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Almost completely decomposable groups with two critical types and their endomorphism rings

Lewis, Wayne Steven, Ph.D.

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ALMOST COMPLETELY DECOMPOSABLE GROUPS WITH TWO CRITICAL TYPES AND THEIR ENDOMORPHISM RINGS

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BY
Wayne Lewis

Dissertation Committee:

Adolf Mader, Chairperson
Lee Lady
Wayne Smith
Ron Brown
Wesley Peterson
ABSTRACT

An almost completely decomposable group with two critical types is a direct sum of rank-one groups and indecomposable rank-two groups. A complete set of near isomorphism invariants for an acd group with two critical types is the isomorphism class of the regulator and the isomorphism class of the regulator quotient; with one additional invariant, namely an element of a certain quotient group of $(\mathbb{Z}/m\mathbb{Z})^\times$, a complete set of isomorphism invariants for an acd group with two critical types is obtained. Finally, the endomorphism ring of an acd group with two critical types is computed and the resulting structure is used to give an example of two nearly isomorphic groups with non-isomorphic endomorphism rings.
# TABLE OF CONTENTS

ABSTRACT ......................................................... iii
CHAPTER 1: INTRODUCTION ........................................ 1
  1.1 Preview of Dissertation ....................................... 1
  1.2 Definitions .................................................. 2
  1.3 Notation .................................................... 4
CHAPTER 2: REPRESENTING ACD GROUPS ........................... 6
  2.1 Technicalities ............................................... 6
  2.2 The Representation ......................................... 9
CHAPTER 3: THE ORIGINAL PROOF ................................. 12
  3.1 Introduction ............................................... 12
  3.2 Outline of Proof of Theorem 1 ............................... 12
  3.3 Normalization ............................................... 13
  3.4 Results .................................................... 21
CHAPTER 4: A NEW APPROACH TO CLASSIFICATION ............... 26
  4.1 Introduction ............................................... 26
  4.2 Structure of ACD Groups with Two Critical Types .......... 26
  4.3 Isomorphism Results ........................................ 31
CHAPTER 5: THE ENDOMORPHISM RING ............................... 38
  5.1 Structure of the Endomorphism Ring ......................... 38
  5.2 Nearly Isomorphic Groups with Non-Isomorphic Endomorphism
     Rings ...................................................... 42
REFERENCES ....................................................... 48
CHAPTER 1

INTRODUCTION

1.1. Preview of Dissertation. The formal study of almost completely decomposable groups began in the early 1970’s with the pioneering paper [La74] of Lee Lady. For about ten years the subject went without much attention and then in the early 1980’s Burkhardt, Krapf, and Mutzbauer started the ball rolling again with the publications [KM], [Bt83], and [Bt84]. Now in the early 1990’s the subject is more active than ever with important contributions by Arnold, Dugas, Mader, Mutzbauer, Oxford, and Vinsonhaler: see, for example, [AD], [DO], [MV1,2], and [MM].

This dissertation is concerned with almost completely decomposable groups with two critical types and their endomorphism rings.

Chapter 2 consists of a lemma of Mader (2.1.1), a new proof of the Brenner-Pierce Lemma (2.1.2), and a representation theorem for almost completely decomposable groups. For an alternative method of representing acd groups, see [MV1].

Chapters 3 and 4 give two independent approaches toward settling the classification of acd groups with two critical types. Chapter 3 contains what most would agree to be a ‘brute force’ approach to the problem of classification. Chapter 4 gives a more elegant approach which depends on the fact that near isomorphism preserves direct sums.
In Chapter 5 we compute the endomorphism ring of an arbitrary acd group with two (incomparable) critical types. This is then used to give an example of two nearly isomorphic groups with non-isomorphic endomorphism rings.

It is pretty clear that, although acd groups have been researched for over twenty years, the subject is still in the early stages of development. The recent surge of interest in the subject indicates that research of acd groups will continue to attract the attention of the algebra community. Apparently, the subject has been avoided in the past due to its seeming inherent difficulty. However, with connections to representation theory of partially-ordered sets and advancements in the development of the foundations of acd groups recently established, the ‘difficulty’ is presently not quite as forbidding. It is hoped that this dissertation will not only play a role in furthering the development of the theory of acd groups, but will also demonstrate the approachability of the subject.

1.2. Definitions. Here we give a list of definitions, a few of which will be repeated later for the sake of continuity. For any remaining undefined terms, we refer the reader to [Fuchs].

A finite rank torsion-free abelian group $G$ is almost completely decomposable, abbreviated acd, if there exists a completely decomposable group $C$ and a positive integer $n$ with $nG \subseteq C \subseteq G$.

If $H$ is any subgroup of $\mathbb{Q}G$, the divisible hull of $G$, then the purification of
$H$ in $G$, denoted $H^G$, is $QH \cap G$. When it is clear from the context that the purification of $H$ is taken in $G$, the superscript is omitted and we simply write $H_\ast$.

Define $G(\tau)$ to be the subgroup of $G$ generated by the elements of type greater than or equal to $\tau$—it is assumed that the reader is familiar with types. For the definition of type, see [Ar82].

Define $G^*(\tau)$ to be the subgroup of $G$ generated by all elements with type greater than $\tau$. Define $G^\#(\tau)$ to be the purification of $G^*(\tau)$: $G^\#(\tau) = G^*(\tau)_\ast$.

The typeset of $G$ is denoted $T(G)$. The critical typeset of $G$, denoted $T_{cr}(G)$, is the set of types $\tau$ with $G(\tau)/G^\#(\tau) \neq 0$. If $A$ is a rank-one group then the type of $A$ is denoted $t(A)$. Homogenous completely decomposable groups of type $\tau$ are abbreviated $\tau$-hcd groups. A pure subgroup of a $\tau$-hcd group is a $\tau$-hcd summand [Fuchs, 86.8].

For an acd group $G$, a regulating subgroup of $G$ is a completely decomposable subgroup of minimal index in $G$ [La74].

The regulator of $G$, denoted $R(G)$, is the intersection of all the regulating subgroups of $G$. It is shown in [Bt84] that $R(G)$ is a completely decomposable subgroup of finite index in $G$.

Two torsion-free groups $G$ and $H$ are quasi-isomorphic if there is a positive integer $r$ and $\phi: G \to H$, $\psi: H \to G$ such that $\psi\phi = r \cdot 1_G$ and $\phi\psi = r \cdot 1_H$; in this case we write $G \sim H$.

$G$ is nearly isomorphic to $H$ if for each non-zero integer $n$ there is a non-zero
integer \( r \) relatively prime to \( n \) and \( \phi: G \to H, \psi: H \to G \) such that \( \psi \phi = r \cdot 1_G \) and \( \phi \psi = r \cdot 1_H \) [La75]; in this case we write \( G \sim_n H \). For equivalent formulations of this definition, see [Ar82, 7.16]. It is shown in [MV1] that two \( \mathbb{Z} \)-modules \( G, H \) with \( G/R(G) \approx H/R(H) \) are nearly isomorphic if there is a monomorphism \( \phi: G \to H \) such that \( |H/\phi(G)| \) is finite and relatively prime to \( |H/R(H)| \).

Two decompositions \( H_1 \oplus \cdots \oplus H_m \) and \( K_1 \oplus \cdots \oplus K_n \) of a torsion-free abelian group are nearly equivalent if \( m = n \) and there is a permutation \( \sigma \in S_n \) such that \( H_i \sim_n K_{\sigma(i)} \) for \( 1 \leq i \leq n \). The relation 'is nearly equivalent to' is an equivalence relation.

Two decompositions \( H_1 \oplus \cdots \oplus H_m \) and \( K_1 \oplus \cdots \oplus K_n \) of an abelian group are equivalent if \( m = n \) and there is a permutation \( \sigma \in S_n \) such that \( H_i \approx K_{\sigma(i)} \) for \( 1 \leq i \leq n \). The relation 'is equivalent to' is an equivalence relation.

Let \( G \) be a finite abelian group. The isomorphism class of \( G \) is denoted \( \text{IsoCl} G \) and the \( p \)-socle of \( G \) is denoted \( G[p] \). The width of \( G \), denoted \( \text{width} G \), is \( n \) if \( G \) is generated by \( n \), but not less than \( n \), elements; this is equivalent to \( n = \max \{ \dim G[p] : p \text{ prime} \} \).

A subset \( \{ c_1, c_2, \ldots \} \) of an abelian group is called independent if \( \langle c_1, c_2, \ldots \rangle = \bigoplus_i \langle c_i \rangle \). Here we allow \( c_i = 0 \).

1.3. Notation. For a rank-one group \( A, S_A = \bigcap_{p \neq A} \mathbb{Z}_p \) where \( \mathbb{Z}_p \) is the localization of \( \mathbb{Z} \) at the prime \( p \). If \( D \) is a ring with identity then the group of units of
$D$ is denoted $D^\times$. If $D_1, D_2$ are subrings of the same ring then $D_1 D_2$ denotes the ring generated by $D_1 \cup D_2$.

If $D$ is a subring of $\mathbb{Q}$—recall that such a ring is a principal ideal domain—and $\alpha, \beta \in D$ then $\gcd_D(\alpha, \beta)$ is the least positive integral gcd in $D$ of $\alpha, \beta$. If $\tau = t(D)$ and $1 < m \in \mathbb{Z}$ with $pD \neq D$ for all primes $p \nmid m$ then $\mathbb{Z}(m, \tau)^\times$ is the image of $D^\times$ in $(\mathbb{Z}/m\mathbb{Z})^\times$ under the ring homomorphism

$$\theta: D \to \mathbb{Z}/m\mathbb{Z} \quad \text{given by} \quad \frac{u}{v} \mapsto (u + m\mathbb{Z})(v + m\mathbb{Z})^{-1}$$

where $\frac{u}{v}$ is reduced. For $s \in \theta^{-1}(\mathbb{Z}/m\mathbb{Z})^\times$, denote the image of $\theta(s)$ in $(\mathbb{Z}/m\mathbb{Z})^\times$ by $[s]$. 


CHAPTER 2

REPRESENTING ACD GROUPS

2.1. Technicalities. We start out with a very useful lemma.

2.1.1. Lemma (Mader). Suppose that \( G \cong (\mathbb{Z}/m\mathbb{Z})^n \) and \( \{x_1, \ldots, x_k\} \), \( 1 \leq k \leq n \), is an independent subset of \( G \). Let \( \frac{m}{d_i} = \text{order}(x_i) \). Then there exist \( y_1, \ldots, y_k \in G \) such that \( x_i = d_i y_i \) and \( G = \langle y_1 \rangle \oplus \cdots \oplus \langle y_k \rangle \oplus H \) for some subgroup \( H \) of \( G \).

