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Finite element modeling of nonsymmetrical cable cross-sections considering nonuniform radial loadings

Le, Tung Tuan, Ph.D.
University of Hawaii, 1992
FINITE ELEMENT MODELING OF NONSYMMETRICAL CABLE CROSS-SECTIONS CONSIDERING NONUNIFORM RADIAL LOADINGS

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DOCTOR OF PHILOSOPHY IN MECHANICAL ENGINEERING

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The author also wishes to thank his mother for her invaluable support while this research was being carried out.
ABSTRACT

A new two-dimensional finite element model, named the REMCC (ring element model with contact constraint), is proposed for the analysis of radial deformations in an umbilical cable. This model accounts for material orthotropy and unsymmetrical geometry and loads. Each component of the cable is assumed to possess a circular cross-section and is modeled as a REMCC element having nodal degrees-of-freedom at all contacting points with adjacent components. A new numerical scheme is developed for forming the stiffness matrices of the REMCC elements. Axisymmetric two-dimensional ring elements are used to form the geometrical configuration of the REMCC element. With the aid of the penalty method for imposing displacement constraints, the unit displacement theorem is applied numerically for obtaining the stiffness matrix of the REMCC element with respect to its contact nodal points. Once the stiffness matrices of all the REMCC elements are generated, the normal global coordinate transformations and assembly procedures of the finite element method are applied to form the system of equations for the entire cable cross-section. Loads are then applied and the system equations are solved for the unknown displacements of the contact nodal points. The model is verified by several examples involving exact classical solutions and the test results obtained for an as-built cable.
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LIST OF SYMBOLS

\([\{\}], \{\}\) \text{ transpose of a matrix or column vector}

\(A_{ij}\) \text{ constants defined by nodal displacements and nodal radial coordinates}

\(b_i\) \text{ unknown constants of displacement shape functions}

\(\{b\}\) \text{ column vector of constants } b_i

\([C]\) \text{ material constitutive matrix}

\([CS]\) \text{ diagonal constant matrix of strain-displacement relation}

\([D]\) \text{ constant matrix defined by Hooke's law}

\(E\) \text{ Elastic modulus}

\([E_i]\) \text{ error function defined by equation (86)}

\(\{f\}\) \text{ column vector of element nodal forces}

\(\{f_0\}\) \text{ equivalent nodal forces of initial strain}

\(f_u\) \text{ radial reaction forces}

\(f_v\) \text{ circumferential reaction force}

\([F_{eq}]\) \text{ row vector of element equivalent nodal forces}

\(\{g\}\) \text{ strain-displacement matrix}

\(G_\theta\) \text{ shear modulus}

\([h]\) \text{ constant matrix relating constant } b_i \text{ with element nodal displacement relations}

\([k]_e\) \text{ condensed ring element stiffness matrix}

\([K]\) \text{ cable cross-section stiffness matrix}

\(N\) \text{ upper limit of Fourier series}

\([p_i]\) \text{ matrix dependent on contact angle, } \beta_i
$P_c$ concentrated force

$P_r, P_o$ internal and external pressures

$\{q\}$ local displacement vector

$\{Q\}$ global displacement vector

$r$ radius in polar coordinates

$\{r\}$ local load vector

$\{R\}$ global load vector

$R_i, R_o, R_m$ inner, outer, and mean radii, respectively

$[T]$ transformation matrix

$u$ radial displacement

$u_o$ radial displacement corresponding to initial strain

$U$ strain energy

$v$ circumferential displacement

$V$ potential force energy

$\alpha_r, \alpha_\theta, \alpha_z$ thermal expansion coefficients in $r, \theta, \text{and } z$ directions, respectively

$\beta$ contact angle

$\Delta T$ temperature change

$\{\varepsilon\}$ strain column vector

$\{\varepsilon_o\}$ initial strain column vector

$[\kappa]$ dependent matrix of stiffness matrix

$[\lambda_i]$ diagonal matrix of the constant penalty parameter, $\lambda_i$

$\nu$ Poisson’s ratio

$\Omega$ penalty term of displacement constraint.

$\Pi$ total potential energy
\{\sigma\} \quad \text{stress column vector}

\theta \quad \text{circumferential angle in polar coordinates}

\textbf{Subscripts:}

\begin{align*}
1,2,3 & \quad \text{ring element nodal points} \\
a & \quad \text{antisymmetric} \\
c & \quad \text{cable component} \\
\text{cond} & \quad \text{condensed} \\
e & \quad \text{condensed ring element} \\
eq & \quad \text{equivalent} \\
i,j & \quad \text{indices of a component of a matrix, vector or summation.} \\
k & \quad \text{Fouries series harmonic} \\
m & \quad \text{respect to element local polar coordinates} \\
M & \quad \text{respect to global coordinates} \\
N & \quad \text{net of} \\
r & \quad \text{respect to radial direction} \\
s & \quad \text{symmetric} \\
T & \quad \text{total of} \\
\theta & \quad \text{respect to circumferential direction} \\
z & \quad \text{respect to axial direction} \\
V & \quad \text{element volume}
\end{align*}

\textbf{Superscript:}

\begin{align*}
* & \quad \text{related to displacement constraints.}
\end{align*}
CHAPTER 1
INTRODUCTION

1.1. Background

Today, there are many applications requiring sophisticated cables such as umbilical cables which have the capability to perform a number of different functions. Umbilical cables are classified into different categories including power cables, control cables, communication cables, tow cables, remotely-operated vehicle cables (tether or ROV), subsea wellhead cables, mining cables, etc. These cables must also possess sufficient structural strength to withstand all the operating and handling loads.

The structure of the umbilical cable is much more complex than its ancestor, the wire rope, which provided only the strength function. Basically, the umbilical cable consists of layers of wires helically arranged around some type of cylindrical core. Usually, the cylindrical core contains many different components used to perform different tasks. A typical umbilical cable is shown in Figure 1.1 (see Section 4.5 for full descriptions of this cable components).

Although, umbilical cables are increasingly in demand by the ocean industry, a comprehensive design methodology does not yet exist due, primarily, to the lack of cable models which accurately represent stress and strain in all of the cable components. Today, industry still relies on
experimental data to develop adequate designs. This approach, however, leads to large development costs and precludes achieving an optimal design. What is needed is a cable model which can be used to predict stress distributions in cables under complex loadings so that design parametric studies can be conducted quickly and efficiently.

Figure 1.1. A typical umbilical cable cross-section

An extensive review of the existing body of cable literature was conducted to determine the state-of-the-art of cable modeling. Current structural cable analysis methods
originated with earlier models developed for wire ropes. The pioneering work by Hruska [1,2,3] is widely cited in the cable literature. Hruska developed approximate procedures for straight wire ropes that are subjected to tension and torsion. Since then, numerous studies have extended this early effort. Machida and Durelli [4], and Phillips [5] developed more exact models of cables with fixed and free ends; however, these models assumed a rigid cylindrical core element. Chi [6,7] developed geometrically exact, nonlinear equations that do account for a deformable core, but no solutions for the state of deformation of the core was offered. Nowak [8] and Knapp [9] further extended these models to more realistic models that represent practical cable constructions, including treatment of core deformation. Additional work dealing with straight cables subjected to tension and torsion are cited in [9,10,11]. Still, these models simply treat the cable core as a single elastic, homogeneous and isotropic component and further restrict the model to geometrical and load symmetry about the cable axis. Recently, Knapp [12] has extended his basic work in [9] with a finite layer model in which the cable core can be treated as the composition of independent layers. Nevertheless, this latest model is also restricted by the symmetry assumptions, stated above. In general, these models limit the practical designs of umbilical cables for the following reasons. First, as revealed in Figure 1.1., the geometry of components in an umbilical cable core are most
often unsymmetric. Second, components sharing the same layer might well have different material properties. Finally, the material properties of umbilical cable components are frequently anisotropic. It is doubtful that the foregoing models can be extended to handle such complications. This marks the point of departure of the present study. It is proposed that the Finite Element Method (FEM), used so successfully in other areas of structural engineering, can be used to develop a more accurate and realistic cable model. Some limited application of the FEM has been suggested by Carlson [13], who used finite element software, NASTRAN, to model two different electro-mechanical cables with limited success. The problem with using commercial software is the great amount of time required to develop a particular model which is then not easily modified for carrying out parametric studies.

1.2. Finite Element Analysis

Since the middle 1950's, the finite element method has been used successfully in structural mechanics, thermodynamics, heat transfer, fluid mechanics, mathematics, electromagnetics, etc. The finite element method is well-known as a numerical procedure for obtaining accurate, albeit approximate, solutions of complex physical problems. In the area of structural analysis, a structure is discretized into small elements whose shapes can be selected to match complex structural boundaries. Each of these elements is
interconnected with adjacent elements at nodal points. The deformations and stresses acting within each element are approximated by assumed shape functions. Each nodal point is represented by generalized degrees-of-freedom (DOF) which represent displacements and nodal forces. Depending on which method (stiffness or flexibility) is used, either nodal displacements or nodal forces can be considered as unknowns for the structure. For each element under static equilibrium, a set of simultaneous linear equations for these unknown quantities can be found by various methods such as principle of minimization of total potential energy, variational principles, principle of virtual work (Castigliano theorems). The equations for all elements in the structure are systematically assembled into a common set which represents the stiffness or the flexibility of the entire structure. Upon implementing boundary conditions and loadings at nodal points of the structure, this common system of equations can be solved to yield nodal displacements (or nodal forces) of the structure. Substituting these values back into interpolating functions for each element provides the distributions of stress and displacement everywhere within each element. In this way, the deformation and stress distribution of the structure are obtained.

Although many general-purpose commercial finite element codes do exist, they are expensive to operate and often require staff who are expert in finite element modeling. For
this reason, the goal of this research is to simplify the finite element process to a sufficient degree that a cable designer untrained in the use of the finite element method can complete a cable design with greater accuracy than is presently possible.

1.3. Objectives

The primary objective of this research is to attain a simple and efficient way to employ the finite element method in designing umbilical cables. The use of finite elements will permit the handling of unsymmetrical geometries and loads. This is especially significant for designing umbilical cables which consist of numerous dissimilar components such as optical fibers, copper conductors, hydraulic tubes and strength members. Existing cable models cannot handle nonsymmetry.

In order to use finite element analysis in modeling umbilical cables, a number of finite element papers and textbooks were reviewed. This review revealed that one possible approach to the problem is to represent each cable component as a substructure. The use of substructures in the finite element method has been successfully applied to large structures [14]. After the large structure is partitioned into a number of substructures, each substructure is modeled with finite elements and assembled to yield the stiffness of the substructure. The stiffness of the substructure must be then rearranged to obtain an equivalent stiffness of this
structure corresponding only to nodal points at its boundaries, where contact is made with adjacent substructures. This process can be done by the condensation method [15]. Once these equivalent stiffnesses of substructures are determined they can be treated as complex structural elements in forming the stiffness of the entire structure. The system of equations is then solved for the unknown displacements of the nodal points on the boundaries of the substructures. When these displacements are known, each substructure can be separately analyzed under the known substructure-boundary displacements. Due to the large amount of data to be handled by this process, computer time may be excessive.

Finite sector elements [16,17,18,19,20], triangular and quadrilateral elements in polar coordinate systems [21], linear strain element with curved edges [22], and the family of isoparametric elements [23] were also reviewed. Examples documented in these papers indicate that a large number of these elements will be needed to model each cable component. This suggests that a considerable burden will be placed on the designer to provide geometrical data including nodal coordinates, labels of nodes and elements for a finite element mesh. Besides these elements there exists another class of finite elements referred to as axisymmetric solid ring elements. These appear to be better suited for cable modeling as discussed below.

The method for forming stiffness matrices of axisymmetric
ring elements is known as the semi-analytical method or method of separation \([23,24]\). By taking advantage of the orthogonality property of the Fourier series, a three-dimensional problem can be broken into a series of two-dimensional problems. The solution of the three-dimensional problem is obtained by superimposing the solutions of the two-dimensional problems. This approach can be applied also to two-dimensional problems, i.e., the problem is broken into a series of one-dimensional problems. This class of elements has been well-known for its excellent performance since it was introduced by Wilson \([25]\) and Clough and Radish \([26]\) in 1965. Radish \([27]\) continued his work in using axisymmetric ring elements to analyze composite structures. Many publications followed which extended their work. Dunham \([28]\) presented the finite element analysis of axisymmetric solids under arbitrary loadings. Utku \([29]\) provided the explicit stiffness matrix for triangular torus elements. Oden and Key \([30]\) performed numerical analysis of finite axisymmetric deformations of incompressible elastic solids of revolution. Charcour \([31]\) proposed an 11 degree-of-freedom axisymmetric triangular element used in modeling hydraulic turbine components. Crose \([32]\) provided a method for analyzing stresses of axisymmetric solids with asymmetric material properties. Belystchko \([33]\) suggested an analytical stiffness formulation of axisymmetric solid elements with nodes on the axis of revolution. Pedersen and Megahed \([34]\) investigated the numerical instability of
axisymmetric elements located too far from the axis of revolution. Pardoen, Falco and Crose [35] modified Crose’s work in [32] to account for asymmetric thermal stress of axisymmetric solids with rectangularly orthotropic properties. Pardoen [36] went further to extend the work in [35] to deal with axisymmetric solids having generally anisotropic material properties. Material and geometric nonlinearities of axisymmetric solid elements were studied by Hartzman in [37]. Zak, Craddock and Drysdale [38] proposed an approximate finite-element method of stress analysis for elastic non-axisymmetric structures. They broadened this method to apply for elastic-plastic non-axisymmetric structures in [39]. Rich [40] provided the closed form elastic-plastic stiffness matrix for axisymmetric finite elements. Surana [41] formulated the transition finite elements for axisymmetric solid and shell elements. These elements will permit transition from the axisymmetric portion of the structure to the axisymmetric shell portion. He extended this work to account for geometrically nonlinear behavior of the transition finite elements in [42]. Sedaghat and Herrmann [43] developed a theory for the analysis of nearly axisymmetric solids. Gong, Nelson, Chiu, and Wang [44] proposed an analytic formulation of a stiffness matrix for axisymmetric solids. Tian and Pian [45] used the hybrid assumed stress method, which is based on an assumed stress field within the element and assumed displacement patterns on the element boundaries, to formulate
axisymmetric solid elements. The isoparametric axisymmetric finite elements and their applications are well documented by Zienkiewicz in [23]. Numerical comparisons of axisymmetric solid elements are found in [46,47]. Many other papers about axisymmetric solid elements and their applications in other areas are cited in [23,24,47].

Fig. 1.2. Two- & three-dimensional axisymmetric ring elements

Nodal points of axisymmetric ring elements are complete circles or single points (if they are located on the axis of revolution). Ideally, they are used to model a structure of a solid of revolution. In two-dimensional analysis, the axisymmetric ring elements which are used to model solid cylinders and hollow tubes, are composed by two or more circles whose circumferences are nodal points of the elements as depicted in Figure 1.2. It would appear to be well-suited to model the cross-section of circular cable components (a good approximation for helical components having small lay angles). Unfortunately, the circle nodes of ring elements in
contact with nodes on adjacent cable components cannot be joined since that would violate the symmetry condition; however, there are some extensions of this method mentioned in [23,24]. The literature search was unable to reveal any publications related to joining adjacent ring elements.

It is proposed in this study that the axisymmetric ring element, based on a generalized Fourier displacement shape function, be employed to treat the nonsymmetrical deformations occurring between adjacent components, rigidly connected at the contact points. The idea of substructures is then used to create REMCC elements (ring element model with contact constraints) whose nodal points are the contact points on its boundary surface. Thus, each component in the cable can be represented by such REMCC elements having nodal degrees-of-freedom at all points of the inner and outer surfaces that contact adjacent components.

To achieve the task of forming stiffness matrices for the REMCC elements, a numerical scheme will be developed. A number of ring elements are to be used to form the geometrical configuration of each cable component. With the aid of the penalty method [23,24], used to form necessary displacement constraints, the unit-displacement theorem [14] is then applied to obtain the stiffness matrix of the REMCC element with respect to its contact nodal points. Once the stiffness matrices of all the REMCC elements representing cable components are generated, the normal global coordinate
transformations and assembly procedure of the finite element method [48] are applied to form the system equation for the entire cable cross-section. The system loads are applied at the nodal points of REMCC elements. Besides the concentrated forces acting at contact points of the REMCC elements, other loads such as internal and external pressures, uniform temperature change, initial strain and stress are converted into equivalent concentrated forces at the contact nodes.

The system equations are then to be solved for the unknown nodal contact displacements in the radial and circumferential directions. The displacements of the nodal points are used to compute the reaction contact forces at all nodal points. Applying the reaction contact forces to each REMCC element, the local deformations and stresses of each cable component can be obtained.

The process described in the foregoing is to be implemented in a computer program. In addition to the requirements of basic layer geometry, material properties, and loads, the locations of contact points between cable components, node and element labels are required. This computer implementation should require substantially less effort than existing commercial finite element codes, since far fewer elements will be required. The program is to be verified using experimental data obtained for the umbilical cable shown in Figure 1.1.
CHAPTER 2
MODEL FORMULATION

Assumption:

In developing the finite element model for an umbilical cable, the following assumptions have been made:

(a) only components having a circular cross-section are considered;

(b) a two-dimensional analysis is used (only radial and circumferential deformations of the cable transverse cross-section are considered);

(c) the material is linearly elastic;

(d) the Love-Kirchhoff hypothesis applies; i.e., initially plane sections of the cable cross-section remain plane before and after deformation;

(e) the theory is valid only for small deformations;

(f) any each component may undergo a constant temperature change;

(g) the solution is based on a constant cable axial strain;

(h) each component is rigidly connected at its contact points with adjacent components.

Overview:

In the proposed finite element model, each component in the cable core is modeled as a REMCC element (ring element model with contact constraints) having nodal degrees-of-
freedom at all points of the inner and outer surfaces that contact adjacent components. A typical REMCC element with its nodal points C, D and E is shown in Figure 2.1.

![Figure 2.1. A typical REMCC element](image)

The number of ring elements shown by solid and dashed lines in Figure 2.1. are used to model the geometric configuration of the REMCC element. A typical ring element having three circle nodal points, 1, 2, and 3, is shown in Figure 2.2.

![Fig. 2.2. Axisymmetric ring element with 3 circle nodal points](image)
With the aid of the penalty method \([23, 24]\), the unit displacement theorem \([14]\) is applied numerically to obtain the stiffness matrix of the REMCC element with respect to its nodal points. Once the stiffness matrices of all the REMCC elements representing cable components are generated, the normal global coordinate transformations and assembly procedure \([48]\) of the finite element method are applied to form the system equations for the entire cable cross-section. Next, the system loads are applied at the nodal points of REMCC elements. Besides the concentrated forces, other loads such as internal and external pressures, uniform temperature change, and initial strain applied to REMCC elements are converted into equivalent concentrated loads, which are applied at the nodal points of the REMCC elements.

The system equations are solved for the unknown displacements of the nodal points and thus the deformations of the cable can be predicted. The displacements of the nodal points are used to compute the contact forces at these nodal points. These contact forces are then applied to element at the contact points in order to find local deformations, strains and stresses.

2.1. Axisymmetric Ring Element

An explicit derivation for a two-dimensional axisymmetric ring element subjected to nonsymmetrical loading was not found in the literature. In this section, the stiffness matrices for a typical two-dimensional ring element is developed. The
basic outline of the stiffness method [24] is followed:

(a) Based on the geometric configuration of the object select the most suitable element whose shape can be used to model the object as closely as possible. Depending on the shape of the element, a number of nodal points for the purpose of interconnecting adjacent elements is assigned to the element.

(b) Based on the nature of deformation of the object, the displacement field of the element is assumed. This displacement field is described by a number of shape functions, which are used to approximate the deformations of the element under all loads. The shape functions contain a number of unknown constants which relate to the nodal displacements of the element. The unknown constants of the displacement functions are then algebraically related to the nodal displacements of the element.

(c) The displacement and strain relations are taken from the theory of elasticity. These relationships between strain and nodal displacements are based on the relationships between nodal displacements and the unknown constants of the shape function given by (b).

(d) The stress-strain relations are given by Hooke's law and then expressed in terms of the nodal displacements.

(e) The strain energy and the total force potential energy of the finite element are introduced. Both of these energies can be expressed in terms of the nodal displacements.
(f) The principle of minimum total potential energy is applied to obtain the stiffness matrix of the element.

2.1.1. Displacement Shape Functions

The ring element is an axisymmetric solid element. Each ring nodal point is formed by a complete circle. Due to the unsymmetrical deformation expected for a typical ring element used in modeling the REMCC element, radial and circumferential displacement shape functions, \( u(r, \theta) \) and \( v(r, \theta) \), respectively, are expressed in polar coordinates according to the following Fourier series [24] with quadratic variations in the radial direction:

\[
\begin{align*}
    u(r, \theta) &= \sum_{k=0}^{N} [ u_{sk}(r) \cos k \theta + u_{ak}(r) \sin k \theta ] \\
    v(r, \theta) &= \sum_{k=0}^{N} [ v_{sk}(r) \sin k \theta - v_{ak}(r) \cos k \theta ]
\end{align*}
\]

where

\[
\begin{align*}
    u_{sk}(r) &= b_{1k} + b_{2k}r + b_{3k}r^2 \\
    v_{sk}(r) &= b_{4k} + b_{5k}r + b_{6k}r^2 \\
    u_{ak}(r) &= b_{7k} + b_{8k}r + b_{9k}r^2 \\
    v_{ak}(r) &= b_{10k} + b_{11k}r + b_{12k}r^2
\end{align*}
\]

the subscripts "s" and "a" represent the symmetric and antisymmetric terms relative to the \( \theta=0 \) axis shown in Figure 2.2, respectively. \( u_k \) and \( v_k \) are the symmetrical components of the nodal displacements, \( u \) and \( v \), respectively, and \( u_{ak} \) and \( v_{ak} \) are the antisymmetrical components of the nodal displacements.
displacements $u$ and $v$. $b_{1k}$ thru $b_{3k}$ are unknown constants of the shape functions of the harmonic term $k$. The upper limit, $N$, of the summations in equations (1) and (2) is chosen to satisfy convergence of the series. In addition to the quadratic variations in radial direction used in equation (3), linear and cubic variations were also considered. Performance comparisons detailed in Section 2.2.5 reveal that the quadratic element is superior. The shape functions and the stiffness matrices of the two-dimensional ring elements with linear and cubic variations in radial direction are given in Appendix B.

As shown in Figure 2.3, each quadratic ring element has three circle nodal points 1, 2 and 3, which correspond to the quadratic variations in radial direction of the displacement functions $u_{sk}(r)$, $u_{ak}(r)$, $v_{sk}(r)$, and $v_{ak}(r)$. For each harmonic term, each circle nodal point consists of four unknown displacements, $u_{sk}$, $v_{sk}$, $u_{ak}$, and $v_{ak}$.

Substituting the radial coordinates of the three nodal points 1, 2, and 3 into the first equation of (3), the relationship between the nodal displacements $u$ of the symmetric term at node 1, 2, and 3 and the constants $b_{1k}$, $b_{2k}$, and $b_{3k}$ follows as
\[
\begin{bmatrix}
    u_{1sk} \\
    u_{2sk} \\
    u_{3sk}
\end{bmatrix}
= 
\begin{bmatrix}
    1 & R_i & R_i^2 \\
    1 & R_{av} & R_{av}^2 \\
    1 & R_o & R_o^2
\end{bmatrix}
\begin{bmatrix}
    b_{1k} \\
    b_{2k} \\
    b_{3k}
\end{bmatrix}
\] (4)

where \( u_{1sk}, u_{2sk}, \) and \( u_{3sk} \) are the nodal displacements \( u \) at nodes 1, 2, and 3, respectively, of the symmetric term \( k \). \( R_i \) and \( R_o \) are the inner and outer radii of the ring element, respectively. \( R_{av} \) is the mean radius of the ring element, i.e., \( R_{av} = (R_i + R_o) / 2 \).

