematics for their support and encouragement.

1. Preliminaries

In this section, we will recall some fundamental concepts of Probability Theory. For a more proper and complete measure-theoretic introduction, the reader is referred to the book titled *Probability with Martingales* by David Williams (see [11]).

Definition 1.1. A *probability space* is a non-negative measure space such that the measure of the whole space is equal to 1.

In other words, a probability space is a triple $(\Omega, \mathcal{F}, \mathbb{P})$ where

- Ω, called the *sample space*, is a non-empty set. A point ω of Ω is called a *sample point*.
- $\mathcal{F}$, called the *family of events*, is a σ-algebra on Ω. An *event* is an element of $\mathcal{F}$; that is, an $\mathcal{F}$-measurable subset of Ω,
- $\mathbb{P}$ is a probability measure on $(\Omega, \mathcal{F})$.

Example 1.2. Toss a coin twice.
Our sample space $\Omega = \{HH, HT, TH, TT\}$ consists of all the possible outcomes when tossing a coin twice. Set $\mathcal{F} := \mathcal{P}(\Omega) =$ the powerset of Ω. In this example, the event "At least one head is obtained" is the element $\{HH, HT, TH\} \in \mathcal{F}$.

Example 1.3. Choose a point between 0 and 1 uniformly at random. We take $\Omega = [0,1], \mathcal{F} = \mathcal{B}[0,1]$, and ω signifying the point chosen. In this case, we can take $\mathbb{P} =$ Leb. Here, $\mathcal{B}[0,1]$ is the collection of the Borel subsets of $[0,1]$ and Leb is the usual Lebesgue measure on $\mathbb{R}$.

Definition 1.4. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A *random variable* is an element of $\mathrm{m}\mathcal{F}$, the set of functions $X : \Omega \to \mathbb{R}$ such that $X^{-1}(B) \in \mathcal{F}$ for all Borel subsets[1] $B \subset \mathbb{R}$. In symbols, let $\mathcal{B}$ be the σ-algebra of Borel subsets of $\mathbb{R}$ then, for $X \in \mathrm{m}\mathcal{F}$,

$$X : \Omega \to \mathbb{R}, \quad X^{-1} : \mathcal{B} \to \mathcal{F}.$$

Definition 1.5. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. The *cumulative distribution function* (CDF), or just *distribution function*, of a real-valued random variable X is given by $F_X(x) = \mathbb{P}(X \leq x), \forall x \in \mathbb{R}$.

The reader can find a more detailed description of a cumulative distribution function in [11].

Definition 1.6. A random variable X is called *constant* if $X(\omega) = c$ for all $\omega \in \Omega$, where $c \in \mathbb{R}$.

Example 1.7. The cumulative distribution function for the constant random variable $X(\omega) = c$ is

$$F_X(x) = \begin{cases} 0, & \text{if } x < c, \\ 1, & \text{if } x \geq c. \end{cases}$$

We call the CDF of a constant random variable a *point-mass CDF*.

[1]The Borel σ-algebra of $\mathbb{R}$, $\mathcal{B} = \sigma(\pi(\mathbb{R}))$, is the σ-algebra generated by the π-system on $\mathbb{R}$, where $\pi(\mathbb{R}) := \{(-\infty, x] : x \in \mathbb{R}\}$. See [11].

Definition 1.8. A random variable X is *discrete* if and only if there is a finite sequence $\{x_i\}_{i=1}^{N}$ or an infinite sequence $\{x_i\}_{i=1}^{\infty}$ of distinct real numbers such that

$$\sum_{i=1}^{N} \mathbb{P}(X = x_i) = 1 \quad \text{or} \quad \sum_{i=1}^{\infty} \mathbb{P}(X = x_i) = 1,$$

respectively. When X is discrete, the *probability mass function* is the function $\rho_X : \mathbb{R} \to [0,1]$ such that

$$\rho_X(x) = \mathbb{P}(X = x),\ \forall x \in \mathbb{R}.$$

Example 1.9. The possible events for a coin toss are heads or tails, thus $\Omega = \{H, T\}$. The number of heads appearing in one toss can be described using the following random variable:

$$X(\omega) = \begin{cases} 0, & \text{if } \omega = T, \\ 1, & \text{if } \omega = H. \end{cases}$$

This is an example of a discrete random variable.
If the coin is fair, then the probability mass function $\rho_X : \mathbb{R} \to [0,1]$ for X is defined as

$$\rho_X(x) = \mathbb{P}(X = x) = \begin{cases} \frac{1}{2}, & \text{if } x \in \{0,1\}, \\ 0, & \text{if } x \notin \{0,1\}. \end{cases}$$

The CDF F_X for this random variable is

$$F_X(x) = \begin{cases} 0, & \text{if } x < 0, \\ \frac{1}{2}, & \text{if } 0 \le x < 1, \\ 1, & \text{if } x \ge 1. \end{cases}$$

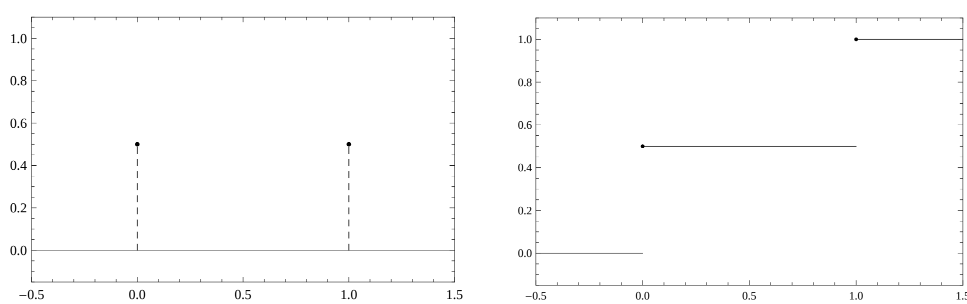


FIGURE 1. The probability mass function ρ_X and cumulative distribution function F_X

Definition 1.10. A random variable X is continuous if F_X is a continuous function. A random variable X is absolutely continuous if F_X is an absolutely continuous function.

If X is an absolutely continuous random variable, then X has *probability density function* f_X where

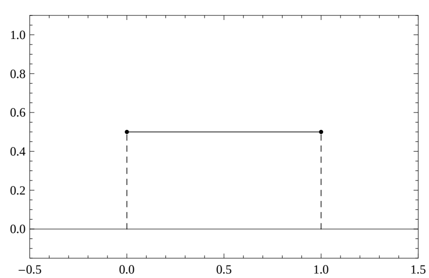
$$\mathbb{P}(a < X \leq b) = \int_a^b f_X(t)dt = F_X(b) - F_X(a).$$

Example 1.11. For *absolutely continuous random variables*, note that $\mathbb{P}(X = x) = 0$ for all $x \in \mathbb{R}$. A random variable X with the *uniform distribution* $U(a, b)$ is one such example of an absolutely continuous random variable. The probability density function f_X of X is given as

$$f_X(x) = \begin{cases} \frac{1}{b-a}, & \text{for } x \in [a, b], \\ 0, & \text{for } x \notin [a, b]. \end{cases}$$

And the CDF is

$$F_X(x) = \begin{cases} 0, & \text{for } x < a, \\ \frac{x-a}{b-a}, & \text{for } a \leq x < b, \\ 1, & \text{for } x \geq b. \end{cases}$$



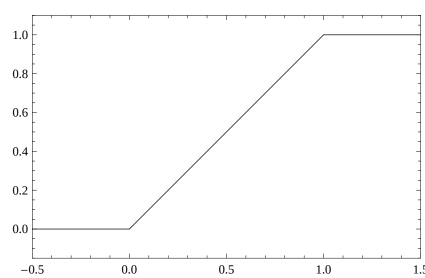


FIGURE 2. The probability density function f_X and cumulative distribution function F_X for the standard uniform distribution $U(0, 1)$

Proposition 1.12. *Suppose that F is the cumulative distribution function of some random variable X. Then*

- **CDF1:** $F : \mathbb{R} \to [0, 1]$
- **CDF2:** $F(x) \leq F(y), \forall x \leq y$ *(F is non-decreasing)*
- **CDF3:** $\lim_{x\to-\infty} F(x) = 0, \quad \lim_{x\to+\infty} F(x) = 1$
- **CDF4:** *F is right-continuous*

Remark 1.13. In fact, F is the CDF of a random variable if and only if F satisfies CDF1 through CDF4 (see Theorem 3.11 in [11]).

Proof.

CDF1: This is clear from the definition, as the image of F_X is contained in the image of $\mathbb{P}$.

