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Problems in hyperbolic geometry

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PROBLEMS IN HYPERBOLIC GEOMETRY

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By

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DEDICATION

To my mother

ACKNOWLEDGEMENTS

I would like to thank the following people for helping me with this thesis: Marvin Ortel, Joel Weiner, Les Wilson, and most of all David Bleecker.

ABSTRACT

In this thesis, we discuss the proof that all convex polyhedral metrics can be realized in euclidean and hyperbolic 3–space. This result is accredited to A.D. Alexandrov and is fundamental in modern synthetic differential geometry. Nevertheless, gaps appear in currently acknowledged proofs:

(1) It is necessary to prove that strictly convex metrics with 4 real vertices can be realized.

(2) It must be shown that, within manifolds of convex polyhedra in E^3 or H^3, there exist submanifolds of degenerate polyhedra which are "thin" when mapped into manifolds of (abstract) strictly convex metrics.

In this thesis we prove these statements.

The remainder of the thesis is devoted to general hyperbolic geometry with emphasis on the synthetic point of view. We first construct horocyclic coordinates and use these to derive the Poincare model for the hyperbolic plane. Then we compute useful formulas for the curvature of a surface, and use these formulas to study C^2 surfaces in H^3, infinitesimal deformations of the horosphere, and curves of constant curvature in H^2. Finally, we also prove that certain surfaces of rotation in E^3 isometrically imbed in H^3. These results, some of which are new, provide a background for synthetic methods underlying the theorem of Alexandrov.

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LIST OF SYMBOLS

$\overleftrightarrow{AB}$: The line through the points A and B.

$\overline{AB}$: The line segment through A and B.

H(A,B): The horocycle through A and B.

H^2: The hyperbolic plane.

E^2: The Euclidean plane.

$\overrightarrow{AB}$: The ray determined by the points A and B.

∇: Covariant differentiation.

d_u: The partial derivative with respect to u.

d_v: The partial derivative with respect to v.

D_x: The tangent vector to the curve in which only the x coordinate varies.

D_y: The tangent vector to the curve in which only the y coordinate varies.

D_z: The tangent vector to the curve in which only the z coordinate varies.

$|V|$: The norm of vector V.

E: $D_u \cdot D_u$ in coordinates (u,v).

F: $D_u \cdot D_v$ in coordinates (u,v).

G: $D_v \cdot D_v$ in coordinates (u,v).

B: The second fundamental form.

[X,Y]: The vector field X bracketed with the vector field Y.

CHAPTER 1: BASIC FACTS OF HYPERBOLIC GEOMETRY

In this chapter we compare the axiom systems of hyperbolic and euclidean geometry, discuss the consistency and completeness of these systems, and record the basic facts of hyperbolic geometry which will be used throughout this thesis.

The axiom which separates hyperbolic plane geometry from euclidean plane geometry is the the parallel postulate: Through a point P not on a given line l, there exactly one line which passes through P that does not intersect l. Let ψ_p denote the parallel postulate and let Ψ_E and Ψ_H denote the set of axioms for euclidean and hyperbolic geometry (see Appendix 2). Then $\Psi_E - \{\psi_p\} = \Psi_H - \{\sim\psi_p\}$, is the set of axioms for neutral geometry (here, $\sim\psi_p$ is the formal negation of ψ_p). Thus, euclidean and hyperbolic geometry share the 14 axioms of neutral geometry, which accounts for their similar structure. Moreover, models of the hyperbolic plane and the euclidean plane are models of neutral geometry.

Now, the issue of consistency arises. A set of statements Ψ is said to be consistent if it is impossible to derive a contradiction from Ψ, and normally consistency is proven by constructing a model within another system. The usual practice is to assume that euclidean geometry is consistent, and within euclidean construct the Poincare model for hyperbolic geometry, hence proving that the axioms of hyperbolic geometry are consistent. It is less well known that one can also assume the consistency of the hyperbolic plane and deduce the consistency of euclidean geometry. This is simple if one assumes the consistency of three–dimensional hyperbolic geometry, since there is a model of E^2 lying in H^3. (later this will be explained in more detail).

An alternate approach, to this problem is to begin with the assumption that set theory is consistent. The real numbers are constructed within this set theory. From the real numbers, one constructs $\mathbb{R} \times \mathbb{R}$. Now place the metrics $ds^2 = dx^2 + dy^2$ or $ds^2 = dx^2 + e^{-2x}dy^2$ on $\mathbb{R} \times \mathbb{R}$, along with tangent spaces at each point (a tangent space can be identified with $\mathbb{R} \times \mathbb{R}$). Use the first metric for euclidean geometry and the second metric for hyperbolic geometry (the second metric will be justified later). Hence, the question of consistency of the geometries can be reduced to that of set theory (or at least the existence of the real numbers).

Euclidean and hyperbolic geometry are categorical systems, that is in any model of set theory it is possible to find an isomorphism between any two models of euclidean geometry and between any two models of hyperbolic geometry. Moreover, these isomorphisms lies within the same set theoretical universe. In fact, the isomorphisms are derived from coordinate systems. In the euclidean case, one constructs cartesian coordinates; in the hyperbolic case one constructs horocyclic coordinates. (This shall be done in Chapter 2.) Hence, by pairing coordinates or ordered pairs of real numbers, we pair points on the different models of E^2 or H^2. This implies that the axiom systems are categorical relative to the given model of set theory.

An axiom system ϕ is complete if, given any statement ψ, then $\phi \cup \{\psi\}$ or $\phi \cup \{\sim\psi\}$ is consistent, but not both. It is well known that number theory is not complete. To prove the completeness of euclidean or hyperbolic geometry we argue as follows: Let M_1 and M_2 be two models of hyperbolic or euclidean geometry. Suppose there was a statement ψ which held in model M_1 and $\sim\psi$ held in M_2.

Let $i: M_1 \to M_2$ be the isomorphism which holds between the models. This isomorphism exists in the model of set theory in which we assume the two geometries live. Since M_1 and M_2 are isomorphic, ψ is true in M_1 if and only if the corresponding statement is true in M_2. This implies ψ and $\sim\psi$ hold in M_1 and M_2 which is a contradiction.

Neutral geometry is not categorical, since any model of euclidean or hyperbolic geometry is a model of neutral geometry. These models are not isomorphic, because the parallel postulate holds in euclidean geometry and the negation of the parallel postulate holds in hyperbolic geometry. Any theorem which is true in euclidean geometry and whose proof does not use the parallel postulate is also true in hyperbolic geometry. For example, the following theorem is of this type: In any triangle, the exterior angle of any vertex is greater then either alternate interior angle (see [Gans]). A consequence of this theorem is:

Theorem. Suppose P is a point not on the line l. Then there exists a line m such that $P \in m$ and $m \cap l = \phi$.

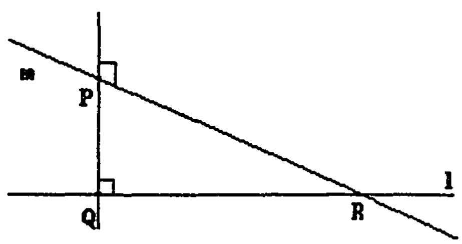


Figure 1.1

Proof. Drop a perpendicular from P to l and call the base of this perpendicular Q (Figure 1.1). Now erect a perpendicular from P to the line $\overleftrightarrow{PQ}$, call this line m. Then $m \cap l = \phi$, for otherwise we would have the situation in Figure 1.1. This is a contradiction, since the exterior angle of triangle PQR is not greater than one of its alternate interior angles. □

The existence of a perpendicular to a given line through a given point, or the existence of a perpendicular from a point on a line, is also a theorem of neutral geometry. If one reflects upon the proofs of these facts from euclidean geometry, one will realize that the parallel postulate was never utilized. Therefore, these are also theorems of neutral geometry.

Let P be a point which is not on a given line l. Then $\sim\psi_p$ implies that there are infinitely many lines through P which do not intersect l.

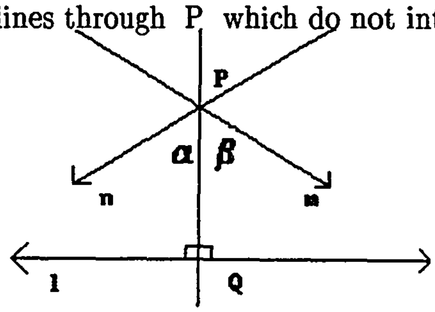


Figure 1.2

The lines through P are partitioned into three distinct classes. Some intersect l, and some do not intersect l. To obtain three classes we further partition the lines. Some lines which do not intersect l share a common perpendicular with l, and some do not. Those which do not are **boundary parallel** (or simply, **parallel**) to l in a given direction. The axiom of betweenness allows for the definition of direction on a given line. All one needs to do is define a ray $\overrightarrow{OP}$.

By definition,

$$\overrightarrow{OP} = \{ Q \in H^2, \text{ such that } Q = O, \text{ or } Q = P, \text{ or } O{\circ}Q{\circ}P, \text{ or } O{\circ}P{\circ}Q \}.$$

If A,B and C are points $A{\circ}B{\circ}C$ means B is between A and C. The lines m and n (Figure 1.2) are boundary parallel to l in direction δ and δ'. Drop a perpendicular from P to l with base point Q. Out of all the lines that do not intersect l, m and n make the smallest angle with the $\overleftrightarrow{PQ}$ when moving in a clockwise or counter clockwise direction. This is an intrinsic criterion for saying whether m or n is boundary parallel to l in direction δ or δ'. Call this angle α and β. The angles $\alpha = \beta < 90$. The lines through P that pass through α and β intersect l. The lines m and n are boundary parallels to l and the remaining lines are the non–intersecting lines relative to l. When speaking of parallel lines the direction is imperative. The betweeness axioms allow for the notion of direction.

Let l and m be directed lines. Define $l \approx m$ if and only if l is parallel to m in a given direction or $l = m$. This defines an equivalence relation on the set of lines in H^2, that is to say $\approx$ is reflexive, symmetric, and transitive. An equivalence class is called a point at infinity and is usually denoted by δ. Note that both m and n are parallel to l, but in different directions, hence they do not have to be parallel to each other.

If the given line l is fixed and P varies one has two families of boundary parallel lines. One of the families is in the direction of δ, the other in the direction δ', as in Figure 1.3.

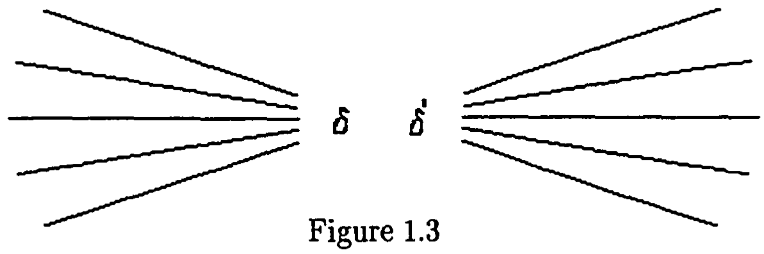


Figure 1.3

In Figure 1.4 the distance between points on boundary parallels m and l goes to zero as point P moves on m toward δ, and the distance goes to infinity as P moves in the opposite direction.

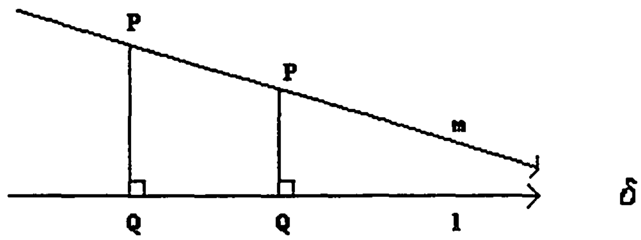


Figure 1.4

Assume l and m are non–intersecting lines, sharing a common perpendicular through A and B. As you move to the right or left of A, the distance between points on m and l gets arbitrarily large.

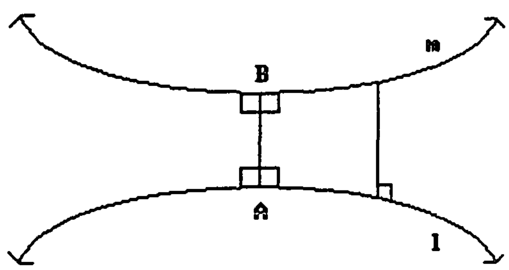


Figure 1.5

A horocycle is characterized by a locus of "corresponding points" with respect to δ. Points P and Q on two separate parallel lines in the direction δ are corresponding if and only if $\measuredangle PQ\delta \cong \measuredangle QP\delta$ or $P = Q$ (Figure 1.6). If δ is a fixed point at infinity write $P \approx Q$ if and only if P and Q are corresponding points. This defines an equivalence relation on the set of points in the hyperbolic plane.

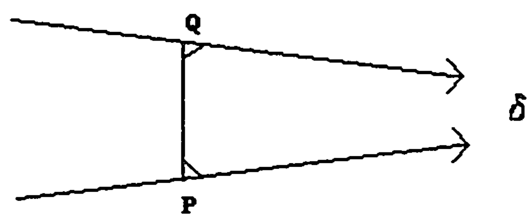


Figure 1.6

Given a point P and a point δ at infinity, this uniquely determines a line l. Suppose m is a line that also goes through δ, then there is a unique point Q_m on m that corresponds to P. The horocyle through P in the direction δ is the locus of all the points Q_m (see Figure 1.7) corresponding to P as m varies within the family of lines parallel to l. Hence a point $P \in H^2$ and a point δ at infinity determine a horocycle through P and the ideal point δ.

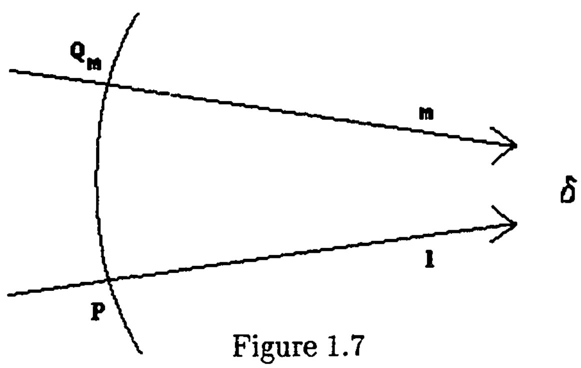


Figure 1.7

If P' is a point to the right or left of P on the line l it is possible to form another horocycle that will be to the right or left of the initial horocycle determined by P' and δ, as in Figure 1.8.

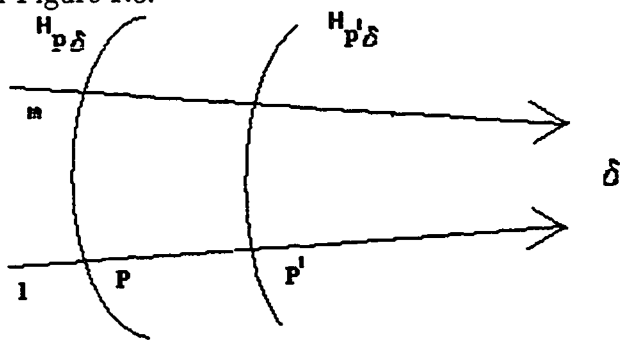


Figure 1.8

If a line n is perpendicular to l at P it is tangent to the horocycle through P. In Figure 1.9, the line n being tangent at P means $H_{p\delta} - \{P\}$ is contained in the half plane that does contain $\overrightarrow{P\delta}$. The horocycle is a curve with arclength (This is proved in Appendix 1). The distance between the pair of corresponding points P and P', and Q and Q', are equal.

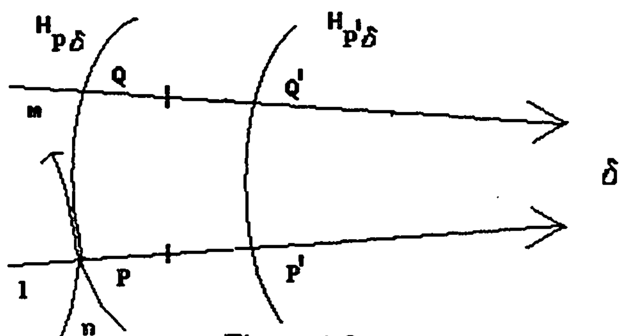


Figure 1.9

Let $\overline{AB}$ denote the chord or line segment determined by the points A and B, and $H(A,B)$ denote the horocycle through A and B in a given direction. Then $\overline{AB} \cong \overline{A'B'}$ if and only if $H(A,B) \cong H(A',B')$.

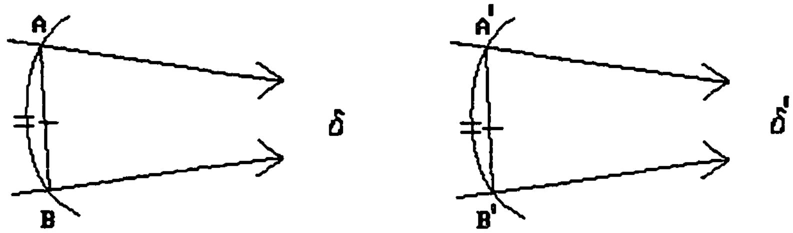


Figure 1.10

Suppose l and m are parallel lines in the direction δ, as in Figure 1.11. Let P be a point on m, Q the projection of P on l, d the distance between P and Q, and $\alpha = \measuredangle\, QP\delta$. Then

$$\alpha = 2\arctan(e^{-d/k}), \tag{1.1}$$

for some constant $k > 0$.

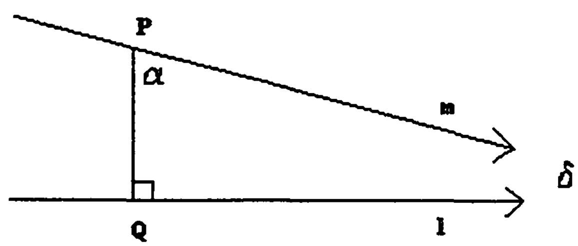


Figure 1.11

The following construction in Figure 1.12 is useful. It can be used in deriving the trigonometric formulas for hyperbolic geometry.

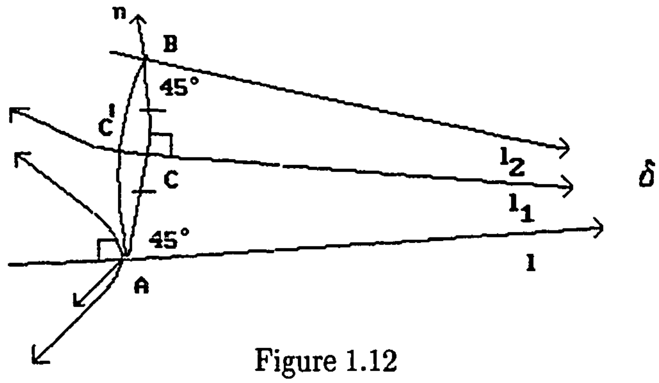


Figure 1.12

Suppose A is a point on line l, let m be perpendicular to l and suppose n is drawn at 45 degrees with respect to l. Let d be the distance which corresponds to 45 degrees in formula (1.1), so when a perpendicular is erected at the point C it will be parallel to l. Call this line l_1. Travel another distance d on the line n to the point B. Draw l_2 parallel to l_1. The angle formed by the lines n and l_2 will also be 45 degrees. Points A and B are corresponding points. Now draw the horocycle that connects the two points. Line l_1 will divide the horocycle arc H(A,B) into 2 congruent pieces at the point C′. The curves H(A,C′) and H(C′,B) are called K arcs. All K arcs are of equal length k (the same k as in formula (1.1)).

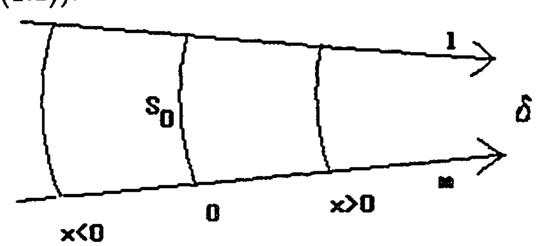


Figure 1.13

Assume that lines l and m are parallel in the direction δ, and m has coordinates as in Figure 1.13. If S_0 is the length of a horocyclic arc at $x = 0$, then

$$S_x = S_0 e^{-x/k} \qquad k > 0 \tag{1.2}$$

where S_x is the length of the horocylic arc at the point with coordinate x.

The above formula relates the arclength S_x to the distance x. When $x > 0$, S_x is to the right of S_0 and if $x < 0$ S_x is to the left of S_0. This formula will be derived in Appendix 1. This formula is important since it is used in deriving the Poincare model of hyperbolic geometry from horocyclic coordinates.

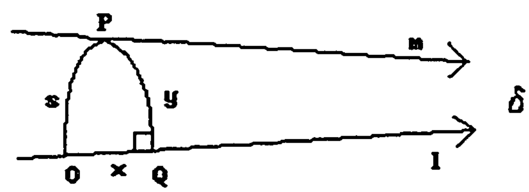


Figure 1.14

If $H(O,P)$ is an arc of a horocycle and Q is the projection of P on the line l which is parallel to m, then

$$e^{x/k} = \cosh(y/k) \qquad\qquad s/k = \sinh(y/k) \tag{1.3}$$

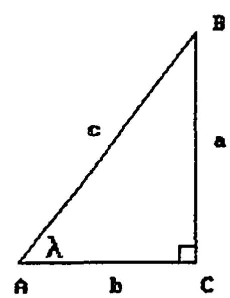


Figure 1.15

For the right triangle ABC in Figure 1.15, the following relationships hold.

$$\cosh(a/k) = \cosh(b/k) \cdot \cosh(c/k) - \sinh(b/k) \cdot \sinh(c/k) \cdot \cos\lambda \tag{1.4}$$

and

$$\sin\lambda = \sinh(a/k)/\sinh(c/k)\ . \tag{1.5}$$

CHAPTER 2: HOROCYCLIC COORDINATES

The goal of this chapter is to construct a global coordinate system for the hyperbolic plane and derive a formula for the distance between two points in terms of their coordinates.

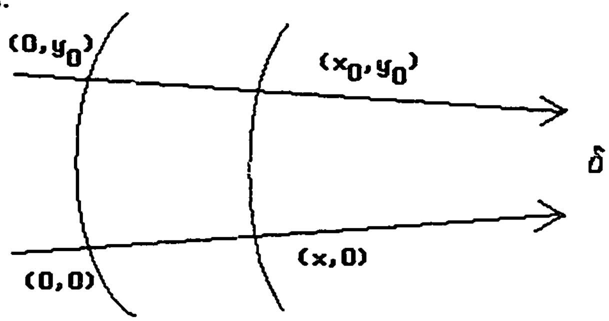


Figure 2.1

It is possible to place a natural global coordinate system on H^2. We will define a map $P: \mathbb{R}^2 \rightarrow H^2$ as follows. The curves $y =$ constant will be lines from a pencil of parallel lines in a given direction δ. The curves $x =$ constant will be horocycles which are orthogonal to the given pencil of lines. First choose the origin O and assign it coordinates $P(0,0)$. The curve $y = 0$ is the line through the origin that passes through the ideal point δ at infinity. The curve $x = 0$ is the unique horocycle determined by O and the point at infinity δ. The constant k from formula (1.2) determines a natural unit length. Assign $P(x,0)$ to the point P, a distance x from O, in the direction δ if $x > 0$ and in the opposite direction if $x < 0$. The line $y = 0$ divides the hyperbolic plane into two half planes H^+, and H^-. Assign $P(0,y)$ to the point A on the horocycle $x = 0$ whose arclength along this horocycle is y. The point $A \in H^+$, if $y > 0$, and $A \in H^-$, if $y < 0$. Let $y = y_0$ correspond to be the geodesic determined by the point $(0,y_0)$ and δ.

Let the curve $x = x_0$ be the horocycle determined by $(x_0,0)$ and δ. The curves $y = y_0$, and $x = x_0$ intersect in a point P. This point P is the image of $P(x_0,y_0)$. For convenience drop the P so $P(x_0,y_0)$ is identified with (x_0,y_0). From the results stated in the previous section the map P is one to one and onto.

Now we will find a formula for the distance along the $\alpha(t) = (x,t)$ $y_1 < t < y_2$. This horocyclic arc connects $A = (x,y_1)$ and $B = (x,y_2)$ (Figure 2.2). If $x = 0$ then since we are assuming the horocycle $y = 0$ is arclength, it follows that $|H(A,B)| = |y_2 - y_1|$.

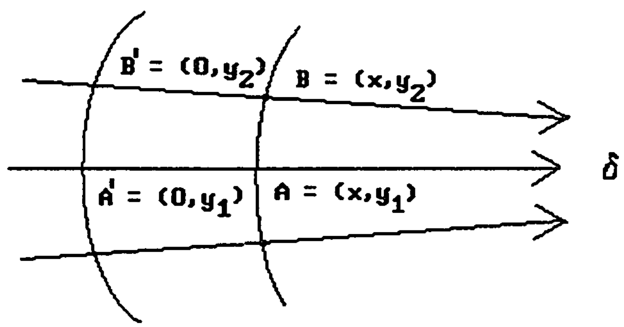


Figure 2.2

If $x \neq 0$ from formula (1.2) it follows that

$$|H(A,B)| = |y_2 - y_1| \cdot e^{-x/k}. \tag{2.1}$$

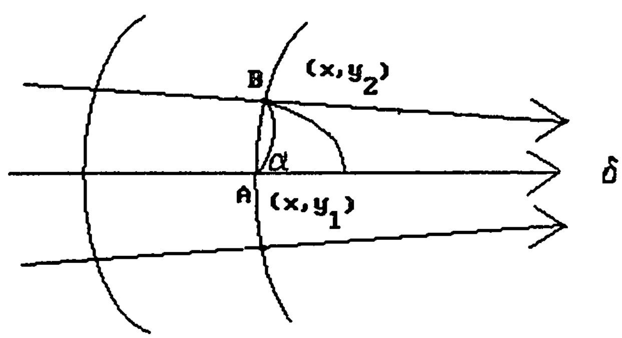


Figure 2.3

From formula (1.3) we derive a formula for $d(A,B)$ with $A = (x,y_1)$, and $B = (x,y_2)$ (Figure 2.3). Let $A' = (0,y_1)$ and $B' = (0,y_2)$. Draw a line connecting C and C', the midpoints of $\overline{AB}$ and $\overline{A'B'}$, and call this line l. The line l is in the pencil of lines $y =$ constant and perpendicular to $\overleftrightarrow{AB}$ and $\overleftrightarrow{A'B'}$. The midpoints D and D' of the curves $H(A,B)$ and $H(A',B')$ lie on l.

$$H(D,B) = H(D',B')\cdot e^{-x/k} = 1/2|\Delta y|e^{-x/k}, \text{ and}$$

$$H(D,B) = k\cdot \sinh(|\overline{BC}|/k).$$

Thus

$$|\Delta y|e^{-x/k}/2k = \sinh(|\overline{BC}|/k),$$

and

$$\overline{AB}/k = 2\overline{BC}/k = 2\sinh^{-1}a, \quad \text{where } a = |\Delta y|e^{-x/k}/2k.$$

Using the identity

$$\sinh 2a = 2\sinh a\cdot\cosh a\ ,$$

we have

$$\sinh(2\sinh^{-1}a) = 2a(1 + a^2)^{1/2}.$$

Hence

$$\sinh(\overline{AB}/k) = 2a(1 + a^2)^{1/2}.$$

Substituting for a we obtain

$$\sinh(\overline{AB}/k) = (|\Delta y|e^{-x/k}/k)(1 + (|\Delta y|e^{-x/k}/2k)^2)^{1/2}. \tag{2.2}$$

Let A and B be corresponding points with respect to δ, and l the line determined by A and B. Now we find a formula for $\measuredangle AB\delta = \measuredangle BA\delta$.