The constants \( b_{1k}, b_{2k}, \) and \( b_{3k} \) of the harmonic term \( k \), are found by taking the inverse of the constant matrix \([3x3]\) in equation (4) as

\[
\begin{bmatrix}
    b_{1k} \\
    b_{2k} \\
    b_{3k}
\end{bmatrix} = \frac{1}{\Delta} \begin{bmatrix}
    A_{11} & A_{12} & A_{13} \\
    A_{21} & A_{22} & A_{23} \\
    A_{31} & A_{32} & A_{33}
\end{bmatrix} \begin{bmatrix}
    u_{1sk} \\
    u_{2sk} \\
    u_{3sk}
\end{bmatrix}
\] (5)

where

\[
\Delta = (R_{av} R_o^2 - R_o R_{av}^2) - (R_i R_o^2 - R_o R_i^2) + (R_i R_{av}^2 - R_{av} R_i^2); \\
A_{11} = R_{av} R_o^2 - R_o R_{av}^2; \quad A_{12} = R_i R_o^2 - R_o R_i^2; \quad A_{13} = R_i R_{av}^2 - R_{av} R_i^2; \\
A_{21} = R_{av}^2 - R_o^2; \quad A_{22} = R_o^2 - R_i^2; \quad A_{23} = R_i^2 - R_{av}^2; \\
A_{31} = R_o - R_{av}; \quad A_{32} = R_i - R_o; \quad A_{33} = R_{av} - R_i; 
\] (6)

The relations between the nodal displacements \( v \) of the symmetric term at nodes 1, 2, and 3 and the constants \( b_{4k}, b_{5k}, \) and \( b_{6k} \) of harmonic term \( k \) can be obtained in the same manner and are expressed as follows:
Combining equations (5) and (7), the relationship between the nodal displacements $u$ and $v$ of the symmetric term $k$ with the constant $b$'s is obtained as

$$
\begin{bmatrix}
\frac{b_4}{b_5}
\frac{b_6}{b_5}
\end{bmatrix} = \frac{1}{\Delta} \begin{bmatrix}
A_{11} & A_{12} & A_{13} \\
A_{21} & A_{22} & A_{23} \\
A_{31} & A_{32} & A_{33}
\end{bmatrix} \begin{bmatrix}
V_{1sk} \\
V_{2sk} \\
V_{3sk}
\end{bmatrix}
$$

(7)

The relationship between constant $b_7$ thru $b_{12k}$ and the nodal displacements $u$ and $v$ at nodes 1, 2, and 3 of the antisymmetric term $k$ is obtained by using the same procedure as above and is given by,

$$
\begin{bmatrix}
\frac{b_7}{b_8}
\frac{b_9}{b_8}
\frac{b_{10k}}{b_8}
\frac{b_{11k}}{b_{12k}}
\end{bmatrix} = \frac{1}{\Delta} \begin{bmatrix}
A_{11} & 0 & A_{12} & 0 & A_{13} & 0 \\
A_{21} & 0 & A_{22} & 0 & A_{23} & 0 \\
A_{31} & 0 & A_{32} & 0 & A_{33} & 0 \\
0 & A_{11} & 0 & A_{12} & 0 & A_{13} \\
0 & A_{21} & 0 & A_{22} & 0 & A_{23} \\
0 & A_{31} & 0 & A_{32} & 0 & A_{33}
\end{bmatrix} \begin{bmatrix}
U_{1sk} \\
V_{1sk} \\
U_{2sk} \\
V_{2sk} \\
U_{3sk} \\
V_{3sk}
\end{bmatrix}
$$

(8)

(9)

Equations (8) and (9) can be written in the compact forms:
\{b\}_{sk} = [h] \{q\}_{sk}
\{b\}_{ak} = [h] \{q\}_{ak}

where

\{b\}_{sk} = [b_{1k} \ldots b_{6k}]^T
\{b\}_{ak} = [b_{7k} \ldots b_{12k}]^T

and

\{q\}_{sk} = [u_{1sk} \quad v_{1sk} \quad u_{2sk} \quad v_{2sk} \quad u_{3sk} \quad v_{3sk}]^T
\{q\}_{ak} = [u_{1ak} \quad v_{1ak} \quad u_{2ak} \quad v_{2ak} \quad u_{3ak} \quad v_{3ak}]^T

and [h] is the constant matrix [6x6]:

\[
[h] = \frac{1}{\Delta}
\begin{bmatrix}
A_{11} & 0 & A_{12} & 0 & A_{13} & 0 \\
A_{21} & 0 & A_{22} & 0 & A_{23} & 0 \\
A_{31} & 0 & A_{32} & 0 & A_{33} & 0 \\
0 & A_{11} & 0 & A_{12} & 0 & A_{13} \\
0 & A_{21} & 0 & A_{22} & 0 & A_{23} \\
0 & A_{31} & 0 & A_{32} & 0 & A_{33} \\
\end{bmatrix}
\] (13)

2.1.2. Strain-Displacement Relations

From the theory of elasticity, Cauchy linear strains in polar coordinates [50] are:
where \( \varepsilon_{rr} \), \( \varepsilon_{\theta\theta} \), and \( \varepsilon_{\theta\phi} \) are the radial, tangential, and shearing strains, respectively. Substituting the displacement shape functions, \( u(r,\theta) \) and \( v(r,\theta) \), of equations (1) and (2) into equation (14), we obtain,

\[
\varepsilon_{rr} = \frac{1}{r} \frac{\partial}{\partial \theta} \left( \frac{u}{r} \right) + \frac{1}{r} \frac{\partial}{\partial r} \left( \frac{v}{r} \right)
\]

\[
\varepsilon_{\theta\theta} = \frac{1}{r} \frac{\partial}{\partial \theta} \left( \frac{v}{r} \right) + \frac{1}{r} \frac{\partial}{\partial r} \left( \frac{v}{r} \right)
\]

\[
\varepsilon_{\theta\phi} = \frac{1}{r} \frac{\partial}{\partial \theta} \left( \frac{v}{r} \right) - \frac{1}{r} \frac{\partial}{\partial r} \left( \frac{v}{r} \right)
\]

These equations above can be assembled in matrix form of as
\[
\begin{align*}
\begin{bmatrix}
\epsilon_{rr} \\
\epsilon_{\theta\theta} \\
\epsilon_{r\theta}
\end{bmatrix} &= \sum_{k=0}^{N} \begin{bmatrix}
cos k \theta & 0 & 0 \\
0 & cos k \theta & 0 \\
0 & 0 & -sin k \theta
\end{bmatrix} \begin{bmatrix}
0 & 1 & 2r & 0 & 0 \\
1 & 1 & r & k & kr \\
k & r & -k & -kr & -1/r & 0 & r
\end{bmatrix} \\
&+ \begin{bmatrix}
b_{1k} \\
b_{2k} \\
b_{3k} \\
b_{4k} \\
b_{5k} \\
b_{6k}
\end{bmatrix}
\end{align*}
\]

\[
\begin{align*}
\sum_{k=0}^{N} \begin{bmatrix}
sin k \theta & 0 & 0 \\
0 & sin k \theta & 0 \\
0 & 0 & -cos k \theta
\end{bmatrix} \begin{bmatrix}
0 & 1 & 2r & 0 & 0 \\
1 & 1 & r & k & kr \\
k & r & -k & -kr & -1/r & 0 & r
\end{bmatrix} \\
&+ \begin{bmatrix}
b_{7k} \\
b_{8k} \\
b_{9k} \\
b_{10k} \\
b_{11k} \\
b_{12k}
\end{bmatrix}
\end{align*}
\]

Note that both terms can be expressed as a function of the same matrix involving \( r \). This is a consequence of choosing the negative sign in the second term in equation (2). This result is expressed more simply as

\[
\begin{align*}
\begin{bmatrix}
\epsilon_{rr} \\
\epsilon_{\theta\theta} \\
\epsilon_{r\theta}
\end{bmatrix} &= \sum_{k=0}^{N} [CS]_{sk} [g]_k \{ b \}_s + \sum_{k=0}^{N} [CS]_{sk} [g]_k \{ b \}_a \\
&= \{ \epsilon \}_s + \{ \epsilon \}_a
\end{align*}
\]

where the symmetrical and antisymmetrical strains are given by,
\[
\{\epsilon\}_s = \sum_{k=0}^{N} [CS]_{sk} [g]_k \{b\}_{sk} \\
\{\epsilon\}_a = \sum_{k=0}^{N} [CS]_{ak} [g]_k \{b\}_{ak}
\]

and

\[
[CS]_{sk} = \begin{bmatrix}
\cos k\theta & 0 & 0 \\
0 & \cos k\theta & 0 \\
0 & 0 & \sin k\theta
\end{bmatrix}
\]

(18)

\[
[CS]_{ak} = \begin{bmatrix}
\sin k\theta & 0 & 0 \\
0 & \sin k\theta & 0 \\
0 & 0 & -\cos k\theta
\end{bmatrix}
\]

\[
[g]_k = \begin{bmatrix}
0 & 1 & 2r & 0 & 0 & 0 \\
\frac{1}{r} & 1 & r & \frac{k}{r} & kkr \\
-\frac{k}{r} & -k & -kr & \frac{1}{r} & 0 & r
\end{bmatrix}
\]

(19)

\{b\}_{sk} and \{b\}_{ak} are defined in equation (11).

Using the relations between the constant b's and the nodal displacements \{q\}_{sk} and \{q\}_{ak} in equation (10), the symmetric and antisymmetric strain terms can be rewritten as

\[
\{\epsilon\}_s = \sum_{k=0}^{N} [CS]_{sk} [g]_k [h] \{q\}_{sk}
\]

(20)

\[
\{\epsilon\}_a = \sum_{k=0}^{N} [CS]_{ak} [g]_k [h] \{q\}_{ak}
\]

where \{q\}_{sk}, \{q\}_{ak}, and [h] are defined by equations (12) and
For most practical cable applications, material properties are either isotropic or orthotropic. For the case of plane strain of a two-dimensional, linearly elastic structure in cylindrical coordinates, Hooke’s law [50] states that the stress-strain relationship of an orthotropic material is given by,

\[
\begin{align*}
\varepsilon_{rr} &= \frac{1}{E_r} \sigma_{rr} - \frac{\nu_{r\theta}}{E_\theta} \sigma_{\theta\theta} - \frac{\nu_{r\theta}}{E_z} \sigma_{zz} + \alpha_\Delta T \\
\varepsilon_{\theta\theta} &= -\frac{\nu_{r\theta}}{E_r} \sigma_{rr} + \frac{1}{E_\theta} \sigma_{\theta\theta} - \frac{\nu_{r\theta}}{E_z} \sigma_{zz} + \alpha_\Delta T \\
\varepsilon_{r\theta} &= \frac{1}{G_{r\theta}} \sigma_{r\theta} \\
\varepsilon_{zz} &= -\frac{\nu_{rz}}{E_r} \sigma_{rr} - \frac{\nu_{rz}}{E_\theta} \sigma_{\theta\theta} + \frac{1}{E_z} \sigma_{zz} + \alpha_\Delta T
\end{align*}
\]

where \(\sigma_{rr}, \sigma_{\theta\theta}, \sigma_{r\theta},\) and \(\sigma_{zz}\) are the radial, tangential (hoop), shearing, and axial stresses, respectively. \(E_r, E_\theta,\) and \(E_z\) are rigidity moduli in the radial, circumferential, and axial directions, respectively. \(G_{r\theta}\) is the shear modulus in the plane of the cable cross-section defined by [53],

\[
G_{r\theta} = \frac{E_r E_\theta}{E_r + E_\theta + 2 \nu_{r\theta} E_r}
\]

Poisson’s ratio, \(\nu_{ij} (i, j = r, \theta, z)\), characterizes the strain in the j-direction produced by the stress in the i-direction. \(G_{r\theta}\) is the shear modulus which characterizes the changes of
angle between direction \( r \) and \( \theta \). Because of the symmetry requirement and Betti's reciprocal theorem [24], we have

\[
E_r \nu_{\theta r} = E_\theta \nu_{r \theta} ; \quad E_r \nu_{zz} = E_z \nu_{rz} ; \quad E_\theta \nu_{z\theta} = E_z \nu_{z\theta}
\]

Thus for orthotropic materials, only six elastic constants are needed: namely, \( E_r, E_\theta, E_z, \nu_{r\theta}, \nu_{rz}, \) and \( \nu_{z\theta} \). The shear modulus, \( G_{r\theta} \), is computed by equation (22) and the other Poisson's ratios such as \( \nu_{\theta r}, \nu_{rz}, \) and \( \nu_{z\theta} \) can be computed by the equation above. To account for thermal loads, the additional three thermal expansion coefficients \( \alpha_r, \alpha_\theta, \) and \( \alpha_z \) are required.

Solving for the axial stress, \( \sigma_{zz} \), from the last expression of equation (21) yields,

\[
\sigma_{zz} = E_z \left[ \varepsilon_{zz} + \frac{\nu_{rz}}{E_r} \sigma_{rr} + \frac{\nu_{z\theta}}{E_\theta} \sigma_{\theta\theta} - \alpha_z \Delta T \right]
\]

Substituting \( \sigma_{zz} \) in equation (23) into the first two expressions of equation (21) results in

\[
\{ \varepsilon \} = \{ \varepsilon_o \} + [D] \{ \sigma \}
\]

where

\[
\{ \varepsilon \} = \begin{bmatrix} \varepsilon_{rr} \\ \varepsilon_{r\theta} \\ \varepsilon_{\theta\theta} \end{bmatrix} ; \quad \{ \sigma \} = \begin{bmatrix} \sigma_{rr} \\ \sigma_{r\theta} \\ \sigma_{\theta\theta} \end{bmatrix}
\]

\[
\{ \varepsilon_o \} = \begin{bmatrix} -\nu_{rr} \varepsilon_{zz} + (\alpha_r + \alpha_z \nu_{z\theta}) \Delta T \\ -\nu_{r\theta} \varepsilon_{zz} + (\alpha_\theta + \alpha_z \nu_{z\theta}) \Delta T \\ 0 \end{bmatrix}
\]

and
Note that $[D]$ is a symmetric matrix since $D_{12} = D_{21}$.

For the case of isotropic materials, we have

$$\nu_{rz} = \nu_{r\theta} = \nu_{z\theta} = \nu_{r\phi} = \nu_{\theta\phi} = \nu$$

$$E_r = E_\theta = E_z = E$$

$$\alpha_r = \alpha_\theta = \alpha_z = \alpha$$

$$G_{r\theta} = \frac{E}{2(1+\nu)}$$

and matrix $[D]$ becomes

$$[D] = \begin{bmatrix}
\frac{1 - \nu_{zz}}{E} & \frac{-\nu_{z\theta} + (1+\alpha)\Delta T}{E} \\
\frac{-\nu_{r\theta} + \nu_{z\theta}}{E} & \frac{1 - \nu_{zz}}{E} \\
0 & 0 & \frac{1}{G_{r\theta}}
\end{bmatrix}$$

The initial strain matrix reduces to

$$\{\varepsilon_0\} = \begin{bmatrix}
-\nu \varepsilon_{zz} + (1 + \alpha)\Delta T \\
-\nu \varepsilon_{\theta\phi} + (1 + \alpha)\Delta T \\
0
\end{bmatrix}$$

From equation (24), the stress-strain relationship of an element can be rewritten as

$$\{\sigma\} = [C]\{\varepsilon - \varepsilon_0\}$$
where \([C]\) is the inverse of the 3x3 matrix, \([D]\), defined by equation (27) for the case of orthotropic materials and equation (28) for the case of isotropic materials. \([C]\) is also a symmetric matrix resulting from the fact that \([D]\) is a symmetric matrix.

In the computer implementation (chapter 3), the more general orthotropic case, equation (27) is used. If a component is specified as isotropic then all the necessary conversions to equation (28) are automatically performed. The matrix \([C]\) is referred to as the constitutive matrix. Its explicit form is easily found in terms of the coefficients of matrix \([D]\) as

\[
[C] = \begin{bmatrix}
    \frac{D_{22}}{D_{11}D_{22} - D_{12}^2} & \frac{-D_{12}}{D_{11}D_{22} - D_{12}^2} & 0 \\
    \frac{-D_{12}}{D_{11}D_{22} - D_{12}^2} & \frac{D_{11}}{D_{11}D_{22} - D_{12}^2} & 0 \\
    0 & 0 & \frac{1}{D_{33}}
\end{bmatrix}
\]

The term \(\{\varepsilon - \varepsilon_0\}\) in equation (30) is understood to mean the net resulting elastic strain, \(\{\varepsilon_N\}\), of the element due to the applied loads.

2.1.4. Strain Energy

The strain energy, \(U\) of a linear elastic structure is defined by,
where \( \{\epsilon_N\} \) is the net elastic strain due to the applied loads defined by

\[
\{\epsilon_N\} = \{\epsilon - \epsilon_0\}
\]

where \( \{\epsilon\} \) is the total strain of the element and \( \{\epsilon_0\} \) is the initial strain defined by equation (26). \( \{\sigma\} \) is the stress vector due to the elastic deformation of the element and defined by equation (30).

Substituting equations (30) and (33) into equation (32) we obtain

\[
U = \frac{1}{2} \int_V \{\epsilon - \epsilon_0\}^T [C] \{\epsilon - \epsilon_0\} dV
\]

(34)

since matrix \([C]\) is symmetric

\[
\{\epsilon\}^T [C] \{\epsilon_0\} = \{\epsilon_0\}^T [C] \{\epsilon\}
\]

and thus we obtain

\[
U = \frac{1}{2} \int_V \{\epsilon\}^T [C] \{\epsilon\} dV - \int_V \{\epsilon\}^T [C] \{\epsilon_0\} dV + \frac{1}{2} \int_V \{\epsilon_0\}^T [C] \{\epsilon_0\} dV
\]

(35)

The strain energy of the ring element is due to both symmetric and antisymmetric strain components given by equation (20). Thus we can write

\[
U = U_s + U_a + U_o
\]

(36)

where \( U_s \) and \( U_a \) are the strain energies due to symmetric and
antisymmetric strains and \( U_0 \) is the constant. According to equation (35) these quantities can be expressed by

\[
U_r = \frac{1}{2} \int_V \{\epsilon_r\}^T [C] \{\epsilon_r\} \, dV - \int_V \{\epsilon_r\}^T [C] \{\epsilon_o\} \, dV
\]

\[
U_a = \frac{1}{2} \int_V \{\epsilon_a\}^T [C] \{\epsilon_a\} \, dV - \int_V \{\epsilon_a\}^T [C] \{\epsilon_o\} \, dV
\]

\[
U_s = \frac{1}{2} \int_V \{\epsilon_s\}^T [C] \{\epsilon_s\} \, dV
\]

2.1.5. Force Potential Energy

The total force potential energy, \( V \), is defined by

\[
V = -[F_{eq}] \{q\}
\]

where \([F_{eq}]\) is the equivalent nodal force vector and \( \{q\} \) is the unknown nodal displacement vector. Since the nodal point of the ring element is a complete circle, the equivalent nodal force vector is obtained below for a number of load cases such as internal and external pressures, partially distributed loads, and concentrated forces.

It is necessary to express the loads in a Fourier series in following the conventions for the displacement shape functions according to

\[
R(\theta) = \sum_{k=0}^{N} [R_{sk} \cos k\theta + R_{ak} \sin k\theta]
\]

\[
T(\theta) = \sum_{k=0}^{N} [T_{sk} \sin k\theta - T_{ak} \cos k\theta]
\]

where \( R \) and \( T \) are the loads in radial and tangential directions, respectively, which depend only on the \( \theta \)
coordinate. The coefficients, $R_{sk}$, $R_{nk}$, $T_{sk}$, and $T_{nk}$, can be found as follows. A given function, $f(\theta)$, is defined by the following Fourier series

$$f(\theta) = \sum_{k=0}^{N} [a_k \cos k\theta + b_k \sin k\theta]$$  \hspace{1cm} (40)$$

The coefficients, $a_k$ and $b_k$, are obtained by [51],

$$a_0 = \frac{1}{2\pi} \int_{0}^{2\pi} f(\theta) d\theta \hspace{1cm} \text{for} \hspace{0.5cm} k=0$$

$$a_k = \frac{1}{\pi} \int_{0}^{\pi} f(\theta) \cos k\theta d\theta \hspace{1cm} \text{for} \hspace{0.5cm} k \neq 0$$  \hspace{1cm} (41)$$

$$b_k = 0 \hspace{1cm} \text{for} \hspace{0.5cm} k=0$$

$$b_k = \frac{1}{\pi} \int_{0}^{\pi} f(\theta) \sin k\theta d\theta \hspace{1cm} \text{for} \hspace{0.5cm} k \neq 0$$

2.1.5.1. Pressure Loads

For the case of internal or external pressure, $P_i$ or $P_o$, this load represents the radial component of force per unit length of the circumference of a node at inner or outer radius of the ring element and can be expressed in terms of the Fourier series as

$$R(\theta) = \sum_{k=0}^{N} [a_k \cos k\theta + b_k \sin k\theta]$$  \hspace{1cm} (42)$$

Using equation (41), the coefficients of the Fourier series representing this load are given as
\[ a_k = R_P P_i, \quad -R_P P_o \quad \text{for } k=0 \]
\[ a_k = 0 \quad \text{for } k \neq 0 \]  \hspace{1cm} (43)
\[ b_k = 0 \quad \text{for all } k \]

since the pressure is uniform around the circumference.

2.1.5.2. Partially Distributed Loads

For the case of a uniform load, \( P \), partially distributed along the circumference of the ring element over a range from \( \beta_1 \) to \( \beta_2 \) as shown in Figure 2.3, equation (41) is used to determine coefficients of the Fourier series representing this load. The coefficients of the series representing this load are determined as follows:

\[
a_o = \frac{1}{2\pi} \int_{\beta_1}^{\beta_2} (-PR_o) d\theta = \frac{-PR_o(\beta_2 - \beta_1)}{2\pi}
\]

\[
a_k = \frac{1}{\pi} \int_{\beta_1}^{\beta_2} (-PR_o) \cos k\theta d\theta = \frac{-PR_o}{\pi k} (\sin k\beta_2 - \sin k\beta_1) \quad \text{for } k \neq 0
\]

\[ b_o = 0 \]

\[
b_k = \frac{1}{\pi} \int_{\beta_1}^{\beta_2} (-PR_o) \sin k\theta d\theta = \frac{PR_o}{\pi k} (\cos k\beta_2 - \cos k\beta_1) \quad \text{for } k \neq 0
\]  \hspace{1cm} (44)
2.1.5.3. Concentrated Forces

For the case of a concentrated force, \( P_c \), applied radially inward at the angular position, \( \beta \), with respect to the axis, \( \theta = 0 \), as shown in Figure 2.4, equation (41) becomes,

\[
\begin{align*}
\alpha &= \frac{1}{2\pi} \int_0^{2\pi} -P_c \delta(\theta - \beta) d\theta = -\frac{P_c}{2\pi} \\
\alpha_k &= \frac{1}{\pi} \int_0^{2\pi} -P_c \delta(\theta - \beta) \cos k\theta d\theta = -\frac{P_c}{\pi} \cos k\beta \quad \text{for } k \neq 0 \\
\beta &= 0 \\
\beta_k &= \frac{1}{\pi} \int_0^{2\pi} -P_c \delta(\theta - \beta) \sin k\theta d\theta = -\frac{P_c}{\pi} \sin k\beta \quad \text{for } k \neq 0
\end{align*}
\]

where \( \delta \) is the Dirac Delta Function.
The above load cases are used mostly for the loading on ring elements. In general, other load cases can be evaluated by equation (39).

Since applied loads are assumed to vary in only in the circumferential direction, \( \theta \), their equivalent nodal force for each harmonic term will be obtained by multiplying its harmonic coefficient with the corresponding shape functions of the displacements defined in equations (1) and (2); i.e., \( \cos k \theta \) and \( \sin k \theta \) for loads applied in the radial direction and \( \sin k \theta \) and \( -\cos k \theta \) for loads applied in the tangential direction \([3]\). From equations (1), (2) and (39), for each harmonic term the equivalent nodal forces can be equated as follows:

\[
[f]_k = [f]_{sk} + [f]_{tk}
\]  

where
\[ [f]_{sk} = \int_{0}^{2\pi} \begin{bmatrix} R_{sk} \cos^2 k\theta \\ T_{sk} \sin^2 k\theta \end{bmatrix}^T d\theta = 2\pi \begin{bmatrix} R_{sk} \\ 0 \end{bmatrix}^T \text{ for } k = 0 \]
\[ = \pi \begin{bmatrix} R_{sk} \\ T_{sk} \end{bmatrix}^T \text{ for } k \neq 0 \] (47)

\[ [f]_{ak} = \int_{0}^{2\pi} \begin{bmatrix} R_{ak} \sin^2 k\theta \\ T_{ak} \cos^2 k\theta \end{bmatrix}^T d\theta = 2\pi \begin{bmatrix} 0 \\ T_{ak} \end{bmatrix}^T \text{ for } k = 0 \]
\[ = \pi \begin{bmatrix} R_{ak} \\ T_{ak} \end{bmatrix}^T \text{ for } k \neq 0 \] (48)

thus for each harmonic term, the equivalent nodal forces of \([F_{eq}]_k\) can be expressed as

\[ [F_{eq}]_k = [f]_{sk} [f]_{ak} \] (49)

which corresponds to nodal displacements, \(\{q\}_k\), of a ring element, where

\[ \{q\}_k = \begin{bmatrix} q_{sk} \\ q_{ak} \end{bmatrix} \] (50)

and \(q_{sk}\) and \(q_{ak}\) are defined in equation (12). From equations (38), (49), and (50), the total force potential energy of ring elements can be put into the symbolic form,

\[ V = V_s + V_a \] (51)

where
\[ V_s = - \sum_{k=0}^{N} [f]_{sk} \{q\}_{sk} \]  
\[ V_a = - \sum_{k=0}^{N} [f]_{ak} \{q\}_{ak} \]  

(1) (52)

2.1.6. Total Potential Energy

The total potential energy of the element is defined as the sum of the strain energy and the total force potential energy of that element,

\[ \Pi = U + V \]  

From equations (36) and (51), the total potential energy can be expressed as the sum of the symmetric and antisymmetric components:

\[ \Pi = \Pi_s + \Pi_a + U_o \]  

where

\[ \Pi_s = U_s + V_s \]
\[ \Pi_a = U_a + V_a \]  

Substituting \( U_s \) and \( U_a \) of equation (37) and \( V_s \) and \( V_a \) of equation (52) into equation (55) we obtain,

\[ \Pi_s = \frac{1}{2} \int_V [\epsilon_s]^T [C] \{\epsilon_s\} dV - \int_V [\epsilon_s]^T [C] \{\epsilon_o\} dV - \sum_{k=0}^{N} [f]_{sk} \{q\}_{sk} \]  

(56.a)

\[ \Pi_a = \frac{1}{2} \int_V [\epsilon_a]^T [C] \{\epsilon_a\} dV - \int_V [\epsilon_a]^T [C] \{\epsilon_o\} dV - \sum_{k=0}^{N} [f]_{ak} \{q\}_{ak} \]  

(56.b)

Substituting the relations between the strains and the nodal
displacements from equation (20) into equations (56.a,b) we have

\[
\Pi_g = \sum_{k=0}^{N} \frac{1}{2} \int_V \{q\}^T_{sk} [h] \{q\}_{sk} \{q\}_{sk} dV - \sum_{k=0}^{N} \frac{1}{2} \int_V \{q\}^T_{sk} [h] \{q\}^T_{sk} \{q\}_{sk} dV - \sum_{k=0}^{N} \{f\}_{sk} \{q\}_{sk}
\]

\[
\Pi_a = \sum_{k=0}^{N} \frac{1}{2} \int_V \{q\}^T_{ak} [h] \{q\}^T_{ak} \{q\}_{ak} dV - \sum_{k=0}^{N} \frac{1}{2} \int_V \{q\}^T_{ak} [h] \{q\}^T_{ak} \{q\}_{ak} dV - \sum_{k=0}^{N} \{f\}_{ak} \{q\}_{ak}
\]

(57)

where \{q\}_{sk} and \{q\}_{ak} are defined by equation (12); [h] is defined by equation (13); [CS]_{sk} and [CS]_{ak} are defined by equation (18); [g]_{k} is defined by equation (19); and [f]_{sk} and [f]_{ak} are defined by equation (47) and (48). From equation (57) we can see that the total potential energy is a quadratic functions of \{q\}_{sk} and \{q\}_{ak}.