PROOF. We induct on the number of distinct prime factors of \( m \).

(a) The induction starts with the case that \( m \) is a prime power: \( m = p^\alpha \). Let \( I = \{i : x_i \neq 0\} \) and \( J = \{j : x_j = 0\} \). For \( i \in I \) there exist \( y_i \in G \) such that \( x_i = d_i y_i \). It is easily verified that \( \langle y_i : i \in I \rangle = \bigoplus_{i \in I} \langle y_i \rangle \) and that \( G = \bigoplus_{i \in I} \langle y_i \rangle \oplus H' \) for some subgroup \( H' \) of \( G \). Also, \( H' = \bigoplus_{j \in J} \langle y_j \rangle \oplus H \) for some subgroup \( H \) of \( G \) and some \( y_j \in G \) with \( \text{order}(y_j) = m \) (if \( J = \emptyset \) we take \( H = H' \)). This proves the lemma in the primary case.

(b) Suppose that \( m \) contains at least two distinct prime factors and write \( m = m_1 m_2 \) where \( \gcd(m_1, m_2) = 1 \) and \( m_j > 1 \). Then \( G \) decomposes into its \( "m_j\text{-parts}" \) \( G = G_1 \oplus G_2, G_j \cong (\mathbb{Z}/m_j\mathbb{Z})^n \). There exist integers \( u_j \) such that \( u_1 m_1 + u_2 m_2 = 1 \). Decompose the \( x_i \) as \( x_i = x_{i1} + x_{i2}, x_{ij} \in G_j \). Let \( s_i = \frac{m}{d_i} \) and \( s_{ij} = \text{order}(x_{ij}) \); note that \( s_i = s_{i1} s_{i2} \) and \( s_{ij} \mid m_j \). By induction hypothesis there exist elements
\( y_{ij} \in G_j \) such that

\[
x_{ij} = \frac{m_j}{s_{ij}} y_{ij}, \quad \text{and} \quad G_j = \langle y_{ij} \rangle \oplus \cdots \oplus \langle y_{kj} \rangle \oplus H_j.
\]

We set

\[
y_i = u_2 s_{i2} y_{i1} + u_1 s_{i1} y_{i2} \in G_1 \oplus G_2, \quad 1 \leq i \leq n.
\]

Then

\[
\frac{m}{s_i} y_i = \frac{m_1}{s_{i1}} \frac{m_2}{s_{i2}} u_2 s_{i2} y_{i1} + \frac{m_2}{s_{i2}} \frac{m_1}{s_{i1}} u_1 s_{i1} y_{i2}
\]

\[
= \frac{m_1}{s_{i1}} (1 - u_1 m_1) y_{i1} + \frac{m_2}{s_{i2}} (1 - u_2 m_2) y_{i2}
\]

\[
= \frac{m_1}{s_{i1}} y_{i1} + \frac{m_2}{s_{i2}} y_{i2} = x_{i1} + x_{i2} = x_i.
\]

Since \( \gcd(u_2, m_1) = \gcd(s_{i2}, m_1) = \gcd(u_1, m_2) = \gcd(s_{i1}, m_2) = 1 \) we have

\[
\langle y_i \rangle = \langle u_2 s_{i2} y_{i1} \rangle \oplus \langle u_1 s_{i1} y_{i2} \rangle = \langle y_{i1} \rangle \oplus \langle y_{i2} \rangle,
\]

so with \( H = H_1 \oplus H_2 \) we have the desired decomposition

\[
G = \langle y_1 \rangle \oplus \cdots \oplus \langle y_k \rangle \oplus H. \quad \square
\]

In the literature there are two virtually identical proofs of 2.1.2: [Br, Lemma 13] and [KM, Lemma 1.2]. Both of these proofs use the ‘Smith normal form’ of a matrix which is essentially a generalization of the Stacked Basis Theorem for free abelian groups [Fuchs, 15.4] to the case of finite rank free modules over a PID. The Smith normal form of a matrix with entries in a PID is used as a key step in the
proof of the Fundamental Structure Theorem for Finitely Generated Modules over a PID [J, 3.8]. The proof we give of 2.1.2 is direct; in particular, it does not use the Smith normal form of a matrix. Interestingly, it is not difficult to see that our proof of 2.1.2 still works if we replace \( \mathbb{Z} \) by a PID \( D \) and \( m \) by a non-zero non-unit of \( D \).

2.1.2. Lemma. Let \( n > 0 \) and \( m > 1 \). If \( [y_{ij}] \in M_n(\mathbb{Z}) \), \( \gcd(m, \det[y_{ij}]) = 1 \), and \( s \equiv \det[y_{ij}] \pmod{m} \) then there exist \( [y'_{ij}] \equiv [y_{ij}] \pmod{mM_n(\mathbb{Z})} \) such that \( \det[y'_{ij}] = s \).

PROOF. We first prove by induction on \( n \) that there exist \( y'_{ij} \equiv y_{ij} \pmod{m} \) for \( 2 \leq i \leq n, 1 \leq j \leq n \) such that the prime factors of \( \gcd(D'_1, \ldots, D'_n) \) divide \( m \) where \( D'_j \) is the determinant of the matrix obtained by deleting the first row and the \( j^{th} \) column of \( [y_{ij}] \) (take \( y'_{1j} = y_{1j} \) for \( 1 \leq j \leq n \)):

The result is vacuously true for \( n = 1 \). Suppose that the result is true for \( n - 1 \). Let \( N = [y_{ij}]_{2 \leq i, j \leq n} \). Applying the induction hypothesis to the transpose of \( N \), we see that there exist \( y'_{ij} \equiv y_{ij} \pmod{m} \) for \( 2 \leq i \leq n, 3 \leq j \leq n \) such that the prime factors of \( g = \gcd(d'_1, \ldots, d'_{n-1}) \) divide \( m \) where \( d'_k = (-1)^{k+1} \det N'_{k1} \), \( N'_{k1} = [y'_{ij}]_{2 \leq i \neq k \leq n} \). Set \( y'_{i2} = y_{i2} \) for \( 2 \leq i \leq n \). Let \( d = y_{21}d'_1 + \cdots + y_{n1}d'_{n-1} \) and write \( d = rs \) where \( (p \mid r \Rightarrow p \mid m) \) and \( (p \mid s \Rightarrow p \mid m) \). Let \( e = \gcd(mg, r) \). Write \( mg = em' \) and \( r = er' \). Since \( \gcd(r's, m') = 1 \) there exists \( t \in \mathbb{Z} \) such that \( \gcd(D'_1, r's + tm') = 1 \) where \( D'_1 = \det[y'_{ij}]_{2 \leq i, j \leq n} \). Choose \( r_2, \ldots, r_n \in \mathbb{Z} \)
such that $r_2d_1' + \cdots + r_nd_{n-1}' = tg$. Set $y_{i1}' = y_{i1} + mr_i$ for $2 \leq i \leq n$. Then $D_2' = y_{21}'d_1' + \cdots + y_{n1}'d_{n-1}' = (y_{21} + mr_2)d_1' + \cdots + (y_{n1} + mr_n)d_{n-1}' = (y_{21}d_1' + \cdots + y_{n1}d_{n-1}') + (r_2d_1' + \cdots + r_nd_{n-1}')m = d+mgt = rs+mgt = er's + e'm't = e(r's + m't).

Hence, all prime factors of $\gcd(D_1', D_2') = \gcd(D_1', e)$ divide $m$ so all prime factors of $\gcd(D_1', \ldots, D_n')$ divide $m$, as desired.

Now we prove the lemma. Choose $y_{ij}' = y_{ij} \pmod{m}$ for $2 \leq i \leq n$, $1 \leq j \leq n$ as above. Since $\gcd(m, \det[y_{ij}]) = 1$, $\gcd(m, \det[y_{ij}']) = 1$ so $\gcd(D_1', \ldots, D_n') = 1$. Choose $u_1, \ldots, u_n \in \mathbb{Z}$ such that $(y_{11} + u_1m)D_1' - (y_{12} + u_2m)D_2' + \cdots + (-1)^{n+1}(y_{1n} + u_nm)D_n' = (y_{11}D_1' - y_{12}D_2' + \cdots + (-1)^{n+1}y_{1n}D_n') + (u_1D_1' - u_2D_2' + \cdots + (-1)^{n+1}u_nD_n')m = s$. Set $y'_{ij} = y_{ij} + u_jm$ for $1 \leq j \leq n$. This completes the proof. \( \square \)

2.2. The Representation. Here we give a concrete representation of a general acd group. We contend that the conclusions of the following proposition are as strong as possible under the given hypotheses.

2.2.1. Proposition. If $R$ is a completely decomposable subgroup of finite index in $G$ and $R \neq G$ then up to isomorphism $G = R + \sum_{i=1}^{t} \mathbb{Z} \frac{e_i}{m}$ where

(1) $1 < m = \exp(G/R)$ and $1 \leq t = \text{width}(G/R) \leq \text{rk} G$;

(2) $G/R = \bigoplus_{i=1}^{t} \left( \frac{e_i}{m} + R \right)$ with $\frac{m}{d_i} = \text{order} \left( \frac{e_i}{m} + R \right) > 1$ where $d_i|d_{i-1}| \cdots |d_1|m$, $d_1 \neq 1$, and $d_1 \neq m$;

(3) $T_{cr}(G) = \{ \tau_1, \tau_2, \ldots, \tau_n \}$ and $R = \bigoplus_{j=1}^{n} A_j^{r_j}$ with
(a) \( \mathbb{Z} \subseteq A_j \subseteq \mathbb{Q} \), \( t(A_j) = \tau_j \), and \( \tau_j > 0 \);

(b) \( \frac{1}{p} \notin A_j \) for primes \( p | m \) such that \( pA_j \neq A_j \);

(4) \( x_i = d_iy_i \) with \( x_i, y_i \in \mathbb{Z}^{\text{rk} G} \), \( 1 \leq i \leq t \);

(5) \( \{y_1, \ldots, y_t\} \) is a subbasis of \( \mathbb{Z}^{\text{rk} G} \).