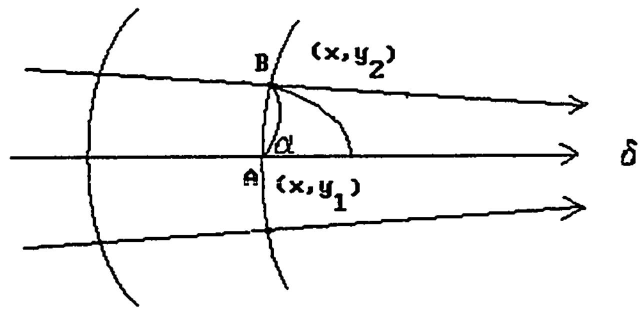


Figure 2.4

Let E be the projection of B on $y = y_1$. From the formula (1.5)

$$\sin\alpha = \sinh(\overline{BE}/k)/\sinh(\overline{AB}/k).$$

The formula for the $\sinh(\overline{AB}/k)$ is given by formula (2.2), and from formula (2.1)

$$H(A,B) = k\cdot\sinh(\overline{BE}/k).$$

From formula (1.2)

$$\sinh(\overline{BE}/k) = |\Delta y|e^{-x/k}/k,$$

and hence

$$\sin\alpha = (1 + (|\Delta y|e^{-x/k}/2k)^2)^{-1/2}. \qquad (2.3)$$

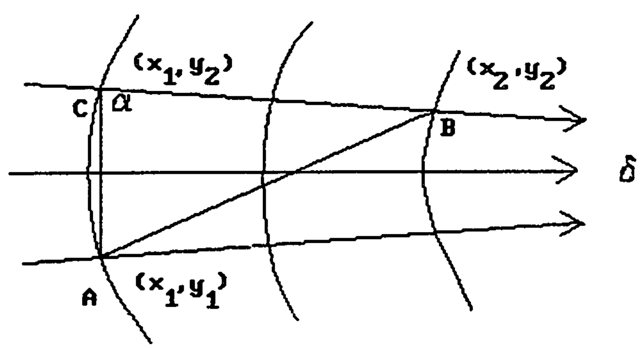


Figure 2.5

Now we derive the distance formula for two points A, B with coordinates (x_1,y_1) and (x_2,y_2) (Figure 2.5). Let C be the point with coordinates (x_1,y_2). Without loss of generality suppose $x_1 < x_2$ and $y_1 < y_2$. From formula (1.4)

$$\cosh(\overline{AB}/k) = \cosh(\Delta x/k)\cosh(\overline{AC}/k) - \sinh(\overline{AC}/k)\cdot\sinh(\Delta x/k)\cdot\cos\alpha.$$

Let $a = (|\Delta y|e^{-x_1/k})/2k$. From formula (2.2)

$$\sinh(\overline{AC}/k) = 2a(1 + a^2)^{1/2}.$$

Then

$$\cosh^2(\overline{AC}/k) = 1 + \sinh^2(\overline{AC}/k) = 1 + 4a^2 + 4a^4,$$

and so

$$\cosh(\overline{AC}/k) = 1 + 2a^2.$$

Also, from formula (2.3),

$$\sin\alpha = (1 + a^2)^{-1/2}.$$

Therefore

$$\cos^2\alpha = a^2/(1 + a^2).$$

Since $\alpha < 90$

$$\cos\alpha = a/(1 + a^2)^{1/2}.$$

Substituting all this into the previous equation for $\cosh(\overline{AB}/k)$, we obtain

$$\begin{aligned}\cosh(\overline{AB}/k) &= \cosh(\Delta x/k)\cdot(1 + 2a^2) - (2a(1 + a^2)^{1/2})\sinh(\Delta x/k)\cdot(a/(1 + a^2)^{1/2})\\ &= \cosh(\Delta x/k)\cdot(1 + 2a^2) - 2a^2\sinh(x/k)\\ &= \cosh(\Delta x/k) + 2a^2e^{-\Delta x/k}.\end{aligned}$$

Since $a = |\Delta y|e^{-x_1/k}/2k$,

$$\cosh(\overline{AB}/k) = \cosh(\Delta x/k) + |\Delta y|^2 e^{-2x_1/k}/2k^2 \cdot e^{-\Delta x/k}$$

or

$$\cosh(\overline{AB}/k) = \cosh(\Delta x/k) + (|\Delta y|^2 e^{-x_1/k} \cdot e^{-x_2/k})/2k^2. \tag{2.4}$$

CHAPTER 3: THE POINCARE MODEL

From the horocyclic coordinate system it is not very difficult to derive the Poincare model for Hyperbolic geometry. First we give a heuristic proof to verify that the metric for the horocyclic coordinate system is

$$ds^2 = dx^2 + (e^{-x/k})^2 dy^2.$$

Assume the existence of a Riemannian metric for the hyperbolic plane in horocyclic coordinates. Define $\alpha\colon (t_0,t) \to H^2$ by $\alpha(t) = (x,y + t)$. The arclength along this curve from $\alpha(t_0)$ to $\alpha(t)$ is given by

$$s(t) = \int_{t_0}^{t} (\alpha'(s)\cdot\alpha'(s))^{1/2}\, ds.$$

The $\cdot$ occurs at the point $(x,y + s)$. From (2.1),

$$s(t) = e^{-x/k}(t - t_0).$$

Differentiating, we obtain

$$s'(t) = (\alpha'(s)\cdot\alpha'(s))^{1/2} = e^{-x/k}, \text{ or } D_y \cdot D_y = (e^{-x/k})^2.$$

Let $\alpha\colon (t_0,t) \to H^2$ be defined by $\alpha(t) = (x + t,y)$. This time $s(t) = t - t_0$. Differentiating, we obtain

$$s'(t) = (\alpha'(s)\cdot\alpha'(s))^{1/2} = 1, \quad \text{or} \quad D_x \cdot D_x = 1.$$

Since reflection in the line $y = y_0$ is an isometry, it follows that the family of horocycles and the family of parallel lines are mutually perpendicular to one another, and so $D_x \cdot D_y = 0$. Therefore the metric is

$$ds^2 = dx^2 + (e^{-x/k})^2 dy^2.$$

Given a metric $ds^2 = dx^2 + g(x)^2 dy^2$, the Gaussian curvature K is given by the formula $K(p) = -g''(x)/g(x)$ (this will be derived later). For the above metric it follows that $K(p) = -1/k^2$.

Now we will give a rigorous proof of the above. Suppose α: $(t_o,t_1) \to H^2$, $\alpha(t) = (x(t),y(t))$, and assume x and y are C^1. If H^2 has horocyclic coordinates, then the arclength $s(t)$ of α is given by

$$s(t) = \int_{t_o}^{t} (x'^2 + y'^2 e^{-2(x/k)}) \, dt.$$

Suppose $P = (x(t_o),y(t_o))$ and $Q = (x(t),y(t))$. Then $s(t)$ is the distance along α from P to Q, and

$$\frac{ds}{dt} = (x'^2 + y'^2 e^{-2(x/k)})^{1/2}.$$

To verify this last formula we shall use formula (2.4). Once this formula is verified the above integral formula immediately follows. Since $x(t)$ and $y(t)$ are C^1 by the mean value theorem $x(t+\Delta t) - x(t) = x'(\bar{t})\cdot\Delta t$, $t < \bar{t} < t+\Delta t$, and $y(t+\Delta t) - y(t) = y'(\tau)\cdot\Delta t$, $t < \tau < t+\Delta t$. Substituting this into formula (2.4) we obtain

$$\cosh(\Delta s/k) = \cosh(x'(\bar{t})\cdot\Delta t/k) + (y'(\tau)\cdot\Delta t/k)^2 e^{-2x/k}/2\cdot e^{-x'(\bar{t})\cdot\Delta t/k}.$$

Expanding both sides using Taylor series, we obtain

$$1 + \Delta s^2/2k^2 + \Delta s^4/4! + \dots$$

$$= 1 + (x'(\bar{t})\cdot\Delta t/k)^2/2 + \Delta t^3\cdot S + (y'(\tau)\cdot\Delta t/k)^2 e^{-2x/k}/2(1 + \Delta t\cdot S').$$

Here S and S' are convergent power series. Therefore,

$$(\Delta s/\Delta t)^2 + (\Delta s/\Delta t)^4/4!\cdot\Delta t^2 + \dots$$

$$= (x'(\bar{t})^2 + (y'(\tau))^2\, e^{-2x/k} + \Delta t\cdot S + (y'(\tau))^2 e^{-2x/k}\cdot\Delta t\cdot S'.$$

Hence

$$\lim_{\Delta t\to 0}\ (\Delta s/\Delta t)^2 = (x'(t)^2 + (y'(t))^2 e^{-x/k}, \text{ and so}$$

$$\frac{ds}{dt} = (x'^2 + y'^2 e^{-2(x/k)})^{1/2}.$$

Note that we have derived the result

$$ds^2 = dx^2 + (e^{-x/k})^2 dy^2 = dx^2 + (e^{-x/k}dy)^2.$$

This is an infinitesimal version of the theorem of Pythagoras. The length of the (infinitesimal) curve $y = c$ from (x_1,c) to (x_2,c) is dx, and the length of the curve $x = c$ from (c,y_1) to (c,y_2) is $e^{-c/k}dy$.

Now we will derive the Poincare model for H^2. Define

$$F\colon \{(x,y) \in \mathbb{R}^2 : y > 0\} \to [\mathbb{R}^2,\ ds^2 = dx^2 + (e^{-x/k})^2 dy^2]$$

by

$$F(x,y) = (k\cdot \ln y, kx)\ , \quad k > 0.$$

The pair $[\mathbb{R}^2,\ ds^2 = dx^2 + (e^{-x/k})^2 dy^2]$ denotes H^2 with horocyclic coordinates. It is easy to verify F is a one–to–one and onto. We pull back this metric and apply " · " at the appropriate point. Then

$$
\begin{aligned}
F_*(D_x)\cdot F_*(D_x) &= F(x+t,y)'|_0 \cdot F(x+t,y)'|_0 \\
&= (k\ln y, k\cdot(x+t))'|_0 \cdot (k\ln y, k\cdot(x+t))|_0' = kD_y \cdot kD_y \\
&= k^2(e^{-k\ln y/k})^2 = 1/(y/k)^2 .
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
F_*(D_y)\cdot F_*(D_y) &= F(x,y+t)'|_0 \cdot F(x,y+t)'|_0 \\
&= (k\ln(y+t),kx)'|_0 \cdot (k\ln(y+t),kx)'|_0 \\
&= k/yD_x \cdot k/yD_x = 1/(y/k)^2.
\end{aligned}
$$

Also

$$F(x+t,y)'|_0 \cdot F(x,y+t)'|_0 = k/yD_x \cdot kD_y = 0.$$

Hence, in a natural way, on the upper half plane, we have derived the metric

$$ds^2 = (dx^2 + dy^2)/(y/k)^2.$$

Now that we have this metric on the upper half plane, we can use it to verify the axioms of H^2 [Greenberg]. The point of this derivation was to motivate the metric for the upper half plane model of H^2.

CHAPTER 4: USEFUL CALCULATIONS

In this chapter some useful formulas will be derived, and Christoffel symbols are computed in various coordinate systems. The formulas and Christoffel symbols will be used in the chapters to follow.

Theorem 4.1. Let M be a two–dimensional manifold, with local coordinates (u,v). Suppose the metric on M is given by $ds^2 = a_1^2 du^2 + a_2^2 dv^2$, where a_1 and a_2 are C^2 functions. Then

$$K = -\frac{1}{a_1 a_2}\left(\partial_v[(\partial_v a_1)/a_2] + \partial_u[(\partial_u a_2)/a_1 \right).$$

Here ∂_u and ∂_v denote the partial derivatives with respect to u and v.

Proof. The Gaussian curvature K is uniquely determined by

$$d\omega_1^2 = -K\omega^1 \wedge \omega^2$$

where the one–forms ω^1 and ω^2 are defined by

$$\omega^1 = a_1 du \qquad\qquad \omega^2 = a_2 dv$$

and the Riemannian connection one–forms ω_i^j are uniquely determined by

$$d\omega^i = \omega^j \wedge \omega_j^i \quad \text{and} \quad \omega_i^j = -\omega_j^i$$

or equivalently

$$[d\omega^1\ d\omega^2] = [\omega^1\ \omega^2] \cdot \begin{bmatrix} 0 & \omega_1^2 \\ \omega_2^1 & 0 \end{bmatrix}. \tag{4.1}$$

We have $d\omega^1 = -\partial_v a_1 du \wedge dv$ and $d\omega^2 = \partial_u a_2 du \wedge dv$, and so (4.1) becomes

$$[-\partial_v a_1 du \wedge dv \quad \partial_u a_2 du \wedge dv] = [a_1 du \quad a_2 dv] \wedge \begin{bmatrix} 0 & \omega_1^2 \\ -\omega_1^2 & 0 \end{bmatrix}.$$

It is now clear that

$$\omega_1^2 = -(\partial_v a_1/a_2)du + (\partial_u a_2/a_1)dv.$$

Now

$$\begin{aligned} d\omega_2^1 &= d(-(\partial_v a_1/a_2))\, du + d(\partial_u a_2/a_1)\, dv \\ &= \partial_v(\partial_v a_1/a_2) du \wedge dv + \partial_u(\partial_u a_2/a_1)\, du \wedge dv \\ &= 1/(a_1 a_2)\, (\, \partial_v(\partial_v a_1/a_2) + \partial_u(\partial_u a_2/a_1)\,)\, \omega^1 \wedge \omega^2 \\ &= -K\, d\omega^1 \wedge d\omega^2. \end{aligned}$$

Therefore,

$$K = -\frac{1}{a_1 a_2}\,(\, \partial_v(\partial_v a_1/a_2) + \partial_u(\partial_u a_2/a_1)\,). \quad \square$$

Lemma. Let M be a two–dimensional manifold, with local coordinates (u,v) and metric $ds^2 = du^2 + f(u)^2 dv^2$. Also assume f is a C^2 function. Then

$$K = -f''/f.$$

Proof. Apply the above theorem. Set $a_1 = 1$, and $a_2 = f(u)$. $\square$

Theorem 4.2. Let M be a C^2 surface in H^3. If $p \in M$ then $K(p) = \det B - 1$, where B is the second fundamental form of S at p.

Proof. Suppose E_1, E_2, E_3, ω^1, ω^2, and ω^3 form an adapted orthonormal frame and coframe of M, where $\omega^i(E_j) = \delta_{ij}$, with E_3 orthogonal to M. The connection one–forms ω_j^i are uniquely determined by

$$d\omega^i = \omega^j \wedge \omega_j^i \tag{4.2}$$

and

$$\omega_j^k = -\omega_k^j \, . \tag{4.3}$$

The curvature two–forms are determined by [Boothby, p.386]

$$\Omega_i^j = \tfrac{1}{2} R_{ikl}^j \omega^k \wedge \omega^l$$

Using $g_{hj} = E_h \cdot E_j = \delta_{hj}$, we have

$$R_{ijkl} = R_{ikl}^h g_{hj} = R_{ikl}^h \delta_{hj} = R_{ikl}^j .$$

Since H^3 is a space of constant curvature -1 [Boothby, p. 382],

$$R_{ijkl} = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk} .$$

Thus,

$$\begin{aligned}
\Omega_i^j &= \tfrac{1}{2} R_{ijkl} \omega^k \wedge \omega^l = \tfrac{1}{2} (\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) \omega^k \wedge \omega^l \\
&= \tfrac{1}{2} \delta_{ik}\delta_{jl} \omega^k \wedge \omega^l - \tfrac{1}{2} \delta_{il}\delta_{jk} \omega^k \wedge \omega^l \\
&= \tfrac{1}{2} \omega^i \wedge \omega^j - \tfrac{1}{2} \omega^j \wedge \omega^i = \omega^i \wedge \omega^j .
\end{aligned}$$

Hence,

$$\Omega_i^j = \omega^i \wedge \omega^j . \tag{4.4}$$

Now restrict the above ω_i^j to S. We have

$$d\omega_1^2 = -K \omega^1 \wedge \omega^2 . \tag{4.5}$$

Also [Boothby, p. 386]

$$d\omega_1^2 \;=\; \Omega_1^2 + \omega_1^k \wedge \omega_k^2 \;=\; \Omega_1^2 + \omega_1^3 \wedge \omega_3^2 \;=\; \Omega_1^2 + \omega_2^3 \wedge \omega_1^3.$$

Let $\omega_i^3 = \omega_{ik}^3 \omega^k$. Note that $\omega^3 = 0$ when ω^3 is restricted to S. So we have

$$0 \;=\; d\omega^3 \quad \text{and} \quad d\omega^3 \;=\; \omega^j \wedge \omega_{jk}^3 \omega^k = \omega_{jk}^3 \omega^j \wedge \omega^k.$$

Therefore,

$$\omega_{jk}^3 = \omega_{kj}^3.$$

Let ∇ be the Levi Civita connection on H^3. The second fundamental form $B(X,Y) \;=\; (\nabla_X Y)\cdot E_3$. Let $[B(E_i,E_j)]$ be the matrix of $B(X,Y)$ with respect to E_1 and E_2. We have

$$B(E_i,E_j) \;=\; \nabla_{E_i} E_j \cdot E_3 = \omega_j^k(E_i)E^k \cdot E_3 \;=\; \omega_j^3(E_i) \;=\; \omega_{ji}^3 \;=\; \omega_{ij}^3.$$

Hence,
$$[B(E_i,E_j)] \;=\; [\omega_{ij}^3].$$

From before we have

$$\begin{aligned} d\omega_1^2 &= \Omega_1^2 + \omega_2^3 \wedge \omega_1^3 \;=\; \Omega_1^2 + \omega_{2k}^3 \omega^k \wedge \omega_{1j}^3 \omega^j \\ &= \omega^1 \wedge \omega^2 + \omega_{2k}^3 \omega_{1j}^3 \omega^k \wedge \omega^j \\ &= \omega^1 \wedge \omega^2 + \omega_{21}^3 \omega_{12}^3 \omega^1 \wedge \omega^2 - \omega_{22}^3 \omega_{11}^3 \omega^1 \wedge \omega^2 \\ &= (1 - \det(B))\omega^1 \wedge \omega^2. \end{aligned}$$

We know $d\omega_1^2 \;=\; -K\omega^1 \wedge \omega^2$, and so $K \;=\; \det B - 1$. □

Theorem 4.3. The metric $ds^2 = dx^2 + e^{-2x}(dy^2 + dz^2)$ has constant curvature -1. The vector field $E_1 = D_x$, $E_2 = e^x D_y$, and $E_3 = e^x D_z$ is an orthonormal frame. The Christoffel symbols Γ_{ij}^k $(1 \le i,j,k \le 3)$ with respect to E_1, E_2 and E_3

are given by

$$[\Gamma_{ij}^1] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad [\Gamma_{ij}^2] = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad [\Gamma_{ij}^3] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix},$$

and the Christoffel symbols $\overline{\Gamma}_{ij}^k$ $(1 \leq i,j,k \leq 3)$ with respect to D_x, D_y and D_z are

$$[\overline{\Gamma}_{ij}^1] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & e^{-2x} & 0 \\ 0 & 0 & e^{-2x} \end{bmatrix} \qquad [\overline{\Gamma}_{ij}^2] = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad [\overline{\Gamma}_{ij}^3] = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}.$$

Proof. Since $ds^2 = dx^2 + e^{-2x}(dy^2 + dz^2)$ it follows immediately that E_1, E_2 and E_3 are an orthogonal frame. Define $\omega^1 = dx$, $\omega^2 = e^{-x}dy$, $\omega^3 = e^{-x}dz$, and note that ω^1, ω^2 and ω^3 are the dual covectors of E_1, E_2 and E_3. The connection one–forms ω_i^j $(1 \leq i,j \leq 2)$ are uniquely determined by

$$d\omega^i = \omega^j \wedge \omega_j^i \tag{4.6}$$

and

$$\omega_i^j = -\omega_j^i\,. \tag{4.7}$$

We have $d\omega^1 = d(dx) = 0$, $d\omega^2 = d(e^{-x}dy) = -e^{-x}dx\wedge dy = -\omega^1\wedge\omega^2$ and $d\omega^3 = d(e^{-x}dz) = -e^{-x}dx\wedge dz = -\omega^1\wedge\omega^3$. Notice that

$$[d\omega^1\ d\omega^2\ d\omega^3] = [0\ \ -\omega^1\wedge\omega^2\ \ -\omega^1\wedge\omega^3] = [\omega^1\ \ \omega^2\ \ \omega^3] \wedge \begin{bmatrix} 0 & -\omega^2 & -\omega^3 \\ \omega^2 & 0 & 0 \\ \omega^3 & 0 & 0 \end{bmatrix}.$$

Therefore,

$$\omega = [\omega_i^j] = \begin{bmatrix} 0 & -\omega^2 & -\omega^3 \\ \omega^2 & 0 & 0 \\ \omega^3 & 0 & 0 \end{bmatrix}.$$

Since $\Omega = d\omega - \omega\wedge\omega$,

$$\Omega = \begin{bmatrix} 0 & \omega^1\wedge\omega^2 & \omega^1\wedge\omega^3 \\ -\omega^1\wedge\omega^2 & 0 & 0 \\ -\omega^1\wedge\omega^3 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & -\omega^2 & -\omega^3 \\ \omega^2 & 0 & 0 \\ \omega^3 & 0 & 0 \end{bmatrix} \wedge \begin{bmatrix} 0 & -\omega^2 & -\omega^3 \\ \omega^2 & 0 & 0 \\ \omega^3 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \omega^1\wedge\omega^2 & \omega^1\wedge\omega^3 \\ \omega^2\wedge\omega^1 & 0 & \omega^2\wedge\omega^3 \\ \omega^3\wedge\omega^1 & \omega^3\wedge\omega^2 & 0 \end{bmatrix}.$$

Hence,

$$\Omega^{ij} = (d\omega - \omega\wedge\omega)^{ij} = \omega^i\wedge\omega^j.$$

Thus, $\mathbb{R}^3$ with the given metric has constant curvature -1 [Boothby, p. 399]. To calculate $(\Gamma_{ij}^{\ k})$ with respect to E_i, we note that

$$0 = \omega_1^1 = \Gamma_{i1}^{\ 1}\omega^i \Rightarrow \Gamma_{11}^{\ 1} = \Gamma_{21}^{\ 1} = \Gamma_{31}^{\ 1} = 0,$$

$$\omega^2 = \omega_2^1 = \Gamma_{i2}^{\ 1}\omega^i \Rightarrow \Gamma_{12}^{\ 1} = \Gamma_{32}^{\ 1} = 0 \text{ and } \Gamma_{22}^{\ 1} = 1$$

$$\omega^3 = \omega_3^1 = \Gamma_{i3}^{\ 1}\omega^i \Rightarrow \Gamma_{13}^{\ 1} = \Gamma_{23}^{\ 1} = 0 \text{ and } \Gamma_{33}^{\ 1} = 1.$$

Finishing the calculations, we obtain

$$[\Gamma_{ij}^{\ 1}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad [\Gamma_{ij}^{\ 2}] = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad [\Gamma_{ij}^{\ 3}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}.$$

Now we compute $\overline{\Gamma}_{ij}^{\ k}$ with respect to D_x, D_y and D_z. For instance,

$$\nabla_{D_y} D_y = \nabla_{e^{-x}E_2} e^{-x}E_2 = e^{-2x}\nabla_{E_2} E_2 = e^{-2x}\Gamma_{22}^{\ k} E_k = e^{-2x}E_1 = e^{-2x}D_x.$$

Hence $\overline{\Gamma}_{22}^{1} = e^{-x}$, $\overline{\Gamma}_{22}^{2} = 0$ and $\overline{\Gamma}_{22}^{3} = 0$. Finishing these calculations, we have

$$[\overline{\Gamma}_{ij}^{1}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & e^{-2x} & 0 \\ 0 & 0 & e^{-2x} \end{bmatrix} \qquad [\overline{\Gamma}_{ij}^{2}] = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad [\overline{\Gamma}_{ij}^{3}] = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}. \ \square$$

Theorem 4.4. The metric $ds^2 = dx^2 + e^{-2x}dy^2$ has constant curvature -1. The vector fields $E_1 = D_x$ and $E_2 = e^x \cdot D_x$ form an orthonormal frame. The Γ_{ij}^{k} $(1 \le i,j,k \le 2)$ with respect to E_1, E_2 are given by

$$[\Gamma_{ij}^{1}] = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \qquad [\Gamma_{ij}^{2}] = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix},$$

and the $\overline{\Gamma}_{ij}^{k}$ with respect to D_x and D_y are given by

$$[\overline{\Gamma}_{ij}^{1}] = \begin{bmatrix} 0 & 0 \\ 0 & e^{-2x} \end{bmatrix} \quad \text{and} \quad [\overline{\Gamma}_{ij}^{2}] = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}.$$

Proof. Proceed as in the previous case, or just notice that this is the metric for the totally geodesic submanifold $z = 0$ of Theorem 4.3. $\square$

Lemma. Let $\alpha(t) = (c,t)$ and $\beta(t) = (t,ce^t)$, $c \in \mathbb{R}$. Suppose α' and β' are unit vector fields along these curves and that $\mathbb{R}^2$ has the metric $ds^2 = dx^2 + e^{-2x}dy^2$. Then $|\nabla_{\alpha'}\alpha'| = 1$ and $|\nabla_{\beta'}\beta'| = c/(1 + c^2)^{1/2}$. The curve β is also equidistant from $y = 0$.

Proof. Since we want $|\alpha'| = 1$, $\alpha' = E_2$, and $\nabla_{\alpha'}\alpha' = \nabla_{E_2}E_2 = \Gamma_{22}^{k}E_K = E_1$.

Therefore $|\nabla_{\alpha'}\alpha'| = 1$. Since $|\beta'| = 1$, $\beta' = 1/(1 + c^2)^{1/2}(E_1 + cE_2)$,

$$\begin{aligned}\nabla_{\beta'}\beta' &= 1/(1 + c^2)(\nabla_{E_1+cE_2}(E_1 + cE_2)\\ &= 1/(1 + c^2)(\Gamma_{11}^{\ k}E_K + c\Gamma_{21}^{\ k}E_K + c\Gamma_{12}^{\ k}E_K + c^2\Gamma_{22}^{\ k}E_K)\\ &= 1/(1 + c^2)(c^2E_1 - cE_2).\end{aligned}$$

Hence $|\nabla_{\beta'}\beta'| = c/(1 + c^2)^{1/2}$. From formula (2.1) the horocyclic arc from $(t,0)$ to (t,ce^t) has arclength c, and by formula (1.3) the distance from (t,ce^t) to the line $y = 0$ is $\sinh^{-1}c\ \forall t$. □

Theorem 4.5. The the surface $x(u,v) = (u,v,ce^u)$, c a constant, has constant curvature $-1/(1+c^2)$, in $\mathbb{R}^3$ with the metric $ds^2 = dx^2 + e^{-2x}(dy^2 + dz^2)$. This is H^3 with horocyclic coordinates.