CDF2: Suppose that $x \leq y$,

$$\begin{aligned} F(y) - F(x) &= \mathbb{P}(X \leq y) - \mathbb{P}(X \leq x) \\ &= \mathbb{P}(X \leq x) + \mathbb{P}(x < X \leq y) - \mathbb{P}(X \leq x) \\ &= \mathbb{P}(x < X \leq y) \\ &\geq 0. \end{aligned}$$

CDF3: By CDF2, $\lim_{x\to-\infty} F(x) = \lim_{n\to-\infty} F(n)$ where $n \in \mathbb{Z}$. By Lemma 1.10b of [11],

$$\lim_{n\to-\infty} F(n) = \lim_{n\to-\infty} \mathbb{P}(X \leq n) = \mathbb{P}(\emptyset) = 0.$$

Similarly, by Lemma 1.10a of [11],

$$\lim_{x\to\infty} F(x) = \lim_{n\to\infty} F(n) = \lim_{n\to\infty} \mathbb{P}(X \leq n) = \mathbb{P}(\Omega) = 1.$$

CDF4: Another application of Lemma 1.10 of [11],

$$\lim_{a\to 0^+} F_X(x+a) = \lim_{n\to\infty} \mathbb{P}(X \leq x + \frac{1}{n}) = \mathbb{P}(X \leq x) = F_X(x).$$

□

Definition 1.14. The *non-trivial domain* of a cumulative distribution function F is the interior of $\{x \mid 0 < F(x) < 1\}$.

This definition is not in any of the papers on the Ω function listed in the reference page. Its motivation will be discussed in greater detail in the next section (see Remark 2.5).

Lemma 1.15. *The non-trivial domain of a cumulative distribution function is either empty or is equal to (A, B) where $-\infty \leq A < B \leq \infty$.*

Proof. Let D denote the non-trivial domain. Suppose that $D \neq \emptyset$. Since D is open and non-empty, it contains more than one point. Suppose that $x_1, x_2 \in D$ and $x_1 < x_2$. Let $y \in (x_1, x_2)$. Since F is non-decreasing, we have

$$0 < F(x_1) \leq F(y) \leq F(x_2) < 1.$$

So $y \in \{x \mid 0 < F(x) < 1\}$, hence $(x_1, x_2) \subseteq \{x \mid 0 < F(x) < 1\}$. Since the interior of a set is the union of all its open subsets, $(x_1, x_2) \subseteq D$. Because x_1 and x_2 are chosen arbitrarily, D is an interval. Since it is an open set, it is an open interval. □

For examples 1.9 and 1.11, the non-trivial domains are $(0, 1)$ and (a, b) respectively.

Lemma 1.16. *The non-trivial domain is empty if and only if F is a point-mass CDF.*

Proof. ($\Leftarrow$) Obvious. Conversely, suppose that the non-trivial domain is empty and F is not a point-mass CDF. Then there exists $c \in \mathbb{R}$ such that $0 < F(c) < 1$. Since F is right continuous, for any $\varepsilon > 0$, there exists $\delta > 0$ such that $0 \leq F(x) - F(c) < \varepsilon$, for all $x \in [c, c+\delta)$. Set $\varepsilon = \text{Min}(F(c), 1 - F(c))$, we have a contradiction. $\square$

We say that a random variable X has finite mean if and only if $\sum_i |x_i|\ \rho(x_i) < \infty$ (for discrete X) or $\int_{-\infty}^{\infty} |x|\ f(x)dx < \infty$ (for absolutely continuous X) (see pp. 68-69 of [11]). If X has finite mean, then the *expected value* (or the *mean*) of X is defined as $E(X) = \sum_i x_i\, \rho(x_i)$ or $E(X) = \int_{-\infty}^{\infty} x\, f(x)dx$.

Lemma 1.17. *Let F be a CDF with a finite mean. Then for any $a \in \mathbb{R}$,*

$$\int_{-\infty}^{a} F(t)dt = \int_{-\infty}^{a} (a-x)dF(x) < \infty$$

and

$$\int_{a}^{\infty} [1 - F(t)]dt = \int_{a}^{\infty} (x-a)dF(x) < \infty.$$

Proof. Let ρ be the probability measure on $\mathbb{R}$ that is induced by F. Let $\tau = m \times \rho$ on $\mathbb{R}^2$ where m is Lebesgue measure. Let $P = \{(t, x) \mid x \leq t \leq a\}$. By Tonelli's Theorem,

$$\begin{aligned} \tau(P) &= \int_{(-\infty,a]} \left(\int_{[x,a]} \mathbf{1}\, dm(t) \right) d\rho(x) \\ &= \int_{(-\infty,a]} m([x,a])d\rho(x) \\ &= \int_{(-\infty,a]} (a-x)dF(x) \\ &< \infty. \end{aligned}$$

Because the last integrand above, $(a-x)dF(x)$, is zero for $x = a$, the last integral can be written as $\int_{-\infty}^{a}(a-x)dF(x)$ without ambiguity. Another application by Tonelli's Theorem gives

$$\begin{aligned} \tau(P) &= \int_{(-\infty,a]} \left(\int_{(-\infty,t]} \mathbf{1}\, d\rho(x) \right) dm(t) \\ &= \int_{(-\infty,a]} \rho((-\infty,t])dm(t) \\ &= \int_{-\infty}^{a} F(t)dt. \end{aligned}$$

Let $Q = \{(t, x) \mid a \leq t < x\}$. Then

$$\tau(Q) = \int_{(a,\infty)} m([a, x))d\rho(x) = \int_{(a,\infty)} (x - a)dF(x) < \infty.$$

Again, because the integrand $(x - a)dF(x)$ is zero for $x = a$, we can write the last integral as $\int_a^\infty (x-a)dF(x)$ without ambiguity. We apply Tonelli's Theorem again to obtain

$$\tau(Q) = \int_a^\infty \rho((t, \infty))dm(t) = \int_a^\infty (1 - F(t))dm(t).$$

$\square$

2. The Omega Function

We are now ready for the main definition of this paper.

Definition 2.1. If F is a cumulative distribution with non-trivial domain (A, B) and if F has finite mean μ then the *Omega function*, Ω, of F is defined as

$$\Omega(x) = \frac{I_1(x)}{I_2(x)}$$

for $x \in (A, B)$, where

$$I_1(x) = \int_A^x F(t)dt = \int_{-\infty}^x (x-t)dF(t)$$

and

$$I_2(x) = \int_x^B [1-F(t)]dt = \int_x^\infty (t-x)dF(t)$$

for all $x \in \mathbb{R}$.

By Lemma 1.17, $I_1(x)$ and $I_2(x)$ are finite for all $x \in \mathbb{R}$. For $x \in (A, B)$, the non-trivial domain, $I_1(x) > 0$ and $I_2(x) > 0$.

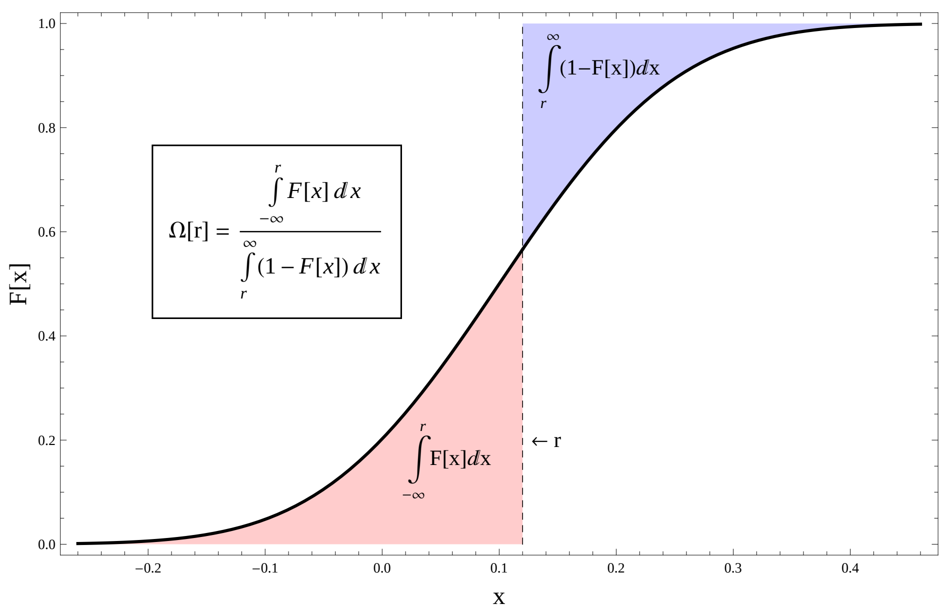


Figure 3. The Omega Ratio

Note 2.2. Many papers in the literature, such as [1, 2, 3, 4, 6, 7, 8, 9, 10], define $\Omega = \frac{I_2}{I_1}$. The current definition is in agreement with [4], which inspired this paper.

