THE INFLUENCE OF LATERAL MASS EFFLUX ON FREE CONVECTION BOUNDARY LAYERS IN A SATURATED POROUS MEDIUM

TECHNICAL REPORT NO. 13

April 28, 1976

Prepared Under

NATIONAL SCIENCE FOUNDATION
Research Grant No. GI-38319

and

ENERGY RESEARCH AND DEVELOPMENT ADMINISTRATION
Research Grant No. E(04-3)-1093

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The Influence of Lateral Mass Efflux on Free Convection Boundary Layers in a Saturated Porous Medium

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The effects of lateral mass efflux with prescribed temperature and velocity on vertical free convection boundary layers in a saturated porous medium at high Rayleigh numbers is studied analytically in this paper. Within the framework of boundary layer theory, similarity solutions are obtained for the special case where the prescribed temperature and velocity of the fluid vary as x^{λ} and $x^{(\lambda-1)/2}$ respectively. The effects of mass efflux on surface heat transfer rate and boundary layer thickness are shown. Application to warm water discharge along a well or fissure to an aquifer of infinite extent is discussed.

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NOMENCLATURE

A	constant defined by Eq. (3a)
a	constant defined by Eq. (3b)
C_p	specific heat of the convective fluid
f	dimensionless stream function defined by Eq. (10)
f_w	lateral mass efflux parameter
g	acceleration due to gravity
K	permeability of the porous medium
k	thermal conductivity of the porous medium
L	length of the source or sink
$\dot{m}$	mass transfer rate
Q	over-all surface heat transfer rate
q	local heat transfer rate
Ra_x	modified local Rayleigh number, $Ra_x \equiv \rho_\infty g\beta K\|T_w - T_\infty\|x/\mu\alpha$
T	temperature
u	velocity component in vertical direction
v	velocity component in horizontal direction
x	vertical coordinate
y	horizontal coordinate
α	equivalent thermal diffusivity
β	coefficient of thermal expansion
δ	boundary layer thickness
η	dimensionless similarity variable defined by Eq. (9)
η_δ	value of η at the edge of boundary layer
θ	dimensionless temperature defined by Eq. (11)

λ	constant defined by Eq. (3a)
μ	viscosity of convective fluid
ρ	density of convective fluid
ψ	stream function

Subscript

∞	condition at infinity
w	condition at the wall

Introduction

The effects of blowing and suction along a vertical flat plate on free convection in air or water have been the subject of numerous investigations. Eichhorn [1] studied these effects for a class of problems where both wall temperature and the blowing or suction velocity are prescribed power functions of distance from the leading edge. Based on the boundary layer approximations, Eichhorn shows that similarity solutions for the problem are possible if the exponents in the prescribed power functions are related in a particular manner. Sparrow and Cess [2] studied the more general problem with arbitrary values of exponents by a perturbation method. The problem was also studied by Mabuchi [3] who used an integral method.

The analogous problem of the effects of recharge or withdrawal of fluid along a vertical line source or sink on free convection in a porous medium at high Rayleigh numbers, where both the temperature of the fluid along the line source or sink (T_w) and the blowing and suction velocity (v_w) are prescribed power functions of distance, is studied in the present paper. If the boundary layer approximations similar to those employed by Wooding [4], McNabb [5], Cheng & Minkowycz [6], and Cheng & Chang [7] are invoked, and if the prescribed power functions are given by $T_w = T_\infty \pm Ax^\lambda$ and $v_w = ax^n$, it is found that similarity solutions are possible if $n = (\lambda-1)/2$. The problem has a number of important engineering and geophysical applications. For example, the residual warm water discharged from a geothermal power plant is usually disposed of through subsurface reinjection wells which can be idealized as vertical line sources in a porous medium. If the temperature of the injected fluid differs from that of the receiving groundwater in the rock formation, the injected fluid would experience a positive or negative buoyancy force (depending on the relative temperature difference) which results in a convective movement of groundwater near the well. Similarly, convection of groundwater also occurs near the vertical fissures or cracks during the natural recharge of aquifer, whenever the temperature of the groundwater discharged from the fissures and cracks differs from that of the receiving water in the aquifer.

