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Solar sailcraft motion in sun-earth-moon space with application
to lunar transfer from geosynchronous orbit

Salvail, James Ronald, Ph.D.

University of Hawaiʻi, 1991
SOLAR SAILCRAFT MOTION IN SUN-EARTH-MOON SPACE WITH
APPLICATION TO LUNAR TRANSFER FROM GEOSYNCHRONOUS ORBIT

A DISSERTATION SUBMITTED TO THE GRADUATE DIVISION OF THE
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By

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ABSTRACT

A three dimensional model of the dynamics of a solar sailcraft in the earth-sun-moon system is developed. The model includes the following features: (1) a development of the physics of radiation pressure; (2) the derivation of a unit vector describing the direction of the resultant radiation pressure force for two types of sails; (3) the derivation of equations of motion for the sailcraft and the moon in spherical coordinates; (4) the derivation of generalized equations for the initial conditions of the sailcraft in terms of orbit parameters; (5) the development of attitude control equations for the sail; (6) an analytical criterion to account for the periodic eclipsing of the sun by the earth, and (7) an analysis of reflection of solar radiation by planetary bodies.

A computer program based on the above model and including a search routine is developed and described. The program is used, together with a search strategy for searching through a four dimensional parametric space of initial orbit parameters, to investigate the problems of transfer to the moon (impact or close flybys) from a geosynchronous orbit. Results are obtained for three types of lunar encounters and three values of the sailcraft's area to mass ratio. Results are presented for the orbits of the
sailcraft and the moon for the lunar encounter and the post encounter sailcraft trajectories when applicable. It is found that a lunar impact or close flyby can take place in 63 to 67 days for an area to mass ratio of 100 and that the type of lunar encounter can result in significantly different post encounter sailcraft trajectories.
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<tr>
<td>A</td>
<td>area of sail</td>
</tr>
<tr>
<td>A_i</td>
<td>apparent area of intersection of earth and sun disks</td>
</tr>
<tr>
<td>A_s</td>
<td>reflectivity of planet</td>
</tr>
<tr>
<td>A_t</td>
<td>apparent area of solar disk</td>
</tr>
<tr>
<td>a</td>
<td>semimajor axis of elliptical orbit</td>
</tr>
<tr>
<td>c</td>
<td>speed of light</td>
</tr>
<tr>
<td>d</td>
<td>apparent distance between centers of earth and sun</td>
</tr>
<tr>
<td>d_x</td>
<td>distance of sailcraft from earth-sun line</td>
</tr>
<tr>
<td>E</td>
<td>energy</td>
</tr>
<tr>
<td>E_n</td>
<td>energy of photon packet</td>
</tr>
<tr>
<td>f</td>
<td>photon frequency</td>
</tr>
<tr>
<td>f_a</td>
<td>ratio of solar area not obscured by earth</td>
</tr>
<tr>
<td>F_r</td>
<td>radiation pressure force</td>
</tr>
<tr>
<td>F_rx</td>
<td>x-component of radiation pressure force</td>
</tr>
<tr>
<td>F_ry</td>
<td>y-component of radiation pressure force</td>
</tr>
<tr>
<td>F_rz</td>
<td>z-component of radiation pressure force</td>
</tr>
<tr>
<td>G</td>
<td>gravitational constant</td>
</tr>
<tr>
<td>h</td>
<td>dimensionless parameter for projection of sun-sailcraft distance into the ecliptic plane</td>
</tr>
<tr>
<td>h_e</td>
<td>distance from apex of diverging cone to earth</td>
</tr>
<tr>
<td>h_p</td>
<td>Planck's constant</td>
</tr>
<tr>
<td>h_s</td>
<td>length of shadow cone</td>
</tr>
<tr>
<td>i</td>
<td>unit vector in x direction centered at earth</td>
</tr>
<tr>
<td>i_o</td>
<td>unit vector in the initial orbit frame</td>
</tr>
<tr>
<td>i_rs</td>
<td>unit vector from the sun in the direction of the sailcraft</td>
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LIST OF SYMBOLS

\( \mathbf{i}_{ss} \) unit vector from the sun in the ecliptic plane normal to the sun-earth line

\( \mathbf{i}_{0s} \) unit vector from the sun normal to the ecliptic plane

\( \mathbf{I} \) unit vector in the inertial X direction

\( \mathbf{j} \) unit vector in y direction centered at earth

\( \mathbf{j}_o \) unit vector in the initial orbit frame

\( \mathbf{J} \) unit vector in the inertial Y direction

\( \mathbf{k} \) unit vector in z direction centered at earth

\( \mathbf{k}_o \) unit vector in the initial orbit frame

\( \mathbf{K} \) unit vector in the inertial Z direction

\( k \) dimensionless parameter for sun to sailcraft distance

\( m \) mass of the sailcraft

\( M \) mass of generalized planetary body

\( M_e \) mass of earth

\( M_m \) mass of moon

\( M_s \) mass of sun

\( n \) number of photons

\( \mathbf{n} \) unit vector normal to the sail

\( n_x \) x-component of \( \mathbf{n} \)

\( n_y \) y-component of \( \mathbf{n} \)

\( n_z \) z-component of \( \mathbf{n} \)

\( P \) total power radiated by the sun

\( P_e \) period of earth's rotation

\( \Delta P_f \) instantaneous power of photon packet of frequency \( f \)

\( P_n \) momentum of a photon packet
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Definition</th>
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<tr>
<td>P&lt;sub&gt;r&lt;/sub&gt;</td>
<td>dimensionless parameter for radiation pressure</td>
</tr>
<tr>
<td>q&lt;sub&gt;r&lt;/sub&gt;</td>
<td>emitted radiation flux</td>
</tr>
<tr>
<td>Q</td>
<td>optical coefficient</td>
</tr>
<tr>
<td>r</td>
<td>distance from earth's center to sailcraft</td>
</tr>
<tr>
<td>r&lt;sub&gt;c&lt;/sub&gt;</td>
<td>radius of earth's shadow cone</td>
</tr>
<tr>
<td>r&lt;sub&gt;d&lt;/sub&gt;</td>
<td>distance from earth sun line to diverging cone</td>
</tr>
<tr>
<td>r&lt;sub&gt;em&lt;/sub&gt;</td>
<td>distance between earth-moon centers</td>
</tr>
<tr>
<td>r&lt;sub&gt;i&lt;/sub&gt;</td>
<td>radial distance of sailcraft's initial orbit</td>
</tr>
<tr>
<td>r&lt;sub&gt;m&lt;/sub&gt;</td>
<td>distance from moon's center to the sailcraft</td>
</tr>
<tr>
<td>r&lt;sub&gt;o&lt;/sub&gt;</td>
<td>distance between sun-earth centers (definition of astronomical unit)</td>
</tr>
<tr>
<td>r&lt;sub&gt;p&lt;/sub&gt;</td>
<td>radius of planet</td>
</tr>
<tr>
<td>r&lt;sub&gt;s&lt;/sub&gt;</td>
<td>distance from sun's center to sailcraft</td>
</tr>
<tr>
<td>r&lt;sub&gt;sm&lt;/sub&gt;</td>
<td>distance between sun-moon centers</td>
</tr>
<tr>
<td>r&lt;sub&gt;xy&lt;/sub&gt;</td>
<td>projection of sun-sailcraft distance into the ecliptic plane</td>
</tr>
<tr>
<td>r&lt;sub&gt;1&lt;/sub&gt;</td>
<td>apparent radius of sun at earth's distance</td>
</tr>
<tr>
<td>r&lt;sub&gt;2&lt;/sub&gt;</td>
<td>apparent radius of earth</td>
</tr>
<tr>
<td>R&lt;sub&gt;e&lt;/sub&gt;</td>
<td>radius of earth</td>
</tr>
<tr>
<td>R&lt;sub&gt;m&lt;/sub&gt;</td>
<td>radius of moon</td>
</tr>
<tr>
<td>R&lt;sub&gt;s&lt;/sub&gt;</td>
<td>radius of sun</td>
</tr>
<tr>
<td>S</td>
<td>solar flux at 1 astronomical unit</td>
</tr>
<tr>
<td>S&lt;sub&gt;r&lt;/sub&gt;</td>
<td>total reflected radiation from planet</td>
</tr>
<tr>
<td>t</td>
<td>time</td>
</tr>
<tr>
<td>T</td>
<td>temperature</td>
</tr>
<tr>
<td>v&lt;sub&gt;sm&lt;/sub&gt;</td>
<td>vector from moon's center to sailcraft</td>
</tr>
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</table>
LIST OF SYMBOLS

\( \mathbf{v}_{rx} \) x-component of sailcraft velocity relative to the moon
\( \mathbf{v}_{ry} \) y-component of sailcraft velocity relative to the moon
\( \mathbf{v}_{rz} \) z-component of sailcraft velocity relative to the moon
\( \mathbf{v}_{r} \) dimensionless radial component of velocity of sailcraft
\( \mathbf{v}_{p} \) dimensionless azimuth component of velocity of sailcraft
\( \mathbf{v}_{\theta} \) dimensionless zenith component of velocity of sailcraft
\( \mathbf{x} \) x-component of geocentric distance
\( \mathbf{y} \) y-component of geocentric distance
\( \mathbf{y}_{p} \) value of \( \mathbf{y} \) coordinate in the sailcrafts orbital plane
\( \mathbf{z} \) z-component of geocentric distance

**Greek Letters**

\( \alpha \) angle between the sun-sail line and its projection in the ecliptic plane
\( \alpha_{p} \) angle between planet-sailcraft line and its projection on the ecliptic plane.
\( \beta \) angle between the projection of the sun-sail line in the ecliptic plane and the projection of the normal to the sail in the ecliptic plane
\( \beta_{p} \) the projection into the ecliptic plane of the angle between the planet-sailcraft line and the normal to the sail
\( \gamma \) angle between the sun-sail line and the normal to the sail
\( \gamma_{p} \) angle between the planet-sailcraft line and the normal to the sail
\( \delta \) longitude of the ascending node of the initial sailcraft orbit on the ecliptic plane
\( i \) angle between the initial orbit and the ecliptic plane
LIST OF SYMBOLS

\( \lambda \) angle between the inertial reference direction and the sun in the ecliptic plane

\( \eta \) angle defined as the difference between \( \delta \) and \( \lambda \)

\( \rho \) dimensionless distance from earth's center to sailcraft

\( \rho_{em} \) dimensionless distance between earth-moon centers

\( \rho_m \) dimensionless distance from moon's center to sailcraft

\( \phi \) azimuth angle of sailcraft from positive \( x \) axis

\( \phi_{em} \) azimuth angle of moon from positive \( x \) axis

\( \phi_p \) longitude on planet relative to subsolar point

\( \theta_p \) latitude on planet referred to the ecliptic plane

\( \theta \) latitude angle of sailcraft from ecliptic plane

\( \theta_{em} \) latitude of moon from ecliptic plane

\( \tau \) dimensionless time

\( \Omega \) nominal angular frequency of the earth in its orbit

\( \omega \) angular velocity vector

\( \omega_o \) angular frequency of the sailcraft in its initial orbit (same as the angular frequency of earth's rotation)

\( \sigma \) Stephan-Boltzmann constant

\( \varepsilon \) emissivity of sail

\( \psi \) angle between the sun-earth line and the sun-sail line

\( \psi_p \) angle between the projection of the sun-sail line in the ecliptic plane and the earth-sun line

\( \psi_\theta \) angle between the sun-sail line and its projection in the ecliptic plane

\( \mu_e \) dimensionless mass of earth

\( \mu_m \) dimensionless mass of moon

\( \mu_s \) dimensionless mass of sun
A. Solar Sailing Concept and Research History

The idea of using radiation pressure from the sun to propel spacecraft has been known since the early days of space travel. This virtually unlimited source of energy can be harnessed by attaching large sheets of a low density material with a high area to mass ratio and a high reflectivity to a spacecraft and "sailing" through space. The trajectory of the spacecraft can be controlled by adjusting the orientation of the sail with respect to the solar direction. Solar sailcraft can play an important role in the future of space travel and exploration (Friedman, 1988). Spacecraft powered by solar sails are capable of missions that would be impossible for spacecraft powered by conventional means. Such missions include circular orbits synchronous with the solar rotation or levitation over the solar poles. A principal advantage of using solar sails is that large quantities of fuel for propulsion are not carried by the spacecraft, allowing for smaller, lighter craft. A small internal power source is needed to control the orientation of the sail and to perform mission functions but even this power could, in principle, come from the sun through the use of solar cells.

Much work has already been done in the area of radiation pressure and solar sailcraft. Garwin (1958) showed that commercially available metallized plastic film can be
used as a solar radiation pressure sail for propulsion of space vehicles within the solar system. Tsu (1959) examined the characteristics of a solar sail, developed and solved equations of motion of a solar sail in heliocentric orbit, optimized the sail tilt angle and computed travel times for trips to Mars and Venus. London (1960) obtained some exact solutions of the equations of motion of a solar sail in heliocentric orbit for special cases of constant sail setting. Sands (1961) developed a relation between the sail tilt angle to the sun and the angular position about a planetary body that facilitates the escape of a solar sail from a planet's gravitational field. It was found that this sail control function would allow a solar sailcraft to escape from Earth's gravitational field in a period of time of the order of several months. Fimple (1962) analyzed the planetary escape trajectory of a solar sail from an original circular satellite orbit oriented perpendicularly to the planet-sun line subject to a steering program that provides maximum time rate of energy increase. Through the use of several simplifying assumptions, a generally applicable simplified solution was obtained that is useful as a means of rapidly estimating solar sail escape performance in preliminary design studies. Buckingham et al. (1965) considered the use of solar radiation pressure as a means of orbit position control for spherical satellites with large area to mass ratios. They showed that solar radiation
pressure could be used to maintain the angular spacing of a system of such satellites placed in the same orbit. Georgevic (1973) derived general expressions for the solar radiation force and torques for reflecting surfaces of any given reflecting characteristics and applied them to computing the solar radiation force and torques on parabolic and cylindrical reflectors. Zee (1973) derived equations of motion for a satellite in a near-equatorial synchronous earth orbit including the effects of the oblate earth, the sun, the moon and radiation pressure. He obtained analytical solutions in terms of orbital elements describing the perturbative effects on the orbit. Ahmad and Stuiver (1974) determined the effects of solar radiation on the orbits of flat plates. Jacobson and Thornton (1978) devised a strategy for solar sail navigation and applied it to a Halley's comet rendezvous mission. Van der Ha and Modi (1977, 1978 and 1979) investigated solar radiation induced perturbations on arbitrarily shaped satellites and their effects on the orbital elements. They derived equations for the variable orbital elements and applied them to determine orbital behavior for special cases. They also modelled and evaluated three-dimensional heliocentric solar sail trajectories with a fixed sail setting. Enomoto (1979) computed earth escape trajectories in the ecliptic plane for a solar sailcraft departing from a geosynchronous orbit using a two dimensional model neglecting the effect of the moon.
Jayaraman (1980) analyzed the problem of optimally controlling the sail steering angle of a solar sail spacecraft to execute a minimum time coplanar orbit transfer from the mean orbital distance of earth to that of Mars. Lukyanov (1981) developed and analyzed designs for space sail "liners" with solar sails from dozens to hundreds of km$^2$ for the purpose of transporting hundreds of tons to supply future space settlements. Tang (1981) built on the work of Georgevic by deriving an expression for the generalized solar radiation pressure force on a circular cylinder in arbitrary orientation. Lunscher and Modi (1984) developed a strategy and design for three-axis librational control of satellites using solar radiation pressure that can produce any desired satellite orientation. Yu (1986) developed a two-dimensional model of a solar sailcraft departing from a geosynchronous orbit in the ecliptic plane neglecting the effect of the moon, and he used it to compute trajectories for a flyby flight of the sun-earth transterrestrial libration point. Kakaria (1988) built on the work of Yu by searching for and computing solar sailcraft trajectories that arrive at the same libration point with zero relative velocity. Forward (1990) analyzed the radiation pressure force magnitudes and resultant directions for solar sails with different optical properties and a combination of these types known as "grey" solar sails.
B. Problem to be Investigated

The problem to be investigated concerns the transfer of a solar sailcraft from a synchronous earth orbit to an impact or close flyby of the moon and possibly a lunar orbit. This problem has a current practical application, as there is a project underway to have a race to the moon and possibly Mars with solar radiation pressure powered sailcraft on the 500th anniversary of Columbus' discovery of America in 1992 (the so-called Columbus 500 Space Sail Cup Race) (Fjermeland, 1991). My investigation will begin with a development of the physics of radiation pressure, leading to equations for the radiation pressure force for two types of sails having different surface optical properties. Following this, coordinate systems involved in the analysis will be defined, and equations for the direction of the resultant radiation force will be derived in terms of an earth-centered coordinate system for both types of sails. The analysis will then proceed with a development of the equations of motion for the sailcraft and the moon in dimensionless earth-centered spherical coordinates. These equations of motion will include the gravitational forces of the earth, the sun and the moon, the solar radiation pressure force and the forces associated with the relative motion of a translating and rotating reference frame. The following second-order effects will be neglected: (1) the small eccentricity of the earth's orbit, (2) perturbations
due to the oblateness of the planetary bodies, (3) reflection of solar radiation from the planetary surfaces, (4) magnetic fields and solar wind. The solar wind consists primarily of hydrogen nuclei emitted from the sun. The pressure due to the solar wind is about two orders of magnitude less than the radiation pressure of the sun at the earth's distance. The sail is assumed to be a plane surface that does not deform under load. Generalized equations for the initial conditions will be derived in terms of initial orbit parameters. These will include the inclination of the orbit and the longitude of the ascending node, both with respect to the ecliptic plane, and the elapsed time in the initial orbit from the anti-solar direction to the position of sail deployment. An analytical criterion will then be developed to account for the eclipsing of the sun by the earth since this would disrupt the radiation incident on the sail. The model development will conclude with equations expressing the sail orientation angles in terms of the angular position coordinates for the purpose of controlling the attitude of the sail. A search algorithm and the structure of the computer program will be described as well as a strategy for minimizing the exploratory computer runs searching through a four-dimensional parametric space. This includes the three initial orbit parameters described above and the date of the month or position of the moon in its orbit. Trajectories of the sailcraft to the vicinity of the
moon and beyond the lunar encounter will be computed for
three values of the sailcraft's area-to-mass ratio. The
effect of the lunar encounter on the post-encounter
trajectory of the sailcraft will be examined as well as the
possibility of stable orbits around the moon. The control
of the attitude of the sailcraft and the sail will also be
discussed.
A. Radiation Pressure

Radiation can be thought of as consisting of discrete packets of massless particles called photons that have an energy \( E \) that is proportional to the photon frequency \( f \),

\[ E = h_p f, \]

where \( h_p \) is Planck's constant, equal to \( 6.63 \times 10^{-34} \text{ J-s} \).

A photon has a momentum defined as

\[ P = \frac{E}{c} = \frac{h_p f}{c}, \]

where \( c \) is the speed of light. For \( n \) photons having frequency \( f \), the energy and momentum are

\[ E_n = nh_p f \quad \text{and} \quad P_n = \frac{nh_p f}{c}. \]

The instantaneous power measured in a plane normal to a beam of light due to the passage of \( n \) photons having frequency \( f \) is

\[ \Delta P_f = \frac{dE_p}{dt} = h_p f \frac{dn}{dt}. \]

The momentum impulses applied to the sail by the stopping of the photons of frequency \( f \) by the sail produce a radiation pressure force that is equal to the rate of change of momentum given by

\[ \Delta F_f = \frac{dp_n}{dt} = \frac{h_p f}{c} \frac{dn}{dt}. \]
which can also be written as
\[ \Delta F_t = \frac{\Delta P_t}{C}. \]

The total radiation force due to all the photons at the different frequencies is obtained by summing the incremental radiation forces over all frequencies:
\[ F_t = \frac{1}{C} \sum_{t=0}^{\infty} \Delta F_t = \frac{P}{C}, \]

where \( P \) is the radiated power due to the entire solar spectrum. This can also be written in terms of a solar flux, \( S \), normalized to one astronomical unit (AU). Since the solar power exerted on the sail is equal to the solar flux at distance \( r_s \) times the sail area, \( A \), the radiation force per unit mass, \( m \), of the sailcraft is
\[ F_r = \frac{SA}{Cm} \left( \frac{r_o}{r_s} \right)^2, \]

which is the familiar inverse square law. \( r_o \) is the length of an AU, and \( r_s \) is the distance to the center of the sun.

The above analysis includes the assumption that all the radiation is traveling in the same direction. If radiation strikes the sail from different directions, as would occur from reflection from a nearby planet, or leaves the sail in different directions, as in diffuse reflection or emission from a rough surface, an exact analysis would have to sum the contribution from each direction vectorially.
The above expression for the radiation force applies only to incoming radiation striking the sail perpendicularly. If incoming radiation strikes a sail at an angle, \(\gamma\), to the normal to the sail, the above expression must be multiplied by the factor \(\cos \gamma\) to account for the reduction in the sail's projected area. If, in addition, the sail has a specularly reflecting surface, photons are reflected at the same angle as the incoming radiation but on the opposite side of the surface normal. In general, the direction of the radiation force is in the same direction as the light velocity when photons enter the sail material and in the opposite direction when photons leave the sail material.

When the vector components of the forces due to the incoming and outgoing photons are added for the specularly reflecting surface, the force components parallel to the sail cancel out, and the components normal to the sail add so that the resultant force has twice the magnitude of the normal component due to the incoming radiation. Thus the resultant radiation force must be multiplied by an additional factor, \(\cos \gamma\), to account for the oblique incidence, and the resulting expression for a specularly reflecting sail is

\[
F_r = \frac{2SA}{cm} \left(\frac{r_o}{r_o} \right)^2 (\cos^2 \gamma) \mathbf{n}. \quad (1)
\]

If the sail is an ideal blackbody, all the radiation is absorbed and reemitted at longer wavelengths from both sides of the sail according to the Stephan-Boltzmann law
where \( \varepsilon \) is the emissivity equal to 1.0 for an ideal black-body, \( \sigma \) is the Stephan-Bolzmann constant and \( T \) is the equilibrium temperature of the sail. The emitted radiation produces a force that is directed normal to the sail. If the surfaces have different emissivities, a net force will be produced normal to the sail. However, both sides of a solar sail are usually made of the same material so that the forces due to thermal emission cancel out. In this case the only force per unit mass is due to the incoming radiation in the direction of the antisolar vector and is given by

\[
F_I = \frac{SA}{cm} \left( \frac{I_0}{I_s} \right)^2 \cos \gamma \hat{i}_{rs} \quad (2)
\]

where \( \hat{i}_{rs} \) is the unit vector at the sail in the antisolar direction. Numerical results will be obtained for these two types of sails. There are two other photometric types of "ideal" solar sails, diffuse reflecting and back reflecting sails and a general combination of all four ideal types, so called "grey" solar sails, that are analyzed and discussed by Forward (1990).