The following theorem will establish some properties of the Omega function.

Theorem 2.3. *Suppose that F is a CDF with finite mean μ and non-trivial domain $(A, B) \neq \emptyset$.*

OF1: *I_1, I_2 are Lipschitz continuous on $\mathbb{R}$.*
OF2: *Ω is continuous on (A, B).*
OF3: *Ω is strictly increasing on (A, B).*
OF4: $\lim_{x \to A^+} \Omega(x) = 0$ *and* $\lim_{x \to B^-} \Omega(x) = \infty$.
OF5: $\mathrm{Image}(\Omega) = (0, \infty)$.
OF6: $I_2(x) - I_1(x) = \mu - x$ *for* $x \in \mathbb{R}$.
OF7: $\Omega(x) = 1 + \frac{x-\mu}{I_2(x)}$ *for* $x \in (A, B)$, *and* $\Omega(\mu) = 1$.
OF8: $\Omega'(\mu^+) = \frac{1}{I_2(\mu)}$, *where $\Omega'(\mu^+)$ denotes the right-hand derivative of Ω at μ.*

Proof.

OF1: Since $0 < F(x) < 1, \ \forall x \in (A, B)$,

$$|I_1(x_2) - I_1(x_1)| = \left| \int_{x_1}^{x_2} F(x) dx \right| \leq \left| \int_{x_1}^{x_2} dx \right| = |x_2 - x_1|,$$

and, since $0 < 1 - F(x) < 1, \ \forall x \in (A, B)$,

$$|I_2(x_2) - I_2(x_1)| = \left| \int_{x_1}^{x_2} (1 - F(x)) dx \right| \leq \left| \int_{x_1}^{x_2} dx \right| = |x_2 - x_1|.$$

OF2: Obviously, since I_1, I_2 are continuous and $I_2 > 0$ on (A, B), so is their quotient $\frac{I_1}{I_2}$.

OF3: From their definitions, it is easy to see that I_1 is strictly increasing on (A, B), and I_2 is strictly decreasing on (A, B). So, for $A < x_1 < x_2 < B$,

$$I_1(x_1) < I_1(x_2) \text{ and } I_2(x_1) > I_2(x_2).$$

Thus,

$$\Omega(x_1) = \frac{I_1(x_1)}{I_2(x_1)} < \frac{I_1(x_2)}{I_2(x_2)} = \Omega(x_2).$$

OF4: Let $c \in (A, B)$. Since $I_2(x)$ is a decreasing function, for all $x \in (A, c)$, $0 < \Omega(x) = \frac{I_1(x)}{I_2(x)} < \frac{I_1(x)}{I_2(c)}$. We only need to show that $\lim_{x \to A^+} I_1(x) = 0$. However, because $I_1(A) = \int_A^A F(t) dt = 0$ and $I_1(x)$ is continuous on $\mathbb{R}$ (by OF1), it must be the case that $\lim_{x \to A^+} I_1(x) = 0$. Thus, $\lim_{x \to A^+} \Omega(x) = 0$.

Since $I_1(x)$ is an increasing function, for all $x \in (c, B)$, $\Omega(x) = \frac{I_1(x)}{I_2(x)} > \frac{I_1(c)}{I_2(x)}$. We want to show that $\lim_{x \to B^-} I_2(x) = 0$. This is also true, because $I_2(B) = 0$ and $I_2(x)$ is continuous on $\mathbb{R}$. Hence, $\lim_{x\to B^-} \Omega(x) = \infty$.

OF5: By OF2, OF3 and OF4.

OF6: By Lemma 1.17,

$$\begin{aligned} I_2(x) - I_1(x) &= \int_x^\infty (t-x)dF(t) - \int_{-\infty}^x (x-t)dF(t) \\ &= \int_{(-\infty,x]} (t-x)\,dF(t) - \int_{(x,\infty)} (x-t)\,dF(t) \\ &= \int_{-\infty}^\infty t dF(t) - x\int_{-\infty}^\infty dF(t) \\ &= \mu - x \end{aligned}$$

OF7: We first show that $\mu \in (A, B)$. Suppose that $-\infty < \mu < A < B \leq \infty$. Then $I_1(\mu) = 0$ and, by OF6, $I_2(\mu) = \int_\mu^B 1 - F(t)\,dt = 0$. But this is a contradiction since $1 - F(x) > 0$ for all $x \in (A, B)$. A similar contradiction arises if $-\infty \leq A < B < \mu < \infty$. Thus $\mu \in (A, B)$.

From the definition of $\Omega(x)$ and by OF6,

$$\begin{aligned} \Omega(x) = \frac{I_1(x)}{I_2(x)} &= \frac{I_2(x)}{I_2(x)} + \frac{I_1(x) - I_2(x)}{I_2(x)} \\ &= 1 + \frac{x-\mu}{I_2(x)}. \end{aligned}$$

Consequently, $\Omega(\mu) = 1 + \frac{\mu-\mu}{I_2(\mu)} = 1$.

OF8: We first show that the right derivatives of $I_1(x)$ and $I_2(x)$ are equal to $F(x)$ and $F(x) - 1$, respectively.

$$\begin{aligned} &\lim_{h\to 0^+} \frac{I_1(x+h) - I_1(x)}{h} \\ = &\lim_{h\to 0^+} \frac{\int_A^{x+h} F(t)dt - \int_A^x F(t)dt}{h} \\ = &\lim_{h\to 0^+} \frac{\int_x^{x+h} F(t)dt}{h} \\ = &\lim_{h\to 0^+} \frac{\int_x^{x+h}[F(t) - F(x) + F(x)]dt}{h} \\ = &\lim_{h\to 0^+} \frac{\int_x^{x+h}[F(t) - F(x)]dt}{h} + F(x) \cdot \lim_{h\to 0^+} \frac{\int_x^{x+h} dt}{h} \\ = &\lim_{h\to 0^+} \frac{\int_x^{x+h}[F(t) - F(x)]dt}{h} + F(x). \end{aligned}$$

Since F is right continuous, for any $\varepsilon > 0$, there exists $\delta > 0$ such that $0 \leq F(t) - F(x) < \varepsilon$ whenever $0 \leq t - x < \delta$. Thus, for all $0 < h < \delta$

$$
\begin{aligned}
0 \leq \lim_{h\to 0^+} \frac{\int_x^{x+h}[F(t)-F(x)]dt}{h} &< \lim_{h\to 0^+} \frac{\int_x^{x+h}\varepsilon\, dt}{h} \\
&= \varepsilon \cdot \lim_{h\to 0^+} \frac{\int_x^{x+h} dt}{h} \\
&= \varepsilon.
\end{aligned}
$$

Since ε is arbitrary, $\lim_{h\to 0^+} \frac{\int_x^{x+h}[F(t)-F(x)]dt}{h} = 0$. So, $I_1'(x^+) = F(x)$.
A similar argument gives $I_2'(x^+) = F(x) - 1$. Hence

$$
\begin{aligned}
\Omega'(x^+) &= \frac{I_1'(x^+)I_2(x) - I_1(x)I_2'(x^+)}{I_2^2(x)} \\
&= \frac{F(x)I_2(x) - I_1(x)[F(x)-1]}{I_2^2(x)} \\
&= \frac{[I_2(x) - I_1(x)]F(x) + I_1(x)}{I_2^2(x)}.
\end{aligned}
$$

Thus, by OF7,

$$
\Omega'(\mu^+) = \frac{[I_2(\mu) - I_1(\mu)]F(\mu) + I_1(\mu)}{I_2(\mu)^2} = \frac{1}{I_2(\mu)}.
$$

□

Proposition 2.4. *The Ω function has left and right derivatives at all x in (A, B) and*

$$
\Omega'(x^+) - \Omega'(x^-) = \frac{[F(x) - F(x^-)][I_2(x) - I_1(x)]}{I_2(x)^2}
$$

Here $F(x^-)$ denotes the left limit at x.