Analysis

Figure 1 shows the problem of recharge or withdrawal of fluids along a vertical line source or sink embedded in a saturated porous medium, where the temperature along the source or sink is given by $T_w = T_\infty \pm Ax^\lambda$ (with T_∞ denoting temperature at infinity and $A>0$) and the discharge or withdrawal rate is given by $v_w = ax^n$ where $a>0$ for discharge of fluid and $a<0$ for withdrawal of fluid. If we assume that (i) the convective flow is due to the density difference between the source (or sink) and at infinity, (ii) the temperature of the fluid is everywhere below the boiling point, (iii) the convective fluid and the porous medium are everywhere in local thermodynamic equilibrium, (iv) properties of the fluid and the porous medium are constant, and (v) the Boussinesq approximation is employed, it can be shown that the governing equations with boundary layer simplifications are given by [6]

$$\frac{\partial^2\psi}{\partial y^2} = \pm \frac{\rho_\infty \beta g K}{\mu}\frac{\partial T}{\partial y} \quad , \tag{1}$$

$$\frac{\partial^2 T}{\partial y^2} = \frac{1}{\alpha}\left(\frac{\partial\psi}{\partial y}\frac{\partial T}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial T}{\partial y}\right) \quad , \tag{2}$$

where the "+" and "-" signs in Eq. (1) are for the cases of $T_w>T_\infty$ (Fig. 1a) and $T_w<T_\infty$ (Fig. 1b) respectively. In Eqs. (1) and (2), ρ, μ, and β are the

density, viscosity, and the thermal expansion coefficient of the fluid; g is the gravitational acceleration; K is the permeability of the porous medium; $\alpha = k/(\rho_\infty C_p)_f$ is the equivalent thermal diffusivity with k denoting the thermal conductivity of the porous medium and $(\rho_\infty C_p)_f$ the density and specific heat of the fluid; ψ is the stream function defined in the usual manner, i.e., $u = \frac{\partial\psi}{\partial y}$ and $v = -\frac{\partial\psi}{\partial x}$.

For the coordinate system shown in Fig. 1, the boundary conditions are given by

$$y = 0 \quad , \quad T = T_\infty \pm Ax^\lambda \quad , \quad v = -\frac{\partial\psi}{\partial x} = ax^n \quad , \tag{3a,b}$$

$$y \to \infty \quad , \quad T = T_\infty \quad , \quad u = \frac{\partial\psi}{\partial y} = 0 \quad . \tag{4a,b}$$

With the exception of boundary condition (3b), Eqs. (1)-(4) are identical to the governing equations and boundary conditions for the problem of free convection about a vertical flat plate embedded in a porous medium where similarity solutions have been found [6]. It can be shown that similarity solutions to Eqs. (1)-(4) exist if $n = (\lambda-1)/2$, and that under such a restricted condition the governing equations and boundary conditions can be transformed into

$$f'' - \theta' = 0 \quad , \tag{5}$$

$$\theta'' + \frac{1+\lambda}{2} f\theta' - \lambda f'\theta = 0 \quad , \tag{6}$$

with boundary conditions given by

$$\eta = 0 \quad , \qquad \theta = 1 \quad , \qquad f = f_w \quad , \tag{7a,b}$$

$$\eta \to \infty \qquad \theta = 0 \quad , \qquad f' = 0 \quad , \tag{8a,b}$$

where the similarity variables, η, f, and θ are defined by

$$\eta = \sqrt{\frac{\rho_\infty g\beta K(T_w-T_\infty)}{\mu\alpha x}}\, y \quad , \tag{9}$$

$$\psi = \sqrt{\frac{\alpha\rho_\infty g\beta K(T_w-T_\infty)x}{\mu}}\, f(\eta) \quad , \tag{10}$$

$$\theta(\eta) = \frac{T-T_\infty}{T_w-T_\infty} \quad , \tag{11}$$

and $f_w \equiv -2a\Big/\left[\frac{\alpha\rho_\infty g\beta KA}{\mu}\right]^{1/2}(1+\lambda)$ which is positive for the withdrawal of fluid and negative for the discharge of fluid. In terms of the new variables, the vertical and horizontal velocity components are given by

$$u = \frac{\rho_\infty g\beta(T_w-T_\infty)K}{\mu}\, f'(\eta) \quad , \tag{12}$$

and

$$v = 1/2\sqrt{\frac{\alpha\rho_\infty g\beta K(T_w-T_\infty)}{\mu x}}\,[(1-\lambda)\eta f' - (1+\lambda)f] \quad . \tag{13}$$

With the aid of boundary condition (8), Eq. (5) can be integrated to give

$$f'-\theta = 0 \quad , \tag{14}$$

which shows that vertical velocity and temperature have the same shape.

Results and Discussion

Equations (14), (6), (7) and (8a) or (8b) are the governing equations and boundary conditions for the problem, which can be integrated numerically by means of the Runge-Kutta method incorporated with the shooting technique for a systematic guessing of $\theta'(0)$. Numerical results for $f(\eta)$, $\theta'(\eta)$, $f'(\eta)$ or $\theta(\eta)$ for selected values of λ with $f_w = -1.0$ to 1.0 are shown in Figs. 2-4. It is noted that $f_w = 0$ corresponds to an impermeable vertical flat plate embedded in a porous medium.