PROOF. 0 \( \neq G/R \) is finite so \( \exp(G/R) = m > 1 \) and \( \text{width}(G/R) = t \geq 1 \) for some \( m, t \in \mathbb{Z} \); \( t \leq \text{rk} G \) since \( G/R \) is a subgroup of \( \frac{1}{m} R/R \) and \( \frac{1}{m} A/A \) is cyclic for a rank-one group \( A \). There exist \( x_1, \ldots x_t \in R \) such that \( G/R = \bigoplus_{i=1}^{t} (\frac{x_i}{m} + R) \) with \( \frac{m}{d_i} = \text{order}(\frac{x_i}{m} + R) \) where \( d_t | d_{t-1} | \cdots | d_1 | m \) (note that \( d_t \neq 1 \) and \( d_i \neq m \) by (1)). Write \( T_{cr}(G) = \{\tau_1, \ldots, \tau_n\} \). For \( 1 \leq j \leq n \) there exist \( A_j \) with \( \mathbb{Z} \subseteq A_j \subseteq \mathbb{Q} \) such that \( R(\tau_j)/R^\#(\tau_j) \approx A_j^{n_j} \). Therefore, there is an isomorphism \( \phi: R \rightarrow \bigoplus_{j=1}^{n} A_j^{n_j} \) and this map extends to a map from \( G \) to some subgroup of \( \mathbb{Q}^{\text{rk} G} \) and \( \bigoplus_{j=1}^{n} A_j^{n_j} \) is a completely decomposable subgroup of finite index in this subgroup with properties (1) and (2). Applying a further isomorphism of the same sort, we see that the \( A_j \) can be assumed to satisfy (3)(b).

Now if \( p^a \) is the highest power of a prime \( p \) dividing the denominator of a \( \tau_j \) entry of \( x_i \) and \( pA_j = A_j \) then multiplication of the \( \tau_j \) component of \( R \) by \( p^a \) induces an isomorphism as above; furthermore, this isomorphism preserves properties (1), (2), and (3). In this manner we obtain \( x_i \) with the property that if \( p \) is a prime with \( pA_j = A_j \) then \( p \) does not divide the denominator of any \( \tau_j \) entry of \( x_i \). If \( p | m \) and \( pA_j \neq A_j \) then \( \frac{r}{s} \notin A_j \) whenever \( \gcd(r, s) = 1 \) with \( p | s \); if \( a, b \in \mathbb{Z} \) with
ar + bs = 1 then
\[
\frac{as}{p} \cdot \frac{r}{s} + \frac{bs}{p} = \frac{1}{p} \notin A_j
\]
by (3)(b) so \( \frac{r}{s} \notin A_j \) since \( \frac{as}{p}, \frac{bs}{p} \in \mathbb{Z} \subseteq A_j \) by (3)(a). Therefore, any prime that divides a denominator of an entry (reduced fraction) of \( x_i \) is relatively prime to \( m \).

Multiplying \( x_i \) by an appropriate integer relatively prime to \( m \) gives an integral representative for a generator of the cyclic group \( \langle \frac{x_i}{m} + R \rangle \); hence, there is no loss of generality in assuming that \( x_i = (x_{i1}, \ldots, x_{ir} \mathbb{G}) \in \mathbb{Z}^{rk} \mathbb{G} \).

Let \( e_i = \gcd(x_{i1}, \ldots, x_{it}, m) \). Since \( \text{order}(\frac{x_i}{m} + R) = \frac{m}{d_i}, e_i \mid d_i \) and \( \frac{x_i}{d_i} \in R \); thus, \((d_i/e_i)(x_i)^* = (x_i)^* \) by (3)(b). Hence, there is no loss of generality in assuming that \( e_i = d_i \). This establishes (4).

For (5), observe that

\[
\bigoplus_{i=1}^{t} \langle \frac{x_i}{m} + R \rangle \approx \bigoplus_{i=1}^{t} \langle \frac{x_i}{m} + \mathbb{Z}^{rk} \mathbb{G} \rangle \leftrightarrow \frac{1}{m} \mathbb{Z}^{rk} \mathbb{G} \bigg/ \mathbb{Z}^{rk} \mathbb{G}
\]

via \( \frac{x_i}{m} + R \mapsto \frac{x_i}{m} + \mathbb{Z}^{rk} \mathbb{G} \). By 2.1.1, there exist \( y_1, \ldots, y_{rk} \mathbb{G} \in \mathbb{Z}^{rk} \mathbb{G} \) such that

\[
\frac{1}{m} \mathbb{Z}^{rk} \mathbb{G} \big/ \mathbb{Z}^{rk} \mathbb{G} = \bigoplus_{i=1}^{rk} (\frac{y_i}{m} + \mathbb{Z}^{rk} \mathbb{G}) \text{ where } \frac{d_i}{m}y_i + \mathbb{Z}^{rk} \mathbb{G} = \frac{x_i}{m} + \mathbb{Z}^{rk} \mathbb{G} \text{ for } 1 \leq i \leq t.
\]

It follows that \( \det[y_{ij}] \) is relatively prime to \( m \). There is no loss of generality in assuming that \( \det[y_{ij}] \equiv 1 \pmod{m} \). By 2.1.2, there is no loss of generality in assuming that \( \det[y_{ij}] = 1 \). This completes the proof. \( \square \)
CHAPTER 3

THE ORIGINAL PROOF

3.1. Introduction. In this chapter our aim is to prove the following two theorems. Theorem 1 settles the most difficult part of the classification problem for acd groups with two critical types. An incorrect proof of Theorem 1 appears in [Ar73] where the problem of classification was originally addressed. The existence and uniqueness (up to near isomorphism) of an indecomposable rank-two acd group is established in Theorem 2.

**Theorem 1.** Every indecomposable acd group with two critical types has rank two.

**Theorem 2 (Lady).** An indecomposable acd rank-two group $G$ is uniquely determined up to near isomorphism by $\text{T}_{\text{cr}}(G)$ and $\exp(G/R(G))$. If $\text{T}_{\text{cr}}(G) = \{\sigma, \tau\}$ and $\exp(G/R(G)) = m$ for such a $G$ then $\sigma, \tau$ are incomparable and $pG(\sigma) \neq G(\sigma)$, $pG(\tau) \neq G(\tau)$ for all primes $p | m$.

Conversely, given two rank-one groups $A, B$ with incomparable types and a positive integer $m$ such that $pA \neq A, pB \neq B$ for all primes $p | m$, there exists an indecomposable acd rank-two group $G \sim A \oplus B$ with $\exp(G/R(G)) = m$.

3.2. Outline of Proof of Theorem 1. First, it is shown that an indecomposable acd group $G$ with two critical types and $\text{width}(G/R(G)) = n$ is quasi-isomorphic to $A^n \oplus B^n$ for some $A, B$ (3.3.10). Next, a few technical lemmas (3.3.1-3.3.5,3.3.7-
3.3.9) and a variant of the Stacked Basis Theorem (3.3.6) are employed to present $G$ as a certain subgroup of $Q^{2n}$ (3.3.11): up to isomorphism,

$$G = (A^n \oplus B^n) + \sum_{i=1}^{n} Z \left( \frac{x_i^A, x_i^B}{m} \right)$$

where, among other things, $Z \subseteq A, B \subseteq Q; R(G) = A^n \oplus B^n; m = \exp(G/R(G)); x_i^A, x_i^B \in Z^n; \sum_{i=1}^{n} \langle x_i^A \rangle_\ast = A^n$; and $\text{rk}(x_1^B, \ldots, x_n^B) = n$. If $\sum_{i=1}^{n} \langle x_i^B \rangle_\ast = B^n$ then $G = \oplus_{i=1}^{n} (x_i^A, x_i^B)_\ast$ so $n = 1$ and we're done; otherwise, it is shown that if $p$ is a prime dividing

$$\left| B^n / \sum_{i=1}^{n} \langle x_i^B \rangle_\ast \right|$$

then there exists $i_0$ and $(\tilde{x}_{i_0}^A, \tilde{x}_{i_0}^B) \in Z^{2n}$ with

$$G/R(G) = \langle \frac{x_1^A, x_1^B}{m} \rangle + R(G), \ldots, \langle \frac{x_{i_0}^A, \tilde{x}_{i_0}^B}{m} \rangle + R(G), \ldots, \langle \frac{x_n^A, x_n^B}{m} \rangle + R(G) \rangle$$

such that $\langle \tilde{x}_{i_0}^A \rangle_\ast + \sum_{i \neq i_0} \langle x_i^A \rangle_\ast = A^n$ and

$$\left| B^n / (\langle \tilde{x}_{i_0}^B \rangle_\ast + \sum_{i \neq i_0} \langle x_i^B \rangle_\ast) \right|$$

divides $\frac{1}{p} \left| B^n / \sum_{i=1}^{n} \langle x_i^B \rangle_\ast \right|$, thereby inductively giving the desired result.

3.3. Normalization. The following three facts on $cd$ groups are well-known and easy to prove.

3.3.1. Lemma. If $T_{cr}(G)$ consists of (pairwise) incomparable types then

$$R(G) = \bigoplus_{\tau \in T_{cr}(G)} G(\tau)$$

is the unique regulating subgroup of $G$. 13
3.3.2. **Lemma.** If $C$ is a homogenous completely decomposable subgroup of $H$ with finite index then $H$ is completely decomposable and $\text{width}(H/C) \leq \text{rk } H$.

3.3.3. **Lemma.** If $\phi : G \to G'$ is an isomorphism then $\phi(\text{R}(G)) = \text{R}(G')$; in particular, $\exp(G/\text{R}(G)) = \exp(G'/\text{R}(G'))$.

The next three lemmas are used to normalize the generating set of $G/\text{R}(G)$ where $G \sim A^n \oplus B^n$. The proof of 3.3.5 is easy and will be omitted.

3.3.4. **Lemma.** Let $b, c$ be relatively prime integers and let $0 < a \in \mathbb{Z}$. Then there exists $k \in \mathbb{Z}$ such that $\gcd(a, b + kc) = 1$.

**Proof.** Write $a = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_n^{\alpha_n}$ where the $p_i$'s are distinct primes and $\alpha_i > 0$ for each $i$. By the Chinese Remainder Theorem there is a $k \in \mathbb{Z}$ such that $b + kc \equiv 1 \pmod{p_i}$ for each $p_i \mid c$. If $p_i \mid c$ then $p_i \mid b$ so $p_i \mid (b + kc)$. Hence, $p_i \mid (b + kc)$ for all $i$ and $\gcd(a, b + kc) = 1$. \(\square\)

3.3.5. **Lemma.** If $G = \langle x_1, \ldots, x_n \rangle$ is a finite abelian group and $[t_{ij}] \in \text{GL}_n(\mathbb{Z})$ then $G = \langle \sum_{j=1}^{n} t_{1j}x_j, \ldots, \sum_{j=1}^{n} t_{nj}x_j \rangle$.