Proof. Define

$$D_u = x_u(u,v) = D_x + ce^uD_z = E_1 + cE_3 \quad\text{and}\quad D_v = x_v(u,v) = D_y = e^{-u}E_2.$$

We have

$$D_u \cdot D_u = 1 + c^2 \quad\text{and}\quad D_v \cdot D_v = e^{-2u}.$$

Define
$$\overline{E}_1 = (1 + c^2)^{-1/2}D_u \quad\text{and}\quad \overline{E}_2 = e^uD_v.$$

The vectors $\overline{E}_1$ and $\overline{E}_2$ form an orthonormal frame on M with dual covectors $\theta_1 = (1 + c^2)^{1/2}du$ and $\theta_2 = e^{-u}dv$. One can see that the metric on M is given by $ds^2 = (1 + c^2)du^2 + e^{-2u}dv^2$. By the lemma to Theorem 4.1,

$$K = -((e^{-u})'(1 + c^2)^{-1/2})' \cdot (1 + c^2)^{-1/2}\, e^u = -1/(1 + c^2). \ \square$$

Theorem 4.5 shows H^3 has a foliation into two–dimensional surfaces of constant negative curvature between 0 and -1. Namely the surface $x(u,v) = (u,v,ce^u)$, c a constant, has curvature $-1/(1+c^2)$. As $c \to 0$ the curvature goes to -1 and as $c \to \infty$ the curvature goes to 0. Also since H^3 has the metric $ds^2 = dx^2 + e^{-2x}(dy^2 + dz^2)$ the surface $x(u,v) = (c,u,v)$ has curvature 0. Hence H^3 can be foliated into surfaces of curvature 0.

Theorem 4.6. The metric $ds^2 = dr^2 + \sinh^2 r\, d\theta^2$, $r > 0$, has constant curvature -1. The vector field $E_1 = D_r$, $E_2 = 1/\sinh r\, D_\theta$ is an orthonormal frame, and Γ_{ij}^{k} $(1 \le i,j,k \le 2)$ with respect to E_1, E_2 are

$$[\Gamma_{ij}^1] = \begin{bmatrix} 0 & 0 \\ 0 & -\coth(r) \end{bmatrix} \qquad [\Gamma_{ij}^2] = \begin{bmatrix} 0 & 0 \\ -\coth(r) & 0 \end{bmatrix}.$$

Proof. Since $D_r \cdot D_r = 1$, $D_r \cdot D_\theta = 0$, and $D_\theta \cdot D_\theta = \sinh^2 r$ it follows immediately that E_1, and E_2 are an orthogonal frame. Define $\omega^1 = dr$, and $\omega^2 = \sinh r\, d\theta$, ω^1 and ω^2 are the covectors of E_1, and E_2. The connection one–forms ω_i^j $1 \le i, j \le 2$ are uniquely determined by

$$[d\omega^1\ d\omega^2] = [\,\omega^1\ \omega^2\,].\begin{bmatrix} \omega_1^1 & \omega_1^2 \\ \omega_2^1 & \omega_2^2 \end{bmatrix} \tag{4.8}$$

and

$$\omega_i^j = -\omega_j^i\,. \tag{4.9}$$

We have $d\omega^1 = d(dr) = 0$, $d\omega^2 = d(\sinh r\, d\theta) = \cosh r\, dr \wedge d\theta$. Therefore, (4.8) and (4.9) become

$$[0\ \cosh r \cdot dr \wedge d\theta] = [dr\ \sinh r \cdot d\theta] \wedge \begin{bmatrix} 0 & \omega_1^2 \\ -\omega_1^2 & 0 \end{bmatrix}.$$

It is easy to note

$$\omega_1^1 = 0,\ \ \omega_1^2 = \cosh r\, d\theta = \coth r\ \omega^2,\ \ \omega_2^1 = -\coth r\ \omega^2,\ \ \omega_2^2 = 0.$$

To calculate Γ_{ij}^{k}, we observe

$$\begin{aligned}
0 &= \omega_1^1 = \Gamma_{i1}^{1}\omega^i \Rightarrow \Gamma_{11}^{1} = \Gamma_{21}^{1} = 0,\\
\omega^2 &= \omega_2^1 = \Gamma_{i2}^{1}\omega^i \Rightarrow \Gamma_{12}^{1} = 0,\ \Gamma_{22}^{1} = -\coth r,\\
-\omega^2 &= \omega_1^2 = \Gamma_{i1}^{2}\omega^i \Rightarrow \Gamma_{11}^{2} = 0,\ \Gamma_{21}^{2} = \coth r,\ \text{and}\\
0 &= \omega_2^2 = \Gamma_{i2}^{2}\omega^i \Rightarrow \Gamma_{12}^{2} = 0,\ \Gamma_{22}^{2} = 0.
\end{aligned}$$

Therefore,

$$[\Gamma_{ij}^{1}] = \begin{bmatrix} 0 & 0 \\ 0 & -\coth r \end{bmatrix} \quad \text{and} \quad [\Gamma_{ij}^{2}] = \begin{bmatrix} 0 & 0 \\ \coth r & 0 \end{bmatrix}.$$

It is easy to check the curvature is -1. From the Lemma of Theorem 4.1, we have $K(p) = -(\sinh r)''/(\sinh r) = -1$. □

This metric is in polar coordinates of H^2. Fix a point p and a line l in the hyperbolic plane p. Give the point p coordinates $(0,0)$ and let (r,θ) represent the point in H^2 which is obtained by first moving to the geodesic which makes an angle θ with l, and then travel on this geodesic a directed length r.

From the metric $ds^2 = dr^2 + \sinh^2 r\, d\theta^2$ one can notice that the set of points $r = c$, c a constant, is equidistant from $(0,0)$ and that the circumference of a circle of radius r is $2\pi \cdot \sinh r$, or $s = \theta \cdot \sinh r$. Here s is the length of the arc on a circle of radius r subtending an angle θ. The curves $r = r_0$, and the curves

$\theta = \theta_0$ are mutually perpendicular at their points of intersection. This is true, since reflection in the curve $\theta = \theta_0$ is an isometry.

Lemma. Given a hyperbolic circle of radius r

$$\nabla_{\alpha'} \alpha' = -\coth r \; d/dr = -\coth r \; E_1 ,$$

where α' is a unit vector field on the circle of radius r.

Proof. This lemma follows easily from the above. Let $\alpha(t) = (r,t)$ be a parameterization of the circle. We are in polar coordinates for H^2. Then $\alpha'(t) = 0 \cdot D_r + D_\theta$, and so the unit tangent vector field is E_2. Thus

$$\nabla_{\alpha'} \alpha' = \nabla_{E_2} E_2 = \Gamma_{22}^k E_k = -\coth r \; E_1 . \; \square$$

Theorem 4.7. For a sphere S^2 in H^3 of radius r, $K(p) = 1/\sinh^2 r$, $p \in S^2$, and the second fundamental form $S\colon T_p(S^2) \to T_p(S^2)$ is defined by

$$S(X) = -\nabla_X E_3 = \coth r \cdot I ,$$

where I is the identity map.

Proof. Let E_1 and E_2 be the principal directions. It is well known that

$$S(E_1) = -\nabla_{E_1} E_3 = \lambda_1 E_1 \qquad \text{and} \qquad S(E_2) = -\nabla_{E_2} E_3 = \lambda_2 E_2 .$$

As in E^3, a plane perpendicular to a sphere of radius r at a point p intersects the sphere in a circle of radius r. Since $E_1 \cdot E_3 = 0$, from Theorem 4.6 we have

$$\nabla_{E_1} E_3 \cdot E_1 = -\nabla_{E_1} E_1 \cdot E_3 = -(-\coth r\, E_3) \cdot E_3 = \coth r.$$

Also

$$\nabla_{E_1} E_3 \cdot E_1 = \lambda_1 E_1 \cdot E_1 = \lambda_1 .$$

Therefore $\lambda_1 = \coth r$. By a similar argument $\lambda_2 = \coth r$. We know $K(p) = \det s_p - 1 = \coth^2 r - 1 = (\cosh^2 r - \sinh^2 r)/\sinh^2 r = 1/\sinh^2 r$. □

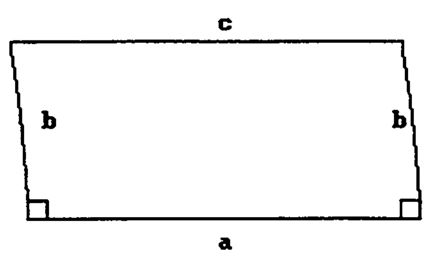


Figure 4.1

In the Figure 4.1, the curves labeled with a and b are geodesics of length a and b. The curve labeled with the c is a curve which is equidistant from line a of length c.

Theorem 4.8. In the above Figure 4.1,

$$c = a \cdot \cosh b.$$

Proof. To prove this theorem use horocyclic coordinates.

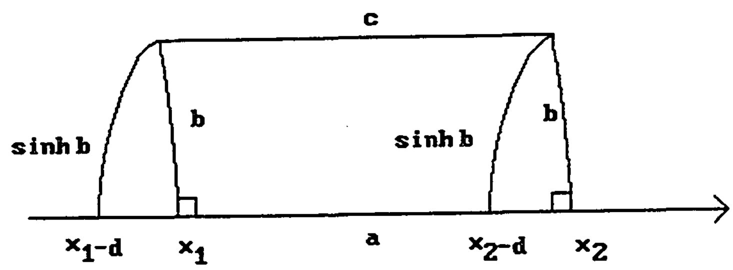


Figure 4.2

Parameterize the the geometric shape in Figure 4.2 as $\alpha(t) = (t, \sinh b \cdot e^t)$, $x_1 - d < t < x_2 - d$, $x_2 - x_1 = a$, $d = \ln(\cosh b)$. Differentiating, $\alpha' = D_x + \sinh b \cdot e^t D_y$, hence $\alpha' \cdot \alpha' = 1 + \sinh^2 b$. Then

$$c = \int_{x_1-d}^{x_2-d} (1 + \sinh^2 b)^{1/2} dt = a \cdot \cosh b. \quad \square$$

From this formula it is possible to form another coordinate system for H^2, namely Fermi coordinates. Let $P: \mathbb{R}^2 \to H^2$ be defined as follows. Fix a line l and a point O. Assign the coordinates $(0,0)$ to O. Suppose a unit length, a direction δ, and a right–hand orientation is given. The ordered pair (z,r) will be assigned to the unique point obtained by traveling a directed distance z along the line l, erecting a perpendicular, and traveling a directed distance r on this perpendicular. It is straightforward to show this map is one to one and onto. The equidistant curves $r = r_0$ are perpendicular to the lines $z = z_0$, because the equidistant curves remain invariant when reflected about the lines $z = z_0$, and this reflection is an isometry. Hence in the above coordinates $D_r \cdot D_r = 1$, $D_r \cdot D_z = 0$, and $D_z \cdot D_z = \cosh^2 r$. The last inequality follows from Theorem 4.8. Therefore H^2 can be viewed as $\mathbb{R}^2$ with the metric $ds^2 = dr^2 + \cosh^2 r\, dz^2$.

Theorem 4.9. When $\mathbb{R}^2$ has the metric $ds^2 = dr^2 + \cosh^2 r\, dz^2$, it has constant curvature -1.

Proof. This follows from the lemma to Theorem 4.1, or the above heuristic argument. □

The above coordinate system can be generalized to three dimensions. It is the analog of E^3 with cylindrical coordinates. Let $P: \mathbb{R}^3 \to H^3$ be defined as follows.

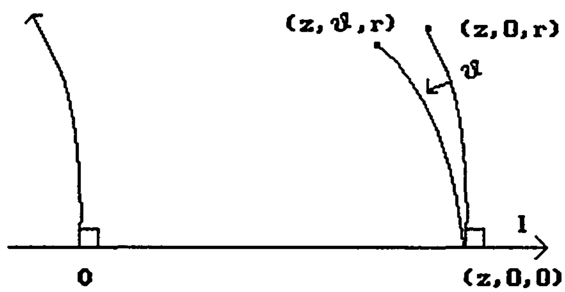


Figure 4.3

Fix a line l, a point O, a direction δ and an orientation, as in Figure 4.3. Then $P(z,r,\theta)$ is the unique point obtained by traveling a directed distance z on the line l, erecting a perpendicular in the $\theta = 0$ plane, and traveling a directed distance r on this perpendicular line, then rotating an angle θ. The given curves $r = r_0$, $z = z_0$ and $\theta = \theta_0$ are mutually perpendicular, and from the trigonometry one can deduce the given metric is $ds^2 = \cosh^2 r dz^2 + \sinh^2 r d\theta + dr^2$.

Theorem 4.10. When $\mathbb{R}^3$ is given the metric $ds^2 = \cosh^2 r dz^2 + \sinh^2 r d\theta + dr^2$, it has constant curvature -1.

Proof. Proceed as in Theorem 4.3.

There is one more coordinate system which is worth mentioning, its is H^3 with the analog of spherical coordinates. It is $\mathbb{R}^3$ with the metric $ds^2 = \sinh^2 r \, d\varphi^2 + \sinh^2 r \cdot \sin^2 \varphi \, d\theta^2 + dr^2$. This can be deduced from the trigonometry formulas for hyperbolic geometry.

CHAPTER 5 CONVEX BODIES

Convex bodies in H^3 are studied in this chapter. We prove that if a C^2 surface M has gaussian curvature strictly greater than -1, then it must be homeomorphic to S^2. It is also demonstrated that given any genus greater than or equal to 1, there exists a surface of that genus in E^3 which cannot be isometrically imbedded in H^3.

Theorem 5.1. Suppose M is a surface parameterized by $(x,y,f(x,y))$ and that $f(0,0) = 0$, $f_x(0,0) = 0$ and $f_y(0,0) = 0$. The surface M is located in $\mathbb{R}^3$, with the metric $dx_1^2 + e^{-2x_1}(dx_2^2 + dx_3^2)$ (This is H^3 with the horocyclic coordinates). Let

$$E_1 = D_{x_1} \qquad E_2 = e^{x_1} D_{x_2} \qquad E_3 = e^{x_1} D_{x_1}.$$

Define

$$U = (f_x^2 e^{-2x_1} + f_y^2 + 1)^{-1/2} (-f_x e^{-x_1} E_1 - f_y E_2 + E_3).$$

The vector field U is a unit normal vector field on M. Let $S_0(E_1) = -\nabla_{E_1} U$ and $S_0(E_2) = -\nabla_{E_2} U$ $((0,0,0) = 0)$, where E_1, E_2 are the standard vectors at 0. We are taking the covariant derivative of the vector field U in the direction E_1 and E_2 at 0. Then (dropping 0 for convenience)

$$S(E_1) = -\nabla_{E_1} U = f_{xx}(0,0)\, E_1 + f_{xy}(0,0)\, E_2$$

$$S(E_2) = -\nabla_{E_2} U = f_{yx}(0,0)\, E_1 + f_{yy}(0,0)\, E_2.$$

Proof. Since $ds^2 = dx_1^2 + e^{-2x_1}(dx_2^2 + dx_3^2)$, it is easy to see $E_1 = D_{x_1}$, $E_2 = e^{x_1}D_{x_2}$ and $E_3 = e^{x_1}D_{x_3}$ form an orthonormal frame. Define $\omega(x,y) = (x,y,f(x,y))$; ω is a parameterization of M. Let

$$D_x = D_{x_1} + f_x D_{x_3} = E_1 + f_x e^{-x} E_3$$

and

$$D_y = D_{x_2} + f_x D_{x_3} = e^{-x}E_2 + f_y e^{-x} E_3 .$$

For each point $p \in M$, the vectors D_x and D_y are a basis for $T_p(M)$. Let $D_x \times D_y$ denote the cross product of D_x and D_y. Since the E_i are orthonormal compute $D_x \times D_y$ just as in linear algebra, namely

$$D_x \times D_y = -f_x e^{-2x} E_1 - f_y e^{-x} E_2 + e^{-x} E_3 .$$

Thus,

$$U = D_x \times D_y / \| D_x \times D_y \|$$

One can also see that U is a unit normal vector field on M by noting $U \cdot U = 1$, $U \cdot D_x = 0$, and $U \cdot D_y = 0$. Suppose

$$f^1(x,y) = -f_x e^{-x}/(f_x{}^2 e^{-2x} + f_y{}^2 + 1)^{1/2}$$

$$f^2(x,y) = -f_y/(f_x{}^2 e^{-2x} + f_y{}^2 + 1)^{1/2}$$

$$f^3(x,y) = 1/(f_x{}^2 e^{-2x} + f_y{}^2 + 1)^{1/2}$$

Since $f_x(0,0) = f_y(0,0) = 0$ it is straightforward to check $f^1_x(0,0) = -f_{xx}(0,0)$, $f^1_y(0,0) = -f_{yx}(0,0)$, $f^2_x(0,0) = -f_{xy}(0,0)$, $f^2_y(0,0) = -f_{yy}(0,0)$, $f^3_x(0,0) = 0$, and $f^3_y(0,0) = 0$. Define

$$\nabla_{E_i} E_j = \Gamma_{ij}^{k} E_k .$$

By a previous calculation (Chapter 4), we have

$$[\Gamma_{ij}^{1}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad [\Gamma_{ij}^{2}] = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad [\Gamma_{ij}^{3}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}.$$

Computing $S(E_1)$, we have

$$-S(E_1) = \nabla_{E_1} U = \nabla_{E_1} f^1E_1 + f^2E_2 + f^3E_3$$

$$= (f^1_x)_0E_1 + (f^2_x)_0E_2 + (f^3_x)_0\, E_3 + f^1(0)\Gamma_{11}^{k}\, E_k + f^2(0)\Gamma_{12}^{k}\, E_k + f^3(0)\Gamma_{13}^{k} E_k$$

$$= -f_{xx}(0,0)\, E_1 - f_{xy}(0,0)\, E_2$$

Therefore, $S(E_1) = f_{xx}(0,0)\, E_1 + f_{xy}(0,0)\, E_2$. By a similar calculation, $S(E_2) = f_{yx}(0,0)\, E_1 + f_{yy}(0,0)\, E_2$. □

Theorem 5.2. Let M be the surface $z = f(x,y)$, where f is C^2 and $f(0,0) = f_x(0,0) = f_y(0,0) = 0$. If D_x, D_y are the principal directions of M at 0,

$$f(x,y) \approx \tfrac{1}{2}(\, k_1x^2 + k_2y^2\,),$$

where k_1 and k_2 are the principal curvatures in the $E_1 = D_{x_1}$ and $E_2 = D_{x_2}$ directions.

Proof. By Taylor's theorem

$$f(x,y) \approx 1/2(f_{xx}(0,0)x^2 + 2f_{xy}(0,0)xy + f_{yy}(0,0)y^2)$$

By the previous Theorem 5.1

$$S(E_1) = f_{xx}(0,0)\, E_1 + f_{xy}(0,0)\, E_2$$

and

$$S(E_2) = f_{yx}(0,0)\, E_1 + f_{yy}(0,0)\, E_2.$$

Since the E_1, E_2 are principal directions $f_{xy}(0,0) = f_{yx}(0,0) = 0$, $f_{xx}(0,0) = k_1$, and $f_{yy}(0,0) = k_2$. Therefore, $f(x,y) \approx \frac{1}{2}(k_1x^2 + k_2y^2)$. □

Theorem 5.3. Assume M is the same surface as before with the added hypothesis $K(0) > -1$, where $K(0)$ is curvature of M at the origin. Then for the appropriate choice of coordinates $\exists\, r > 0$ such that $d((x,y),(0,0)) < r$ implies $f(x,y) > 0$.

Proof. Since $k_1(0)\cdot k_2(0) - 1 = K(0)$, then $k_1(0)\cdot k_2(0) > 0$. So without loss of generality, assume $k_1(0)$ and $k_2(0) > 0$ (suppose D_{x_3} is in the direction of U). The theorem now follows from Theorem 5.2 and Taylor's theorem. □

Theorem 5.4. Suppose $M \subset H^3$ is a C^2 surface, and for a point p, $K(p) > -1$. Then there is a neighborhood U_p of p, such that $p^* \in U_p$ implies that p^* lies on one side of the tangent plane.

Proof. Construct the horocyclic coordinate system at the point p setting

$p = (0,0,0)$, and $T_p(M) = \{ (x_1,x_2,x_3) \in \mathbb{R}^3,\ x_3 = 0 \}$. The result now follows from Theorem 5.3. □

Theorem 5.5. Suppose M is a connected, immersed, compact, C^2 surface in H^3, such that for $p \in M$, $K(p) > -1$. Then M is homeomorphic to S^2.

Proof. Let $n: M \to T_p(H^3)$ be the Gauss map. That is for each $p \in M$ let $n(p)$ be the normal vector to $T_p(M)$ so that $n(p)$ points in the direction opposite of M. This is allowed by Theorem 5.4.

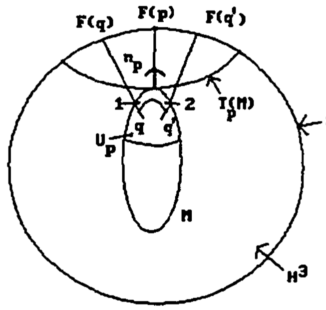


Figure 5.1

Let $\overrightarrow{n_p}$ be the geodesic ray determined by $\overrightarrow{n_p}$ and P. Without loss of generality, suppose we are in the Poincare ball model for H^3 with $d(H^3) = S^2$. Define $F: M \to S^2$, by $F(p) = \overrightarrow{n_p} \cap S^2$. The map F is clearly continuous. The map F is also locally one–to–one. To see this suppose $p \in M$. Choose a U_p from Theorem 5.4, such that M is locally convex on U_p. Suppose $q, q' \in U_p$. The angles 1 and 2 are obtuse, hence F is one–to–one near p. Since M is compact it follows that F is a covering map. Then M must be homeomorphic to S^2. □

Theorem 5.6. Suppose M is a C^2 compact immersed surface in H^3. Then there exists a point p such that $K(p) > -1$.

Proof. Since M is compact, it lies within the interior of some hyperbolic sphere S. Slowly contract this sphere until it becomes tangent to M. At the first point of contact the surface M shall lie in S and its interior. Let p be this point of contact. At p, $k_1(p) \cdot k_2(p) > 0$, so $K(p) = k_1(p) \cdot k_2(p) - 1 > -1$. □

Theorem 5.7. If M is a C^2 compact, immersed surface in H^3 of constant curvature, then its curvature is positive and M is homeomorhic to S^2.

Proof. By the previous theorem the constant curvature must be greater than -1, now apply Theorem 5.5. □

From Theorem 5.7 it follows that there are no double tori M of constant negative curvature $-1/k^2$, $k > 0$ in H^3. Indeed, as in E^3, there are no compact surfaces of constant curvature in H^3 other than spheres.

Theorem 5.8. For all $n \geq 2$ there exist C^∞ surfaces in E^3, which do not have a C^2 isometric imbedding in H^3.

Proof. Let M be a C^∞ surface in E^3 of genus $n \geq 1$. Dilate M with a similarity map so that $|k_1|, |k_2| < 1$ $\forall p \in M$, where the k_i are the principal curvatures. Then $K(p) > -1$. If M were to exist in H^3 as a C^2 surface, then by Theorem 5.5, M would be homeomorpic to a sphere, a contradiction. □

CHAPTER 6: CURVES OF CONSTANT CURVATURE

The following theorem is true for curves in E^2. For a proof of this theorem see [Spivak]. Let $c:[a,b] \to \mathbb{R}$ be a C^2 curve parameterized by arclength. If $c''(s) \neq 0$, then for s_1, s_2, s_3 sufficiently close to s, the points $c(s_1), c(s_2), c(s_3)$ do not lie on a line. As $s_1, s_2, s_3 \to s$ the unique circle through the points $c(s_i)$ approaches a circle passing through $c(s)$, whose radius is $1/|c''(s)|$, and whose center lies on the line through $c(s)$ perpendicular to the tangent line through $c(s)$. In this chapter we shall prove an analogous theorem to the above for curves in H^2.

Theorem 6.1. Let $k: [a,b] \to \mathbb{R}$. Suppose $t_1^2(a) + t_2^2(a) = 1$. Then there is a unique curve $c: [a,b] \to H^2$, parameterized by arclength with $c'(a) = t_1(a)E_1 + t_2(a)E_2$, whose curvature at s is $k(s)$, and $c(a) = (x_o, y_o)$.

Proof. Let D_s denote the derivative of c with respect to arclength. Let t_1 and t_2 be two real valued functions defined on $[a,b]$. Let $D_s = t_1E_1 + t_2E_2$, so that we consider the following differential equation

$$\nabla_{D_s} D_s = k(s)\cdot(-t_2(s)E_1 + t_1(s)E_2). \tag{6.1}$$

Here $E_1 = D_x$, $E_2 = e^x \cdot D_y$, where D_x and D_y are the tangent vectors along the coordinate curve $y = \text{constant}$, $x = \text{constant}$, and ∇ denotes covariant differentiation. We also want condition

$$(t_1(a), t_2(a)) = (\cos\alpha, \sin\alpha) \quad \alpha \in \mathbb{R}. \tag{6.1$_a$}$$

Assuming (6.1) and (6.1_a) are true we have, by the Levi–Civita connection

$$D_s(D_s \cdot D_s) = \nabla_{D_s} D_s \cdot D_s + D_s \cdot \nabla_{D_s} D_s = 2\nabla_{D_s} D_s \cdot D_s$$

$$= 2(-t_2(s)E_1 + t_1(s)E_2) \cdot (t_1(s)E_1 + t_2(s)E_2) = 0.$$

Since $(D_s \cdot D_s)_a = 1$, we have $(D_s \cdot D_s)_s = 1$ for $s \in [a,b]$, or $t_1^2(s) + t_2^2(s) = 1$ for $s \in [a,b]$. Now we have

$$\nabla_{D_s} D_s = \nabla_{D_s} t_1E_1 + t_2E_2 = t_1'E_1 + t_2'E_2 + t_1\nabla_{D_s} E_1 + t_2\nabla_{D_s} E_2 ,$$

and

$$\nabla_{D_s} E_1 = \nabla_{t_1E_1+t_2E_2} E_1 = t_1\Gamma_{11}^k E_k + t_2\Gamma_{21}^k E_k = -t_2E_2.$$

$$\nabla_{D_s} E_2 = \nabla_{t_1E_1+t_2E_2} E_2 = t_1\Gamma_{12}^k E_k + t_2\Gamma_{22}^k E_k = t_2E_1.$$

Therefore,

$$\begin{aligned} \nabla_{D_s} D_s &= t_1'E_1 + t_2'E_2 - t_1t_2E_2 + t_2^2E_1 \\ &= (t_1' + t_2^2)E_1 + (t_2' - t_1t_2)E_2 . \end{aligned}$$

From (6.1) we have

$$(t_1' + t_2^2)E_1 + (t_2' - t_1t_2)E_2 = -k \cdot t_2E_1 + k \cdot t_1E_2.$$

The differential equation (6.1) with (6.1_a) becomes the system of differential equations

$$\begin{aligned} t_1' &= -t_2^2 - k \cdot t_2 \\ t_2' &= t_1t_2 + k \cdot t_1 , \end{aligned} \tag{6.2}$$

with the same initial condition

$$(t_1(a),t_2(a)) = (\cos\alpha,\sin\alpha) \quad \alpha \in \mathbb{R}. \tag{6.2$_a$}$$

For $s \in [a,b]$, $t_1^2(s) + t_2^2(s) = 1$. Let $t_1(s) = \cos\theta(s)$ and $t_2(s) = \sin\theta(s)$. The differential equation

$$t_1' = -t_2^2 - k \cdot t_2$$
$$(\text{with } t_1(a) = \cos\alpha)$$

becomes

$$-\sin\theta(s) \cdot \theta'(s) = -\sin^2\theta(s) - k\sin\theta(s)$$

or

$$\theta'(s) = k(s) + \sin\theta(s), \quad \theta(a) = \alpha.$$

Set $f(s,\theta) = k(s) + \sin\theta$. The function f is uniformly lipschitz on $[a,b] \times \mathbb{R}$. Hence the solution $\theta(s)$ for the above differential equation and initial condition exists for all $s \in [a,b]$ and is unique. Now given this $\theta(s)$ one deduces $t_1(s) = \cos\theta(s)$ and $t_2(s) = \sin\theta(s)$ solves (6.1) and (6.1$_a$).