Figure 2 shows that the value of θ or f' decreases from 1 to 0 as η is increased from zero at different values of f_w. If the edge of the boundary layer thickness (denoted by η_δ) is defined as the value of η where θ (or f') has a value of 0.01, it follows from Eq. (9) that the expression for the boundary layer thickness δ is

$$\frac{\delta}{x} = \frac{\eta_\delta}{\sqrt{Ra_x}} \quad , \tag{15}$$

where $Ra_x = \rho_\infty g K \beta x (T_w - T_\infty)/\mu\alpha$ is the modified Rayleigh number, and the value of η_δ for selected λ is presented in Table 1 and plotted in Fig. 5, where it is shown that the boundary layer thickness decreases as the value of f_w increases from -1.0 to 1.0. The expression for local heat flux can be shown to be

$$q(x,y) = -k\frac{\partial T}{\partial y} = kA^{3/2}\sqrt{\frac{\rho_\infty g\beta K}{\mu\alpha}}\, x^{\frac{3\lambda-1}{2}} \left[-\theta'(\eta)\right] \quad , \tag{16}$$

where the value of $-\theta'(\eta)$ is plotted in Fig. 3, which shows that its value decreases from a maximum value to zero as η is increased from zero. The expression for surface heat flux $q(x)$ is given by Eq. (16) with $\eta = 0$ and the total surface heat rate (per unit width perpendicular to the x-y plane) along the line source or sink with a height L is

$$Q = \int_0^L q(x)dx = kA^{3/2}\sqrt{\frac{\rho_\infty g\beta K}{\mu\alpha}}\left(\frac{2}{1+3\lambda}\right) L^{\frac{1+3\lambda}{2}} [-\theta'(0)] \quad , \tag{17}$$

where the value of $[-\theta'(0)]$ for selected values of λ is tabulated in Table 1 and is plotted in Fig. 6, which shows that heat transfer rate increases as the value of f_w is increased. Consequently, the values of $q(x)$ and Q increase as the value of f_w is increased.

With the aid of Eqs. (13) and (8b), the horizontal velocity component at infinity is given by

$$v_\infty = -\frac{1}{2}\sqrt{\frac{\alpha\rho_\infty g\beta K(T_w-T_\infty)}{\mu x}}\,(1+\lambda)f(\infty) \quad , \tag{18}$$

which can be positive or negative depending on the sign of $f(\infty)$, which in turn, depending on the value of f_w. It is shown in Fig. 4 that although $f(\infty)$ is positive for the range of parameters considered, it could be negative for sufficiently large negative values of f_w, i.e., for strong discharge rates.

Finally, the total mass transfer rate (per unit width perpendicular to the x-y plane) along a vertical line source or sink with a length L is

$$\dot{m} = \rho_\infty \int_0^L v(x)dx = 2\rho_\infty a L^{\frac{\lambda+1}{2}}\Big/(1+\lambda) \quad , \tag{19}$$

where we have made use of Eq. (3b).

As is discussed in Ref. 6, the range of λ for which the problem is physically realistic is $0<\lambda<1$ which follows from the simultaneous consideration of Eqs. (13) and (15). We shall now discuss the variation of δ and $q(x)$, as given by Eqs. (15) and (16), for the special cases of $\lambda = 0$, 1/3, and 1.

(a) $\lambda = 0$ corresponding to the case of uniform wall temperature with $v_w \sim x^{-1/2}$, $\delta \sim x^{1/2}$ and $q \sim x^{-1/2}$.

(b) $\lambda = 1/3$ corresponding to the case of uniform heat flux with $v_w \sim x^{-1/3}$, $T_w \sim x^{1/3}$ and $\delta \sim x^{1/3}$.

(d) $\lambda = 1$ corresponding to the case of uniform wall velocity with $T_w \sim x$, δ = constant and $q \sim x$.

To gain some insight on the magnitude of various physical quantities, consider the discharge of warm geothermal water at 90°C (i.e., $\lambda = 0$) from a fissure crack or well of 500m to an aquifer at 15°C with $f_w = 1.0$. We used the following value of physical properties for computations: $\rho_\infty = 0.92\text{X}10^6 \text{gm/m}^3$, $C = 1$ cal/gm°C, $k = 0.58$ cal/sec-c-m, $\psi = 0.3$, $g = 9.8\text{m/sec}^2$, $\mu = 0.68$ gm/sec-m and $K = 10^{-10}\text{m}^2$. From the definition of f_w and Eq. (19), we found that the discharge rate is approximately 45 gal/hr per meter width perpendicular to the x-y plane. The corresponding boundary layer thickness as given by Eq. (15) is approximately 30m at x = 500m.