2.1.7. Principle of Minimum Potential Energy

Applying the principle of minimum potential energy [23] to the linearly elastic structure, equilibrium is achieved if

\[
\frac{\partial \Pi}{\partial q_i} = 0
\]

(58)

Applying the stationary condition in equation (58) to equation (54) whose components are given explicitly in equation (57)
yields two sets of simultaneous linear equations for each harmonic term, \( k \), expressed by,

\[
\int_V [h]^T ([CS]_{sk}[g]_k)^T [C][CS]_{sk}[g]_k [h] \{q\}_{sk} dV - \int_V [h]^T ([CS]_{sk}[g]_k)^T [C]\{\epsilon_o\} dV - \{f\}_{sk} = 0
\]

\[
\int_V [h]^T ([CS]_{ak}[g]_k)^T [C][CS]_{ak}[g]_k [h] \{q\}_{ak} dV - \int_V [h]^T ([CS]_{ak}[g]_k)^T [C]\{\epsilon_o\} dV - \{f\}_{ak} = 0
\]

These two equations can be put into the forms,

\[
[k]_{sk} \{q\}_{sk} = \{f_o\}_{sk} + \{f\}_{sk} \quad (59)
\]

\[
[k]_{ak} \{q\}_{ak} = \{f_o\}_{ak} + \{f\}_{ak} \quad (60)
\]

where

\[
[k]_{sk} = [h]^T \int_V ([CS]_{sk}[g]_k)^T [C][CS]_{sk}[g]_k dV [h] \quad (61)
\]

\[
[k]_{ak} = [h]^T \int_V ([CS]_{ak}[g]_k)^T [C][CS]_{ak}[g]_k dV [h] \quad (62)
\]

\[
\{f_o\}_{sk} = \int_V [h]^T ([CS]_{sk}[g]_k)^T [C]\{\epsilon_o\} dV \quad (63)
\]

\[
\{f_o\}_{ak} = \int_V [h]^T ([CS]_{ak}[g]_k)^T [C]\{\epsilon_o\} dV \quad (64)
\]

for \( k = 0, N \). The element stiffness matrices of the ring element representing symmetric and antisymmetric terms are 6x6 square matrices, \([k]_{sk}\) and \([k]_{ak}\) defined by equations (61) and
\( (62) \) respectively. \( \{f_o\}_sk \) and \( \{f_o\}_sk' \) are the symmetric and antisymmetric equivalent nodal force vectors of initial strain, i.e., thermal and axial strains, defined by equations \( (63) \) and \( (64) \). \( \{f\}_sk \) and \( \{f\}_sk' \) are the symmetric and antisymmetric equivalent nodal force vectors of applied loads and computed by formulas given in section 2.1.5.1. thru 2.1.5.3. The resultant stiffness equations \( (59) \) and \( (60) \) reveal the important characteristic of the ring element or, in general, of the class of axisymmetric solid elements. Due to the orthogonality property of the Fourier series used in the displacement shape functions, the unknowns of these equations, \( \{q\}_sk \) and \( \{q\}_sk' \), are decoupled for every harmonic term. Thus, these equations can be solved independently for the nodal displacement vectors \( \{q\}_sk \) and \( \{q\}_sk' \). The displacements are obtained by superimposing the solutions of the nodal displacements in these vectors into equations \( (1) \) and \( (2) \). The use of the Fourier series in the displacement shape functions, which leads to this consequence, is known as the semi-analytical method or method of separation [23,24]. With this method, a two-dimensional problem is reduced into a series of one-dimensional problems.

2.1.8. Stiffness Matrices for the Ring Element

For a two-dimensional analysis, by taking a unit depth, the volume integral of equations \( (61) \), \( (62) \), \( (63) \), and \( (64) \) in polar coordinates becomes
\[ \int_V dV = 2\pi \int_0^{2\pi} r dr d\theta \]

Substitute the expressions of the matrices \([CS]_s, [CS]_a, [g]_k,
and \([C]\) into equations (61) and (62) and manually multiply
these matrix products. With the aid of the trigonometric
identities,

\[
\int_0^{2\pi} \sin(m \theta) \sin(n \theta) d\theta = 0 \quad \text{for } m = n = 0 \\
\frac{\pi}{\pi} \quad \text{for } m = n \neq 0
\]

\[
\int_0^{2\pi} \cos(m \theta) \cos(n \theta) d\theta = 0 \quad \text{for } m \neq n \\
\frac{\pi}{2\pi} \quad \text{for } m = n \neq 0
\]

\[
\int_0^{2\pi} \sin(m \theta) \cos(n \theta) d\theta = 0 \quad \text{for } m = n
\]

where \(n, m\) are integers; the explicit stiffness matrices of the
ring element can then be obtained by performing direct
integrations of the resultant matrices with respect to \(r\) and
\(\theta\) (See Appendix C for the details of algebric manipulations).
The stiffness matrices of the ring element for each harmonic
term can then be given in the forms of,

\[
[k]_s = [h]^T [\kappa]_s [h] \\
[k]_a = [h]^T [\kappa]_a [h]
\]

(66.a,b)

where \([k]_s, [k]_a, [\kappa]_s,\) and \([\kappa]_a\) are \([6x6]\) symmetric matrices
resulting from the fact that the constitutive matrix \([C]\) is
symmetric. The coefficients of matrix \([\kappa]_s\) are given as,
\[ \kappa_{11} = (C_{22}C^2 + k^2C_{33}S^2) \ln(R) \]
\[ \kappa_{12} = ((C_{12} + C_{22})C^2 + k^2C_{33}S^2) R1 \]
\[ \kappa_{13} = ((2C_{12} + C_{22})C^2 + k^2C_{33}S^2) R2 \]
\[ \kappa_{14} = k(C_{22}C^2 + C_{33}S^2) \ln(R) \]
\[ \kappa_{15} = kC_{22}C^2 R1 \]
\[ \kappa_{16} = k(C_{22}C^2 - C_{33}S^2) R2 \]
\[ \kappa_{22} = ((C_{11} + 2C_{12} + C_{22})C^2 + k^2C_{33}S^2) R2 \]
\[ \kappa_{23} = ((2C_{11} + 3C_{12} + C_{22})C^2 + k^2C_{33}S^2) R3 \]
\[ \kappa_{24} = k((C_{12} + C_{22})C^2 + C_{33}S^2) R1 \]
\[ \kappa_{25} = k(C_{12} + C_{22})C^2 R2 \]
\[ \kappa_{26} = k((C_{12} + C_{22})C^2 - C_{33}S^2) R3 \]
\[ \kappa_{33} = ((4C_{11} + 4C_{12} + C_{22})C^2 + k^2C_{33}S^2) R4 \]
\[ \kappa_{34} = k((2C_{12} + C_{22})C^2 + C_{33}S^2) R2 \]
\[ \kappa_{35} = k(2C_{12} + C_{22})C^2 R3 \]
\[ \kappa_{36} = k((2C_{12} + C_{22})C^2 - C_{33}S^2) R4 \]
\[ \kappa_{44} = (k^2C_{22}C^2 + C_{33}S^2) \ln(R) \]
\[ \kappa_{45} = k^2C_{22}C^2 R1 \]
\[ \kappa_{46} = (k^2C_{22}C^2 - C_{33}S^2) R2 \]
\[ \kappa_{55} = k^2C_{22}C^2 R2 \]
\[ \kappa_{56} = k^2C_{22}C^2 R3 \]
\[ \kappa_{66} = (k^2C_{22}C^2 + C_{33}S^2) R4 \]

where \( \kappa_{ij} = \kappa_{ji} \) for \( i,j=1,6 \). \( c_{ij} \) are the elements of the constitutive matrix \([C]\) defined in equation (31), and
\[ \ln(R) = \ln(R_o) - \ln(R_i) \]
\[ R1 = R_o - R_i \]
\[ R2 = \frac{1}{2} (R_o^2 - R_i^2) \]
\[ R3 = \frac{1}{3} (R_o^3 - R_i^3) \]
\[ R4 = \frac{1}{4} (R_o^4 - R_i^4) \]

and

\[ S^2 = \int_0^{2\pi} \sin^2(k\theta) d\theta = \frac{0}{\pi} \quad \text{for } k = 0 \]
\[ \quad \quad \quad \text{for } k \neq 0 \]
\[ C^2 = \int_0^{2\pi} \cos^2(k\theta) d\theta = \frac{2\pi}{\pi} \quad \text{for } k = 0 \]
\[ \quad \quad \quad \text{for } k \neq 0 \]

The coefficients of \([\kappa]\) in equation (67) can be used for computing the stiffness matrix of the ring element corresponding to the symmetric term, i.e., \([\kappa]\). For the antisymmetric term, the coefficients of \([\kappa]\) are found by interchanging \(C^2\) with \(S^2\) and vice-versa in equation (67); see Appendix C. Because of equation (69), the stiffness matrices, \([\kappa]\) and \([\kappa]\) are identical for all harmonic terms except for the term \(k=0\). The advantage of having identical stiffness matrices for both of the symmetric and antisymmetric terms is to save computer time used in formulating the stiffness matrices for each harmonic term. This advantage is a consequence of the use of the minus sign in the Fourier series representing the displacement shape function \(v(r,\theta)\) in equation (2); see Cook [24]. The last step in obtaining the
explicit stiffnesses of \([k]_{sk}\) and \([k]_{ak}\) is achieved by performing the matrix multiplications of the terms \([h]^T[k]_{ak}[h]\) and \([h]^T[k]_{sk}[h]\) in equation (66). This is accomplished numerically.

2.1.9. Equivalent Nodal Forces for Initial and Thermal Strains

The equivalent nodal force vector \(\{f_0\}_{ak}\) and \(\{f_0\}_{sk}\) of the symmetric and anti-symmetric terms due to initial and thermal strains are given by equations (63) and (64) respectively. Substituting the expression of \([CS]\) and \([g]\) from equations (18) and (19) into these equations we have

\[
[f_0]_{sk} = [h]^T \left[ \begin{array}{ccccc} \cos \theta & 0 & 0 \\ 0 & \cos \theta & 0 \\ 0 & 0 & \sin \theta \end{array} \right] \left[ \begin{array}{cccc} 0 & 1 & 2r & 0 & 0 \\ \frac{1}{r} & 1 & r & k & kr \\ -\frac{k}{r} & -k & -kr & \frac{1}{r} & 0 \\ 0 & 0 & -\cos \theta \end{array} \right]^T \int_0^{2\pi} \int_{r_1}^r \epsilon_{a1} \epsilon_{a2} \left[ \begin{array}{cc} \cos \theta & 0 \\ \frac{1}{r} & 1 \\ -\frac{k}{r} & -k \\ 0 & 0 \end{array} \right] \epsilon_{a1} \epsilon_{a2} \left[ \begin{array}{cc} -\cos \theta & 0 \\ 0 & 0 \end{array} \right] \right] r \theta dr \left[ \begin{array}{c} \epsilon_{a1} \\ \epsilon_{a2} \\ 0 \end{array} \right]
\]

\[
[f_0]_{ak} = [h]^T \left[ \begin{array}{ccccc} \sin \theta & 0 & 0 \\ 0 & \sin \theta & 0 \\ 0 & 0 & -\cos \theta \end{array} \right] \left[ \begin{array}{cccc} 0 & 1 & 2r & 0 & 0 \\ \frac{1}{r} & 1 & r & k & kr \\ -\frac{k}{r} & -k & -kr & \frac{1}{r} & 0 \\ 0 & 0 & -\cos \theta \end{array} \right]^T \int_0^{2\pi} \int_{r_1}^r \epsilon_{a1} \epsilon_{a2} \left[ \begin{array}{cc} \cos \theta & 0 \\ \frac{1}{r} & 1 \\ -\frac{k}{r} & -k \\ 0 & 0 \end{array} \right] \epsilon_{a1} \epsilon_{a2} \left[ \begin{array}{cc} -\cos \theta & 0 \\ 0 & 0 \end{array} \right] \right] r \theta dr \left[ \begin{array}{c} \epsilon_{a1} \\ \epsilon_{a2} \\ 0 \end{array} \right]
\]

Equations (70.a,b) can be directly integrated to yield the explicit forms of the equivalent nodal forces due to the initial strain. Note that,
and the third component of vector \( \{ \varepsilon_s \} \) is equal to zero. From the above equations it can be seen that the equivalent nodal forces for the initial and thermal strains only exists (nonzero) for the first symmetric load term, \( k = 0 \); otherwise, it is equal to 0 for \( k \neq 0 \). Substituting \([CS]_k\), \([g]_k\), and \( \{ \varepsilon_s \} \) with \( k = 0 \) into equation (70.a) and performing the direct integrations respect to \( r \) and \( \theta \), we obtain,

\[
[f_0]_{sk} = 2\pi \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}^T \begin{bmatrix}
C_{11} & C_{12} & 0 \\
C_{12} & C_{22} & 0 \\
0 & 0 & C_{13}
\end{bmatrix} \begin{bmatrix}
\varepsilon_{o1} \\
\varepsilon_{o2} \\
0
\end{bmatrix}
\]

R1, R2 and R3 as defined in equation (68), and \([h]\) of equation (13), and performing matrix multiplications, we can find the load term of the first harmonic term, \( k = 0 \), due to the initial strain. In summary then, the equivalent nodal forces for the initial strains are:
45

for \( k = 0 \)

\[ \{ f_o \}_{sk} = \begin{cases} \frac{1}{\Delta} \begin{bmatrix} A_{11}T_1 + A_{21}T_2 + A_{31}T_3 \\ 0 \\ 0 \\ A_{13}T_1 + A_{23}T_2 + A_{33}T_3 \\ 0 \end{bmatrix} & \text{for } k = 0 \\ \{ \theta \} & \text{for } k \neq 0 \end{cases} \]  

(71)

\[ \{ f_o \}_{sk} = \{ \theta \} \quad \text{for all } k \]

where \( \Delta \) and \( A_{ij} \) are given by equation (6) and

\[
T_1 = 2\pi(C_{12}\varepsilon_{01} + C_{22}\varepsilon_{02})R1 \\
T_2 = 2\pi((C_{11} + C_{12})\varepsilon_{01} + (C_{21} + C_{22})\varepsilon_{02})R2 \\
T_3 = 2\pi((2C_{11} + C_{12})\varepsilon_{01} + (2C_{12} + C_{22})\varepsilon_{02})R3
\]

where \( \varepsilon_{01} \) and \( \varepsilon_{02} \) are the first and second components of vector \( \{ \varepsilon_0 \} \) defined in equation (26). \( C_{ij} \) are the elements of the constitutive matrix defined in equation (31). \( R1, R2, \) and \( R3 \) are defined in equation (68).

2.1.10. Displacements, Strain and Stress Calculations

If the forces acting on a cable component in static equilibrium are known, then the deformation and stress distribution can be obtained by using the finite element method. With an axisymmetric ring element, the following steps of the finite element procedure are taken to accomplish the task:

(a) Select a number of ring elements to model the geometric shape of the component;
the stiffness matrices of all the ring elements are formed by equations (66);

(c) assemble all stiffness matrices of all elements into the component stiffness matrices, $[K_c]_k$ and $[K_c]_k'$, with respect to the corresponding symmetric and antisymmetric terms, $k$;

(d) for a particular load set, form the required equivalent load vectors $\{R_c\}_k$ and $\{R_c\}_k'$ for the symmetric and antisymmetric terms, respectively, based on sections 2.1.5.1 thru 2.1.5.3 and 2.1.8.

This would yield:

$$[K_c]_k \{Q_c\}_k = \{R_c\}_k$$

$$[K_c]_k' \{Q_c\}_k' = \{R_c\}_k'$$

(72)

where $[K_c]_k$ and $[K_c]_k'$ are component stiffness matrices corresponding to the symmetric and antisymmetric terms, $k$. $\{Q_c\}_k$ and $\{Q_c\}_k'$ consist of all the nodal displacements of all the ring elements with respect to the symmetric and antisymmetric terms, $k$. Similarly, $\{R_c\}_k$ and $\{R_c\}_k'$ are the equivalent load vectors of symmetric and antisymmetric terms, $k$.

For each harmonic term, the system of linear equations in (72) are solved independently for the displacements vectors $\{Q_c\}_k$ and $\{Q_c\}_k'$. $\{q\}_k$ and $\{q\}_k'$ of every harmonic term, which consists of the nodal displacements, $u_k$, $u_k'$, $v_k$ and $v_k'$, of the particular element under study can be extracted from these
two displacement vectors. The total displacements, \( u \) and \( v \), of this element at any particular desired location within the element can be computed by using equations (1) and (2). The strains at any particular location within the element are obtained by substituting the relationship between constant vectors \( \{b\}_k \) and \( \{b\}_a \) and the nodal displacements vectors \( \{q\}_k \) and \( \{q\}_a \) (equation (10)) into equation (15), repeated below,

\[
\begin{bmatrix}
\varepsilon_{rr} \\
\varepsilon_{\theta\theta} \\
\varepsilon_{r\theta}
\end{bmatrix} = \sum_{k=0}^{N} [CS]_{sk}[g]_k[h]\{q\}_k + \sum_{k=0}^{N} [CS]_{ak}[g]_k[h]\{q\}_a
\]

(15)

Recall that the relations between strains and stresses are defined by equation (30). Substituting this into equation (15) above, we obtain,

\[
\begin{bmatrix}
\sigma_{rr} \\
\sigma_{\theta\theta} \\
\sigma_{r\theta}
\end{bmatrix} = \sum_{k=0}^{N} [C][CS]_{sk}[g]_k[h]\{q\}_k + \sum_{k=0}^{N} [C][CS]_{ak}[g]_k[h]\{q\}_a - [C]\{\varepsilon_0\}
\]

(73)

The axial stress, \( \sigma_{zz} \), of the element at the same location is computed from equation (22) with the corresponding radial and tangential stresses given by equation (73),

\[
\sigma_{zz} = E_z[\varepsilon_{zz} + \frac{\nu_z}{E_z}\sigma_{rr} + \frac{\nu_z}{E_\theta}\sigma_{\theta\theta} - \alpha_z\Delta T]
\]

(22)
2.2. REMCC Element

A number of solid ring elements used in forming the geometrical configuration of a typical component are shown by the dashed lines in Figure 2.1. A cable component, which might be a composite of concentric layers having different material properties, can be modeled with concentric ring elements. The geometry of the REMCC element is formed by condensing these concentric ring elements into one ring element whose circle nodal points are the inner and outer diameters of the cable component.

2.2.1. Method of Condensation

Recall that each of the quadratic ring elements consists of three nodal circle points, (see Section 1.1). It can be seen that for each harmonic term the number of unknown nodal displacements is four for each circle nodal point, two are the symmetric terms and the other two are the antisymmetric terms. Thus for $N_h$ harmonic terms and $N_e$ elements, the total number of unknowns is $4N_h(2N_e+1)$. Since in the formulation of the REMCC element, the symmetric and antisymmetric displacements of each nodal point are coupled among all the harmonic terms by displacement constraints (to be discussed in the following section), a large number of linear equations must be solved simultaneously. In order to reduce the number of unknowns in the model, all the circle nodal points of the ring elements are condensed to leave only two circle nodal points, A and B, which coincide with the inner and outer diameters of the
component as shown in Figure 2.1. The method of condensation based on partition of matrices (Wilson [15]) will be used. Depending on the node which needs to be condensed, the necessary partition matrices must be explicitly developed and inserted into the condensation algorithm. This is demonstrated in the below procedure.

First, the middle node of each ring element is condensed to yield only two nodal points at its inner and outer diameters. Finally, condensation is applied consecutively to the common nodes between adjacent elements so that only two nodal points at inner and outer diameters of the component remain. Mathematically, the two steps are similar and can be demonstrated as follows. For a typical harmonic term, the equation expressing the static equilibrium state of an element is \( [k][q] = \{r\} \), where \([k]\) is the stiffness matrix, \([q]\) is the displacement vector and \([r]\) is the load vector. For the quadratic ring element having 6 nodal displacements at its three nodal points for each symmetric and antisymmetric term, this equation can be written in the general form:

\[
\begin{bmatrix}
  k_{11} & k_{12} & k_{13} & k_{14} & k_{15} & k_{16} \\
  k_{21} & k_{22} & k_{23} & k_{24} & k_{25} & k_{26} \\
  k_{31} & k_{32} & k_{33} & k_{34} & k_{35} & k_{36} \\
  k_{41} & k_{42} & k_{43} & k_{44} & k_{45} & k_{46} \\
  k_{51} & k_{52} & k_{53} & k_{54} & k_{55} & k_{56} \\
  k_{61} & k_{62} & k_{63} & k_{64} & k_{65} & k_{66}
\end{bmatrix} \begin{bmatrix}
  q_1 \\
  q_2 \\
  q_3 \\
  q_4 \\
  q_5 \\
  q_6
\end{bmatrix} = \begin{bmatrix}
  r_1 \\
  r_2 \\
  r_3 \\
  r_4 \\
  r_5 \\
  r_6
\end{bmatrix}
\]
We wish to condense node 2, ([q₃ q₄]), from this equation. Rearranging the matrix elements in the above equation, we have,

\[
\begin{bmatrix}
 k_{11} & k_{12} & k_{15} & k_{16} & k_{13} & k_{14} \\
 k_{21} & k_{22} & k_{25} & k_{26} & k_{23} & k_{24} \\
 k_{31} & k_{32} & k_{35} & k_{36} & k_{33} & k_{34} \\
 k_{41} & k_{42} & k_{45} & k_{46} & k_{43} & k_{44} \\
 k_{51} & k_{52} & k_{55} & k_{56} & k_{53} & k_{54} \\
 k_{61} & k_{62} & k_{65} & k_{66} & k_{63} & k_{64}
\end{bmatrix}
\begin{bmatrix}
 q_1 \\
 q_2 \\
 q_3 \\
 q_4 \\
 q_5 \\
 q_6
\end{bmatrix} =
\begin{bmatrix}
 r_1 \\
 r_2 \\
 r_3 \\
 r_4 \\
 r_5 \\
 r_6
\end{bmatrix}
\] (74)

The matrices in equation (74) are partitioned to yield,

\[
\begin{bmatrix}
 [K_{11}] & [K_{12}] \\
 [K_{21}] & [K_{22}]
\end{bmatrix}
\begin{bmatrix}
 \{Q_1\} \\
 \{Q_2\}
\end{bmatrix} =
\begin{bmatrix}
 \{R_1\} \\
 \{R_2\}
\end{bmatrix}
\] (75)

where

- [K₁₁] is an 4x4 matrix
- [K₁₂] is an 4x2 matrix
- [K₂₁] is an 2x4 matrix
- [K₂₂] is an 2x2 matrix

\{Q₁\} = \{ q₁, q₂, q₅, q₆ \}^T

\{Q₂\} = \{ q₃, q₄ \}^T

\{R₁\} = \{ r₁, r₂, r₅, r₆ \}^T

\{R₂\} = \{ r₃, r₄ \}^T

From the partitioned matrix, equation (75), we can write the two equations:

\[
[K_{11}]\{Q_1\} + [K_{12}]\{Q_2\} = \{R_1\}
\] (76.a, b)

\[
[K_{21}]\{Q_1\} + [K_{22}]\{Q_2\} = \{R_2\}
\]

Using equation (76.b) to solve for \{Q₂\} we obtain
\( \{ q_2 \} = -[K_{22}]^{-1}([K_{21}]\{ q_1 \} - \{ R_2 \}) \)

and substituting this result into matrix equation (76.a) yields the result,

\[
([K_{11}] - [K_{12}] [K_{22}]^{-1} [K_{11}]) \{ q_1 \} = \{ R_1 \} - [K_{12}] [K_{22}]^{-1} [K_{11}] \{ q_2 \}
\]

Thus for each harmonic term we have the condensed stiffness matrix of the ring element corresponding to the four nodal displacements of the two nodal points 1 and 3,

\[
[k_{\text{cond}}]_k = [K_{11}] - [K_{12}] [K_{22}]^{-1} [K_{11}]
\]

(77)

The process above is repeated to condense interconnected nodes between elements. This leads to the stiffness matrix \([k]_e\) of the condensed ring element having two nodal points A and B at the inner and outer diameters of the component as

\[
[k]_e = \begin{bmatrix}
[k]_{k0} \\
\vdots \\
[k]_{kN}
\end{bmatrix}
\]

(78)

This is an \(8(N+1)\) square matrix with all harmonic terms decoupled. In equation (78),

\[
[k]_{sk} = [k_{ij}]_{sk} \quad (i,j=1,4; k=0,N)
\]

\[
[k]_{ak} = [k_{ij}]_{ak} \quad (i,j=1,4; k=0,N)
\]

(79)

These matrices correspond to the nodal displacement vectors,

\[
\{ q \}_{sk} = \begin{bmatrix} u_A \\ v_A \\ u_B \\ v_B \end{bmatrix}_{sk} \quad ; \quad \{ q \}_{ak} = \begin{bmatrix} u_A \\ v_A \\ u_B \\ v_B \end{bmatrix}_{ak}
\]

(80)
Consequently, the stiffness matrix of the condensed ring element defined in equation (78) corresponds to the nodal displacement vector,

\[
\{q\} = \begin{bmatrix}
\{q\}_{s0} \\
\vdots \\
\{q\}_{sN}
\end{bmatrix}
\]

(81)

where \(\{q\}_{s0}\) and \(\{q\}_{sN}\) are defined by equation (80). This is an \(8(N+1)\) column vector.