**B. Coordinate System and Sail Orientation**

The origin of the coordinate system is taken to be at the center of the earth. The \( x \) axis is along the sun-earth line with the positive direction pointing away from the sun.
The z axis is normal to the ecliptic plane with the positive direction pointing in the direction of the north celestial pole. The y axis is normal to the plane defined by the x and z axes, and the positive direction points in such a way as to form a right-handed coordinate system.

A diagram of the earth-sun-moon-sail system is shown in Figure 1. \( r_o \) and \( r_s \) have been defined previously; \( r_{em} \) is the distance between the centers of the earth and the moon and \( r_{xy} \) is the projection of the vector \( r_s \) onto the ecliptic plane. Angles \( \phi, \theta, \phi_{em}, \theta_{em} \) are the azimuth and zenith coordinates for the sail and the moon respectively. \( \psi_\phi \) and \( \psi_\theta \) are the azimuth and zenith angles of the sail relative to the sun; \( i, j, k \) are the cartesian unit vectors of the sail relative to the earth, and \( i_{rs}, i_{gs}, i_{gs} \) are the spherical unit vectors of the sail relative to the sun. The unit vectors \( i_{rs}, i_{gs} \) and \( i_{gs} \) can be expressed in terms of the unit vectors \( i, j, k \) by first rotating about the Z axis through the angle \( \psi_\phi \) and then about the intermediate Y' axis through the angle \( \psi_\theta \), where the X, Y, Z axes are parallel to the x, y, z axes defined earlier but with the origin at the sun. These rotations can be expressed by the following matrix multiplications

\[
\begin{align*}
\begin{bmatrix}
i_{rs} \\
i_{gs} \\
i_{gs}
\end{bmatrix} &= \begin{bmatrix}
cos\psi_\theta & 0 & sin\psi_\theta \\
0 & 1 & 0 \\
-sin\psi_\theta & 0 & cos\psi_\theta
\end{bmatrix} \begin{bmatrix}
cos\psi_\phi & sin\psi_\phi & 0 \\
-sin\psi_\phi & cos\psi_\phi & 0 \\
0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
i \\
j \\
k
\end{bmatrix}
\end{align*}
\]
Figure 1. Illustration of the sailcraft-sun-earth-moon system together with the coordinates, position vectors and angles used in the analysis.
yielding

\[
\begin{pmatrix}
I_{rs} \\
I_{\phi s} \\
I_{\theta s}
\end{pmatrix} = \begin{pmatrix}
\cos\psi_\theta [\cos\psi_\phi I + \sin\psi_\phi J] + \sin\psi_\theta K \\
-\sin\psi_\theta I + \cos\psi_\phi J \\
-\sin\psi_\phi [\cos\psi_\phi I + \sin\psi_\phi J] + \cos\psi_\theta K
\end{pmatrix}.
\]

Using spherical coordinates, one may write

\[
\sin \psi_\phi = \frac{r \cos \theta \sin \phi}{r_{xy}}, \quad \cos \psi_\phi = \frac{r_o + r \cos \theta \cos \phi}{r_{xy}},
\]

\[
\sin \psi_\theta = \frac{r \sin \theta}{r_s}, \quad \cos \psi_\theta = \frac{r_{xy}}{r_s},
\]

and

\[
I_{xy} = [r_s^2 - (r \sin \theta)^2]^k. \quad (3)
\]

Substitution of these expressions into the expression for the unit vectors gives

\[
\begin{pmatrix}
I_{rs} \\
I_{\phi s} \\
I_{\theta s}
\end{pmatrix} = \begin{pmatrix}
\frac{r_{xy}}{r_s} \left[ \frac{r_o + r \cos \theta \cos \phi}{r_{xy}} I + \frac{r \cos \theta \sin \phi}{r_{xy}} J \right] + \frac{r \sin \theta}{r_s} K \\
- \frac{r \cos \theta \sin \phi}{r_s} I + \frac{r_o + r \cos \theta \cos \phi}{r_{xy}} J \\
- \frac{r \sin \theta}{r_s} \left[ \frac{r_o + r \cos \theta \cos \phi}{r_{xy}} I + \frac{r \cos \theta \sin \phi}{r_{xy}} J \right] + \frac{r_{xy}}{r_s} K
\end{pmatrix}
\]

From this it follows that

\[
I_{rs} = \frac{r_o + r \cos \theta \cos \phi}{r_s} I + \frac{r \cos \theta \sin \phi}{r_s} J + \frac{r \sin \theta}{r_s} K \quad (4)
\]

which is the unit vector representing the direction of the radiation force for the case of the fully absorbing "black" sail given in equation (2).
To express the unit vector normal to the sail in terms of the cartesian unit vectors, it is necessary to perform two additional rotations on the unit vectors $i_{rs}$, $i_{\theta s}$, $i_{\phi s}$. Before doing this, the angular momentum vector of the sail is constrained to always point in a direction parallel to the z axis. This constraint and methods that can be used to maintain it against various destabilizing factors are discussed in chapter 5.

The first of the two rotations referred to above is about the z axis through an angle, $\beta$, which is the angle between the normal vector to the sail and the projection of the antisolar vector in the ecliptic plane. The second rotation is about the intermediate $y'$ axis through an angle, $\alpha$, which is the vertical component of the angle between the antisolar vector and the normal vector to the sail. The unit vectors defining the orientation of the sail are designated as $n$, $t$, $b$. The two rotations are expressed in matrix form as:

$$
\begin{bmatrix}
    n_x \\
    t_x \\
    b_x 
\end{bmatrix} = 
\begin{bmatrix}
    \cos \alpha & 0 & \sin \alpha \\
    0 & 1 & 0 \\
    -\sin \alpha & 0 & \cos \alpha 
\end{bmatrix} 
\begin{bmatrix}
    \cos \beta & \sin \beta & 0 \\
    -\sin \beta & \cos \beta & 0 \\
    0 & 0 & 1 
\end{bmatrix} 
\begin{bmatrix}
    i_{rs} \\
    i_{\phi s} \\
    i_{\theta s} 
\end{bmatrix}
$$

yielding

$$
\begin{bmatrix}
    n_x \\
    t_x \\
    b_x 
\end{bmatrix} = 
\begin{bmatrix}
    \cos \alpha \cos \beta i_{rs} + \cos \alpha \sin \beta i_{\phi s} + \sin \alpha i_{\theta s} \\
    -\sin \beta i_{rs} + \cos \beta i_{\phi s} \\
    -\sin \alpha \cos \beta i_{rs} - \sin \alpha \sin \beta i_{\phi s} + \cos \alpha i_{\theta s} 
\end{bmatrix}
$$

Here only the unit vector $n$ is of interest. Substituting
the expressions for $i_{rs}, i_{\phi s}, i_{\theta s}$, in $n$ leads to

\[
\begin{align*}
n &= \cos \alpha \cos \beta \left[ \frac{r_o + r \cos \theta \cos \phi}{r_s} + \frac{r \cos \theta \sin \phi}{r_s} \right] i + \frac{r \sin \theta}{r_s} k + \\
&+ \cos \alpha \sin \beta \left[ -\frac{r \cos \theta \sin \phi}{r_{xy}} i + \frac{r_o + r \cos \theta \cos \phi}{r_{xy}} j \right] + \\
&+ \sin \alpha \left[ -\frac{r \sin \theta (r_o + r \cos \theta \cos \phi)}{r_s r_{xy}} i - \frac{r^2 \sin \theta \cos \theta \sin \phi}{r_s r_{xy}} j + \frac{r_{xy}}{r_s} k \right]
\end{align*}
\]

Multiplying matrices and grouping unit vectors yields,

\[
\begin{align*}
n &= \left[ \cos \alpha \cos \beta \frac{r_o + r \cos \theta \cos \phi}{r_s} - \frac{r \cos \alpha \sin \beta \cos \theta \sin \phi}{r_s} \right] i + \left[ \frac{r \cos \alpha \cos \beta \cos \theta \sin \phi}{r_s} + \right. \\
&+ \left. \cos \alpha \sin \beta \frac{r_o + r \cos \theta \cos \phi}{r_{xy}} - \frac{r^2 \sin \alpha \sin \theta \cos \theta \sin \phi}{r_s r_{xy}} \right] j + \\
&+ \left[ \frac{r \cos \alpha \cos \beta \sin \theta}{r_s} - \frac{r_{xy} \sin \alpha}{r_s} \right] k. \tag{5}
\end{align*}
\]

This expression represents the unit vector specifying the orientation of the normal vector to the sail, and it is also the direction of the resultant radiation force for a spectrally reflecting sail.

C. Equations of Motion For Sailcraft

The equations of motion will now be derived for the sailcraft based on the coordinate system defined in the previous section. Since the Earth is not identified with an inertial reference frame, the acceleration of the sailcraft must include terms appropriate for a translating and rotating coordinate system. According to Newton's second
law the acceleration of the sailcraft must be balanced by the sum of the external forces per unit mass acting on the sailcraft. For this case Newton's second law is written in vector form as

\[ \ddot{\mathbf{r}} + \dot{\mathbf{r}} \times \mathbf{\omega} + 2 \mathbf{\omega} \times \dot{\mathbf{r}} + \mathbf{\omega} \times (\mathbf{\omega} \times \mathbf{r}) = -\frac{GM_\text{e} \mathbf{r}}{r^3} - \frac{GM_\text{s} \mathbf{r}}{r_\text{e}^3} - \frac{GM_\text{m} \mathbf{r}_\text{m}}{r_\text{m}^3} + \frac{GM_\text{e} \mathbf{r}_\text{e}}{r_\text{e}^3} + \frac{GM_\text{m} \mathbf{r}_\text{m}}{r_\text{m}^3} \]

\[ + \mathbf{F}_\text{r}. \quad (6) \]

Let \( \omega = \Omega \), the angular velocity of the earth around the sun. Since the earth's orbit is nearly circular, having an eccentricity near 0.01, it will be assumed that the earth's orbit is circular. This requires that \( \Omega \) be constant and \( \dot{\Omega} = 0 \) so that the term \( \mathbf{\omega} \times \mathbf{r} = 0 \). Also, \( \ddot{\mathbf{r}}_\text{e} \), the acceleration of the earth about the sun, is \( \Omega^2 \mathbf{r}_\text{e} \); \( \ddot{\mathbf{r}} \) is the acceleration of the sailcraft with respect to the earth; \( 2 \mathbf{\omega} \times \dot{\mathbf{r}} \) is the Coriolis acceleration, and \( \mathbf{\omega} \times (\mathbf{\omega} \times \mathbf{r}) \) is the centripetal acceleration, both due to the earth's motion about the sun. The first three terms on the right hand side of the equation represent the gravitational forces per unit mass of the sailcraft for the earth, sun, moon respectively, and the last term is the radiation force per unit mass of the sailcraft, where \( \mathbf{F}_\text{r} \) is given by equations (1) or (2) depending on the optical properties of the sail.

The vector quantities are given in cartesian component form as follows:

\[ \mathbf{r} = x \mathbf{i} + y \mathbf{j} + z \mathbf{k}, \quad \dot{\mathbf{r}} = \dot{x} \mathbf{i} + \dot{y} \mathbf{j} + \dot{z} \mathbf{k}, \quad \ddot{\mathbf{r}} = \ddot{x} \mathbf{i} + \ddot{y} \mathbf{j} + \ddot{z} \mathbf{k}, \]
\[ \ddot{\mathbf{r}}_\text{e} = -\Omega^2 x_\text{e} \mathbf{i}, \quad \mathbf{\omega} = \Omega \mathbf{k}, \quad \mathbf{r}_\text{s} = (x_\text{s} + x) \mathbf{i} + y \mathbf{j} + z \mathbf{k}, \]
\[ \mathbf{r}_{\text{em}} = x_{\text{em}} \mathbf{i} + y_{\text{em}} \mathbf{j} + z_{\text{em}} \mathbf{k}, \quad \text{and} \]
\[ \mathbf{r}_m = (x - x_{\text{em}}) \mathbf{i} + (y - y_{\text{em}}) \mathbf{j} + (z - z_{\text{em}}) \mathbf{k}, \]

where \( x_{\text{em}}, y_{\text{em}}, z_{\text{em}} \) are the coordinates of the moon relative to earth. The vector cross products are expanded using determinants to give

\[
2 \omega \times \dot{\mathbf{r}} = \begin{vmatrix} i & j & k \\ 0 & 0 & \Omega_x \\ \dot{x} & \dot{y} & \dot{z} \end{vmatrix} = 2(-\Omega_y \dot{y} + \Omega_x \dot{j})
\]

\[
\omega \times \mathbf{r} = \begin{vmatrix} i & j & k \\ 0 & 0 & \Omega_z \\ x & y & z \end{vmatrix} = -\Omega_y \dot{y} + \Omega_x \dot{j}
\]

\[
\omega \times (\omega \times \mathbf{r}) = \begin{vmatrix} i & j & k \\ 0 & 0 & \Omega_y \\ -\Omega_x & \Omega_y & 0 \end{vmatrix} = -\Omega^2 x \dot{i} - \Omega^2 y \dot{j}
\]

If the vector quantities in equation (6) are replaced by the vectors in component form given above, the equations of motion can be written in cartesian component form as

\[
-x_0 \Omega^2 + \ddot{x} - 2 \Omega \dot{y} - \Omega^2 x = -\frac{GM_e x}{r^3} - \frac{GM_s (r_e + x)}{r_s^3} - \frac{GM_m (x - x_{\text{em}})}{r_m^3} + \frac{x_{\text{em}}}{r_m^3} + F_{rx},
\]

\[
\ddot{y} + 2 \Omega \dot{x} - \Omega^2 y = \frac{-GM_e y}{r^3} - \frac{GM_s y}{r_s^3} - \frac{GM_m (y - y_{\text{em}})}{r_m^3} + \frac{y_{\text{em}}}{r_m^3} + F_{ry},
\]

\[
\ddot{z} = \frac{-GM_e z}{r^3} - \frac{GM_s z}{r_s^3} - \frac{GM_m (z - z_{\text{em}})}{r_m^3} + \frac{z_{\text{em}}}{r_m^3} + F_{rz},
\]

where \( F_{rx}, F_{ry}, F_{rz} \) are the components of the radiation force per unit mass given by equations (1) or (2), together with equations (4) or (5) depending on the type of sail.
The equations of motion will now be expressed in terms of spherical coordinates, \( r, \phi, \theta \), since the results are more meaningful in this system. Cartesian and spherical coordinates are related by the following equations:
\[
\begin{align*}
    x &= r \cos \theta \cos \phi, \\
    y &= r \cos \theta \sin \phi, \\
    z &= r \sin \theta,
\end{align*}
\]
where the latitude is used instead of the colatitude.
Differentiating these equations yields the cartesian velocity and acceleration components in terms of spherical coordinates:
\[
\begin{align*}
    \dot{x} &= \dot{r} \cos \theta \cos \phi - r \dot{\theta} \sin \theta \cos \phi - r \dot{\phi} \cos \theta \sin \phi, \\
    \dot{y} &= \dot{r} \cos \theta \sin \phi - r \dot{\theta} \sin \theta \sin \phi + r \dot{\phi} \cos \theta \cos \phi, \\
    \dot{z} &= \dot{r} \sin \theta + r \dot{\theta} \cos \theta,
\end{align*}
\]
\[
\begin{align*}
    \ddot{x} &= \ddot{r} \cos \theta \cos \phi - 2 \dot{r} \dot{\theta} \sin \theta \cos \phi - 2 \dot{r} \dot{\phi} \cos \theta \sin \phi + 2 \theta \dot{\phi} \sin \theta \sin \phi - r \ddot{\theta} \cos \theta \cos \phi - r \dot{\theta} \dot{\phi} \cos \theta \sin \phi - r \dot{\phi}^2 \cos \theta \cos \phi - r \phi \dot{\phi} \sin \theta \sin \phi - r \phi \dot{\phi} \cos \theta \cos \phi, \\
    \ddot{y} &= \ddot{r} \cos \theta \sin \phi - 2 \dot{r} \dot{\theta} \sin \theta \sin \phi + 2 \dot{r} \dot{\phi} \cos \theta \cos \phi - 2 \theta \dot{\phi} \sin \theta \cos \phi - r \ddot{\theta} \sin \theta \sin \phi - r \dot{\theta} \dot{\phi} \sin \theta \cos \phi - r \dot{\phi}^2 \sin \theta \sin \phi - r \phi \dot{\phi} \cos \theta \cos \phi - r \phi \dot{\phi} \sin \theta \sin \phi + r \phi \dot{\phi} \cos \theta \cos \phi, \\
    \ddot{z} &= \ddot{r} \sin \theta + 2 \dot{r} \dot{\theta} \cos \theta + r \ddot{\theta} \cos \theta - r \dot{\theta}^2 \sin \theta.
\end{align*}
\]
Bringing all the terms except the second derivative terms to the right hand side of the cartesian form of the equations of motion, and replacing the cartesian coordinates and their derivatives by their spherical coordinate equivalents given above, the equation of motion can be expressed in spherical
coordinates as

\[(\ddot{r} - \dot{r}^2 - r^2)\cos \theta \cos \phi - (2\dot{r} \dot{\theta} + \dot{r} \dot{\phi})\sin \theta \cos \phi - (2\dot{r} \dot{\phi} + r \dot{\phi})\cos \theta \sin \phi + 2r \dot{\theta} \dot{\phi} \sin \theta \sin \phi = r_o \Omega^2 + 2 \Omega (r \dot{\cos \theta} \sin \phi - \dot{r} \sin \theta \sin \phi + r \phi \cos \theta \cos \phi) + \Omega^2 r \cos \theta \cos \phi - \frac{GM_e \cos \theta \cos \phi}{r^2} - \frac{GM_d (r_o + r \cos \theta \cos \phi)}{r_s^3}
\]

\[-GM_m \left(\frac{r \cos \theta \cos \phi - r_{em} \cos \theta_{em} \cos \phi_{em}}{r_m^3} + \frac{r_{em} \cos \theta_{em} \cos \phi_{em}}{r_{em}^3}\right) + F_{rx},
\]

\[(\ddot{r} - \dot{r}^2 - r^2)\sin \theta - (2\dot{r} \dot{\theta} + \dot{r} \dot{\phi})\sin \theta \sin \phi - (2\dot{r} \dot{\phi} + r \dot{\phi})\cos \theta \cos \phi - 2r \dot{\theta} \dot{\phi} \sin \theta \cos \phi - 2 \Omega (r \dot{\cos \theta} \sin \phi - \dot{r} \sin \theta \cos \phi - r \phi \cos \theta \sin \phi) + \Omega^2 r \cos \theta \sin \phi - \frac{GM_e \sin \phi}{r^2} - \frac{GM_d r \cos \phi}{r_s^3}
\]

\[GM_m \left(\frac{r \sin \theta - r_{em} \sin \theta_{em}}{r_m^3} + \frac{r_{em} \sin \theta_{em}}{r_{em}^3}\right) + F_{ry},
\]

\[(\ddot{r} - \dot{r}^2) \sin \theta + (2\dot{r} \dot{\theta} + \dot{r} \dot{\phi}) \cos \theta = -\frac{GM_s \sin \theta}{r^2} - \frac{GM_d \cos \theta}{r_s^3}
\]

\[-GM_m \left(\frac{r \sin \theta - r_{em} \sin \theta_{em}}{r_m^3} + \frac{r_{em} \sin \theta_{em}}{r_{em}^3}\right) + F_{rz}.
\]

Using the law of cosines and Figure 1, \(r_s\) and \(r_m\) are expressed in terms of spherical coordinates as follows:

\[r_s^2 = r_o^2 + r^2 + 2r_o r \cos \psi.
\]

From spherical trigonometry \(\cos \psi = \cos \phi \cos \theta\) so that

\[r_s^2 = r_o^2 + r^2 + 2r_o r \cos \theta \cos \phi. \hspace{1cm} (7)
\]

Substituting the spherical coordinate expressions into the equation for \(r_m\) in cartesian coordinates, squaring, simplifying and using basic trigonometric relations, \(r_m\) can
be expressed as

\[
    r_m = (r^2 + r_{em}^2 - 2rr_{em}[\cos \theta \cos \theta_{em} \cos(\phi - \phi_{em}) + \sin \theta \sin \theta_{em}])^{\frac{1}{2}} \tag{8}
\]

where \(r_{em}, \phi_{em}, \theta_{em}\) are the spherical coordinates of the moon relative to the earth.