ACKNOWLEDGEMENT

The author would like to take this opportunity to thank Mr. W.C. Chau for his assistance in the numerical computations. This study is part of the Hawaii Geothermal Project funded in part by the RANN program of the National Science Foundation of the United States (Grant No. GI-38319), the Energy Research and Development Administration of the United States (Grant No. E(04-3)-1093), and by the State and County of Hawaii.

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TABLE 1 Values of $-\theta'(0)$ and δ for Selected Values of f_w and λ

f_w	$-\theta'(0)$			η_δ		
	$\lambda=0$	$\lambda=1/3$	$\lambda=1$	$\lambda=0$	$\lambda=1/3$	$\lambda=1$
-1.0	.2043	.3971	.6180	7.84	7.58	7.20
- .8	.2432	.4416	.6770	7.58	7.21	6.67
- .6	.2865	.4917	.7440	7.29	6.81	6.12
- .4	.3345	.5476	.8198	6.98	6.40	5.59
- .2	.3870	.6096	.9049	6.65	5.98	5.08
0	.4438	.6776	1.000	6.31	5.57	4.60
.2	.5050	.7517	1.104	5.96	5.17	4.16
.4	.5701	.8316	1.219	5.61	4.79	3.77
.6	.6389	.9169	1.344	5.28	4.44	3.42
.8	.7111	1.007	1.477	4.96	4.11	3.12
1.0	.7863	1.102	1.618	4.65	3.81	2.85