3.3.6. **Lemma.** If $a_1, \ldots, a_n \in \mathbb{Z}^n$ are linearly independent then there exists $[t_{ij}] \in \text{GL}_n(\mathbb{Z})$ and a basis $c_1, \ldots, c_n$ of $\mathbb{Z}^n$ such that $\sum_{j=1}^{n} t_{ij}a_j = m_ic_i$, $1 \leq i \leq n$, where $0 < m_i \in \mathbb{Z}$ with $m_{i-1} \mid m_i$, $2 \leq i \leq n$.

**Proof.** By the Stacked Basis Theorem [Fuchs, 15.4] there exists a basis $a_1', \ldots, a_n'$ of $\sum_{i=1}^{n} \mathbb{Z}a_i$, a basis $c_1, \ldots, c_n$ of $\mathbb{Z}^n$, and $m_i \in \mathbb{Z}$ with $m_{i-1} \mid m_i$, $2 \leq i \leq n$, such
that $m_i c_i = a'_i$, $1 \leq i \leq n$. Let $T$ be the matrix of the basis change from the $a_i$'s to the $a'_i$'s. Then $T$ is the desired matrix. □

Here is a useful fact concerning divisibility in homogenous completely decomposable groups. Recall that a pure subgroup of a $\tau$-hcd group is a $\tau$-hcd summand.

3.3.7. Lemma. Suppose that $H$ is a subgroup of $\mathbb{Q}^n$ and $A \subseteq \mathbb{Q}$; then $H_{\mathbb{Z}^n} = \mathbb{Q}H \cap \mathbb{Z}^n$ is a free subgroup of $\mathbb{Z}^n$ of rank $rk H$. If $a_1, \ldots, a_{rk H}$ is a basis of $H_{\mathbb{Z}^n}$ then $H^A_{\mathbb{Z}^n} = \oplus_{i=1}^{rk H} A a_i = \oplus_{i=1}^{rk H} \langle a_i \rangle^A_{\mathbb{Z}^n}$.

Proof. We first show that $(\oplus_{i=1}^{rk H} \langle a_i \rangle)^A_{\mathbb{Z}^n} \subseteq \oplus_{i=1}^{rk H} A a_i$. Let $w \in (\oplus_{i=1}^{rk H} \langle a_i \rangle)^A_{\mathbb{Z}^n}$; then $w = \frac{r}{s} \sum t_i a_i$ for some $\frac{r}{s} \in \mathbb{Q}$ and $t_i \in \mathbb{Z}$. Without loss of generality, $gcd(t_1, \ldots, t_{rk H}) = 1$. Let $u = (u_1, \ldots, u_n) = \sum t_i a_i$ and let $d = gcd(u_1, \ldots, u_n)$. Write $u_i = d u'_i$ and $u' = (u'_1, \ldots, u'_n)$. Then $u' \in \mathbb{Z}^n$ and $d u' = u \in H_{\mathbb{Z}^n}^A$. Hence $u' \in H_{\mathbb{Z}^n}^A$ and $u' = \sum s_i a_i$ for some $s_i$. It follows that $t_i = d s_i$. But $gcd(t_1, \ldots, t_{rk H}) = 1$ so $d = 1$. Also, $\frac{r}{s} u_i \in A$ for $1 \leq i \leq n$ since $\frac{r}{s} u = w \in A^n$. There exist $k_1, \ldots, k_n \in \mathbb{Z}$ with $\sum_{i=1}^{n} k_i u_i = 1$ so $\frac{r}{s} = \frac{r}{s} \sum_{i=1}^{n} k_i u_i = \sum_{i=1}^{n} (\frac{r}{s} u_i) \in A$ whence $w = \frac{r}{s} \sum t_i a_i = \sum_{i=1}^{rk H} (\frac{r}{s} t_i) a_i \in \sum_{i=1}^{rk H} A a_i$. Therefore,

$H^A_{\mathbb{Z}^n} = (\oplus_{i=1}^{rk H} \langle a_i \rangle)^A_{\mathbb{Z}^n} \subseteq \oplus_{i=1}^{rk H} A a_i \subseteq \oplus_{i=1}^{rk H} \langle a_i \rangle^A_{\mathbb{Z}^n} \subseteq (\oplus_{i=1}^{rk H} \langle a_i \rangle)^A_{\mathbb{Z}^n} = H^A_{\mathbb{Z}^n}$. □

As a result of 3.3.7, if $a_1, \ldots, a_n$ is a basis of $\mathbb{Z}^n$ and $A \subseteq \mathbb{Q}$ then $\sum_{i=1}^{n} \langle a_i \rangle^A_{\mathbb{Z}^n} = A^n$.

The next lemma describes the relationship between the order of the quotient of two homogenous completely decomposable groups of equal rank and type and the determinant of a certain matrix.
3.3.8. Lemma. Let $B \subseteq \mathbb{Q}$. Let $b_1, \ldots, b_n \in \mathbb{Q}^n$ be linearly independent; there are two elements in $\mathbb{Q}b_i \cap \mathbb{Z}^n$ with coordinates having greatest common divisor 1; let $\hat{b}_i$ denote the one whose first non-zero coordinate is positive. Then

$$
\Delta = \left| B^n / \sum_{i=1}^{n}(\hat{b}_i)_* \right| \text{ divides } D = \det \begin{bmatrix} \hat{b}_1 \\
\vdots \\
\hat{b}_n \end{bmatrix}
$$

with $\Delta = \pm D$ if and only if $pB \neq B$ for all primes $p | D$.

Moreover, $e = \exp(\mathbb{Z}^n / \sum_{i=1}^{n}(\hat{b}_i))$ is the greatest integer for which there exist $k_1, \ldots, k_n \in \mathbb{Z}$ with $\gcd(k_1, \ldots, k_n) = 1$ such that $\sum_{i=1}^{n} k_i \hat{b}_i \in c\mathbb{Z}^n$.

PROOF. Note that $pB \neq B$ for all primes $p | \Delta$. Let $\alpha \in \mathbb{Q}$ with $\mathbb{Z} \subseteq \alpha \mathbb{B}$ and $\frac{1}{p} \notin \alpha \mathbb{B}$ for all primes $p | \Delta$. Then $B^n / \sum_{i=1}^{n}(\hat{b}_i)_* \approx (\alpha \mathbb{B})^n / \sum_{i=1}^{n}(ab_i)_*^{(\alpha \mathbb{B})}$ and $\hat{\alpha} \hat{b}_i = \hat{b}_i$ so without loss of generality $\mathbb{Z} \subseteq B$ and $\frac{1}{p} \notin B$ for all primes $p | \Delta$. Also, $(b_i)_* = (\hat{b}_i)_*$ and $\hat{\alpha} \hat{b}_i = \hat{b}_i$ so without loss of generality $b_i = \hat{b}_i \in \mathbb{Z}^n$. By 3.3.6 there exists a basis $c_1, \ldots, c_n \in \mathbb{Z}^n$ such that $\sum_{i=1}^{n}(\hat{b}_i)_* = \sum_{i=1}^{n} m_i (c_i)_*$ where $0 < m_i \in \mathbb{Z}$ with $m_i - 1 | m_i$, $2 \leq i \leq n$. As in the proof of 3.3.7, we see that $\sum_{i=1}^{n}(\hat{b}_i)_* = \sum_{i=1}^{n} m_i (c_i)_*$; also, by 3.3.7, $B^n = \sum_{i=1}^{n}(c_i)_*$. Hence, $B^n / \sum_{i=1}^{n}(\hat{b}_i)_* = \sum_{i=1}^{n}(c_i)_* / \sum_{i=1}^{n} m_i (c_i)_* \approx \mathbb{Z} / r_1 \mathbb{Z} \oplus \cdots \oplus \mathbb{Z} / r_n \mathbb{Z}$ where $r_i = \prod_{\substack{p | m_i \atop p \neq B}} p^\max\{\alpha : p^n | m_i\}$. Therefore, $\Delta = \prod_{i=1}^{n} r_i | \prod_{i=1}^{n} m_i = \pm D$ with $\Delta = \pm D$ if and only if $pB \neq B$ for all primes $p | D$.

Now, suppose that $k_1, \ldots, k_n \in \mathbb{Z}$ with $\gcd(k_1, \ldots, k_n) = 1$ such that $\sum_{i=1}^{n} k_i \hat{b}_i \in \mathbb{Z}^n$; then the order of $\frac{1}{r} \sum_{i=1}^{n} k_i \hat{b}_i + \sum_{i=1}^{n}(\hat{b}_i)$ in $\mathbb{Z}^n / \sum_{i=1}^{n}(\hat{b}_i)$ is $|r|$ so $r | e$. On the other hand, there exists $a = (a_1, \ldots, a_n) \in \mathbb{Z}^n$ such that the order of $a + \sum_{i=1}^{n}(\hat{b}_i)$
in \( \mathbb{Z}^n / \sum_{i=1}^{n} \langle b_i \rangle \) is \( e \). Without loss of generality, \( \gcd(a_1, \ldots, a_n) = 1 \). Write \( ea = \sum_{i=1}^{n} k_i b_i \); by the minimality of \( e \), \( \gcd(k_1, \ldots, k_n) = 1 \) as desired. \( \square \)

The next lemma is just a special case of 2.2.1. We state it here in simplified form as a convenience.

3.3.9. Lemma. If \( G \) is an acd group with \( T_{cr}(G) = \{ \tau_1, \ldots, \tau_n \} \) and \( G \neq R = R(G) \) then up to isomorphism \( G = R + \sum_{i=1}^{t} \mathbb{Z} \frac{z_i}{m} \) where

1. \( m = \exp(G/R) \) and \( \text{width}(G/R) = t \);

2. \( R = \oplus_{j=1}^{n} A_{n_j} \) with

   a. \( \mathbb{Z} \subseteq A_{n_j} \subseteq \mathbb{Q} \), \( t(A_{n_j}) = \tau_j \), and \( n_j > 0 \) for \( 1 \leq j \leq n \);

   b. \( \frac{1}{p} \notin A_{n_j} \) for \( 1 \leq j \leq n \) and all \( p \) prime such that \( p \mid m \) and \( pA_{n_j} \neq A_{n_j} \);

3. \( x_i \in \mathbb{Z}^{rk} G \) for \( 1 \leq i \leq t \).

With all of the technicalities out of the way, we now get to the heart of the matter. For a group \( G \sim A^m \oplus B^n \), let \( \tau_A = t(A), \tau_B = t(B) \), \( R_A = G(\tau_A), R_B = G(\tau_B) \), and \( R = R_A \oplus R_B \) (note that \( R = R(G) \) by 3.3.1). Let \( \pi_A : \mathbb{Q} G \rightarrow \mathbb{Q} R_A \) (\( \pi_B : \mathbb{Q} G \rightarrow \mathbb{Q} R_B \)) be the projection with kernel \( \mathbb{Q} R_B \) (\( \mathbb{Q} R_A \)).