Now set $c(s) = (x(s),y(s))$. We then have

$$c'(s) = x'(s)E_1 + y'(s)e^{-x(s)}E_2.$$

Let

$$x'(s) = t_1(s) \quad \text{and} \quad y'(s)e^{-x(s)} = t_2(s).$$

Hence

$$x(s) = x_o + \int_a^s t_1(r)\,dr,$$

and

$$y(s) = y_o + \int_a^s t_2(r) \cdot \exp\left(x_o + \int_a^r t_1(t)\,dt\right)\,dr.$$

The curve $c(s)$ is the desired curve. From a previous observation, we have $c'(s)\cdot c'(s) = 1$ for $s \in [a,b]$, and since $t_1(s)$, and $t_2(s)$ solve (6.1) and (6.1_a), the curve $c(s)$ has the desired curvature $k(s)$.

Lemma. Any three points in H^2 lie on a curve of constant curvature, that is either a geodesic, hyperbolic circle, horocycle, or an equidistant curve.

Proof. Without loss of generality assume the three points are in the upper half space model for H^2. From a theorem in Euclidean geometry any three points lie on a Euclidean circle or a Euclidean line (just treat the circle and line as a set of points). Now a Euclidean circle or line is interpreted as a geodesic, circle, horocycle, or an equidistant curve, when the upper half plane is given the metric $ds^2 = 1/y^2(\, dx^2 + dy^2)$. Therefore the lemma is proved. $\square$

From Chapter 3 we known that the upper half plane with the metric $ds^2 = 1/y^2(\, dx^2 + dy^2\,)$ has constant curvature -1. Set $E_1 = ydx$, and $E_2 = ydy$. It is easy to see E_1, E_2 is an orthonormal frame and $\omega^1 = 1/y\ dx$, and $\omega^2 = 1/y\ dy$ is the dual coframe. We have

$$d\omega^1 = 1/y^2 dx\wedge dy \qquad\qquad d\omega^2 = 0.$$

We can now find the connection one forms as before,

$$[\ 1/y^2\ dx\wedge dy \quad 0\ ] = [\ 1/ydx \quad 1/ydy]\cdot\begin{bmatrix} 0 & 1/y\ dx \\ -1/y\ dx & 0 \end{bmatrix}.$$

Therefore,

$$(\omega_i^j) = \begin{bmatrix} 0 & \omega^1 \\ -\omega^1 & 0 \end{bmatrix}, \quad \text{where } \omega^1 = 1/y\ dx.$$

We can compute the Christoffel symbols Γ_{ij}^{k}, $1 \le i,j,k \le 2$, to be

$$\Gamma_{ij}^{1} = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} \qquad\qquad \Gamma_{ij}^{2} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

The Γ_{ij}^{k} are with respect to E_1, E_2. For instance

$$\omega^1 = \omega_1^2 = \Gamma_{i1}^{2}\,\omega^i = \Gamma_{11}^{2}\,\omega^1 + \Gamma_{21}^{2}\,\omega^2 \;\Rightarrow\; \Gamma_{11}^{2} = 1 \text{ and } \Gamma_{21}^{2} = 0.$$

Theorem 6.2. Let $\alpha: [-\epsilon,\epsilon] \to H^2$ be a C^2 curve parametrized by arclength s in H^2. If $\nabla_{D_s} D_s \neq 0$, then for t_1, t_2, and t_3 sufficiently close to 0, the points $\alpha(t_1)$, $\alpha(t_2)$, and $\alpha(t_3)$ do not lie on a geodesic. As t_1, t_2, and $t_3 \to 0$, the unique constant curvature curve through the points $\alpha(t_i)$ approaches a hyperbolic circle, a horocycle or an equidistant curve depending on whether

$$|\nabla_{D_s} D_s| > 1,\; |\nabla_{D_s} D_s| = 1, \text{ or } |\nabla_{D_s} D_s| < 1.$$

Proof. Without loss of generality, assume $\alpha(t) = (t, f(t))$ with $f'(0) = 0$, and $\alpha(0) = (0,1)$. Then

$$\alpha'(t) = D_x + f'(t)D_y = 1/f(t)\, E_1 + f'(t)/f(t)\, E_2 .$$

Therefore (for convenience drop the t), $|\alpha'| = (1/f)(1 + f'^2)^{1/2}$. The unit vector field on α is

$$D_s = f/(1 + f'^2)^{1/2}(1/f\, E_1 + f'/f\, E_2) = (1 + f'^2)^{-1/2}(E_1 + f'\, E_2).$$

Set

$$f^1 = (1 + f'^2)^{-1/2} \text{ and } f^2 = f' \cdot (1 + f'^2)^{-1/2}.$$

It is easy to note that

$$f^{1\prime}(0) = 0 \quad \text{and} \quad f^{2\prime}(0) = f''(0).$$

Hence, at $t = 0$,

$$\nabla_{E_1} f^1E_1 + f^2E_2 = f^{1\prime}(0)E_1 + f^{2\prime}(0)E_2 + f^1(0)\Gamma_{11}^{k}E_k + f^2(0)\Gamma_{22}^{k}E_k$$

$$= f''(0)E_2 + f^1(0)\Gamma_{11}^{k}E_k + 0 = (f''(0) + 1)E_2.$$

Therefore,

$$|\nabla_{D_s} D_s| = |f''(0) + 1|$$

Since $|\nabla_{D_s} D_s| \neq 0$, we have $f''(0) \neq -1$, and we can assume $f''(0) < -1$. Since $\alpha(0) = (0,1)$, we can assume $\alpha(t)$ locally lies beneath the geodesic parameterized by $(t,(1 - t^2)^{1/2})$. From the result quoted in the beginning of this chapter, we know that as $t_1, t_2, t_3 \to 0$ the unique Euclidean circle through $\alpha(t_i)$ approaches a circle passing through $(1,0)$ with radius $1/|f''(0)|$ tangent to the curve $y = 1$ and lies in the half plane $y < 1$. This "Euclidean circle" is interpreted as a curve of constant curvature in H^2. Suppose $|\nabla_{D_s} D_s| > 1$. Then $|f''(0) + 1| > 1$, or $|f''(0) - (-1)| = d(f''(0),-1) > 1$, and since $f''(0) < -1$, we have $f''(0) < -2$. The $\alpha(t_i)$ approach the Euclidean circle through $(0, 1 - 1/|f''(0)|)$, of radius $1/|f''(0)| < 1/2$. In this situation, the given circle lies in the upper half plane, and so it can be interpreted as a hyperbolic circle.

If $|\nabla_{D_S} D_S| = 1$ then the above Euclidean circle has center $(0,1/2)$ and radius $1/2$ which is interpreted as a horocycle. If $|\nabla_{D_S} D_S| < 1$ then $1/|f''(0)| > 1/2$. Thus, the above circle has center $(0,c)$ and radius r, where $c < 1/2$ and $r > 1/2$, and so the above circle is interpreted as an equidistant curve. □

CHAPTER 7: ROTATION SURFACES

In this chapter, we will demonstrate that certain rotation surfaces from E^3 have an isometric imbedding in H^3, while other rotation surfaces do not. Also it is demonstrated that surfaces of rotation of constant curvature K where $-1 \leq K \leq 0$ can be isometrically imbedded in H^3.

Problem 1. Given the metric $ds^2 = \cosh^2 r\, dz^2 + \sinh^2 r\, d\theta^2 + dr^2$ on $\mathbb{R}^3$ (H^3 with the analog of cylindrical coordinates) and a surface $S(u,v) = (\overline{\psi}(u),v,\overline{\varphi}(u))$, find $E(u,v)$,$F(u,v)$,$G(u,v)$ and $K(u,v)$, its gaussian curvature.

Assume $(\overline{\psi}(u),0,\overline{\varphi}(u))$ is parameterized by arclength. Let $E_1 = 1/\cosh r\, D_z$, $E_2 = 1/\sinh r\, D_\theta$, and $E_3 = D_r$. E_1, E_2, and E_3 are an orthonormal frame on H^3. Therefore,

$$\begin{aligned} D_uS(u,v) &= \overline{\psi}'(u)\, D_z + \overline{\varphi}'(u)\, D_r \\ &= \overline{\psi}'(u)\cosh \overline{\varphi}(u)\, E_1 + \overline{\varphi}'(u)\, E_3. \end{aligned}$$

Hence,

$$E(u,v) = D_u \cdot D_u = \overline{\psi}'(u)^2 \cosh^2 \overline{\varphi}(u) + \overline{\varphi}'(u)^2 = 1,$$

since the curve $(\overline{\psi}(u),0,\overline{\varphi}(u))$ is parameterized by arclength. Also,

$$D_vS(u,v) = D_\theta = \sinh(\overline{\varphi}(u))\, E_2.$$

Therefore, $G(u,v) = D_v \cdot D_v = \sinh^2 \overline{\varphi}(u)$ and $F(u,v) = D_u \cdot D_v = 0$. Let $\overline{E}_1 = D_u$, $\overline{E}_2 = 1/\sinh \overline{\varphi}(u)\, D_v$, $\overline{\theta}_1 = du$, $\overline{\theta}_2 = \sinh \overline{\varphi}(u)\, dv$. Thus

$ds = du^2 + \sinh^2 \overline{\varphi}(u)\, dv^2$, and by a previous formula,

$$K(u,v) = -\sinh'' \overline{\varphi}(u)/\sinh \overline{\varphi}(u)\,. \qquad (7.1)$$

Problem 2. Given the metric $ds^2 = dz^2 + r^2 d\theta^2 + dr^2$ (cylindrical coordinates in E^3) and a surface $S(u,v) = (\psi(u), v, \varphi(u))$, find its gaussian curvature $K(u,v)$ and find $E(u,v)$, $F(u,v)$, and $G(u,v)$.

Assume the curve $(\psi(u), 0, \varphi(u))$ is parameterized by arclength. Let $E_1 = D_z$, $E_2 = 1/\varphi(u) D_\theta$, and $E_3 = D_r$. The vectors E_1, E_2 and E_3 are an orthonormal frame. Differentiating

$$D_u S(u,v) \;=\; \psi'(u)\, D_z + \varphi'(u)\, D_r \;=\; \psi'(u) E_1 + \varphi'(u) E_3.$$

Thus,

$$E(u,v) \;=\; \psi'(u)^2 + \varphi'(u)^2 \;=\; 1.$$

Differentiating again,

$$D_v\, S(u,v) = D_\theta = \varphi(u) E_2.$$

Hence

$$G(u,v) = D_v \cdot D_v = \varphi^2(u) \quad \text{and} \quad F(u,v) = D_u \cdot D_v = 0.$$

Let $\overline{E}_1 = D_u$, $\overline{E}_2 = 1/\varphi(u)\, D_v$; $\overline{E}_1$ and $\overline{E}_2$ are an adapted orthonormal frame on S. Since $\overline{\theta}_1 = du$, and $\overline{\theta}_2 = \varphi(u) dv$, we have $ds^2 = du^2 + \varphi(u)^2 dv^2$, and by a previous calculation,

$$K(u,v) = -\varphi''(u)/\varphi(u). \qquad (7.2)$$

Theorem 7.1. Any surface of the form $(\psi(u), v, \varphi(u))$, $a < u < b$, that exists in $\mathbb{R}^3$ with the metric $ds^2 = dz^2 + r^2 d\theta^2 + dr^2$ can be isometrically imbedded into $\mathbb{R}^3$ with the metric $ds^2 = \cosh^2 r \cdot dz^2 + \sinh^2 r \cdot d\theta^2 + dr^2$.

Proof. Assume the surface will be of the form $(\overline{\psi}(u), v, \overline{\varphi}(u)) \subset H^3$. Hence there are two surfaces $(\psi(u), v, \varphi(u)) \subset E^3$ and $(\overline{\psi}(u), v, \overline{\varphi}(u)) \subset H^3$. Let $\cdot_E$ and $\cdot_H$ denote the dot product in E^3 and H^3. Assume the curve $(\psi(u), 0, \varphi(u))$ is parameterized by arclength. From a previous problem $D_u \cdot_E D_u = \psi'(u)^2 + \varphi'(u)^2 = 1$, $D_u \cdot_E D_v = 0$ and $D_v \cdot_E D_v = \varphi(u)^2$. In H^3 also suppose the curve $(\overline{\psi}(u), v, \overline{\varphi}(u))$ is parameterized by arclength. Then $D_u \cdot_H D_u =$ $\overline{\psi}'(u)^2 \cosh^2(\overline{\varphi}(u)) + \overline{\varphi}'(u)^2$, $D_u \cdot_H D_v = 0$, and $D_v \cdot_H D_v = \sinh^2 \overline{\varphi}(u)$. Setting $D_v \cdot_E D_v = D_v \cdot_H D_v$ implies $\varphi(u)^2 = \sinh^2 \overline{\varphi}(u)$ or $\overline{\varphi}(u) = \sinh^{-1} \varphi(u)$. Assume $(\overline{\psi}(u), 0, \overline{\varphi}(u))$ is parameterized by arclength

$$\overline{\psi}'(u)^2 \cdot \cosh^2 \overline{\varphi}(u) + \overline{\varphi}'(u)^2 = 1$$

or

$$\overline{\psi}'(u)^2 = (1/\cosh^2 \overline{\varphi}(u)) \cdot (1 - \overline{\varphi}'^2(u)).$$

We have $\cosh^2 \overline{\varphi}(u) = 1 + \sinh^2 \overline{\varphi}(u) = \varphi^2(u) + 1$. Therefore $\cosh \overline{\varphi}(u) \cdot \overline{\varphi}'(u)$ $= \varphi'(u)$, and hence

$$\overline{\varphi}'(u)^2 = \varphi'(u)^2 / \cosh^2 \overline{\varphi}(u) = \varphi'(u)^2 / (\varphi^2(u) + 1).$$

Therefore,

$$1 - \overline{\varphi}'(u)^2 = \varphi^2(u) + 1 - \varphi'^2(u))/(\varphi^2(u) + 1)$$

$$= (\varphi^2(u) + \psi'^2(u))^2/(\varphi^2(u) + 1).$$

Thus,

$$\overline{\psi}'(u)^2 = (\varphi^2(u) + \psi'(u)^2)/(\varphi^2(u) + 1)^2.$$

By integrating the desired curve is obtained. □

Now we will apply Problem 2 to show that there exist surfaces of rotation in E^3 which cannot be imbedded as C^2 surfaces in H^3. Let $S(u,v) = (r\cos(u/r), v, r\sin(u/r) + R)$, $0 < r < R$.

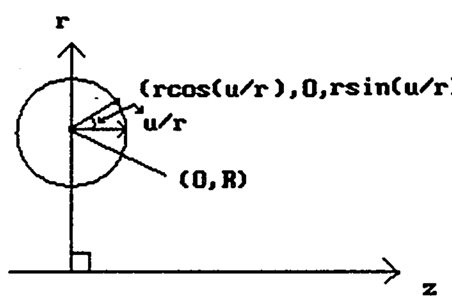


Figure 7.1

Problem 2 implies

$$K(u,v) = -(r\sin(u/r) + R)''/(r\sin(u/r) + R)$$

$$= \sin(u/r)/(r^2\sin(u/r) + rR).$$

Applying calculus, we see that the maximum curvature is $1/(r^2 + rR)$ and that the minimum curvature is $-1/(-r^2 + rR)$. Therefore if $R > r + 1/r$, $K(u,v) > -1$. Hence by Theorem 5.5, the given surface does not have a C^2 isometric imbedding in H^3.

Problem 3. Find a surface of rotation in $\mathbb{R}^3$, with metric $dx^2 + e^{-2x}(dy^2 + dz^2)$, that has constant curvature $-c^2$, where $0 < c \leq 1$.

Let

$$x(u,v) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos v & -\sin v \\ 0 & \sin v & \cos v \end{bmatrix} \begin{bmatrix} h(u) \\ 0 \\ e^{h(u)}g(u) \end{bmatrix}$$
$$= [\, h(u), -e^{h(u)}g(u)\sin v,\ e^{h(u)}g(u)\cos v]^T .$$

Differentiate to obtain

$$\begin{aligned} D_v &= x_v(u,v) = -e^{h(u)}g(u) \cos v\, D_y + -e^{h(u)}g(u) \sin v\, D_z \\ &= -g(u) \cos v\, E_2 + -g(u) \sin v\, E_3. \end{aligned}$$

Therefore, $D_v \cdot D_v = g^2(u)$. Set $\overline{E}_2 = 1/g(u)D_v$ (assume $g(u) > 0$), and let $\overline{\theta}_2 = g(u)dv$. If it is possible to find an $h(u)$ such that $D_u \cdot D_u = 1$, then set $\overline{E}_1 = D_u$, and $\overline{\theta}_1 = du$. The curvature of S (the desired surface) will be $-g''(u)/g(u)$. So if $g''(u) = c^2 g(u)$ where $0 < c \leq 1$, then S will have constant negative curvature $-c^2$. The general solution for this differential equation is

$$g(u) = \alpha \cdot e^{cu} + \beta \cdot e^{-cu}.$$

Since $g(u) > 0$, $\alpha, \beta > 0$. Therefore $\exists$ a and b, such that $e^a = \alpha$ and $e^b = \beta$.

Hence

$$\begin{aligned} g(u) &= \alpha\cdot e^{cu} + \beta\cdot e^{-cu} \\ &= e^a e^{cu} + e^b e^{-cu} \\ &= e^{cu+a} + e^{-cu+b} \\ &= e^{c(t+(b-a)/2c)+a} + e^{-c(t+(b-a)/2c)+b} \end{aligned}$$

where $t = u - (b-a)/2c$. Therefore,

$$\begin{aligned} g(u) &= 2e^{(b+a)/2}(e^{ct} + e^{-ct})/2 \\ &= 2e^{(b+a)/2}\cosh ct \\ &= 2e^{(b+a)/2}\cosh(cu-(b-a)/2) \end{aligned}$$

Without loss of generality assume $g(u) = A\cosh(cu)$, $A > 0$. Differentiating,

$$\begin{aligned} D_u = x_u &= h'(u)D_x - (e^{h(u)}g(u))'\sin v\, D_y + (e^{h(u)}g(u))'\cos v\, D_z \\ &= h'(u)E_1 - (h'(u)g(u) + g'(u))\sin v\, E_2 + (h'(u)g(u) + g'(u))\cos v\, E_3. \end{aligned}$$

Hence,

$$\begin{aligned} D_u\cdot D_u &= h'(u)^2 + (h'(u)g(u) + g'(u))^2 \\ &= h'(u)^2 + (h'(u)g(u))^2 + 2h'(u)g(u)g'(u) + g'(u)^2. \end{aligned}$$

Set $D_u\cdot D_u = 1$. Then (omit u)

$$(1 + g^2)h'^2 + 2h'\cdot g\cdot g' + g'^2 - 1 = 0.$$

If there exists a solution to this equation, then the desired surface of rotation with constant curvature $-c^2$ will be obtained. From the quadratic formula

$$h' = (-gg' \pm (1 + g^2 - g'^2)^{1/2})/(1 + g^2).$$

Since $g(u) = A \cosh(cu)$, $A>0$, make this substitution to obtain

$$h'(u) = l(u)/m(u),$$

where

$$l(u) = -cA^2 \cosh(cu) \sinh(cu) \pm (1+c^2A^2+A^2\cosh(cu)\cdot(1-c^2))^{1/2}$$

and

$$m(u) = 1 + A^2\cosh^2 cu.$$

Integrate to obtain $h(u)$. For $0 \leq c \leq 1$, $h(u)$ is a C^∞ function. Therefore, there is a C^∞ surface of rotation in H^3 of constant curvature $-c^2$. Also for $c > 1$, we can solve the above formula in an interval about 0, which is analogous to the bugle surface in E^3.

CHAPTER 8: TRIANGULATIONS OF S^2

In this chapter we demonstrate that all triangulations of the two sphere can be recursively constructed by three fundamental procedures.

Theorem 8.1. Triangulations of S^2 with at least 4 vertices can be recursively constructed. This means one can start with the tetrahedron (a triangulation of S^2 with 4 vertices) and obtain any other triangulation by doing one of the following operations.

1) Add a vertex to a given triangle and connect it to the remaining vertices of that triangle.

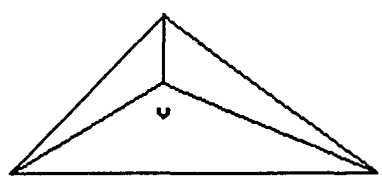


Figure 8.1

2) Place a vertex on an edge and connect it to the vertices opposite the given edge on the two unique triangles which meet at the given edge.

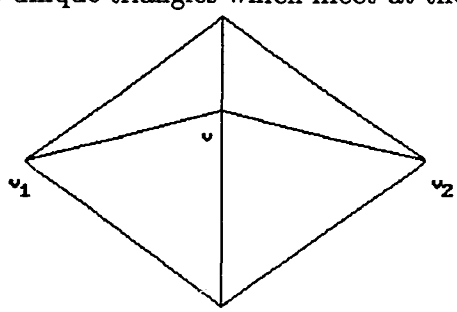


Figure 8.2

3) Given two triangles say T_1 with vertices $v_1, v_2,$ and v_3 and T_2 with vertices $v_1, v_2,$ and v_4 so $T_1 \cap T_2 = \overline{v_1 v_2}$, you can change the edge $\overline{v_1 v_2}$ to $\overline{v_3 v_4}$ when the degrees of v_1 and v_2 are greater than 3.

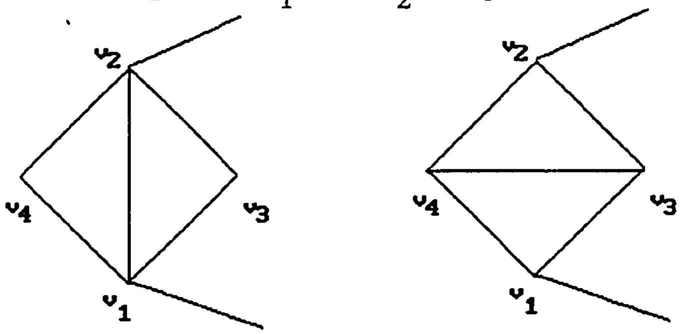


Figure 8.3

Procedures 1 and 2 increase the number of vertices while procedure 3 does not. It is straightforward to prove that procedure 1 and procedure 3 imply procedure 2. Let T_1 be a triangle with vertices v_1, v_2, v_3 and T_2 be a triangle with vertices v_2, v_3, v_4 (Figure 8.4). Place vertex v in the triangle with vertices v_2, v_3, v_4. Now rotate the edge with vertices v_2 and v_3. Therefore from procedures 1 and 3 we have deduced procedure 4.

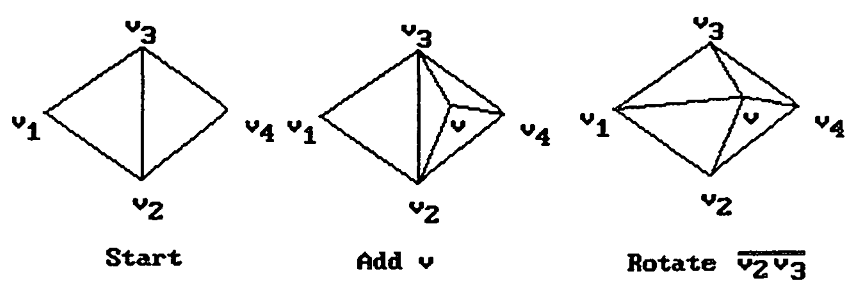


Figure 8.4

We are interested in constructing triangulations recursively by adding one vertex at a time or rotating a given edge. A **triangle** is the homeomorphic image of the simplex with in $\mathbb{R}^2$ with vertices (0,0), (1,0) and (0,1). A **triangulation** of S^2 is a finite number of triangles that satisfy conditions 1 through 7 listed below. A vertex, edge, and face are defined in the obvious manner.

1) Each edge is connected to exactly two vertices.

2) Each edge is incident to exactly two faces. If two faces intersect, the intersection is either an edge or a vertex.

3) Each vertex is incident to at least three faces.

4) Each face has exactly three vertices

5) For any two vertices v_1 and v_2 there exist a sequence of edges $e_1, e_2, ..., e_n$, such that $v_1 \in e_1, v_2 \in e_2$ and $e_i \cap e_{i+1}$ is a vertex.

6) If two triangles T and T′ share a common vertex v, then they intersect along a common side, or there exists two sequences of triangles T_i and T_j^* ,T_i starts with T and ending at T′, T_j^* starts with T′ and ends with T such that T_i and T_{i+1} and T_j^* and T_{j+1}^* intersect one another along a side containing vertex v.

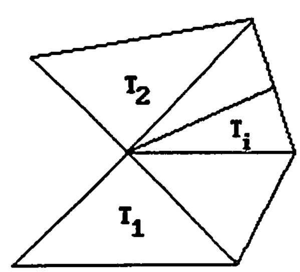


Figure 8.5

7) The union of all the faces must be S^2.

Let F be the number of faces E the number of edges and V the number of vertices of your triangulation. From condition 2 and 3 it follows that $3F = 2E$. Also since the triangulation is of S^2, we have $F - E + V = 2$. One now easily concludes $E = 3V - 6$, and $F = 2V - 4$.

Definition. The *star* of a vertex v is the $\bigcup_{i=1}^{n} T_i$, where the T_i, $1 \leq i \leq n$, are all the triangles with v as a vertex.

Lemma 1. The star of a vertex is homeomorphic to a closed disk.

Proof. This is an immediate consequence of condition 2 and 6, and the definition of the quotient topology. □

Definition. The *degree* of a vertex v, denoted $d(v)$ is the number of edges containing the vertex v.

Lemma 2. $2E = \sum_{i=1}^{V} d(v_i)$.

Proof. Each edge lies on exactly 2 vertices, so by adding up the degrees, you count each edge twice. □

Lemma 3. For every triangulation of S^2 of the above type, there is a vertex with degree 3, 4, or 5.

Proof. Suppose the conclusion of the theorem is not true. Then we have $\forall i$, $d(v_i) \geq 6$. By the above formula, $2E = \sum_{i=1}^{V} d(v_i) \geq 6V$, but $2E = 6V - 12$, and so $6V - 12 \geq 6V$, a contradiction. Hence, there must be some vertex of degree 3, 4, or 5. □

Now we prove Theorem 8.1. If the triangulation has 4 vertices, the triangulation must be the standard tetrahedron. Since it is assumed the degree of all the vertices is at least 3 and $2E = 12$ the degree of each vertex is 3. Thus, we must have the tetrahedron. The theorem will be proved by induction on the number of vertices V, where $V \geq 4$. Suppose the theorem is true for V vertices. Let T be a triangulation with $V + 1$ vertices. From the above lemma, it follows that there is a vertex of degree 3, 4, or 5. If there is a vertex v of degree 3, the following situation holds.

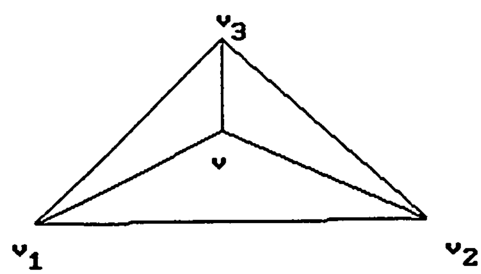


Figure 8.6

Let v_i $1 \leq i \leq 3$ be all the vertices which can be connected to v. It can be assumed that $\forall i$, $d(v_i) \geq 4$. If this were not the case, without loss of generality, suppose $d(v_1) = 3$. Then v_1, v_2, and v_3 would all lie on the same triangle, so $d(v_i) = 3$ for all i. Hence the given triangulation must be a tetrahedron. Remove the vertex v of degree 3. Then create a new triangulation T^* which has V vertices and satisfies conditions 1 through 6. By the induction hypothesis, T^* is obtained by the recursive process. Now replace the vertex that has been removed. This is allowed by rule 1. We then have obtained the triangulation T by recursion.