Proof. Since F is non-decreasing and right-continuous, F has a left limit on $\mathbb{R}$ (i.e., $F(x^-)$ exists and is finite for all $x \in \mathbb{R}$).
Follow the same arguments in OF8 of the previous theorem; the left derivatives of $I_1(x)$ and $I_2(x)$ are $F(x^-)$ and $F(x^-) - 1$, respectively. And thus the left derivative of $\Omega(x)$ is

$$
\Omega'(x^-) = \frac{[I_2(x) - I_1(x)]F(x^-) + I_1(x)}{I_2(x)^2}.
$$

Consequently,

$$\begin{aligned}
&\Omega'(x^+) - \Omega'(x^-) \\
= \;& \frac{[I_2(x) - I_1(x)]F(x) + I_1(x)}{I_2(x)^2} - \frac{[I_2(x) - I_1(x)]F(x^-) + I_1(x)}{I_2(x)^2} \\
= \;& \frac{[F(x) - F(x^-)][I_2(x) - I_1(x)]}{I_2(x)^2}
\end{aligned}$$

□

Remark 2.5. It is worth mentioning the motivation for definition 1.14 (the non-trivial domain), which is also the domain of our Ω function. Consider a point-mass CDF F where, for some $c \in \mathbb{R}$, $F(x) = 0$ if $x < c$, and $F(x) = 1$ otherwise. The non-trivial domain of F is the empty set. According to definition 2.1, we have

$$I_{1,F}(x) = \begin{cases} 0, & \text{if } x \leq c, \\ x - c, & \text{if } x \geq c, \end{cases}$$

and

$$I_{2,F}(x) = \begin{cases} c - x, & \text{if } x \leq c, \\ 0, & \text{if } x \geq c. \end{cases}$$

Then $\frac{I_{1,F}(x)}{I_{2,F}(x)}$ is maximally defined on $(-\infty, c)$, but is 0 for all $x \in (-\infty, c)$.

In Example 1.9, a maximal choice for a domain for $\frac{I_{1,F}(x)}{I_{2,F}(x)}$ is $(-\infty, 1)$, in which case

$$\frac{I_{1,F}(x)}{I_{2,F}(x)} = \begin{cases} 0, & \text{if } x \leq 0, \\ \frac{x}{1-x}, & \text{if } 0 \leq x < 1. \end{cases}$$

If we were to use $[0, 1)$ as the domain for Ω, the range of Ω will be $[0, \infty)$. If we use $(0, 1)$, the range is $(0, \infty)$.

The last example deals with the case of a continuous CDF. Consider the probability density function $f(x) = \frac{1}{\sqrt{2\pi}} \exp(-\frac{x^2}{2})$ for the normal distribution with mean 0 and standard deviation 1. The non-trivial domain for its CDF is $(-\infty, \infty)$ and the range of the corresponding Ω is $(0, \infty)$.

We want to define Ω so that its domain is the largest among the choices of intervals where $\frac{I_1}{I_2}$ make sense. We also want the range of Ω to be predetermined, preferably $(0, \infty)$. The interior of $\{x \,|\, 0 < F(x) < 1\}$ comes up as the best choice.

Corollary 2.6. *Let F be a CDF with non-trivial domain $(A, B) \neq \emptyset$, finite mean μ and Ω function $\Omega(x)$. Then, for $x \in (A, B) \setminus \{\mu\}$,*

$$F(x) = 1 + \frac{1}{\Omega(x) - 1} + \frac{(\mu - x)\Omega'(x^+)}{(\Omega(x) - 1)^2}.$$

Proof. By OF6 and OF8 of Theorem 2.3,

$$\begin{aligned}
\Omega'(x^+) &= \frac{(I_2(x) - I_1(x))F(x) + I_1(x)}{I_2^2(x)} \\
&= \frac{(\mu - x)F(x) + I_1(x)}{I_2^2(x)} \\
&= \frac{(\mu - x)F(x)}{I_2^2(x)} + \frac{I_1(x)}{I_2^2(x)} \\
\iff \quad (\mu - x)F(x) &= I_2^2(x)\Omega'(x^+) - I_1(x) \\
\iff \quad (\mu - x)F(x) &= I_2^2(x)\Omega'(x^+) + (I_2(x) - I_1(x)) - I_2(x) \\
\iff \quad (\mu - x)F(x) &= I_2^2(x)\Omega'(x^+) + (\mu - x) - I_2(x).
\end{aligned}$$

So, for $x \neq \mu$,

$$F(x) = 1 - \frac{I_2(x)}{\mu - x} + \frac{I_2^2(x)\Omega'(x^+)}{\mu - x}$$

Since Ω is strictly increasing and $\Omega(\mu) = 1$, by OF3 and OF8 of Theorem 2.3 , we have $x \neq \mu \Rightarrow \Omega(x) \neq 1$. Therefore, $\frac{I_2(x)}{\mu - x} = \frac{1}{1 - \Omega(x)}$ by OF7 of the same theorem. Consequently,

$$\begin{aligned}
F(x) &= 1 - \frac{I_2(x)}{\mu - x} + \frac{(\mu - x)I_2^2(x)\Omega'(x^+)}{(\mu - x)^2} \\
&= 1 + \frac{1}{\Omega(x) - 1} + \frac{(\mu - x)\Omega'(x^+)}{(\Omega(x) - 1)^2}.
\end{aligned}$$

□

Corollary 2.7. *Let F, G be CDFs with finite means and non-empty non-trivial domains and let Ω_F, Ω_G be their Omega functions, respectively. Then, $F = G$ if and only if $\Omega_F = \Omega_G$.*

Proof. ($\Rightarrow$) is obvious. Conversely, suppose that $\Omega_F = \Omega_G$. Because $\Omega_F = \Omega_G$, their domains are equal. And since their domains are the non-trivial domains of F and G, respectively, the latter two are equal. Let (A, B) be the common non-trivial domain for F and G. Note that $\mu_F = \Omega_F^{-1}(1) = \Omega_G^{-1}(1) = \mu_G$. So let μ denote the common mean. By Corollary 2.6, $F(x) = G(x)$ for all $x \in (A, B) \setminus \{\mu\}$. But since F and G are right-continuous, $F(\mu) = G(\mu)$. So $F(x) = G(x)$ for all $x \in (A, B)$.

For $x < A$, $F(x) = G(x) = 0$ by the definition of non-trivial domain. Likewise, for $x > B$ we have $F(x) = G(x) = 1$. Because $A < B$, if A is finite the right continuity of F and G gives us $F(A) = G(A)$. Likewise, $F(B) = G(B)$ if B is finite. Thus $F(x) = G(x)$ for all real x. □

Definition 2.8. Given a cumulative distribution function F with finite mean and that is not a point-mass CDF (cf. Example 1.7) with associated Ω_F function, the *standard dispersion*, denoted by ω_F, is

$$\omega_F = \frac{1}{\Omega'_F(\mu^+)}.$$

By OF8 and OF9 of Theorem 2.3, we have $\omega_F = I_{2,F}(\mu) = I_{1,F}(\mu)$. For point-mass CDFs, we set $\omega_F = 0$.

Proposition 2.9. *Let F be a cumulative distribution function with finite mean μ and non-empty non-trivial domain. The standard dispersion ω can be defined directly from F as*

$$\omega_F = \frac{1}{2} E_F(|\mu - x|).$$

Here E_F is the expectation with respect to the F-induced probability measure ρ on the Borel σ-algebra $\mathcal{B}$ of $\mathbb{R}$.

Proof. By Lemma 1.17 and OF9 of Theorem 2.3, we have

$$\omega_F = I_2(\mu) = \int_\mu^\infty [1 - F(x)]\, dx = \int_\mu^\infty (t - \mu) dF(t)$$

and

$$\omega_F = I_1(\mu) = \int_{-\infty}^\mu F(x)\, dx = \int_{-\infty}^\mu (\mu - t)\, dF(t).$$

Because $\mu - t = 0$ when $t = \mu$,

$$\begin{aligned}
\omega_F &= \frac{1}{2}\left(\int_{-\infty}^{\mu} (\mu - t) dF(t) + \int_\mu^\infty (t - \mu) dF(t)\right) \\
&= \frac{1}{2}\left(\int_{(-\infty,\mu]} (\mu - t) dF(t) + \int_{(\mu,\infty)} -(\mu - t) dF(t)\right) \\
&= \frac{1}{2}\left(\int_{(-\infty,\mu]} |\mu - t|\, dF(t) + \int_{(\mu,\infty)} |\mu - t|\, dF(t)\right) \\
&= \frac{1}{2}\left(\int_{-\infty}^{\infty} |\mu - t|\, dF(t)\right) \\
&= \frac{1}{2} E_F(|\mu - t|).
\end{aligned}$$

□

Proposition 2.9 shows that the standard dispersion ω measures the variability of a set about its mean. The higher the concentration around the mean, the lower is ω (and the higher the value of $\Omega'(\mu^+)$). Figure 4 illustrates this fact with two normally distributed sets B(lue) and P(urple) where $\mu_B = \mu_P = 0, \sigma_B = 1$ and $\sigma_P = 0.5$.