3.3.10. Lemma (Lady). If \( G \sim A^m \oplus B^n \) \((m, n > 0)\) and \( \text{width}(G/R) = r \) then \( r \leq m, n \) and \( G = G_r \oplus G_A \oplus G_B \) for some \( G_r \sim A^r \oplus B^r \) with no rank-one summands, \( G_A \approx A^{m-r} \), and \( G_B \approx B^{n-r} \).

Proof. If \( \tau_A, \tau_B \) are comparable then \( \{ \tau_A, \tau_B, \inf \{ \tau_A, \tau_B \} \} \) is linearly ordered.
so \( G \) is completely decomposable [Ar82, 2.3] and \( r = 0 \). Assume for the remainder of the proof that \( \tau_A, \tau_B \) are incomparable.

Before considering the general case, we assume that \( G \) has no rank-one summands and show that \( m = n = r \). We may assume without loss of generality that \( m = \text{rk} R_A \leq \text{rk} R_B = n \). \( \pi_A \) induces a map \( G/R \to \pi_A(G)/R_A \) which is monic (in fact an isomorphism) since \( R_B \) is pure in \( G \). Hence, \((\exp(G/R))\pi_A(G) \subseteq R_A \subseteq \pi_A(G)\) so \( r \leq m \) by 3.3.2. Let \( x_1, \ldots, x_r \) be a set of representatives for generators of \( G/R \) and let \( G' = (R_A, x_1, \ldots, x_r)_* \). Then \( G = G' + R_B = G' + G(\tau_B) \) and \( G/G' \approx R_B/(G' \cap R_B) \) is \( \tau_B \)-hcd since a pure subgroup of a hcd group is a summand; by Baer's Lemma [Ar82, 5.6], \( G' \) is a summand of \( G' \) with \( \tau_B \)-hcd complementary summand. But \( G \) has no rank-one summands so \( G = G' \). Thus, \( \text{rk} G = \dim_Q QG \leq \dim_Q QR_A + r = m + r \) so \( n = \text{rk} R_B \leq r \). But \( r \leq m \leq n \) so \( m = n = r \).

Now we consider the general case. Write \( G = G_r \oplus G_A \oplus G_B \) where \( G_r \) has no rank-one summands, \( G_A \) is \( \tau_A \)-hcd, and \( G_B \) is \( \tau_B \)-hcd. \( G_r/R(G_r) \approx G/R \) so \( \text{width}(G_r/R(G_r)) = \text{width}(G/R) = r \); as before, \( \text{rk} G_r(\tau_A) = \text{rk} G_r(\tau_B) = r \). Hence, \( \text{rk} G_r = 2r, \text{rk} G_A = m - r, \) and \( \text{rk} G_B = n - r \). \( \square \)

The last preparatory lemma gives a normalized form for \( G \sim A^n \oplus B^n \) with no rank-one summands. Later, we will see that the representation for \( G \) can be simplified considerably.
3.3.11. Lemma. If \( G \) is an acd group with two critical types and no rank-one summands then up to isomorphism \( G = R + \sum_{i=1}^{n} \mathbb{Z} \frac{x_i}{m} \) where

1. \( m = \exp(G/R), n = \frac{1}{2} \text{rk} G, \) and width\((G/R) = n;\)

2. \( R = R(G) = A^n \oplus B^n \) with \( \mathbb{Z} \subseteq A, B \subseteq \mathbb{Q} \) and \( \frac{1}{p} \notin A, B \) for all primes \( p \mid m; \)

3. \( x_i \in \mathbb{Z}^{2n} \) for \( 1 \leq i \leq n; \)

4. \( \sum_{i=1}^{n} (\pi_A(x_i))_* = A^n \) and \( \text{rk}(\pi_B(x_1), \ldots, \pi_B(x_n)) = n; \)

5. \( x_i = d_i(x_i^A, s_i x_i^B) \) with \( d_i \mid m, \gcd(s_i, m) = 1, x_i^A, x_i^B \in \mathbb{Z}^n, \) and

\[
\gcd(x_i^A, \ldots, x_i^B) = \gcd(x_i^A, \ldots, x_i^B) = 1 \text{ for } 1 \leq i \leq n.
\]

Proof. Let \( R = R(G) \). Write \( m = \exp(G/R), T_{cr}(G) = \{ \sigma, \tau \}, \) and \( \text{rk} G(\sigma) = n. \) By 3.3.10, \( n = \text{rk} G(\tau) = \frac{1}{2} \text{rk} G \) and \( \text{width}(G/R) = n. \) We now show that \( pG(\sigma) \neq G(\sigma), pG(\tau) \neq G(\tau) \) for all primes \( p \mid m. \)

Suppose that \( pG(\sigma) = G(\sigma) \) and \( p \mid m. \) Let \( \frac{y_i}{m} + R \in G/R, 1 \leq i \leq n, \) be generators for \( G/R \) where \( y_i \in R = R(G). \) Let \( y_i^\sigma \in G(\sigma) \) and \( y_i^\tau \in G(\tau) \) with \( y_i = y_i^\sigma + y_i^\tau. \) Then \( \frac{y_i}{m} = \frac{y_i^\sigma + y_i^\tau}{m} \in G \) and \( \frac{y_i}{m} \in G(\sigma) \) so \( \frac{y_i}{m} \in G. \)

By purity, \( \frac{y_i}{m} \in G(\tau); \) hence, \( \frac{y_i}{m} \in R = G(\sigma) \oplus G(\tau). \) Thus, \( \frac{m}{p} G = \frac{m}{p} (R + \sum_{i=1}^{n} \mathbb{Z} \frac{y_i}{m}) = \frac{m}{p} (R + \sum_{i=1}^{n} \mathbb{Z} \frac{y_i}{m}) \subseteq R, \) contradicting the minimality of \( m. \) Similarly we get that \( pG(\tau) \neq G(\tau) \) for all primes \( p \mid m. \)

It follows by 3.3.9 that up to isomorphism \( G = R + \sum_{i=1}^{n} \mathbb{Z} \frac{x_i}{m} \) as in (1), (2), (3).
For $G = R + \sum_{i=1}^{n} \mathbb{Z} \frac{x_i}{m}$ as in (1), (2), (3) we show that there is no loss of generality in assuming (4).

First of all, for any set of generators $y_1 + R, \ldots, y_n + R$ of $G/R$ we have that

$$G = \langle \pi_A(y_1), \ldots, \pi_A(y_n); \pi_B(y_1), \ldots, \pi_B(y_n) \rangle \ast K_A \ast K_B$$

for some $\tau_A$-hcd group $K_A$ and some $\tau_B$-hcd group $K_B$; but $G$ has no rank-one summands so $K_A = K_B = 0$. In particular $\text{rk}(\pi_A(y_1), \ldots, \pi_A(y_n)) = \text{rk}(\pi_B(y_1), \ldots, \pi_B(y_n)) = n$.

Now, by 3.3.6 there exists $T \in \text{GL}_n(\mathbb{Z})$ such that

$$T \begin{bmatrix} \pi_A(x_1) \\ \vdots \\ \pi_A(x_n) \end{bmatrix} = \begin{bmatrix} r_1 x_1^{A_i} \\ \vdots \\ r_n x_n^{A_i} \end{bmatrix}$$

for some $r_i \in \mathbb{Z}, x_i^{A_i} \in \mathbb{Z}^n$ with $\mathbb{Z}^n = \sum_{i=1}^{n} \mathbb{Z} x_i^{A_i}$; 3.3.7 gives that $\sum_{i=1}^{n} \langle r_i x_i^{A_i} \rangle = \sum_{i=1}^{n} \langle x_i^{A_i} \rangle = A^n$. By 3.3.5 we can write $G = R + \sum_{i=1}^{n} \mathbb{Z} \frac{x_i^{B_i}}{m}$ where

$$\begin{bmatrix} x_1' \\ \vdots \\ x_n' \end{bmatrix} = T \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} (r_1 x_1^{A_i}, x_1^{B_i}) \\ \vdots \\ (r_n x_n^{A_i}, x_n^{B_i}) \end{bmatrix}.$$  

Also, $\text{rk}(x_1^{B_i}, \ldots, x_n^{B_i}) = n$. This completes the argument since $\pi_A(x_i') = r_i x_i^{A_i}$ and $\pi_B(x_i') = x_i^{B_i}$.

Finally, if $G = R + \sum_{i=1}^{n} \mathbb{Z} \frac{x_i}{m}$ as in (1), (2), (3), (4) then there is no loss of generality in assuming (5):
Write \( x_i = (r_i x_i^A, s_i x_i^B) \) where

\[
\gcd(x_i^A, \ldots, x_i^A) = \gcd(x_i^B, \ldots, x_i^B) = 1.
\]

By 3.3.4, there exist \( k_i \in \mathbb{Z} \) such that \( \gcd(\frac{r_i + k_i m}{\gcd(r_i, m)}, m) = 1 \),

\[
1 \leq i \leq n,
\]

so

\[
\frac{x_i}{m} + R = \frac{x_i}{m} + m \frac{(k_i x_i^A, 0)}{m} + R = \frac{(d_i r_i' x_i^A, s_i x_i^B)}{m} + R
\]

where \( d_i = \gcd(r_i, m) \), \( r_i' = \frac{r_i + k_i m}{\gcd(r_i, m)} \), and \( \gcd(r_i', m) = 1 \). Repeating the argument with \( s_i x_i^B \), we see that there is no loss of generality in assuming that

\[
\frac{x_i}{m} = \frac{(d_i r_i x_i^A, e_i s_i x_i^B)}{m}
\]

where \( d_i, e_i | m \) and \( \gcd(r_i, m) = \gcd(s_i, m) = 1 \).

Suppose that \( p^k | m \) for some prime \( p \) and some \( k > 0 \). Then \( p^k | d_i \)

if and only if

\[
\frac{e_i s_i x_i^B}{p^k} = \frac{m}{p^k} \cdot \frac{x_i}{m} - \frac{d_i r_i}{p^k} x_i^A \in G \text{ if and only if } p^k | e_i.
\]

Hence, \( d_i = \pm e_i \) for each \( i \). Also, if \( t_i r_i \equiv 1 \pmod{m} \) then

\[
\langle \frac{x_i}{m} + R \rangle = \langle \frac{(d_i r_i x_i^A, e_i t_i s_i x_i^B)}{m} + R \rangle = \langle \frac{(d_i x_i^A, \pm d_i t_i s_i x_i^B)}{m} + R \rangle
\]

so we may assume without loss of generality that

\[
\frac{x_i}{m} = \frac{d_i (x_i^A, s_i x_i^B)}{m}
\]

where \( d_i | m \) and \( \gcd(s_i, m) = 1 \) for each \( i \). Note that only the coefficients of \( x_i^A, x_i^B \)

were altered in the above argument so that \( 4 \) still holds. This completes

the proof. \( \square \)

3.4. Results.

3.4.1. Theorem. Every indecomposable acd group with two critical types has

rank two.
PROOF. Let $G$ be an indecomposable acd group with two critical types. Then up to isomorphism $G = R + \sum_{i=1}^{n} \mathbb{Z} x_i$ as in 3.3.11.