LIST OF FIGURES

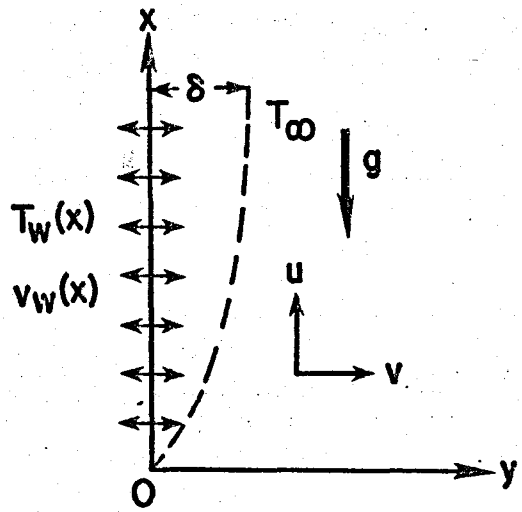

Fig. 1a Coordinate System for $T_w > T_\infty$

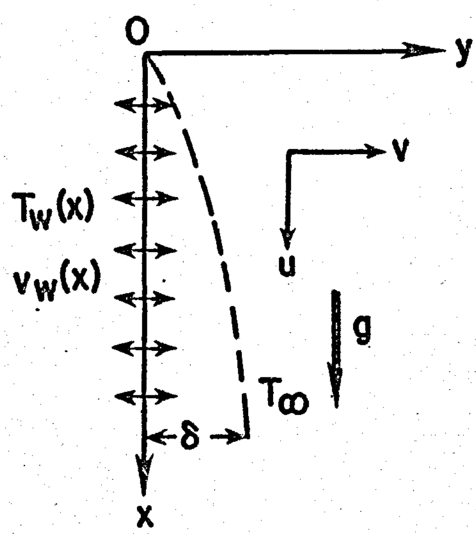