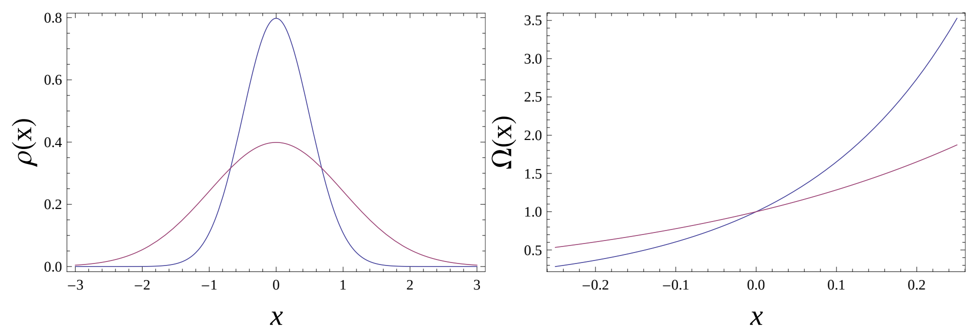


FIGURE 4. Normal Distributions B and P and their associated Ω functions

Proposition 2.10. *If two CDFs F and G have the same finite mean μ, then*

$$\int_{-\infty}^{\infty} (F(x) - G(x))dx = 0.$$

Furthermore, if they have the same ω, then

$$\int_{-\infty}^{\mu} (F(x) - G(x))dx = 0 \text{ and } \int_{\mu}^{\infty} (F(x) - G(x))dx = 0.$$

Proof. Let μ denote the common mean of F and G. Because we can add or subtract absolutely convergent, improper integrals we have

$$I_{1,F}(\mu) - I_{1,G}(\mu) = \int_{-\infty}^{\mu} (F(x) - G(x))\, dx$$

and

$$\begin{aligned} I_{2,G}(\mu) - I_{2,F}(\mu) &= \int_{\mu}^{\infty} ((1 - G(x)) - (1 - F(x)))\, dx \\ &= \int_{\mu}^{\infty} (F(x) - G(x))\, dx. \end{aligned}$$

Recall that, for any distribution F having finite mean μ and non-empty non-trivial domain, $I_{2,F}(\mu) = I_{1,F}(\mu)$. We have

$$\begin{aligned} 0 &= (I_{1,F}(\mu) - I_{2,F}(\mu)) + (I_{2,G}(\mu) - I_{1,G}(\mu)) \\ &= (I_{1,F}(\mu) - I_{1,G}(\mu)) + (I_{2,G}(\mu) - I_{2,F}(\mu)) \\ &= \int_{-\infty}^{\mu} (F(x) - G(x))\, dx + \int_{\mu}^{\infty} (F(x) - G(x))\, dx \\ &= \int_{-\infty}^{\infty} (F(x) - G(x))\, dx. \end{aligned}$$

If, in addition, they have the same ω, then $\omega = I_{1,F}(\mu) = I_{2,F}(\mu) = I_{1,G}(\mu) = I_{2,G}(\mu)$ and the claim is obvious. □

Theorem 2.11. *(Addition formula) Let $\{F_i\}_{i=1}^n$ be a collection of CDFs where each F_i has finite mean μ_i and standard dispersion ω_i. Let $F = \sum_{i=1}^n a_i F_i$ where each $a_i \geq 0$, and $\sum_{i=1}^n a_i = 1$. Then F is a CDF for some random variable with finite mean*

$$\mu = \sum_{i=1}^{n} a_i \mu_i.$$

The standard dispersion ω of F can be calculated as

$$\omega = \sum_{i=1}^{n} a_i \omega_i + \sum_{i=1}^{n} a_i \int_{\mu_i}^{\mu} F_i(x) dx.$$

Proof. We first show that F is the CDF of some random variable. Since

$$\begin{cases} \sum_{i=1}^n a_i \mu_i = \sum_{i=1, a_i \neq 0}^n a_i \mu_i, \\ \sum_{i=1}^n a_i \omega_i = \sum_{i=1, a_i \neq 0}^n a_i \omega_i, \\ \sum_{i=1}^n a_i \int_{\mu_i}^{\mu_F} F_i dx = \sum_{i=1, a_i \neq 0}^n a_i \int_{\mu_i}^{\mu_F} F_i dx, \\ \sum_{i=1, a_i \neq 0}^n a_i = 1, \end{cases}$$

we may assume that $a_i > 0$ for all i. Thus

(1) $\lim_{x \to +\infty} F(x) = \sum_{i=1}^n a_i \lim_{x \to +\infty} F_i(x) = \sum_{i=1}^n a_i \cdot 1 = 1$,
(2) $\lim_{x \to -\infty} F(x) = \sum_{i=1}^n a_i \lim_{x \to -\infty} F_i(x) = \sum_{i=1}^n a_i \cdot 0 = 0$,
(3) For $x \leq y$, $F(x) = \sum_{i=1}^n a_i F_i(x) \leq \sum_{i=1}^n a_i F_i(y) = F(y)$,
(4) because F_i is right-continuous on $\mathbb{R}$ for $i = 1, 2, \ldots, n$, F is also right-continuous on $\mathbb{R}$.

By Theorem 3.11 in [11], F is the CDF of some random variable.

We then show that F has finite mean (i.e. $\int_{-\infty}^{\infty} |x|\, dF(x) < \infty$). Let ρ_i be the probability measure induced on the Borel σ-algebra of $\mathbb{R}$ by

F_i, for $i = 1, \ldots, n$. Let ρ be the corresponding probability measure induced by F. For $x \in \mathbb{R}$, we have

$$\begin{aligned}\rho((-\infty, x]) &= F(x) \\ &= \sum_{i=1}^{n} a_i F_i(x) \\ &= \sum_{i=1}^{n} a_i \rho_i((-\infty, x]).\end{aligned}$$

Thus $\rho = \sum_{i=1}^{n} a_i \rho_i$ on sets of the form $(-\infty, x]$, for all $x \in \mathbb{R}$. By the corollary on page 19 of [11], $\rho = \sum_{i=1}^{n} a_i \rho_i$ on the Borel σ-algebra $\mathcal{B}$ of $\mathbb{R}$. Hence

$$\begin{aligned}\int_{-\infty}^{\infty} |x|\, dF(x) &= \int_{-\infty}^{\infty} |x|\, d\rho(x) \\ &= \int_{-\infty}^{\infty} |x| \sum_{i=1}^{n} a_i d\rho_i(x) \\ &= \sum_{i=1}^{n} a_i \int_{-\infty}^{\infty} |x|\, d\rho_i(x) \\ &= \sum_{i=1}^{n} a_i \int_{-\infty}^{\infty} |x|\, dF_i(x) \\ &< \infty.\end{aligned}$$

Consequently,

$$\begin{aligned}\mu &= \int_{-\infty}^{\infty} x\, dF(x) \\ &= \sum_{i=1}^{n} a_i \int_{-\infty}^{\infty} x\, dF_i(x) \\ &= \sum_{i=1}^{n} a_i \mu_i.\end{aligned}$$

Suppose first that F is a point-mass CDF (i.e., for some $c \in \mathbb{R}$, $F(x) = 0$ if $x < c$ and $F(x) = 1$ otherwise). Then a direct argument shows that for each i, $F_i = F$. In this case, $\omega = 0$ and $\omega_i = 0$ for all i. So trivially, $\sum_{i=1}^{n} a_i \omega_i = 0 = \omega$.

Now assume that F is not a point-mass CDF, then its non-trivial domain $(A, B) \neq \emptyset$. Let (A_i, B_i) be the non-trivial domain of F_i, for each i. We want to show that $(A_i, B_i) \subseteq (A, B)$.

Suppose $x \in (A_i, B_i)$, then $F(x) \geq a_i F_i(x) > 0$. Also,

$$F(x) = \sum_i^n a_i F_i(x) \leq \sum_{j \neq i}^n a_j + a_i F_i(x) < \sum_i^n a_i = 1.$$

Thus $(A_i, B_i) \subseteq \{x \in \mathbb{R} \mid 0 < F(x) < 1\}$. But (A_i, B_i) is an open set, hence $(A_i, B_i) \subseteq \mathrm{Int}\{x \in \mathbb{R} \mid 0 < F(x) < 1\} = (A, B)$ for $i = 1, \dots, n$. Note that $F_i(x) = 0$ for $x \in (A, A_i)$, so $\int_A^{A_i} F_i(x)dx = 0$. We then have,

$$\begin{aligned}
\omega = I_1(\mu) &= \int_A^\mu F(x)dx = \int_A^\mu \sum_{i=1}^n a_i F_i(x)dx \\
&= \sum_{i=1}^n a_i \int_{A_i}^\mu F_i(x)dx \\
&= \sum_{i=1}^n a_i \left(\int_{A_i}^{\mu_i} + \int_{\mu_i}^{\mu} \right) F_i(x)dx \\
&= \sum_{i=1}^n a_i \int_{A_i}^{\mu_i} F_i(x)dx + \sum_{i=1}^n a_i \int_{\mu_i}^{\mu} F_i(x)dx \\
&= \sum_{i=1}^n a_i \omega_i + \sum_{i=1}^n a_i \int_{\mu_i}^{\mu} F_i(x)dx.
\end{aligned}$$

□

The following theorem is new. It bounds the ratio of standard deviation to standard dispersion ($\frac{\sigma}{\omega}$), which Shadwick and Cascon call the *first C-S character*, in the context of an empirical CDF.