Let $l_{ij} = \gcd(d_i s_i, d_j s_j), 1 \leq i, j \leq n$. Let

$$\Delta = \left| B^n / \sum_{i=1}^{n} (x_i^B)^* \right|.$$ 

Suppose that $p | \Delta$; note that $pB \neq B$. By 3.3.8 there exist $k_1, \ldots, k_n \in \mathbb{Z}$ with $\gcd(k_1, \ldots, k_n) = 1$ such that $p | \sum_{i=1}^{n} k_i x_i^B$. Let $I = \{ i : p \nmid k_i \}$ and let $p^{\alpha_i}$ denote the highest power of $p$ dividing $d_i s_i$, $i \in I$ (note that $\gcd(d_i, s_i) = 1$, $\alpha_i$ may be zero, and $I \neq \emptyset$). Choose $i_0 \in I$ with $\alpha_{i_0}$ maximal. Let $k_{i_0}^*$ satisfy $k_{i_0}^* k_{i_0} \equiv 1 \pmod{p}$. For $i \in I$, let $t_i$ satisfy $t_i (d_i s_i / l_{i_0 i}) \equiv 1 \pmod{p}$; for $i \notin I$ let $t_i = 0$. Then $p^{\alpha_{i_0}+1}$ divides

$$\sum_{i=1}^{n} t_i (d_i s_i / l_{i_0 i}) k_{i_0}^* k_i (d_i s_i x_i^B) = d_i s_i k_{i_0}^* \sum_{i=1}^{n} t_i (d_i s_i / l_{i_0 i})(k_i x_i^B)$$

since $p | \sum_{i=1}^{n} k_i x_i^B$. Let $u_{i_0} = 1$ and let $u_i = t_i d_i s_i k_{i_0}^* k_i / l_{i_0 i}$ if $i \neq i_0$. Then $p d_i s_i | \sum_{i=1}^{n} u_i (d_i s_i x_i^B)$ since $k_{i_0}^* k_{i_0} \equiv t_{i_0} \equiv 1 \pmod{p}$. Also, $d_{i_0} | \sum_{i=1}^{n} u_i (d_i x_i^A)$ since $l_{i_0 i} | s_{i_0} d_i$ for $i \neq i_0$ (recall that $\gcd(m, s_i) = 1$).

Write

$$\bar{x}_{i_0} = \sum_{i=1}^{n} u_i x_i = \sum_{i=1}^{n} u_i d_i (x_i^A, s_i x_i^B) = d_{i_0} (\bar{x}_{i_0}, s_{i_0} p u \bar{x}_{i_0}^B)$$

where $v \in \mathbb{Z}$ is chosen so that $\gcd(\bar{x}_{i_0 1}, \ldots, \bar{x}_{i_0 n}) = 1$. Since $u_{i_0} = 1$,

$$G/R = \langle \frac{x_1}{m} + R, \ldots, \frac{x_{i_0-1}}{m} + R, \frac{\bar{x}_{i_0}}{m} + R, \frac{x_{i_0+1}}{m} + R, \ldots, \frac{x_n}{m} + R \rangle$$
and \( (\bar{x}_{i_0}^A)_* + \sum_{i \neq i_0} (x_i^A)_* = A^n \) (recall that \( \sum_{i=1}^n (x_i^A)_* = A^n \)). Moreover, \( \Delta = \)

\[
\left| B^n / \sum_{i=1}^n (x_i^B)_* \right| = \frac{B^n / ((\bar{x}_{i_0}^B)_* + \sum_{i \neq i_0} (x_i^B)_*)}{((\bar{x}_{i_0}^B)_* + \sum_{i \neq i_0} (x_i^B)_*)/ \sum_{i=1}^n (x_i^B)_*)}
\]

and \( p \) divides

\[
\left| (\bar{x}_{i_0}^B)_* + \sum_{i \neq i_0} (x_i^B)_* \right| = \frac{\left( (\bar{x}_{i_0}^B)_* \cap \sum_{i=1}^n (x_i^B)_* \right)}{\sum_{i=1}^n (x_i^B)_*}
\]

so \( B^n / ((\bar{x}_{i_0}^B)_* + \sum_{i \neq i_0} (x_i^B)_*) \) divides \( \frac{\Delta}{p} \).

Without loss of generality, \( \bar{x}_{i_0} = d_{i_0}(\bar{x}_{i_0}^A, s_{i_0} p v \bar{x}_{i_0}^B) \) can be replaced by \( x_{i_0}' = d_{i_0}(\bar{x}_{i_0}^A, s_{i_0}' \bar{x}_{i_0}^B) \) where \( \gcd(s_{i_0}', m) = 1 \) (see the proof of 3.3.11(5)).

It now follows by induction that there exist \( x_i = d_i(x_i^A, s_i x_i^B) \in \mathbb{Z}^{2n}, 1 \leq i \leq n \), with \( d_i | m, \gcd(s_i, m) = 1, \sum_{i=1}^n (x_i^A)_* = A^n, \sum_{i=1}^n (x_i^B)_* = B^n \), and \( G = R + \sum_{i=1}^n \mathbb{Z} \frac{x_i}{m} \).

Therefore, \( G = \bigoplus_{i=1}^n (x_i^A, x_i^B)_* \) and \( n = 1 \). \( \square \)

Note that in proving Theorem 1 we showed that if \( G \) is an acd group with two critical types and no rank-one summands then up to isomorphism \( G = G_1 \oplus \cdots \oplus G_n \) where \( n = \frac{1}{2} \text{rk } G \) and \( G_i = A \oplus B + \mathbb{Z} \frac{(1, s_i)}{m_i} \) for some incomparable rank-one groups \( A, B \) containing \( \mathbb{Z} \) and some \( s_i, m_i \in \mathbb{Z} \) with \( \frac{1}{p} \notin A, B \) for all primes \( p | \text{lcm}(m_1, \ldots, m_n) = \exp(G/R) \) and \( \gcd(s_i, \exp(G/R)) = 1 \).

The following theorem gives a complete set of near isomorphism invariants for an indecomposable acd rank-two group.
3.4.2. **Theorem (Lady).** An indecomposable acd rank-two group $G$ is uniquely determined up to near isomorphism by $T_{cr}(G)$ and $\exp(G/R)$. If $T_{cr}(G) = \{\sigma, \tau\}$ and $\exp(G/R) = m$ for such a $G$ then $\sigma, \tau$ are incomparable and $pG(\sigma) \neq G(\sigma)$, $pG(\tau) \neq G(\tau)$ for all primes $p \mid m$.

Conversely, given two rank-one groups $A, B$ with incomparable types and a positive integer $m$ such that $pA \neq A, pB \neq B$ for all primes $p \mid m$, there exists an indecomposable acd rank-two group $G \sim A \oplus B$ with $\exp(G/R) = m$.

**Proof.** Let $G$ be an indecomposable acd rank-two group. Up to isomorphism, $G = (A \oplus B) + \mathbb{Z}^{(1,s)}_m$ for some rank-one groups $A, B$ containing $\mathbb{Z}$ and some $s, m \in \mathbb{Z}$ with $\frac{1}{p} \notin A, B$ for all primes $p \mid m = \exp(G/A \oplus B)$ and $\gcd(s, m) = 1$. $A$ and $B$ have incomparable types since $G$ is indecomposable.

Now suppose that $H$ is another indecomposable acd group with $\exp(H/R(H)) = m$ and $T_{cr}(H) = \{t(A), t(B)\}$. Without loss of generality, $H = (A \oplus B) + \mathbb{Z}^{(1,t)}_m$ for some $t \in \mathbb{Z}$ with $\gcd(t, m) = 1$. Hence $t \equiv k s \pmod{m}$ for some $k \in \mathbb{Z}$ relatively prime to $m$.

Let $0 \neq n \in \mathbb{Z}$ and choose $r \equiv k \pmod{m}$ with $\gcd(n, r) = 1$ (3.3.4). Define $\phi: G \to H$ by $\phi(a, b) = (a, rb)$ and $\psi: H \to G$ by $\psi(a, b) = (ra, b)$. Then $\phi, \psi$ are well-defined, $\psi \phi = r \cdot 1_G$, and $\phi \psi = r \cdot 1_H$. This shows that $G$ is nearly isomorphic to $H$.

Finally, suppose that $A, B, m$ are given. Let $G = R + \mathbb{Z}^{(a,b)}_m$ where $R = A \oplus B$.
and \( a \in A, b \in B \) with \( \frac{a}{p} \notin A, \frac{b}{p} \notin B \) for all primes \( p \mid m \). The only possible decomposition of \( G \) is \( A_\ast \oplus B_\ast \), but \( A, B \) are pure in \( G \) and \( G \neq A \oplus B \). It is clear that \( G \sim A \oplus B \). Also, the order of \( \frac{(a,b)}{m} + R \) is \( m \) so \( \exp(G/R) = m \). \( \square \)
CHAPTER 4
A NEW APPROACH
TO CLASSIFICATION

4.1. Introduction. Herein a detailed analysis of almost completely decomposable groups with two critical types is presented. The approach given in this chapter differs from that of Chapter 3 in that we first derive a complete set of near isomorphism invariants (4.2.2) and then use this information to show that an indecomposable acd group with two critical types has rank two (4.2.2); in Chapter 3, we proceeded in the reverse order. The (lack of) uniqueness in a decomposition of such a group is quantified in 4.2.3. Further, by fixing two particular subgroups of $\mathbb{Q}$ representing the respective types, a complete set of isomorphism invariants for such a group is given (4.3.3). Finally, it is shown that the set of isomorphism classes contained in the near isomorphism class of such a group (without rank-one summands) has the structure of a finite abelian group and this group is computed explicitly (4.3.3).

4.2. Structure of ACD Groups with Two Critical Types. We proceed to show that an acd group $G$ with two critical types $\tau_A, \tau_B$ is determined uniquely up to near isomorphism by $\mathrm{rk} G(\tau_A), \mathrm{rk} G(\tau_B)$, and $\mathrm{IsoCl}(G/R(G))$. We then settle the question concerning the degree to which a decomposition of an acd group with two critical types is unique.
The following is just a reformulation of 3.3.8 in a simpler setting. We state it here for convenience.