If T has a vertex of order 4, we can now suppose that all the other vertices have degree at least 4 (Figure 8.7). If this were not the case, we would be in the previous case. Now remove two nonadjacent edges and let the remaining two edges become one edge. We again have reduced the number of vertices by one, so we obtain a triangulation T* with V vertices which satisfy conditions 1 through 6. The triangulation T* is obtained by recursion. Replace the vertex, which is allowed by rule 2, hence T is obtained by the recursive construction.

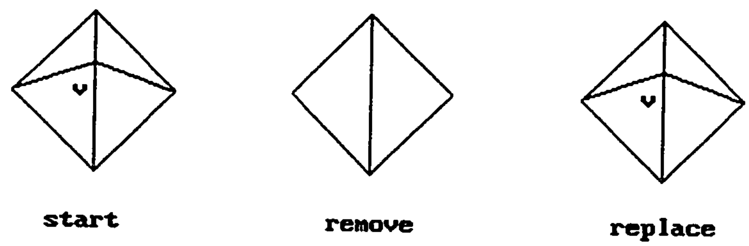


Figure 8.7

If T has a vertex v of order 5, we can now suppose that all the other vertices have degree at least 5, or else we are in one of the the previous cases. Let the star of v have vertices v_1, v_2, v_3, v_4 and v_5 (Figure 8.8). Remove the vertex and retriangulate the star of v, connect v_1 to v_3 and v_1 to v_4. This is allowable, since the star of a vertex v is homeomorphic to the closed unit disk. We obtain a triangulation T* which satisfies conditions 1 through 6.

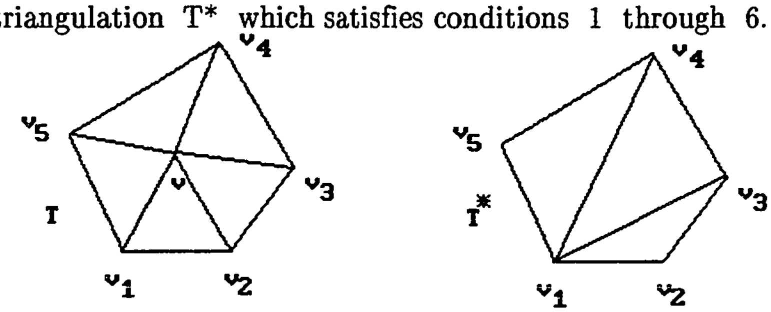


Figure 8.8

From T^* the net T can be obtained (Figure 8.9). Place a vertex v on $\overline{v_1 v_3}$. Connect v to v_4 and v_2. This is permissible by step 2. Replace $\overline{v_4 v_1}$ by $\overline{v v_5}$. This is allowed from step 3. Therefore T is obtained by recursion and the theorem has been proved.

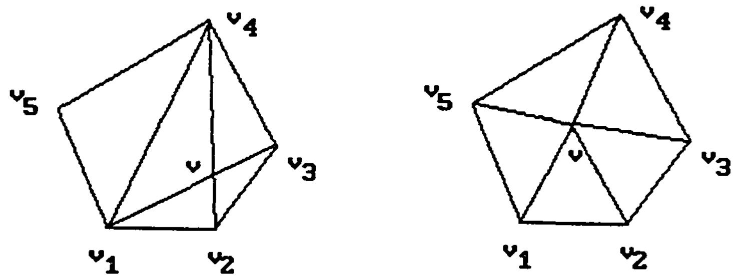


Figure 8.9

CHAPTER 9: METRICS ON ABSTRACT POLYHEDRA

By assigning numbers to the edges of a triangulation of S^2 and the same constant curvature to the all the faces, it is possible to define a metric on S^2. In this chapter and in chapters 10 and 11 a triangulation means the following

1) If T_1 and T_2 are two triangles of your triangulation then $T_1 \cap T_2$ is the union of edges or vertices belonging to both T_1 and T_2. For instance $T_1 \cap T_2$ may be an edge, an edge and a vertex, or three vertices.

2) One edge belongs to exactly two triangles.

3) If T_1 and T_2 are two triangles that meet at a given vertex, then there exist a sequence of triangles starting with T_1 ending at T_2 which intersect each other in an edge.

4) Any two triangles are joined by a chain of triangles glued along sides.

5) The union of the triangles is S^2, and $F - E + V = 2$.

If the number e is assigned to the edge connecting vertices v_1 and v_2, it is to be be interpreted as the length of that edge, which is a geodesic, but not necessarily a minimal geodesic. A **geodesic** is a polygonal path that locally minimizes distance, which means that if $p \in g$ (g is a geodesic), then there exists an $r_p > 0$, such that if $d(q,p) < r_p$ and $q \in g$, then $l(\overline{pq}) = d(p,q)$ ($l(\overline{pq})$ is the length of g which lies between p and q; d(p,q) will soon be defined). A **minimal geodesic** g_{pq} connecting p to q is a geodesic such that $l(g_{pq}) = d(p,q)$. We want the geometry of each face (a triangle) to be same as a triangle with constant negative curvature, constant positive curvature, or 0 curvature. Therefore, if e_1, e_2 and e_3 are three numbers assigned to edges which all belong to the same face, then $\forall i\ e_i > 0$ and $e_i < e_j + e_k$, for any permutation (ijk) of $\{1,2,3\}$. Metrics of this type are called polyhedral metrics. The curvature on all the faces is the same constant $0, -1/k^2$ or $+1/k^2$, where $k > 0$. When the curvature of a face is $1/k^2$, we must also assume $\forall i,\ 0 < e_i < 2\pi k$. Let p and q be two points of S^2. The distance between two points $p,q \in S^2$ is $d(p,q) = \inf\{\ l(P_{pq})$, where P_{pq} is a polygonal path connecting p and q and $l(P_{pq})$ denotes its length$\}$. It is possible to define $l(P_{pq})$ by adding up all the lengths of all segments of P_{pq}. Each segment consists of segments inside one or more triangles, and therefore can be assigned a length. When discussing a polyhedral metric, one should remember that a triangulation has been placed on S^2.

To prove this construction of assigning numbers to the edges in the appropriate fashion gives rise to a metric, the axioms of a metric space must be verified. For all $p,q \in S^2$, $d(p,q) = 0$ if and only if $p = q$. This is clear. For all $p,q \in S^2$ $d(p,q) = d(q,p)$, since the set of polygonal paths from p to q is equal to the set of polygonal paths from q to p. Suppose $\epsilon > 0$, and p,r and $q \in S^2$. There exist polygonal paths P_{pr} and P_{rq} such that $l(P_{pr}) < d(p,r) + \epsilon/2$ and $l(P_{rq}) < d(r,q) + \epsilon/2$. Let $P_{pq} = P_{pr} \cup P_{rq}$, then $d(p,q) \leq l(P_{pr}) + l(P_{rq}) < d(p,r) + d(r,q) + \epsilon$, so $d(p,q) \leq d(p,r) + d(r,q)$. *Therefore, a triangulation of the sphere, with numbers assigned to the edges in the appropriate fashion and the same constant curvature assigned to all the faces, gives rise to a metric.* Since the topology of the sphere is being formed by the quotient topology, the metric is continuous. Suppose that a polyhedral metric m on S^2 is given, but S^2 is retriangulated with different geodesics whose edge lengths, determined by m, satisfy the triangle inequalities. This triangulation will also give rise to a metric m'. Since the length of a polygonal path in one metric is the same as that in another, these triangulations lead to the same metric, or $d_m(p,q) = d_{m'}(p,q)$. Hence this metric has been defined independent of its triangulation with numbers assigned to the edges.

We will prove that any two points p and q can be joined by a minimal geodesic. It is appropriate to first prove a lemma and two theorems. This lemma is the polygonal version of the exponential mapping theorem.

Lemma 1. For all points $p \in S^2$, $\exists r > 0$, such that a minimal geodesic of length r extends from p in all directions.

Proof. To see this, break down the cases. Either the point p is interior to a triangle, p is on an edge, or p is a vertex. In any case, it is possible to verify the above, and the geodesic will be a line segment. □

Theorem 9.1. Given $p, q \in S^2$ and $p \neq q$, suppose that $r < d(p,q)$, and it is possible to extend geodesics from p at least a distance r. Then $\exists p' \in S^2$, $d(p,p') = r$ and $d(p,q) = d(p,p') + d(p',q)$.

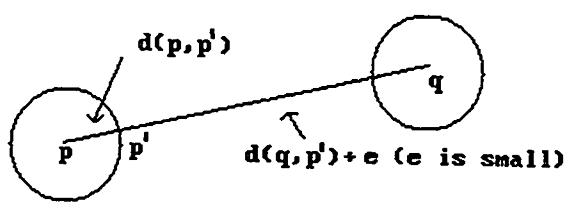


Figure 9.1

Proof. By definition of the metric there exist a sequence of polygonal paths P^i from p to q, such that $\lim_{i\to\infty} l(P^i) = d(p,q)$. Suppose P^i: $\overline{pp_1^i} \cup \ldots \cup \overline{p_{i_n}^i q}$; that is,

P^i is the polygonal path that connects the points p and q with line segments and has interior vertices $p_1^i,\ldots,p_{i_n}^i$. Without loss of generality, assume $d(p,p_1^i) = r$. Since the sphere is compact, there is a subsequence i_k such that $\lim_{k\to\infty} p_1^{i_k} = p'$.

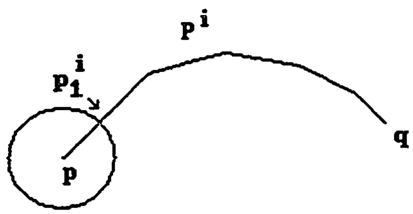


Figure 9.2

By continuity of the metric, $d(p,p') = r$. Also by the triangle inequality, $d(p,q) - r \le d(q,p')$. Let $\overline{P}^{i_k} = p_1^{i_k}\ldots q$. From above, it follows that $\lim_{k\to\infty} l(\overline{P}^{i_k}) = d(p,q) - r$, and $\lim_{k\to\infty} d(p',p_1^{i_k}) = 0$. Now $\overline{P}^{i_k} \cup p'p^{i_k}$ is a polygonal path from p' to q. Thus, by definition of the metric, $d(p',q) \le \lim_{i\to\infty} l(\overline{P}^{i_k}) + d(p',p_1^{i_k}) = d(p,q) - r$. Hence, $d(p',q) = d(p,q) - r$, and $d(p,q) = d(p,p') + d(p',q)$. □

Theorem 9.2. Suppose that g_{pq} is a minimal geodesic between p and q. If $r \in g_{pq}$ then $\overline{p\,r}$ (the polygonal path of g_{pq} which stops at r) is a minimal geodesic, and if $r,s \in g_{pq}$ then $\overline{r\,s}$ is a minimal geodesic.

Proof. If $\overline{p\,r}$ were not a minimal geodesic then there would a shorter path connecting p and q. Argue similarly in the $\overline{r\,s}$ case. □

Definition. Suppose $T_1, \dots, T_i, \dots, T_n$ are all the triangles of the given triangulation, such that $v \in T_i$ and α_i is the angle of T_i at vertex v. The **angle sum** Ψ_v of v, by definition, equals $\sum_{i=1}^{n} \alpha_i$.

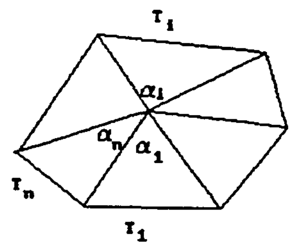


Figure. 9.3

Definition. A **real vertex** is a vertex of the triangulation whose angle sum Ψ_v is not 2π. The **curvature** $K(v)$ **of a real vertex** v is $K(v) = 2\pi - \Psi_v$. If $p \in S^2$ and p is not a real vertex, the curvature $K(p)$ of p is the curvature that has been assigned to the faces of the triangulation. Hence the curvature K has been assigned for all points of the sphere. The real vertices can be thought of as point masses of curvature.

From Theorem 9.1, it follows that any two points on the sphere with a polyhedral metric can be connected by a minimal geodesic.

Theorem 9.3. Any two points p and q of S^2, with a polyhedral metric, can be joined by a minimal geodesic g_{pq}. The set $g_{pq} - \{p,q\}$ contains no real vertices with positive curvature. If $\overline{x_1x_2}$ and $\overline{x_2x_3}$ are two segments of g_{pq} and x_2 is not a real vertex, then the angle between these segments is π.

Proof. As in the proof of Theorem 9.1, choose a sequence of polygonal paths P^i: $\overline{pp_1^i} \cup \ldots \cup \overline{p_{n_i}^i q}$ from p to q, such that $\lim_{i\to\infty} l(P^i) = d(p,q)$, and $d(p,p_1^i) = r$. Assume that the set of points $\overline{B}_r(p) = \{p' \in S^2, d(p',p) = r\} - \{p\}$ has no real vertices, and a geodesic from p extends in all directions at least a distance r. Since the sphere is compact, there is a subsequence i_k such that $\lim_{k\to\infty} p_1^{i_k} = p_b$. Let

$$S = \{ p' \in S^2, \text{ such that } p \text{ and } p' \text{ can be joined by a polygonal path } g_{pp'}, g_{pp_b} \subset g_{pp'}, l(g_{pp'}) = d(p,p'), g_{pp'} - \{p,p'\} \text{ contains no real vertices, and } d(p,p') + d(p',q) = d(p,q) \}.$$

From Theorem 9.2, $S \neq \phi$. Let $|S| = \{d(p,p'): p' \in S\}$. It is straightforward to verify that if $\rho \in |S|$ and $r \le \rho' < \rho$, then $\rho' \in |S|$. Suppose that $p_1, p_2 \in S$, and $l(g_{pp_1}) < l(g_{pp_2})$. Then $g_{pp_1} \subset g_{pp_2}$. If not, then we would have the situation in Figure 9.4.

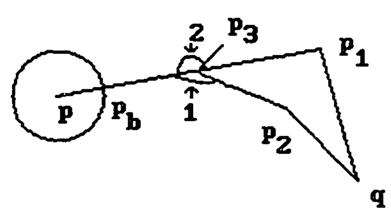


Figure 9.4

Let $D = \{ p'$, such that $p' \in g_{pp_1}$ implies $p' \in g_{pp_2}$, and $p^* \in g_{pp_1}$ with $d(p,p^*) < d(p,p')$ implies $p^* \in g_{pp_2}\}$. Let $|D| = \{ r \in \mathbb{R}$, such that $r = d(p,p')$ for some $p' \in D\}$, and d = least upper bound of $|D|$. The set $|D|$ is a closed set. If $d < l(g_{pp_1})$, let $p_3 \in g_{pp_1} \cap g_{pp_2}$ and $d(p,p_3) = d$. We must have either $\measuredangle 1$ or $\measuredangle 2$ in Figure. 9.4 less than π. If $\measuredangle 1 < \pi$, then by cutting across $\measuredangle 1$ in Figure 9.4, one could connect p and p_2 with a path of length less than $d(p,p_2)$.

Let $p_i \in S$, such that $d(p,p_i) \to l$, where l = l.u.b S. Since S^2 is compact, we can assume $p_i \to v_1$, $v_1 \in S^2$. Choose an r_{v_1} that satisfies Theorem 9.1 for the point v_1. Since $p_i \to v_1$, $\exists i$ such that $d(p_i,v_1) < r$. Hence we can choose a point $\overline{p} \in g_{pp_i}$, such that $d(\overline{p},v_1) = r_{v_1}$; see Figure 9.5.

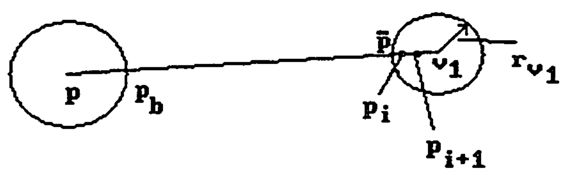


Figure 9.5

Let $g_{p\overline{p}}$ be all points on g_{pp_i} whose distance from p is less than or equal to $d(p,\overline{p})$. Connect $\overline{p}$ to v_1. It follows that $v_1 \in S$ and the path in Figure 9.5 is a geodesic connecting p to q of length l. If $l = d(p,q)$ the theorem has been proved. Suppose $l < d(p,q)$. The point v_1 must be a real vertex. If it were not a real vertex, then l would not be the l.u.b. of $|S|$. If v_1 had positive curvature, then it would be possible to connect p and q with a path of distance less than $d(p,q)$. Hence v_1 is a real vertex with negative curvature. Now repeat this argument with v_1 and q, obtaining a point v_2 which is either q or a real vertex of negative curvature.

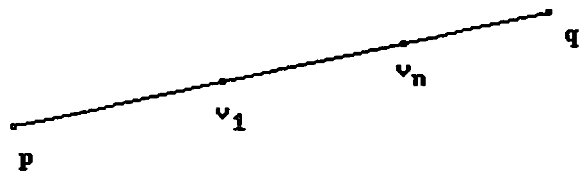


Figure 9.6

Since the number of real vertices is finite this process stops at some point v_n. Joining p to v_1, v_1 to v_2,... and v_n to q, we get the desired geodesic (see Figure 9.6).

If α is the angle between segments $s_1 = \overline{p_i p_{i+1}}$ and $s_2 = \overline{p_{i+1} p_{i+2}}$, p_{i+1} is not a real vertex, and $\alpha \neq \pi$, then $\alpha < \pi$. Choose a point A and B between p_i, p_{i+1}, and p_{i+1}, p_{i+2} sufficiently close to p_{i+1}. Replace $\overline{p_i p_{i+1}} \cup \overline{p_{i+1} p_{i+2}}$ with $\overline{p_i A} \cup \overline{AB} \cup \overline{B p_{i+2}}$ in geodesic g_{pq}. Therefore it is possible to find a polygonal path between p and q of smaller length (as in Figure 9.7). By construction g_{pq} cannot have a point of positive curvature. □

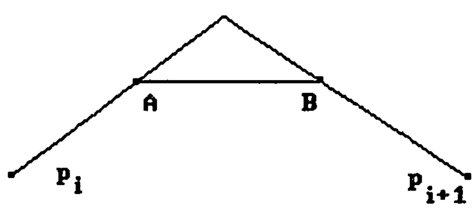


Figure 9.7

Definition. A polygonal region of the sphere is one of the two components of the complement of a simple, closed polygonal path.

Remark. Suppose P is a polygonal region whose boundary consists of minimal geodesics. Each pair of these geodesics intersect each other in at most one point. To restrict the metric to P means the following. If $p, q \in P$, then $d_P(p,q) = \inf\{ l(P_{pq}) \in \mathbb{R}$, where P_{pq} is a polygonal path in P connecting p and q and $l(P_{pq})$ denotes its length$\}$. Analogous results to the previous theorems hold when the metric is restricted to P, since P is compact like S^2. Note that some geodesics may contain vertices on the boundary of P. This will be possible if the vertices has an interior angle sum $\geq \pi$. No geodesic can contain a vertex with an interior angle sum $< \pi$.

From the beginning we have assumed that the Gaussian curvature on the interior of each triangle is the same and constant. We have already defined curvature for a point of S^2. It is also possible to define the total curvature on S^2 given a polyhedral metric. In the Euclidean case the total curvature of a triangle is 0, in the nonzero curvature case it is $\underline{+}k^2(\sum_j \alpha_j - \pi)\cdot\underline{+}1/k^2 = (\sum_j \alpha_j - \pi)$, where the α_j are the interior angles of the triangles. The total curvature of a triangle is given by the above formula, since the area of a triangle in a space of constant curvature $\underline{+}1/k^2$ is $\underline{+}k^2(\sum_j \alpha_j - \pi)$. There are also point masses of curvature at the real vertices, namely $K(v) = 2\pi - \Psi_v$. The total curvature of S^2 can now be defined as the sum of the total curvature of the faces added to the sum of the curvatures of the vertices. Now we will prove a polygonal version of the Gauss–Bonnet Theorem.

Theorem 9.4 The total curvature of the sphere with a polyhedral metric is 4π.

Proof. In the Euclidean case the total curvature is

$$2\pi V - \sum_{i=1}^{V} \Psi_i = 2\pi V - \sum_{i=1}^{F} \pi = 2\pi V - \pi(2V-4) = 4\pi.$$

When the curvature of the faces is nonzero, the total curvature is

$$2\pi V - \sum_{i=1}^{V} \Psi_i + \sum_{i=1}^{F} \sum_{j} (\alpha_{ij} - \pi)$$

$$= 2\pi V - \sum_{i=1}^{V} \Psi_i + \sum_{i=1}^{V} \Psi_i + (2V-4)\cdot(-\pi) = 4\pi.$$

Note for any triangulation we obtained 4π. □

Remark. For a surface of genius g ($\chi = 2-2g$, χ = Euler characteristic), $F = 2V + 4g - 4$. Substituting this for F in the above theorem, one finds that the total curvature is $4\pi(1-g)$ in general. The results about the geodesics also would be true for triangulations of any compact surface. If S^2 were retriangulated, then the total curvature would still be 4π. Thus, the total curvature is independent of a triangulation.

Theorem 9.5. Given a polyhedral metric m whose faces all have curvature 0 or $-1/k^2$, then there exist at least three vertices whose curvature is greater than 0.

Proof. This follows from the above theorem, since the curvature of each face is less than or equal to zero. □

Definition. The **total curvature** K **of a polygonal region** is the total curvature of its interior. This can be defined once the polygonal region has been triangulated. The following theorem shows this notion is well defined.

Remark. It is always possible to triangulate the interior of a polygonal region some of whose vertices may not belong to the original triangulation. This seems obvious but a little technical to prove. Here is an outline of a proof. Suppose T_i is a triangle of the original triangulation that determines the convex metric on S^2, and T_i intersects the interior of P. This intersection will be a finite number of polygonal regions. Triangulate these regions. You may add vertices. Do this for all triangle that intersect the interior of P until the interior of P is completely triangulated.

Theorem 9.6. Let P be a polygonal region contained in the sphere. Then

$$\sum_{i=1}^{V^B} v_i^B = (V^B - 2)\cdot\pi + K.$$

Here V^B denotes the number of vertices on the boundary of P, v_i^B is the interior measure of the angle at the i–th vertex on the boundary of P, and K is the total curvature of P.

Proof. Assume the polygonal region has a triangulation already. It is straightforward to deduce that $F = V^B + 2(V^I - 1)$, where F is the number of triangles and V^I are the number of interior vertices. First we handle the case in which the faces have 0 curvature. Let k_i be the curvature at the interior vertices Then $K = \sum_{1=1}^{V^I} k_i = \sum_{1=1}^{V^I} (2\pi - v_i^I) = 2\pi V^I - \sum_{1=1}^{V^I} v_i^I$, where v_i^I is the angle sum at the interior vertex v_i. Also

$$\pi \cdot V^B + 2\pi(V^I - 1) = \pi \cdot F = \sum_{i=1}^{V^B} v_i^B + \sum_{i=1}^{V^I} v_i^I$$
$$= \sum_{i=1}^{V^B} v_i^B + 2\pi V^I - K.$$

Therefore,

$$\sum_{i=1}^{V^B} v_i^B = (V^B - 2) \cdot \pi + K.$$

When the faces have constant curvature $\pm 1/k^2$, then the total curvature of the faces is

$$K = \sum_{1=1}^{V^I} (2\pi - v_i^I) + \sum_{1=i}^{F} (\sum_{j=1}^{3} \alpha_j^i - \pi) .$$

The second term comes from the curvature of the triangles, and α_j^i is the j–th angle of the i–th triangle. Breaking these sums up, we obtain

$$K = 2\pi V^I - \sum_{i=1}^{V^I} v_i^I + \sum_{i=1}^{V^I} v_i^I + \sum_{i=1}^{V^B} v_i^B - F \cdot \pi$$
$$= 2\pi V^I + \sum_{i=1}^{V^B} v_i^B - \pi \cdot V^B - 2\pi V^I + 2\pi.$$

Therefore, $$\sum_{i=1}^{V^B} v_i^B = (V^B - 2) \cdot \pi + K. \quad \square$$

Corollary. Let P be a polygonal region contained in the sphere. Then

$$\sum_{i=1}^{V^B} \bar{v}_i^B = 2\pi - K.$$

Here V^B denotes the number of vertices on the boundary of P, $\bar{v}_i^B$ is the measure of the exterior angle at the i–th vertex on the boundary of P, and K is the total curvature. By definition $\bar{v}_i^B = \pi - v_i^B$ (v_i^B is the interior angle).

Proof. This corollary follows from the previous theorem, since

$$\sum_{i=1}^{V^B} \bar{v}_i^B = \sum_{i=1}^{V^B} (\pi - v_i^B) = \pi V^B - ((V^B - 2)\cdot\pi + K) = 2\pi - K. \quad \square$$

Theorem 9.7. Suppose that a polyhedral metric on S^2 is formed by faces of curvature 0 or $-1/k^2$. Given a polygonal region P of S^2, whose interior has no real vertices from the given triangulation that determined the polyhedral metric, it is possible to retriangulate the region so it has all of its vertices on the boundary. Note that the vertices of P may or may not be real vertices.

Proof. This will be proved by induction on V^B, the number of vertices on the boundary. If $V^B = 3$, then there is nothing to prove. Suppose $V^B = k + 1$. From the above corollary there must be three vertices on the boundary with exterior angle measure greater than 0. Hence one can choose four vertices v_1, v_2, v_3, and v_4, such that v_1 and v_3 lie between v_2 and v_4, with v_2 and v_4 having positive exterior angles, so the interior angle at v_2 and v_4 are between 0 and π (Figure 9.8).

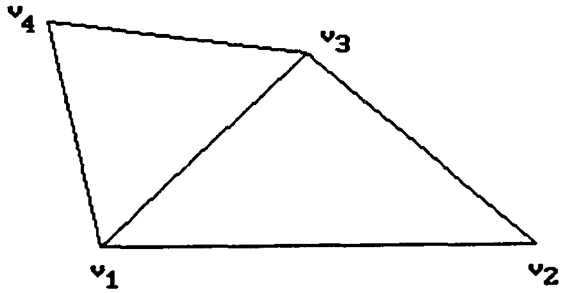


Figure 9.8

Connect v_1 and v_3 with a geodesic that lies in the given polygonal region. This geodesic cannot pass through v_2 or v_4, because if the geodesic did pass through v_1 or v_4 it could be made shorter. Therefore the polygonal region has been divided into at least two polygonal regions with $V^B \leq k$. Hence the theorem is proved by the induction hypothesis. □

Lemma. Assume g_1 and g_2 are two minimal geodesic connecting v_1 to v_2 and v_1 to v_3. Then $g_1 \cap g_2 = \{v_1\}$.

Proof. If g_1 and g_2 intersected at another point besides v_1 then it would be possible to find a path from v_1 to v_2 of length less then $d(v_1,v_2)$.

Theorem 9.8 Suppose that S^2 has a polyhedral metric with no real vertices of negative curvature and the curvature of all its faces is either 0 or $-1/k^2$. Then it is possible to retriangulate S^2 in such a way that all of its vertices are real.

Proof. Choose three real vertices v_1,v_2 and v_3, this is allowed by Theorem 9.5. Connect these vertices with minimal geodesics. By the previous lemma we will have a triangle as in Figure 9.9.

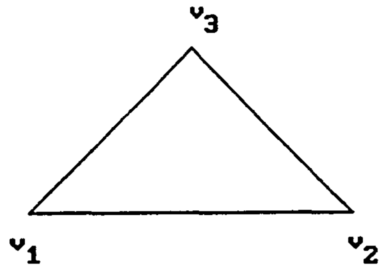


Figure 9.9

If there are no more real vertices then we are done. Let v_4 be another real vertex. It must lie in one of the two triangles formed by v_1, v_2 and v_3. If the angles at v_1, v_2 and v_3 are all less then π, then we can form the triangulation in Figure 9.10.