Fig. 1b Coordinate System for $T_w < T_\infty$

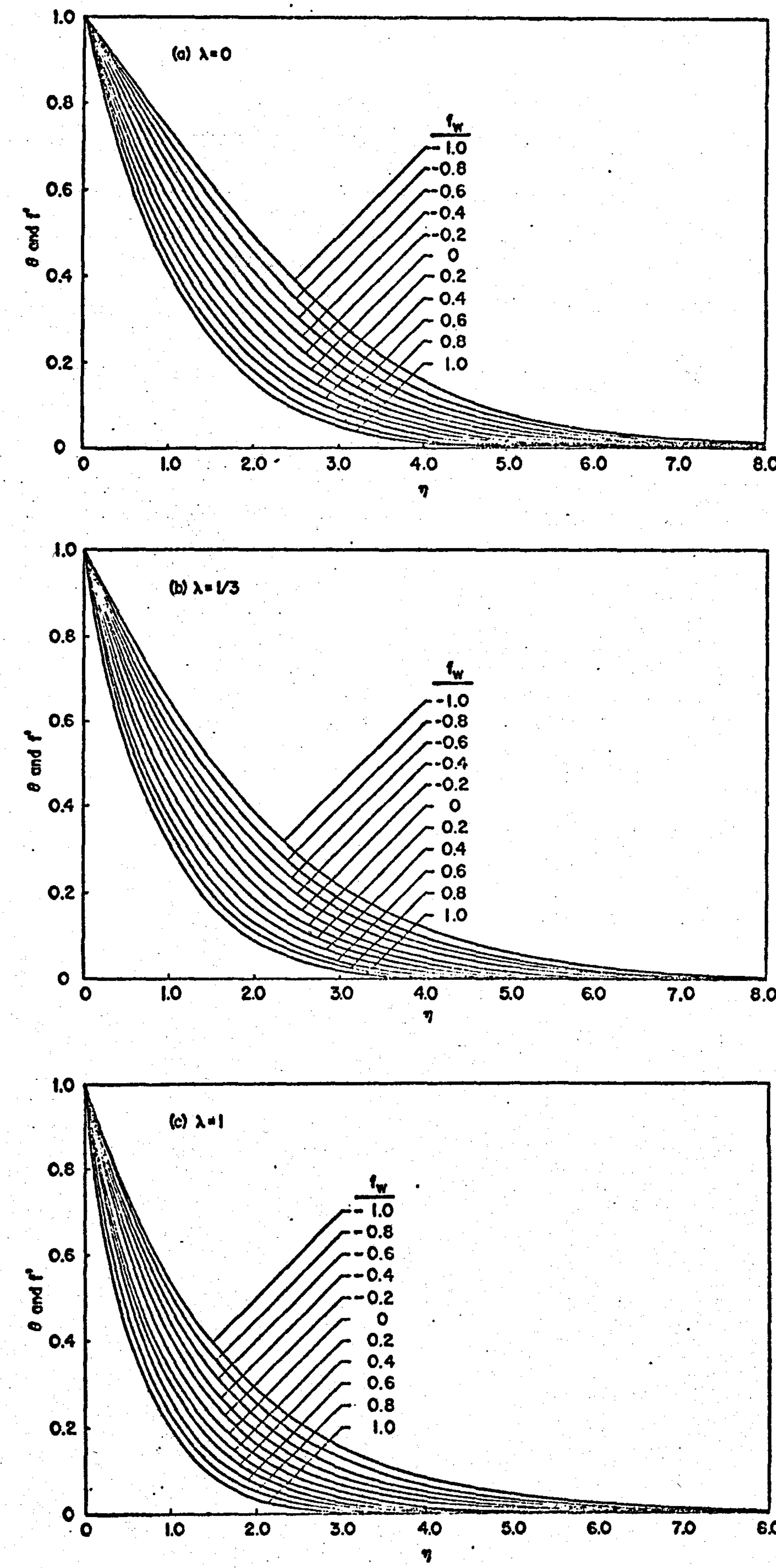

Fig. 2 Values of θ and f' Versus η

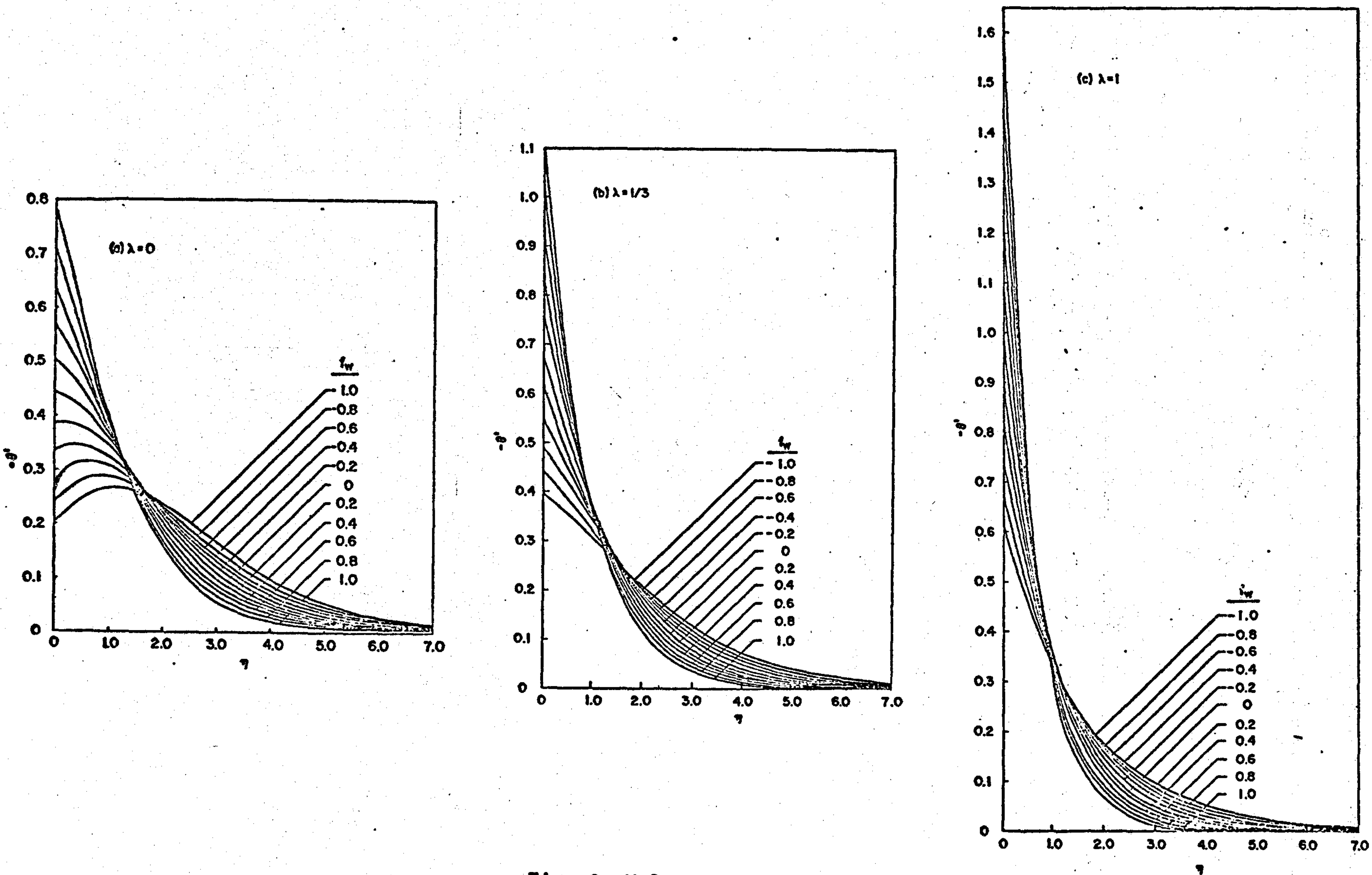