Another advantage of using the condensation method is that regardless of the number of ring elements used in composing a cable component, the number of displacement unknowns, \(\{q\}\) defined by (81), remain the same, i.e., the desired accuracy of solutions in obtaining stiffnesses of REMCC elements can be attained.

2.2.2. Displacement Constraints

The condensed ring element that represents the component shown in Figure 2.1. can now be treated as a linear elastic structure in equilibrium with the applied forces at its contact points C, D and E. The unit displacement theorem [14] is used to obtain the stiffness matrix, \([k]_m\), of this component with respect to the \(u\) and \(v\) displacements at all contact points. According to this theorem, a column vector of \([k]_m\), corresponding to a unit displacement (\(u\) or \(v\)) at any contact point, is generated by determining the attendant reaction forces for the degree-of-freedom at all contact points to maintain static equilibrium. A typical example of
a cable component having three contact points, C, D, and E at the outer diameter (Figure 2.1) will be used throughout the subsequent sections.

If a unit radial displacement, \( u_c = 1 \), is imposed at the contact point, C, making an angle, \( \beta_c \), with the axis line, \( \theta = 0 \), from equations (1) and (2) the following conditions must be satisfied,

\[
\sum_{k=0}^{N} \left[ u_{B_k} \cos(k\theta_i) + u_{B_k} \sin(k\theta_i) \right] = 1 \quad \text{for} \quad \theta_i = \beta_c
\]

\[ \quad 0 \quad \text{for} \quad \theta_i = \beta_{D,E} \]

\[
\sum_{k=0}^{N} \left[ v_{B_k} \sin(k\theta_i) - v_{B_k} \cos(k\theta_i) \right] = 0 \quad \text{for} \quad \theta_i = \beta_{C,D,E}
\]

The above conditions can be put into matrix form for each contact point.

**Point C**

\[
\begin{bmatrix}
1 & 0 & \ldots & \cos(N\beta_c) & \sin(N\beta_c) \\
0 & 1 & \ldots & \sin(N\beta_c) & -\cos(N\beta_c)
\end{bmatrix}
\begin{bmatrix}
u_{B_0} \\
u_{B_1} \\
\vdots \\
u_{B_n} \\
u_{B_{n+1}}
\end{bmatrix}
= 1 \quad \text{for} \quad \theta_i = \beta_c
\]

\[
\begin{bmatrix}
u_{B_0} \\
u_{B_1} \\
\vdots \\
u_{B_n} \\
u_{B_{n+1}}
\end{bmatrix}
= 0 \quad \text{for} \quad \theta_i = \beta_{C,D,E}
\]
Point D

\[
\begin{bmatrix}
1 & 0 & \ldots & \cos(N\beta_D) & \sin(N\beta_D) \\
0 & \ldots & & & \\
\end{bmatrix}
\begin{bmatrix}
U_{Bso} \\
U_{Bsg} \\
U_{Bsn} \\
\end{bmatrix}
= 0 \quad (83.a)
\]

\[
\begin{bmatrix}
V_{Bso} \\
V_{Bsg} \\
V_{Bsn} \\
\end{bmatrix}
= 0 \quad (83.b)
\]

Point E

\[
\begin{bmatrix}
1 & 0 & \ldots & \cos(N\beta_E) & \sin(N\beta_E) \\
0 & \ldots & & & \\
\end{bmatrix}
\begin{bmatrix}
U_{Bso} \\
U_{Bsg} \\
U_{Bsn} \\
\end{bmatrix}
= 0 \quad (84.a)
\]

\[
\begin{bmatrix}
V_{Bso} \\
V_{Bsg} \\
V_{Bsn} \\
\end{bmatrix}
= 0 \quad (84.b)
\]

Note that all the contact angles are related to the element polar coordinate system as shown in Figure 2.1.
Each pair of the equations from (82) thru (84) can be expressed in the form,

\[ [p_i]{q} = {r_i} \] (85)

where the subscript "i" represents the constraint at contact point "i". \([p_i]\) is the matrix 4 by 8(N+1) consisting of constant terms related to the angles at \(i = C, D\) or \(E\) and zeroes; \({q}\) is the unknown displacement vector defined in equation (81). \({r_i}\) is a 4x1 column vector whose elements equal 0 or 1.

### 2.2.3. Penalty Method

Currently, there are two popular methods used in handling constraints in the finite element method; viz., the penalty method and the Lagrangian multiplier method. The advantage of the penalty method over the other will be discussed in the forthcoming.

The error function, \({E_i}\), representing each displacement constraint above is defined as

\[ \{E_i\} = [p_i]{q} - {r_i} \] (86)

such that \({E_i} = \{0\}\) when equation (85) is satisfied. Following equation (86), the penalty term, \(\Omega_i\), of each displacement constraint is defined as

\[ \Omega_i = \frac{1}{2} \{E_i\}^T [\lambda] \{E_i\} \] (87)

where \([\lambda]\) is a diagonal 4x4 matrix with constant element \(\lambda\). The magnitude of this constant will be discussed in more
details in the next chapter, Computer Implementation.

Using the stiffness matrix of the condensed element and its corresponding nodal displacement vector, \(\{q\}\), defined in equations (78) and (81), respectively, and assuming that there are no external forces applied to the element (i.e., the force potential energy of the element is equal to zero), the total potential energy, \(\Pi_e\), of the condensed element can be written as

\[
\Pi_e = \frac{1}{2} \{q\}^T [k]_e \{q\} 
\]  

(88)

This total potential energy of the condensed element can be augmented by the penalty terms, \(\Omega_i\), for \(i = C, D,\) and \(E\) to yield,

\[
\Pi_e^* = \frac{1}{2} \{q\}^T [k]_e \{q\} + \frac{1}{2} \sum_i \{E_i\}^T \lambda \{E_i\} 
\]  

(89)

From this equation we can see that the augmented total potential energy of the condensed element does not change its value as the penalty terms \(\Omega_i\) approaches \(\{0\}\) for \(i = C, D, E\). Substituting the error function \(\{E_i\}\) from equation (86) into equation (89), we obtain

\[
\Pi_e^* = \frac{1}{2} \{q\}^T [k]_e \{q\} + \frac{1}{2} \sum_i \left[\{p_i\} \{q\} - \{r_i\}\right]^T \lambda \left[\{p_i\} \{q\} - \{r_i\}\right]
\]

According to the principle of minimum potential energy applied to the condensed element, equilibrium is achieved for the stationary condition, \(\partial \Pi_e^*/\partial q_j = 0\), which leads ultimately to the result,
\[
[k_e] + \sum_i [P_i]^T[\lambda][P_i]\{q\} = \sum_i [P_i]^T[\lambda]\{r_i\}
\]  
(90)

where \([k_e]\) is given by equation (78) and \(\{q\}\) is given by 
equation (81). Equation (90) reveals the static equilibrium condition of the condensed element subjected to the imposed unit displacement, \(u_c=1\), applied at \(C\). The term on the right-hand side of equation (90) can be interpreted as the equivalent nodal forces for the unit displacement applied on the condensed element at point "i". Due to the displacement constraint term (the second term on the left-hand side of equation (90)), the nodal displacements between harmonic terms are no longer decoupled. This contrasts with solving the decoupled axisymmetric ring element equation in section 2.1.10 where the displacements of each harmonic term can be solved independently and the total nodal displacements is obtained by superimposing them into equations (1) and (2).

As mentioned at the beginning of this section, the penalty method utilizes only the number of unknown displacement variables, \(\{q\}\). This contrasts with the Lagrangian multiplier method [23,24] where, in addition, a number of unknown multipliers corresponding to the number of constraints is involved. Consequently, the penalty method requires less computer time and has been used in this study.

2.2.4. Stiffness Matrix

Following the penalty method [23,24], the parameter, \(\lambda\), is incrementally determined such that \(\{q\}\) found by equation
(90) satisfies equation (85); viz., \{E_i\} = \{0\}. Having determined \{q\}, the reaction forces acting on the element with respect to the element local coordinate system at any of contact points can be found by the equations,

\[
f_u(r, \theta) = \sum_{k=0}^{N} \left[ [k_{ij}]_s \{q\}_s \cos k \theta + [k_{ij}]_s \{q\}_s \sin k \theta \right] \quad (91.\text{a})
\]

\[
(i = 1 \text{ at } r = R; \ i = 3 \text{ at } r = R_e)
\]

\[
f_v(r, \theta) = \sum_{k=0}^{N} \left[ [k_{ij}]_s \{q\}_s \sin k \theta - [k_{ij}]_s \{q\}_s \cos k \theta \right] \quad (91.\text{b})
\]

\[
(i = 2 \text{ at } r = R; \ i = 4 \text{ at } r = R_e)
\]

and \(j = 1, 4\) for both equations. \([k_{ij}]_s\), \([k_{ij}]_s\) and \{q\}_s, \{q\}_s are defined by equations (79) and (80), respectively. These reaction forces are evaluated at all contact points. This solution represents one column of the stiffness matrix, \([k]_m\), corresponding to the imposed unit displacement, \(u = 1\) at \(C\). The process is repeated with the displacement constraints at \(C\), applying a unit displacement \(v = 1\) at \(C\). Again, equation (90) is formed corresponding to the new set of displacement constraints. Solving the updated equation (90) and using equations (91.a,b) to obtain another column of the stiffness matrix, \([k]_m\), corresponding to the imposed unit displacement, \(v = 1\) at \(C\). This process is repeated for other contact points, \(D\) and \(E\), to obtain the other four columns of the stiffness matrix, \([k]_m\) to form a 6x6 square matrix. In general, for a component having a number of contact points, \(N_c\), the size of the stiffness matrix will be \(2N_c\) by \(2N_c\).
The reaction forces at the contact points are understood as equivalent nodal forces applied at these points. Recall that the equivalent nodal force of the concentrated force, \( P_c \), which acts radially inward on the ring element at an angle \( \beta \) is determined by substituting its Fourier series coefficients given in equation (45) into equations (47) and (48) to yield,

\[
\begin{align*}
    f_{sk} &= -P_c \quad \text{for } k=0 \\
    f_{sk} &= -P_c \cos k\beta \quad \text{for } k \neq 0 \\
    f_{ak} &= 0 \quad \text{for } k=0 \\
    f_{ak} &= -P_c \sin k\beta \quad \text{for } k \neq 0
\end{align*}
\]  

(92)

Comparison of equations (92) and equation (91.a) reveals that

\[
f_u(r, \theta) = \sum_{k=0}^{N} P_r
\]  

(93.a)

where \( P_r \) is the actual magnitude of the reaction force in the radial direction at the contact point located at angle \( \theta = \beta \).

Similarly, for the reaction force in the tangential direction at the contact point located at angle \( \theta = \beta \), we have,

\[
f_v(x, \theta) = \sum_{k=0}^{N} P_t
\]  

(93.b)

where \( P_t \) is the actual magnitude of the reaction force in the radial direction at the contact point located at angle \( \theta = \beta \).

According to equations (93.a,b), in order to obtain true stiffnesses of a REMCC element stiffness matrix corresponding to actual loads, the magnitudes of all the elements of the
REMCC element stiffness matrix obtained by equations (91.a,b) need to be re-scaled by multiplying a factor of \(1/(N+1)\).

Another advantage of the penalty method over the Lagrangian multiplier method is revealed in this process. A careful study of equation (90) reveals that for all sets of displacement constraints corresponding to unit displacements at each contact points, the left-hand side of equation (90) remains the same and only the summation term on the right-hand side of this equation is changed. In the finite element analysis, it is analogous to the problem of a same structure under study for different load cases. This means that the term on the left-hand side needs to be formed only one time. Depending on a particular set of displacement constraints, the term corresponding to this set is placed into the right-hand side of equation (90). For example, for a unit displacement, \(u_c=1\), is applied at \(C\). The term on the right-hand side of equation (90) becomes,

\[
\sum_i [p_i][\lambda]{r_i} = \lambda \begin{bmatrix} 0 & 1 & 0 & \cos \beta_c & \cdots & \cos k \beta & 0 & 0 & \sin \beta_c & \cdots & \sin k \beta_c \end{bmatrix}^T
\]

If a unit displacement, \(v_c=1\), is applied at \(C\). The term on the right-hand side of equation (90) becomes,

\[
\sum_i [p_i][\lambda]{r_i} = \lambda \begin{bmatrix} 0 & 0 & \sin \beta_c & \cdots & \sin k \beta & 0 & 0 & -\cos \beta_c & \cdots & -\cos k \beta_c \end{bmatrix}^T
\]

This results in saving a significant reduction of computer time. In this manner, the entire stiffness matrix for the REMCC macro-element of the component shown in Figure
2.1. \([k]_m\), is generated in a very efficient way.

2.2.5. Alternative Ring Elements

In the process of searching for the best element to model the REMCC element, three element types were examined. In the previous sections, a quadratic ring element was developed. For comparisons, this section presents an overview of the other two elements whose displacement shape functions are linear and cubic polynomials in the radial direction. It can be shown that stiffness matrices of the REMCC elements modeled with all these three ring element types gave nearly identical results. Although the cubic ring element provided the same results with the use of fewer elements, it requires more computer time in forming its stiffness matrix. On the other hand, since the assumed strain field in the radial direction of the linear ring element is constant, a larger number of elements to achieve accurate results for the strain and stress distribution is required. Consequently, the quadratic ring element was selected for use in this study because it combines the good characteristics of the linear and cubic elements. Only an adequate number of elements which results in the least computer time usage is needed to attain the desirable accuracy in building the stiffness matrices of the REMCC elements (forming equation (90)). The derivations of stiffness matrices for the linear and cubic ring elements are presented in Appendix B.
2.3. Model Solution

An umbilical cable normally consists of different types of round components, each one of which can be treated with the REMCC element described in the previous sections. Elements sharing the same geometrical (including contact geometry) and material properties can be put into the same element group. Then only one stiffness matrix for this group needs to be generated. The remaining steps follow the normal procedure of the finite element method.

Nodes for all the contact points of the model must be numbered. Based on these node numbers, element stiffness are developed in a common global Cartesian coordinates system. The next step is to assemble the stiffness matrices of all elements into the system stiffness matrix, $[K]$. The equivalent load vector corresponding to the loads applied to the model is then formed. The resulting system equation is,

$$[K]\{Q\} = \{R\}$$

(94)

where $[K]$ is stiffness matrix for all cable components; $\{Q\}$ is the unknown global displacement vector of all component contact points; and $\{R\}$ is the global load vector consisting of equivalent concentrated loads applied at the contact points. All the steps mentioned above will be discussed in detail in the following sections.

2.3.1. Node Labeling

With REMCC elements, the finite element model of a typical umbilical cable consists of fewer elements and nodal
points than would be required using conventional finite elements. Consequently, it is usually not necessary to implement any bandwidth optimization scheme for the system equation of the model.

2.3.2. Coordinate Transformation

Recall that each nodal point of a REMCC element consists of two displacements, $u$ and $v$, in the radial and tangential directions, respectively. Let's take the REMCC element in Figure 2.1. as an example to illustrate how the displacements, $u$ and $v$, in the local polar coordinate system are transformed to the displacements, $U$ and $V$, in the global Cartesian coordinate system. Assume that the center of local Cartesian coordinate system of the REMCC element coincides with the center of the element's local polar coordinate system and its local $x$-axis coincides with the axis $\theta=0$. Also assume that its local $x$- and $y$-axes are parallel to the global $X$- and $Y$-axes, respectively. The contact points, $C$, $D$, and $E$, form the angles $\beta_C$, $\beta_D$, and $\beta_E$, with the global $X$-axis, respectively. Following the standard coordinate transformations which can be found in many finite element textbooks, the expression below is required for the transformation.
or in the other form,

\[
\{q\}_M = [T]\{q\}_m
\]  \hspace{1cm} (95)

where subscript "m" denotes properties of a REMCC element, m, with respect to the element local coordinate system and subscript "M" denotes properties of a REMCC element in the global coordinate system. \(\{q\}_M\) is the displacement vector of the REMCC element in the X_Y global coordinate system. \([T]\) is the transformation matrix; and \(\{q\}_m\) is the displacement vector of the REMCC element in the element local coordinate system.

The static equilibrium condition of a single macro-element can be expressed in the form,

\[
[k]_m\{q\}_m = \{r\}_m
\]  \hspace{1cm} (96)

where \([k]\_m\) is the element stiffness in the element local coordinate system and \(\{r\}_m\) is the column vector of the local nodal forces. Transforming the local force vector into the global coordinate system requires a similar expression to equation (95),

\[
\{r\}_M = [T]\{r\}_m
\]  \hspace{1cm} (97)
Substituting equations (95) and (97) into equation (96), we obtain

$$[k]_m[T]^T\{q\}_M = [T]^T\{r\}_M$$

Multiplying both sides of the equation above with the transformation matrix, $[T]$, we find

$$[T][k]_m[T]^T\{q\}_M = \{r\}_M \quad (98)$$

From this equation, the transformed stiffness matrix of an REMCC element in the global coordinate system is:

$$[k]_M = [T][k]_m[T]^T \quad (99)$$

where $[k]_M$ denotes the stiffness matrix of REMCC element in the global coordinate system.

2.3.3. Assembly Procedure

After the stiffness matrices for all REMCC elements are transformed to the same global coordinate system following the assembly procedure in [48], they are assembled into the system stiffness matrix, $[K]$, of equation (94). The matrix $[K]$ can be interpreted as the stiffness of entire cable cross-section.

2.3.4. Equivalent Nodal Forces of REMCC Element

With the conventional finite element approach, equivalent nodal forces for radial pressure, axial strain and temperature change are based on the assumed displacement shape functions. A different approach is required to determine equivalent nodal forces for REMCC elements. Assume that the displacements at nodal points and the stiffness matrix of a
particular REMCC element are known. From equation (96), we can then calculate exactly the magnitude of the forces, \( \{r\}_m' \), applied at these nodal points to cause the same amount of displacements at these nodes.

The closed-form solution of a composite orthotropic plate subjected to pressures was derived by Lekhnitskii [50]. The model is made of an arbitrary number of layers in a form of concentric rings of identical thickness. It is assumed that each layer is orthotropic, the orthotropic poles of all layers are located at the center and all layers are rigidly connected, i.e., bonded along the contacting surfaces. Based on kinematic constraints, the radial displacements over two contacting surfaces must be identical, this leads to closed-form solutions for radial displacements, radial and hoop stresses related to the applied pressures (i.e., internal and external pressures) are obtained. Since this is purely a plane-stress solution, conversion to a plane-strain solution is straight-forward by modifying the elastic constants; viz., moduli of rigidity and Poisson's ratios, in the expressions of the solutions. Since these solutions are valid only for the case of applied pressures, to account for thermal and axial strains these solutions must be re-derived with the additional terms in the radial displacement constraints. The radial displacement, \( u_0 \), of a cylinder caused by axial strain and temperature change can be obtained from equation (26). At the inner and outer surfaces of the cylinder, we have
The radial displacement in equation (100) is incorporated into the kinematic constraint equation of Lekhnitskii. The derivation of the new expressions which take into account of both applied pressures, axial and thermal strains is given in Appendix A. A suggested procedure in obtaining numerical results for these closed-form solutions is also mentioned in this Appendix.

Since the radial displacements under uniform pressure or initial strain are uniform along a surface at any particular radial distance, r, the radial displacements at $R_i$ and at $R_o$ are assigned for all nodal points on the inner and outer surfaces of the components, respectively. The radial displacements refer to the local polar coordinate system. With known global contact angles at these nodes, radial displacements are transformed into the global displacements by using equation (95). With both the known global displacements and stiffness matrix of an REMCC element obtained from equation (99), the equivalent nodal forces with respect to the global coordinate system of a REMCC element due to radial pressure(s), and axial and thermal strains are defined by

$$\{r_{e_f}\}_M = [k]_M\{q\}_M$$

(101)

2.3.5. System Solutions

As mentioned above, the system stiffness matrix \([K]\),
and the equivalent nodal force vector, \( \{R\} \) are used to form the system equation for the entire cable cross-section,

\[
[K]\{Q\} = \{R\}
\]  

(94)

where \( \{Q\} \) is the unknown global displacement vector of all component contact points. Equation (94) above is a system of linear equations and can be solved in a number of ways. The Gauss elimination method [48] was selected to be implemented as the equation solver of equation (94). Once this equation is solved, the radial deformation of the cable model can be determined.

2.3.6. Local Deformations, Strain and Stress Distributions of Cable Components

The study of deformations and stress distributions in each individual cable component is very important in the design. The global displacements at contact points of a component, \( \{q\}_M \), are extracted from the global displacement vector \( \{Q\} \) after equation (94) is solved. The global reaction forces of a component can be computed by

\[
\{r\}_M = [k]_M \{q\}_M
\]  

(102)

The reaction forces at contact points of that element in the element coordinate system are obtained by the coordinate transformation,

\[
\{r\}_m = [T]^T\{r\}_M
\]  

(103)
Once the reaction forces of an element are known, the procedure outlined in section 2.1.10. can be used to obtain the local deformations, strain and stress distributions of the component.
CHAPTER 3
COMPUTER IMPLEMENTATION

The objective of this chapter is to describe the computer implementation of the finite element model of umbilical cables developed in the previous chapter. The computer program, CABLES, was written in FORTRAN and documented fully in References [54]. Subroutine and file names appearing in this chapter refer to this program. Due to its low memory requirement, this program can operate on personal computers running under the DOS operating system. Using a modular approach, the program was broken down into dozens of subroutines which are described in the forthcoming.

This chapter is divided into four main sections describing the modeling procedures outlined in Figure 3.1. The first section of this chapter will be used for data preparation needed to model an umbilical cable model. The remaining three sections discuss the three basic operations of the main program, CABLES. Each operation is similar to standard operations of a conventional finite element program. Following data preparation, the stiffness matrices of the REMCC (Ring Element Model with Contact Constraints) elements (cable components) are formed according to equation (90). Next, the global displacements of the nodal contact points of the model are calculated by solving equation (94). Finally, the local deformations, strains, and stresses of cable
components are calculated by the outline given in Section 2.2.10.

Figure 3.1. Implementation procedures of the finite element model of umbilical cables.
3.1. Data Preparation

In any particular finite element program, data for every element in the model must be supplied to the program. The idea of using one REMCC macro-element for each component in an umbilical cable results in a substantial reduction of input data required by the program compared with a conventional finite element model. As with conventional finite element meshes, each existing node in a model must be given a distinct number. The model nodal displacements and elements refer to the nodal numbers.

The following data are required by the program to build the finite element mesh for an umbilical cable model:

(a) To minimize computer time, it is important to allocate identical components to a single group. This eliminates the need to compute stiffness matrices for all components. Identical components are characterized as having the same geometrical and material properties as well as the same contact points. For example, Figure 3.2 shows the cross-section of a typical cable strand consisting of identical circular wires. The four contact points on each wire have the same angular contact positions if the local axis, $\theta=0$ of each wire is placed along the dashed lines as shown in Figure 3.2. These wires satisfy the geometric and contact requirements.
Figure 3.2. A typical cable strand

(b) Each element consists of a number of nodal points equivalent to the contact points with adjacent elements or where concentrated forces act. It is not necessary to assign node numbers in any order; however, it is recommended that nodes which are on the same surface be numbered sequentially following a counter-clockwise direction.

(c) To determine the nodal coordinates, the set of contact angles of the element in the local polar coordinate system is required. The set of contact angles with respect to the global coordinate system of each element in the group must be also supplied. An algorithm to automate this process is reported by Iska [55].