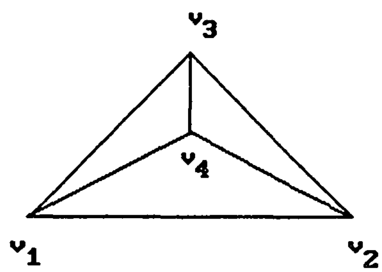


Figure 9.10

If one of the vertex angles is greater than π, then we do the following. We connect v_4 to v_1 with a minimal geodesic with respect to the metric restricted to the given triangle. One of two things can happen. Either $g_{v_1 v_4}$ lies completely in the given triangle or it passes through some vertex, say v_2, whose interior angle is greater than π. The situation in Figure 9.11 holds.

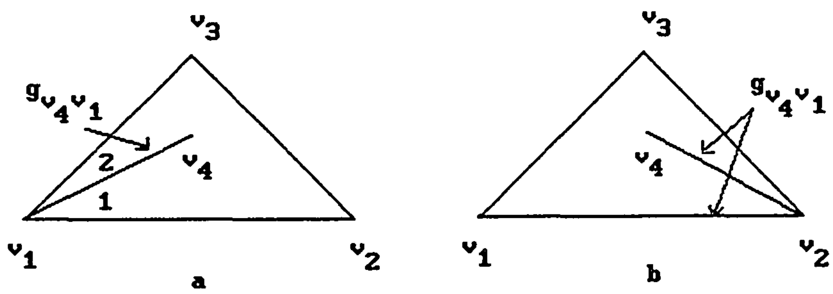


Figure 9.11

Note the part of the geodesic which lies in the interior of the triangle does not contain any real vertices, because if it did it could be made shorter. Assume the situation in Figure 9.11a holds. Either $\measuredangle 1$ or $\measuredangle 2$ is less then π. If $\measuredangle 1$ is less than π, then connect v_4 to v_2 with a minimal geodesic. The path $v_4v_1 \cup v_1v_3 \cup v_3v_2$ is not a minimal geodesic, hence then we have one of the following situations. Either the minimal geodesic $g_{v_4v_2}$ passes through v_2 or it passes through v_3 (Figure 9.12). In either case we have now divided the sphere into three polygonal regions, two triangles and one four–sided polygon. Assume there is a fifth vertex v_5. The vertex v_5 lies in one of three polygonal regions. By repeating the above argument it is possible to connect v_5 to two other vertices with a minimal geodesic with respect to the given polygon in which v_5 lies.

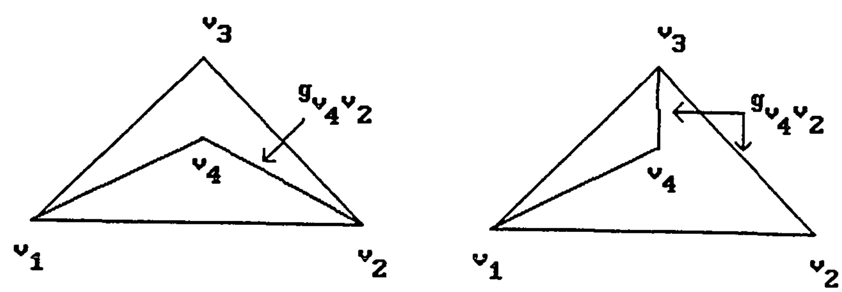


Figure 9.12

The sphere has now been partitioned into four polygons. Continue this procedure until there are no more vertices. By Theorem 9.6 it is possible to triangulate all the polygons which were formed. Therefore all of S^2 has been triangulated.

Remark. By construction, the triangle inequality holds for the lengths of the edges of any triangle of this triangulation.

CHAPTER 10: THE TETRAHEDRON METRIC

Given three positive numbers which satisfy the triangle inequality, it is possible to construct a triangle in E^3 or H^3 which has these numbers as the lengths of its sides. By assigning these numbers to edges of a triangle, and the curvature 0 or -1 to the triangle's face, it is possible to place a metric on the closed triangle. We therefore have proved that any metric of this type is realizable in E^2 or H^2. We will now generalize this idea to the triangulation of S^2, by assigning curvature 0 or -1 to the faces of the triangulation, numbers to the edges in the appropriate fashion (described shortly), thus placing a convex polyhedral metric (defined shortly) on S^2. Then we will prove these metrics are realizable by convex polyhedra in E^3 or H^3.

First let's suppose the triangulation of S^2 is isomorphic to the standard tetrahedron, or equivalently, suppose S^2 has a simplicial complex with 4 faces, 6 edges, 4 vertices, and the degree of each vertex is 3. Assign six positive numbers $e_1, \ldots, e_6$ to the edges such that $e_i < e_j + e_k$, whenever e_1, e_2, and e_3 belong to the same triangle (face). In Chapter 9, we showed that this assignment of numbers to the edges and the assignment of curvature of 0 or -1 to the faces gave rise to a polygonal metric on S^2. If α_{ij} is the j–th angle at the i–th vertex, assume $\Psi_i = \sum_j \alpha_{ij} \le 2\pi$. When $\Psi_i = \sum_j \alpha_{ij} \le 2\pi$, the metric is defined to be convex. When the faces are assigned 0 curvature the metric will be realized in E^3, and when the faces have curvature -1 the metric will be realized in H^3. This realization will be unique. If there were two realizations of this metric, then there exists an isometry which carries one tetrahedron onto the other, by Cauchy's Rigidity Theorem on convex bodies in E^3 or H^3.

Cauchy's Rigidity Theorem. Let C_1 and C_2 be two convex polyhedra in E^3 or H^3, with topologically equivalent triangulations. Suppose that corresponding faces of C_1 and C_2 are isometric. Then C_1 is congruent to C_2.

From Theorem 9.5, there exist three vertices whose angle sum is strictly less than 2π. If a vertex v has an angle sum of 2π construct a double covered triangle (Figure 10.1).

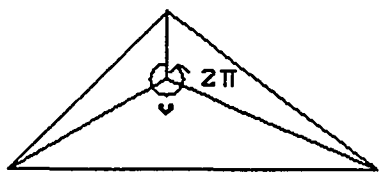


Figure 10.1

The double covered triangle in Figure 10.1 is considered to be a convex body and Cauchy's Rigidity Theorem can be extended to polyhedra of this type. Now suppose that the angle sum at each vertex is less than 2π. This case breaks up into various cases. If at a given vertex the angles satisfied the triangle inequality, then the embedding problem is again simple. Indeed, at vertex v_1, suppose α_{11}, α_{12} and α_{13} were the given angles.

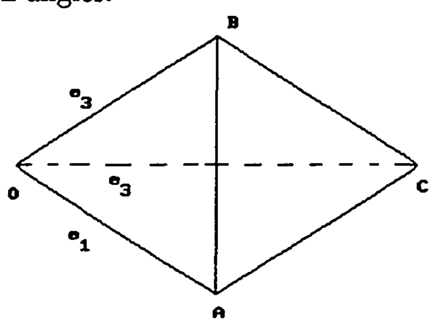


Figure 10.2

If $\alpha_{1i} < \alpha_{1j} + \alpha_{1k}$ for all i, j, k, just construct the dihedral angle with angles α_{11}, α_{12} and α_{13}, which is always possible (Figure 10.2). On the given rays emanating from O move out lengths e_1, e_2, and e_3 to points A, B and C. Then connect the points A, B and C, as in Figure 10.2. The constructed tetrahedron will be the desired tetrahedron. The remaining sides have lengths e_4, e_5 and e_6 by side–angle–side. If there were angle equality at a given vertex say $\alpha_{1i} = \alpha_{1j} + \alpha_{1k}$ at vertex v_1, then it is possible to construct a double covered polyhedron (Figure 10.3). In this case, one can view the sphere as two disks connected along their boundary.

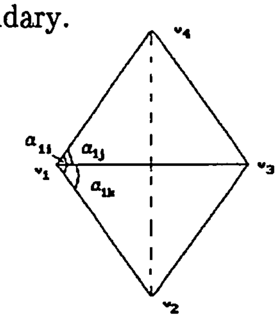


Figure 10.3

Now suppose the given metric has no vertex which satisfies the angle inequality. The idea of the proof is to retriangulate the abstract tetrahedron with minimal geodesics. This retriangulation will be isomorphic to the standard tetrahedron and an angle inequality will be forced at a vertex. Then one proceeds as in the previous cases, and the proof will be completed. To prove that this retriangulation can be done, do the following. Suppose v_1, v_2, and v_3 are three of the given vertices. Connect v_1 to v_2 with a minimal geodesic. This can be done by Theorem 9.3. Call this given geodesic segment e_3. Now connect v_2 to the vertex v_3 with a minimal geodesic. Call this geodesic segment e_1. It is straightforward to verify $e_1 \cap e_3 = \{v_2\}$. Now connect v_3 to

v_1 with a minimal geodesic e_2. By the lemma preceding Theorem 9.8, no minimal geodesics emanating from the same real vertex and ending up at two different real vertices intersect. Using this fact it is straightforward to verify $e_2 \cap e_3 = \{v_1\}$ and $e_2 \cap e_1 = \{v_3\}$. Now the sphere can be viewed as two triangles joined along their boundaries. The vertex v_4 must lie inside one of these triangles. This follows from Theorem 9.3. By cutting along the edges e_2 and e_3 and opening the sphere up, we obtain the following (Figure 10.4).

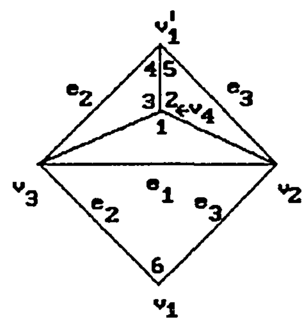


Figure 10.4

Now connect v_4 to the remaining vertices with minimal geodesics, where these minimal geodesic are minimal with respect to the triangle in which v_4 lies. This is allowed by an argument similar to that in Theorem 9.8. Without loss of generality, in Figure 10.4, we can assume $\measuredangle 1$ is greater than or equal to $\measuredangle 2$ and $\measuredangle 3$. Now we will prove $\measuredangle 1 \leq \measuredangle 2 + \measuredangle 3$. We can assume $\measuredangle 2 + \measuredangle 3 < \pi$, because if $\measuredangle 2 + \measuredangle 3 \geq \pi$, then $\measuredangle 1 < \measuredangle 2 + \measuredangle 3$, and we would be done by a previous case. One of the following three cases must hold: $\measuredangle 4 + \measuredangle 5 < \pi$, $\measuredangle 4 + \measuredangle 5 = \pi$, $\measuredangle 4 + \measuredangle 5 > \pi$. Each case is represented in one of the following figures.

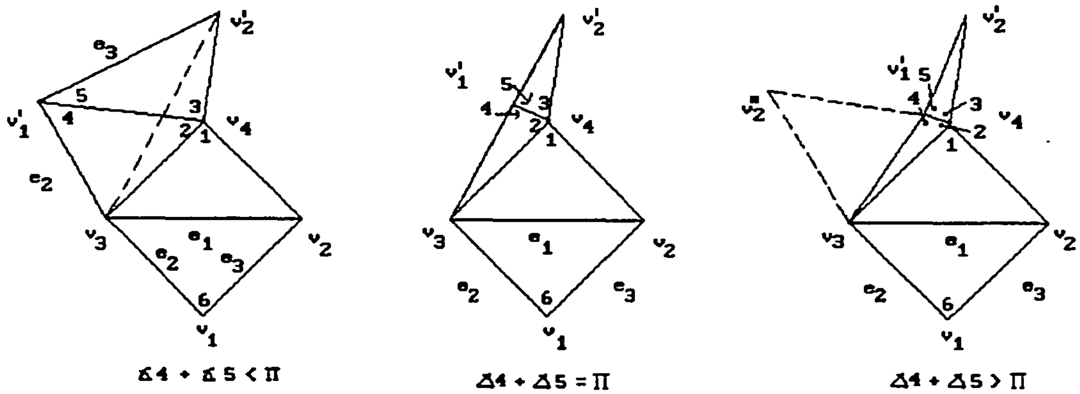


Figure 10.5

In each of the above figures, the tetrahedron has been cut along edge e_4. Since the angle sum at each vertex is less then 2π, one can lie the tetrahedron down in the euclidean or hyperbolic plane. First assume $\measuredangle 4 + \measuredangle 5 < \pi$. Then the situation in Figure 10.5 holds. If $\measuredangle 2 + \measuredangle 3 < \measuredangle 1$, then deduce the length of chord $\overline{v_3 v_2'}$ is less than e_1 which contradicts the minimality of e_1. If $\measuredangle 4 + \measuredangle 5 = \pi$, then deduce $e_2 + e_3 < e_1$, another contradiction. If $\measuredangle 4 + \measuredangle 5 > \pi$ then in a counterclockwise direction from edge $\overline{v_3 v_1'}$ at vertex v_1' with an angle of measurent of $\measuredangle 6$ construct edge $\overline{v_1' v_2''}$ of length e_3 (Recall $\measuredangle 4 + \measuredangle 5 + \measuredangle 6 < 2\pi$). Deduce $e_1 < \overline{v_3 v_2''}$, but we also deduce from the triangle with vertices v_3, v_4 and v_2' that $\overline{v_3 v_2''} < e_1$, another contradiction. To summerize the above, the following have been proved.

Theorem 10.1. Suppose m is a convex metric on S^2 which comes from an abstract tetrahedron. Then this metric is realizeable in E^3 or H^3.

Theorem 10.2. Suppose m is a convex metric on S^2 which arises from an abstract tetrahedron. Then there exists a triangulation with minimal geodesics.

CHAPTER 11: ALEXANDROV

In Chapter 9 a polyhedral metric was defined. Given a fixed triangulation T of S^2, it is possible to make the set of all the polyhedral metrics which come from this triangulation a manifold M, by allowing the edges to vary. This idea is due to the Russian mathematician Alexandrov.

Assume that the edges E and vertices V of the triangulation are ordered. To each edge $e_i \in E$ assign a positive number. Also denote this number by e_i. If $e_{i_1}, e_{i_2}, e_{i_3}$, are edges of the same triangle, then the triangle inequality must hold, that is, $e_{i_1} < e_{i_2} + e_{i_3}, e_{i_2} < e_{i_1} + e_{i_3}, e_{i_3} < e_{i_1} + e_{i_2}$. Assume either euclidean or hyperbolic geometry holds for each face (the faces have curvature 0 or -1), then from the discussion in Chapter 9, it is possible to define a metric m from this triangulation. We will make further restrictions on the numbers e_i. Let ϕ_{ij} be the j–th angle at vertex v_i, which is implicitly defined by

$$c^2 = a^2 + b^2 - 2ab\cos(\phi_{ij})$$

or

$$\cosh c = \cosh a \cdot \cosh b - (\sinh a)\cdot(\sinh b)\cdot\cos(\phi_{ij}).$$

The side c is opposite angle ϕ_{ij} and a and b are the remaining sides. Use the first formula in the euclidean case and the second formula for the hyperbolic case.

Define $\Psi_i = \Sigma_k \phi_{ik}$. Also assume $\Psi_i \leq 2\pi$ $\forall$i. An abstract polygonal metric satisfying these conditions will be called an abstract convex metric. If $\Psi_i < 2\pi$ $\forall$i, the metric is defined to be **strictly convex**.

Triangulating the faces of any convex polyhedron in E^3 or H^3 gives rise to a abstract convex metric on S^2. Also, a double covered convex polygon (2 polygonal regions glued together along their boundary) gives rise to a convex polyhedron when a triangulation is placed on it. We will prove all the possible abstract convex metrics that arise on S^2 are realized by concrete polyhedra.

Given two points p and q on S^2, with a polyhedral metric, there exists a minimal geodesic connecting p and q (Theorem 9.2). A vertex will be called a **real vertex** if $K(v_i) = 2\pi - \Psi_i > 0$. Call $K(v_i)$ the curvature at v_i. All other points on S^2 are defined to have curvature 0 or -1, depending on the case being considered. From Theorem 9.3, no geodesic goes through a real vertex. From Theorem 9.4, there must be at least 3 vertices whose curvature is less than 2π.

Suppose T is a fixed triangulation, with vertices V(T) and edges E(T). E will denote the number of edges, and V will denote the number of vertices of T. Assume the vertices and edges are ordered. $V(T) = \{v_1, v_2, ..., v_V\}$ and $E(T) = \{e_1, e_2, ..., e_E\}$. Set

$$M = \{(e_1, ..., e_E) \in \mathbb{R}^E, \text{ such that if } e_i, e_j, e_k \text{ are on the same face, then the triangle inequality holds } e_i < e_j + e_k\}$$

Define $E_i((e_1, e_2, ..., e_E)) = e_i$ and $f_{i,j,k} ((e_1, e_2, ..., e_E)) = e_i + e_j - e_k$. The functions $f_{i,j,k}$ are only defined when i, j, k belong to the same face. Then $M = \bigcap_i E_i^{-1}(x>0) \cap \bigcap_{i,j,k} f_{ijk}^{-1}(x>0)$. This intersection is finite. The sets $E_i^{-1}(x>0)$ and $f_{ijk}^{-1}(x>0)$ are open sets (inverse images of open sets of continuous functions). Therefore M is an open set of $\mathbb{R}^E$, and hence a differentiable manifold. Note that no conditions on Ψ_i were assumed.

Define $\quad M_{<2\pi} = \{m \in M, \text{ such that } \forall i, \Psi_i = \sum_j \phi_{ij}(m) < 2\pi\}.$

Suppose the triangle at vertex v_i with angle ϕ_{ij} has sides e, f, e_{ij} where e_{ij} is opposite ϕ_{ij}. Since on each triangle euclidean geometry or hyperbolic geometry holds, ϕ_{ij} is implicitly defined by

$$e_{ij}^2 = e^2 + f^2 - 2ef\cos(\phi_{ij})$$

or

$$\cosh(e_{ij}) = \cosh(e)\cdot\cosh(f) - \sinh(f)\cdot\sinh(e)\cdot\cos(\phi_{ij}).$$

By the above formula, one can deduce that ϕ_{ij} is a differentiable a function from M to $\mathbb{R}$. The function $\Psi_i = \sum_j \phi_{ij}$ is also a differentiable function from M to $\mathbb{R}$. By definition, the set $M_{<2\pi} = \bigcap_i \Psi_i^{-1}(\, 0 < x < 2\pi\,)$. Therefore $M_{<2\pi}$ is an open set of M, and so $M_{<2\pi}$ is a manifold. Define

$M_{v_i} = \{\, m \in M, \text{ such that } \sum_j \phi_{ij} = 2\pi, \text{ and for } k \neq i,\ \Psi_k = \sum_j \phi_{kj} < 2\pi\,\}$, and

$M_{v_{i_1} v_{i_2}} = \{\, m \in M, \text{ such that } \sum_j \phi_{i_1 j} = 2\pi\,,\ \sum_j \phi_{i_2 j} = 2\pi\,, \text{ and for}$

$\qquad k \neq i_1, i_2,\ \Psi_k = \sum_j \phi_{kj} < 2\pi\,\}.$

Define $M_{v_{i_1} \dots v_{i_n}}$ analogously.

Lemma 1. Let $\partial(M_{<2\pi})$ denote the boundary of $M_{<2\pi}$. Then $\partial(M_{<2\pi}) \subset (\bigcup_i M_{v_i}) \cup (\bigcup_{ij} M_{v_i v_j}) \cup \dots \cup (\bigcup_{i_1 \dots i_n} M_{v_{i_1} \dots v_{i_n}})$, and $V - n > 3$. Since $V - n > 3$, there are at least 3 vertices that have a total angle less than 2π.

Proof. Suppose $m \in \partial(M_{<2\pi})$. Then $\exists m_n \in M$ such that $\lim_{n\to\infty} m_n = m$. Since $m_n \in M$, we have $\Psi_i(m_n) = \sum_j \phi_{ij}(m_n) < 2\pi$, for all n. By the continuity of Ψ_i, $\lim_{n\to\infty} \Psi_i(m_n) = \Psi_i(m) \leq 2\pi$, for all i. Since M is open there exists an i such that $\Psi_i(m) = 2\pi$. By Theorem 9.5 $V - n \geq 3$. Therefore, $m \in$ R.H.S. □

Theorem 11.1. Let $M_{<2\pi} = \bigcup_{i\in I} M_i$, where the M_i are the path connected components of $M_{<2\pi}$. For each $i \in I$, $\exists m \in \partial M_i$ and an open set $U_m \subset M$, such that $m \in U_m$, and $U_m \cap M_{<2\pi} \subset M_i$.

Proof. Define $M^o_{v_i} = \{m \in M$, such that $\sum_j \phi_{ij} = 2\pi\}$. Note that $M^o_{v_i}$ may not equal M_{v_i}, since for $m \in M^o_{v_i}$ there may exist an $i_o \neq i$ such that $\Psi_{i_o}(m) \geq 2\pi$. Let M_i be a connected component of $M_{<2\pi}$. Clearly, $\partial(M_{<2\pi}) \neq \phi$ and $\partial M_i \neq \phi$. Thus, there is some

$$m \in \partial M_i \subset \partial M_{<2\pi} \subset (\bigcup_i M_{v_i}) \cup (\bigcup_{ij} M_{v_i v_j}) \ldots \cup (\bigcup_{i_1 \ldots i_n} M_{v_{i_1} \ldots v_{i_n}}).$$

It suffices to show by Lemma 1 that if $m \in M_{v_{i_1} \ldots v_{i_n}}$ and $m \in \partial M_i$, then there exists an m^* with the property stated the lemma. This will be proved by induction on n, the number of vertices v_i of m such that $\sum_j \phi_{ij} = 2\pi$. Define $C^E_\epsilon(x) = \{y \in \mathbb{R}^E$, such that $| x^i - y^i | < \epsilon$, for $i = 1, 2, \ldots n\}$. The set $C^E_\epsilon(x)$ is an open cube in $\mathbb{R}^E$ with center x and edge length 2ϵ. Assume $n = 1$, and let $m \in M_{v_1} \subset M^o_{v_1}$. Recall $M^o_{v_1} = \{m \in M$, such that $\sum_j \phi_{1j} = 2\pi\}$. Without loss of generality, assume edge e_1 is opposite vertex v_1 in the triangulation. Since $D_{e_1}(\sum_j \phi_{ij}) > 0$, M_{v_1} is a (E–1)–dimensional submanifold of M.

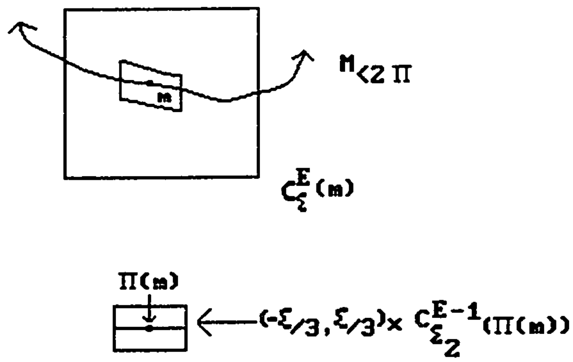


Figure 11.1

We have $m \in M_{v_1}$, $\sum_j \phi_{1j} = 2\pi$ and $\sum_j \phi_{ij} < 2\pi$ for $i = 2, 3, \ldots, V$. By continuity of the functions $\sum_j \phi_{ij}$ ($\forall i \geq 2$) and the fact M is open is $\mathbb{R}^E$, $\exists \epsilon > 0$ such that if $x \in C^E_\epsilon(m)$, then $\sum_j \phi_{ij}(x) < 2\pi$, for $\forall i \geq 2$. Define $\pi(e^1,\ldots,e^E) = (e^2,\ldots,e^E)$. By the implicit function theorem, there exists $\epsilon_1 > 0$, such that M_{v_1} is represented by the graph $(f(e^2, \ldots, e^E), e^2, \ldots, e^E)$, for $(e^2,\ldots, e^E) \in C^{E-1}_{\epsilon_1}(\pi(m))$. The function f is also differentiable. By continuity $\exists\, \epsilon_2 > 0$, such that $x \in C^{E-1}_{\epsilon_2}(\pi(m))$ implies $|f(x) - f(\pi(m))| < \epsilon/3$ (assume $\epsilon_2 < \epsilon_1$). Define the function $F\colon (-\epsilon/3,\epsilon/3) \times C^{E-1}_{\epsilon_2}(\pi(m)) \to M$ by $F(e^1, e^2,\ldots, e^E) = (f(e^2,\ldots,e^E) + e^1, e^2,\ldots, e^E)$. This is a homeomorphism. The set $F((-\epsilon/3,\epsilon/3) \times C^{E-1}_{\epsilon_2}(\pi(m))) \subset C^E_\epsilon(m)$, is open ($\epsilon_2 < \epsilon_1$). Since $m \in \partial M_i$ and $D_{e_1}(\sum_j \phi_{ij}) > 0$, everything of the form $F(e^1,\ldots,e^E)$ with $e^1 \geq 0$ lies outside $M_{<2\pi}$. Therefore, the lemma holds for m, and the neighborhood U_m is $F((-\epsilon/3,\epsilon/3) \times C^{E-1}_{\epsilon_2}(\pi(m)))$.

Now for the induction step. If $m \in M_{v_{i_1}\ldots v_{i_{n+1}}}$, then $m \in M^o_{v_{i_1}} \cap \ldots \cap M^o_{v_{n+1}}$. Reasoning as in the $n = 1$ case (assume edge e_1 is opposite v_{i_1}), there exist an $\epsilon > 0$ and a homeomorphism

$$F: (-\epsilon/3,\epsilon/3) \times C_\epsilon^{E-1}(\pi(m)) \to M$$

(for an appropriate $\epsilon > 0$), such that if $-\epsilon < e^1 < 0$, then $\sum_j \phi_{i_1 j}(F(e^1,....e^E)) < 2\pi$ and if $0 < e^1 < \epsilon$, then $\sum_j \phi_{i_1 j}(F(e^1,....e^E)) > 2\pi$.

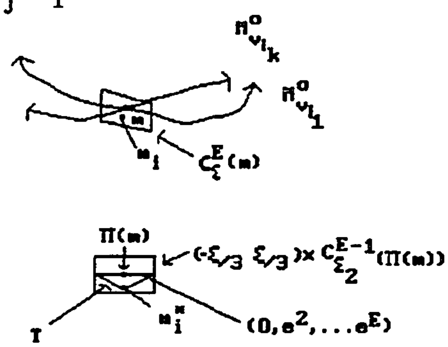


Figure 11.2

We also have $\sum_j \phi_{ij}(p) < 2\pi$ if $i \notin (i_1,...i_{n+1})$ and $p \in C_\epsilon^E(m)$. Since $m \in \partial M_i$, there exists $m_i \in M_i$ such that $m_i \in F((-\epsilon/3,\epsilon/3) \times C_\epsilon^{E-1}(\pi(m))) \subset M$. Let $m_i^* = F^{-1}(m_i)$; define $T = \{\, p \in (-\epsilon/3,\epsilon/3) \times C_\epsilon^{E-1}(\pi(m)\}$, such that $\exists x \in \{0\} \times C_\epsilon^{E-1}(\pi(m))$ and $p \in \overline{m_i^* x}$, $\overline{m_i^* x}$ is the set of points on the line segment connecting m_i to x not including $x\}$. Define $SQ = \{p \in (-\epsilon/3,\epsilon/3) \times C_\epsilon^{E-1}(\pi(m)\}$, where $p = (x_1,....x_E)$ and $x_1 \geq 0\}$. If $F(T) \subset M_i$, then we are done because $m \in F((T \cap SQ)^o)$ ($(T \cap SQ)^o$ is the interior of $T \cup SQ$). If $F(T)$ is not contained in M_i, then $\exists p \in (-\epsilon/3,\epsilon/3) \times C_\epsilon^{E-1}(\pi(m))$, $p \in \overline{m_i^* x}$ and $x \in \{0\} \times C_\epsilon^{E-1}(\pi(m)$ and $F(p) \notin M_i$. Let $\alpha: [0,1] \to M$ be defined by $\alpha(t) = m_i + t(x-m_i)$, α is the line segment connecting m_i and x. Let $t_o =$

$\{\inf t \in [0,1]$, such that, $\alpha(t) \notin M_i\}$. We have $0 < t_o < 1$, $\alpha(t_o) \in \partial M_i$ and $\sum_j \phi_{ij}(\alpha(t_o)) < 2\pi$ if $i \notin \{i_2,\ldots,i_{n+1}\}$. Therefore $\alpha(t_o)$ has at most n deletable vertices. By the induction hypothesis, $\exists m \in M_i$ and an open set U_m, such that $m \in \partial M_i$ and $U_m \cap M \subset M_i$. □

In [Efimov] the following theorem which is due to Alexandrov was proved. The proof Efimov gives seems to lack some rigor. This theorem will be discussed.