Fig. 3 Values of $-\theta'$ Versus η

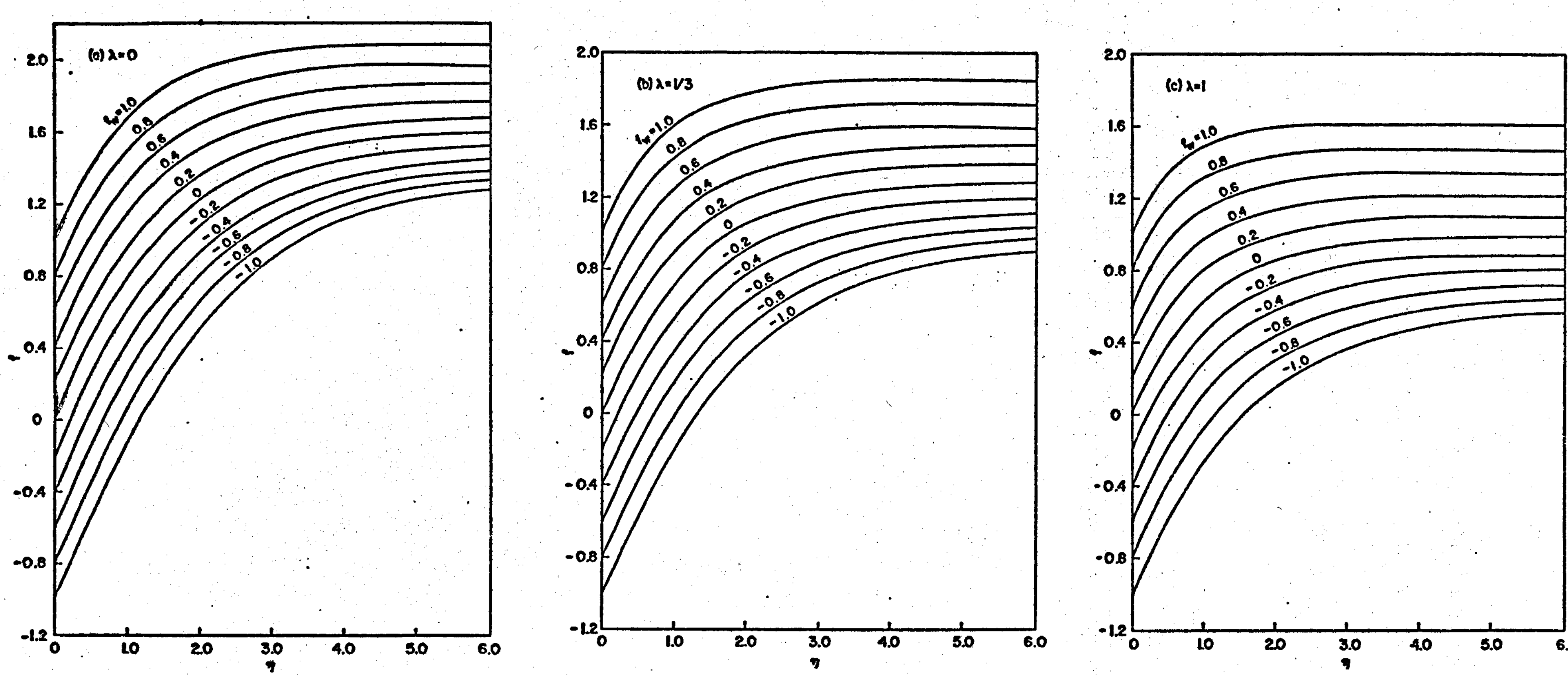

Fig. 4 Values of f Versus η

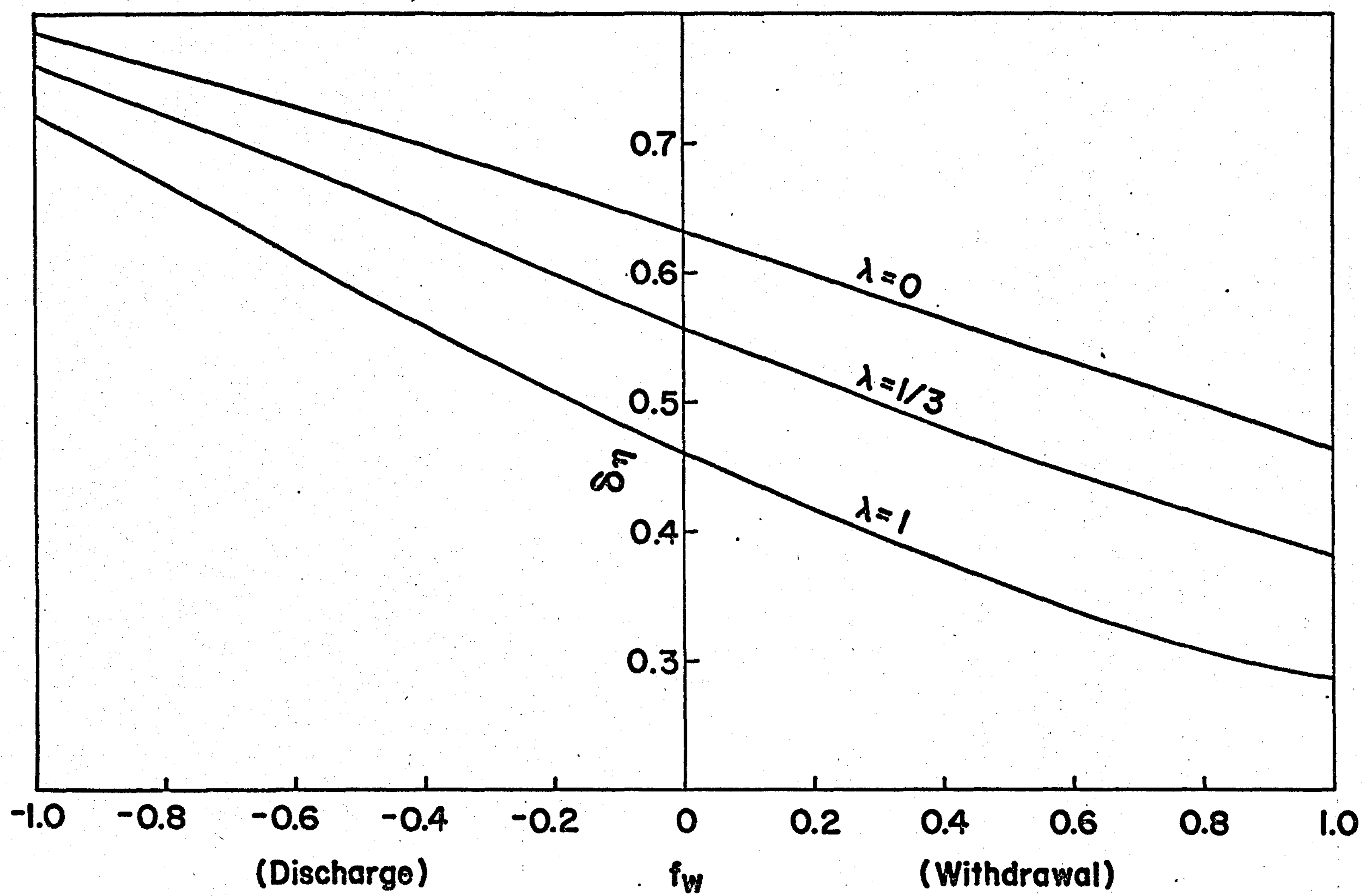