Also of interest is the velocity of the sailcraft relative to the moon. This can be a critical factor in determining whether the sailcraft encounters the moon in a hyperbolic orbit or whether it goes into orbit around the moon. The relative velocity of the sailcraft with respect to the moon is written in vector form as,

\[
    v_{sa} = \dot{r} - \dot{r}_{em} = (\dot{x} - \dot{x}_{em}) \hat{i} + (\dot{y} - \dot{y}_{em}) \hat{j} + (\dot{z} - \dot{z}_{em}) \hat{k} + (x - x_{em}) \hat{i} + (y - y_{em}) \hat{j} + (z - z_{em}) \hat{k},
\]

where the rotation of the coordinate axes is taken into account. Using

\[
    \hat{i} = \omega \times \hat{\omega} = \Omega \hat{j}, \quad \hat{j} = \omega \times \hat{j} = -\Omega \hat{i}, \quad \hat{k} = \omega \times \hat{k} = 0
\]

and replacing the cartesian coordinates by spherical coordinates, the relative velocity components can be expressed as

\[
    v_{rx} = \dot{r} \cos \theta \cos \phi - r \dot{\theta} \sin \theta \cos \phi - r \dot{\phi} \cos \theta \sin \phi - \\
    r_{em} \dot{\theta} \cos \theta_{em} \cos \phi_{em} + r_{em} \dot{\frac{\phi}{\theta}} \sin \theta_{em} \cos \phi_{em} + r_{em} \dot{\phi}_{em} \cos \theta_{em} \sin \phi_{em} - \\
    \Omega (r \cos \theta \sin \phi - r_{em} \cos \theta_{em} \sin \phi_{em}).
\]
\[ v_{ry} = r \cos \theta \sin \phi \sin \theta - r \sin \theta \sin \phi \cos \theta + r \phi \cos \theta \cos \phi - r e_m \cos \theta e_m \sin \phi e_m + \\
+ r e_m \sin \phi e_m \sin \phi e_m - r e_m \phi e_m \cos \phi e_m + \\
\Omega (r \cos \theta \cos \phi - r e_m \cos \theta e_m \cos \phi e_m), \]
\[ v_{i z} = \dot{r} \sin \theta + r \dot{\theta} \cos \theta - \dot{r} e_m \sin \phi e_m - r \dot{e}_m \cos \theta e_m. \]

The magnitude of the velocity is
\[ v_r = (v_{rx}^2 + v_{ry}^2 + v_{i z}^2)^{1/2}. \]

These equations will not be used in the differential equations of motion but only to obtain auxiliary information that will be useful.

The equations of motion and related equations will be put into dimensionless form as follows:

Let \( \rho = \frac{r}{r_o} \), \( \rho_{em} = \frac{r_{em}}{r_o} \), \( \rho_m = \frac{r_m}{r_o} \), and \( \tau = \Omega t \).

Also let \( \mu_s = \frac{M_s}{M_s + M_e} \), \( \mu_e = \frac{M_e}{M_s + M_e} \), \( \mu_m = \frac{M_m}{M_s + M_e} \),
\[ G = \frac{\Omega^2 r_o^3}{M_s + M_e} \] and \( P_r = \frac{Q S_o}{c \Omega^2 r_o^2} \),

where \( Q \) is a coefficient that depends on the optical properties of the sail (2 for specularly reflecting, 1 for pure absorbing). Using the chain rule for derivatives, the dimensional derivatives are put into dimensionless form as follows:
\[ \dot{r} = \frac{dr}{dt} = r_o \Omega \frac{d\rho}{dt} = \rho \Omega \overline{\rho}, \quad \ddot{r} = \frac{d^2r}{dt^2} = r_o \Omega^2 \frac{d\rho}{dt} = \rho r_o \Omega^2 \overline{\rho}, \]
\[ \dot{\theta} = \frac{d\theta}{dt} = \Omega \frac{d\rho}{dt} = \rho \overline{\theta}, \quad \ddot{\theta} = \frac{d^2\theta}{dt^2} = \Omega^2 \overline{\theta}, \]
\[ \dot{\phi} = \frac{d\phi}{dt} = \Omega \frac{d\rho}{dt} = \rho \overline{\phi}, \quad \ddot{\phi} = \frac{d^2\phi}{dt^2} = \Omega^2 \overline{\phi}. \]
where the bars indicating dimensionless variables will be omitted in what follows. Using equation (7) $r_s$ is expressed in dimensionless form as

$$r_s^2 = r_0^2(1 + \rho^2 + 2 \rho \cos \theta \cos \phi) = r_0^2 k$$

where

$$k = 1 + \rho^2 + 2 \rho \cos \theta \cos \phi.$$

Also, from equation (3),

$$r_{xy} = r_s \left[ 1 - \left( \frac{r}{r_s} \right)^2 \sin^2 \theta \right]^{\frac{3}{4}} = r_0 k^{\frac{3}{4}} \left[ 1 - \frac{\rho^2 \sin^2 \theta}{k} \right] = r_0 k^{\frac{3}{4}} h^{\frac{3}{4}},$$

where

$$h = 1 - \frac{\rho^2 \sin^2 \theta}{k}.$$

The unit vector to the sail in the antisolar direction, equation (4), is written in terms of dimensionless variables as,

$$\mathbf{l}_{rs} = \frac{1 + \rho \cos \theta \cos \phi}{k^{\frac{3}{4}}} \mathbf{i} + \frac{\rho \cos \theta \sin \phi}{k^{\frac{3}{4}}} \mathbf{j} + \frac{\rho \sin \theta}{k^{\frac{3}{4}}} \mathbf{k}, \quad (9)$$

where the parameter $k$ is not to be confused with the unit vector $\mathbf{k}$. The unit vector normal to the sail, equation (5), is written in terms of dimensionless variables as

$$\mathbf{n} = \left[ \cos \alpha \cos \beta \left( \frac{1 + \rho \cos \theta \cos \phi}{k^{\frac{3}{4}}} \right) - \frac{\rho \cos \alpha \cos \beta \cos \theta \sin \phi}{(kh)^{\frac{3}{4}}} \right] \mathbf{i} + \left[ \frac{\rho \cos \alpha \cos \beta \cos \theta \sin \phi}{k^{\frac{3}{4}}} \right] \mathbf{j} + \left[ \frac{\rho \cos \alpha \cos \beta \sin \theta}{k^{\frac{3}{4}}} - \frac{\rho \sin \alpha \sin \theta \cos \phi}{h^{\frac{3}{4}}} \right] \mathbf{k}.$$
Dividing through the equations of motion by $r_0 \Omega^2$, substituting the dimensionless variables and parameters and bringing all terms not containing a second order derivative to the right hand side results in

$\dot{\rho} \cos \theta \cos \phi - \rho \dot{\theta} \sin \theta \cos \phi - \rho \dot{\phi} \cos \theta \sin \phi = \rho (\dot{\theta}^2 + \dot{\phi}^2) \cos \theta \cos \phi$

$+ 2 \dot{\theta} \sin \theta \cos \phi + 2 \dot{\phi} \cos \theta \sin \phi - 2 \rho \dot{\theta} \sin \theta \sin \phi + 1 + 2(\rho \cos \theta \sin \phi - \rho \dot{\theta} \sin \theta \sin \phi + \rho \dot{\phi} \cos \theta \cos \phi) + \rho \cos \theta \cos \phi - \frac{\mu \cos \theta \cos \phi}{\rho^2} - \frac{\mu}{k^{3/2}} (1 + \rho \cos \theta \cos \phi)$

$\frac{p_2}{\rho^2} \frac{\cos \theta \cos \phi - \rho \sin \theta \cos \phi - \rho \dot{\phi} \cos \theta \sin \phi}{P_m}$

$\frac{\rho \cos \theta \cos \phi - \rho \sin \theta \cos \phi + \rho \dot{\phi} \cos \theta \sin \phi}{P_{m}}$

$\frac{\mu_m}{P_{m}^3} \left[ \frac{\rho \cos \theta \cos \phi - \rho \sin \theta \cos \phi + \rho \dot{\phi} \cos \theta \sin \phi}{P_{m}^2} \right] + \frac{P_{\xi}}{k} \cos \gamma \bar{n}_x,$

$\dot{\rho} \sin \theta + \rho \dot{\theta} \cos \theta = \rho (\dot{\theta}^2 \sin \theta - 2 \dot{\theta} \cos \theta - \frac{\mu \cos \theta}{\rho^2} - \frac{\mu}{k^{3/2}} \rho \sin \theta)$

$- \frac{\mu_m}{P_{m}^3} \left[ \frac{\rho \cos \theta \cos \phi - \rho \sin \theta \cos \phi + \rho \dot{\phi} \cos \theta \sin \phi}{P_{m}^2} \right] + \frac{P_{\xi}}{k} \cos \gamma \bar{n}_z.$

Using the definition of $\rho_m$ and equation (8) for $r_m$, $\rho_m$ is readily expressed in terms of dimensionless variables as,

$\rho_m = \left[ \rho^2 + \rho_{em}^2 - 2 \rho \rho_{em} [\cos \theta \cos \theta \cos (\phi - \phi_{em}) + \sin \theta \sin \theta_{em}] \right]^{1/2}.$

In the radiation force terms the exponent $\eta$ and the $\eta$ in the definition of the dimensionless parameter $\rho$ are the same and have the value 2 for a spectrally reflecting sail and 1 for...
a pure absorbing "black" sail. \( \overrightarrow{n}_x, \overrightarrow{n}_y \) and \( \overrightarrow{n}_z \) represent the components of the vectors \( n \), or \( i_r \) in normalized form, equations (10) or (9), depending on the optical properties of the sail.

The three dimensionless equations of motion of the sail are coupled with respect to the accelerations or second derivatives of the variables. The next step is to solve the equations explicitly for the accelerations, \( \dot{\rho}, \dot{\phi}, \dot{\theta} \). This is done by writing the equations in the form

\[
\begin{align*}
\dot{\rho} \cos \theta \cos \phi - \rho \dot{\theta} \sin \theta \cos \phi - \rho \dot{\phi} \cos \theta \sin \phi &= -R_1, \\
\dot{\rho} \cos \theta \sin \phi - \rho \dot{\theta} \sin \theta \sin \phi + \rho \dot{\phi} \cos \theta \cos \phi &= R_2, \\
\dot{\rho} \sin \theta + \rho \dot{\theta} \cos \theta &= R_3,
\end{align*}
\]

where \( R_1, R_2, R_3 \) consist of all the terms on the right hand side of the normalized equations of motion. If \( \dot{\rho}, \rho \dot{\phi} \) and \( \rho \dot{\theta} \) are considered as unknowns, they can be solved explicitly using determinants and Cramer's Rule. Evaluating the determinants called for in Cramer's Rule and simplifying, the unknowns can be expressed in terms of \( R_1, R_2, R_3 \) as,

\[
\begin{align*}
\dot{\rho} &= R_1 \cos \theta \cos \phi + R_2 \cos \theta \sin \phi + R_3 \sin \theta, \\
\rho \dot{\phi} &= \frac{R_2 \cos \phi - R_3 \sin \phi}{\cos \theta}, \\
\rho \dot{\theta} &= R_3 \cos \theta - R_1 \sin \theta \cos \phi - R_2 \sin \theta \sin \phi.
\end{align*}
\]

When the substitutions are made for \( R_1, R_2 \) and \( R_3 \), the accelerations are obtained explicitly after a great deal of algebra and simplification. For the case of a spectrally
reflecting sail, the final form of the equations of motion is obtained using equation (10) for $n$ and $q = 2$:

$$\dot{\rho} = \rho \ddot{\phi}^2 + \rho \phi^2 \cos^2 \theta + \cos \theta \cos \phi + 2 \rho \phi \cos^2 \theta + \rho \cos^2 \theta - \frac{\mu_s}{\rho^2} - \frac{\mu_m}{k^{3/2}} (\cos \theta \cos \phi + \rho) - \frac{\mu_m}{\rho_m} [p - \rho_{em} (\cos \theta \cos \theta_{em} \cos (\phi - \phi_{em}) + \sin \theta \sin \theta_{em}) - \frac{\mu_m}{\rho_{em}} [\cos \theta \cos \theta_{em} \cos (\phi - \phi_{em}) + \sin \theta \sin \theta_{em}] + \frac{P_r \cos^2 \gamma}{k} \left[ \frac{\cos \alpha \cos \beta (\cos \theta \cos \phi + \rho)}{k^{1/2}} + \frac{\cos \alpha \sin \beta \cos \theta \sin \phi}{(kh)^{1/2}} \right] - \frac{\rho \sin \alpha \sin \theta \sin \theta \sin \phi}{kh^{1/2} (\cos \theta \sin \phi + \rho \sin \theta \cos^2 \theta) - h^{1/2} \sin \alpha \sin \theta}] \right), \quad (11)$$

$$\dot{\phi} = \frac{1}{\rho} \left\{ - 2 \rho \phi \cos \theta + 2 \rho \phi \sin \phi - \sin \phi - 2 \rho \cos \theta + 2 \rho \sin \theta + \frac{\mu_s}{k^{3/2}} \sin \phi - \frac{\mu_m \rho_{em}}{\rho_m} \cos \theta_{em} \sin (\phi - \phi_{em}) + \frac{\mu_m}{\rho_{em}} \cos \theta_{em} \sin (\phi - \phi_{em}) + \frac{P_r \cos^2 \gamma}{k} \left[ - \frac{\cos \alpha \cos \beta \sin \theta}{k^{1/2}} + \frac{\cos \alpha \sin \beta \cos \theta \sin \phi}{(kh)^{1/2}} \right] \right\}, \quad (12)$$

$$\dot{\theta} = \frac{1}{\rho} \left\{ - \rho \phi^2 \sin \theta \cos \theta - 2 \rho \phi - \sin \theta \cos \phi - 2 \rho \phi \sin \theta \cos \theta + \frac{\mu_s}{k^{3/2}} \sin \theta \cos \phi - \frac{\mu_m \rho_{em}}{\rho_m} [\sin \theta \cos \theta_{em} \cos (\phi - \phi_{em}) - \cos \theta \sin \theta_{em}] + \frac{\mu_m}{\rho_{em}} [\sin \theta \cos \theta_{em} \cos (\phi - \phi_{em}) - \cos \theta \sin \theta_{em}] + \frac{P_r \cos^2 \gamma}{k} \left[ - \frac{\cos \alpha \cos \beta \sin \theta \cos \phi}{k^{1/2}} + \frac{\cos \alpha \sin \beta \sin \theta \sin \phi}{(kh)^{1/2}} \right] \right\}. \quad (13)$$
For the case of a fully absorbing "black" sail, the final form of the equations of motion are obtained using equation (9) for \( i_r \) and \( Q = 1 \):

\[
\begin{align*}
\dot{\rho} &= \rho \dot{\theta}^2 + \rho \dot{\phi}^2 \cos^2 \theta + \cos \theta \cos \phi + 2 \rho \dot{\phi} \cos^2 \theta + \rho \cos^2 \theta \\
&- \frac{\mu_e}{\rho^2} \frac{\mu_s}{k^{3/2}} (\cos \theta \cos \phi + \rho) - \frac{\mu_m}{\rho_m^3} \left[ \rho - \rho \cos \theta \cos \theta \cos \phi \cos (\phi - \phi_m) \right] \\
&+ \sin \theta \sin \theta_m] - \frac{\mu_m}{\rho_m^2} \left[ \cos \theta \cos \theta \cos \phi \cos (\phi - \phi_m) + \sin \theta \sin \theta_m \right] + \\
&\frac{P_c \cos \gamma}{k^{3/2}} (\cos \theta \cos \phi + \rho) , \\
\end{align*}
\] (14)

\[
\begin{align*}
\dot{\phi} &= \frac{1}{\rho \cos \theta} \left\{ -2 \dot{\rho} \cos \theta + 2 \rho \dot{\phi} \sin \theta - \sin \phi - 2 \dot{\rho} \cos \theta + 2 \rho \dot{\theta} \sin \theta \\
&+ \frac{\mu_s}{k^{3/2}} \sin \phi - \frac{\mu_m \rho \cos \theta}{\rho_m^3} \cos \theta \cos \theta \cos \phi \cos (\phi - \phi_m) + \frac{\mu_m}{\rho_m^2} \cos \theta \cos \theta \cos \phi \cos (\phi - \phi_m) \\
&- \frac{P_c \cos \gamma \sin \phi}{k^{3/2}} \right\} , \\
\end{align*}
\] (15)

\[
\begin{align*}
\dot{\theta} &= \frac{1}{\rho} \left\{ -\rho \phi^2 \sin \theta \cos \theta - 2 \rho \dot{\theta} - \sin \phi \cos \phi - 2 \rho \phi \sin \theta \cos \theta \\
&+ \frac{\mu_s}{k^{3/2}} \sin \theta \cos \phi - \frac{\mu_m}{\rho_m^3} \left[ \rho \cos \theta \cos \theta \cos \phi \cos (\phi - \phi_m) + \cos \theta \sin \theta \cos \theta \cos \phi \cos (\phi - \phi_m) \right] + \\
&\frac{\mu_m}{\rho_m^2} \left[ \sin \theta \cos \theta \cos \phi \cos (\phi - \phi_m) - \cos \theta \sin \theta \cos \theta \cos \phi \cos (\phi - \phi_m) \right] - \frac{P_c \cos \gamma}{k^{3/2}} (\sin \theta \cos \phi) \right\} , \\
\end{align*}
\] (16)

The equations of motion are expressed in state variable form for the purpose of numerical computation:

\[
\begin{align*}
\nu_\rho &= \dot{\rho} , & \nu_\phi &= \dot{\phi} , & \nu_\theta &= \dot{\theta} , & \nu_\rho &= \ddot{\rho} , & \nu_\phi &= \ddot{\phi} , & \nu_\theta &= \ddot{\theta} ,
\end{align*}
\]

where the right hand sides of the last three state variable equations are symbolic for the equations of motion.
D. Equations of Motion for Moon

In the preceding section the coordinates of the moon relative to the earth were treated as known quantities. In fact, they are unknown, and they must be computed simultaneously along with the computations of the sailcraft's coordinates and velocity components. The moon's coordinates can be calculated approximately using invariant orbital elements. However, the effect of solar perturbations on the moon's orbit can be significant, and the use of invariant orbital elements to compute the moon's orbit could lead to results that are too imprecise for our purpose here. Therefore, the moon's coordinates relative to the earth will be computed numerically in a way similar to that of the sailcraft. The equations of motion for the moon can be readily obtained from the equations of motion of the sailcraft by dropping the terms concerning the lunar gravitational force and the radiation force and replacing the sailcraft's coordinates, velocities, accelerations and related parameters with those of the moon. To do this it is necessary to introduce a new quantity, the distance of the moon from the sun, \( r_{ms} \), which can be written in a way similar to \( r_s \),

\[
\begin{align*}
r_{sm}^2 &= r_s^2 (1 + 2 \rho_{em} \cos \theta_{em} \cos \phi_{em} + \rho_{em}^2) = r_s^2 k_{em}, \\
\text{where} \quad k_{em} &= 1 + 2 \rho_{em} \cos \theta_{em} \cos \phi_{em} + \rho_{em}^2.
\end{align*}
\]
Then the equations of motion of the moon can be written as,

\[
\ddot{\rho}_{en} = \rho_{en}\dot{\theta}_{en} + \cos \theta_{en} \cos \phi_{en} + \rho_{en}\cos^2 \theta_{en}(\dot{\phi}_{en} + 1)^2 - \frac{\mu_e}{\rho^2} - \frac{\mu_e}{k_{en}^{3/2}}(\rho_{en} + \cos \theta_{en} \cos \phi_{en}), \tag{17}
\]

\[
\dot{\phi}_{en} = \frac{1}{\rho_{en} \cos \theta_{en}} \left\{ - \sin \phi_{en} - 2 \rho_{en} \cos \theta_{en}(\dot{\phi}_{en} + 1) + 2 \rho_{en} \dot{\theta}_{en} \sin \theta_{en}(\dot{\phi}_{en} + 1) + \frac{\mu_e \sin \phi_{en}}{k_{en}^{3/2}} \right\}, \tag{18}
\]

\[
\dot{\theta}_{en} = \frac{1}{\rho_{en}} \left\{ - 2 \rho_{en} \dot{\theta}_{en} - \sin \theta_{en} \sin \phi_{en} - \rho_{en} \dot{\phi}_{en} \sin \theta_{en} \cos \theta_{en}(\dot{\phi}_{en} + 2) + \frac{\mu_e \sin \theta_{en} \cos \phi_{en}}{k_{en}^{3/2}} \right\}. \tag{19}
\]

\textbf{E. Initial Conditions for Sailcraft and Moon}

The three components of position and velocity must be specified for both the sailcraft and the Moon at the point where time is considered to begin for the problem. It will be assumed that the sailcraft is transported by a rocket into a circular orbit of distance, \( r_i \), from the center of the earth. This circular orbit can have many possible orientations with respect to the ecliptic plane and the earth-sun line, depending on the specified values of certain orbital parameters. To allow for the possibility of choosing any desired orientation for the initial orbit of the sailcraft about the earth, it will be convenient to express the initial conditions in parametric form, using time, \( t \), as the only variable, along with the specified
orbital parameters. Two of the orbital parameters define the orientation of the orbit. The first is the position of the ascending node, $\delta$, which is the angle from a stationary reference direction, known by astronomers as the first point of the constellation Aries, to the ascending node of the sailcraft's orbit on the ecliptic plane. The second is the angle of inclination, $i$, from the ecliptic plane to the orbital plane. A third parameter, $\lambda'$, is the angle from the reference direction to the sun. A fourth angle specifies the instantaneous position in the orbit with respect to the ascending node and is a function of time.