Theorem 2.12. *Let $x_1, \dots, x_n$ be data points. Set $\bar{x} = \frac{1}{n}\sum_{i=1}^n x_i$ and suppose that $x_i \neq \bar{x}$ for at least one i. Then*

$$2 \leq \frac{\sqrt{\frac{1}{n}\sum_{i=1}^n (x_i - \bar{x})^2}}{\frac{1}{2n}\sum_{i=1}^n |x_i - \bar{x}|} \leq \sqrt{2n}$$

Proof. (The following proof was suggested by Professor Thomas Ramsey.) Without loss of generality, we may assume $\bar{x} = 0$; that is $x_1 + \cdots + x_n = 0$. Fix $D := \frac{1}{2n}\sum_{i=1}^n |x_i| > 0$. We will maximize the square of the numerator $N := \frac{1}{n}\sum_{i=1}^n x_i^2$ subjecting to these two constraints.

The two constraints are continuous functions on $\mathbb{R}^n$, so the set S satisfying both constraints is the intersection of two closed sets and thus closed. Because $D > 0$, this closed set excludes $(0, ..., 0)$. Also, for each j,

$$|x_j| \leq \sum |x_i| = 2nD.$$

Thus, $S \subseteq [-2nD, 2nD]^n$ and thus bounded. As S is a closed and bounded subset of R^n, it is compact. Since $f(x_1, \dots, x_n) = \sum x_i^2$ is continuous on $\mathbb{R}^n$, the Extreme Value Theorem asserts the existence of a maximum and a minimum in S. Let the maximum occur at $x_1, \dots, x_n$. Let $P = \{i \mid x_i > 0\}$, $Q = \{i \mid x_i < 0\}$, and $T = \{i \mid x_i = 0\}$. We claim that the maximum of $\sum_i x_i^2$ is bounded by $n^2 D^2$.
Fix $x_i = 0$ for $i \in T$. We will maximize $\sum x_i^2$ subject to $\sum x_i = 0$ and $2nD = \sum_{i \in P} x_i - \sum_{i \in Q} x_i$ for $i \in P \cup Q$. Using the Lagrange multiplier method (for more detail, see [5]), we have

$$\frac{\partial N}{\partial x_i} = \lambda_1 \frac{\partial (x_1 + \cdots + x_n)}{\partial x_i} + \lambda_2 \frac{\partial \left(\sum_{i \in P} x_i - \sum_{i \in Q} x_i \right)}{\partial x_i}, i \in P \cup Q.$$

We obtain,

$$\begin{cases} 2x_i = \lambda_1 + \lambda_2 \text{ if } i \in P, \\ 2x_i = \lambda_1 - \lambda_2 \text{ if } i \in Q. \end{cases}$$

In particular, there exist constants a and b such that $x_i = a$ for all $i \in P$, and $x_i = b$ for all $i \in Q$. Let p, q be the numbers of elements in P, Q, respectively, then

$$\begin{cases} p \cdot a + q \cdot b = 0, \\ p \cdot a - q \cdot b = 2nD. \end{cases}$$

This system has solution

$$\begin{cases} a = \frac{nD}{p}, \\ b = \frac{-nD}{q}. \end{cases}$$

Thus,

$$\begin{aligned} \sum_i x_i^2 &= p \cdot a^2 + q \cdot b^2 \\ &= p \cdot \frac{n^2 D^2}{p^2} + q \cdot \frac{n^2 D^2}{q^2} \\ &= n^2 D^2 \left(\frac{1}{p} + \frac{1}{q} \right) \\ &\leq n^2 D^2 \left(\frac{1}{1} + \frac{1}{1} \right) \\ &= 2n^2 D^2, \end{aligned}$$

and so

$$\begin{aligned}\frac{\sqrt{\frac{1}{n}\sum_{i=1}^{n}x_i^2}}{\frac{1}{2n}\sum_{i=1}^{n}|x_i|} &\leq \frac{\sqrt{\frac{1}{n}2n^2D^2}}{D}\\ &= \sqrt{2n}.\end{aligned}$$

By Hölder's inequality, we have

$$\begin{aligned}\frac{1}{2n}\sum|x_i| &\leq \frac{1}{2n}\sqrt{\sum_{i=1}^{n}1^2}\sqrt{\sum_{i=1}^{n}x_i^2}\\ &= \frac{\sqrt{n^2}}{2n}\sqrt{\frac{1}{n}\sum_{i=1}^{n}x_i^2}\\ \iff 2 &\leq \frac{\sqrt{\frac{1}{n}\sum_{i=1}^{n}x_i^2}}{\frac{1}{2n}\sum_{i=1}^{n}|x_i|}.\end{aligned}$$

□

It has been shown that the first C-S characters of the Uniform, the Normal, and the Laplace distributions are $\frac{4}{\sqrt{3}}$, $\sqrt{2\pi}$, and $2\sqrt{2}$, respectively. For a more detailed treatment of the first C-S character, see [2, 3, 4].

Shadwick also introduced (see [4]) the analogues of the Markov and Chebyshev inequalities for ω.

Lemma 2.13. *Let X be a random variable with finite mean μ. For every $b > 0$,*

$$\mathbb{P}(X-\mu \geq b) \leq \frac{\omega}{b}.$$

Proof. Let F be the CDF for X. By Proposition 2.9, for every $b > 0$,

$$\begin{aligned}\omega &= \int_{\mu}^{\infty}(t-\mu)dF(t)\\ &\geq \int_{[\mu+b,\infty)}(t-\mu)dF(t)\\ &\geq b\int_{[\mu+b,\infty)}dF(t)\\ &= b\left(\int_{(-\infty,\infty)}dF(t)-\int_{(-\infty,\mu+b)}dF(t)\right)\\ &= b\,(1-\mathbb{P}(X<\mu+b))\\ &= b\,\mathbb{P}(X\geq\mu+b).\end{aligned}$$

Thus, $\mathbb{P}(X - \mu \geq b) \leq \frac{\omega}{b}$. $\square$

Lemma 2.14. *Let X be a random variable with finite mean μ. For every $b > 0$,*

$$P(|X - \mu| \geq b) \leq \frac{2\omega}{b}.$$

Proof. Similarly,

$$\begin{aligned} \omega &= \int_{-\infty}^{\mu} (\mu - t) dF(t) \\ &\geq \int_{(-\infty,\, \mu - b]} (\mu - t) dF(t) \\ &\geq b \int_{(-\infty,\, \mu - b]} dF(t) \\ &= b\, \mathbb{P}(X \leq \mu - b). \end{aligned}$$

So, $\mathbb{P}(X - \mu \leq -b) \leq \frac{\omega}{b}$. Consequently, with the result from Lemma 2.13, we obtain

$$\mathbb{P}(|X - \mu| \geq b) \leq \frac{2\omega}{b}.$$

$\square$

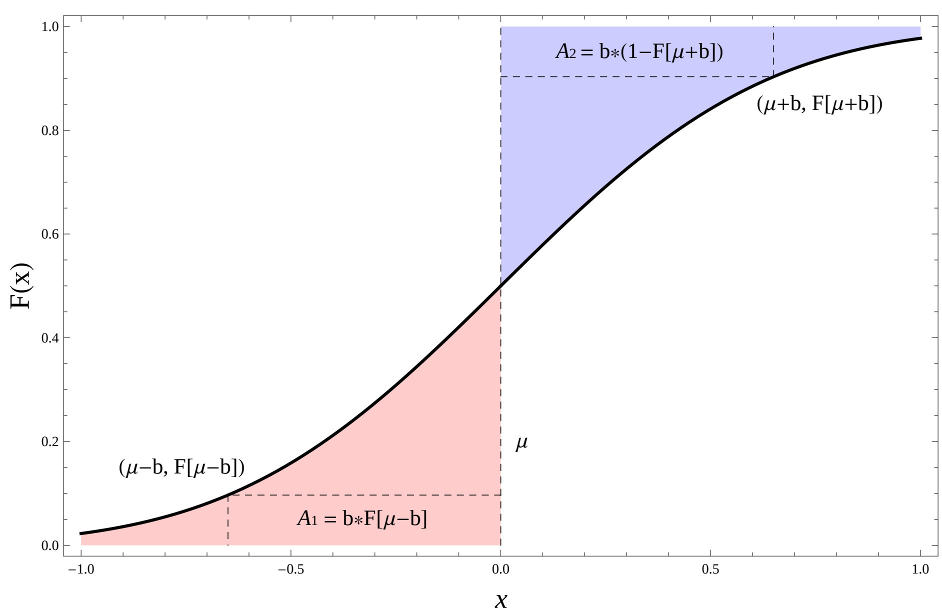


FIGURE 5. A more intuitive proof of lemmas 2.13 and 2.14

3. Applications

In this section, we work with real data to demonstrate some of the properties of the Ω function and its derived statistic, the standard dispersion ω. For a more thorough investigation on the applications of Ω, refer to [2, 3, 6, 7].