(d) Material properties for each element or element groups are organized by material sets. For isotropic materials, only the modulus of elasticity, \( E \), and the
Poisson's ratio, $\nu$, are required. If thermal loads are involved, the thermal expansion coefficient, $\alpha$, of the material is required as well. For polar orthotropic materials, three elastic constants are required; viz., $E_r$, $E_\theta$, and $E_z$, the Young's Moduli in the radial ($r$), circumferential ($\theta$), and axial ($z$) directions, respectively. The three Poisson's ratios, $\nu_r$, $\nu_\theta$, and $\nu_z$ are also needed. For thermal loads, the three thermal expansion coefficients $\alpha_r$, $\alpha_\theta$, $\alpha_z$ in $r$, $\theta$, and $z$ directions are required.

(d) As mentioned in Section 2.3.4., axial strain, pressure and temperature loads are handled in a special manner. Concentrated forces are applied directly to the element nodes. Note that the cable axial strain (initial strain) is assigned as the constant parameter, AXSTRN, in the main program.

3.2. Forming REMCC Element Stiffness Matrices

After the main program, CABLES, is initialized it transfers the control to subroutine FORMK, which is used to form stiffness matrices of REMCC elements used for the model (Figure 3.3). Based on equation (90) repeated below, a stiffness matrix for each REMCC element representing a group is built. For an imposed unit displacement, this equation represents a system of linear equations with the unknown nodal displacement vector $\{q\}$ defined by equation (81).
Figure 3.3. Forming REMCC element stiffness matrix.
ULOAD
FORM EQUIVALENT NODAL FORCES DUE TO IMPOSED UNIT DISPLACEMENTS

STIFF
SOLUTIONS OF NODAL DISPLACEMENTS FROM DISPLACEMENT CONSTRAINTS

SRING
FORM A COLUMN VECTOR OF REMCC ELEMENT STIFFNESS MATRIX

COEFF
FORM REMCC ELEMENT STIFFNESS MATRIX

Figure 3.3 (cont.). Forming REMCC element stiffness matrix.
\[
[k]_r + \sum_m [p_m]^T[\lambda][p_m]\{q\} = \sum_m [p_m]^T[\lambda]\{r_m\}
\] (90)

The flow chart in Figure 3.3. shows the operations used in forming REMCC element stiffness matrices. Subroutine FORMK repeats these operations for all element groups.

3.2.1. Read Data Input

Element data of the model are read from a data input file, DATIN. This data consists of the geometrical and material properties, the number of nodal contact points, and a set of adjacent element contact angles in the local coordinate system. Next, element numbering, node numbering, and a set of contact angles in the global coordinate system, OXY, for all elements are read into the program. These data are saved into the scratch file, IELMNT, for later use.

3.2.2. Set Penalty Parameter, \( \lambda \)

In practice, the determination of \( \lambda \) as discussed in Section 2.2.3 results in programming complexity and lengthy solution time. The system equation (90) are solved by setting the penalty parameter of each element group to a constant whose magnitude is equal to the largest modulus of elasticity given for a group multiplied by a large number of order \( 10^8 \). This factor has been experimentally determined to satisfy displacement constraints without altering solutions of equation (90). This factor can be adjusted manually over a numerical range in the program to double-check convergence of solutions of the system equations (90).
3.2.3. Build Ring Element Stiffness Matrices

In the next step, subroutine FORMK transfers control to subroutine STIFF where the REMCC element stiffness matrix is formed. First, this subroutine forms the nodal displacement vector \( \{q\} \) of equation (90). Next, it calls subroutine KRING to build ring element stiffness matrices used to model the geometric configuration of the REMCC element. Based on equation (67), the stiffness matrices of a ring element for each harmonic term, \( k \), are built.

3.2.4. Form and Assemble Condensed Ring Element

After stiffness matrices of a ring element is formed for each harmonic term \( k \), subroutine REMOVE is called to condense the middle nodes of ring elements from the stiffness equation. This yields stiffness matrix having only 2 nodal points, 1 and 3, as defined by equation (77). Figure 3.4 shows nodes 2 and 4 of elements 1 and 2, respectively, which have been condensed. Note that the two consecutive ring elements share one common node, viz., nodal point 3 as shown in Figure 3.4. The process of condensation is applied again to remove this common node in order to merge two elements into one element. Thus the resulting element has the stiffness matrix with respect to nodes 1 and 5 as shown in Figure 3.4. The condensation of stiffness matrices of the two ring elements is also done on the basis of each harmonic term. This process is done by calling subroutine STATIC [15]. The processes above are repeated for each subsequent ring element.
until the stiffness matrix \([k_e]\) of the condensed ring element defined in equation (78) of the previous chapter is formed. Next, using the identifications of unknown nodal displacements of vector \(\{q\}\), the stiffness matrix of the condensed element is assembled into the system of linear equations (90) and then saved to the scratch file, SAVE.DAT, for later use.

![Figure 3.4. Two ring elements are condensed into one.](image)

3.2.5. Form and Assemble Displacement Constraints

Once the stiffness matrix of the condensed element, \([k_e]\), is assembled into the system equations (90), subroutine STIFF then calls subroutine PELE to form displacement constraints of all nodal contact points of the REMCC element (see Figure 3.3). The displacement constraints at a nodal contact point are expressed in equation (85),
By using the penalty method, the constraints at a particular nodal point is presented as the term, \([p_i]^T[\lambda][p_i]\), in the summation term on the right-hand side of equation (90). The matrix \([p]\) is formed in the same manner as illustrated for point C, D, and E in Section 2.2.2. Based on the contact angle of a particular node in the local coordinate system, the matrix \([p_i]\) is formed. With the given constant diagonal matrix, \([\lambda]\), the term \([p_i]^T[\lambda][p_i]\) can be computed. All these steps is done automatically in subroutine PELE. This term is then assembled into equation (90) with respect to unknown nodal displacements vector \(\{q\}\). Similarly, the terms \([p_m]^T[\lambda][p_m]\) are formed and assembled into equation (90) for all nodal points of the REMCC element.

3.2.6. Form Equivalent Nodal Forces for Imposed Unit Displacements

The column vector \(\{r_m\}\) of the summation term on the right hand side of equation (90) is defined by equation (85). This column vector results from displacement constraints at each nodal point. In another words, it is the equivalent nodal forces for the corresponding imposed unit displacement. It has been found in Section 2.3.4. that, the resultant of right-hand summation term in equation (90) consists of only one term, \([p_m]^T[\lambda][r_m]\), and the subscript "m" denotes the nodal point where a unit displacement is imposed. With a given

\[
[p_i]\{q\} = \{r_i\} \tag{85}
\]
constant diagonal matrix \([\lambda]\) and unity column vector \([r_m]\), the term above is simply equal to \(\lambda [P_m]^T\), where \(\lambda\) is the constant penalty parameter. This term is formed automatically by a call to subroutine ULOAD.

### 3.2.7. Solutions of Nodal Displacements Resulting from Displacement Constraints

All the terms in equation (90) are now formed as described in the above sections. This equation simply represents a set of \(8(N+1)\) linear equations with the unknown nodal displacement vector, \(\{q\}\), where \(N\) is the upper limit of the summations in equations (1) and (2). Since loads applied on REMCC elements presumably are concentrated loads at its nodal points, a number of twenty-five harmonic terms, i.e., \(N=0,24\), are used in the process to ensure the convergence of the solution. This number has been verified to yield excellent solutions for various number of REMCC elements having from two to thirty-six contact nodes. As a result, the number of linear equations which must be solved for an imposed unit displacement at a nodal point is 208. Many equation solvers as indicated in [23,24,48] can be used to solve equation (90). The direct method, Gauss Elimination, is used in this program to solve equation (90).

### 3.2.8. Form Stiffness Matrix Columns

According to the unit displacement theorem, a column of the stiffness matrix of the REMCC element, \([k]\)\(_m\), corresponding to an imposed unit displacement (u or v) at any contact point,
is determined by the attendant reaction forces for the degree-of-freedom at all contact points of the element to maintain static equilibrium. Once the unknown displacement vector \( \{ q \} \) has been solved from equation (90) corresponding to the imposed unit displacement at the contact points, the reaction forces at all of the contact points of the element can be determined by equations (91) and (92), repeated below,

\[
f_u(r, \theta) = \sum_{k=0}^{N} \left[ [k_{ij}]_{sk} \{ q \}_{sk} \cos \theta + [k_{ij}]_{ak} \{ q \}_{ak} \sin \theta \right]
\]

\[
(i = 1 \text{ at } r=R_i; \ i = 3 \text{ at } r=R_o)
\]

\[
f_v(r, \theta) = \sum_{k=0}^{N} \left[ [k_{ij}]_{sk} \{ q \}_{sk} \sin \theta - [k_{ij}]_{ak} \{ q \}_{ak} \cos \theta \right]
\]

\[
(i = 2 \text{ at } r=R_i; \ i = 4 \text{ at } r=R_o)
\]

and \( j = 1, 4 \) for both equations. \([k_{ij}]_{sk}\), \([k_{ij}]_{ak}\) and \(\{q\}_{sk}\), \(\{q\}_{ak}\) are sub-matrices of the condensed ring element and the displacement vector \( \{ q \} \), defined in equations (79) and (80), respectively. With the known displacement vector \( \{ q \} \) and the stiffness matrix of the condensed element \([k]_e\) retrieved from the scratch file, SAVE.DAT, the reaction forces in both radial and circumferential directions at all contact nodes can be computed. These computations are done in subroutine SRING. All the reaction forces are put into a column vector corresponding to the nodal displacement vector of the REMCC element stiffness matrix.

The other column vectors of the REMCC element stiffness matrix are formed in the same manner as above. In turn, an
imposed unit displacement at all the contact points with respect to their nodal displacement direction (u or v) yields another set of linear equations as expressed by equation (90). As revealed in Section 2.3.4, with the use of penalty method, the left-hand side of equation (90) remains unchanged and the right-hand side of equation (90) becomes $\lambda [p_m]^T$ as indicated in the above section. This term is again formed in subroutine ULOAD. Equation (90) is solved again with the new right-hand term to yield the solutions of the new displacement vector \{q\}. This displacement vector together with the condensed element stiffness matrix are used in equations (91) and (92) to determine the attendant forces at all contact points. In other words, another column vector of the REMCC element stiffness matrix has been found. The process is repeated until all columns of the stiffness matrix are generated. Subroutine STIFF collects all these column vectors and put them together in a $2(\text{NC})$ square matrix, where NC is the number of contact points of the REMCC element. This matrix is identified as the REMCC element stiffness matrix.

3.2.9. Considerations for Using REMCC Elements

Subroutine STIFF passes the REMCC element stiffness matrix to subroutine FORMK. As mentioned above, this matrix represents the stiffness matrix of REMCC element in the same group. A study of numerical values of elements in the REMCC stiffness matrices having different sets of contact points reveals the following facts:
(a) for a REMCC element having contact points symmetric about the axis, $\theta=0$, its stiffness matrix is perfectly symmetric.

(b) for a REMCC element having contact points too close together (about 10 degree apart or closer), the symmetry of the stiffness matrix is not preserved. The differences between some stiffness elements, $k_{ij}$ and $k_{ji}$, are as much as twenty percent. This result is due mainly to the displacement constraints imposed at these points. To preserve the symmetric characteristic of a linear structure according to Betti's theorem [14] (stiffnesses of a linear structure must be symmetric, i.e., $k_{ij}=k_{ji}$). In another word, the reaction force at node $i$ due to a unit displacement at node $j$ must be equal to the reaction force of node $j$ due to a unit displacement at node $i$.) the approximate value of stiffness terms, $k_{ij}$ and $k_{ji}$, are formed by averaging. This process is done in subroutine COEFF in order to attain approximate symmetric stiffness matrices of REMCC elements. As proved in Section 2.3.4, the equivalent nodal force for a concentrated force resulting from the Fourier coefficients and the displacement shape functions is expressed by equation (93). Therefore; stiffnesses, $k_{ij}$, of a REMCC element stiffness matrix must be multiplied by a factor of $1/(N+1)$, where $N$ is the upper limit of the Fourier series representing the concentrated force. This process is also done in subroutine COEFF.
MATROT
COORDINATE TRANSFORMATIONS OF
REMCC ELEMENT STIFFNESS MATRICES

CONSTRAINT ELIMINATIONS

KMACRO
ASSEMBLE REMCC ELEMENT STIFFNESS
MATRICES INTO SYSTEM STIFFNESS \([K]\)

EXACT & CALFXY
NODAL FORCES FOR
PRESSURE LOADS AND INITIAL STRAIN

LOADS & XLOAD
FORM SYSTEM LOAD VECTOR \([R]\)

Figure 3.5. Compute global displacements of nodal points.
Figure 3.5 (cont.). Compute global displacements of nodal points.

3.3. Calculate Global Displacements of Model Nodal Points

Next, the displacements of all nodal points in the model are computed with respect to the global coordinate system, OXY. Equation (94) representing the system equations of the entire cable cross-section is repeated here for subsequent discussion,

\[ [K] \{Q\} = \{R\} \quad (94) \]

where \([K]\) is the cable core stiffness matrix, \(\{R\}\) is the equivalent load vector, and \(\{Q\}\) consists of nodal displacements of all contact points of the model. Equation (94) represents a set of linear equations with the unknown nodal displacements defined in vector \(\{Q\}\). The total number of linear equations given by equation (94) is 2 times the
total number of contact nodes less the number of nodal displacement constraints. The flow chart for these operations are shown in Figure 3.5.

3.3.1. Coordinate Transformations of REMCC Element Stiffness Matrices

Stiffness matrices of REMCC elements built by subroutine FORMK as described above correspond to the nodal displacements at contact points in local polar coordinate systems. In order to solve the system equations (94) for the unknown nodal displacement vector \( \{Q\} \), these stiffness matrices must be transformed into the same global coordinate system. With the given input data of the contact angles in the global coordinate system, OXY, of each REMCC element, the coordinate transformation is performed for every element by a call to subroutine MATROT. In this subroutine the transformation matrix, \( T \), defined by equation (95), is formed automatically based on global contact angles of each REMCC element. Equation (99) is repeated below for subsequent discussion,

\[
[k]_M = [T][k]_m[T]^T \tag{99}
\]

The matrix multiplications in equation (99) are performed in subroutine MATROT to obtain an equivalent stiffness matrix of an REMCC element corresponding to its nodal displacements in the global coordinate system, OXY. This process is done for every REMCC element in the model. Each transformed stiffness matrix is sequentially stored in the scratch file, KKK.
3.3.2. Constraint Eliminations

After the process of coordinate transformation is completed, subroutine FORMK hands over the control back to the main program, CABLES. The program continues to read in the total number of nodal points and displacement constraints of the model from the input data file. The existence of the displacement constraints (fixed or known nodal displacements) is used to prevent rigid body motions of the model. If there are any displacement constraints they are eliminated from the unknown nodal displacement vector \( \{Q\} \) defined in Equation (94).

3.3.3. Equivalent Nodal Forces for Pressures, Axial Strain and Thermal Strain

Subroutine KMACRO is called from the main program to perform the computation of equivalent nodal forces for pressure loads, axial strain and thermal strain applied to REMCC elements. Subroutine EXACT uses equations (A.11), (A.15), and (A.18) thru (A.20) to calculate local radial displacements at the inner and outer surface of the REMCC element due to pressures loads, axial and thermal strains applied to it. With known values of element contact angles in the global coordinate system, the local displacements are transformed into the global displacements for every nodal points of the element by equation (95). The equivalent nodal forces due to these loads are computed by a call to subroutine CALFXY. Knowing the REMCC element stiffness matrix and its
displacements in the global coordinate system, the equivalent
global nodal forces are obtained from equation (102),

\[ \{r_{eq}\}_M = [k]_M \{g\}_M \] (102)

The equivalent nodal force vector \( \{r_{eq}\} \) of each REMCC element
are stored in the scratch file, ISAVE, for later use.

3.3.4. Assemble Stiffness Matrices of REMCC Elements

Transformed stiffness matrices of REMCC elements are
retrieved from the scratch file, KKK, and are then assembled
into the system equations (94). This process is performed in
subroutine, KMACRO.

3.3.5. Form System Load Vector

System loads which consists of concentrated forces at
nodal points of the modal are read from the data input file.
Subroutine LOADS is called to perform this task. The loads
are identified by node numbers and are assembled in load
vector \( \{R\} \) in which magnitude and direction are specified.
Additional loads (equivalent nodal forces) on element nodal
points obtained from pressure loads, temperature change on
each element are read from the scratch file, ISAVE. These
loads are superimposed onto the system load vector \( \{R\} \). This
step is done by a call to subroutine XLOAD from the main
program.

3.3.6. Solve for Global Displacements

The system stiffness matrix \( [K] \) and the system load
vector \( \{R\} \) have been formed as described in the above two
sections. The direct Gauss’s elimination method is applied again to solve the system equation (94) for \( \{Q\} \). Subroutine WRDISP extracts the nodal displacements \( U \) and \( V \) of every nodal points from the displacement vector \( \{Q\} \) and writes them to the output file, ANS.

3.4. Local Deformations, Strain and Stress Distributions of Cable Components

The local deformation, strain and stress distributions of a cable component can be computed when the forces acting on it are known. Each cable component can be modeled as a single elastic structure under static equilibrium with reaction forces at its contact points. The conventional finite element approach can be used for each cable component to obtain its local deformations, strains, and stresses as described in Section 2.1.10. A number of axisymmetric ring elements is used again to model each cable component. The flow chart shown in Figure 3.6. shows these final operations.
Figure 3.6. Compute local deformations, strains and stresses.
3.4.1. Compute Reaction Forces at Contact Points

As stated above, the forces acting on a cable component are equivalent to the reaction forces at the contact points of the REMCC element representing the component. Since the global displacements of contact points of each element have been computed previously, the reaction forces can be computed by the following basic equation,

\[ \{r_T\}_M = [k]_M \{q\}_M \]  \hspace{1cm} (104)

where subscript "M" denotes properties of the REMCC element, M, with respect to the global coordinate system, OXY. \( \{r_T\}_M \) is the equivalent nodal force vector acting on contact points of a REMCC element; \([k]_M \) is the REMCC element stiffness matrix; \( \{q\}_M \) is the nodal displacement vector of the element.

The each component in the force vector \( \{r_T\}_M \) is the sum of the forces due to contact between elements and the equivalent nodal forces due to the presence of pressures and initial strain applied to the element as defined by equation (99). Therefore, the forces, \( \{r_c\}_M \), due to contact is the difference of forces of \( \{r_T\}_M \) and \( \{r_q\}_M \), that is,

\[ \{r_c\}_M = \{r_T\}_M - \{r_q\}_M \]  \hspace{1cm} (105)

The force vector due to contact between elements ,\( \{r_c\}_M \), is transformed into local polar coordinate system of the REMCC element as shown in Figure 3. From equation (95), we obtain
\[ \{r_c\}_m = [T]^T \{R_c\}_M \]  

(106)

where subscript "m" denotes properties of a REMCC element respect to its local coordinate system. \( \{r_c\}_m \) is contact forces of the REMCC element, m. \( \{r_c\}_M \) of each REMCC element is read from the scratch file, ISAVE. Equations (104) and (105), equation (106) are used for computations of \( \{r_c\}_m \) for every REMCC element. The column vector \( \{r_c\}_m \) is then stored together with applied pressures, temperature change of every cable component in the scratch file, FUVDAT, for later use. These processes are done in subroutine KMACRO. At this point, the main program, CABLES, gives the control to subroutine SUB3 to perform the remaining operations.

3.4.2. Build Ring Element Stiffness Matrices

A number of ten ring elements are used to model each cable component. For a composite component formed by concentric layers having different material properties (e.g., a solid wire or a hollow tube surrounded by a jacket), five ring elements are used for each layer. Subroutine FRING is called to build the stiffness matrices of ring elements. Based on equations (67), the stiffness matrices corresponding to all the symmetric terms are built. The stiffness matrices corresponding to the antisymmetric terms are exactly identical to the ones of the symmetric terms for all harmonic terms, except for the harmonic term, k=0. Following the fact discussed at the end of Section 2.2.7, the harmonic term k=0
the elements of antisymmetric stiffness matrix of a ring element is computed by using expressions in (67) with the interchanges of \( S^2 \) and \( C^2 \) and vice-versa. Combining all stiffness matrices of all harmonic terms into one, the stiffness matrix of a ring element, \( n \), as shown in Figure 2.2, can be written in the form of,

\[
[k]_n = \begin{bmatrix}
[k]_{s0} \\
\vdots \\
[k]_{aN}
\end{bmatrix}
\]  \hspace{1cm} (107)

where

\[
[k]_{sk} = [k]_{yk} (i, j = 1, 6; k = 0, N) \\
[k]_{ak} = [k]_{yk} (i, j = 1, 6; k = 0, N)
\]  \hspace{1cm} (108)

The stiffness matrix \([k]_n\) is a square matrix \(12(N+1)\) with all harmonic terms decoupled. The stiffness matrix corresponds to its nodal displacement vector \(\{q\}_n\),

\[
\{q\}_n = \begin{bmatrix}
\{q\}_{s0} \\
\vdots \\
\{q\}_{aN}
\end{bmatrix}; \quad \{q\}_{(\alpha, \eta) k} = \begin{bmatrix}
u_1 \\
v_1 \\
u_2 \\
\vdots \\
u_3 \\
\vdots \\
u_3 \\
\end{bmatrix}
\]  \hspace{1cm} (109)

3.4.3. Assemble Ring Element Stiffness Matrices

Stiffness matrices \([k]_n\) of all ring elements are assembled together to form the stiffness, \([k]_e\), of the component. This process is also done in the subroutine FRING.
3.4.4. Form Equivalent Nodal Forces for Concentrated Loads, Pressures and Initial and Thermal Strains

The equivalent nodal forces for each harmonic term are obtained from equations (45), (47) and (48). As mentioned above, the forces applied on a cable component consists of contact forces which are equivalent to concentrated loads and might include pressures or initial strain. The contact forces, \{r_c\}_m, pressure loads, temperature change of each REMCC element are retrieved from the scratch file, FUVDAT. The equivalent nodal forces of initial strain of a ring element is given explicitly in equation (70) of Section 2.2.8. Based on this equation, the equivalent nodal forces for axial and thermal strains are computed in subroutine FRING and their numerical values are initially stored in the column vector \{r\}_c, where \{r\}_c is defined as the component load vector corresponding to the stiffness matrix of a cable component, \[k\]_c. The equivalent nodal forces for pressure loads and contact forces are obtained by determining first their Fourier series coefficients from equations (43) and (45), respectively. Second, substituting these coefficients into equation (47) and (48), the equivalent nodal forces of a cable component for pressure loads and reaction forces are found. This process is done in subroutine FLOAD. The equivalent nodal forces obtained from the last step are superimposed onto the component load vector \{r\}_c. These computations are done in subroutine FLOAD.
3.4.5. Solve for Local Nodal Displacements

The static equilibrium condition of a cable component yields,

\[ [k]_c \{q\}_c = \{r\}_c \]  \hspace{1cm} (110)

where \([k]_c\) is the component stiffness obtained from assembling all ring element stiffness as described above. \({q}\}_c\) is the unknown nodal displacements of the component. \({r}\}_c\) is the equivalent nodal force vector obtained by loads acting on the component. The total number of unknowns in vector \({q}\}_c\) is equal to \(4(N+1)(2N_e+1)\). Where \(N\) is the upper limit of the Fourier series defined in equation (1) and (2) and \(N_e\) is the number of ring elements used to model the cable component. For \(N=24\) and \(N_e=10\), equation (110) represents a system of 2,100 of linear equations. Since equations in equation (109) are decoupled for each symmetric and antisymmetric harmonic terms. \(N+1\) sets of systems of equations of size \(4(2N_e+1)\) can be solved independently to obtain the solution of equation (110) as mentioned in Section 2.1.10. However, as an alternate choice to reduce the computer time, the system of 2,100 linear equations are solved at once in this program.

3.4.6. Compute Displacements, Strains and Stresses:

The outline in Section 2.1.10 describes all the necessary equations used to compute displacements, strains, and stresses at any point within a component. At any point, equations (1) and (2) are used to compute displacements \(u\) and \(v\). This
process is done in subroutine WRITS. Equations (15), (22) and (73) are used to calculate strains and stresses. This process is done inside subroutine STRESS. The numerical of values of these results at specified locations in a component are also written to the output file ANS.
CHAPTER 4
VERIFICATION EXAMPLES

In this chapter, a number of examples demonstrate the ability of the REMCC solid ring elements and macro-elements to model umbilical cables. The first three examples consist of problems from the classical theory of elasticity. These examples involve different types of cylinders subjected to various loadings such as internal and external pressures, axial strain, temperature changes and concentrated radial forces. REMCC solutions are compared with exact solutions or with other finite element solutions. The fourth example involves three contacting cylinders subjected to opposing forces. The final example is of an as-built fiber-optic cable. The REMCC solution is compared with the test data obtained for this cable [56].