To derive equations for the initial conditions it will be necessary to define three sets of unit vectors. The first set of unit vectors, $I, J, K$, define the inertial direction, where $I$ points to the first point of Aries; $K$ is normal to the ecliptic plane and points toward the north celestial pole, and $J$ is normal to the plane defined by $I$ and $K$ and points in a direction that defines a right handed coordinate system. A second set of unit vectors, $i, j, k$, is such that $k = K$, but $i$ and $j$ are oriented by rotating $I$ and $J$ counterclockwise through an angle $\lambda'$ so that $i$ and $j$ point in the direction of the x and y axes. This is the same set of unit vectors that has been used in the previous sections. A third set of unit vectors, $i_o, j_o, k_o$, defines the orientation of the orbital plane. $i_o$ points toward the
ascending node of the orbit on the ecliptic plane; \( \mathbf{k}_o \) is normal to the orbital plane and the points into the north celestial hemisphere, and \( \mathbf{j}_o \) is normal to the plane defined by \( \mathbf{i}_o \) and \( \mathbf{k}_o \) and points in a direction that defines a right handed coordinate system. Two rotations are required to go from the \( \mathbf{I}, \mathbf{J}, \mathbf{K} \) set of unit vectors to the \( \mathbf{i}_o, \mathbf{j}_o, \mathbf{k}_o \) set. One rotation is counterclockwise about the \( z \) axis through the angle, \( \delta \), and the second rotation is counterclockwise about the intermediate, \( x' \) axis through the angle of inclination, \( i \). These rotations can be described by the following matrix multiplications:

\[
\begin{bmatrix}
    \mathbf{i}_o \\
    \mathbf{j}_o \\
    \mathbf{k}_o
\end{bmatrix} =
\begin{bmatrix}
    1 & 0 & 0 \\
    0 & \cos i & \sin i \\
    0 & -\sin i & \cos i
\end{bmatrix}
\begin{bmatrix}
    \cos \delta & \sin \delta & 0 \\
    -\sin \delta & \cos \delta & 0 \\
    0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
    \mathbf{I} \\
    \mathbf{J} \\
    \mathbf{K}
\end{bmatrix}.
\]  

(20)

The \( \mathbf{I}, \mathbf{J}, \mathbf{K} \) set must be expressed in terms of the \( i, j, k \) set because the latter set coincides with the coordinate system used in the equations of motion. The \( z \) axis is rotated counterclockwise through an angle, \( \lambda \) where \( \lambda = \lambda' + \pi \). This rotation is expressed by

\[
\begin{bmatrix}
    \mathbf{I} \\
    \mathbf{J} \\
    \mathbf{K}
\end{bmatrix} =
\begin{bmatrix}
    \cos \lambda & -\sin \lambda & 0 \\
    \sin \lambda & \cos \lambda & 0 \\
    0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
    \mathbf{i} \\
    \mathbf{j} \\
    \mathbf{k}
\end{bmatrix} =
\begin{bmatrix}
    \cos \lambda \mathbf{i} - \sin \lambda \mathbf{j} \\
    \sin \lambda \mathbf{i} + \cos \lambda \mathbf{j}
\end{bmatrix}.
\]  

(21)

Performing the matrix multiplication in equation (20) and substituting the result given by equation (21) gives
Performing the multiplication and rearranging gives,

\[
\begin{bmatrix}
\mathbf{i}_o \\
\mathbf{j}_o \\
\mathbf{k}_o
\end{bmatrix} = 
\begin{bmatrix}
\cos \delta & \sin \delta & 0 \\
-cos \delta \sin \delta & \cos \delta \cos \delta & \sin \delta \\
\sin \delta \sin \delta & -\sin \delta \cos \delta & 1
\end{bmatrix}
\begin{bmatrix}
\cos \lambda \mathbf{i} - \sin \lambda \mathbf{j} \\
\sin \lambda \mathbf{i} + \cos \lambda \mathbf{j} \\
\mathbf{k}
\end{bmatrix}.
\]

Defining an angle \( \eta = \delta - \lambda \) and recognizing fundamental trigonometric identities allows the above result to be written more concisely as

\[
\begin{bmatrix}
\mathbf{i}_o \\
\mathbf{j}_o \\
\mathbf{k}_o
\end{bmatrix} = 
\begin{bmatrix}
\cos \eta \mathbf{i} + \sin \eta \mathbf{j} \\
-cos \eta \sin \mathbf{i} + \cos \delta \mathbf{j} + \sin \mathbf{k} \\
\sin \eta \sin \mathbf{i} - \sin \delta \cos \mathbf{k}
\end{bmatrix}.
\]

The position vector of the sailcraft in the initial orbital plane is given by

\[
\mathbf{r}_i = \mathbf{r}_i(\cos \omega_o t \mathbf{i}_o + \sin \omega_o t \mathbf{j}_o),
\]

where \( \omega_o \) is the angular velocity of the sailcraft in the orbit. Substituting the expression for \( \mathbf{i}_o \) and \( \mathbf{j}_o \) and rearranging yields,

\[
\mathbf{r}_i = \mathbf{r}_i(\cos \omega_o t \cos \eta - \sin \omega_o t \sin \eta \cos \mathbf{i} + \\
(\cos \omega_o t \sin \eta + \sin \omega_o t \cos \eta \cos \mathbf{i}) \mathbf{j} + \sin \omega_o t \sin \mathbf{k}].
\]
Thus, the initial position coordinate can be written as

\[ x_i = r_i \cos \omega_o t \cos \eta - \sin \omega_o t \sin \eta \cos i, \]  
\[ y_i = r_i \cos \omega_o t \sin \eta + \sin \omega_o t \cos \eta \cos i, \]  
\[ z_i = r_i \sin \omega_o t \sin i. \]  

Differentiating the expressions for the initial coordinates yields the expressions for the initial velocity components,

\[ \dot{x}_i = \omega_o r_i (-\sin \omega_o t \cos \eta - \cos \omega_o t \sin \eta \cos i), \]  
\[ \dot{y}_i = \omega_o r_i (-\sin \omega_o t \sin \eta + \cos \omega_o t \cos \eta \cos i), \]  
\[ \dot{z}_i = \omega_o r_i \cos \omega_o t \sin i. \]  

The initial circular orbit of the sailcraft is specified to be a geosynchronous orbit, or one having a period equal to the earth's rotational period. Therefore, with the period of the orbit specified the radius of the orbit can be calculated from

\[ r_i = \left( \frac{G M_e P_e^2}{4 \pi^2} \right)^{1/3}, \]

where \( P_e \) is the rotational period of the earth in seconds. The angular velocity, \( \omega_o \), in radians per second is found from

\[ \omega_o = \frac{2 \pi}{P_e}. \]

The parameters \( i, \eta \) and \( t \), already defined, are variable and must be determined from computer searches that will be discussed.
The above expressions give the six initial conditions of the sailcraft in cartesian coordinates. The cartesian and spherical coordinates are related by

\[ x = r \cos \theta \cos \phi, \]
\[ y = r \cos \theta \sin \phi, \]
\[ z = r \sin \theta. \]

They are differentiated with respect to time to give

\[ \dot{x} = \dot{r} \cos \theta \cos \phi - r \dot{\theta} \sin \theta \cos \phi - r \dot{\phi} \cos \theta \sin \phi, \quad (28) \]
\[ \dot{y} = \dot{r} \cos \theta \sin \phi - r \dot{\theta} \sin \theta \sin \phi + r \dot{\phi} \cos \theta \cos \phi, \quad (29) \]
\[ \dot{z} = \dot{r} \sin \theta + r \dot{\theta} \cos \theta. \quad (30) \]

The equations relating the cartesian and spherical coordinates can be solved explicitly for the spherical coordinates to give,

\[ r = \sqrt{x^2 + y^2 + z^2}, \]
\[ \phi = \tan^{-1}(y/x), \]
\[ \theta = \sin^{-1}(z/r). \]

The time derivatives of the variables, the spherical velocities are found by considering \( \dot{r}, \phi, \theta \) as unknowns in the equations (28) through (30), and then solving the equations for the unknowns in terms of the cartesian velocities using determinants and Cramer's rule to get

\[ \dot{r} = x \cos \theta \cos \phi + y \cos \theta \sin \phi + z \sin \theta, \]
\[ \dot{\phi} = \frac{1}{r \cos \theta} [-x \sin \phi + y \cos \phi], \]
\[ \dot{\theta} = \frac{1}{r} [-x \sin \theta \cos \phi - y \sin \theta \sin \phi + z \cos \theta]. \]
φ must be modified by subtracting from it the angular velocity of the earth-sun line, \( \Omega \), which is equal to the angular velocity of the earth about the sun.

Performing a similar procedure for the moon would be slightly more complicated since the moon's orbit is not circular. However, it is not necessary to obtain such general expressions for the initial conditions of the moon because the moon's orbit is generally fixed except for solar perturbations, which change its orbit only slightly. Initial conditions of the moon can be obtained for any desired date from an astronomical almanac. The moon's initial condition can also be obtained for various dates following an initial target date by numerical computation. This was done by storing the values of the position coordinates and velocity components of the moon that are calculated at chosen points of an initial lunar orbit, which use initial conditions obtained from an astronautical almanac. Lunar orbits starting at later dates use stored values of the position coordinates and velocity components obtained during the initial orbit as initial conditions.

F. **Eclipsing of Sun by Earth**

For a large range of specified initial orbits, the sailcraft will be periodically totally or partially eclipsed by the earth up to a certain distance prior to earth escape, causing the radiation force to drop to zero or be reduced for the duration of each eclipse. This will increase the
amount of time required for earth escape. Therefore, this effect must be accounted for in any accurate model.

Equations can readily be derived from the geometry of the problem as depicted in Figure 2. In this figure $R_s$ is the radius of the sun; $R_e$ is the radius of the earth; $r_o$ is the distance between the centers of the earth and the sun; $h_s$ is the length of the converging shadow cone; $r_c$ is the radius of the converging shadow cone.

For the case of total eclipse, similar triangles in the figure yields the proportion

$$
\frac{R_s}{h_s} = \frac{r_c}{h_s - x},
$$

from which $r_c$ can be solved for to get

$$
r_c = \frac{R_e}{h_s}(h_s - x).
$$

A second proportion yields an expression for $h_s$ in terms $r_o$, $R_e$ and $R_s$,

$$
\frac{h_s}{R_s} = \frac{r_o + h_s}{R_e},
$$

from which $h_s$ can be solved for to get

$$
h_s = \frac{R_o R_e}{R_s - R_o}.
$$
Figure 2. Illustration of the geometry of the sun, earth and earth's shadow involved in the eclipsing of the sun by the earth.
Substituting the expression for $h$ into the equation for $r_c$ gives,

$$r_c = R_e \left[ 1 - x \frac{(R_g - R_e)}{R_e r_o} \right] . \quad (31)$$

The distance of the sailcraft from the earth-sun line in cartesian coordinates is

$$d_x = (y^2 + z^2)^{1/2}.$$  

If the $x$ coordinate of the sailcraft is less than $h_e$, and if $d_x < r_c$, the sailcraft is in the earth's shadow. In this case, the solar flux at 1 AU, $S$, is set equal to zero, and the radiation force drops to zero.

There is a much larger zone in which the sun is partially eclipsed by the earth. This is the zone outside the converging cone of total eclipse but inside a diverging cone that extends theoretically to infinity along the $x$-axis. However, the partial eclipsing of the sun is significant for only a relatively small portion of this diverging cone. Using figure 2 and the geometry of similar triangles, the following relations can be written:

$$\frac{h_e}{R_e} = \frac{r_o - h_e}{R_g},$$

from which $h_e$ can be solved for to get

$$h_e = \frac{R_g r_o}{R_g + R_o};$$
and

\[
\frac{h_o}{R_o} = \frac{h_e + x}{r_d},
\]

from which \(r_d\) can be solved for to get

\[
r_d = \frac{R_o (h_o + x)}{h_o}. \]

Substituting the expression for \(h_e\) into the one for \(r_d\) gives the final equation for the boundary of the diverging cone as

\[
r_d = \frac{R_o r_o + x (R_e + R_o)}{r_o}. \]

The mathematical conditions for both total and partial eclipses can now be given as follows: If \(x < h_s\) and \(d_x < r_e\), the eclipse is total and \(S = 0\). If \(x < h_s\) and \(r_e < d_x < r_d\), or if \(x > h_s\) and \(d_x < r_d\), the eclipse is partial; and the solar flux at 1 AU, \(S\), is replaced by \(f_s S\), where \(f_s\) is the fraction of the area of the sun that is not obscured by the earth. This assumes that the luminosity of the sun is uniform over its disk. If the earth's disk is apparently inside the sun's disk, \(f_s\) is equal to the difference in the apparent areas of the sun and the earth, divided by the apparent area of the sun as viewed from the sailcraft. This is given as

\[
f_s = \frac{\frac{R_s x}{r_o + x}^2 - \left(\frac{R_e x}{x}\right)^2}{\pi \left(\frac{R_s x}{r_o + x}\right)^2} = 1 - \left[\frac{R_e (r_o + x)}{R_s x}\right]^2. \]
This occurs when

\[ \frac{R_o(z_e + x)}{R_o x} < 1 \text{ so that } f_a > 0. \] 

If the sun is partially eclipsed, and if the disk of the earth apparently intersects the disk of the sun, the analysis is more difficult and proceeds as follows:

The problem consists of deriving an equation that gives the area common to two intersecting circles in terms of the radii of the circles and the distance between their centers. Consider two circles of radii \( r_1 \) and \( r_2 \) and the distance, \( d \), between their centers. \( r_1 \) is associated with the sun, and \( r_2 \) is associated with the earth. The origin of a coordinate system is taken to be at the center of the circle \( r_1 \). The \( y \)-axis is taken to be in the plane of the sailcraft's orbit, and the \( z \)-axis is normal to the sailcraft's orbit plane. The \( y \) and \( z \) axes defined here are not the same as the \( y \) and \( z \) axes used in the previous analysis. The two sets of axes differ in orientation by the angle of inclination of the sailcraft's orbit to the ecliptic. The two circles are described by the equations,

\[ y^2 + z^2 = r_1^2 \] \quad \text{and} \quad \text{(} y - d \text{)}^2 + z^2 = r_2^2. \]

The differential area of the intersecting circles is \( z \ dy \). Solving the above equations for \( z \), the area of intersection of the two circles is given by the integrals
where \( y_p \) is the value of \( y \) where the circles intersect, and the factor 2 accounts for the areas above and below the \( y \)-axis. Using the variable transformations, \( y = r_1 \sin \theta \) and \( y - d = r_2 \sin \theta \), where \( \theta \) is an integrating variable not related to the \( \theta \) used previously, the integrals can be evaluated to get

\[
A_i = \frac{r_1^2}{2} \left[ \frac{\pi}{2} - \sin^{-1}\left( \frac{y_p}{r_1} \right) + \frac{y_p}{r_1^2} (r_1^2 - y_p^2)^{1/2} \right] +
\frac{r_2^2}{2} \left[ \frac{\pi}{2} + \sin^{-1}\left( \frac{y_p - d}{r_2} \right) + \frac{(y_p - d)}{r_2^2} [r_2^2 - (y_p - d)^2]^{1/2} \right].
\]

The quantities in the equation for \( A_i \) must be related to quantities of interest in the larger problem. Using the distance of the earth from the sailcraft as the reference distance, \( r_2 = R_e \). The apparent radius of the sun at this distance is

\[
r_1 = \frac{R_s x}{R_o + x}.
\]

Using similar triangles, the apparent distance between the centers of the sun and the earth is

\[
d = d_x \left( \frac{r_o}{r_o + x} \right) = (y^2 + z^2)^{1/2} \left( \frac{r_o}{r_o + x} \right).
\]
The point of intersection, \( y_p \), is expressed in terms of \( r_1 \), \( r_2 \) and \( d \) by solving the equations of the circles for \( z \) and equating to get

\[
x_1^2 - y_p^2 = x_2^2 - (y_p - d)^2.
\]

Solving for \( y_p \) yields

\[
y_p = \frac{x_1^2 - x_2^2 + d^2}{2d}.
\]

The fraction of the sun's projected area not obscured by the earth is

\[
f_a = 1 - \frac{A_i}{A_t},
\]

where the area of intersection, \( A_i \), is given above, and the apparent area of the sun, \( A_t \), is

\[
A_t = \pi r_1^2 = \pi \left( \frac{R_s X}{r_o + X} \right)^2.
\]

If \( d + r_2 \leq r_1 \), the earth's disk is inside the sun's disk, and \( f_a \) is calculated from the simple relation given earlier. If \( d + r_2 > r_1 \) and \( d < r_1 + r_2 \), the disks of the sun and earth apparently intersect, and the calculation of \( f_a \) proceeds as follows: \( r_1, r_2 \) and \( d \) are calculated in terms of \( R_s, R_e, r_o, x, y \) and \( z \). \( y_p \) is then calculated in terms of \( r_1, r_2 \) and \( d \). \( A_i \) is next calculated in terms of \( r_1, r_2, d \) and \( y_p \), and \( A_t \) is calculated from the simple relation. Then \( f_a \) is calculated.
from $A_i$ and $A_t$. The solar flux, $S$, is then multiplied by the value of instantaneous $f_s$ to get the effective solar flux incident on the sail. (The results do not include the effect of partial eclipses).

**G. Attitude Control Laws**

The equations of motion include three angles, $\alpha$, $\beta$ and $\gamma$, that specify the orientation of the sail with respect to the solar direction. These angles are not chosen arbitrarily nor are they constant. They must be related to position coordinates relative to the earth and the sun by control laws. One control law has already been mentioned implicitly, that the angular momentum vector is constant and always normal to the ecliptic plane. This provides the relation,

$$\sin \alpha = \frac{z}{r_s} = \frac{r_s \sin \theta}{r_s}.$$

In dimensionless form this becomes

$$\sin \alpha = \frac{\rho \sin \theta}{k}, \quad (32)$$

where the symbols have been defined earlier. This control law keeps $\alpha$ very small, almost negligible. The primary control that will be used was first given by Sands (1961),

$$\beta = \frac{\phi}{2} + \frac{\pi}{4}.$$

This law relates the component in the ecliptic plane of the angle between the sun-sail line and the normal to the sail
to the azimuth angle. The coefficient, $1/2$, and the phase angle, $\pi/4$, provide the maximum work per orbit, at least for this simple form of a control law, that can be performed on the sail by the radiation force. This law provides for the sail to be normal to the solar direction as it crosses the negative $y$ axis going away from the sun and parallel to the solar direction as it crosses the positive $y$-axis going toward the sun. The energy of the orbit, as measured by its semi-major axis, is increased through each revolution about the earth until earth escape occurs. The control law must be modified for programming by replacing $\phi$ with $\phi - 2n\pi$.

The modified control law then becomes

$$\beta = \frac{\Phi}{2} + \frac{\pi}{4} - n\pi.$$  \hspace{1cm} (33)

The integer $n$ is incremented by one each time the sailcraft crosses the positive $y$ axis where the sail becomes parallel to the solar direction. The angle, $\gamma$, between the sun-sail line and the normal to the sail is given by spherical trigonometry as

$$\cos \gamma = \cos \alpha \cos \beta.$$  \hspace{1cm} (34)

$\cos \gamma$ computed here is used in the radiation force term in the equations of motion. The angles $\alpha$, $\beta$ and $\gamma$ are recomputed at each time step along with the variables computed in the equations of motion.
The importance of the sail orientation control relations can be illustrated by considering the relative magnitudes of the radiation force and the gravitational forces. The gravitational force of the earth per unit mass at the geosynchronous distance (\(4.2 \times 10^6 \text{ km}\)) is \(-2.29 \times 10^{-1} \text{ Nt/kg}\); the gravitational force of the sun per unit mass at the earth's distance is \(-5.93 \times 10^{-3} \text{ Nt/kg}\), and the radiation force per unit mass due to the sun at the earth's distance for an area-to-mass ratio of 100 is \(-9.02 \times 10^{-4}\text{ Nt/kg}\). This shows that the radiation force is more than two orders of magnitude less than the earth's gravitational force at the geosynchronous distance and several times less than the gravitational force of the sun at the earth's distance. The only way that the radiation force can have a significant effect on the sailcraft's trajectory and be useful as a method of propulsion is by transforming it into a nonconservative force. The sail control relations presented above do that by allowing net positive work to be done on the sail during each planetary orbit, gradually increasing the orbital energy of the sailcraft. Without the sail control relations, the radiation force would be quite useless as a source of propulsion to escape from planetary gravitational fields.

H. Reflection by Planetary Bodies

Earlier it was stated that the reflection of radiation from planetary bodies would be neglected as unimportant.
This is something that cannot be deduced intuitively, nor is it universally true. One case in nature where the reflection and emission of radiation from a planetary body is an important energy contribution concerns the satellites of Mars, Phobos and Deimos, which receive a significant fraction of their incident radiant energy from Mars. Therefore, it is appropriate to analyze this effect and justify my previous assumption for the case being studied.

Solar radiation incident on a planetary body is either absorbed or reflected. Most of the energy that is absorbed by the body is emitted as infrared radiation. This can be just as important or even more important than the reflected radiation from a planetary body on the thermal environment of a natural or artificial satellite. However, infrared or thermal radiation has considerably less energy than visible or ultraviolet radiation. Reflected radiation, however, contains the entire solar spectrum of energies, and it is much more important than emitted infrared radiation in producing radiation pressure. Therefore, the emitted radiation will not be considered further.

To evaluate the contribution of reflected radiation from planetary bodies, consider a differential area, $dA$, at coordinates $(\phi_p, \theta_p)$ on the sunlit face of a planetary object of radius, $r_p$ with $\phi_p$, the longitude, referred to the subsolar point and $\theta_p$ the latitude referred to the ecliptic plane. A satellite (sailcraft) is located at radial
distance, $r$, from the center of the planetary body, at azimuth angle, $\phi$, referred to the subsolar point and latitude $\theta$ referred to the ecliptic. At the point $(\phi_p, \theta_p)$ the radiation reflected from $dA$ is

$$S_A_s \cos \phi_p \cos \theta_p dA,$$

where $S$ is the solar flux at the planetary body; $A_s$ is the average albedo or reflectivity of the planetary body (0.39 for the earth) and

$$dA = (r_p \cos \theta_p d\phi_p)(r_p d\theta_p) = r_p^2 \cos \theta_p d\theta_p d\phi_p.$$

This allows the reflected radiation from $dA$ to be written as

$$S_A_s r_p^2 \cos^2 \theta_p \cos \phi_p d\theta_p d\phi_p.$$

Using Lambert's law of diffuse radiation, the fraction of this reflected flux intercepted by unit area of the satellite is

$$dS_I = S_A_s r_p^2 \cos^2 \theta_p \cos \phi_p d\theta_p d\phi_p \cdot \frac{\cos \psi}{r_{ps}^2},$$

where $\psi$ is the angle between the radii to the differential area, $dA$, and the satellite, and $r_{ps}$ is the distance between the satellite and the differential area, $dA$. $\cos \psi$ can be expressed as

$$\cos \psi = \cos(\phi_p - \phi) \cos(\theta_p - \theta).$$
This can be expanded using trigonometric relations to give

\[
\cos \psi = \cos \phi_p \cos \phi \cos \theta_p \cos \phi + \cos \phi_p \cos \sin \theta_p \sin \phi + 
\sin \phi_p \sin \phi \cos \theta_p \cos \phi + \sin \phi_p \sin \phi \sin \theta_p \sin \phi .
\]

Using the law of cosines, \( r_{ps}^2 \) is written as

\[
r_{ps}^2 = r^2 + r_p^2 - 2rr_p \cos \psi . \quad (35)
\]

The reflected radiant flux intercepted per unit area of the satellite can now be written as

\[
dS_t = \frac{\cos^2 \theta_p \cos \phi_p \cos \psi(\phi_p, \theta_p, \phi, \theta)}{r^2 + r_p^2 - 2rr_p \cos \psi(\phi_p, \theta_p, \phi, \theta)} \, d\theta_p \, d\phi_p . \quad (36)
\]

The total reflected flux received per unit area of the satellite is found by integrating over the sunlit face of the planet to get

\[
S_t(r, \phi, \theta) = \frac{\cos^2 \theta_p \cos \phi_p \cos \psi(\phi_p, \theta_p, \phi, \theta)}{r^2 + r_p^2 - 2rr_p \cos \psi(\phi_p, \theta_p, \phi, \theta)} \, d\theta_p \, d\phi_p .
\]

If for any element \( \cos \psi < 0 \), that element is invisible to the satellite and does not contribute any radiant flux. Therefore, if \( \cos \psi < 0 \), then set \( \cos \psi = 0 \). This double integral must be integrated numerically for each set of satellite coordinates, \( r, \phi, \theta \). \( \phi \) used in this analysis is referenced to the solar direction, whereas in the analysis up to this section, it was referred to the anti-solar direction. The double integral was solved using Simpson's rule by a computer program for the range of values of \( \phi \) and \( \theta \). It was found that the maximum reflected flux occurred at
\[ \phi = 0, \ \theta = 0, \] when the satellite was directly above the subsolar point. At this point the ratio of the reflected flux to the solar flux at the sailcraft distance, \( 4.2171 \times 10^7 \text{ m} \), was found to be \( S_r(r_i,0,0)/S = 0.0236. \) This would seem to justify my original assumption to neglect the reflected radiation from the earth. If the sailcraft had been given an initial orbit radius at half the above distance, the ratio of the reflected radiation to the solar flux would have been \( S_r(r_i,0,0)/S = 0.121, \) perhaps too large to neglect.