4.2.1. **Lemma.** Let $A \subseteq \mathbb{Q}$ and let $0 \neq a_1, \ldots, a_n \in \mathbb{Z}^n$ be linearly independent. Then

$$|A^n / \sum_{i=1}^{n} Aa_i| \text{ divides } \det \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}.$$ 

The following theorem has an interesting history. To begin with, part (b) of the theorem is implied by Theorem 1 of [Ar73] but the proof given there is admittedly flawed. The validity of (b) was posed as a question by Lee Lady in 1989. It was established by the author while participating in a seminar in 1990—this is the content of Chapter 3. A result similar to the theorem appearing below was proven independently by Manfred Dugas (also in 1990) in an unpublished manuscript. In addition, it is a consequence of [AD, Theorem 3.2] which employs representations of partially ordered sets. The proof given below is elementary; the heart of the proof lies in the fact that near isomorphism preserves direct sum decompositions [Ar82, 12.9].

4.2.2. **Theorem.** Let $G$ be an acd group with two incomparable critical types $\tau_A, \tau_B$. Then

(a) \( \text{rk } G(\tau_A), \text{rk } G(\tau_B), \text{ and } \text{IsoCl}(G/R) \) give a complete set of near-isomorphism invariants for $G$ and
(b) \(G\) is a direct sum of indecomposable groups of rank \(\leq 2\). In particular, an indecomposable acd group with two critical types has rank two.

**Proof.** Suppose that \(G \sim_n H\) and \(T_{cr}(G) = \{\tau_A, \tau_B\}\). By [La74, Theorem 11], \(G \oplus R(G) \approx H \oplus R(H)\). It follows that \(G/R(G) \approx H/R(H)\). Also, \(\text{rk } G(\tau_A) = \text{rk } H(\tau_A)\) and \(\text{rk } G(\tau_B) = \text{rk } H(\tau_B)\) since \(G \sim H\).

Conversely, suppose that \(G/R(G) \approx H/R(H), \text{rk } G(\tau_A) = \text{rk } H(\tau_A),\) and \(\text{rk } G(\tau_B) = \text{rk } H(\tau_B)\) for acd groups \(G, H\) with \(T_{cr}(G) = T_{cr}(H) = \{\tau_A, \tau_B\}\). By 3.3.10, we can assume without loss of generality that \(G, H\) have no rank-one summands, \(n = \text{rk } G(\tau_A) = \text{rk } G(\tau_B) = \text{rk } H(\tau_A) = \text{rk } H(\tau_B) = \text{width}(G/R(G))\) and \(G/R(G) \approx H/R(H)\) with invariants \(m_1|m_2|\cdots|m_n\) (\(1 < m_1\) and \(m_n = \exp(G/R(G))\)). Let \(m = m_n\) and \(R = R(G)\).

Let \(R_A = G(\tau_A)\) and \(R_B = G(\tau_B)\)—note that \(R = R_A \oplus R_B\). Let \(\pi_A : \mathbb{Q}G \rightarrow \mathbb{Q}R_A\) (\(\pi_B : \mathbb{Q}G \rightarrow \mathbb{Q}R_B\)) be the projection with kernel \(\mathbb{Q}R_B\) (\(\mathbb{Q}R_A\)). In proving 3.3.10 we showed that \(\pi_A(G)/R_A \approx G/R \approx \pi_B(G)/R_B\); in particular, if \(p \mid m\) then \(pR_A \neq R_A\) and \(pR_B \neq R_B\). Hence, there exist \(Z \subset A, B \subset \mathbb{Q}\) with \(t(A) = \tau_A, t(B) = \tau_B\), and \(\frac{1}{p} \notin A, B\) for all primes \(p \mid m\). Let \(K = \bigoplus_{i=1}^{n} K_i\) where \(K_i = A \oplus B + \mathbb{Z}[\frac{(1)}{m_i}]\) and note that \(R(K) = (A \oplus B)^n, R(K_i) = A \oplus B, K_i/R(K_i) \approx \mathbb{Z}/m_i\mathbb{Z}\), and \(K_i\) is indecomposable.

It suffices to show that \(G \sim_n K\). We can assume without loss of generality that \(G = (A \oplus B)^n + \sum_{i=1}^{n} \mathbb{Z}[x_i]^{\frac{1}{m_i}}\) with \(x_i \in \mathbb{Z}^2, R = (A \oplus B)^n,\) and \(G/R = \bigoplus_{i=1}^{n} (\frac{x_i}{m_i} + R) \approx \mathbb{Z}/m_i\mathbb{Z}\).
\[ \bigoplus_{i=1}^{n} \left( \frac{m}{m_i} \bar{x}_i \right) \subseteq (\mathbb{Z}/m\mathbb{Z})^{2n} \] where \( \bar{x}_i \) is the image of \( x_i \) under the natural epimorphism \( \mathbb{Z}^{2n} \to (\mathbb{Z}/m\mathbb{Z})^{2n} \). As indicated above,

\[ \pi_A(G)/R_A = \bigoplus_{i=1}^{n} \left( \frac{\pi_A(x_i)}{m_i} + R_A \right) \approx G/R \approx \bigoplus_{i=1}^{n} \left( \frac{\pi_B(x_i)}{m_i} + R_B \right) = \pi_B(G)/R_B, \]

whence \( \frac{m}{m_1} \pi_A(x_1), \ldots, \frac{m}{m_n} \pi_A(x_n) \) (respectively, \( \frac{m}{m_1} \pi_B(x_1), \ldots, \frac{m}{m_n} \pi_B(x_n) \)) are independent. By 2.1.1, we can assume without loss of generality that \( \pi_A(x_1), \ldots, \pi_A(x_n) \) (respectively, \( \pi_B(x_1), \ldots, \pi_B(x_n) \)) are independent. It follows that

\[ p \nmid d_A = \det \begin{bmatrix} \pi_A(x_1) \\ \vdots \\ \pi_A(x_n) \end{bmatrix}, d_B = \det \begin{bmatrix} \pi_B(x_1) \\ \vdots \\ \pi_B(x_n) \end{bmatrix} \]

for each prime \( p \mid m \).

Define \( \phi: K \to G \) by \( \phi(e_i + f_i) = x_i \) where \( e_i \) is 0 in every component except for 1 in the \( 2i - 1 \) coordinate and \( f_i \) is 0 everywhere except 1 in the \( 2i \) coordinate, \( 1 \leq i \leq n \). It is easy to see that \( \phi(K) \subseteq G \). Note that \( \phi(R_A) = \sum_{i=1}^{n} A\pi_A(x_i) \) and \( \phi(R_B) = \sum_{i=1}^{n} B\pi_B(x_i) \). By 4.2.1, \( |R_A/ \sum_{i=1}^{n} A\pi_A(x_i)| \) divides \( d_A \)

and \( |R_B/ \sum_{i=1}^{n} B\pi_B(x_i)| \) divides \( d_B \). Since \( G/\phi(K) = (\phi(K) + R)/\phi(K) \approx R/(R \cap \phi(K)) = R/\phi(R) \approx (R_A/\phi(R_A)) \oplus (R_B/\phi(R_B)), |G/\phi(K)| \) is relatively prime to \( m \). Thus, \( G \sim_n K \). This completes the proof of (a). This also establishes (b) since \( K \) is a direct sum of rank-two groups and near isomorphism respects direct sums [Ar82, 12.9].

Bearing in mind the extreme non-uniqueness of a decomposition of a finite abelian group of width \( n \) into \( n \) cyclic summands, the following fact illustrates the degree to which the Krull-Schmidt property fails for acd groups.
4.2.3. Corollary. Let $G$ be an acd group with two critical types and no rank-one summands. Then there is a one-to-one correspondence between near equivalence classes of decompositions of $G$ into indecomposable summands and equivalence classes of decompositions of $G/R$ into $\text{rk } G/2$ cyclic summands.

PROOF. The correspondence is given by $\{\text{near equivalence class of } H_1 \oplus \cdots \oplus H_n\} \mapsto \{\text{equivalence class of } H_1/R(H_1) \oplus \cdots \oplus H_n/R(H_n)\}$. The correspondence is clearly well-defined and injective. We now show that the correspondence is surjective. Suppose that $G/R \approx K_1 \oplus \cdots \oplus K_n$ with each $K_i$ cyclic and $2n = \text{rk } G$. Let $H = H_1 \oplus \cdots \oplus H_n$ where $T_{\text{cr}}(H_i) = T_{\text{cr}}(G)$, $\text{rk } H_i = 2$, and $H_i/R(H_i) \approx K_i$. By 4.2.2, $G \sim_n H$ so $G \approx G_1 \oplus \cdots \oplus G_n$ for some $G_i \sim_n H_i$ [Ar82, 12.9]. Then $\{\text{near equivalence class of } G_1 \oplus \cdots \oplus G_n\} \mapsto \{\text{equivalence class of } K_1 \oplus \cdots \oplus K_n\}$. □

4.2.4. Example. Here we show that the number of indecomposable summands in a decomposition of an acd group is not unique. Let $G = G_1 \oplus G_2$ where $G_i = A \oplus B + \mathbb{Z}(\frac{1}{p_i})$ with $\mathbb{Z} \subseteq A, B \subseteq \mathbb{Q}$; $A, B$ incomparable; and $p_1, p_2$ distinct primes such that $\frac{1}{p_1}, \frac{1}{p_2} \notin A, B$. Then $G_1, G_2$ are indecomposable and $G \approx A \oplus B \oplus (A \oplus B + \mathbb{Z}(\frac{p_2 p_1}{p_1 p_2}))$.

The reader may have noticed that the pathological nature of the decompositions indicated by (4.2.3) and (4.2.4) arise from the manner in which the $p$-primary parts of the regulator quotient are pieced together. It is thus apparent that the difficulty
can be avoided by considering acd groups with two critical types and $p$-primary regulator quotient.

4.2.5. Corollary. If $G$ is an acd group with two critical types and $p$-primary regulator quotient then any two decompositions of $G$ into indecomposable summands are nearly equivalent.