4.1. Isotropic Cylinders

4.1.1. Thick-Wall Cylinder Subjected to Internal and External Pressures

A thick-walled isotropic cylinder having 1 and 5 inch inner, \( R_i \), and outer radii, \( R_o \), respectively, is subjected to an internal pressure, \( P_i \), of 100 psi and an external pressure, \( P_o \), of 200 psi. The elastic modulus, \( E \), and Poisson’s ratio, \( \nu \), of this cylinder are \( 5.0 \times 10^6 \) psi and 0.35, respectively. The exact solution of this problem is known as Lamé’s solution [49] given by
\[ u(r) = \frac{1}{E(R_0^2 - R_i^2)} \left[ \left( P_i R_i^2 - P_o R_0^2 \right) (1 - \nu) r + R_0^2 R_i^2 \left( P_i - P_o \right) (1 + \nu) \frac{1}{r^2} \right] \]

\[ \sigma_r(r) = \frac{1}{(R_0^2 - R_i^2)} \left[ R_i^2 R_0^2 (P_0 - P_i) \frac{1}{r^2} + (P_i R_i^2 - P_o R_0^2) \right] \]

\[ \sigma_\theta(r) = \frac{1}{(R_0^2 - R_i^2)} \left[ R_i^2 R_0^2 (P_i - P_0) \frac{1}{r^2} + (P_i R_i^2 - P_o R_0^2) \right] \]

\[ u(\theta) = 0; \quad \sigma_\theta(r) = 0 \]

(111)

where \( r \) is the radial distance from the cylindrical axis; \( u(r) \) and \( u(\theta) \) are the radial and tangential displacements, respectively; \( \sigma_r(r), \sigma_\theta(r) \) and \( \sigma_\phi(r) \) are the radial, hoop (tangential) and shearing stresses, respectively. The expressions above can also be obtained by setting \( k=1 \) in Lekhnitskii's expressions [50] (equations (A.1) and (A.2) in Appendix A).

This cylinder is modeled with ten quadratic ring elements. Because the equivalent nodal forces of a ring element for uniform pressure, axial and thermal strains result in a nonzero term for the symmetric term, \( k=0 \), as noted in Chapter 2, only this term, \( k=0 \), is necessarily used in equation (61) to formulate the stiffness matrix of the ring element. The solutions of displacements and stresses obtained are in excellent agreement with the exact solutions obtained from equation (111). The radial and circumferential displacements, \( u(r) \) and \( u(\theta) \), and the shearing stress, \( \sigma_\theta \), are...
matched with the exact solutions. A maximum error of 0.6% for the hoop stress occurs at the inner surface of the cylinder. A maximum error of 3.5% for the radial stress also occurs at the inner surface of the cylinder. This small error is due mainly to the large thickness of the cylinder. Note that the solutions obtained with the quadratic ring elements converge with the use of only five elements. The solutions of the radial displacement, radial and hoop stresses are plotted against the exact solutions in Figures 4.1., 4.2. and 4.3., respectively.

The cylinder was also modeled with linear and cubic ring elements whose stiffness matrices are given in Appendix B. With ten cubic ring elements for this model, slightly improved solutions were obtained in comparison with the quadratic ring elements. The displacements and shearing stresses are in nearly exact agreement. A maximum error of only 0.3% for the hoop stress was also obtained at the inner surface of the cylinder. The radial stress is within 1.8% agreement at the same location. Also, convergence was achieved with only five elements. The solutions obtained with the linear ring elements required at least 20 elements to achieve the same accuracy as the quadratic ring elements.
Figure 4.1. Radial displacement of an isotropic cylinder subjected to pressure.

Figure 4.2. Radial stress distribution of an isotropic cylinder subjected to pressure.
4.1.2. Composite Cylindrical Tube Subjected to Internal and External Pressures and Axial Strain

In this example, a cylindrical tube composed of three concentric layers of different materials is modeled with the quadratic ring elements. It is assumed that the material of each layer is isotropic and these layers are rigidly connected; i.e., tightly bonded over the contact surfaces. This model represents cable components which consists of different material layers, e.g., a conduit carrying hot hydraulic fluid and supported by a structural sheath and insulation jacket.

The geometric and material properties of the three layers

Figure 4.3. Hoop stress distribution of an isotropic cylinder subjected to pressure.
of the cylindrical tube are given in Table 4.1.

Table 4.1. Properties for Composite Isotropic Tube.

<table>
<thead>
<tr>
<th>Layer</th>
<th>I.D. (inches)</th>
<th>O.D. (inches)</th>
<th>Young Modulus (psi)</th>
<th>Poisson’s Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>2</td>
<td>5x10^6</td>
<td>.35</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>3</td>
<td>2x10^6</td>
<td>.40</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>4</td>
<td>1x10^6</td>
<td>.45</td>
</tr>
</tbody>
</table>

Internal and external pressures of 500 psi and 1,000 psi, are applied at the inner and outer diameters of layers 1 and 3. In addition, an axial strain 10^{-3} in/in is applied to the cylinder. The exact solution of this problem can be obtained by setting \( k=1; \) i.e., \( E_r=E_\theta=E, \) and the Poisson’s ratio \( \nu_{r\theta} \) of each layer equal to the corresponding isotropic Poisson’s ratio in equations (A.14) and (A.15), which are modified from Lekhnitskii equations [50] to account for axial and thermal strains.

Each layer of the cylinder is modeled with ten quadratic ring elements for the symmetric term, \( k=0. \) The plots of the radial displacements, radial and hoop stresses obtained from the exact solutions and the REMCC model (see Figures 4.4., 4.5. and 4.6.) show excellent behavior of the quadratic ring element. The radial displacements within and between contact
surfaces of all layers shown in Figure 4.4. are identical with the exact solutions of equation (A.15). For the radial stresses, there is a negligible difference (less than 0.1%) in numerical values at the surfaces of layers 1 and 2. The maximum error of 1.4% as compared with the exact solution of this stress is also found at the inner surface of layer 1.

Figure 4.6. shows the discontinuities of the hoop stresses along the contact surfaces of layers 1 and 2, and layers 2 and 3. This reveals the fact that the material discontinuities between these layers do exist. One again, the quadratic ring element has shown its ability to model this type of component. There are negligible errors as compared with the exact solutions.

Figure 4.4. Radial displacement of a composite isotropic tube subjected to internal and external pressures and axial strain.
Figure 4.5. Radial stress distribution of a composite isotropic tube subjected to pressures and axial strain.

Figure 4.6. Hoop stress distribution of a composite isotropic tube subjected to pressures and axial strain.
4.1.3. Composite Solid Cylinder Subjected to External Pressure

To model the solid cylinder, the model for the previous example is modified slightly. The geometric and material properties are the same as in the previous example except that the inner radius of layer 1 is set to zero. An external pressure of 1,000 psi is applied at outer surface of layer 3. The exact solutions for this model are obtained also by setting $k=1$ and $c_o=0$ in equations (A.14) and (A.15) in Appendix A.

Again, each layer of the cylinder is modeled with 10 quadratic ring elements for the symmetric term, $k=0$. The radial and circumferential displacements are perfectly matched with the exact solutions. The maximum errors in radial and hoop stresses at the contact surface between layers 1 and 2 are less than 0.1%. It is noted that both stresses obtained from the exact solution and the REMCC model encounter a singularity on the cylindrical axis where $r=0$. The stress singularities are resolved by recognizing that the radial and circumferential stresses of a solid cylinder from its center throughout to its surface must be equal to the stress acting on its surface. Figures 4.7, 4.8, and 4.9. show the variations of the radial displacements, radial and hoop stresses, respectively, for both of the analytical and finite element solutions.
Figure 4.7. Radial displacement of a solid composite cylinder subjected to external pressure.

Figure 4.8. Radial stress distribution of a solid composite cylinder subjected to external pressure.
Figure 4.9. Hoop stress distribution of a solid composite cylinder subjected to external pressure.

4.2. Orthotropic Cylinders

Examples of orthotropic cylinders subjected to pressures, axial and thermal strains modeled with the quadratic ring elements are presented in this section.

4.2.1. Thin-Wall Cylinder Subject to Temperature Change

Consider an orthotropic cylinder with properties described as follow:

\[ R_i = 49.0 \text{ inches} \]
\[ R_o = 50.0 \text{ inches} \]
\[ E_r = 29.0 \times 10^6 \text{ psi} \quad G_{\theta \phi} = 1.2 \times 10^6 \text{ psi} \quad \nu_{\phi \theta} = 0.30 \]
\[ E_\theta = 29.0 \times 10^6 \text{ psi} \quad G_{\theta z} = 0.8 \times 10^6 \text{ psi} \quad \nu_{\theta z} = 0.15 \]
\[ E_z = 4.40 \times 10^6 \text{ psi} \quad G_{rz} = 0.8 \times 10^6 \text{ psi} \quad \nu_{rz} = 0.15 \]
\[ \alpha_r = \alpha_z = \alpha_z = 1.0 \times 10^{-6} \text{ in/in } ^\circ F \]

The cylinder is subjected to a temperature change of -100 \(^\circ F\). This simple example is used to confirm that the expressions derived from Hooke’s law for the axial stress and initial strain, equations (23) and (26) of Chapter 2, respectively, are correct. The solution is also compared with the finite element code, ANSYS [53] (modeled with plane strain and axisymmetric ANSYS quadrilateral elements).

This example can be modeled with one quadratic ring element to yield the exact solution. From an analytical point-of-view, the radial, hoop, and shearing stresses must be zero everywhere within the cylinder and the axial stress, \( \sigma_{zz} \), is a constant 440.0 psi obtained by the formula,

\[ \sigma_{zz} = -E_z (\alpha_z \Delta T) \]

This is a special case obtained from equation (23) by setting the axial strain and the radial and hoop stresses equal to zero. Although, ANSYS can not provide the solutions of the radial displacements of the model, these solutions can be computed from the expression of the initial strain, equation (26),

\[ u(r) = [(\alpha_r + \alpha_z \nu_{rz}) \Delta T] r \]  \hspace{1cm} (111)

where the Poisson’s ratio, \( \nu_{rz} \), is computed from the relationship \( \nu_{rz} = (E_z \nu_{rz})/E_t \). Figure 4.10. compares the plot of the radial displacements obtained from the finite element
solution with the analytical solutions obtained from equation (111).

Figure 4.10. An orthotropic cylinder subjected to a temperature change.

4.2.2. Composite Cylindrical Tube Subjected to Internal and External Pressures, Axial Strain, and Temperature Change

In this example, a cylindrical tube made of an isotropic layer sandwiched between two identical orthotropic layers is modeled with the quadratic ring elements. It is assumed that these layers are rigidly connected; i.e., tightly bonded over the contact surfaces. The geometric and material properties of the three layers of the composite cylinder are given in Table 4.2.
Table 4.2. Properties for Composite Orthotropic Tube.

Layer 1 and 3:

<table>
<thead>
<tr>
<th>Layer</th>
<th>Ri</th>
<th>Ro</th>
<th>Er</th>
<th>Go</th>
<th>νrΩ</th>
<th>Ez</th>
<th>Go</th>
<th>νrz</th>
<th>αr</th>
<th>αθ</th>
<th>αz</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1 inches</td>
<td>2 inches</td>
<td>29.0x10^6 psi</td>
<td>1.2x10^6 psi</td>
<td>0.30</td>
<td>4.40x10^6 psi</td>
<td>0.8x10^6 psi</td>
<td>0.15</td>
<td>1.0x10^-6 in/in °F</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>3 inches</td>
<td>4 inches</td>
<td>29.0x10^6 psi</td>
<td>0.8x10^6 psi</td>
<td>0.15</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Layer 2:

<table>
<thead>
<tr>
<th>Layer</th>
<th>Ri</th>
<th>Ro</th>
<th>E</th>
<th>ν</th>
<th>α</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>2 inches</td>
<td>3 inches</td>
<td>5.0x10^6 psi</td>
<td>0.35</td>
<td>1.0x10^-6 in/in °F</td>
</tr>
</tbody>
</table>

The composite cylinder is subject to a 100 psi pressure at the inner and outer diameters of layers 1 and 3, respectively. An axial strain 10^-3 in/in is applied to the cylindrical tube. Also, all three layers experience a uniform temperature change of 100 °F. The analytical solutions of this model are obtained from equations (A.14) and (A.15) in Appendix A.

In the REMCC model, each layer of the composite tube is modeled with ten quadratic ring elements. The solutions obtained from this model confirm the reliability of the quadratic ring element. Even, with the large changes of the
radial displacements, radial and hoop stresses along the contact surfaces agree well with the analytical solutions (see Figures 4.11., 4.12. and 4.13.). The radial displacements are nearly indistinguishable from the analytical solutions. A maximum error of 3% was found at the inner surface of layer 1 for the radial stress in contrast to the errors found along the surface contacts of layers 1 and 2, and layers 2 and 3, which are almost negligible (less than 0.2%). Despite the dissimilar material properties over the contact surfaces, the hoop stress errors are always less than 0.1%.

Figure 4.11. Radial displacement of a composite orthotropic tube subjected to pressure, axial strain and temperature change.
Figure 4.12. Radial stress distribution of a composite orthotropic tube subjected to pressure, axial strain and temperature change.

Figure 4.13. Hoop stress distribution of a composite orthotropic tube subjected to pressure, axial strain and temperature change.
4.3. Cylinder Subjected to an Equal and Opposing Line Load

Consider the problem of an infinitely long cylindrical tube compressed between two equal and opposing uniform line forces along its length as shown in Figure 4.14. This problem is discussed by Timoshenko [49] and the exact solution was found with a stress function expressed in two-dimensional polar coordinates. This problem demonstrates the capability of the axisymmetric solid element in treating an unsymmetrical load.

The problem can be treated as a plane-strain problem. Ten quadratic ring elements are used to model this portion of the cylinder. Due to symmetry about the axis, $\theta=0$, as shown in Figure 4.14., the equivalent nodal forces obtained from equations (45), (47) and (48) result in nonzero numerical
values for symmetric terms and zero for all antisymmetric
terms. Consequently, only equation (61) is needed to
formulate the stiffness matrices of the ring element
corresponding to each harmonic term. Ten harmonic terms
\((k=0, 2, 4, \ldots)\) are needed to achieve convergence toward the
analytical solution. The geometric and material properties
used for the model are:

\[ R_i = 10"; \quad R_o = 20"; \quad E = 30.0 \times 10^6 \text{ psi}; \quad \nu = 0; \]

and the magnitude of each the concentrated force is 10,000\(\pi\)
lbs.

The radial displacements along the section \(n-n_1\) (Figure
4.14) as a function of the number of elements shows the
convergence characteristic. The radial displacement
distribution shown in Figure 4.15 displays monotonic
convergence with 10 quadratic ring elements. No further
improvement in the solution was observed by increasing number
of elements. The analytical distribution of the hoop stresses
along sections \(m-m_1\) and \(n-n_1\) of the cylinder obtained are
shown in Figure 4.16. From the distribution of hoop stress
along section \(m-m_1\), good agreement between the analytical and
the REMCC solutions is seen. At section \(n-n_1\), the maximum
hoop stress occurs at the inner surface of the cylinder. At
this location, the REMCC solution has an error of only 0.13%;
however; moving away from the location of the concentrated
force in Figure 4.16., we can see that the solution is only
valid for \(r\) less than or equal \(0.9R_o\). Near the singularity
point where the concentrated force is applied, \( r=R_a \), the hoop stress has a large value as shown in Figure 4.16. This problem is expected for any numerical procedure [49].

![Figure 4.15. Radial displacement of a cylinder subjected to equal and opposing line load.](image)
Figure 4.16. Hoop stress distribution of a cylinder subjected to equal and opposing line load.

4.4. Three Contacting Cylinders Subjected to a Concentrated Force

Consider the problem of three contacting cylinders subjected to a concentrated force as shown in Figure 4.17. The cylinders have the same geometric and material properties as the cylinder in section 4.4. It’s assumed that the cylinders are rigidly connected only at the contact points. This assumption is also applied for the lowest cylinder and the rigid base. Also, it is assumed that no indentation occurs at the contact points. The solution of this problem
Figure 4.17. Three contacting cylinders subjected to a concentrated force.

can be found by the method of superposition; i.e., each cylinder is subjected to a pair of equal and opposite forces acting through the center of each cylinder. The solution of
this subproblem is obtained from the previous example. The total displacement at each contact point is found by superimposing the displacement components obtained from all contact points. However, to verify that the theory presented in Chapter 2 is valid, this problem is solved by using REMCC elements. This also serves for the purpose of showing how the forces are transmitted at contact points of the REMCC elements.

The global coordinate system is placed at the rigid base shown in Figure 4.17. Each cylinder is modeled by a REMCC element which has two nodal points at the contact points of the cylinders. The nodal points of each macro-element are labeled as shown in Figure 4.17. The stiffness matrix of the macro-element is formed by using the method described in Chapter 2. Since these cylinders have the same geometric and material properties, the stiffness matrix of each REMCC element must be identical. Once the stiffness matrices of all REMCC elements are formed, they can be assembled into the system stiffness equations. Note that there is no coordinate transformation necessary for this problem since the local angular locations of the contact points of each element are the same in the global coordinate system. To keep the system in static equilibrium, the vertical and horizontal displacements at node 1 must be constrained (fixed). The system equations are then solved with the applied force at node 4. The resulting displacements are listed in Table 4.3.
The vertical Y-displacements at nodes 2, 3, and 4 represent the total displacement relative to the reference node 1. It is easy to show that the Y-displacement relative to each cylindrical axis is the same as given in the previous example. To verify that this solution yields identical concentrated forces at node 1, 2 and 3, the stiffness matrix of the REMCC element according to the equation, \([k]{q}={r}\), is used. As expected, these attendant forces are equal in magnitude to the concentrated force applied at node 4.

Table 4.3. Displacements of the Model Nodal Points.

<table>
<thead>
<tr>
<th>Node</th>
<th>X-displacement</th>
<th>Y-displacement</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(inches)</td>
<td>(x10^-2 inches)</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>-1.14032</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>-2.28064</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>-3.42095</td>
</tr>
</tbody>
</table>

4.5. Umbilical Cable Modeled with REMCC Elements

The cross-section of a remotely-operated vehicle (ROV) umbilical cable, manufactured by Sumitomo Electric Industries, Ltd. of Osaka, Japan, is shown in Figure 1.1. This cable consists of a core of three electrical conductor, three
jacketed optical fibers, and filler wires. The core is encased in a cylindrical core jacket. Two round wire layers are helically-served around the core to provide strength and external armor. With the radial pressure on layer 5 (core jacket) obtained from the KNAPP-SAC program [12], only the cable core is modeled. Figure 1.1 shows that the geometry and material properties of the cross-section is unsymmetric. A total of 14 REMCC elements are used to model the fourteen components of the core cross-section (see Figure 4.18). Twenty-four evenly-spaced nodal points are assigned to the outermost layer of the cable core, layer 5, for placing equivalent radial forces, which approximate the uniform pressure produced by the armor wires. The finite element mesh for this model is shown in Figure 4.18.

By taking advantage of symmetrical geometry, five element groups corresponding to the core components require definition. Properties for these groups are listed in Table 4.4.

Table 4.4. Properties for REMCC Element Groups.

<table>
<thead>
<tr>
<th>Group I: Filler wire</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of component(s)</td>
</tr>
<tr>
<td>Component inner radius</td>
</tr>
<tr>
<td>Component outer radius</td>
</tr>
<tr>
<td>Young's Modulus</td>
</tr>
<tr>
<td>Poisson's ratio</td>
</tr>
<tr>
<td>Number of contact points</td>
</tr>
<tr>
<td>-------------------------</td>
</tr>
<tr>
<td>Contact angles at O.D.</td>
</tr>
</tbody>
</table>

### Group II: Copper Conductors

<table>
<thead>
<tr>
<th>Number of component(s)</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Component inner radius</td>
<td>0 mm</td>
</tr>
<tr>
<td>Component outer radius</td>
<td>1.08204 mm</td>
</tr>
<tr>
<td>Young's modulus</td>
<td>98,595 MPa</td>
</tr>
<tr>
<td>Poisson's ratio</td>
<td>0.33</td>
</tr>
<tr>
<td>Jacket inner radius</td>
<td>1.08204 mm</td>
</tr>
<tr>
<td>Jacket outer radius</td>
<td>2.14122 mm</td>
</tr>
<tr>
<td>Young's modulus (Jacket)</td>
<td>186.16 MPa</td>
</tr>
<tr>
<td>Poisson's ratio (Jacket)</td>
<td>0.45</td>
</tr>
<tr>
<td>Number of contact points</td>
<td>8</td>
</tr>
<tr>
<td>Contact angles at O.D.</td>
<td>3.68, 102.43, 150, 180, 210, 257.57, 286.32</td>
</tr>
</tbody>
</table>

### Group III: Optical Fibers

<table>
<thead>
<tr>
<th>Number of component(s)</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Component inner radius</td>
<td>0 mm</td>
</tr>
<tr>
<td>Component outer radius</td>
<td>1.03124 mm</td>
</tr>
<tr>
<td>Young's modulus</td>
<td>2,118.24 MPa</td>
</tr>
<tr>
<td>Poisson's ratio</td>
<td>0.40</td>
</tr>
<tr>
<td>Number of contact points</td>
<td>5</td>
</tr>
<tr>
<td>Contact angles at O.D.</td>
<td>60, 114.67, 197.57</td>
</tr>
</tbody>
</table>
### Group IV: Filler Wires

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of component(s)</td>
<td>6</td>
</tr>
<tr>
<td>Component inner radius</td>
<td>0 mm</td>
</tr>
<tr>
<td>Component outer radius</td>
<td>0.51562 mm</td>
</tr>
<tr>
<td>Young's modulus</td>
<td>882.60 MPa</td>
</tr>
<tr>
<td>Poisson's ratio</td>
<td>0.45</td>
</tr>
<tr>
<td>Number of contact points</td>
<td>3</td>
</tr>
<tr>
<td>Contact angles at O.D.</td>
<td>38.24, 158.33, 253.68</td>
</tr>
</tbody>
</table>

### Group V: Core Jacket

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of component(s)</td>
<td>1</td>
</tr>
<tr>
<td>Component inner radius</td>
<td>4.6228 mm</td>
</tr>
<tr>
<td>Component outer radius</td>
<td>5.57276 mm</td>
</tr>
<tr>
<td>Young's modulus</td>
<td>882.60 MPa</td>
</tr>
<tr>
<td>Number of contact points</td>
<td>36</td>
</tr>
<tr>
<td>Contact angles at I.D.</td>
<td>0, 38.24, 60, 81.76, 120.0, 158.24, 180.0, 201.76, 240.0, 240.0, 278.24, 300.0, 321.76</td>
</tr>
<tr>
<td>Contact angles at O.D.</td>
<td>at 24 nodes, 15° apart and starts at 0°.</td>
</tr>
</tbody>
</table>
4.5.1. Comparison with Experimental Data

Three experimental data sets were obtained [56] for the initial and loaded radial deformations of the cable. With an initial radial indentation of 0.02540 mm, fifteen incremental tension loads from 4,450 N to 66,723 N were applied to the cable specimen. This data is used to validate the REMCC model.

With all the data given in Table 4.4., the REMCC model for the test cable consisted of 60 nodal points as shown in Figure 4.18. Since each nodal point has 2 translational degrees-of-freedom, the total system displacements of the model are 120. The stiffness matrices of the REMCC elements are formed by using the method discussed in Chapter 2. Coordinate transformations and assembly procedures are performed to form the system stiffness matrix of the entire cable cross-section. The radial pressure applied to layer 5 is converted into equally equivalent nodal forces acting at the twenty-four nodes on the outer surface of layer 5 for each tension increment. Thus, for each of the fifteen tension increments the corresponding axial strain and equivalent nodal forces are applied to the model. Due to the asymmetries of material and geometry of the cable cross-section, the radial displacements at the outer diameter of the core jacket which represent the radial deformation of the cable cross-section are not uniform (although, symmetric deformation about the global X-axis does exist). By taking the average of the
radial displacements of the nodal points on the outer diameter of the core jacket for each tension load from the REMCC model and taking into account the measured wire indentation, the curve of these values is plotted against the mean curve of the experiment data as shown in Figure 4.19. These curves are in excellent agreement.

Figure 4.18. ROV umbilical cable modeled with REMCC elements.
4.5.2. Comparison with Conventional Finite Element Analysis

The test cable above was modeled with the commercial finite element software, I-DEAS, for further verification. Since this software does not have a plane strain element, triangular and quadrilateral thin shell elements were used with Poisson's ratios set to zero to simulate the plane strain condition. Due to symmetry of cable components about the...
global X-axis, only a half of the cable cross-section needed to be modeled. A total of 1,322 thin shell elements were used as shown in Figure 4.20. The total number of nodes for this model is 950. Since each node of the thin shell element has 6 degrees-of-freedom (3 translations and 3 rotations), the total number of unknown displacement is about 5,700. The uniform pressure, 24.65 MPa, is applied at the outer surface of the cable core jacket. To be able to equally compare the REMCC and I-DEAS models, axial cable strain and initial wire indentation are neglected.

The radial displacements of the nodal points on the outer surface of the cable core jacket obtained from both models are plotted in Figure 4.21. This plot reveals that for both models, the displacement shapes of the radial deformation of the test cable are similar. Although, the numerical values of both solutions are close, the I-DEAS cross-section deformation plot (see Figure 4.22) reveals that the overlapping does occur between the elements adjacent to the inner surface of the core jacket. This indicates that the radial deformation predicted by the I-DEAS model overestimates the true deformation of the cable cross-section. Considering this, the two solutions are in very close agreement.
Figure 4.20. ROV umbilical cable modeled with I-DEAS thin shell elements.