The sailcraft gets close enough to the moon to receive a very significant contribution of reflected radiation. However, the sailcraft is very close to the moon for only a few hours at most, and the variation of the sail's orientation is not synchronized to the sailcraft's position with respect to the moon. Reflected radiation from the moon could significantly affect the sailcraft's orbit only if the sailcraft were in a stable orbit about the moon and the sail control relations were synchronized with the lunar orbit. Neither of these conditions exist.

If the radiation force for the reflected radiation is to be included in the equations of motion, an expression similar to equation (1) can be obtained. In that equation the optical coefficient, \( Q \), the area to mass ratio, \( A/m \),
the speed of light, \( c \), and the normal vector \( n \), are
independent of the source of radiation and are unchanged. The quantity, \( S(r_c/r_s)^2 \), is replace by \( S_i(r_i, \theta, \phi) \).

So far only the magnitude of the resultant radiation force due to planetary reflection has been derived. The direction of this resultant force and the angle it makes with the normal to the sail must also be derived. The resultant radiative flux vector due to planetary reflection can be written as
\[
S_x = S_i'(\cos \xi_x i' + \cos \xi_y j' + \cos \xi_z k'), \tag{37}
\]
where \( \xi_x, \xi_y \), and \( \xi_z \) are the direction cosines and \( i', j', k' \)
are unit vectors related to unit vectors \( i, j, k \) by \( i' = -i, j' = -j, k' = k \).

The differential radiative flux vector can be written as
\[
dS_x = dS_i' \left[ \frac{(X - X_p)}{r_{ps}} i' + \frac{(Y - Y_p)}{r_{ps}} j' + \frac{(Z - Z_p)}{r_{ps}} k' \right], \tag{38}
\]
where \( r_{ps} \) is given by equation (35) and \( dS_i \) is given by equation (36) and
\[
x = r \cos \theta \cos \phi, \quad y = r \cos \theta \sin \phi, \quad z = r \sin \theta
x_p = r_p \cos \theta_p \cos \phi_p, \quad y_p = r_p \cos \theta_p \sin \phi_p, \quad z_p = r_p \sin \theta_p.
\]

If equation (38) is integrated similar to the total reflected flux equation and set equal to equation (37), the
direction cosines can be expressed as

\[
\cos \xi_x = \frac{S \rho_i r_p^2}{S_i(r, \phi, \theta)} \int_{-\pi/2}^{\pi/2} \int_{-\pi/2}^{\pi/2} (x-x_p) \cos^2 \theta \cos \phi \cos \psi \, d\theta \, d\phi_p,
\]

\[
\cos \xi_y = \frac{S \rho_i r_p^2}{S_i(r, \phi, \theta)} \int_{-\pi/2}^{\pi/2} \int_{-\pi/2}^{\pi/2} (y-y_p) \cos^2 \theta \cos \phi \cos \psi \, d\theta \, d\phi_p,
\]

\[
\cos \xi_z = \frac{S \rho_i r_p^2}{S_i(r, \phi, \theta)} \int_{-\pi/2}^{\pi/2} \int_{-\pi/2}^{\pi/2} (z-z_p) \cos^2 \theta \cos \phi \cos \psi \, d\theta \, d\phi_p.
\]

The cosine of the angle between the resultant radiative flux unit vector due to planetary reflection and the normal to the sail is given by the scalar product,

\[
\cos \gamma_p = -n_x \cos \xi_x - n_y \cos \xi_y + n_z \cos \xi_z,
\]

where \( n_x, n_y, n_z \) are the components of \( n \) given in equation (5). The negative signs in the first two terms arise from \( i \cdot i' = -1 \) and \( j \cdot j' = -1 \).

The radiation force due to the planet's reflection adds to the radiation force of the sun when \( \cos \xi_x < 0 \), and it subtracts from the radiation force of the sun when \( \cos \xi_x > 0 \). With these changes the radiation force due to the planet's reflection can be written as

\[
F_{rp} = \pm \frac{Q S_i(r, \phi, \theta) A}{c} \frac{A}{m} \cos^2 \gamma_p \mathbf{n}. \quad (+, \cos \xi_x < 0; -, \cos \xi_x > 0)
\]

If this term is added to the corresponding term due to the sun, a single radiation force term can be written as

\[
F_{r} = \frac{Q}{c} \frac{A}{m} \left[ S \left( \frac{x}{r_s} \right)^2 \cos^2 \gamma \pm S_i(r, \phi, \theta) \cos^2 \gamma_p \right] \mathbf{n}.
\]
CHAPTER 3 - NUMERICAL COMPUTATION

A. Numerical Integration Procedure

The systems of differential equations for the sailcraft and the moon were integrated numerically in state variable form using the fifth order Runge-Kutta method, also known as Butcher's method (James et. al., 1977). The Runge-Kutta type methods have good stability characteristics, and the step size can be changed as desired without any complications. Step size changes are often required in astro­nautical applications. Changes to smaller step sizes are required to enhance the accuracy of results at certain points in a trajectory and to maintain stability in the solution of the differential equations when the object of interest approaches a large attracting body. The nominal step size used was 1/40 of a day, but this was reduced by as much as a factor of ten when the sailcraft approached within a certain distance of the moon.

B. Search Method

A search was programmed to find the correct initial values of the angular position coordinates that would be required to attain a near lunar flyby or impact trajectory. For a given set of orbital parameters defining the sail­craft's initial orbit, the time variable is incremented by a constant amount, each time yielding a consistent set of
initial conditions and resulting trajectories. When the differences between the angular position coordinates of the sailcraft and the moon change sign for both angular position coordinates, the program bisects the latest interval of time to obtain the new time, based on its position in the orbit. Then the initial conditions are calculated from the equations derived earlier. After each bisection, the left and right limits of the current time interval are tested using the above criterion to determine which adjacent subinterval will be subsequently bisected. This procedure will continue until a trajectory is found that comes within a specified distance from the moon's center.

C. Computer Program Organization

A computer program was written in FORTRAN incorporating all the features previously discussed. The main program contains input values, the initial position coordinates and velocity components of the moon, and it calls a subroutine which calculates the initial position coordinates and velocity components of the sailcraft in both cartesian and spherical coordinates using specified orbit parameters. It also contains the search algorithm for finding the correct angle of departure as discussed in section 3-B. After the initial conditions are computed but prior to the search algorithm, a subroutine is called to compute the trajectory of the sailcraft and the orbit of the moon by solving the two systems of differential equations simultaneously using
Butcher's method. While in the BT subroutine, subroutines are called to compute the accelerations of the sailcraft and the moon, respectively, using the equations of motion in normalized form derived in section 2-C and 2-D. Output for a given quantity is directed to a separate output file in block form for use in computer plotting. A printout of the program is provided as an appendix. Values of the constant input parameters (Halliday and Resnick, 1974) are given in Table 1.

An attempt was made to verify the equations of motion and the computer by duplicating the results of Yu (1986). Yu used a two-dimensional model to compute flyby trajectories of the sun-earth transterrestrial libration point in the ecliptic plane. Setting the angles $\theta$ and $\alpha$ equal to zero in the equations of motion and removing the terms involving the lunar gravitational force, the equations of motion reduce to the two two-dimensional equations of Yu. The program in the appendix was run with $\theta$, $\alpha$ and the mass of the moon equal to zero, with the search routine modified to search for flyby trajectories of the sun-earth transterrestrial libration point. Close flyby trajectories of this point were achieved for area-to-mass ratios of 50, 100 and 200, and travel times were identical to those of Yu to the nearest day for all three cases.
<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gravitational constant (G)</td>
<td>$6.6704 \times 10^{-11}$ N·m²/kg⁻²</td>
</tr>
<tr>
<td>Speed of light (c)</td>
<td>$3.0 \times 10^8$ m/s</td>
</tr>
<tr>
<td>Mass of sun ($M_s$)</td>
<td>$1.9891 \times 10^{30}$ kg</td>
</tr>
<tr>
<td>Mass of earth ($M_e$)</td>
<td>$5.9764 \times 10^{24}$ kg</td>
</tr>
<tr>
<td>Mass of moon ($M_m$)</td>
<td>$7.347 \times 10^{22}$ kg</td>
</tr>
<tr>
<td>Average heliocentric distance of earth ($r_e$)</td>
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</tr>
<tr>
<td>Solar flux at 1 AU (S)</td>
<td>$1.353 \times 10^3$ J/m²-s</td>
</tr>
<tr>
<td>Radius of sun ($R_s$)</td>
<td>$6.955 \times 10^8$ m</td>
</tr>
<tr>
<td>Radius of earth ($R_e$)</td>
<td>$6.378 \times 10^6$ m</td>
</tr>
<tr>
<td>Radius of moon ($R_m$)</td>
<td>$1.738 \times 10^6$ m</td>
</tr>
<tr>
<td>Rotational period of earth ($P_e$)</td>
<td>$8.64 \times 10^4$ s</td>
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<tr>
<td>Optical constant (Q)</td>
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</tr>
<tr>
<td>Geocentric distance of SEL2 ($r_f$)</td>
<td>$1.5 \times 10^9$ m</td>
</tr>
<tr>
<td>Nominal time step ($\Delta t$)</td>
<td>1/40 days</td>
</tr>
</tbody>
</table>
CHAPTER 4 - RESULTS

A. Initial Orbit Parameters and Search Strategies

The task of getting the sailcraft from its initial circular orbit to the near vicinity of the moon depends on finding the correct initial conditions of the sailcraft and the moon. For the sailcraft two of the six initial conditions are known, the radial distance and the zero radial velocity due to the circular orbit. The remaining four initial conditions, the azimuth and zenith angles and the angular velocities are unknown. These initial conditions depend in turn on three parameters. One is the elapsed time from a reference point in the initial orbit to the time when the sail is deployed or, equivalently, an angle in the ecliptic plane from the earth-sun line in the antisolar direction. The second is the inclination of the orbit, which is the absolute value of the angle between the orbit and the ecliptic plane, and the third is the longitude of the ascending node which is defined in section 2-E. For the moon the three initial coordinates are given in astronomical almanacs. The initial velocity components are not given explicitly in the almanacs that I have access to. Therefore, it was necessary to use finite differences to calculate the initial velocities from the position coordinates given. Since inaccuracies could result from these approximations of the velocities, the above procedure for
computing the initial velocities was used only for the first day of the target month. The initial conditions for the remaining days of the target month were obtained by solving the equations of motion of the moon in section 2-D using the initial conditions obtained from the almanac for the first day of the target month and then printing out the results for each whole day and reading the data files into the computer program. Initial conditions of the moon for each day in the lunar cycle were needed because the sail must be initially deployed on an optimum date during the target month to make it possible for the sailcraft to intercept the moon. Thus, there must be four independent parameters that must be searched for, the three orbital parameters for the sailcraft and the correct date of the month corresponding to the location of the moon in its orbit. Section 3-B describes a search method for finding the correct time from a given reference point to the time the sail is deployed. This time will be referred to as the time of departure. Similar algorithms could be written to find correct values for the other three independent parameters using nested do loops in the program. However, it would be practically impossible to search systematically by computer through a four dimensional parametric space because of the astronomically large computing time that would be required. Therefore, it was necessary to develop and implement a strategy that would eliminate large portions of the four
dimensional parametric space. Before describing this strategy, it will be necessary to discuss a couple of preliminary items that will affect the diagnostic and final results. The only free parameter that will be investigated in this work is the ratio of the area of the sail to the mass of the sailcraft, which includes the sail and the small craft from which the sail is deployed and on which it remains attached. This was previously and will henceforth be referred to as the area to mass ratio (A/m). Three values will be used for this parameter in obtaining the results: 50, 100 and 200. These values bracket the values of the area to mass ratio that have been proposed for the various sailcraft designs for the Columbus 500 Space Sail Cup race that was discussed briefly in the introduction. It is not the purpose of this work to present and discuss the details of the sailcraft designs. However, it is worth mentioning that the sails will be made of materials such as aluminized mylar, which is very light, tough and reflective and will result in area to mass ratios within the range of values that will be used in this work. Since this race commemorates the 500th anniversary of Columbus' discovery of the Americas, it is scheduled to begin on or shortly after October 12, 1992, depending upon the physical constraints placed on the date by the position of the moon in its orbit. Therefore, the initial conditions of the moon that will be used to obtain the results in this work are those that are
found in the astronautical almanac for that date and those that are computed for one month past that date. This will include one full orbit of the moon so that all the azimuth angles of the moon relative to the earth can be tested by numerical computation as an initial condition that will make a sailcraft-moon encounter possible.

The first part of the strategy referred to earlier sought to reduce the number of possible starting dates within the target month. This was done by computing a sequence of orbits, one for each date of the target month and each with the initial orbit in the ecliptic plane (0° inclination). The ecliptic plane was chosen for the initial orbit so that the effects of the starting date could be isolated. The starting date is the date on which the sail is deployed, not the date of the launch, as the sailcraft is assumed to be orbiting without sail in a circular, geosynchronous orbit.

The principal result of these computer runs was the final or terminal azimuth angle of the sailcraft. This is the value of the azimuth angle when the sailcraft reaches the moon's instantaneous geocentric distance. The final azimuth angle of the sailcraft was plotted as a function of the initial azimuth angle or angle of departure of the sailcraft for the various dates during the target month. The relation between the final azimuth angles and the initial azimuth angles was found to be irregular primarily because of the nonlinear variation of the radiation pressure force with the initial
azimuth angle. In addition to this, the moon's geocentric distance constantly changes, varying by as much as $5 \times 10^4$ km. Since a sailcraft trajectory for these results is defined to terminate when the instantaneous lunar distance is reached, each trajectory represents a different distance. However, the important fact learned from this is that the range of variation of the final azimuth angle is small relative to 360°. All the final azimuth angles were in the first quadrant, and the maximum variation was less than 40° for any value of the area to mass ratio. However, the range of values of the final azimuth angle shifted to higher values within the first quadrant as the area to mass ratio increased. This means that the sailcraft cannot intercept the moon unless the moon's azimuth angle is within this narrow range of azimuth angles when the sailcraft reaches the moon's distance. The transfer time was found to be between 62 and 80 days for $A/m = 100$ and to vary with an approximate inverse linear relationship with area to mass ratio. For any given starting date and transfer time, the position of the moon at the time the sailcraft reaches the moon's distance can be determined from an astronautical almanac. The azimuth angle of the moon given in the almanac is based on the standard reference direction, the first point of Aries, whereas the azimuth angle computed by the program is referred to the constantly changing antisolar direction, which is the positive x axis of my chosen
coordinate system. The azimuth angle of the moon given in the almanac is converted to an azimuth angle referred to the antisolar direction using the relation

$$\phi_m = \pi + \phi_a - \phi_s,$$

where $\phi_a$ is the azimuth angle of the moon given in the almanac and $\phi_s$ is the longitude of the sun, also given in the almanac. Comparing the final azimuth angle of the sailcraft and $\phi_m$ computed for each starting date, one can determine whether the moon will be in the general vicinity of the sailcraft at the time the sailcraft reaches the moon's distance. Using this procedure, it was possible to rule out a significant fraction of the target month as suitable days on which to deploy the sail. This resulted in a reduction of the number of search trajectories by nearly 50%.

The second part of the strategy sought to identify suitable values of the inclination and the longitude of the ascending node (LAN) of the initial orbit of the sailcraft. To do this, computer runs were made for a given angle of departure for a range of initial orbit inclinations ranging from one to ten degrees and repeated for each of four values of the longitude of the ascending node. The four values corresponded to the positive and negative x axes (the antisolar and solar directions) and the positive and negative y axes. The principal diagnostic resulting from these runs was the ecliptic latitude, which is the angle of the sailcraft above or below the ecliptic plane when it reaches
the moon's instantaneous distance. The ecliptic latitude was plotted as a function of inclination for the four values of the LAN. The ecliptic latitude of the moon is also given in the astronomical almanac. The ecliptic latitude of the moon was compared to the final ecliptic latitude of the sailcraft for the terminal date of each trajectory corresponding to a suitable starting date determined in step one of the strategy and the transfer time. This would allow the final ecliptic latitude of the sailcraft to be near that of the moon when it reaches the moon's distance. By implementing this two part search strategy to greatly diminish the ranges of suitable inclinations, LANs and starting dates and then by systematically searching over the initial azimuth angle, near lunar flyby and impact trajectories were obtained as implicit functions of time.

B. Lunar Flyby and Impact Trajectories

All the results to be presented below have been obtained for a spectrally reflecting sail, which is the type of sail that has been commonly used by other investigators. Several computer runs were made for a black absorbing sail, and the results were unfavorable. Travel times were much longer than those obtained for a spectrally reflecting sail, by a factor of at least four or five. Problems of orbital instability were also encountered. Therefore it was decided to discontinue the computer runs for this type of sail.
The results first presented are for an area to mass ratio of 100. Orbits of the sailcraft and the moon are projected into the x-y plane for a lunar impact trajectory in figure 3. The initial orbit of the sailcraft is inclined at 7.5° to the ecliptic plane; the LAN is in the solar direction at 226.2° to the reference direction; the angle of departure is 275° from the antisolar earth-sun line, and the starting date is day 28 of the target month (a 30 day period following October 12, 1992). The transfer time is 63.5 days. In this and all plots of this type the sailcraft moves in a constantly expanding elliptical orbit, and it intercepts the moon in the first quadrant. The same orbits are projected onto the x-z plane in figure 4 and onto y-z plane in figure 5. There is nothing remarkable about these figures or in the x-z and y-z projections that will be presented for other cases. However, they illustrate the three dimensional nature of the problem, and a graphic representation would be incomplete without them. An enlargement of this plot in the vicinity of the moon is shown in figure 6. It shows the sailcraft crashing onto the trailing hemisphere of the moon. The velocity of the sailcraft relative to the moon is shown as a function of its distance from the moon in figure 7. The sailcraft is shown accelerating to its escape speed, about 2.3 km/s for the moon, before impact.

Orbits of the sailcraft and the moon for a leading hemisphere lunar flyby are projected onto the x-y plane in
Figure 3. Projection of the orbits of the sailcraft and the moon on the x-y plane for case 1.
Figure 4. Projections of the orbits of the sailcraft and the moon on the x-z plane for case 1.
Figure 5. Projections of the orbits of the sailcraft and the moon on the y-z plane for case 1.
Figure 6. Projection of the orbits of the sailcraft and the moon on the x-y plane for case 1 using enlarged scale in the vicinity of the moon.
Figure 7. Velocity of the sailcraft relative to the moon vs. distance from the moon for case 1.
The initial orbit of the sailcraft is inclined at $6^\circ$ to the ecliptic; the LAN is in the solar direction at $225.9^\circ$ to the reference direction; the angle of departure is $264.1^\circ$ from the antisolar earth-sun line, and the starting date is day 27 of the target month. The transfer time is 64.4 days. The same orbits are projected onto the $x$-$z$ plane in figure 9 and onto the $y$-$z$ plane in figure 10. An enlargement of figure 8 in the vicinity of the moon is shown in figure 11. It shows the sailcraft's trajectory curving around the moon until it intersects its initial path of approach to the moon. The approach trajectory first encountered the moon on its leading hemisphere, coming to within about 11,410 km of the moon's surface at the point where the geocentric distances of the sailcraft and the moon were equal. After this point the sailcraft was pulled around the moon toward the trailing hemisphere while at the same time being pulled closer to the moon, coming to within 5.6 km of the moon's surface on the trailing hemisphere. This particular trajectory was chosen for presentation because it was found to be approximately the closest trajectory intercepting the moon on its leading hemisphere that would make a complete flyby of the moon without ultimately crashing onto the moon. A trajectory only slightly closer to the moon on the side of the leading hemisphere was found to cause the sailcraft to impact the back side of the moon. The curving trajectory around the moon in figure 11 prompts
Figure 8. Projection of the orbits of the sailcraft and the moon on the x-y plane for case 2.
Figure 9. Projection of the orbits of the sailcraft and the moon on the x-z plane for case 2.
Figure 10. Projection of the orbits of the sailcraft and the moon on the y-z plane for case 2.
Figure 11. Projection of the orbits of the sailcraft and the moon on the x-y plane using an enlarged scale in the vicinity of the moon for case 2.
the question of what affect the moon might have on the subsequent trajectory of the sailcraft. This would not be a concern for a lunar impact, only for a flyby. To answer this question, the sailcraft's trajectory is extended well beyond the point of the lunar encounter, and it is shown, along with the lunar orbits, projected onto the x-y plane in figure 12. The encounter with the moon radically alters the direction of the sailcraft so that it travels at about 90° to its original direction at lunar encounter, as shown. The velocity of the sailcraft relative to the moon is shown as a function of its distance from the moon in figure 13. This shows the sailcraft acquiring a large increase in velocity due to its encounter with the moon. This type of interaction between a spacecraft and a planetary body is known as a gravity assist. The moon accelerates the sailcraft by its gravitational pull; and as long as the sailcraft does not impact the moon, there will be a large increase in velocity. The magnitude of velocity increase becomes greater as the distance of the closest approach decreases. However, as indicated above, there is a minimum distance of closest approach below which the sailcraft will impact the moon. The above description can be generalized to any spacecraft encountering any planetary body. The minimum distance of closest approach for a gravity assist depends on the mass of the planetary body and the relative velocity of the spacecraft at the encounter. Gravity assists are commonly used
Figure 12. Projection of the orbits of the sailcraft and the moon on the x-y plane for a time past the lunar encounter for case 2.
Figure 13. Velocity of the sailcraft relative to the moon vs. distance from the moon for case 2.
by space mission planners to increase the energy of spacecraft for more remote missions. A recent example is the flight of the Galileo spacecraft to Venus in order to get a gravity assist from Venus to enable it to go to Jupiter.