3.1. **Interpretation.** Recall from Definition 2.1 that, for $x \in (A, B)$,

$$I_1(x) = \int_A^x F(t)dt = \int_{-\infty}^x (x-t)dF(t)$$

and

$$I_2(x) = \int_x^B [1-F(t)]dt = \int_x^\infty (t-x)dF(t).$$

The functions $I_1(x)$ and $I_2(x)$ may be interpreted as the probability weighted loss and the probability weighted gain, respectively, at a certain level x. Then $\Omega(x)$, as the ratio of loss over gain, will give a performance measure (see [6, 8, 9]). Furthermore, from the definition of the standard dispersion,

$$\omega = \frac{1}{\Omega'(\mu^+)} = \frac{1}{2}E(|x-\mu|)$$

we see that the Omega function of a riskier distribution (bigger ω) is flatter (smaller $\Omega'(\mu^+)$) than that of a less risky one, and vice versa (smaller ω, bigger $\Omega'(\mu^+)$). Figures 6 and 7 illustrate this observation using three distributions.

The distributions used here have common mean 0. The blue graph is of a normal distribution with standard deviation 0.5. The purple graph is of another normal distribution with standard distribution 1. And the olive graph is of a Laplace distribution with location parameter 0 and scale parameter 0.5.

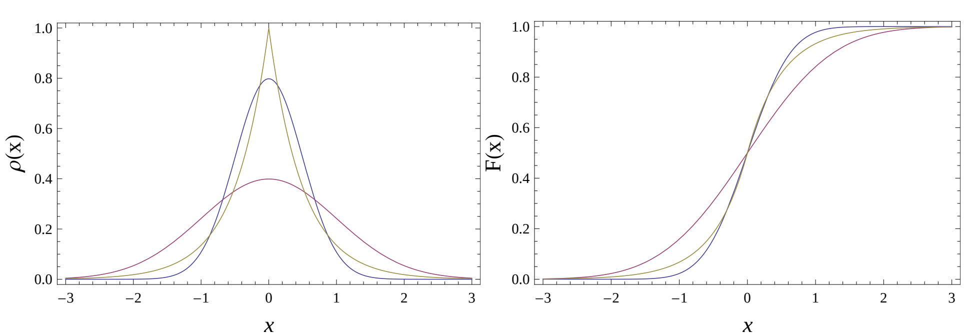


Figure 6. Graphs of the probability density functions (left) and cumulative distribution functions (right).

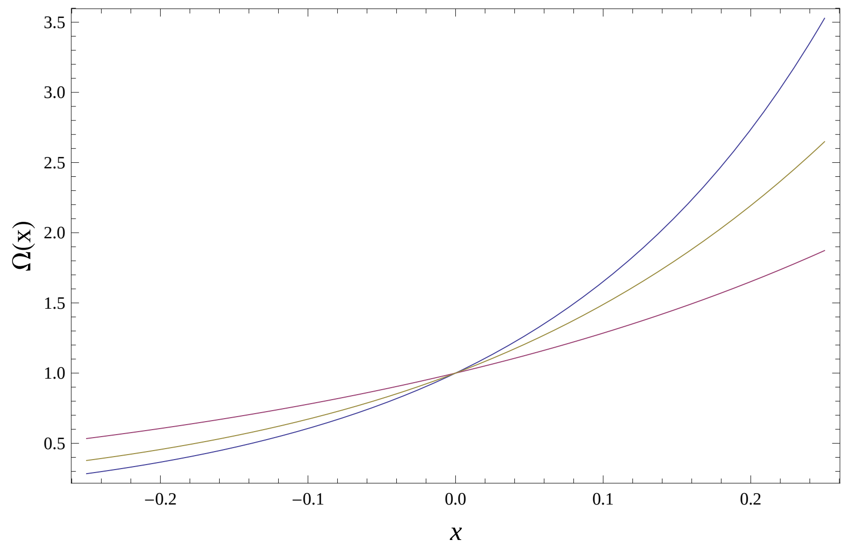


FIGURE 7. Graphs of the Omega functions.

3.2. **Sensitivity to Outliers.** Outliers, which are extreme values to the left or right, affect statistics. Here we compare the effect of outliers on the standard dispersion ω to the effect on the standard deviation σ.

We consider the set MS of daily returns from Microsoft Corporation stocks since January 1st, 2000.

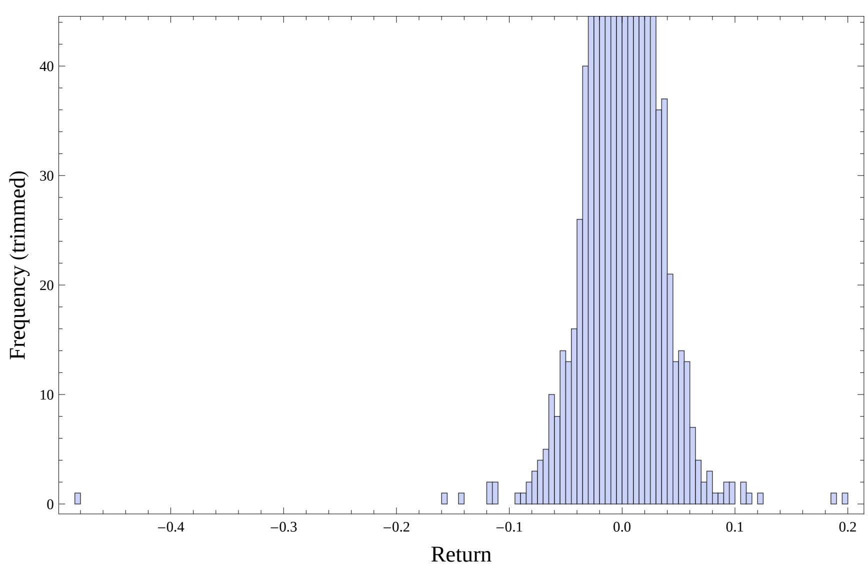


FIGURE 8. Histogram of Microsoft's daily returns since Jan. 1, 2000.

The two statistics, standard deviation σ_{MS} and standard dispersion ω_{MS}, are 0.0250939 and 0.00774291, repectively. The outlier is a loss of -0.48323 on February 18th of 2003, while the mean of this set of returns is -0.000382191. After removing this outlier, the new statistics are $\sigma_{\mathrm{MS}'} = 0.0230086$ and $\omega_{\mathrm{MS}'} = 0.00764251$. So,

$$\frac{|\sigma_{\mathrm{MS}} - \sigma_{\mathrm{MS}'}|}{|\sigma_{\mathrm{MS}}|} = 0.083099$$

whereas,

$$\frac{|\omega_{\mathrm{MS}} - \omega_{\mathrm{MS}'}|}{|\omega_{\mathrm{MS}}|} = 0.0129672.$$

Observe that in this case, the percent change of the standard deviation is about 6.4 times more affected than the percent change of the standard dispersion.

We provide one more instance: the S&P 500 index since January 1, 1987 (this example was introduced in [4]). The biggest loss was on October 19, 1987 and it was -0.204669, while the mean is 0.000252815. The percent changes after removing the outlier are

$$\frac{|\sigma_{\mathrm{SP}} - \sigma_{\mathrm{SP}'}|}{|\sigma_{\mathrm{SP}}|} = 0.0262401$$

and

$$\frac{|\omega_{\mathrm{SP}} - \omega_{\mathrm{SP}'}|}{|\omega_{\mathrm{SP}}|} = 0.00461926.$$

In this case, the standard deviation is, again, much more affected than the standard dispersion. It is about 5.7 times more affected.