Theorem 11.2. For each convex metric on a (euclidean or hyperbolic) abstract polyhedron, there exists a unique (up to euclidean or hyperbolic isometries) closed convex polyhedron in E^3 or H^3 realizing it.

If the theorem has been proven for the strictly convex metrics, it then can be proven for all convex metrics, as follows. Let m be a convex metric. By Theorem 9.8, it is possible to retriangulate the sphere with geodesics using only vertices of positive curvature. Call the metric which arises from this new triangulation $\overline{m}$. For all $p, q \in S^2$, $d_m(p,q) = d_{\overline{m}}(p,q)$. The metric $\overline{m}$ is realizable. Since $\overline{m}$ is isometric to m, it follows m is realizable by a retriangulation of $\overline{m}$ in E^3 or H^3.

To prove Theorem 11.2, we will use induction on the number of vertices. For the $V = 3$ case, just construct the triangles with sides of the desired lengths and make a double covered triangle. The $V = 4$ case was proved in Chapter 10.

Given convex polyhedron $\overline{p}$, with a triangulation placed on it, so that the triangle inequality holds for the lengths of the edges, with V vertices in E^3 or H^3, one may construct a manifold P. Order the vertices of $\overline{p}$. Place the first vertex of $\overline{p}$ at $(0,0,0)$, the second vertex on the ray $\{(x,0,0) \in \mathbb{R}^3 \; x > 0\}$, the third vertex in the half plane $\{(x,y,0) \in \mathbb{R}^3, y > 0\}$. The polyhedron $\bar{p}$ lies in E^3 or H^3 in one of two ways. Choose one of them. Now perturb the vertices $v_2,\ldots,v_V$ slightly, but the vertex v_2 must remain on the ray $\{(x,0,0) \in \mathbb{R}^3\ x > 0\}$, and v_3 must remain on the half–plane $\{(x,y,0) \in \mathbb{R}^3, y > 0\}$. The vertex v_1 has coordinates $(0,0,0)$, v_2 has coordinates of the form $(x_1,0,0)$, $x_1 > 0$, v_3 has coordinates of the form $(x_2,x_3,0)$ $x_3 > 0$, v_4 has coordinates (x_4,x_5,x_6), etc. Continue this process till v_V has been assigned coordinates (x_{E-1},x_{E-2},x_E). The last index is E since $E = 3V - 6$. One is allowed to move the vertices of $\bar{p}$ as long as the perturbed polyhedron $\bar{p}'$ remains strictly convex, and the starting triangulation is moved to another triangulation whose edge lengths preserve the triangle inequality. The manifold P is the set of all polygons obtained by perturbing the vertices of $\bar{p}$. The manifold P can be viewed as a subset of $\mathbb{R}^E$, and it is an open subset since the angles at each vertex and the lengths of the edges of the triangulation depend continuously on $x_1,\ldots,x_E$. The dimension of P is E.

Lemma 2. Let M be a manifold of polyhedral metrics (euclidean or hyperbolic). Each $m \in M$ determines a metric d_m on S^2. The metrics d_m are continuously dependent on m, in the following sense. The manifold M can be identified with an open subset of $\mathbb{R}^E$. So let d_M denote distance between two points in $\mathbb{R}^E$ with the usual euclidean metric. To say the metric is continuous at m means, given $\epsilon > 0$, then there exist a $\delta > 0$, such that $\forall m' \in M$ and $\forall p, q \in S^2$, $d_M(m,m') < \delta$ implies $|d_m(p,q) - d_{m'}(p,q)| < \epsilon$. Later in this chapter, we may identify m with d_m.

Note: In order to have this theorem make sense it is necessary to pair the triangles and line segments of metric m to the triangles and line segments of metric m'.

Given three sides of a euclidean triangle it is possible to map it to three sides of another euclidean triangle using an affine transformation. This also maps geodesics to geodesics. Given three sides of one hyperbolic triangle it is possible to map it to three sides of another hyperbolic triangle using the model of H^2 defined on the set of points $\{(x_1,x_2,x_3) \in \mathbb{R}^3, -x_1{}^2 + x_2^2 + x_3^2 = -1, x_1 > 0\}$ when $\mathbb{R}^3$ has the metric $ds^2 = -dx_1^2 + dx_2^2 + dx_3^2$. Assume the two triangles we would like to compare lie in this model. Map one of the triangles to the other using a linear transformation. This is possible since the vertices of the triangles can be viewed as vectors emanating from the origin. Since geodesics are planes through the origin intersected with the upper half of the given hyperboloid defined by the equation $-x_1{}^2 + x_2^2 + x_3^2 = -1$, geodesics map to geodesics. Now one can naturally pair triangles and geodesic segments in both the euclidean and hyperbolic case.

Proof. Suppose $m \in M$ and $p, q \in S^2$. Let $L = \sup(\{d_m(r,s), r, s \in S^2\} \cup \{1\})$. P_{pq} is the set of all polygonal paths from p to q, and l^m is the length of segment l in the metric d_m. Let $\epsilon > 0$, and also assume that $\epsilon < L/2$. Now suppose that the segment l lies completely inside a closed triangle from the triangulation used to construct m. Choose a $\delta > 0$, such that $(1 - \epsilon/2L)l^m < l^{m'} < (1 + \epsilon/2L)l^m$, if $d_M(m,m') < \delta$ and l lies completely inside a closed triangle. Here $l^{m'}$ is the of the geodesic segment in the metric m' corresponding to l in m. Let $P \in P_{pq}$, and $P = l_1 \cup l_2 \cup \ldots \cup l_n$, where the l_i are geodesic segments. Without loss of generality, we can suppose all the l_i are contained in the triangles of the triangulation. Then $\sum_{i=1}^{n} (1 - \epsilon/2L)l_i^m < \sum_{i=1}^{n} l_i^{m'} < \sum_{i=1}^{n} (1 + \epsilon/2L)l_i^m$. From this inequality, we have $|\{\inf\{l_m(P), P \in P_{pq}\} - \{\inf\{l_{m'}(P), P \in P_{pq}\}| < \epsilon$ or $|d_m(p,q) - d_{m'}(p,q)| < \epsilon$. □

Lemma 3. Let $m \in M_{<2\pi}$, and let $M'_{<2\pi}$ be a manifold obtained from retriangulating $M_{<2\pi}$ with minimal geodesics (this will be explained in the proof). Then there exist an open neighborhood $U_m \subset M_{<2\pi}$, $m \in U_m$, and a continuous, one–to–one function $F_m\colon U_m \to M'_{<2\pi}$.

Proof. Let $m \in M_{<2\pi}$. Retriangulate m with minimal geodesics $g_1, \dots, g_E$ joining real vertices. From this triangulation, it is possible to create a new manifold $M'_{<2\pi}$, as before. Let the lengths of the minimal geodesics in metric d_m also be denoted by $g_1, \dots, g_E$. Define $F_m(m) = (g_1, \dots, g_E)$. If g_i, g_j, and g_k are edges on the same face then since d_m is a metric $g_i < g_j + g_k$. By slightly changing the lengths of the edges the triangle inequality will be preserved. Also since the Ψ_i do not change much, the map is into $M'_{<2\pi}$. For m' sufficiently close to m, define $F_m(m') = (g'_1, \dots, g'_E)$, where g'_i is the length of the i–th edge in metric $d_{m'}$. The result now follows from Lemma 2 and its Note. □

We now return to the proof of Theorem 11.2. Let M_i be an arbitrary connected component of $M_{<2\pi}$. There exist by Theorem 11.1 an $m \in \partial M_i$ and an open set U_m, such that $U_m \cap M_{<2\pi} \subset M_i$. Delete all vertices v_i from m, where $\Psi_i = \sum_j \phi_{ij}(m) = 2\pi.$, and retriangulate m with geodesics that pass through the real vertices (Theorem 9.8). Call this metric m^*. By induction, m^* is realized by a convex polygon p^* in E^3 or H^3. Now place the old triangulation from m on this polyhedron. Perturb all the vertices v_k slightly that have $\sum_j \phi_{kj} = 2\pi$, and obtain a convex polyhedron $\overline{p}$. The metric obtained from this triangulation on $\overline{p}$ lies in M_i. From this polyhedron $\overline{p}$, construct a manifold P, as was previously done. Assume the edges and vertices of m are ordered. This orders the edges and vertices of the corresponding triangulation on $\bar{p}$. Define a map $\phi: P \to M_{<2\pi}$ by $\phi(p) = (E_1(p),\dots,E_E(p))$, where $E_i(p)$ is the length of the i–th edge on $\bar{p}$. The map ϕ is continuous. Small variation of the vertices of $\bar{p}$ causes small variation of the lengths of the real edges of $\bar{p}$ (by real edges of $\bar{p}$, one means those edges which come from E^3 or H^3), and this causes small variations of the lengths of the edges of the triangulation placed on $\bar{p}$. Therefore the map ϕ is continuous (ϕ is probably analytic). ϕ also has a closed image. To show ϕ has a closed image, suppose there exists a sequence of metrics $m_n \in M_{<2\pi}$, such that $m_n \to m \in M_{<2\pi}$, and for each m_n, there is a $p_n \in P$ such that $\phi(p_n) = m_n$. It must be shown that $\exists p \in P$ with $\phi(p) = m$. By definition of P, the first vertex of each p_n is (0,0,0). The second vertex is on the positive x–axis, and the third vertex belongs to $\{(x,y,z) \in \mathbb{R}^3,$ where $y > 0, z = 0\}$. The rest of the vertices of p_n are sent to E^3 or H^3. Since $m_n \to m$, it follows that the the p_n are bounded. Let v_i^n = the vertices of p_n ($v_1^n = (0,0,0)$). There exists a subsequence $v_2^{n_k}$ with $\lim_{k\to\infty} v_2^{n_k} = v_2$. From this sequence, there exists a subsequence $v_3^{n_{k_j}}$, such that $\lim_{k\to\infty} v_3^{n_{k_j}} = v_3$. Continuing this process, we obtain a subsequence n_j of n, such that $\lim_{j\to\infty} v_i^{n_j} = v_i$. The polyhedron $p = \lim_{j\to\infty} p_{n_j}$ is formed by the vertices v_i. It belongs to P, since the angle inequality conditions and triangle inequality conditions are satisfied by continuity of the dot product. Also, $\phi(p) = m$, since $\lim_{j\to\infty} m_{n_j} = m$. Therefore, ϕ is a closed map. The inverse images of bounded sets under ϕ are bounded. Therefore, ϕ is a proper map (by definition a function is proper if the inverse image of compact sets are compact).

It is possible to partition P into two sets P_d and $P_r = P - P_d$. The set P_d consists of the degenerate polyhedra of P, that is double covered polyhedra of P. More precisely,

$P_d = \{p \in P$, such that $p = ((0,0,0),(x_1,0,0),(x_2,x_3,0),(x_4,x_5,0),\ldots,(x_{E-2},x_{E-1},0))\}$.

Now, P_r is the set of real convex polyhedron of P that are not degenerate. The set P_d is relatively closed, and hence P_r is open in P. By Cauchy's Rigidity Theorem for convex bodies ϕ is locally one–to–one on the set P_r. Assume $p, p' \in P_r$ and $\phi(p) = \phi(p')$. Then p and p' have congruent nets. Refine the net on p by adding the real edges of p in E^3 or H^3. Now, do the same for p'. Use ϕ to map the new net on p to p' and use ϕ^{-1} to map the new net on p' to p. From a slightly more general version of Cauchy's rigidity theroem it follows p is congruent to p'. Therefore there exists an isometry $i_{pp'}$ that takes p to p'. The first, second, and third vertices of p must remain fixed by $i_{pp'}$ since they are of the form $(0,0,0)$, $(x_1,0,0)$ $x_1 > 0$ and $(x_2,x_3,0)$ $x_3 > 0$. Therefore, $i_{pp'}$ is either a reflection in the plane determined by the first three vertices or $i_{pp'}$ is the identity. Hence ϕ is locally one–to–one on P_r. By the invariance of domain theorem, ϕ will be an open map when restricted to the open set P_r. Therefore $\phi(P_r)$ is open. We have $\phi(P) = \phi(\overline{P\;\;}) = \phi(P_r) \cup \phi(P_d)$. We will now show that $\phi(P)$ is open. It then follows that $\phi(P) = M_i$, and since M_i was an arbitrary component of $M_{<2\pi}$, Theorem 11.2 will be proved. For the sake of argument, assume that ϕ is a C^1 map (the function ϕ seems analytic, but this looks to be tedious to prove). Since P_d has codimension ≥ 2 for $V \geq 5$, from Sard's Theorem, $\phi(P_d)$ is locally thin in $M_{<2\pi}$ (see Appendix 3). To say that $\phi(P_d)$ is locally thin means that for each $m \in \phi(P_d)$, for some $\epsilon > 0$,

$B_\epsilon(m) - \phi(P_d)$ is path connected in $B_\epsilon(m)$. We are assuming $B_\epsilon(m) \subset M_{<2\pi}$. To prove that $\phi(P)$ is open, it suffices to prove $\forall m \in \phi(P_d)$ there exist an open ball $B_m(\epsilon) \subset M_{<2\pi}$, such that $m \in B_m(\epsilon) \subset \phi(P)$. Let $m \in \phi(P_d)$. Then there exists an open ball $B_m(\epsilon)$, such that $m \in B_m(\epsilon) \subset M_{<2\pi}$ and $B_\epsilon(m) - \phi(P_d)$ is path connected in $B_\epsilon(m)$. By the continuity of ϕ there exists $m' \in B_m(\epsilon)$, such that $m' \in \phi(P_r)$. Suppose that $m^* \in B_m(\epsilon)$. Connect m^* to m' with a path that misses $\phi(P_d)$. It is straightforward to show that all points on this path are in $\phi(P_r)$. Let α be a path that connects m to m^*, that is α: $[0,1] \to M_{<2\pi}$, and $\alpha(0) = m'$ and $\alpha(1) = m^*$. Let $S = \{t \in [0,1]$, such that $\alpha(t) \in \phi(P_r)\}$. One can prove that S is an open and closed set of $[0,1]$. Therefore, $\alpha([0,1]) \subset \phi(P_r)$. Hence, $B_m(\epsilon) \subset \phi(P)$, and so $\phi(P)$ is open. Since $\phi(P)$ is connected, $\phi(P) = M_i$. As M_i was an arbitrary path connected component, all of M is realizable.

Now we will give an argument to show $\phi(P)$ is open in $M_{<2\pi}$, which avoids assuming ϕ is C^1. On the polygon $\bar{p}$ (this was the polygon used in the above proof) instead of placing the triangulation which comes from m, place a real triangulation on $\bar{p}$, or a triangulation which comes from E^3 or H^3. By this we mean the triangulation contains all the real edges of $\bar{p}$, along with geodesics along the faces of $\bar{p}$. As in Lemma 3 create a manifold $M'_{<2\pi}$ which comes from this triangulation. Now, as before we have a map ϕ': $P \to M'_{<2\pi}$, which is definitely analytic, since the edges of the triangulation start out as real geodesic segments from E^3 or H^3, and contain all the real edges. When perturbing the vertices of $\bar{p}$, the lengths of all the new edges and angle between geodesic segments are C^1 functions of the coordinates. As before, we can prove $\phi(P)$ is both an open and closed in $M'_{<2\pi}$. Let $m \in \phi(P_d)$. From Lemma 3, there exist an open neighborhood $U_m \subset M_{<2\pi}$, $m \in U_m$, and a continuous one–to–one function

$$\begin{tikzcd} & M_{<2\pi} \supseteq U \arrow[d, "F_m"] \\ P \arrow[ur, "\Phi"] \arrow[r, "\Phi'"'] & M'_{<2\pi} \end{tikzcd}$$

Figure 11.3

$F_m\colon U_m \to M'_{<2\pi}$. We now have the commutative diagram in Figure 11.3. By the invariance of domain theorem, F_m is a homeomorphism onto its image. Therefore, there exist an $\epsilon > 0$ and an open ball $B_\epsilon(F_m(m)) \subset F_m(M_{<2\pi})$, such that $m \in F_m^{-1}(B_\epsilon(F_m(m)) = \phi \circ \phi'^{-1}(B_\epsilon(F_m(m))$. Thus, ϕ is open. □

CHAPTER 12: INFINITESIMAL DEFORMATIONS

Suppose there is a smooth surface S in E^3 or H^3. Let S vary smoothly or deform smoothly in time. For instance imagine S is a small sheet of metal which starts out flat and is slowly bent into a cylinder. Let C be an arbitrary curve on S, and C_t the deformation of C at time t on S_t. If $l(C_t)$, the length of C_t, is constant through out time t, the mapping, $t \to S_t$ is called an **isometric deformation**, abbreviated ID. If the surface S_t is congruent to S throughout time, the ID is called **trivial**. The above example is a non trivial ID. Examples of trivial IDs are obtained by rotating or translating a surface in E^3 or H^3. If there is no nontrivial ID of S, the surface is said to be **rigid**. Early in the 1900's Hilbert, Liebmann, Minkowski, and Weyl proved that the sphere is rigid, and it is clear the plane is not rigid, since it can be deformed into a parabolic cylinder.

The partial differential equations for ID are nonlinear. There is a simpler concept than an ID, namely an **infinitesimal isometric deformation**, abbreviated IID. By definition this means $(l(C_t))'|_0 = 0$. Now the partial differential equations become linear. In [Bleecker], the problem is solved for S^2 minus a point in E^3. The surface is assumed to be sitting in E^3. A natural analog to S^2 minus a point is a horosphere, which is similar to a sphere in H^3 minus a point at infinity. In this chapter we will study this problem for a horosphere in H^3, but before we begin, it is necessary to do some preliminary calculations.

Let M and $\overline{M}$ be manifolds with $\dim(\overline{M}) - \dim(M) = 1$, (think of M as a surface S and $\overline{M}$ as H^3), and let $I(M,\overline{M})$ be the set of all C^∞ immersions

$f: M \to \overline{M}$. For $f \in I(M,\overline{M})$, let f_* be the derivative map $f_*: TM \to T\overline{M}$. If $\overline{g}$ is a Riemannian metric on $\overline{M}$, then $f^*\overline{g}$ is a metric on M provided f is a immersion. Let $f(t) : M \to \overline{M}$ be a smooth family of immersions with $f(0) = f$. Then $f'(0) : M \to T\overline{M}$ is a vector field along f (that is for each $p \in M$, $f'(0)_p \in T_{f(p)}\overline{M}$). Let I_f denote the space of vector fields along f (that is I_f consists of all maps $V: M \to T\overline{M}$ such that $V(p) \in T_{f(p)}\overline{M}$ for all $p \in M$). For $V \in I_f$, we can define a "standard" deformation $\overline{V} : M\times\mathbb{R} \to \overline{M}$ by $\overline{V}(p,t) = \mathrm{Exp}_{f(p)}(tV(p))$. We have a linear map $L: I_f \to S^2(M) \equiv$ space of covariant symmetric 2–tensors, defined via $L(V)_p = [\overline{V}(\cdot,t)^*\overline{g}]_p'\Big|_0$, where $\overline{V}(\cdot,t)^*\overline{g}$ is the pull–back of $\overline{g}$. We say that $V \in I_f$ is an infinitesimal isometric deformation of f if $L(V) = 0$. We will derive a more explicit formula for $L(V)$.

First consider the case where $V_{f(p)}$ is the unit normal $N_{f(p)}$ to $T_{f(p)}f(M)$ for all $p \in f(M)$. It can be shown that $L(N)$ is twice negative of the the second fundamental form of $f(M) \subset \overline{M}$ in this case. When f is an embedding, the tangent vectors to the curves $t \to \overline{N}(p,t)$ will then define a vector field, say $\tilde{N}$, on a neighborhood U of $f(M)$ in $\overline{M}$.

For X and Z vector fields locally defined about p on M, let $\overline{X}$ and $\overline{Z}$ be extensions of f_*X and f_*Z to a neighborhood of $f(p)$ in $\overline{M}$. In the following calculation "$\mathcal{L}$" denotes Lie differentiation.

$$
\begin{aligned}
L(N)_p(X,Z) &= [\overline{N}(\cdot,t)^*\overline{g}]_p'(X,Z)\Big|_0 \\
&= \left[\overline{g}_{\overline{N}(p,t)}(\overline{N}(\cdot,t)_{*p}(Z),\overline{N}(\cdot,t)_{*p}(Z))\right]'\Big|_0 \\
&= (\mathcal{L}_{\tilde{N}}\overline{g})_{f(p)}(\overline{X},\overline{Z}) = \left[\tilde{N}[\overline{g}(\overline{X},\overline{Z})] - \overline{g}(\mathcal{L}_{\tilde{N}}\overline{X},\overline{Z}) - \overline{g}(\overline{X},\mathcal{L}_{\tilde{N}}\overline{Z})\right]_{f(p)} \\
&= \left[\tilde{N}[\overline{g}(\overline{X},\overline{Z})] - \overline{g}([\tilde{N},\overline{X}],\overline{Z}) - \overline{g}(\overline{X},[\tilde{N},\overline{Z}])\right]_{f(p)} \\
&= \left[\overline{g}(\overline{\nabla}_{\tilde{N}}\overline{X},Z) + \overline{g}(\overline{X},\overline{\nabla}_{\tilde{N}}\overline{Z}) - \overline{g}([\tilde{N},\overline{X}],\overline{Z}) - \overline{g}(\overline{X},[\tilde{N},\overline{Z}])\right]_{f(p)} \\
&= \left[\overline{g}(\overline{\nabla}_{\tilde{N}}\overline{X} - [\tilde{N},\overline{X}],\overline{Z}) + \overline{g}(\overline{X},\overline{\nabla}_{\tilde{N}}\overline{Z} - [\tilde{N},\overline{Z}])\right]_{f(p)} \\
&= \left[g(\overline{\nabla}_X N,Z) + g(X,\overline{\nabla}_Z N)\right]_{f(p)} = -2B(X_p,Z_p),
\end{aligned}
$$

where B is the second fundamental form of $f(M) \subset \overline{M}$. Thus, for $N \equiv$ the unit field along f, we have $L(N) = -2B$. For a C^∞ function h on M, replacing N by hN in the above yields $L(hN) = -2hB$, since

$$
\begin{aligned}
&\overline{g}(\overline{\nabla}_X hN,\, Z) + \overline{g}(X,\, \overline{\nabla}_Z hN) \\
&\quad = \overline{g}(h\overline{\nabla}_X N,\, Z) + g(X,\, h\overline{\nabla}_Z N) + \overline{g}(\overline{X}[h]N,Z) + g(X,\overline{Z}[h]N) \\
&\quad = g(h\overline{\nabla}_X N,Z) + g(X,h\overline{\nabla}_Z N).
\end{aligned}
$$

Now suppose that V is tangent to $f(M)$, and let $\tilde{V}$ be an extension of V to a neighborhood of $f(M)$. The same computation as above with "N" replaced by "V" yields

$$L(V)_p = \left[g(\overline{\nabla}_X V,Z) + g(X,\overline{\nabla}_Z V)\right]_p = g_p(\nabla_X V,Z) + g_p(X,\nabla_Z V),$$

where $g = f^* \overline{g}$ and ∇ is the Riemannian connection of g on M. Thus, for an arbitrary $V \in I$ with tangential and normal components V^T and hN, we have

$$L(V)(X,Z) = g_p(\nabla_X V^T,Z) + g_p(X,\nabla_Z V^T) - 2hB(X,Z),$$

or more compactly,

$$V = V^T + hN \to L(V) = \mathcal{L}_{(V^T)} g - 2hB. \tag{12.1}$$

Note that $\mathcal{L}_{(V^T)} g$ is intrinsic to M and $-2hB$ only depends on B and the normal component $h = \langle V,N \rangle$ of V. No derivatives of h enter.

Suppose that the surface S is now in H^3 with horocyclic coordinates, or $\mathbb{R}^3$ with the metric $\overline{g} = ds^2 = dx_1^2 + e^{-2x_1}(dx_2^2 + dx_3^2)$. Parameterize the horosphere by $s(u,v) = (c,u,v)$. Note that $D_u = D_{x_2}$ and $D_v = D_{x_3}$. Let $V = v_1 D_{x_1} + v_2 D_{x_2} + v_3 D_{x_3}$ be a C^1 vector field. Here the tangential component of V is $V^T = v_2 D_{x_2} + v_3 D_{x_3}$, and the normal component is $N = v_1 D_{x_1}$. From formula (12.1), we want

$$L(V)(X,Z) = g(\nabla_X V^T,Z) + g(X,\nabla_Z V^T) - 2hB(X,Z) = 0.$$

By linearity of $L(V)$, it suffices to evaluate the above for (X,Z) equal to (D_u,D_u), (D_v,D_v), and (D_u,D_v), and see what partial differential equations arise. Suppose

$$0 = L(V)(D_u,D_u) = 2g(\nabla_{D_{x_2}}(v_2D_{x_2} + v_3D_{x_3}), D_{x_2}) - 2v_1B(D_{x_2},D_{x_2}).$$

Now

$$2g(\nabla_{D_{x_2}}(v_2D_{x_2} + v_3D_{x_3}), D_{x_2}) = 2(v_2)_u e^{-2c} + v_2\Gamma_{22}^{2}\cdot e^{-2c} = 2(v_2)_u e^{-2c}$$

and

$$2v_1B(D_{x_2},D_{x_2}) = 2v_1 g(\nabla_{D_{x_2}}D_{x_2}, D_{x_1}) = 2v_1\Gamma_{22}^{1} = 2v_1e^{-2c}.$$

Hence,

$$v_1 = (v_2)_u. \tag{12.2}$$

Now suppose

$$0 = L(V)(D_v,D_v) = 2g(\nabla_{D_{x_3}}(v_2D_{x_2}+ v_3D_{x_3}), D_{x_3}) - 2v_1B(D_{x_3}, D_{x_3}).$$

Then

$$2g(\nabla_{D_{x_3}}(v_2D_{x_2} + v_3D_{x_3}), D_{x_3}) = 2(v_3)_v e^{-2c} + v_3\Gamma_{33}^{3}\cdot e^{-2c} = 2(v_2)_u e^{-2c}$$

and

$$2v_1B(D_{x_3}, D_{x_3}) = 2v_1\, \bar{g}(\nabla_{D_{x_3}}D_{x_3}, D_{x_1}) = 2v_1\Gamma_{33}^{1} = 2v_1e^{-2c}.$$

Therefore,

$$(v_3)_v = v_1. \tag{12.3}$$

From the equation $0 = L(V)(D_u,D_v)$, in a similar fashion as the above, one finds

$$(v_2)_v = -(v_3)_u. \tag{12.4}$$

From (12.2) and (12.3) we have $(v_2)_u = (v_3)_v$. By (12.4) it follows that v_2 and v_3 are harmonic conjugates of each other. Putting all of this together, the following theorem has been proved.