Orbits of the sailcraft and the moon are projected onto the x-y plane for a trailing hemisphere lunar flyby in figure 14. The initial orbit of the sailcraft is inclined at 5° to the ecliptic; the LAN is in the solar direction at 222.9° from the reference direction; the angle of departure is 202.9° from the antisolar earth-sun line, and the starting date is day 24 of the target month. The transfer time is 66.575 days. The same orbits are projected onto the x-z plane in figure 15 and onto the y-z plane in figure 16. An enlargement of figure 14 in the vicinity of the moon is shown in figure 17. It shows the sailcraft passing the moon on its trailing hemisphere. The distance of closest approach in this case is within 83 km of the moon's surface. An extension of the same trajectory for a considerable time past the lunar rendezvous point is shown in figure 18. The sailcraft is pulled gravitationally by the moon on a course that has a greater component in the direction of the moon's orbit motion but outside the moon's orbit. The velocity of the sailcraft relative to the moon is shown as a function of its distance from the moon in figure 19. This shows the sailcraft acquiring a significant increase in velocity due to its close flyby of the moon's trailing hemisphere. A comparison of
Figure 14. Projection of the orbits of the sailcraft and the moon on the x-y plane for case 3.
Figure 15. Projection of the orbits of the sailcraft and the moon on the x-z plane for case 3.
Figure 16. Projection of the orbits of the sailcraft and the moon on the y-z plane for case 3.
Figure 17. Projection of the orbits of the sailcraft and the moon on the x-y plane for case 3 using an enlarged scale in the vicinity of the moon.
Figure 18. Projection of the orbits of the sailcraft and the moon on the x-y plane for a time past the lunar encounter for case 3.
Figure 19. Velocity of the sailcraft relative to the moon vs. distance from the moon for case 3.
figures 13 and 19 is instructive. The trajectory approaching the moon on the trailing hemisphere (figure 19) has an approach velocity significantly greater than the one approaching on the leading hemisphere (figure 13) (about 55% greater at a distance of $1 \times 10^8$ km). However, the gravity assist for the trajectory approaching on the leading hemisphere is more effective, increasing the relative velocity by a factor of approximately three in comparison with the increase in the relative velocity for the trajectory approaching on the trailing hemisphere at $1 \times 10^8$ m. The final result is that the velocity of the latter is only slightly greater than that of the former at $1 \times 10^8$ m after encounter. The only significant difference resulting from the two types of gravity assists is the difference in the direction of the final trajectories rather than in the magnitude of the final velocities.

The following results are for an area to mass ratio of 50. Orbits of the sailcraft and the moon for a lunar impact trajectory are projected onto the x-y plane in figure 20. The initial orbit is inclined at $6^\circ$ to the ecliptic; the LAN is in the direction of the negative y axis at $302.9^\circ$ from the reference direction; the angle of departure is $18.1^\circ$ from the antisolar earth-sun line, and the starting date is day 25 of the target month. The transfer time is 123.475 days. An enlargement of this plot in the vicinity of the moon is shown in figure 21. It shows the sailcraft crashing
Figure 20. Projection of the orbits of the sailcraft and the moon on the x-y plane for case 4.
Figure 21. Projection of the orbits of the sailcraft and the moon on the x-z plane for case 4.
onto the leading hemisphere of the moon. The same orbits of
the sailcraft and the moon are projected onto the x-z plane
in figure 22 and onto y-z plane in figure 23. The velocity
of the sailcraft relative to the moon is shown as a function
of its distance from the moon in figure 24. This plot is
practically identical to figure 7.

Orbits of the sailcraft and the moon for a leading
hemisphere lunar flyby are projected onto the x-y plane in
figure 25. The initial orbit of the sailcraft is inclined
at 6° to the ecliptic; the LAN is in the direction of the
negative y axis at 312.9° from the reference direction; the
angle of departure is 348.3° from the antisolar earth-sun
line, and the starting date is day 24 of the target month.
The transfer time is 124.27 days. The same orbits are
projected onto the x-z plane in figure 26 and onto the y-z
plane in figure 27. An enlargement of figure 25 in the
vicinity of the moon is shown in figure 28. It shows the
sailcraft approaching the moon on its leading hemisphere.
When the sailcraft first reaches the moon's geocentric
distance it is about 10,760 km from the moon's surface. It
is then pulled gravitationally around the backside of the
moon and inward toward the moon until it attains a closest
approach distance of only 0.2 km from the moon's surface.
The extended trajectory of the sailcraft well beyond the
point of the lunar encounter is shown in figure 29. The
encounter with the moon alters the direction of the
Figure 22. Projection of the orbits of the sailcraft and the moon on the y-z plane for case 4.
Figure 23. Projection of the orbits of the sailcraft and the moon on the x-y plane for case 4 using an enlarged scale in the vicinity of the moon.
Figure 24. Velocity of the sailcraft relative to the moon vs. distance to the moon for case 4.
Figure 25. Projection of the orbits of the sailcraft and the moon on the x-y plane for case 5.
Figure 26. Projection of the orbits of the sailcraft and the moon on the x-z plane for case 5.
Figure 27. Projection of the orbits of the sailcraft and the moon on the y-z plane for case 5.
Figure 28. Projection of the orbits of the sailcraft and the moon on the x-y plane for case 5 using an enlarged scale in the vicinity of the moon.
Figure 29. Projection of the orbits of the sailcraft and the moon on the x-y plane for a time past the lunar encounter for case 5.
sailcraft by about 90° from its pre-lunar encounter heading, similar to that of figure 12. The velocity of the sailcraft relative to the moon is shown as a function of its distance from the moon in figure 30. This again shows the large increase in velocity due to the lunar encounter, similar to that of figure 13.

Orbits of the sailcraft and the moon for a trailing hemisphere lunar flyby trajectory are projected onto the x-y plane in figure 31. The initial orbit of the sailcraft is inclined at 6° to the ecliptic; the LAN is in the direction of the negative y axis at 302.9° from the reference direction; the angle of departure is 74.1° from the antisolar earth-sun line, and the starting date is day 14 of the target month. The transfer time is 133.475 days. The same orbits are projected onto the x-z plane in figure 32 and onto the y-z plane in figure 33. An enlargement of figure 31 in the vicinity on the moon is shown in figure 34. It shows the sailcraft passing the moon on its trailing hemisphere. The distance of closest approach is within 195 km of the moon's surface. The extended trajectory of the sailcraft for a considerable time past the lunar encounter is shown in figure 35. It shows the sailcraft being pulled gravitationally by the moon on a course that has a greater component in the direction of the moon's orbital motion but outside the moon's orbit. The velocity of the sailcraft relative to the moon is shown as a function of its distance
Figure 30. Velocity of the sailcraft relative to the moon vs. distance to the moon for case 5.
Figure 31. Projection of the orbits of the sailcraft and the moon on the x-y plane for case 6.
Figure 32. Projection of the orbits of the sailcraft and the moon on the x-z plane for case 6.
Figure 33. Projection of the orbits of the sailcraft and the moon on the y-z plane for case 6.
Figure 34. Projection of the orbits of the sailcraft and the moon on the x-y plane for case 6 using an enlarged scale in the vicinity of the moon.
Figure 35. Projection of the orbits of the sailcraft and the moon on the x-y plane for a time past the lunar encounter for case 6.
from the moon in figure 36. This shows only a small increase in relative velocity due to the gravitational assist of the moon, compared to that in figure 19. The maximum relative velocity attained is also considerably less than that shown in figure 19. Both of these differences can be attributed to the fact that the distance of closest approach to the moon's surface is about 2.3 times greater than that of the corresponding case for $\frac{A}{m} = 100$.

The following results are for an area to mass ratio of 200. Orbits of the sailcraft and the moon for a lunar impact trajectory are projected onto the x-y plane in figure 37. The initial orbit of the sailcraft is inclined at 3° to the ecliptic; the LAN is in the solar direction at 200° from the reference direction; the angle of departure is 216° from the antisolar earth-sun line, and the starting date is day 1 of the target month. The transfer time to impact is 32.975 days. The same orbits are projected onto the x-z plane in figure 38 and onto the y-z plane in figure 39. An enlargement of figure 37 in the vicinity of the moon is shown in figure 40. It shows the sailcraft crashing onto the front side of the moon slightly toward the trailing hemisphere. The velocity of the sailcraft relative to the moon is shown as a function of its distance from the moon in figure 41. This plot is also practically identical to the corresponding plots for the other area to mass ratios.
Figure 36. Velocity of the sailcraft relative to the moon vs. distance to the moon for case 6.
Figure 37. Projection of the orbits of the sailcraft and the moon on the x-y plane for case 7.
Figure 38. Projection of the orbits of the sailcraft and the moon on the x-z plane for case 7.
Figure 39. Projection of the orbits of the sailcraft and the moon on the y-z plane for case 7.
Figure 40. Projection of the orbits of the sailcraft and the moon on the x-y plane for case 7 using an enlarged scale in the vicinity of the moon.
Figure 41. Velocity of the sailcraft relative to the moon vs. distance to the moon for case 7.
Orbits of the sailcraft and the moon for a leading hemisphere lunar flyby are projected onto the x-y plane in figure 42. The initial orbit of the sailcraft is inclined at 3° to the ecliptic; the LAN is in the solar direction at 227° from the reference direction; the angle of departure is 342.5° from the antisolar earth sun-line, and the starting date is day 28 of the target month. The transfer time is 34.069 days. The same orbits are projected onto the x-z plane in figure 43 and onto the y-z plane in figure 44. An enlargement of figure 42 in the vicinity of the moon is shown in figure 45. It shows the sailcraft approaching the moon on its leading hemisphere. When the sailcraft first reaches the moon's geocentric distance, it is about 1900 km from the moon's surface. It is then pulled gravitationally around the backside of the moon and inward toward the moon until it reaches a closest approach distance of about 1.2 km from the moon's surface. The extended trajectory of the sailcraft for a considerable time past the lunar encounter is shown in figure 46. The nature of this extended trajectory is different from all others presented because it shows a double planetary encounter. The encounter with the moon already described provides a large gravity assist to the sailcraft, hurling it back in the general direction of the earth. Nearing the earth once again, it reaches a minimum geocentric distance approximately equal to its original distance in synchronous orbit, and then it
Figure 42. Projection of the orbits of the sailcraft and the moon on the x-y plane for case 8.
Figure 43. Projection of the orbits of the sailcraft and the moon on the x-z plane for case 8.
Figure 44. Projection of the orbits of the sailcraft and the moon on the y-z plane for case 8.
Figure 45. Projection of the orbits of the sailcraft and the moon on the x-y plane for case 8 using an enlarged scale in the vicinity of the moon.
Figure 46. Projection of the orbits of the sailcraft and the moon on the x-y plane for a time past the lunar encounter for case 8.
experiences a second gravity assist which sends it out of earth's sphere of influence in an approximate antisolar direction. The velocity of the sailcraft relative to the moon is shown as a function of its distance from the moon in figure 47. Again, it shows the significant increase in velocity due to the lunar gravity assist. The velocities of approach and recession are greater than those of the corresponding cases for the other area to mass ratios because the velocity is roughly directly proportional to the area to mass ratio.

Orbits of the sailcraft and the moon are projected onto the x-y plane for a trailing hemisphere lunar flyby in figure 48. The initial orbit of the sailcraft is inclined at 3° to the ecliptic; the LAN is in the solar direction at 200° from the reference direction; the angle of departure is 16.2° from the antisolar earth-sun line, and the starting date is day 29 of the target month. The transfer time to the moon is 33.425 days. The same orbits are projected onto the x-z plane in figure 49 and onto the y-z plane in figure 50. An enlargement of figure 48 in the vicinity of the moon is shown in figure 51. It shows the sailcraft passing the trailing hemisphere of the moon. The distance of closest approach is about 129 km. The extended trajectory of the sailcraft for a considerable time past the lunar encounter is shown in figure 52. It shows the sailcraft being pulled gravitationally by the moon on a course that has a greater
Figure 47. Velocity of the sailcraft relative to the moon vs. distance to the moon for case 8.
Figure 48. Projection of the orbits of the sailcraft and the moon on the x-y plane for case 9.
Figure 49. Projection of the orbits of the sailcraft and the moon on the x-z plane for case 9.
Figure 50. Projection of the orbits of the sailcraft and the moon on the y-z plane for case 9.
Figure 51. Projection of the orbits of the sailcraft and the moon on the x-y plane for case 9 using an enlarged scale in the vicinity of the moon.
Figure 52. Projection of the orbits of the sailcraft and the moon on the x-y plane for a time past the lunar encounter for case 9.
component in the direction of the moon's orbital motion but outside the moon's orbit. The velocity of the sailcraft relative to the moon is shown as a function of its distance from the moon in figure 53. It is very similar to that of the corresponding case for $A/m = 50$.

Comparing the sailcraft trajectories for the three values of the area to mass ratio, it can be seen that initial encounters on the moon's leading hemisphere result in one type of resultant trajectory while those on the trailing hemisphere produce a different type of resultant trajectory. Also, the major axes of the expanding elliptical orbits are oriented in different directions relative to the earth-sun line, which defines the $x$ axis of the earth centered coordinate system. The angle between the sun-earth line and the major axes of the expanding elliptical orbits increases as the area to mass ratio is increased. For $A/m = 50$, this angle is about 45°; for $A/m = 100$, it increases to about 60°, and for $A/m = 200$, it is near 75°. The value of the azimuth angle of the sailcraft when it intercepts the moon similarly increases as the area to mass ratio is increased. Transfer times are roughly inversely proportional to the area to mass ratio, but this relationship is only approximately linear. Transfer times can vary significantly for trajectories resulting from the same area to mass ratio. This is due primarily to the significant variation in the geocentric distance of the
Figure 53. Velocity of the sailcraft relative to the moon vs. distance to the moon for case 9.
moon. Two secondary causes are different angles of
departure and different angular positions of interception.

Near lunar flyby and impact trajectories could have been obtained for other combinations of the sailcraft's initial orbit parameters and the starting date. A complete search through the entire range of LANs would certainly have yielded other acceptable combinations of initial orbit parameters, but the time required to do this would have been prohibitively large. Even using the four values of LAN used in this work, other acceptable trajectories were obtained, but I decided to present only three solutions for each value of area to mass ratio representing three types of sailcraft-lunar interactions. Near flyby trajectories could have been obtained by varying the angle of departure by 1° or less from that of an impact trajectory, but these would not have been noticeably different from the impact trajectories. For this reason the flyby and impact trajectories presented were chosen to be noticeably different, having different starting dates and significantly different angles of departure. It is not the purpose of this work to present all or many lunar impact and near flyby trajectories. These results were presented to demonstrate the utility of the computer model, to show that it is possible through a search strategy to intercept a planetary body with a solar sailcraft without undue difficulty and to illustrate the three types of interactions between a sailcraft and the moon or a general
planetary body and their effects, in the case of the flybys, on the subsequent sailcraft trajectories. The computer program could be easily adapted to have the sailcraft intercept a planet, such as Mars, instead of the moon. The moon was chosen as the object of interception to minimize computing time. A summary of the variable input parameters for the nine sample trajectories presented and certain characteristic results are given in Table 2.

C. Lunar Orbits

One objective of this work was to obtain lunar orbits for the sailcraft. This was to be done by dismantling the sail at some point in the trajectory to reduce the velocity below a value that would allow the sailcraft to attain a lunar orbit. However, simply reducing the velocity of a sailcraft in its orbit to a prescribed value is not a sufficient condition for obtaining an orbit about the moon or any other body because it cannot guarantee that the velocity once reduced is sufficient to bring the sailcraft to the vicinity of the body of interest. It turns out that this is an insurmountable problem. I now show that it is impossible for a sailcraft without retro rocket brakes to enter into an elliptical orbit around any planetary body.

If the semimajor axis of an orbit is known, the total mechanical energy per unit mass of the object in orbit is
Table 2

<table>
<thead>
<tr>
<th>A/m</th>
<th>Day #</th>
<th>Incl.</th>
<th>L.A.N.</th>
<th>Departure Angle</th>
<th>Transfer Time (days)</th>
<th>Type of Encounter</th>
<th>Closest Distance (km)</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>28</td>
<td>7.5</td>
<td>226.95</td>
<td>275.0</td>
<td>63.5</td>
<td>impact</td>
<td>-</td>
</tr>
<tr>
<td>100</td>
<td>27</td>
<td>6.0</td>
<td>225.94</td>
<td>264.1</td>
<td>64.4</td>
<td>f-flyby</td>
<td>0.2</td>
</tr>
<tr>
<td>100</td>
<td>24</td>
<td>5.0</td>
<td>222.93</td>
<td>202.9</td>
<td>66.575</td>
<td>b-flyby</td>
<td>82.8</td>
</tr>
<tr>
<td>50</td>
<td>25</td>
<td>6.0</td>
<td>313.94</td>
<td>18.1</td>
<td>123.475</td>
<td>impact</td>
<td>-</td>
</tr>
<tr>
<td>50</td>
<td>24</td>
<td>6.0</td>
<td>312.93</td>
<td>348.3</td>
<td>124.27</td>
<td>f-flyby</td>
<td>5.6</td>
</tr>
<tr>
<td>50</td>
<td>14</td>
<td>6.0</td>
<td>302.9</td>
<td>77.1</td>
<td>133.40</td>
<td>b-flyby</td>
<td>28.9</td>
</tr>
<tr>
<td>200</td>
<td>1</td>
<td>3.0</td>
<td>200.0</td>
<td>216.0</td>
<td>32.915</td>
<td>impact</td>
<td>-</td>
</tr>
<tr>
<td>200</td>
<td>28</td>
<td>3.0</td>
<td>226.95</td>
<td>342.5</td>
<td>34.069</td>
<td>f-flyby</td>
<td>1.1</td>
</tr>
<tr>
<td>200</td>
<td>29</td>
<td>3.0</td>
<td>227.95</td>
<td>18.2</td>
<td>33.395</td>
<td>b-flyby</td>
<td>26.6</td>
</tr>
</tbody>
</table>
given by
\[ E = -\frac{GM}{2a}, \]
where \( M \) is the mass of the planetary body, and "\( a \)" is the semimajor axis of the orbit. The change in mechanical energy per unit mass of the sailcraft from the geosynchronous or any prescribed earth orbit to the vicinity of the moon is
\[ \Delta E_1 = -\frac{GM_\oplus}{2a_m} + \frac{GM_s}{2a_s} = \frac{GM_\oplus}{2} \left( \frac{1}{a_s} - \frac{1}{a_m} \right). \]
a\( s \) is the semimajor axis of the sailcraft in its initial orbit, and \( a_m \) is the semimajor axis of its orbit at the time that it intercepts the moon. \( a_m \) is not the same as the semimajor axis of the moon's orbit, although it should not be greatly different. Using appropriate values, \( \Delta E_1 \) is found to be \( 4.2 \times 10^6 \text{ J/kg} \) if \( a_m \) is equal to the nominal semimajor axis of the moon. This is the work that must be done by the radiation pressure force to bring the sailcraft from its initial earth orbit to the vicinity of the moon.
If the sailcraft goes into orbit around the moon, the change in mechanical energy per unit mass from that at its initial orbit is
\[ \Delta E_2 = -\frac{GM_\oplus}{2a_m} + \frac{GM_m}{2a_c} + \frac{GM_s}{2a_s}, \]
where the second term represents the orbital energy per unit
mass of the sailcraft in its orbit around the moon. It is easy to see that no matter what finite value is chosen for the semimajor axis, $a_c$, of the lunar orbit, $\Delta E_2 < \Delta E_1$. This shows that it always takes more energy for the sailcraft to get from its initial orbit to the vicinity of the moon than it does to additionally go into a lunar orbit. If the sail is withdrawn at a time that will allow the sailcraft's velocity to be reduced sufficiently to enter into a lunar orbit, the sailcraft will not have sufficient energy to reach the moon's distance. The effect of the sun on the orbital energy differences was not considered in this analysis because it would be very much smaller than the primary contributions.
CHAPTER 5 - DISCUSSION

Up to now it has been assumed that the angular momentum vector of the sailcraft could be oriented and maintained in a direction normal to the ecliptic plane and that the orientation of the sail could be controlled as a function of time in such a way that the control relations given in section 2-G could be satisfied. Although a detailed consideration of the design and control systems of a sailcraft is beyond the scope of this work, I will now briefly discuss some methods that can be used to control the angular momentum and the sail orientation of a solar sailcraft.