Figure 4.21. Radial displacements of the ROV cable modeled with I-DEAS thin shell elements and REMCC elements.
Figure 4.22. I-DEAS Deformation Plot of Cable Cross-Section
CHAPTER 5
CONCLUSIONS

An extensive search of the existing literature of cable mechanics and finite element modeling clearly indicates that there is a suitable methodology to model umbilical cables is lacking. Umbilical cables are often complex constructions composed of dissimilar components having nonsymmetrical cross-sectional geometries, anisotropic material properties and nonsymmetrical loads. The primary goal of this research has been to develop a finite element model suitable for cable design. The model is able to analyze cables having nonsymmetrical properties and loadings so that the deformations, strains and stresses within each cable component can be determined.

Commercially available general purpose finite element software can perform the same job but finite element experts are needed to operate the program and much time is required to build the models. Also in this research effort, a number of elements were reviewed including sector elements [16,17,18,19,20], triangular, quadrilateral elements [21,22] and isoparametric elements [23]. These elements only yielded good results when a large number of elements were used as indicated by documented examples in these references. Current commercial finite element software and conventional finite element models are impractical for designing cables. For
example, the I-DEAS software used to model the ROV cable in Chapter 4 is a typical case. More than ten hours were required for data preparation and solution time in contrast to with only two hours needed to solve the same problem with the software described in Chapter 3.

After examining the body of the finite element literature, the only viable candidate was the axisymmetric solid ring element. This element can be used to model exactly the geometrical shape of circular components, which is a good approximation for cables having small lay angles (equal or less than 20 degrees). Using the general Fourier series shape function, this element can handle nonsymmetrical loads. The examples in Chapter 4 show that this element yields very close results to exact solutions.

In order to allow contact between adjacent ring elements which do not share the same axis of symmetry, the displacement constraints at contact points of these elements were introduced into modeling the REMCC macro-element. A REMCC element of a cable component is formed by imposing unit displacements, in turn, at each contact node and solving for the reaction forces. The resulting stiffness matrix for this component is then assembled into the stiffness matrix for the entire cable. The use of ring elements and contact points as the base of the REMCC macro-element greatly reduces the amount of data preparation. Only basic geometric and material data such as component radii, Young's moduli, Poisson's ratios,
etc. are required to be specified.

The excellent performance of the REMCC macro-element has been proven in the verification examples of Section 4.5. The radial deformation obtained from the REMCC macro-element model in comparison with experimental test data yields good agreement for the entire range of tension loadings. In comparison with the IDEAS-model, which requires about 5,700 degrees-of-freedom, the REMCC element model, with only 120 degrees-of-freedom, achieves higher accuracy. The REMCC macro-element can be implemented easily on personal computers and run with minimum requirements from a system such as 500 Kbytes of system memory and 1 Megabyte of disk space.

In conclusion, the REMCC macro-element is a promising candidate to model cables having nonsymmetrical geometric and material properties, and unsymmetrical loads. This model promises to offer cable designers a design tool unencumbered by the complexity of the conventional finite element approach. It can also be used to handle anisotropic material properties of cable components.
APPENDIX A

Solutions of Composite Orthotropic Cylinders Subjected to Internal and External Pressures, Axial and Thermal Strains

As mentioned in Section 2.3.4, with the conventional finite element approach, equivalent nodal forces for radial pressure, axial and thermal strains are based on the assumed displacement shape functions. A different approach is required to determine the equivalent nodal forces for REMCC macro-elements. Assume that the displacements at nodal points and the stiffness matrix of a particular REMCC are known. From equation (96) we can calculate exactly the magnitude of the forces, \( \{r\}_m \), applied at these nodal points to cause the same amount of displacements at these nodes. Thus, the forces, \( \{r\}_m \), obtained from this approach can be considered as equivalent nodal forces of pressures, axial and thermal strains.

The problem of a composite orthotropic plate subjected to pressures was addressed by Lekhnitskii [50] and its closed-form solution was given also in [50]. The problem is reformulated to account for axial and thermal strains. Following Lekhnitskii approach, the solution is obtained first for a composite orthotropic plate; i.e., a plane-stress problem. The solution for a composite orthotropic cylinder; i.e., a plane strain problem, can be found by modifying elastic constants in the solution of the plane-stress problem.
For an orthotropic annular plate having inner and outer radii \(a\) and \(b\), respectively, subjected to the internal and external pressures \(p\) and \(q\), the solutions of radial and hoop stresses are given in [49] as

\[
\sigma_r(r) = \frac{pc^{k+1}-q}{1-c^{2k}} \left( \frac{r}{B} \right)^{k-1} - \frac{pc^{k+1}}{1-c^{2k}} \left( \frac{r}{B} \right)^{k-1}
\]

\[
\sigma_\theta(r) = \frac{pc^{k+1}-q}{1-c^{2k}} k \left( \frac{r}{B} \right)^{k-1} + \frac{pc^{k+1}}{1-c^{2k}} k \left( \frac{r}{B} \right)^{k-1}
\]

\[\tau_{r\theta}(r) = 0\]

from which Lekhnitskiii arrived at

\[
u(r) = \frac{b}{E_y(1-c^{2k})} \left[ (pc^{k+1}-q)(k-\nu_{\theta r})(\frac{r}{B})^k + (p-qc^{k+1})c^{k+1}(k+\nu_{\theta r})(\frac{b}{R})^k \right]
\]

(A.2)

where \(u(r)\) is the radial displacement and the constants, \(c\) and \(k\), are defined as

\[
c = \frac{a}{b}, \quad k = \sqrt{\frac{E_y}{E_x}}
\]

(A.3)

Note that for \(k=1\), gives Lamé's solution, the case of an isotropic annular plate. For the case of \(a=0\), the solution of a solid plate can be obtained from these equations.

If thermal and axial strains are present, the solution above can be modified as described as follow. For a plane-stress problem, the strains produced by axial strain and temperature change [23] are
The total radial displacement, \( u_T(r) \) is found as

\[
\varepsilon_{r0} = \alpha_r \Delta T - \nu_{zr} \varepsilon_z \\
\varepsilon_{\theta 0} = \alpha_{\theta} \Delta T - \nu_{z\theta} \varepsilon_z
\]  \hspace{1cm} (A.4)

The total radial displacement, \( u_T(r) \) is found as

\[
u_T(r) = u(r) + r\varepsilon_{r0}
\]  \hspace{1cm} (A.5)

where \( u(r) \) is defined in equation (A.2) and the term \( r\varepsilon_{r0} \) in

equation (A.5) is the additional radial displacement produced by thermal and axial strains.

Figure A.1. A composite orthotropic plate subjected to pressures.

Next, consider a composite orthotropic plate subjected to pressures as shown in Figure A.1. The model consists of an arbitrary number of layers in the form of concentric rings of identical thickness. It is assumed that the material of each layer is orthotropic; the orthotropic poles of all layers are
located at the center and all layers are rigidly connected. The following notations apply: "n" is the number of layers; "a" and "b" are inner and outer radii of the composite plate; "p" and "q" are the internal and external pressures per unit area; "am-1", and "am" are the inner and outer radii of layer "m"; σr and ur(m) are the radial stress and displacement of layer "m", respectively. Also the constants, "c" and "k"", are defined as

\[ c_m = \frac{a_{m-1}}{a_m}, \quad k_m = \frac{E_0^{[m]}}{E_r^{[m]}} \]  \hspace{1cm} (A.6)

At the inner and outer surface we have the following boundary conditions:

\[ \sigma_r = -p \quad \text{ ( } r = a \text{ )} \]
\[ \sigma_r = -q \quad \text{ ( } r = b \text{ )} \]  \hspace{1cm} (A.7)

The radial displacements and stresses along the contact surfaces between each layer must be the same. This provides the following additional conditions:

\[ \sigma_{r(r-1)}^{(m-1)} = \sigma_{rr}^{(m)} \]  \hspace{1cm} (A.8.a,b)
\[ u_{(r-1)}^{(m-1)} = u^{(m)} \]

when \( r=a_{m-1} \). For a particular layer "m", equation (A.5) can be rewritten as

\[ u_r^{(m)} = u^{(m)} + r \varepsilon_{\text{r0}}^{(m)} \]  \hspace{1cm} (A.9)

At the outer surface of layer "m-1"; i.e., the inner surface
of layer "m", the displacement constraint in equation (A.8.b) must be satisfied. Thus, based on equation (A.9), we can write this constraint as

\[ U_{(r=a_m-1)}^{(m-1)} + \alpha_{m-1} \varepsilon_{r0}^{(m-1)} = U_{(r=a_m-1)}^{(m)} + \alpha_{m-1} \varepsilon_{r0}^{(m)} \quad (A.10) \]

Note that \( a_0 = a \) and \( a_n = b \), the inner and outer radii, respectively, of the composite plate.

Let's denote \( q_{m-1} \) and \( q_m \) the normal forces acting on the inner and outer surfaces of layer "m". Based on the radial stress condition in equation (A.8.a), equation (A.2) is used to evaluate \( u^{(m)} \) and \( u^{(m+1)} \) at \( r = a_m \) to obtain the following expressions:

\[ u^{(m)} = \frac{a_m}{E_m^{(m)}(1-C_m^{2m})} \left[ (q_{m-1} c_{m+1}^{k+1} - q_m) (k_m - \nu_{\theta r}^m) + (q_{m-1} - q_m c_{m}^{k-1}) c_m^{k+1} (k_m + \nu_{\theta r}^m) \right] + a_m \varepsilon_{r0}^m \]

\[ u^{(m+1)} = \frac{a_{m+1}}{E_{m+1}^{(m+1)}(1-C_{m+1}^{2m+1})} \left[ (q_m c_{m+1}^{k_{m+1}} - q_{m+1}) (k_{m+1} - \nu_{\theta r}^{m+1}) (c_{m+1}^{k_{m+1}} + \frac{1}{c_{m+1}^{k_{m+1}}}) + a_{m+1} \varepsilon_{r0}^{m+1} \right] \]

From equation (A.10) and these two expressions, the following equation is found

\[ a_m \beta_m q_m + a_m \alpha_{m-1} q_{m-1} = a_m (\varepsilon^{(m)} \quad (A.11) \]

where
\[ \alpha_m = \frac{2k_m}{E_0^{(m)}} \frac{C_m^k}{1 - C_m^{2k_m}} \]
\[ \beta_m = \frac{1}{E_0^{(m)}} (\mu^{(m)} - k_m \frac{1 + C_m^{2k_m}}{1 - C_m^{2k_m}}) \]
\[ \frac{1}{E_0^{(m+1)}} (\mu^{(m+1)} - k_{m+1} \frac{1 + C_{m+1}^{2k_{m+1}}}{1 - C_{m+1}^{2k_{m+1}}}) \]

For \( m = 1, 2, \ldots, n-1 \), equation (A.11) forms a system of \( (n-1) \) simultaneous linear equations in which \( q_0 \) thru \( q_{n-1} \) are unknowns. Note that \( q_0 = p \) and \( q_{n-1} = q \). Equation (A.11) is almost identical to Lekhnitskii equation (27.6) except the right-hand side of this equation is no longer equal to zero. The term on the right-hand side of equation (A.11) obviously results from the present of axial and thermal strains.

In matrix notation, the system of linear equations of equation (A.11) can be expressed as
\[ [TD] \{q\} = \{R\} \]  
(A.13)

where \([TD]\) is a tri-diagonal square matrix. A special algorithm, the Thomas method [52] is the most efficient algorithm for solving this type of the system of equations. Once equation (A.11) is solved for \( q_m \)'s, the radial, hoop stresses and the radial displacement can be computed by equations (A.1) and (A.2), respectively. Lekhnitskii gave the closed form expressions of these components as
Equations (A.14) and (A.15) are identical with Lekhnitskii's equations (27.4) and (27.5). Note that in equation (A.15) the plus sign in front of the second \( v_{tr} \) is corrected from typographical error in Reference [50]. These equations are still valid provided that the \( q_m \)'s be calculated from equation (A.11).

Equations (A.11), (A.14), and (A.15) are the solutions of the plane-stress problem of a composite orthotropic plate. To
make these equations valid for the plane-strain problem of a composite orthotropic multi-layered cylinder, the conversions take place among elastic constants by the formula [50],

$$\beta_{ij} = a_{ij} - \frac{a_{i3}a_{j3}}{a_{33}} \quad \text{for } i,j = 1,2,3 \quad (A.17)$$

Where $a_{ij}$ for $i,j=1,6$, are the thirty-six constants defined in the generalized Hooke's law. For polar orthotropic materials,

$$[a] = \begin{bmatrix}
\frac{1}{E_r} & -\frac{\nu_{r\theta}}{E_r} & -\frac{\nu_{r\phi}}{E_r} & 0 & 0 & 0 \\
-\frac{\nu_{r\theta}}{E_r} & \frac{1}{E_{\theta}} & -\frac{\nu_{\theta\phi}}{E_{\theta}} & 0 & 0 & 0 \\
-\frac{\nu_{r\phi}}{E_r} & -\frac{\nu_{\theta\phi}}{E_{\theta}} & \frac{1}{E_z} & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{G_{\theta\phi}} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{G_{z\phi}} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{G_{z\phi}}
\end{bmatrix}$$

To obtain the equivalent $E_r^{(eq)}$ of $E_r$, let's set $i=j=1$ in equation (A.17), this gives

$$\frac{1}{E_r^{(eq)}} = \frac{1}{E_r} - \frac{\nu_{r\theta}}{E_r} - \frac{\nu_{r\phi}}{E_r} - \frac{\nu_{\theta\phi}}{E_{\theta}} - \frac{\nu_{r\phi}}{E_z}$$

Recall from Betti's reciprocal theorem; viz., $\nu_{\phi}E_{\phi} = \nu_{\theta}E_{\theta}$, the equation above becomes
\[
\frac{1}{E_r^{(eq)}} = \frac{(1 - \nu_{rz}\nu_{rr})}{E_r}
\]

or simply

\[
E_r^{(eq)} = \frac{E_r}{(1 - \nu_{rz}\nu_{rr})} \tag{A.18}
\]

Similarly, we have

\[
E_\theta^{(eq)} = \frac{E_\theta}{(1 - \nu_{\theta r}\nu_{\theta \theta})} \tag{A.19}
\]

To find the equivalent \(\nu_{\theta r}^{(eq)}\) for \(\nu_{\theta r}\), set \(i=1\) and \(j=2\) in equation (A.17) to obtain

\[
\frac{\nu_{\theta r}^{(eq)}}{E_\theta^{(eq)}} = \frac{\nu_{\theta r}}{E_\theta} + \frac{\nu_{zr} \nu_{z\theta}}{E_z E_z} \frac{1}{E_z}
\]

Again, from Betti's theorem, \(\nu_{\theta z} E_z = \nu_{\theta \theta} E_\theta\), and with the expression of \(E_\theta^{(eq)}\) in equation (A.19), the above equation finally results in

\[
\nu_{\theta r}^{(eq)} = \frac{(\nu_{\theta r} + \nu_{zr} \nu_{\theta z})}{1 - \nu_{\theta z} \nu_{\theta \theta}} \tag{A.20}
\]

Using the equivalent values of these elastic constants in equations (A.11), (A.14), and (A.15) will yield the solutions of a composite orthotropic cylinder. Note that Poisson's ratio \(\nu_{zx}\) in equation (A.4) must not be converted because the
term, $- \nu \varepsilon_z$, is the compressive strain in the radial direction caused by the axial strain, $\varepsilon_z$, as proved in the derivation of the stress-strain relation in Section 2.1.3. The Thomas method [52] is employed to solve equation (A.11) with the implementation of the conversion formulas of $E_r$, $E_\theta$, and $\nu$. Once the column vector of radial pressures at the surfaces of layers, \{q\}, in equation (A.11) is known, it is substituted into equation (A.15) to compute the radial displacements produced by pressures, axial and thermal strains for multi-layered cable components. With the known radial displacement and the stiffness matrix of a component, the equivalent nodal forces of applied pressures, axial and thermal strains are computed as described above.
APPENDIX B

Stiffness Matrices of Axisymmetric Ring Elements

Three different axisymmetric ring elements used in modeling REMCC element were experimentally carried out in this research for the purposes of verifying solutions and optimizing computer time used to form stiffness matrices of REMCC elements. In general, the displacement shape functions of these ring elements are the same as expressed by equation (1) and (2),

\[
 u(r, \theta) = \sum_{k=0}^{N} \left[ u_{sk}(r) \cos k \theta + u_{ak}(r) \sin k \theta \right] 
\]

\[
 v(r, \theta) = \sum_{k=0}^{N} \left[ v_{sk}(r) \sin k \theta - v_{ak}(r) \cos k \theta \right] 
\]

The only difference between these elements is that the variations in the radial direction of the displacements, \( u \) and \( v \), are assumed to be linear, parabolic, and cubic. The stiffness matrices of REMCC elements obtained from these three elements are almost identical; however, the quadratic ring element yields the best performance in saving computer time among these three elements. The derivation of the stiffness matrices of ring element with quadratic variations in the radial direction have been presented in Chapter 2. Following the same procedure, the stiffness matrices of the linear and cubic ring elements were derived. In the following two
sections, the derivations of the stiffness matrices of these two elements are briefly presented and their explicit forms are given.

B.1. Linear Ring Element

The linear ring element consists of two circle nodal points 1 and 2 as shown in Figure B.1.

![Linear ring element with 2 circle nodal points](image)

Figure B.1. Linear ring element with 2 circle nodal points.

The displacement shape functions \( u_k(r) \), \( v_k(r) \), \( u_a(r) \) and \( v_a(r) \) in equations (1) and (2), are defined as follow:

\[
\begin{align*}
  u_k(r) &= b_{1k} + b_{2k} r \\
  v_k(r) &= b_{3k} + b_{4k} r \\
  u_a(r) &= b_{5k} + b_{6k} r \\
  v_a(r) &= b_{7k} + b_{8k} r
\end{align*}
\]  

(B.1)

Corresponding to the four constants used for each of the symmetric and antisymmetric terms, the four nodal displacements are assigned to each of the nodal displacement vectors \( \{q\}_k \) and \( \{q\}_a \) as follows
\[
\{q\}_{sk} = \begin{bmatrix} u_{1sk} & v_{1sk} & u_{2sk} & v_{2sk} \end{bmatrix}^T
\]
(B.2)

\[
\{q\}_{ak} = \begin{bmatrix} u_{1ak} & v_{1ak} & u_{2ak} & v_{2ak} \end{bmatrix}^T
\]

Using the principle of minimum potential energy, equations (59) and (60) are again obtained for the linear ring element.

\[
[k]_{sk}\{q\}_{sk} = \{f_o\}_{sk} + \{f\}_{sk}
\] (59)

\[
[k]_{ak}\{q\}_{ak} = \{f_o\}_{ak} + \{f\}_{ak}
\] (60)

where \([k]_{sk}\) and \([k]_{ak}\) are the stiffness matrices of the linear ring element for symmetric and antisymmetric terms, \(k\), defined by equations (61) and (62), respectively as

\[
[k]_{sk} = [h]^T \int_V ([CS]_{sk}[g]_k)^T[C] ([CS]_{sk}[g]_k) dV[h]
\] (61)

\[
[k]_{ak} = [h]^T \int_V ([CS]_{ak}[g]_k)^T[C] ([CS]_{ak}[g]_k) dV[h]
\] (62)

In equations (61) and (62), matrices \([CS]_{sk}\), \([CS]_{ak}\), and \([C]\) remain the same as defined in Chapter 2, only matrices \([h]\) and \([g]_k\) are changed as follow:

\[
[h] = \frac{1}{\Delta} \begin{bmatrix}
A_{11} & 0 & A_{12} & 0 \\
A_{21} & 0 & A_{22} & 0 \\
0 & A_{11} & 0 & A_{12} \\
0 & A_{21} & 0 & A_{22}
\end{bmatrix}
\] (B.3)

\[
\Delta = R_o - R_i; \quad A_{11} = R_o; \quad A_{12} = -R_i; \quad A_{21} = -1; \quad A_{22} = 1;
\]

and
Equations (61) and (62), together with the expressions of \([g]_k\) and \([h]\) given in equations (B.3) and (B.4), can be directly integrated over the volume of the ring element to yield,

\[
[k]_{sk} = [h]^T [\kappa]_{sk}[h]
\]

where the coefficients of matrix \([\kappa]_{sk}\) are

\[
\begin{align*}
\kappa_{11} &= (C_{22}C^2 + k^2C_{33}S^2) \ln(R) \\
\kappa_{12} &= ((C_{12} + C_{22})C^2 + k^2C_{33}S^2) R1 \\
\kappa_{13} &= k(C_{22}C^2 + C_{33}S^2) \ln(R) \\
\kappa_{14} &= kC_{22}C^2 R \\
\kappa_{22} &= ((C_{11} + 2C_{12} + C_{22})C^2 + k^2C_{33}S^2) R2 \\
\kappa_{23} &= k((C_{12} + C_{22})C^2 + 2C_{33}S^2) R1 \\
\kappa_{24} &= k(C_{12} + C_{22})C^2 R2 \\
\kappa_{33} &= (k^2C_{22}C^2 + C_{33}S^2) \ln(R) \\
\kappa_{34} &= kC_{22}C^2 R1 \\
\kappa_{44} &= k^2C_{22}C^2 R2
\end{align*}
\]

where \(\kappa_{ij} = \kappa_{ji}, \ i,j = 1,4\). The constants \(\ln(R)\), \(R1\), and \(R2\) are defined in equation (68). \(C^2\) and \(S^2\) are defined by equation
Similar to the quadratic ring element, the coefficients of the matrix $[\kappa]_k$ for the antisymmetric term, $k$, are obtained by interchanging $C^2$ with $S^2$ and vice-versa in equation (B.6).

In equations (59) and (60), the two column vectors $\{f_o\}_k$ and $\{f_o\}_k^\ast$ are equivalent nodal force vectors of initial strain. Similar to the case of the quadratic ring element, $\{f_o\}_k$ is equal to the zero column vector for every harmonic term, $k$, and $\{f_o\}_k^\ast$ is defined by equation (63) in terms of matrices $[h]$ and $[g]_k$ given in equations (B.3) and (B.4), respectively. This equation is integrated over the volume of the ring element to yield,

$$\{f_o\}_k = \frac{1}{\Delta} \begin{pmatrix} A_{11}T_1 + A_{21}T_2 \\ 0 \\ A_{12}T_1 + A_{22}T_2 \\ 0 \end{pmatrix} \quad \text{for } k = 0$$

and

$$\{f_o\}_k^\ast = \{0\} \quad \text{for } k \neq 0$$

where

$$T_1 = 2\pi(C_{12}\epsilon_{01} + C_{22}\epsilon_{02})R1$$

$$T_2 = 2\pi((C_{11} + C_{12})\epsilon_{01} + (C_{12} + C_{22})\epsilon_{02})R2$$

in which $\Delta$ and $A_{ij}$ for $i,j=1,2$ are defined in equation (B.4), and $R1$ and $R2$ are defined in equation (68). $C_{ij}$ are elements of the constitutive matrix $[C]$, and $\epsilon_{01}$ and $\epsilon_{02}$ are the first and second components of the initial strain column vector $\{\epsilon_0\}$ defined by equation (26).
The column vectors \( \{f\}_{sk} \) and \( \{f\}_{ak} \) of equations (59) and (60) are equivalent nodal force vectors for the symmetric and antisymmetric terms. The same expressions in equations (47) and (48) are used to obtain the equivalent nodal forces of concentrated loads, pressures.

**B.2. Cubic Ring Element**

The cubic ring element also consists of two-circle nodal points, 1 and 2, as the linear ring element. The displacement shape functions \( u(r,\theta) \) and \( v(r,\theta) \) are described by equations (1) and (2), respectively, where

\[
\begin{align*}
    u_{sk}(r) &= b_{1k} + b_{3k}r + b_{3k}r^2 + b_{4k}r^3 \\
    v_{sk}(r) &= b_{5k} + b_{7k}r + b_{7k}r^2 + b_{8k}r^3 \\
    u_{ak}(r) &= b_{9k} + b_{11k}r + b_{11k}r^2 + b_{12k}r^3 \\
    v_{ak}(r) &= b_{13k} + b_{15k}r + b_{15k}r^2 + b_{16k}r^3
\end{align*}
\]

(B.9)

Corresponding to the cubic variations in the radial direction of the displacement functions, eight unknown nodal displacements are assigned to each of the nodal displacement vectors \( \{q\}_{sk} \) and \( \{q\}_{ak} \) for each symmetric and antisymmetric term \( k \), respectively, as

\[
\begin{align*}
    \{q\}_{sk} &= [u_{1,sk} \ u_{1,sk} \ v_{1,sk} \ v_{1,sk} \ u_{2,sk} \ u_{2,sk} \ v_{2,sk} \ v_{2,sk}]^T \\
    \{q\}_{ak} &= [u_{1,ak} \ u_{1,ak} \ v_{1,ak} \ v_{1,ak} \ u_{2,ak} \ u_{2,ak} \ v_{2,ak} \ v_{2,ak}]^T
\end{align*}
\]

(B.10)

where the subscripts "1" and "2" denote nodes 1 and 2, respectively. The subscripts "1,\( \dot{r} \)" and "2,\( \dot{r} \)" denote derivatives with respect to "\( \dot{r} \)" at circle nodes 1 and 2. The
subscripts "sk" and "ak" stand for the symmetric and antisymmetric terms, k.