Theorem 12.1. Let $V = v_1 D_{x_1} + v_2 D_{x_2} + v_3 D_{x_3}$ be a C^1 vector field on a horosphere parameterize by $s(u,v) = (c,u,v)$ in horocyclic coordinates. Then V is an IID if and only if v_2 and v_3 are harmonic conjugates and $v_1 = (v_2)_u$.

The following corollaries follow immediately from the theory of complex variables.

Corollary 1. If $V = v_1 D_{x_1} + v_2 D_{x_2} + v_3 D_{x_3}$ is a C^1 vector field on a horosphere which is an IID then it is real analytic.

Corollary 2. If $V = v_1 D_{x_1} + v_2 D_{x_2} v_3 D_{x_3}$ and $V = v'_1 D_{x_1} + v'_2 D_{x_2} v'_3 D_{x_3}$ are two IID on a horosphere which agree on some set S which has a limit point then $V = V'$.

Corollary 3. If $V = v_1 D_{x_1} + v_2 D_{x_2} v_3 D_{x_3}$ is an IID on a horosphere, and $(v_2^2 + v_3^2)^{1/2}$ is bounded on the horosphere, then v_2 and v_3 are constants, and $v_1 = 0$. In other words, V generates a translation of the horosphere within itself.

APPENDIX 1: DERIVATION OF $S = S_0 e^{-X/K}$.

In this appendix we give a proof of formula (2.1), which was the crucial point in deriving the Poincare model of hyperbolic geometry. While reading this appendix, it may be necessary to refer to Chapters 1 and 2. For more details, the reader may refer to [Gans] or [Wolfe]. Most of the arguments given in this appendix are similar to the arguments in [Gans], except we prove that horocycles have arclength. Gans has written an excellent book on hyperbolic geometry from the synthetic point of view, or in the spirit of Euclid, and is highly recommended by the author.

Definition. Let l and m be two lines which pass through the ideal point δ at infinity. Suppose $A \in l$ and $B \in m$. The cord $\overline{AB}$, in union with the rays $\overrightarrow{A\delta}$ and $\overrightarrow{B\delta}$ is called a **trilateral**.

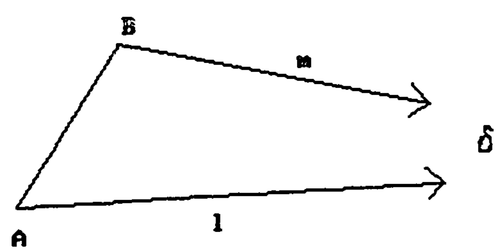


Figure A.1

Theorem A.1. Suppose $AB\delta$ and $A'B'\delta'$ are two trilaterals. If $\measuredangle A \approx \measuredangle A'$ and $\overline{AB} \approx \overline{A'B'}$, then the trilaterals are congruent.

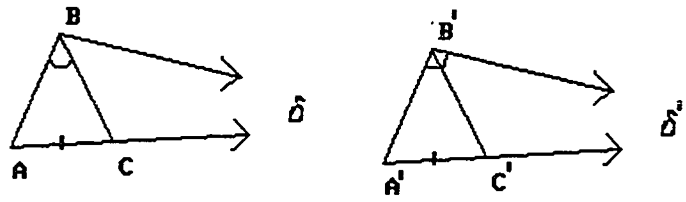


Figure A.2

Proof. In Figure A.2, suppose $\overline{AB} \approx \overline{A'B'}$ and $\measuredangle A \approx \measuredangle A$. Suppose the two trilaterals are not congruent. Without loss of generality, assume $\measuredangle B' < \measuredangle B$. Through $\measuredangle B$ draw a line n which makes an angle of B' with $\overline{AB}$. This line must intersect the ray $\overrightarrow{A\delta}$ at a point. Call this point C. On the line $\overleftrightarrow{A'\delta'}$, find the point C' such that $\overline{AC} \approx \overline{A'C'}$. Deduce triangles ABC and $A'B'C'$ are congruent. Then $\measuredangle B' \approx \measuredangle A'B'C' < \measuredangle B'$, a contradiction. □

Theorem A.2. Let $AB\delta$ and $A'B'\delta'$ be two trilaterals. If $\measuredangle A \approx \measuredangle A'$ and $\measuredangle B \approx \measuredangle B'$, then the trilaterals are congruent.

The proof of this theorem is similar to the proof of the previous theorem and is left as an exercise.

Theorem A.3. Suppose $AB\delta$ is a trilateral and $\measuredangle A \approx \measuredangle B$. Let M be the midpoint of $\overline{AB}$. If n is a perpendicular erected at M, then n goes through δ.

Proof. The proof follows from the following diagram.

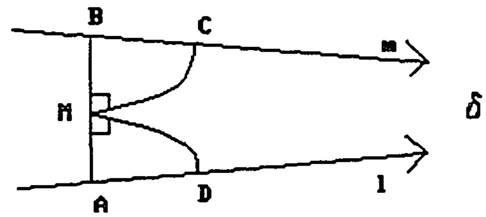


Figure A.3

First suppose $m \cap \overrightarrow{B\delta} \neq \phi$. Let $m \cap \overrightarrow{B\delta} = C$. Choose D to the right of A such that $\overline{AD} \approx \overline{BC}$. Deduce triangle MAD is congruent to triangle MBC. Then $\measuredangle AMD$ is a right angle. Hence $m \cap \overrightarrow{B\delta} = \phi$. Similarly, $m \cap \overrightarrow{A\delta} = \phi$. Since the distance between $\overrightarrow{A\delta}$ and $\overrightarrow{B\delta}$ goes to zero as you approach δ, m must be a boundary parallel to $\overrightarrow{A\delta}$ and $\overrightarrow{B\delta}$. □

Theorem A.4. Given any point A on one of two boundary parallels passing through the ideal point δ, there is a unique point A' on the other line which corresponds to it. The angles in the trilateral $AA'\delta$ are congruent and acute.

Proof. Let l and m be two parallel lines which pass through the ideal point δ at infinity. Choose a point A on l and drop a perpendicular to m. Call the base of the perpendicular B. Choose another point P on l such that B is between P and δ.

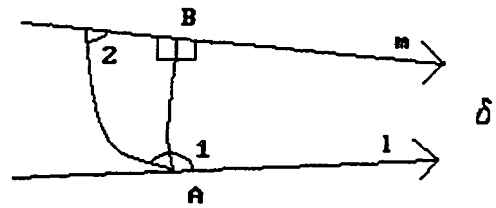
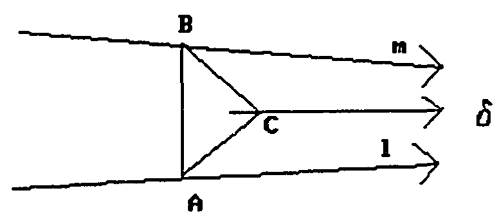


Figure A.4

Let $f = \measuredangle 1 - \measuredangle 2$. The theorem follows from the continuity of f. □

Theorem A.5. Given any two boundary parallels passing through the ideal point δ, there exists a line equidistant from them both passing through δ.

Proof. Let l and m be the two parallel lines in the direction δ. Choose the point A on l, B on m and connect these two points with a line segment. Bisect angle $BA\delta$ and $BC\delta$. They must meet in a point C. Now connect C to δ. It is clear that all points on the line $\overleftrightarrow{C\delta}$ are equidistant from l and m. □



Figue A.5

Theorem A.6. Horocyclic arcs are convex.

Proof. Let H be a horocyclic arc through A and B and the ideal point δ. Let C belong to the line segment determined by A and B. Since angle 1 and angle 2 cannot both be acute, $C \notin H$. Similarly, any point in the interior of trilateral $AB\delta$ is not on H, hence H is convex. □

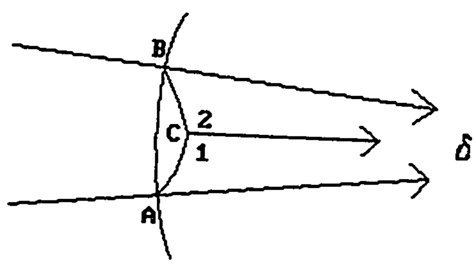


Figure A.6

Definition. A curve C which connects points A and B has arclength, if for all partition P_0: $A = P_0, P_1 \dots P_n = B$ $(\forall P_i \in C)$, we have $\sup \{ \sum_{i=1}^{n} d(P_i, P_{i-1} \} < \infty$. The **arclength** of C is $\sup \{ \sum_{i=1}^{n} d(P_i, P_{i-1}) \} < \infty$, where the supremum is taken over all partitions.

Theorem A.7. Horocyclic arcs have arclength.

Proof. Let l and m be two parallel lines passing through the point δ at infinity. Suppose A and B are two points on l and m, and H(A,B) is the horocyclic arc through these points. Choose a point M on H(A,B), and connect it with geodesics to points E and F, where E is on l, and F is on m. We want A to be between E and δ, and we also want B to be between F and δ.

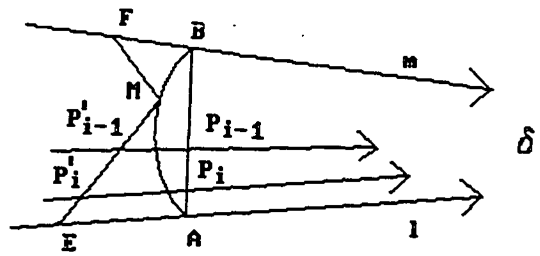


Figure A.7

Let $P_0 = A, P_1, \ldots, P_i, \ldots, P_n = B$ be points on H(A,B). Without loss of generality, assume M is one of these points. For $\forall i$, $\overleftrightarrow{P_i\delta}$ either intersects $\overline{ME}$ or $\overline{MF}$. Again without loss of generality, suppose $\overleftrightarrow{P_{i-1}\delta}$ and $\overleftrightarrow{P_i\delta}$ intersect $\overline{EM}$ in points P'_{i-1} and P'_i. Since the base angles in the quadrilateral $P_{i-1}P_iP'_{i-1}P'_i$ are obtuse, $d(P_{i-1},P_i) < d(P'_{i-1},P'_i)$. Therefore,
$\sum_{i=1}^{n} d(P_{i-1},P_i) < \sum_{i=1}^{n} d(P'_{i-1},P'_i) = d(E,M) + d(M,F)$, and so H(A,B) has arclength. □

Theorem A.8. Suppose H(A,B) and H(A′,B′) are horocycles with $\overline{AB} \approx \overline{A'B'}$. Choose C and D on $\overline{AB}$, and C′ and D′ on $\overline{A'B'}$, such that $\overline{AC} \approx \overline{A'C'}$ and $\overline{DB} \approx \overline{D'B'}$. Then $\overline{\bar{D}\ \bar{C}} \approx \overline{\bar{D}'\bar{C}'}$. Here $\bar{C} = H(A,B) \cap \overrightarrow{C\delta}$, and D, D′ and $\bar{C}'$ are defined similarly.

Proof. The proof follows from the following figure.

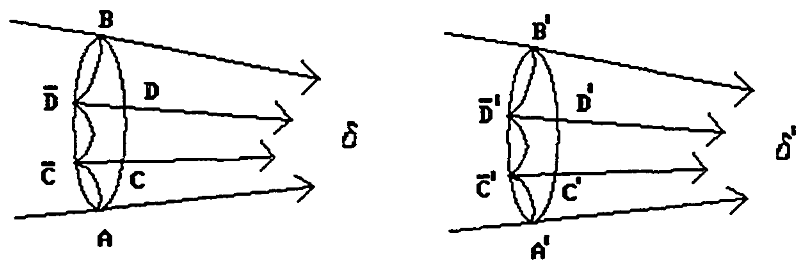


Figure A.8

Deduce trilateral $A\overline{C}\delta \approx$ trilateral $A'\overline{C}'\delta'$ and trilateral $B\overline{D}\delta \approx$ trilateral $B'\overline{D}'\delta'$. This implies trilateral $\overline{C}\,\overline{D}\,\delta \approx$ trilateral $\overline{C}'\overline{D}'\delta'$. Therefore, $\overline{\overline{D}\ \overline{C}} \approx \overline{\overline{D}'\overline{C}'}$. □

Theorem A.8. The arcs subtending two chords of the same or different horocycles are congruent if and only if the chords are equal.

Proof. The proof follows from the previous two theorems. Just take a sequence of partitions P^i of $H(A,B)$, such that the sum of the lengths of its cords converge to $H(A,B)$. The partition P^i corresponds to a partition of P'^i of $H(A',B')$, whose sum of the length of its chords is the same. Therefore, $l(H(A,B)) \leq l(H(A',B'))$. Similarly, $l(H(A',B')) \leq l(H(A,B))$. Hence, if the length of the cords are equal, then the arclengths of the horocyles are equal. The converse is a direct consequence of this. □

Definition. A **radius** of a horocycle $H(A,B)$ which passes through the ideal point δ at infinity, is a ray $\overrightarrow{P\delta}$ such that $P \in H(A,B)$.

Definition. Suppose l and m are two lines which are parallel and pass through the point δ at infinity. Assume A and A′ lie on l, and B and B′ lie on m, with A corresponding to B and A′ corresponding to B′. We say horocycles H(A,B) and H(A′,B′) are **codirectional.**

Theorem A.10. The radius, which bisects an arc of a horocycle, bisects the corresponding arc of any codirectional horocycle and also bisects both subtended chords at right angles.

Proof. Let H(A,B) be the desired horocycle, and H(A′,B′) be any other codirectional horocycle (Figure. A.9). If C is the midpoint of H(A,B), then C is the intersection of H(A,B) and line $\overleftrightarrow{MM'}$, where M and M′ are the midpoints of chords $\overline{AB}$ and chords $\overline{A'B'}$.

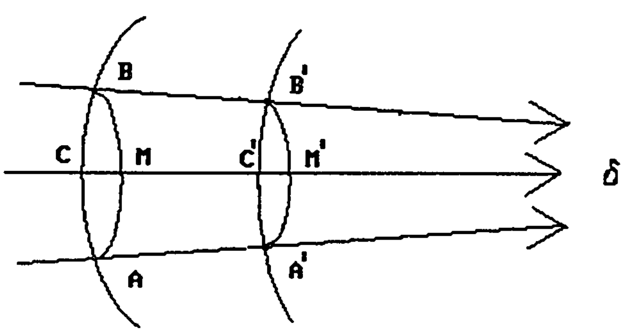


Figure A.9

Line $\overleftrightarrow{MM'}$ passes through δ. The theorem now follows. □

Theorem A.10. Radii which divide an arc of a horocycle into n equal parts do likewise to the corresponding arc of any codirectional horocycle.

This follows from the previous theorem.

Theorem A.11. Let $H(A,B)$ and $H(A',B')$ be two codirectional horocycles. Then $\overline{AB} \approx \overline{A'B'}$.

Proof. The proof follows from Figure A.10.

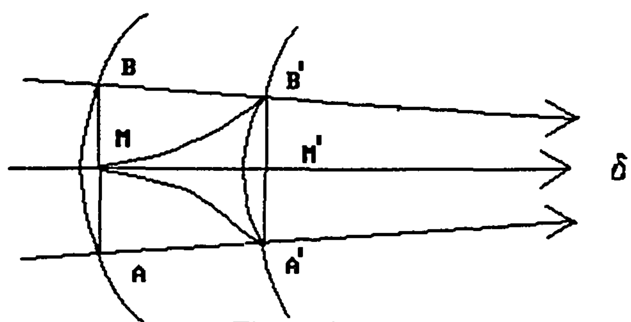


Figure A.10

Try to deduce triangle $MBB' \approx$ triangle MAA'. The points M and M' are the midpoints of $\overline{AB}$ and $\overline{A'B'}$. □

Theorem A.12. A radius which divides an arc of a horocyle will divide the corresponding arc of any codirectional horocycle so that the two ratios are equal.

Proof. Suppose $H(A,B)$ and $H(A',B')$ are your two horocycles which pass through δ. Let C be a point of $H(A,B)$, and let C' be the intersection of $H(A',B')$ and line $\overleftrightarrow{C\delta}$.

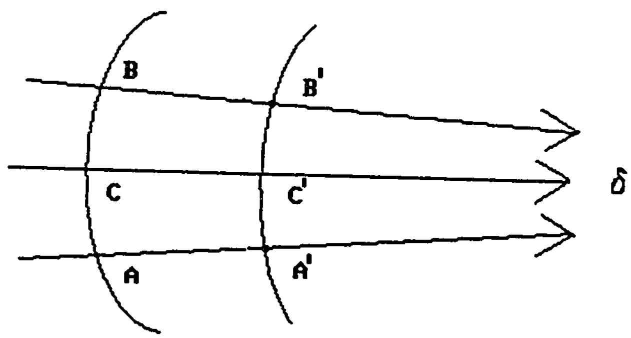


Figure A.11

If $l(H(A,B))/l(H(A,C))$ is a rational number then the result follows from Theorem A.10. If $l(H(A,B))/l(H(A,C))$ is irrational the result follows from the rational case by passing a limit. □

Theorem A.13. The ratio of a pair of corresponding arcs is a function f, depending on x, the distance between the corresponding arcs. The function f is also an increasing function of x.

Proof. The proof of the theorem breaks down to two cases. Case one is when $H(A,B) \approx H(C,D)$ (Figure A.12). If $H(A,B) \approx H(C,D)$, then $\overline{AB} = \overline{CD}$. So quadrilaterals $ABA'B'$ and $CDC'D'$ are congruent. Hence, $H(A',B') \approx H(C',D')$. Therefore,

$$H(A,B)/H(A',B') = H(C,D)/H(C',D').$$

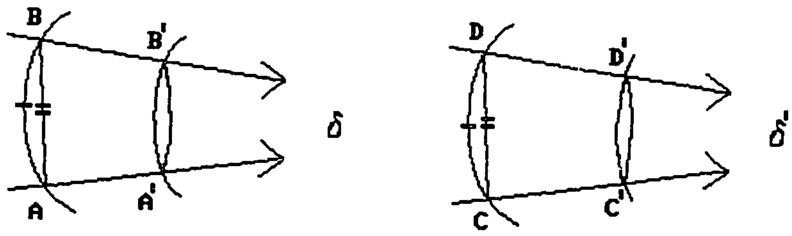


Figure A.12

Case two is when H(A,B) is not congruent to H(C,D) (Figure A.13). Without loss of generality, suppose the arclength of H(A,B) is greater than the arclength of H(C,D).

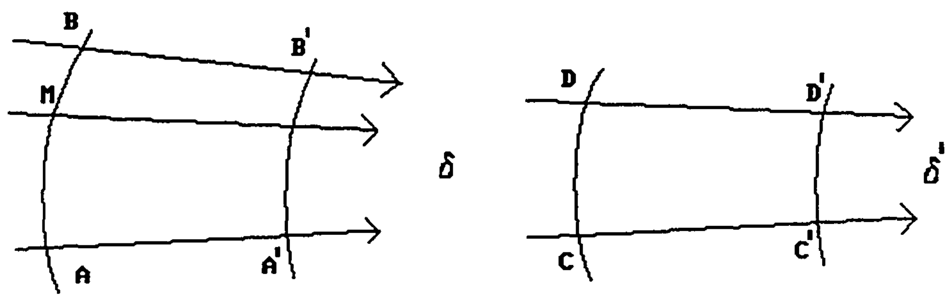


Figure A.13

Then there exists a point M on H(A,B), such that the arclength of H(A,M) equals the arclength of H(C,D). Let M′ be the intersection of line $\overleftrightarrow{M\delta}$ and the horocyclic arc H(A′,B′). From case, we deduce

$$H(A,M)/H(A',M') = H(C,D)/H(C',D').$$

From the previous theorem, we have

$$H(A,B)/H(A,M) = H(A',B')/H(A',M').$$

Therefore,

$$H(A,B)/H(A',B') = H(C,D)/H(C',D').$$

It is straightforward to show this ratio increases as the distance between the arcs increases. □

Theorem A.14. Let $H(A,B)$ be a horocyclic arc passing through the point δ at infinity. Suppose the arclength of $H(A,B)$ is s_0. Then $s(x)$, the length of the horocyclic arc of $H(A',B')$ where A' is a directed length of x from A, is given by

$$s_x = s(x) = s_0 e^{-x/k}.$$

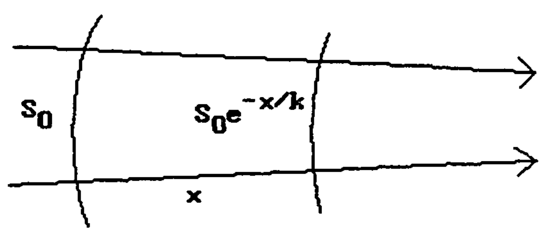


Figure A.14

Proof. First suppose x is an integer. If $x = 1$ then $s_1/s_0 = e^{-1/k}$ for some $k > 0$ or $s_1 = s_0 e^{-1/k}$. If $x = 2$, then $s_2/s_1 = s_1/s_0 = e^{-1/k}$, and so $s_2 = s_0 e^{-2/k}$. In general, if $x = n$, n an integer, it is straightforward to show

$s_n = e^{-n/k}$. Suppose $x = 1/2$. Then $s_0/s_{1/2} = s_{1/2}/s_1$, so $(s_{1/2})^2 = s_0^2 e^{-1/k}$. Therefore, $s_{1/2} = s_0 e^{-1/2 \cdot k}$. In general, if x is rational or $x = m/n$, then $s_{m/n} = s_0 e^{-m/(n \cdot k)}$. If x is irrational, just approximate x with a sequence of rational numbers, and pass the limit. Hence we have proved

$$s_x = s_0 e^{-x/k}. \quad \square$$

APPENDIX 2: HILBERT'S AXIOMS

In this appendix, Hilbert's axioms are stated. Hilbert's axioms can be partitioned into five categories, namely incidence axioms, betweenness axioms, congruence axioms, continuity axioms, and parallelism axioms. This list of axiom can be found in the back of [Greenberg] (*Euclidean and non Euclidean geometries*) by Marvin Jay Greenberg.

Incidence Axioms

Axiom A–1. For every point P and for every point Q not equal to P there exists a unique line l that passes through P and Q.

Axiom A–2. For every line l, there exist at least two distinct points that are incident with l.

Axiom A–3. There exist three points with the property that no line is incident with all three of them.

Betweenness Axioms

Axiom B–1. If A*B*C, then A, B, and C are three distinct points all lying on the same line, and C*B*A.

Axiom B–2. Given any two distinct points B and D, there exist points A, C and E lying on $\overleftrightarrow{BD}$ such that A*B*D, B*C*D and B*D*E.

Axiom B–3. If A,B, and C are three distinct points lying on the same line, then one and only one of the points is between the other two.

Axiom B–4. For every line l and any three points A,B, and C not lying on l:

(i) if A and B are on the same side of l, and B and C are on the same side of l, then A and C are on the same side of l.

(ii) if A and B are on opposite sides of l, and B and C are on opposite sides of l, then A and C are on the same side of l.

Congruence Axioms

Axiom C–1. If A and B are distinct points and if A' is any point, then for each ray r emanating from A', there is a unique point B' on r such that $B' \neq A'$ and $\overline{AB} \cong \overline{A'B'}$.

Axiom C–2. If $\overline{AB} \cong \overline{CD}$ and $\overline{AB} \cong \overline{EF}$, then $\overline{CD} \cong \overline{EF}$. Moreover, every segment is congruent to itself.

Axiom C–3. If A*B*C, A'*B'*C', $\overline{AB} \cong \overline{A'B'}$ and $\overline{BC} \cong \overline{B'C'}$, then $\overline{AC} \cong \overline{A'C'}$.

Axiom C–4. Given any angle $\measuredangle BAC$ (where by definition of "angle" $\overrightarrow{AB}$ is not opposite to $\overrightarrow{AC}$),and given any ray $\overrightarrow{A'B'}$ emanating from a point A', then there is a unique ray $\overrightarrow{A'C'}$ on a given side of line $\overleftrightarrow{A'B'}$ such that $\measuredangle B'A'C' \cong \measuredangle BAC$.

Axiom C–5. If $\measuredangle A \cong \measuredangle B$ and $\measuredangle A \cong \measuredangle C$, then $\measuredangle B \cong \measuredangle C$. Moreover, every angle is congruent to itself.

Axiom C–6 (SAS). If two sides and the included angle of one triangle are congruent respectively to two sides and the include angle of another triangle, then the two triangles are congruent.

Continuity Axioms

Archimedes Axiom. If $\overline{AB}$ and $\overline{CD}$ are any segments, then there is a number n such that if segment $\overline{CD}$ is laid off n times on ray $\overrightarrow{AB}$ emanating from A, then a point E is reached where $n \cdot \overline{CD} \cong AE$ and B is between A and E.

Dedekind's Axiom. Suppose that the set of all points on a line l is the union $\Sigma_1 \cup \Sigma_2$ of two nonempty subsets, such that no point of Σ_1 is between two points of Σ_2 and vice versa. Then there is a unique point O lying on l, such that $P_1 * O * P_2$ if and only if $P_1 \in \Sigma_1$ and $P_2 \in \Sigma_2$ and $O \neq P_1, P_2$.

Parallelism Axioms

Hilbert's Parallel Axiom For Euclidean Geometry. For every line l and every P not lying on l, there is at most one line m through P such that m is parallel to l.

Euclid's Fifth Postulate (which is equivalent to the above). If two lines intersected by a transversal in such a way that the sum of the degree measures of the two interior angles on one side of the transversal is less than 180°, then the two lines meet on that side of the transversal.

Hyperbolic Parallel Axiom. There exist a line l, a point P not on l such that at least two distinct lines parallel to l pass through P.

APPENDIX 3: THINNESS

Theorem 1. Suppose that X is an open subset of $\mathbb{R}^n$. Let $k \geq 2$, and assume that $f: X \to \mathbb{R}^{n+k}$ is a C^1 map. Then $\mathbb{R}^{n+k} - f(X)$ is a path connected set.

Proof. Let $y \in \mathbb{R}^{n+k}$, define $F_y: X \times \mathbb{R} \to \mathbb{R}^{n+k}$, by $F_y(x,t) = y + t(f(x) - y)$. The map F_y is C^1. By Sard's theorem, $F_y(X \times \mathbb{R})$ has measure zero in $\mathbb{R}^{n+k}$. Let $y_1, y_2 \in \mathbb{R}^{n+k}$. There exists a $z \in \mathbb{R}^{n+k}$, such that $z \notin F_{y_1}(X \times \mathbb{R})$ and $z \notin F_{y_2}(X \times \mathbb{R})$. Let $[z,y_2]*[z,y_2]$ be the polygonal path formed by the line segment $[y_1,z]$ followed by the line segment $[z,y_2]$. We have
$[z,y_2]*[z,y_2] \cap (F_{y_1}(X) \cup F_{y_2}(X)) = \phi$. □

The elegant proof of the above theorem is due to Les Wilson. From this theorem, the missing detail in Theorem 11.2 now follows. Let $m \in \phi(P_d)$, (the image of the degenerate polygons under ϕ), and let $\epsilon > 0$ be such that $B_\epsilon(m) \subset M_{<2\pi}$. Let $y_1, y_2 \in B_\epsilon(m)$. By modifying the above proof, we have a $z \in B_\epsilon(m)$, such that $[z,y_2]*[z,y_2] \subset B_\epsilon(m) - \phi(P_d)$.

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