A solar sailcraft would be transported from Earth to space and deployed at the geosynchronous distance ( \( \approx 3.6 \times 10^4 \) km from Earth's Surface) in some initial orbit by a rocket. The initial orbit would be predetermined by analysis and computation to be one which would accomplish some mission objective, as has been done in this work. It is also very likely that the sailcraft could be deployed from the space shuttle with a prescribed attitude and spin. For this problem the sailcraft should be deployed so that the angular momentum vector is perpendicular to the ecliptic plane. The sail would not be deployed until the moon is at a prescribed orbital position on a given date and on that date only when the sailcraft reaches a prescribed azimuth angle. Both the starting date and initial azimuth angle are
predetermined by computation. The sail, when it is deployed, would be attached to a collapsible frame which is itself attached to a rotating shaft. According to my conception, this shaft should be coincident with or at least parallel to the spin axis of the sailcraft, which means that it must also point normal to the ecliptic plane. This restriction is necessary only because it is an assumption used in the development of my control relations (Section 2-G). Also, in order to eliminate or minimize solar radiation torques which would change the direction of the angular momentum vector, the sail should be flat and deployed so that it is symmetrical about the control craft. However, even with such safeguards, various minor effects such as meteoric impacts or energy dissipation due to the presence of non rigid internal components and the sail frame will cause the angular momentum vector to wander from its desired constant direction. It is also possible that the sailcraft may not be deployed from the space shuttle with the correct attitude. In either case some mechanism must be included in the control craft to adjust and maintain the angular momentum vector in the desired direction. The rotational angular velocity or spin of the sailcraft must also be controlled. A large spin velocity is undesirable because the rotating shaft, which controls the tilt angle of the sail, must rotate faster to compensate for the spin velocity in order to perform its function of maintaining the
correct time varying orientation of the sail with respect to the solar direction. A small minimum spin velocity is necessary, however, to maintain attitude stability. The sailcraft should be axisymmetric, with the axis having the maximum moment of inertia, the major axis, as the spin axis. This should be true both before and after sail deployment. This is necessary because a semirigid body, which dissipates energy, is stable only when spinning about its major axis.

Mechanisms for controlling the direction and the magnitude of the angular momentum vector are shown in figure 54 (Kaplan, 1976). Although no thrusters are permitted for propulsion or braking, small thrusters requiring little propellant are permitted for attitude and spin control. A thruster for attitude adjustment is located near the rim and parallel to the spin axis. This is labeled as the precession thruster in figure 54. To reorient the sailcraft this thruster is fired for a short interval through the angle $\Delta \phi$. The thrust is parallel to the spin axis, producing a torque impulse

$$I_{torque} = 2 \int_0^{\Delta \phi/2} FR \cos \phi \, d\phi = 2 \int_0^{\Delta \phi/2} FR \cos \phi \frac{d\phi}{\omega} = \frac{2FR}{\omega} \sin \left( \frac{\Delta \phi}{2} \right).$$

In this equation $F$ is the thrust force; $R$ is the distance of the thruster from the spin axis, and $\omega$ is the spin rate. Since the sailcraft is assumed to have a small spin rate for stability, only one precession thruster is necessary for complete attitude control through any plane. Spin thrusters are shown mounted to the lateral surface of the craft and
Figure 54. Diagram of a spacecraft attitude control system (from Kaplan (1976)).
directed normal to the spin axis. Two thrusters are required in order to adjust the spin rate in either sense. In this case the angular impulse to change the spin rate is simply \( I_{\text{spin}} = FR' \Delta t = I_3 (\omega' - \omega) \), when set equal to the change in the rotational angular momentum. \( R' \) is the radius of the craft; \( I_3 \) is the maximum moment of inertia; \( \omega' \) is the desired spin rate, and \( \Delta t \) is the impulse duration which is easily solved for.

The direction and magnitude of the angular momentum vector of a solar sailcraft can also be controlled without thrusting devices. The angular momentum vector can be changed by deploying and/or withdrawing the sail in segments in such a way that the deployed portion of the sail or sails is distributed asymmetrically around the control craft. This would produce a desired solar torque which if maintained for a certain time would rotate the sailcraft about a new axis, thus changing the resultant direction and magnitude of the angular momentum vector. By displaying portions of the sail asymmetrically in a given configuration for a given time, an angular impulse of any desired magnitude and direction can be produced to change the angular momentum. This method would eliminate the need for thrusters and the weight that goes with them, although the weight required for the thrusters would probably be only a small fraction of the total weight of the craft plus sail. The only drawback I can see for this method is that the
maneuvers would likely require a significantly longer time than if thrusters were used.

The only remaining control problem is controlling the time varying angle of the sail with respect to the sun in such a way that the control relations given in section 2-G can be physically realized. When the sailcraft is in its initial circular orbit about the earth, control of the sail orientation is easy. The shaft on which the sail or sails are mounted must only be made to rotate the sail assembly at half the angular rate of the sailcraft in its orbit around the earth in order to satisfy Sand's control law. This assumes that the sailcraft itself has no spin. The sailcraft will most likely have a small spin rate, and this must be taken into account so that the spin rate of the sailcraft minus the rotation rate of the sail mounting shaft, in whatever sense is necessary, results in the net rate stated above. As the sailcraft's orbit becomes ever larger and more eccentric, a radial component of velocity develops that is comparable to and at times larger than the transverse velocity component, so that the resultant angular velocity about the earth becomes increasingly nonconstant. This greatly complicates the control of the sail. The method I envisage to control the sail orientation is as follows. The position coordinates and velocity components of the sailcraft are computed as implicit functions of time at each time step. For trajectories that intercept the moon
in a desirable way, the position coordinates and velocity components for the entire trip to the moon and even beyond can be stored in a computer along with the corresponding times beginning at the time of sail deployment. The most important dependent variable for control purposes is the angular velocity component in the ecliptic plane, \( \phi(t) \). The rotation rate of the shaft on which the sail is mounted, \( \phi_r(t) \), must satisfy the relation, \( \phi_s \pm \phi_r(t) = \phi(t) \), where \( \phi_s \) is the spin rate of the sailcraft. The spin rate can be considered as a constant, although it can be changed if the angular momentum vector must be adjusted by methods already described. \( \phi_r(t) \) is explicitly given as \( \pm \phi_r(t) = \phi(t) - \phi_s \).

If \( \phi(t) > \phi_s \), the positive sign applies, and the shaft is rotated counterclockwise. If \( \phi(t) < \phi_s \), the negative sign applies, and the shaft is rotated clockwise. The values of \( \phi_r(t) \) can be computed by and stored in the computer. At any given time in the trajectory, the magnitude and sense of \( \phi_r(t) \) in the computer can be transmitted to an electric motor controlling the rate of rotation of the shaft by an electric signal or current proportional to the value of \( \phi_r(t) \) corresponding to the time into the orbit. This is repeated at each time step to give the correct rotation rate of the shaft at each chosen point throughout the trajectory.
CHAPTER 6 - CONCLUSIONS

A three dimensional analytical model of a spacecraft equipped with a sail to utilize the radiation pressure from the sun and under the influence of the gravitational forces of the sun, earth and moon has been developed. This model includes the following: (1) the development of the physics of radiation pressure; (2) the derivation of a unit vector describing the direction of the resultant radiation force for two types of sails; (3) the derivation of dimensional and dimensionless equations of motion for the sailcraft and the moon in an earth-centered coordinate system; (4) the development of generalized equations for the initial conditions of the sailcraft in terms of orbital parameters of the initial orbit; (5) an adaptation of Sands simple control function for controlling the attitude of a sail to a three dimensional system; (6) the development of a simple criterion for taking into account the eclipsing of the sailcraft by the earth, and (7) an analysis of reflection from planetary bodies and the contribution to the total radiation pressure. A computer program based on the above model and including a search method was developed and described. This computer program was used, together with a search strategy for searching through a four dimensional parametric space, to investigate the problem of transfer to the moon from a geosynchronous orbit.
The results obtained for the lunar transfer problem indicate the following conclusions:

(1) Specular reflecting sails are much more efficient than black absorbing sails in terms of transfer time. I cannot see any advantage in using black absorbing sails.

(2) Transfer to the near vicinity of the moon can only be accomplished using a three dimensional model and by knowing correct values of the orbital inclination, the longitude of the ascending node and the angle of departure (the azimuth angle at which the sail is deployed) for a circular initial orbit of prescribed radius.

(3) A successful transfer to the moon also requires knowledge of the correct day of any given target month, or the position of the moon in its orbit, on which to deploy the sail. The orbits of the sailcraft expand into the first quadrant of the earth-centered coordinate system (away from the sun and to the left of the earth-sun line in the antisolar direction) for specular reflecting sails. The sail must be deployed on the correct date, for a given set of initial orbit parameters, so that the moon will be in the first quadrant when the sailcraft reaches the lunar distance.

(4) Transfer times to the moon are approximately inversely proportional to the area to mass ratio of the
sailcraft (see table 2 for typical values for specular reflecting sails). The absence of an exact inverse linear dependence is due to the changing geocentric distance of the moon and also the angle of departure.

(5) Interactions of the sailcraft with the moon are of three basic types, impacts, flybys approaching the moon on its leading hemisphere and flybys approaching the moon on its trailing hemisphere. Impacts occur at a relative velocity equal to the escape velocity of the moon ~ 2.3 km/s. Both types of flybys result in gravity assists for the sailcraft. However, flybys approaching the moon on its leading hemisphere produce significantly greater increases in velocity than those approaching the moon on its trailing hemisphere. The directions of the resulting trajectories for the two types of flybys are also quite different as seen in the plots. Double gravity assists are also possible.

Under certain circumstances the trajectory of a sailcraft approaching the moon on its leading hemisphere can be altered in a way that takes the sailcraft back toward the earth. It will then encounter the earth on a hyperbolic orbit, receive a second gravity assist and escape from the earth's sphere of influence.

(6) The trajectories of the sailcraft interacting with and affected by the gravitational field of the moon or any planetary body can be controlled or "fine tuned" by
making small changes in the angle of departure. This can determine the velocity vector (speed and direction) after a lunar or planetary encounter, and it can be adjusted to meet specific mission requirements. One example would be to use the moon and the earth for gravity assists to go to Mars. It may also be necessary to use a specific value of the area to mass ratio in the design of the sailcraft if mission requirements depend heavily on the travel time. Unfortunately, the correct area to mass ratio and the correct angle of departure, in addition to other initial orbit parameters, cannot be determined a priori but can be found only after much numerical experimentation.
Appendix

Computer Program

DIMENSION V(6), VI(6), VN(6), VMI(6), VM(6), VMN(6), VMS(6,60),
1 F1(200), F2(200), F3(200), F4(200), F5(200), F6(200), F7(200)
1, API(40), APO(40), TRT(40), ATO(20), AIN(20)
DOUBLE PRECISION V, VI, VN, VMI, TM, PI, SM, EM, UM, G, SC, RES, DEL,
1 DELS, RC, RM, PSI, H, DIS, DSL, Q, AMR, DT, RI, VSO, PS, RME, RMS, RMM, RD, VD,
1 TD, OMS, DTO, T0, TM, RMO, PM, PHOM, THOM, AIM, PM, PHD, THD, PHD1, THD1,
1 PHD2, THD2, PHDS, THDS, PHR, PHMR, TS1, TS2, PHI, PHI1, PHI2, DPHI, VMS, ETA,
1 SL, A, ECM, APM, PSI, TRT, RME, RMS, RMM, RD, VD,
1 VR, VRE, RC, VC, VMO, AR, VAF, VA, VKC, VKT, WK, API, APO, ATO, AIN, TRT
1 FORMAT(' R=', D12.5, ' PH=', F9.4, ' TH=', F9.4, ' VR=', D12.5,
12X, ' VP=', D12.5, ' VT=', D12.5)
2 FORMAT(' RM=', D12.5, ' PM=', F9.4, ' THM=', F9.4, ' VMR=', D12.5,
1 ' VMP=', D12.5, ' VMT=', D12.5)
1 ' VY=', D12.5, ' VZ=', D12.5)
8 FORMAT(' TM=', D12.5, ' DIS=', D12.5, ' VR=', D12.5,
1 ' PHD=', F9.4, ' THD=', F9.4, ' WK=', D12.5)
10 FORMAT(' PHD1=', F9.4, ' PHD2=', F9.4)
11 FORMAT(' THD1=', F9.4, ' THD2=', F9.4)
12 FORMAT(' PHI1=', F9.4, ' PHI2=', F9.4)
13 FORMAT(' L=', I3, ' LAN=', F7.4, ' INC=', F7.4)
16 FORMAT(' WKC=', D12.5)
18 FORMAT(9F8.1)
19 FORMAT(T(10F7.2)
20 FORMAT(T(9F8.3)
21 FORMAT(T(10F7.1)
22 FORMAT(T(10F6.2)
24 FORMAT(6D12.5)
25 FORMAT(T(8F10.4)
23 FORMAT(' J=', I3)
50 FORMAT(' I=', I3)
G=6.6704D-11
C=3.0D8
SM=1.9891D30
EM=5.9764D24
UM=7.347D22
SC=1.353D3
RES=1.4959787D11
PI=3.141592654
DEL=5.15*PI/180.
RC=1.4959787D4
DSL=6.0*PI/180.
RE=6.378D6
RSN=6.955D8
RM=1.738D6
PR=8.64D4
H=3.5793D7
DSL=5.0D4
Q=2.0
AMR=5.0D1
DT=2.16D3
DI=0.
RI=RE+H
VSO=(G*EM(RI)**.5
OMS=VSO/RI
PS=2*PI/OMS
RME=EM/(SM+EM)
RMS=SM/(SM+EM)
RMM=UM/(SM+EM)
TD=RES**1.5/(G*(SM+EM))**.5
VD=(G*(SM+EM)/RES)**.5
RD=RES
OMZ=VD/RD
PM=2.4192D6
AR=54.0*PI/180.
M=6
IL=9000
LI=10
JI=60
N1=1
NC=0
DT0=PS/3600.
DPHI=PI/180.
OPEN(5,FILE='f1')
READ(5,24)(F1(J),J=1,J1)
CLOSE(5)
OPEN(5,FILE='f2')
READ(5,24)(F2(J),J=1,J1)
CLOSE(5)
OPEN(5,FILE='f3')
READ(5,24)(F3(J),J=1,J1)
CLOSE(5)
OPEN(5,FILE='f4')
READ(5,24)(F4(J),J=1,J1)
CLOSE(5)
OPEN(5,FILE='f5')
READ(5,24)(F5(J),J=1,J1)
CLOSE(5)
OPEN(5,FILE='f6')
READ(5,24)(F6(J),J=1,J1)
CLOSE(5)
OPEN(5,FILE='f7')
READ(5,20)(F7(J),J=1,J1)
CLOSE(5)
DO 15 J=24,24
WRITE(6,23)J
SL=F7(J)*PI/180.
PSI=SL+PI/2.
IF(PHI.GT.2*PI)PSI=PSI-2*PI
VMS(1,J)=F1(J)
VMS(2,J)=F2(J)
VMS(3,J)=F3(J)
VMS(4,J)=F4(J)
VMS(5,J)=F5(J)
VMS(6,J)=F6(J)
ETA=PI+PSI-SL
ETAM=PSI-SL
NREVP=0
NMREVP=0
DO 17 N=I,N1
T0=70.1*PS/360.
PHI=70.1*PI/180.
DO 3 L=1,L1
WRITE(6,13)L,PSI,DLS
DO 14 K=1,M
14 VM(K)=VMS(K,J)
VMN(1)=VM(1)/RD
VMN(2)=VM(2)
VMN(3)=VM(3)
VMN(4)=VM(4)/VD
VMN(5)=VM(5)/OMZ
VMN(6)=VM(6)/OMZ
PHMR=VM(2)
CALL ICS(VI,V, VN, RD, RI, OMS, OMZ, 1u, DLS, ETA, PHI, PI)
PHR=V(2)
TM=0.
WRITE(6,1) V(1),V(2), V(3), V(4), V(5), V(6)
WRITE(6,2) VM(1), VM(2), VM(3), VM(4), VM(5), VM(6)
API(L)=V(2)*180./PI
CALL BT(V, VN, VM, VMN, PHD, DT, PI, RME, RMS, RMM, OMZ, RD, VD, TD, DIS, DSL, TM, 
1 Q, SC, RES, C, AMR, RE, RM, RSN, VMS, PHR, PHMR, PHC, PHMC, VR, VRE, WKC, WK, J, J1, 
1 L, M, L1, L)
THD=V(3)-VM(3)
WRITE(6,1) V(1), PHC, V(3), V(4), V(5), V(6)
WRITE(6,2) VM(1), PHMC, VM(3), VM(4), VM(5), VM(6)
WRITE(6,8) TM, DIS, VR, PHD, THD, WK
APO(L)=PHC*180./PI
TRT(L)=TM/8.64E4
ARG=V(2)/(2*PI)
NREV=INT(ARG)
ARGM=VM(2)/(2*PI)
NMREV=INT(ARGM)
IF((NC.EQ.0).AND.(L.GT.2))THEN
PHD2=PHD1
THD2=THD1
ENDIF
IF((NC.EQ.0).AND.(L.GT.1))THEN
PHD1=PHD
THD1=THD
ENDIF
IF((NC.EQ.1).AND.((PHD*PHD1).LT.0.))THEN
PHDS=PHD1
THDS=THD1
ENDIF
IF((NC.EQ.1).AND.((PHD*PHD2).LT.0.))THEN
PHD1=PHD
THD1=THD
ENDIF
IF(L.EQ.1)THEN
TSl=TO
PHI1=PHI
T0=T0+DT0
PHI=PHI+DPHI
ENDIF
IF(L.EQ.1)GO TO 7
IF(NC.EQ.1)GO TO 9
IF((PHD*PHDl).GT.0.)THEN
TS2=TS1
PHI2=PHI1
TS1=T0
PHI1=PHI
PHI=PHI+DPHI
T0=T0+DT0
ENDIF
IF((PHD*PHD1).LT.0.)GO TO 7
IF((NREV.NE.NREVp).AND.(NMREV.EQ.NMREVp)).OR.(NREV.EQ.NREVp)
1.AND.(NMREV.NE.NMREVp))GO TO 7
9 IF((PHD*PHD1).LT.0.)THEN
TS2=TS1
PHI2=PHI1
TS1=T0
PHI1=PHI
T0=(TSl+TS2)/2.
PHI=(PHI1+PHI2)/2.
NC=1
ENDIF
IF((PHD*PHD1).LT.0.)GO TO 7
IF((PHD*PHD2).LT.0.)THEN
TS1=T0
PHI1=PHI
T0=(TSl+TS2)/2.
PHI=(PHI1+PHI2)/2.
ENDIF
7 NREVp=NREV
NMREVp=NMREV
IF((DIS.GT.RM_.AND.(DIS.LT.(RM+DSL)))GO TO 5
3 CONTINUE
AIN(N)=DLS*180./PI
ATO(N)=V(3)*180./PI
17 DLS=DLS+DI
15 CONTINUE
GO TO 5
OPEN(6,FILE='api')
WRITE(6,25)(API(L),L=I,L1)
CLOSE(6)
OPEN(6,FILE='apo')
WRITE(6,25)(APO(L),L=I,L1)
CLOSE(6)
OPEN(6,FILE='trt')
WRITE(6,20)(TRT(L),L=I,L1)
CLOSE(6)
OPEN(6,FILE='ain')
WRITE(6,25)(AIN(N),N=I,N1)
CLOSE(6)
OPEN(6,FILE='ato')
WRITE(6,25)(ATO(N),N=I,N1)
CLOSE(6)
STOP
END
SUBROUTINE ICS(VI,V,VN,RD,VD,RI,OMS,OMZ,T0,DLs,ETA,PHI,PI)
DIMENSION VI(6),V(6),VN(6)
DOUBLE PRECISION VI,V,VN,RD,VD,TO,OMS,OMZ,DLs,ETA,AN
VI(1)=RI*(DCOS(OMS*TO)*DCOS(ETA)-DSIN(OMS*TO)*DSIN(ETA)*
1DCOS(DLS))
VI(2)=RI*(DCOS(OMS*TO)*DSIN(ETA)+DSIN(OMS*TO)*DCOS(ETA)*
1DCOS(DLS))
VI(3)=RI*DSIN(OMS*TO)*DSIN(DLS)
VI(4)=RI*OMS*(-DSIN(V(3))*DCOS(V(2))-DSIN(V(3))*DSIN(V(2)))+
1OMS(DLs))
VI(5)=RI*OMS*(-DSIN(V(3))*DSIN(DLS)+DCOS(V(3))*DCOS(V(2)))+
1DCOS(DLS))
VI(6)=RI*OMS*DCOS(OMS*TO)*DSIN(DLS)
AN=DATAN(VI(2)/VI(1))
IF(VI(1).GT.0.).AND.(VI(2).GT.0.))PI=AN
IF(VI(1).GT.0.).AND.(VI(2).LT.0.))PI=2*PI+AN
IF(VI(1).LT.0.)PI=AN+PI
1 V(I)=(VI(I)**2+VI(2)**2+VI(3)**2)**.5
V(2)=PHI
V(3)=DASIN(VI(3)/RI)
V(4)=VI(4)*DCOS(V(3))*DCOS(V(2))+VI(5)*DCOS(V(3))*DSIN(V(2))+
1VI(6)*DSIN(V(3))
V(5)=(VI(5)*DCOS(V(2))-VI(4)*DSIN(V(2)))/(RI*DCOS(V(3)))-OMZ
V(6)=(VI(4)*DSIN(V(3))*DCOS(V(2))-VI(5)*DSIN(V(3))*DSIN(V(2))+
1VI(6)*DCOS(V(3)))/RI
VN(1)=V(1)/RD
VN(2)=V(2)
VN(3)=V(3)
VN(4)=V(4)/VD
VN(5)=V(5)/OMZ
VN(6)=V(6)/OMZ
RETURN
END
SUBROUTINE BT(V,VN,VM,VMN,PHD,DT,PI,RME,RMS,RMM,OMZ,RD,VD,HD,TD,
1DIS,DSL,SC,RES,C,AMR,RE,RSN,VM,PHR,PHRM,PHC,PHMC,VR,
1VRE,WKC,WK,J,J1,L,M,I)
DIMENSION V(6),VM(6),VN(6),VMN(6),V1(6),V2(6),V3(6),V4(6),V5(6),
1V6(6),D1(6),D2(6),D3(6),D4(6),D5(6),D6(6),AC(3),VM1(6),VM2(6),
DOUBLE PRECISION V, VM, VN, VMN, V1, V2, V3, V4, V5, V6, D1, D2, D3, D4, D5, D6, ACM, VM1, VM2, VM3, VM4, VM5, VM6, DM1, DM2, DM3, DM4, DM5, DM6, ACM, PI, RMS, RME, VMM, CRP, DT, DTN, OMZ, RD, VR, TD, ALP, BET, GAM, FK, FH, FKM, PHR, PHP, PHMR, PHMP, PHD, DM, TM, DSL, DIS, RX, X, Q, SC, RES, CA, MR, RE, RM, RSN, DMA, VMS, W7, W8, W9, TMW, PHC, PHMC, DMP, VXR, VYR, YS, ZS, HS, D, YP, ALAT, FA