Again, using the principle of minimum potential energy, stiffness matrices $[k]_{sk}$ and $[k]_{ak}$ of the cubic ring element for symmetric and antisymmetric terms, k, are obtained and they have the same forms of equations (61) and (62) above. The matrices $[CS]_{sk}$, $[CS]_{ak}$, and $[C]$ again remain the same. The matrices $[h]$ and $[g]_k$ corresponding to the nodal displacement vector $\{q\}_{sk}$ and $\{q\}_{ak}$ are found as follows:

$$[h] = \frac{1}{\Delta} \begin{bmatrix}
A_{11} & A_{13} & 0 & 0 & A_{12} & A_{14} & 0 & 0 \\
A_{21} & A_{23} & 0 & 0 & A_{22} & A_{24} & 0 & 0 \\
A_{31} & A_{33} & 0 & 0 & A_{32} & A_{34} & 0 & 0 \\
A_{41} & A_{43} & 0 & 0 & A_{42} & A_{44} & 0 & 0 \\
0 & 0 & A_{11} & A_{13} & 0 & 0 & A_{12} & A_{14} \\
0 & 0 & A_{21} & A_{23} & 0 & 0 & A_{22} & A_{24} \\
0 & 0 & A_{31} & A_{33} & 0 & 0 & A_{32} & A_{34} \\
0 & 0 & A_{41} & A_{43} & 0 & 0 & A_{42} & A_{44}
\end{bmatrix}$$ (B.14)

$$[g]_k = \begin{bmatrix}
0 & 1 & 2r & 3r^2 & 0 & 0 & 0 & 0 \\
\frac{1}{r} & 1 & r & r^2 & \frac{k}{r} & kr & kr^2 & 0 \\
-\frac{k}{r} & -k & -kr & -kr^2 & -\frac{1}{r} & 0 & r & 2r
\end{bmatrix}$$ (B.15)

where

$$\Delta = 4R_iR_o^3 + 4R_oR_i^3 - 6R_i^2R_o^2 - R_o^4 - R_i^4$$

and
Equations (61) and (62), together with the expressions of 
\([h]\) and \([g]_k\) given in equations (B.14) and (B.15), can be
directly integrated over the volume of the ring element to yield,

\[
[k]_{sk} = [h]^T [\kappa]_{sk} [h]
\]  

(B.17)

where the coefficients of matrix \([\kappa]_{sk}\) are given below:
\[ \kappa_{11} = (C_{22}C^2 + k^2C_{33}S^2) \ln(R) \]
\[ \kappa_{12} = ((C_{12} + C_{22})C^2 + k^2C_{33}S^2)R_1 \]
\[ \kappa_{13} = ((2C_{12} + C_{22})C^2 + k^2C_{33}S^2)R_2 \]
\[ \kappa_{14} = ((3C_{12} + C_{22})C^2 + k^2C_{33}S^2)R_3 \]
\[ \kappa_{15} = k(C_{22}C^2 + C_{33}S^2) \ln(R) \]
\[ \kappa_{16} = kC_{22}C^2R_1 \]
\[ \kappa_{17} = k(C_{22}C^2 - C_{33}S^2)R_2 \]
\[ \kappa_{18} = k(C_{22}C^2 - 2C_{33}S^2)R_3 \]
\[ \kappa_{22} = ((C_{11} + 2C_{12} + C_{22})C^2 + k^2C_{33}S^2)R_2 \]
\[ \kappa_{23} = ((2C_{11} + 3C_{12} + C_{22})C^2 + k^2C_{33}S^2)R_3 \]
\[ \kappa_{24} = ((3C_{11} + 4C_{12} + C_{22})C^2 + k^2C_{33}S^2)R_4 \]
\[ \kappa_{25} = k((C_{12} + C_{22})C^2 + C_{33}S^2)R_1 \]
\[ \kappa_{26} = k(C_{12} + C_{22})C^2R_2 \]
\[ \kappa_{27} = k((C_{12} + C_{22})C^2 - C_{33}S^2)R_3 \]
\[ \kappa_{28} = k((C_{12} + C_{22})C^2 - 2C_{33}S^2)R_4 \]
\[ \kappa_{33} = ((4C_{11} + 4C_{12} + C_{22})C^2 + k^2C_{33}S^2)R_4 \]
\[ \kappa_{34} = ((6C_{11} + 5C_{12} + C_{22})C^2 + k^2C_{33}S^2)R_5 \]
\[ \kappa_{35} = k((2C_{12} + C_{22})C^2 + C_{33}S^2)R_2 \]
\[ \kappa_{36} = k(2C_{12} + C_{22})C^2R_3 \]
\[ \kappa_{37} = k((2C_{12} + C_{22})C^2 - C_{33}S^2)R_4 \]
\[ \kappa_{38} = k((2C_{12} + C_{22})C^2 - 2C_{33}S^2)R_5 \]
\[ \kappa_{44} = ((9C_{11} + 6C_{12} + C_{22})C^2 + k^2C_{33}S^2)R6 \]
\[ \kappa_{45} = k((3C_{12} + C_{22})C^2 + C_{33}S^2)R3 \]
\[ \kappa_{46} = k(3C_{12} + C_{22})C^2R4 \]
\[ \kappa_{47} = k((3C_{12} + C_{22})C^2 - C_{33}S^2)R5 \]
\[ \kappa_{48} = k((3C_{12} + C_{22})C^2 - 2C_{33}S^2)R6 \]
\[ \kappa_{55} = (k^2C_{22}C^2 + C_{33}S^2)\ln(R) \]
\[ \kappa_{56} = k^2C_{22}C^2R1 \]
\[ \kappa_{57} = (k^2C_{22}C^2 - C_{33}S^2)R2 \]
\[ \kappa_{58} = (k^2C_{22}C^2 - 2C_{33}S^2)R3 \]
\[ \kappa_{66} = k^2C_{22}C^2R2 \]
\[ \kappa_{67} = k^2C_{22}C^2R3 \]
\[ \kappa_{68} = k^2C_{22}C^2R4 \]
\[ \kappa_{77} = (k^2C_{22}C^2 + C_{33}S^2)R4 \]
\[ \kappa_{78} = (k^2C_{22}C^2 + 2C_{33}S^2)R5 \]
\[ \kappa_{88} = (k^2C_{22}C^2 + 4C_{33}S^2)R6 \]

(B.18)

where \( \kappa_{ij} = \kappa_{ji} \), \( i,j = 1,8 \). \( C^2 \) and \( S^2 \) are defined by equation (69).

The constants \( \ln(R) \), \( R1 \), \( R2 \), \( R3 \), and \( R4 \) are defined in equation (68) and the constants \( R5 \) and \( R6 \) are defined as,

\[ R5 = \frac{1}{5}(R_o^5 - R_i^5) \]
\[ R6 = \frac{1}{6}(R_o^6 - R_i^6) \]

(B.19)

Similar to the quadratic ring element, the coefficients of matrix \([\kappa]_{ik}\), for the antisymmetric term, \( k \), are obtained by
interchanging \( C^2 \) with \( S^2 \) and vice-versa in equation (B.18).

In equations (56) and (57), the two column vectors \( \{f_o\}_{ak} \) and \( \{f_o\}_{sk} \) are the equivalent nodal force vectors of the initial strain. Similar to the case of the quadratic ring element, \( \{f_o\}_{sk} \) is equal to the zero column vector for every harmonic term, \( k \), and \( \{f_o\}_{ak} \) is defined by equation (63) in terms of matrices \([h]\) and \([g]\)_k given by equations (B.14) and (B.15), respectively. This equation is integrated over the volume of the ring element to yield,

\[
\{f_o\}_{sk} = \frac{1}{\Delta} \begin{bmatrix}
A_{11}T_1 + A_{21}T_2 + A_{31}T_3 + A_{41}T_4 \\
A_{12}T_1 + A_{22}T_2 + A_{32}T_3 + A_{42}T_4 \\
0 \\
A_{13}T_1 + A_{23}T_2 + A_{33}T_3 + A_{43}T_4 \\
A_{14}T_1 + A_{24}T_2 + A_{34}T_3 + A_{44}T_4 \\
0
\end{bmatrix}
\]

for \( k = 0 \)

\[
\{0\} \quad \text{for } k \neq 0
\]

(B.19)

where

\[
T_i = 2\pi (C_{12}\epsilon_{01} + C_{22}\epsilon_{02})R_i
\]

\[
T_2 = 2\pi ((C_{11} + C_{12})\epsilon_{01} + (C_{12} + C_{22})\epsilon_{02})R_2
\]

\[
T_3 = 2\pi ((2C_{11} + C_{12})\epsilon_{01} + (2C_{12} + C_{22})\epsilon_{02})R_3
\]

\[
T_4 = 2\pi ((3C_{11} + C_{12})\epsilon_{01} + (3C_{12} + C_{22})\epsilon_{02})R_4
\]

in which \( \Delta \) and \( A_{ij} \) for \( i,j=1,4 \) are defined in equation (B.16). \( R_1, R_2, R_3, \) and \( R_4 \) are defined in equation (68). \( C_{ij} \) are
coefficients of the constitutive matrix $[C]$, and $\epsilon_{o1}$ and $\epsilon_{o2}$ are the first and second components of the initial strain column vector $\{\epsilon_0\}$ defined by equation (26).

The column vectors $\{f\}_{sk}$ and $\{f\}_{ak}$ of equations (59) and (57) are equivalent nodal force vectors for the symmetric and antisymmetric terms. The same expressions in equations (47) and (48) are used to obtain the equivalent nodal forces for concentrated loads, pressures.

B.3. Summary

Based on the stiffness formulations of the linear, quadratic, and cubic axisymmetric ring elements, it is found that the stiffness matrices of these elements always lead to the same expressions as in equations (61) and (62) as long as the shape function in equation (1) and (2) are used. Depending on the order of the element in the formulation, the matrices $[h]$ and $[g]_k$ which are formed by element geometry, the displacement and strain relations from elastic theory are changed. The other matrices such as the constitutive matrix, $[C]$, and the two diagonal matrices $[CS]_k$ and $[CS]_k$ remain unchanged. Therefore, these two equations form the foundation in obtaining stiffness matrices for any higher order axisymmetric two-dimensional ring elements.
APPENDIX C
Formulation Stiffness Matrix for Axisymmetric Quadratic Ring Element

This appendix is used to show the algebraic manipulations in equations (61) and (62) to yield the expressions for the elements of the matrix, \([k]_{sk}\), given in equation (67) and the elements of the matrix, \([k]_{sk}\), which is defined by equation (66.b). Equations (61) and (62) representing the stiffness matrices corresponding to the symmetric and antisymmetric terms of the quadratic ring element are repeated below for subsequent discussion.

\[
[k]_{sk} = [h]^T \int_V ([CS]_{sk}[g]_k)^T [C] ([CS]_{sk}[g]_k) \, dV[h] \quad (61)
\]
\[
[k]_{sk} = [h]^T \int_V ([CS]_{sk}[g]_k)^T [C] ([CS]_{sk}[g]_k) \, dV[h] \quad (62)
\]

Substituting the expression of the volume integral into equations (61) and (62) yields,

\[
[k]_{sk} = [h]^T \int_{\theta} \int_{r_0}^{r_0} ([CS]_{sk}[g]_k)^T [C] ([CS]_{sk}[g]_k) r \, dr \, d\theta[h] \quad (C.1)
\]
\[
[k]_{sk} = [h]^T \int_{\theta} \int_{r_0}^{r_0} ([CS]_{sk}[g]_k)^T [C] ([CS]_{sk}[g]_k) r \, dr \, d\theta[h] \quad (C.2)
\]

where the matrices \([h]\), \([CS]_{sk}\) and \([CS]_{sk}\), and \([g]_k\) are defined in equations (13), (18) and (19), respectively, as
\[ [h] = \frac{1}{\Delta} \begin{bmatrix} A_{11} & 0 & A_{12} & 0 & A_{13} & 0 \\ A_{21} & 0 & A_{22} & 0 & A_{23} & 0 \\ A_{31} & 0 & A_{32} & 0 & A_{33} & 0 \\ 0 & A_{11} & 0 & A_{12} & 0 & A_{13} \\ 0 & A_{21} & 0 & A_{22} & 0 & A_{23} \\ 0 & A_{31} & 0 & A_{32} & 0 & A_{33} \end{bmatrix} \]  

(13)

where \( \Delta \) and \( A_{ij} \) are given in equation (6).

\[
[CS]_{sk} = \begin{bmatrix} \cos k \theta & 0 & 0 \\ 0 & \cos k \theta & 0 \\ 0 & 0 & \sin k \theta \end{bmatrix}
\]

(18)

\[
[CS]_{sk} = \begin{bmatrix} \sin k \theta & 0 & 0 \\ 0 & \sin k \theta & 0 \\ 0 & 0 & -\cos k \theta \end{bmatrix}
\]

\[
[g]_k = \begin{bmatrix} 0 & 1 & 2r & 0 & 0 & 0 \\ \frac{1}{r} & 1 & \frac{k}{r} & k & kr \\ -k & -k & -kr & \frac{-1}{r} & 0 & r \end{bmatrix}
\]

(19)

and the constitutive matrix, \([C]\), is referred as

\[
[C] = \begin{bmatrix} C_{11} & C_{12} & 0 \\ C_{12} & C_{22} & 0 \\ 0 & 0 & C_{33} \end{bmatrix}
\]

where its elements, \( C_{ij} \), are defined by equation (31).

**C.1. Stiffness Matrices for Symmetric Terms**

Let the matrix \([G]_k\) be the product of the matrices \([CS]_{sk}\), \([g]_k\) in equation (C.1). Carrying out the matrix
multiplication yields,

\[
[G]_{sk} = [CS]_{sk} [G]_{k} = \begin{bmatrix} C_k & 2C_r & 0 & 0 & 0 \\
\frac{C_k}{F} & C & C_r & kC_k & kC_k \frac{C_r}{F} \\
-kS_k & -kS_k & -kS_k & -S_k & 0 & S_k \frac{C_r}{F} \end{bmatrix}
\] (C.3)

where \( C_k \) and \( S_k \) denote \( \cos k\theta \) and \( \sin k\theta \), respectively. Defining the matrix \([X]_{sk}\) as

\[
[X]_{sk} = [C] [G]_{sk} = \begin{bmatrix} C_{11} & C_{12} & 0 \\
C_{12} & C_{22} & 0 \\
0 & 0 & C_{33} \end{bmatrix} \begin{bmatrix} 0 & C_k & 2C_r & 0 & 0 & 0 \\
\frac{C_k}{F} & C & C_r & kC_k & kC_k \frac{C_r}{F} \\
-kS_k & -kS_k & -kS_k & -S_k & 0 & S_k \frac{C_r}{F} \end{bmatrix}
\]

\[
= \begin{bmatrix} C_{12}C_k \frac{1}{F} & (C_{11}+C_{12})C_k & (2C_{11}+C_{12})C_r & kC_{12}C_k \frac{1}{F} & kC_{12}C_k & kC_{12}C_k \frac{C_r}{F} \\
C_{22}C_k \frac{1}{F} & (C_{12}+C_{22})C_k & (2C_{12}+C_{22})C_r & kC_{22}C_k \frac{1}{F} & kC_{22}C_k & kC_{22}C_k \frac{C_r}{F} \\
-kC_{33}S_k \frac{1}{F} & -kC_{33}S_k & -kC_{33}S_k & -kC_{33}S_k \frac{1}{F} & 0 & C_{33}S_k \frac{C_r}{F} \end{bmatrix}
\] (C.4)

and the matrix \([X]_{sk}\) as

\[
[X]_{sk} = [G]_{sk}^T [X]_{sk} \] (C.5)

where the matrix, \([G]_{sk}^T\), is the transpose of the matrix, \([G]_{sk}\) defined in equation (C.3). Thus, equation (C.1) can be rewritten as
From equation (66.a),

\[
[k]_{sk} = [h]^{T} \int_{0}^{\frac{2\pi}{R_s}} \int_{\frac{R_s}{h}} \chi_{sk} \, dr \, d\theta \, [h] (C.6)
\]

developed in equation (65) are repeated below,

\[
\begin{align*}
\int_{0}^{2\pi} \sin(m\theta) \sin(n\theta) \, d\theta &= 0 \quad \text{for } m = n = 0 \\
&= \pi \quad \text{for } m = n \neq 0 \\
\int_{0}^{2\pi} \cos(m\theta) \cos(n\theta) \, d\theta &= 0 \quad \text{for } m \neq n \\
&= \frac{\pi}{2} \quad \text{for } m = n \neq 0 \\
&= 2\pi \quad \text{for } m = n = 0 \\
\int_{0}^{2\pi} \sin(m\theta) \cos(n\theta) \, d\theta &= 0 \quad \text{for } m = n
\end{align*}
\]

In order to obtain the matrix \([\kappa]_{sk}\), first the matrix multiplication in equation (C.5) is carried out. Note that by taking advantage of the last integral in equation (65), all the terms involving the product, \(C_{k}S_{k}\), are removed from all the elements of the matrix \([\chi]_{sk}\) in advance. The elements of the matrix \([\chi]_{sk}\) are given below,
\[ X_{11} = \left( C_{22} C_k^2 + k^2 C_{33} S_k^2 \right) \frac{1}{r} \]

\[ X_{12} = \left( C_{12} + C_{22} \right) C_k^2 + k^2 C_{33} S_k^2 \]

\[ X_{13} = \left( (2C_{12} + C_{22}) C_k^2 + k^2 C_{33} S_k^2 \right) r \]

\[ X_{14} = k \left( C_{22} C_k^2 + C_{33} S_k^2 \right) \frac{1}{r} \]

\[ X_{15} = k C_{22} C_k^2 \]

\[ X_{16} = k \left( C_{22} C_k^2 - C_{33} S_k^2 \right) r \]

\[ X_{22} = \left( (C_{11} + 2C_{12} + C_{22}) C_k^2 + k^2 C_{33} S_k^2 \right) r \]

\[ X_{23} = \left( (2C_{11} + 3C_{12} + C_{22}) C_k^2 + k^2 C_{33} S_k^2 \right) r^2 \]

\[ X_{24} = k \left( C_{12} + C_{22} \right) C_k^2 + C_{33} S_k^2 \]

\[ X_{25} = k \left( C_{12} + C_{22} \right) C_k^2 r \]

\[ X_{26} = k \left( (C_{12} + C_{22}) C_k^2 - C_{33} S_k^2 \right) r^2 \]

\[ X_{33} = \left( (4C_{11} + 4C_{12} + C_{22}) C_k^2 + k^2 C_{33} S_k^2 \right) r^3 \]

\[ X_{34} = k \left( (2C_{12} + C_{22}) C_k^2 + C_{33} S_k^2 \right) r \]

\[ X_{35} = k \left( (2C_{12} + C_{22}) C_k^2 r^2 \right) \]

\[ X_{36} = k \left( (2C_{12} + C_{22}) C_k^2 - C_{33} S_k^2 \right) r^3 \]

\[ X_{44} = \left( k^2 C_{22} C_k^2 + C_{33} S_k^2 \right) \frac{1}{r} \]

\[ X_{45} = k^2 C_{22} C_k^2 \]

\[ X_{46} = \left( k^2 C_{22} C_k^2 - C_{33} S_k^2 \right) r \]

\[ X_{55} = k^2 C_{22} C_k^2 r \]

\[ X_{56} = k^2 C_{22} C_k^2 r^2 \]

\[ X_{66} = \left( k^2 C_{22} C_k^2 + C_{33} S_k^2 \right) r^3 \]

where \( X_{ij} = X_{ji} \) for \( i, j = 1, 6 \).
Finally, with all the elements of the matrix \([X]_{\text{sk}}\) given above, the integrations with respect to \(\theta\) and \(r\) in equation (C.7) are performed to yield the matrix, \([\kappa]_{\text{sk}}\) whose elements are defined in equation (67). The stiffness matrices for symmetric terms are then obtained by carrying out the matrix multiplications of the product \([h]^{T}[\kappa][h]\) in equation (66.a).

C.2. Stiffness Matrices for Antisymmetric Terms

Similar to the procedure above, the stiffness matrices for antisymmetric terms can be obtained. Let \([G]_{\text{sk}}\) be the product of the matrices \([CS]_{\text{sk}}, [g]_{k}\) in equation (C.2), and performing the matrix multiplication yields,

\[
[G]_{ak} = [CS]_{ak} [g]_k = \begin{bmatrix}
0 & S_k & 2S_k r & 0 & 0 & 0 \\
S_k \frac{r}{kC_k} & S & S_k r & \frac{kS_k}{k} & kS_k r & \frac{kS_k}{k} \\
S_k \frac{r}{kC_k} & kC_k & kC_k r & \frac{C_k}{k} & 0 & -C_k r
\end{bmatrix}
\] (C.9)

where \(C_k\) and \(S_k\) denote \(\cos k\theta\) and \(\sin k\theta\), respectively. Defining the matrix \([X]_{ak}\) as

\[
[X]_{ak} = [C] [G]_{ak} = \begin{bmatrix}
C_{11} & C_{12} & 0 \\
0 & C_{22} & 0 \\
0 & 0 & C_{33}
\end{bmatrix} \begin{bmatrix}
0 & S_k & 2S_k r & 0 & 0 & 0 \\
S_k \frac{r}{kC_k} & S & S_k r & \frac{kS_k}{k} & kS_k r & \frac{kS_k}{k} \\
S_k \frac{r}{kC_k} & kC_k & kC_k r & \frac{C_k}{k} & 0 & -C_k r
\end{bmatrix}
\]

Matrix multiplication yields,
and the matrix \([X]_{ak}\) as

\[
[X]_{ak} = [G]_{ak}^T [X]_{ak} r
\]  
(C.11)

where the matrix, \([G]_{ak}^T\), is the transpose of the matrix, \([G]_{ak}\) defined in equation (C.9). Thus, equation (C.2) can be rewritten as

\[
[k]_{ak} = [h]^T \int_{\Omega} \int_{R_i} [X]_{ak} dr d\theta [h]
\]  
(C.12)

From equation (66.b),

\[
[k]_{ak} = [h]^T [\kappa]_{ak} [h]
\]  
(66.b)

the matrix \([\kappa]_{ak}\) can be expressed as

\[
[\kappa]_{ak} = \int_{\Omega} \int_{R_i} [X]_{ak} dr d\theta
\]  
(C.13)

Similar to the section above to obtain the matrix \([\kappa]_{ak}\), the matrix multiplication in equation (C.11) is carried out first. The elements of the matrix, \([X]_{ak}\), are given below,
\[ \chi_{11} = (c_{22}S_k^2 + k^2c_{33}C_k^2) \frac{1}{r} \]
\[ \chi_{12} = (c_{12} + c_{22})S_k^2 + k^2c_{33}C_k^2 \]
\[ \chi_{13} = ((2c_{12} + c_{22})S_k^2 + k^2c_{33}C_k^2)r \]
\[ \chi_{14} = k(c_{22}S_k^2 + c_{33}C_k^2) \frac{1}{r} \]
\[ \chi_{15} = k(c_{22}S_k^2) \]
\[ \chi_{16} = k(c_{22}S_k^2 - c_{33}C_k^2)r \]
\[ \chi_{22} = ((c_{11} + 2c_{12} + c_{22})S_k^2 + k^2c_{33}C_k^2)r \]
\[ \chi_{23} = ((2c_{11} + 3c_{12} + c_{22})S_k^2 + k^2c_{33}C_k^2)r^2 \]
\[ \chi_{24} = k(c_{12} + c_{22})S_k^2 + c_{33}C_k^2 \]
\[ \chi_{25} = k(c_{12} + c_{22})S_k^2r \]
\[ \chi_{26} = k((c_{12} + c_{22})S_k^2 - c_{33}C_k^2)r^2 \]  
\[ \chi_{33} = ((4c_{11} + 4c_{12} + c_{22})S_k^2 + k^2c_{33}C_k^2)r^3 \]
\[ \chi_{34} = k((2c_{12} + c_{22})S_k^2 + c_{33}C_k^2)r \]
\[ \chi_{35} = k(2c_{12} + c_{22})S_k^2r^2 \]
\[ \chi_{36} = k((2c_{12} + c_{22})S_k^2 - c_{33}C_k^2)r^3 \]
\[ \chi_{44} = (k^2c_{22}S_k^2 + c_{33}C_k^2) \frac{1}{r} \]
\[ \chi_{45} = k^2c_{22}S_k^2 \]
\[ \chi_{46} = (k^2c_{22}S_k^2 - c_{33}C_k^2)r \]
\[ \chi_{55} = k^2c_{22}S_k^2r \]
\[ \chi_{56} = k^2c_{22}S_k^2r^2 \]
\[ \chi_{66} = (k^2c_{22}S_k^2 + c_{33}C_k^2)r^3 \]

where \( \chi_{ij} = \chi_{ji} \) for \( i, j = 1, 6 \).
Equation (C.14) reveals that the matrix $[\chi]_{sk}$ elements can be obtained also by interchanging $S^2$ and $C^2$ and vice-versa in the expressions of the matrix $[\chi]_{sk}$ elements given by equation (C.8). The integrations with respect to $\theta$ and $r$ in equation (C.13) are performed to yield the matrix, $[\kappa]_{sk}$. With the trigonometric identities given by equation (65), it can be seen that the matrices $[\kappa]_{sk}$ and $[\kappa]_{sk}$ are identical for all harmonic terms, $k$, except for the term $k=0$. This results from the use of a minus sign in the displacement shape function (equation (2)). Consequently, the stiffness matrices for both the symmetric and antisymmetric terms are identical except for the term $k=0$ as mentioned in Section 2.1.8.
REFERENCES


46. Dario, N.P. and Bradley, "A Comparison of First and Second Order Axially Symmetric Finite Elements,"