3.3. **Portfolio Optimization.** We provide here an example of Portfolio Optimization using ω.

We will form a portfolio consisting of 5 different indices: IBM (International Business Machines Corp.), MSFT (Microsoft Corporation), VBMFX (Vanguard Total Bond Market Index), VGSIX (Vanguard Specialized Port Inc.), VTSMX (Vanguard Total Stock Market Index). The adjusted daily prices are obtained directly from the Mathematica data server (using the function FinancialData, see [12]). We calculate quarterly returns for each candidate based on these daily prices. Table 1 shows some characteristics of the quarterly returns.

We will optimize portfolios using 2 methods: (1) the mean-variance method and (2) the mean-standard dispersion method. Considering the mean return of a return distribution the reward, its variance and standard dispersion the risk, we minimize the risk subject to a fixed reward level.

Name	**IBM**	**MSFT**	**VBMFX**	**VGSIX**	**VTSMX**
Mean	0.04093	0.03436	0.01440	0.02228	0.01434
Min	-0.37633	-0.32472	-0.03370	-0.55490	-0.37696
Max	0.35724	0.47662	0.04923	0.22689	0.17606
σ	0.15639	0.16009	0.01747	0.11164	0.09379
ω	0.06299	0.06048	0.00689	0.03841	0.03445

TABLE 1. Characteristics of return series of the candidates

In particular, let $R_1, \dots, R_n$ be the returns. We seek a solution $S = (a_1, \dots, a_n)$ to:

(1) Minimizing $\mathrm{Var}(\sum_i^n a_i R_i)$.
(2) Minimizing $\omega(\sum_i^n a_i R_i)$.

Both subject to $\sum_i^n a_i E(R_i) \geq r$, $a_i \geq 0$, and $\sum_i^n a_i = 1$.

We use the first 10 quarters to gain information for the calculation. We then do a rebalance every 2 quarters afterward. In this scenario, we assume that buying and selling can be done with no delay. Also, buying and selling are done at the exact dates. And we always buy and sell all the resouces we have at each moment. There are a total of 54 quarters, thus we have 23 portfolio rebalances.

Returns from each period of the two portfolios are very similar. At the final period, the portfolio allocation lost 28.46% of the initial investment using the second method (minimizing standard dispersion ω). This is a bit better compared to a loss of 32.03% using the first method (minimizing variance σ^2). Detailed portfolio allocations can be found on the next 2 pages. For a more detailed discussion of portfolio optimization, refer to [6].

Conclusion. In this paper, we properly set up a formal framework for the Omega function. We show its properties, along with formal proofs, and introduced a new theorem (see Theorem 2.12.)

There are some possible future directions to go from here. We need algorithms to optimize portfolios using the Omega function. Furthermore, we want to look for an intuitive interpretation for the Addition formula (see Theorem 2.11), and consequently, a portfolio optimization method based on it. The first C-S character can be used in distribution test. In the case of normality test, Dr. Thomas Ramsey has run simulations to compare the first C-S character method against the Kolmogorov-Smirnov method. The new method has shown some very promising results.

Portfolio					Return %	
IBM	MSFT	VBMFX	VGSIX	VTSMX	_	Cum.
22.4	28.6	0.09	0.02	48.87	0	0
28.12	23.9	0.03	0.01	47.92	11.12	11.12
23.24	35.24	0.08	0.	41.42	-5.07	5.48
20.1	26.22	0.64	0.03	52.99	-9.3	-4.32
32.27	15.58	5.17	0.1	46.85	-9.5	-13.42
29.77	26.21	28.83	15.11	0.05	-3.54	-16.48
32.95	22.45	42.73	1.85	0.	-13.34	-27.62
27.95	23.14	12.6	36.28	0.	-5.8	-31.81
23.12	26.22	15.77	34.85	0.02	-5.09	-35.28
27.84	23.22	42.39	6.52	0.	-5.05	-38.55
19.45	22.94	8.59	49.	0.	-1.13	-39.24
17.62	16.77	5.48	60.11	0.	8.83	-33.87
12.62	18.91	15.92	52.52	0.	9.17	-27.81
11.22	16.74	20.32	51.7	0.	1.08	-27.03
5.37	14.93	25.62	54.06	0.	10.04	-19.7
5.42	14.29	27.28	52.95	0.02	9.73	-11.89
3.88	10.18	38.21	47.7	0.01	-0.33	-12.17
4.45	11.	33.83	50.64	0.06	5.09	-7.71
6.67	11.31	32.96	48.94	0.09	-2.71	-10.21
8.31	14.78	26.45	50.39	0.04	1.72	-8.67
16.17	10.49	26.63	46.68	0.01	9.15	-0.31
29.18	18.47	52.32	0.01	0.	-35.79	-35.99
35.12	14.63	49.94	0.29	0.	6.18	-32.03

TABLE 2. Portfolios according to the first method (Mean-Variance)

Portfolio					Return %	
IBM	MSFT	VBMFX	VGSIX	VTSMX	_	Cum.
14.88	35.94	2.07	0.	47.08	0	0
20.28	30.43	0.	0.	49.27	8.64	8.64
17.45	40.15	0.11	0.97	41.3	-2.9	5.5
17.3	28.35	0.02	0.	54.31	-9.8	-4.82
32.38	19.23	1.23	2.59	44.55	-10.67	-14.97
24.55	32.44	39.26	3.72	0.	-2.78	-17.34
24.16	31.59	42.67	0.12	1.44	-13.4	-28.42
29.23	20.2	4.94	45.6	0.	-5.66	-32.46
30.62	18.18	9.96	41.23	0.	-5.29	-36.04
26.82	24.58	30.09	17.28	1.2	-4.54	-38.94
28.07	14.96	8.73	48.22	0.	0.15	-38.85
22.11	15.86	12.8	49.21	0.	9.2	-33.23
14.02	21.14	21.5	43.14	0.17	7.64	-28.12
15.76	14.93	23.18	45.58	0.52	1.24	-27.23
0.7	19.24	26.01	54.03	0.	9.67	-20.2
0.57	18.57	27.82	53.02	0.	10.09	-12.15
0.	13.91	38.52	47.55	0.	-0.66	-12.73
0.	15.81	34.56	49.61	0.	4.76	-8.58
1.36	16.34	33.08	49.2	0.	-2.76	-11.1
2.88	18.28	25.93	52.9	0.	2.97	-8.46
22.18	8.75	30.44	38.61	0.	8.82	-0.39
27.63	18.71	47.71	5.93	0.	-33.06	-33.32
29.86	15.84	34.21	17.3	2.77	7.29	-28.46

Table 3. Portfolios according to the second method (Min ω)

References

[1] Ana Cascon, Con Keating and William F. Shadwick, *The Omega Function*, The Finance Development Centre, London, England, March 2003.

[2] Ana Cascon, William Shadwick, *The C-S Character and Limitations of the Sharpe Ratio*, Journal of Investment Consulting 8, no. 1 (summer 2006).

[3] Ana Cascon, William Shadwick, *The Standard Dispersion and its Application to Risk Analysis*, Journal of Investment Consulting 9, no. 2 (summer 2007).

[4] Ana Cascon, William Shadwick, *The Geometry of Probability Distributions: A New Central Limit Theorem*, Omega Analysis Limited, Mathematical Sciences Research Institute, 2008.

[5] Encyclopædia of Mathematics, *Lagrange Multipliers*, available at `http://eom.springer.de/L/l057190.htm`.

[6] Alexandre Favre-Bulle, Sébastien Pache, *The Omega Measure: Hedge Fund Portfolio Optimization*, Master thesis, University of Lausanne - Ecole des HEC, January 2003.

[7] Robert J. Frey, *On the Ω-Ratio*, Research Paper, Stony Brook University, April 2009, available at `http://www.ams.sunysb.edu/~frey/Research/Research/OmegaRatio/OmegaRatio.pdf`.

[8] Hossein Kazemi, Thomas Schneeweis, Raj Gupta, *Omega as a Performance Measure*, Research Paper, University of Massachusetts, Amherst, June 2003.

[9] Con Keating and William F. Shadwick, *A Universal Performance Measure*, The Finance Development Center, London, England, 2002, available at `http://www.isda.org/c_and_a/pdf/GammaPub.pdf`.

[10] Con Keating and William F. Shadwick, *An Introduction to Omega*, The Finance Development Center, London, England, 2002.

[11] David Williams, *Probability with Martingales*, Cambridge University Press, Cambridge, England, 1991.

[12] Wolfram Research Inc., *FinancialData*, Wolfram Mathematica Documentation Center, 2009, available at `http://reference.wolfram.com/mathematica/ref/FinancialData.html`.