Fig. 5 Effect of Mass Transfer on Boundary Layer Thickness

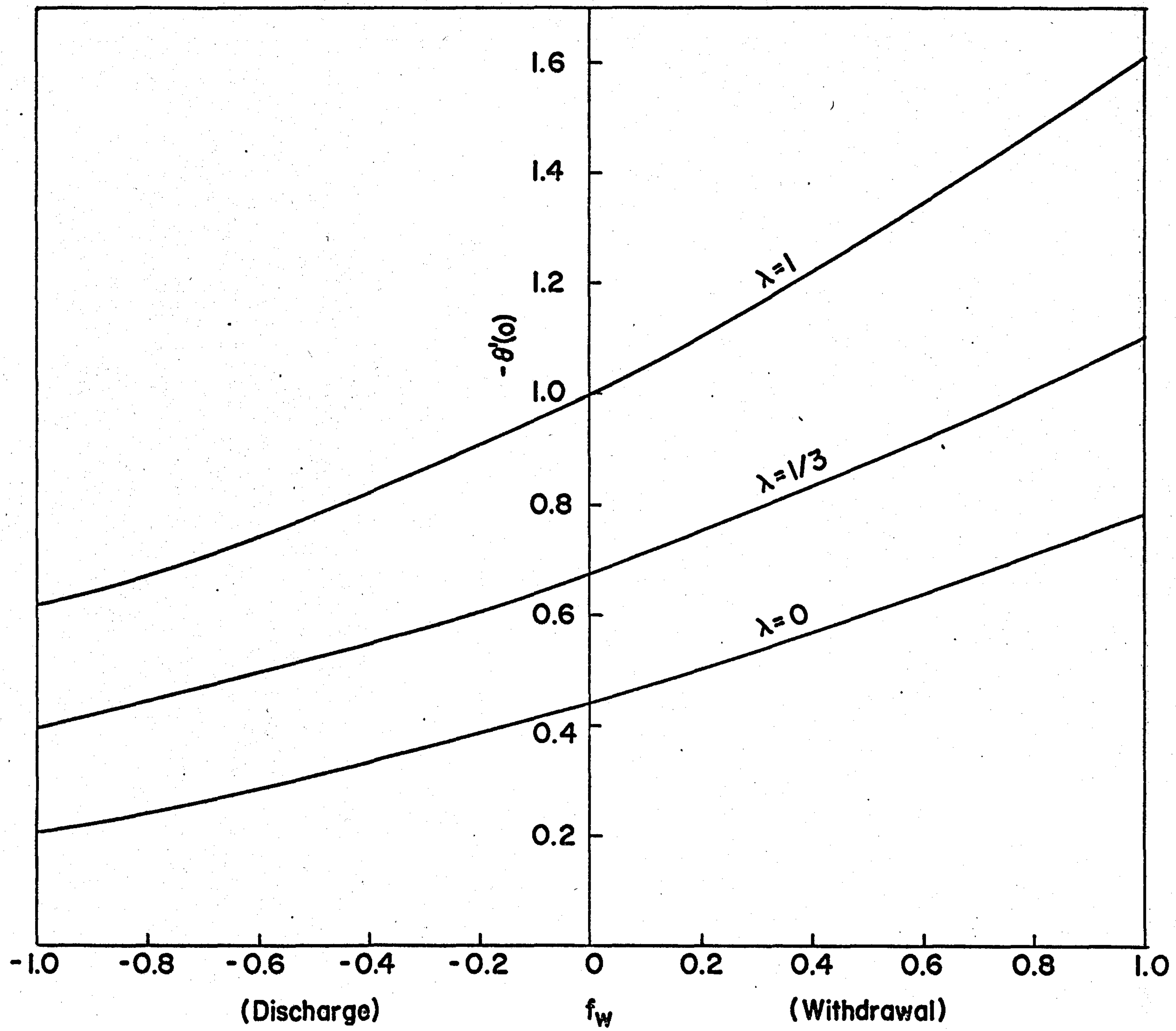

Fig. 6 Effect of Mass Transfer on Surface Heat Transfer Rate