1VZR, VR, VA, TMP, X, Y, Z, XM, YM, ZM, VX, VY, VZ, PN, PNV, CRP, DK, WK, WKC, PNW, VRE


24 FORMAT(6E13.5)

29 FORMAT('O I=', I6)

50 FORMAT('O I=', I3)

IC=0
IMS=0
IS=1
J=1
TM=0.
TMP=0.
NR=0
NRM=0
PH=0.
PHMP=0.
PN=1.0D8
PNV=1.0D3
PNW=1.0D6
PHR=VMN(2)-2*PI*NR
PHMR=VMN(2)-2*NRM*PI
RMF=3.734E8
DTN=DT/TD
DTN1=DTN/50.
DO 3 I=1, IL
SC=1.353D3
XS=V(1)*DCOS(V(3))*DCOS(PHR)
IF(XS.LT.0) GO TO 31
YS=V(1)*DCOS(V(3))*DSIN(PHR)
ZS=V(1)*DSIN(V(3))
HS=RES*RE/(RSN-RE)
RC=(RES*RE-EXS*(RSN-RE))/RES
RD=(RES*RE+EXS*(RSN+RE))/RES
RX=V(1)*((DCOS(V(3))*DSIN(PHR))**2+DSIN(V(3))**2)**.5
R1=RSN*XS/(RES+XS)
R2=RE
D=(Y**2+ZS**2)**.5*(RES/(RES+XS))
YP=(R1**2-R2**2+D**2)/(2*D)
IF(D+R2).LE.R1)AI=PI*RE**2
IF(ABS(RE+R2).GT.(R1-R2)))AI=R1**2*(PI/2.-ASIN((YP/R1)+
1((YP/R1**2)*((R1**2-YP**2)**.5)+R2**2*(PI/2.+ASIN(YD/R2)+
...

\[
1(\frac{\text{YP}-D}{R2**2})\cdot(\frac{\text{R2**2}}{\text{YP}-D})\cdot(\text{YP}-D)^{**2}
\]
\[
\text{AT}=\pi\left(\frac{\text{RSN}\times\text{XS}}{\text{RES}+\text{XS}}\right)^{**2}
\]
\[
\text{FA}=1-\text{AI}/\text{AT}
\]
\[
\text{IF}((\text{XS},\text{LT},\text{HS}),\text{AND},(\text{D},\text{LT},\text{RC}))\text{SC}=0.
\]
\[
\text{IF}((\text{XS},\text{LT},\text{HS}),\text{AND},(\text{D},\text{GT},\text{RC}),\text{AND},(\text{D},\text{LT},\text{RD})),\text{OR}.
\]
\[
1((\text{XS},\text{GT},\text{HS}),\text{AND},(\text{D},\text{RT},\text{RD})))\text{SC}=\text{FA}*1.353\text{D3}
\]
\[
\text{CRPD}=\text{Q}\times\text{SC}\times\text{AMR}/(\text{C}\times\text{RES}\times\text{OMZ}^{**2})
\]
\[
\text{CRP}=\text{Q}\times\text{SC}\times\text{AMR}/(\text{C}\times\text{RES}\times\text{OMZ}^{**2})
\]
\[
\text{DO} \ 4 \ K=1, M
\]
\[
\text{VM1}(K)=\text{VMN}(K)
\]
\[
\text{V1}(K)=\text{VN}(K)
\]
\[
\text{CALL AICL(VM1,RMS,RME,MM,FKM,PI,PHMR,PHMP,PHMC,NRM,ACM)}
\]
\[
\text{CALL ACS(V1,VM1,RMS,RME,MM,ALP,BET,GAM,FH,FK,CRP,DM,PI,PHR,}
\]
\[
1\text{PHP,PHMR,PHC,PHMC,PHD,SR,AC)}
\]
\[
\text{DO} \ 5 \ K=1, M
\]
\[
\text{IF}(K.LE.M/2)\text{D1}(K)=V1(K+M/2)
\]
\[
\text{IF}(K.GT.M/2)\text{D1}(K)=\text{AC}(K-M/2)
\]
\[
\text{IF}(K.LE.M/2)\text{DM1}(K)=\text{VM1}(K+M/2)
\]
\[
\text{IF}(K.GT.M/2)\text{DM1}(K)=\text{ACM}(K-M/2)
\]
\[
\text{DO} \ 7 \ K=1, M
\]
\[
\text{VM2}(K)=\text{VM1}(K)+(\text{DTN}/4.)*\text{DM1}(K)
\]
\[
\text{V2}(K)=V1(K)+(\text{DTN}/4.)*D1(K)
\]
\[
\text{CALL AICL(VM2,RMS,RME,MM,FKM,PI,PHMR,PHMP,PHMC,NRM,ACM)}
\]
\[
\text{CALL ACS(V2,VM2,RMS,RME,MM,ALP,BET,GAM,FH,FK,CRP,DM,PI,PHR,}
\]
\[
1\text{PHP,PHMR,PHC,PHMC,PHD,SR,AC)}
\]
\[
\text{DO} \ 8 \ K=1, M
\]
\[
\text{IF}(K.LE.M/2)\text{D2}(K)=V2(K+M/2)
\]
\[
\text{IF}(K.GT.M/2)\text{D2}(K)=\text{AC}(K-M/2)
\]
\[
\text{IF}(K.LE.M/2)\text{DM2}(K)=\text{VM2}(K+M/2)
\]
\[
\text{IF}(K.GT.M/2)\text{DM2}(K)=\text{ACM}(K-M/2)
\]
\[
\text{DO} \ 10 \ K=1, M
\]
\[
\text{VM3}(K)=\text{VM1}(K)+(\text{DTN}/8.)*\text{DM1}(K)+\text{DM2}(K)
\]
\[
\text{V3}(K)=V1(K)+(\text{DTN}/8.)*D1(K)
\]
\[
\text{CALL AICL(VM3,RMS,RME,MM,FKM,PI,PHMR,PHMP,PHMC,NRM,ACM)}
\]
\[
\text{CALL ACS(V3,VM3,RMS,RME,MM,ALP,BET,GAM,FH,FK,CRP,DM,PI,PHR,}
\]
\[
1\text{PHP,PHMR,PHC,PHMC,PHD,SR,AC)}
\]
\[
\text{DO} \ 12 \ K=1, M
\]
\[
\text{IF}(K.LE.M/2)\text{D3}(K)=V3(K+M/2)
\]
\[
\text{IF}(K.GT.M/2)\text{D3}(K)=\text{AC}(K-M/2)
\]
\[
\text{IF}(K.LE.M/2)\text{DM3}(K)=\text{VM3}(K+M/2)
\]
\[
\text{IF}(K.GT.M/2)\text{DM3}(K)=\text{ACM}(K-M/2)
\]
\[
\text{DO} \ 14 \ K=1, M
\]
\[
\text{VM4}(K)=\text{VM1}(K)+(\text{DTN}/2.)*(\text{DM1}(K)+\text{DM2}(K))
\]
\[
\text{V4}(K)=V1(K)+(\text{DTN}/2.)*D1(K)+D2(K)
\]
\[
\text{CALL AICL(VM4,RMS,RME,MM,FKM,PI,PHMR,PHMP,PHMC,NRM,ACM)}
\]
\[
\text{CALL ACS(V4,VM4,RMS,RME,MM,ALP,BET,GAM,FH,FK,CRP,DM,PI,PHR,}
\]
\[
1\text{PHP,PHMR,PHC,PHMC,PHD,SR,AC)}
\]
\[
\text{DO} \ 16 \ K=1, M
\]
\[
\text{IF}(K.LE.M/2)\text{D4}(K)=V4(K+M/2)
\]
\[
\text{IF}(K.GT.M/2)\text{D4}(K)=\text{AC}(K-M/2)
\]
\[
\text{IF}(K.LE.M/2)\text{DM4}(K)=\text{VM4}(K+M/2)
IF(K.GT.M/2)DM4(K)=ACM(K-M/2)
12 CONTINUE
   DO 13 K=1,M
      VM5(K)=VM1(K)+(DTN/16.)*((3*D1(K)+9)*DM4(K))
   13 V5(K)=V1(K)+(DTN/16.)*((3*D1(K)+9)*D4(K))
   CALL ACL(V5,VM5,RMS,RME,RMM,FKM,PI,PHMR,PHMP,PHMC,NRM,ACM)
   CALL ACS(V5,VM5,RMS,RME,RMM,ALP,BET,GAM,FH,FK,CRP,DM,PI,PHR,IPHP,PHMR,PHC,PHMC,PHD,NR,AC)
   DO 14 K=1,M
      IF(K.LE.M/2)D5(K)=V5(K+M/2)
      IF(K.GT.M/2)D5(K)=AC(K-M/2)
      IF(K.LE.M/2)DM5(K)=VM5(K+M/2)
      IF(K.GT.M/2)DM5(K)=ACM(K-M/2)
   14 CONTINUE
   DO 15 K=1,M
      VM6(K)=VM1(K)+(DTN/7.)*(-3*DM1(K)+2*DM2(K)+12*DM3(K)-12*DM4(K)+8*DM5(K))
   15 V6(K)=V1(K)+(DTN/7.)*(-3*D1(K)+2*D2(K)+12*D3(K)-12*D4(K)+8*D5(K))
   CALL ACL(V6,VM6,RMS,RME,RMM,FKM,PI,PHMR,PHMP,PHMC,NRM,ACM)
   CALL ACS(V6,VM6,RMS,RME,RMM,ALP,BET,GAM,FH,FK,CRP,DM,PI,PHR,IPHP,PHMR,PHC,PHMC,PHD,NR,AC)
   DO 16 K=1,M
      IF(K.LE.M/2)D6(K)=V6(K+M/2)
      IF(K.GT.M/2)D6(K)=AC(K-M/2)
      IF(K.LE.M/2)DM6(K)=VM6(K+M/2)
      IF(K.GT.M/2)DM6(K)=ACM(K-M/2)
   16 CONTINUE
   PHP=PHR
   PHMP=PHMR
   DO 17 K=1,M
      VMN(K)=VM1(K)+(DTN/90.)*((7*DM1(K)+32*DM3(K)+12*DM4(K)+32*DM5(K)+7*DM6(K))
   17 VN(K)=V1(K)+(DTN/90.)*((7*D1(K)+32*D3(K)+12*D4(K)+32*D5(K)+7*D6(K))
   DO 18 K=1,M
      IF(K.EQ.1)V(K)=VN(K)*RD
      IF(K.EQ.2)V(K)=VN(K)
      IF(K.EQ.3)V(K)=VN(K)
      IF(K.EQ.4)V(K)=VN(K)*VD
      IF(K.EQ.5)V(K)=VN(K)*OMZ
      IF(K.EQ.6)V(K)=VN(K)*OMZ
      IF(K.EQ.1)VM(K)=VMN(K)*RD
      IF(K.EQ.2)VM(K)=VMN(K)
      IF(K.EQ.3)VM(K)=VMN(K)
      IF(K.EQ.4)VM(K)=VMN(K)*VD
      IF(K.EQ.5)VM(K)=VMN(K)*OMZ
      IF(K.EQ.6)VM(K)=VMN(K)*OMZ
   18 CONTINUE
   VA=(V(4)**2+(V(1)*V(5)*DCOS(V(3)))*V(4)**2+(V(1)*V(6))**2)**.5
   VX=V(4)*DCOS(V(3))*DCOS(VM(3))**2-V(1)*V(6)*DSIN(V(3))**2*DCOS(VM(3))**2
   VX=V(4)*DCOS(V(3))*DCOS(VM(3))**2-V(1)*V(6)*DSIN(V(3))**2*DCOS(VM(3))**2
   VY=V(4)*DCOS(V(3))*DSIN(VM(3))**2-V(1)*V(6)*DSIN(V(3))**2*DSIN(VM(3))**2
   VY=V(4)*DCOS(V(3))*DSIN(VM(3))**2-V(1)*V(6)*DSIN(V(3))**2*DSIN(VM(3))**2
   VZ=V(4)**2*DCOS(VM(3))**2-V(1)*V(6)**2*DSIN(VM(3))**2
   VZ=V(4)**2*DCOS(VM(3))**2-V(1)*V(6)**2*DSIN(VM(3))**2
1V(1)*V(5)*DCOS(V(3))*DCOS(PHC)-VM(4)*DCOS(VM(3))*DSIN(PHMC)+
1VM(1)*VM(6)*DSIN(VM(3))*DSIN(PHMC)-VM(1)*VM(5)*DCOS(VM(3))*
1DCOS(PHMC)+OMZ*(V(1)*DCOS(V(3))*DCOS(PHC)-VM(1)*DCOS(VM(3)))*
1DCOS(PHMC))
VZR=V(4)*DSIN(V(3))+V(1)*V(6)*DCOS(V(3))-VM(4)*DSIN(VM(3))-
1VM(1)*VM(6)*DCOS(VM(3»)
VR=(VXR**2+VYR**2+VZR**2)**.5
IF(VR.GT.VRE)VR=VRE
GO TO 21
IF((J.NE.1).OR.(L.NE.1))GO TO 21
IF(J1.GT.J1-1)GO TO 21
IF(LNE.40*IMS+1)GO TO 21
DO 22 K=1,M
22 VMS(K,JI+1)=VM(K)
IMS=IMS+1
JI=JI+1
21 TM=TM+DTN*TD
IF((TMP.EQ.0.).AND.(V(1).GT.VM(1)))TMP=TM
DMA=DM*RD
IF(DMA.LT.RM)DMA=RM
IF(V(1).GT.VM(1))DIS=DMA
GO TO 30
X(I)=V(1)*DCOS(V(3))*DCOS(V(2))/PN
Y(I)=V(1)*DCOS(V(3))*DSIN(V(2))/PN
Z(I)=V(1)*DSIN(V(3))/PN
XM(I)=VM(1)*DCOS(VM(3))*DCOS(VM(2))/PN
YM(I)=VM(1)*DCOS(VM(3))*DSIN(VM(2))/PN
ZM(I)=VM(1)*DSIN(VM(3))/PN
W7(I)=DMA/PN
W8(I)=VR/PNV
W9(I)=VA/PNV
30 IF(DMALT.RM)GO TO 26
IF(V(1).GT.VM(1).OR.(DMA.LE.RM))GO TO 25
DMP=DMA
IP=I
3 CONTINUE
25 IP=I
WRITE(6,29)IP
GO TO 28
OPEN(6,FILE='x')
WRITE(6,24)(X(I),I=1,IP)
CLOSE(6)
OPEN(6,FILE='y')
WRITE(6,24)(Y(I),I=1,IP)
CLOSE(6)
OPEN(6,FILE='z')
WRITE(6,24)(Z(I),I=1,IP)
CLOSE(6)
OPEN(6,FILE='xm')
WRITE(6,24)(XM(I),I=I,IP)
CLOSE(6)
OPEN(6,FILE='ym')
WRITE(6,24)(YM(I),I=I,IP)
CLOSE(6)
OPEN(6,FILE='zm')
WRITE(6,24)(ZM(I),I=I,IP)
CLOSE(6)
OPEN(6,FILE='dm')
WRITE(6,24)(W7(I),I=I,IP)
CLOSE(6)
OPEN(6,FILE='vr')
WRITE(6,24)(W8(I),I=I,IP)
CLOSE(6)
OPEN(6,FILE='va')
WRITE(6,24)(W9(I),I=I,IP)
CLOSE(6)
RETURN
END
SUBROUTINE ACS(V,VM,RMS,RME,RMM,ALP,BET,GAM,FH,FK,CRP,DM,PI,
IPHR,PHP,PHMR,PHC,PHMC,PHD,NR,AC)
DIMENSION V(6),VM(6),AC(3)
DOUBLE PRECISION V, VM,RMS,RME,RMM,ALP,BET,GAM,FH,FK,CRP,DM,PI,
IPHR,PHP,PHMR,PHC,PHMC,PHD,AC,ARG1,ARG2,X,Y,AN
PHR= V(2)-2*NR *PI
IF( PHR.LT.PI/2.).AND.(PHR.GT.PI/2.).NR=NR+1
X=V(1)*DCOS(V(2))
Y=V(1)*DSIN(V(2))
AN=DATAN(Y/X)
IF((X.GT.0.).AND.(Y.GT.0.))PHC=AN
IF((X.GT.0.).AND.(Y.LT.0.))PHC=2*PI+AN
IF(X.LT.0.)PHC=AN+PI
PHD=PHC-PHMC
BET= V(2)/2.+PI/4.-NR *PI
FK=1+2*V(1)*DCOS(PHR)*DCOS(V(3))+V(1)**2
FH=1-(V(1)*DSIN(V(3))**2/FK
ARG1=V(1)*DSIN(V(3))/FK**.5
ALP=DASIN(ARG1)
ARG2=DCOS(ALP)*DCOS(BET)
IF(ARG2.GT.1.0)ARG2=1.0
IF(ARG2.LT.-1.0)ARG2=-1.0
GAM=DCOS(ARG2)
DM=(V(1)**2+VM(1)**2-2*V(1)*VM(1)*(DCOS(V(3))*DCOS(VM(3)))*
DCOS(PHR)+DSIN(V(3))*DSIN(VM(3)))**.5
AC(1)=V(1)**2+VM(1)**2+DCOS(V(3))*DCOS(VM(3))**2*(V(3)
**2-RME/V(1)**2-RMS*(V(1)+DCOS(V(3))*DCOS(V(2))/FK**1.5-RMM*
I(V(1)-VM(1)*DCOS(V(3)))*DCOS(VM(3))*DCOS(PHR)+DSIN(V(3))*)
I(1-V(3))**3-RMM*(DCOS(V(3))*DCOS(VM(3))*DCOS(PHR)+
I(1)*DSIN(VM(3)))*VM(1)**2
1+(CRP*DCOS(GAM)**2/FK)*DCOS(ALP)*DCOS(BET)*(V(1)+
DCOS(V(3))*DCOS(PHR))/FK**1.5+DCOS(ALP)*DSIN(BET)*DCOS(V(3))*DSIN
I(PHR)/FK*FH**.5-V(1)*DSIN(ALP)*DSIN(V(3))*DCOS(V(3))*(DCOS(PHR)
+V(1)*DCOS(V(3)))/(FK*FH**.5-FH**.5*DSIN(ALP)*DSIN(V(3))
AC(2)=(-DSIN(V(2))-2*V(4)*DCOS(V(3))*(V(5)+1)+2*V(1)*V(6)*DSIN(V(3))\+1)+(V(5)+1)+RMS*DSIN(V(2))/FK**1.5-RMM*VM(1)*DCOS(VM(3))*DSIN(PHD)\+1/DM**3+RMM*DCOS(VM(3))*DSIN(PHD)/VM(1)**2\+1+(CRP*DCOS(GAM)**2/FK)*(-DCOS(ALP)*DCOS(BET)*DSIN(PHR)/FK**.5\+1+DCOS(ALP)*DSIN(BET)*DCOS(PHR)+V(1)*DCOS(V(3)))/(FK*FH)**.5+VM(1)*DSIN(ALP)*DSIN(PHR)/(FK*FH)**.5)/\(V(1)*DCOS(V(3)))\+1+RMM*DSIN(VM(3))*DCOS(PHR)-DCOS(V(2))/FK**1.5-RMM*(VM(1)*DCOS(VM(3))*DCOS(VM(3))*DSIN(VM(3)))/\(VM(1)**2+(CRP*DCOS(GAM)**2/FK)*(-DCOS(ALP)*DCOS(BET)*DSIN(V(3)))*\+1DCOS(PHR)/FK**.5-DCOS(ALP)*DSIN(BET)*DSIN(PHR)/(FK*FH)\+1*5+V(1)*DSIN(ALP)*DSIN(V(3))*2*(DCOS(PHR)+V(1)*DCOS(V(3)))/\(1*(FK*FH)**.5-FH**.5*DSIN(ALP)*DCOS(V(3)))/V(1)\+RETURN\+END

SUBROUTINE ACL(VM,RMS,RME,RMM,FKM,PI,PHMR,PHMP,PHMC,NRM,ACM)
DIMENSION VM(6),ACM(3)
DOUBLE PRECISION VM,RMS,RME,RMM,FKM,PI,PHMR,PHMP,PHMC,NRM,ACM,
X,Y,AN - RMC=RME+RMM
PHMR=VM(2)-2*NRM*PI
IF((PHMP.LT.PI/2.).AND.(PHMR.GT.PI/2.))NRM=NRM+1
X=VM(1)*DCOS(VM(2))
Y=VM(1)*DSIN(VM(2))
AN=DATAN(Y/X)
IF((X.GT.0.).AND.(Y.GT.0.))PHMC=AN
IF((X.GT.0.).AND.(Y.LT.0.))PHMC=2*PI+AN
IF((X.LT.0.).PHMC=AN+PI
FKM=1+2*VM(1)*DCOS(VM(3))*DCOS(PHR)+VM(1)**2
ACM(1)=VM(1)*VM(6)**2*DCOS(VM(3))*DCOS(VM(2))+VM(1)*DCOS(VM(3))\+1*2*(VM(5)+1)**2-RMC*VM(1)**2-2*RMS*(VM(1)+DCOS(VM(3)))\+DCOS(VM(2)]\+1/FKM**1.5
ACM(2)=(-DSIN(VM(2))-2*V(4)*DCOS(VM(3))*(V(5)+1)+2*V(1)*V(6)*\+DSIN(VM(3))*(V(5)+1)+RMS*DSIN(V(2))/FKM**1.5)/(VM(1)*\+1DCOS(VM(3)))\+ACM(3)=(2*V(4)*VM(6)-DSIN(VM(3))*DCOS(VM(2))-VM(1)*VM(5)*\+1DSIN(VM(3))*DCOS(VM(3))*(V(5)+2)+RMS*DSIN(VM(3))*DCOS(VM(2))/\+1FKM**1.5)/VM(1)
RETURN
END
REFERENCES