PROOF. Evident from (4.2.2), (4.2.3), and the fact that any two decompositions of a primary group into cyclic summands are equivalent. □

4.3. Isomorphism Results. For the remainder of the chapter, let $A, B \subseteq \mathbb{Q}$ denote incomparable rank-one groups containing 1. Let $S_A = \cap_{p \neq A} \mathbb{Z}_p$ and $S_B = \cap_{p \neq B} \mathbb{Z}_p$. Let $S = S_A S_B$ and set $t(S) = \tau$. Let $S^\times$ denote the group of units of $S$. Let $m > 1$ with $pS \neq S$ for all primes $p \mid m$ and let $\mathbb{Z}(m, \tau)^\times$ denote the image of $S^\times$ in $(\mathbb{Z}/m\mathbb{Z})^\times \subseteq \mathbb{Z}/m\mathbb{Z}$ under the ring homomorphism

$$\theta : S \to \mathbb{Z}/m\mathbb{Z} \quad \text{given by} \quad \frac{r}{s} \mapsto (r + m\mathbb{Z})(s + m\mathbb{Z})^{-1}$$

where $\frac{r}{s}$ is reduced. For $s \in \theta^{-1}(\mathbb{Z}/m\mathbb{Z})^\times$, let $[s]$ denote the image of $\theta(s)$ in $(\mathbb{Z}/m\mathbb{Z})^\times / \mathbb{Z}(m, \tau)^\times$. It is shown in [KM, Theorem 1.3] that a unit of $M_n(\mathbb{Z}/m\mathbb{Z})$ lifts to a unit of $M_n(S)$ (under the induced map) if and only if the determinant of the former lies in $\mathbb{Z}(m, \tau)^\times$.

We now proceed to define an invariant which, together with the near isomorphism invariants given in the theorem, give a complete set of isomorphism invariants for an acd group with two critical types modulo the choice of two partic-
ular subgroups of $Q$ representing the respective types. We then show that the set of isomorphism classes contained in a near isomorphism class of such a group with no rank-one summands has the structure of a finite abelian group, namely $(\mathbb{Z}/m\mathbb{Z})^\times / \mathbb{Z}(m, \tau)^\times$ for some $m$.

4.3.1. **Lemma.** Let $m_1, m_2 \in \mathbb{Z}$ with $1 < m_1 \mid m_2$. Let $s_1, s_2 \in \mathbb{Z}$ with $\gcd(s_1, m_1) = 1$ and $\frac{1}{p} \notin A, B$ for all $p \mid m_2$. Then

$$\left( A \oplus B + \mathbb{Z}\left(\frac{1, s_1}{m_1}\right) \right) \oplus \left( A \oplus B + \mathbb{Z}\left(\frac{1, s_2}{m_2}\right) \right) \approx$$

$$\left( A \oplus B + \mathbb{Z}\left(\frac{1, s_1 s_2}{m_1}\right) \right) \oplus \left( A \oplus B + \mathbb{Z}\left(\frac{1, 1}{m_2}\right) \right).$$

**Proof.** It can be shown using the Chinese Remainder Theorem that $\gcd(s_1 + km_1, s_2) = 1$ for some $k \in \mathbb{Z}$ so there is no loss of generality in assuming that $\gcd(s_1, s_2) = 1$. Let $e_1 = (1, 0, 0, 0), e_2 = (0, 0, 1, 0), f_1 = (0, 1, 0, 0), f_2 = (0, 0, 0, 1)$. We identify

$$A \oplus B + \mathbb{Z}\left(\frac{1, s_1}{m_1}\right) \text{ with } Ae_1 \oplus Bf_1 + \mathbb{Z}\frac{e_1 + s_1 f_1}{m_1},$$

$$A \oplus B + \mathbb{Z}\left(\frac{1, s_2}{m_2}\right) \text{ with } Ae_2 \oplus Bf_2 + \mathbb{Z}\frac{e_2 + s_2 f_2}{m_2},$$

$$A \oplus B + \mathbb{Z}\left(\frac{1, s_1 s_2}{m_1}\right) \text{ with } Ae_1 \oplus Bf_1 + \mathbb{Z}\frac{e_1 + s_1 s_2 f_1}{m_1},$$

$$A \oplus B + \mathbb{Z}\left(\frac{1, 1}{m_2}\right) \text{ with } Ae_2 \oplus Bf_2 + \mathbb{Z}\frac{e_2 + f_2}{m_2}.$$
we obtain
\[
\left( A e_1 + B f_1 + Z \frac{e_1 + s_1 f_1}{m_1} \right) \oplus \left( A e_2 + B f_2 + Z \frac{e_2 + s_2 f_2}{m_2} \right) = \\
\left( A (s_2 e_1 + s_1 e_2) \oplus B (f_1 + f_2) + Z \frac{(s_2 e_1 + s_1 e_2) + s_1 s_2 (f_1 + f_2)}{m_1} \right) \oplus \\
\left( A (\gamma \frac{m_2}{m_1} e_1 + \delta e_2) \oplus B (\gamma \frac{m_2}{m_1} e_1 + \delta e_2) + Z \frac{(\gamma \frac{m_2}{m_1} e_1 + \delta e_2) + (\gamma \frac{m_2}{m_1} e_1 + \delta e_2)}{m_2} \right) \\
\left( A e_1 + B f_1 + Z \frac{e_1 + s_1 s_2 f_1}{m_1} \right) \oplus \left( A e_2 + B f_2 + Z \frac{e_1 + f_2}{m_2} \right)
\]
where the last isomorphism is given by \( s_2 e_1 + s_1 e_2 \mapsto e_1, f_1 + f_2 \mapsto f_1, \gamma \frac{m_2}{m_1} e_1 + \delta e_2 \mapsto e_2, \gamma \frac{m_2}{m_1} s_1 f_1 + \delta s_2 f_2 \mapsto f_2. \)

4.3.2. Proposition. Let \( m_1, m_2, \ldots, m_n \in \mathbb{Z} \) with \( 1 < m_1 | m_2 | \cdots | m_n \) and suppose that \( \frac{1}{p} \not\in A, B \) for all primes \( p | m_n \). Let \( s_i, t_i \in \mathbb{Z} \) with \( \gcd(s_i t_i, m_i) = 1 \) for \( 1 \leq i \leq n \). Then

\[
\bigoplus_{i=1}^{n} \left( A \oplus B + Z \frac{1, s_i}{m_i} \right) \approx \bigoplus_{i=1}^{n} \left( A \oplus B + Z \frac{1, t_i}{m_i} \right)
\]

if and only if \( \prod_{i=1}^{n} s_i = \prod_{i=1}^{n} t_i \) in \( \mathbb{Z}/m_1 \mathbb{Z} \times \mathbb{Z}/m_2 \mathbb{Z} \times \cdots \times \mathbb{Z}/m_n \mathbb{Z} \).

**Proof.** (\( \Rightarrow \)) Let \( \phi \) denote an isomorphism from left to right. Let \( e_i \in \mathbb{Z}^{2^n} \) denote the vector with 1 in the \( 2i - 1 \) coordinate and 0's elsewhere; let \( f_i \) denote the vector with 1 in the \( 2i \) coordinate and 0's elsewhere. We identify

\[
A \oplus B + Z \frac{1, s_i}{m_i} \text{ with } Ae_i \oplus B f_i + Z \frac{e_i + s_i f_i}{m_i},
\]

\[
A \oplus B + Z \frac{1, t_i}{m_i} \text{ with } Ae_i \oplus B f_i + Z \frac{e_i + t_i f_i}{m_i}.
\]

Since \( \text{End } A^n \approx M_n(S_A) \) and \( \text{End } B^n \approx M_n(S_B) \), \( e_i^* = \phi(e_i) \in \sum_{i=1}^{n} S_A e_i \) and \( f_i^* = \phi(f_i) \in \sum_{i=1}^{n} S_B f_i \). Also, \( \phi \) restricts to automorphisms of \( \sum_{i=1}^{n} Ae_i = \sum_{i=1}^{n} A e_i^* \).
and $\sum_{i=1}^{n} B_{fi} = \sum_{i=1}^{n} B_{fi}^*$, respectively; since $\text{Aut} A^n \approx \text{GL}_n(S_A)$ and $\text{Aut} B^n \approx \text{GL}_n(S_B)$,

\[
\begin{align*}
\begin{bmatrix}
    e_1^* \\
    \vdots \\
    e_n^*
\end{bmatrix} \in S_A^x \quad \text{and} \quad
\begin{bmatrix}
    f_1^* \\
    \vdots \\
    f_n^*
\end{bmatrix} \in S_B^x.
\end{align*}
\]

There exist $c_{ij} \in \mathbb{Z}$ such that $\sum_{j=1}^{n} c_{ij} \left( \frac{e_{ij} + t_{ij}}{m_{ij}} + R \right) = \frac{s_{i}^* + s_{i}f_{i}^*}{m_{i}} + R$ for $1 \leq i \leq n$; it follows that $\sum_{j=1}^{n} c_{ij} \frac{e_{ij}}{m_{ij}} = \frac{s_i^*}{m_i} + r_i^A$ and $\sum_{j=1}^{n} c_{ij} \frac{t_{ij}}{m_{ij}} = \frac{s_i f_i^*}{m_i} + r_i^B$ for some $r_i^A \in \sum_{j=1}^{n} A_{ej}$, $r_i^B \in \sum_{j=1}^{n} B_{f_j}$. Hence, $m_n r_i^A = (\sum_{j=1}^{n} c_{ij} \frac{m_j}{m_i} e_j) - (\sum_{j=1}^{n} S_A e_j)$ and $m_n r_i^B = (\sum_{j=1}^{n} c_{ij} \frac{m_j}{m_i} t_{ij} f_j) - \frac{m_n s_i f_i^*}{m_i} \in \sum_{j=1}^{n} S_B f_j$. But $m_n A \cap S_A = m_n S_A$ and $m_n B \cap S_B = m_n S_B$ so $r_i^A \in \sum_{j=1}^{n} S_A e_j$ and $r_i^B \in \sum_{j=1}^{n} S_B f_j$.

Let

\[
\Delta = \begin{vmatrix}
    c_{11} & m_1 c_{12} & \cdots & m_1 c_{1n} \\
    m_2 c_{11} & c_{22} & \cdots & m_2 c_{2n} \\
    \vdots & \vdots & \ddots & \vdots \\
    m_n c_{11} & m_n c_{12} & \cdots & c_{nn}
\end{vmatrix} = \begin{vmatrix}
    e_1^* + m_1 r_1^A \\
    \vdots \\
    e_n^* + m_n r_n^A
\end{vmatrix}
\quad \text{and}
\]

\[
\Phi = \begin{vmatrix}
    c_{11} t_1 & m_1 c_{12} t_2 & \cdots & m_1 c_{1n} t_n \\
    m_2 c_{11} t_1 & c_{22} t_2 & \cdots & m_2 c_{2n} t_n \\
    \vdots & \vdots & \ddots & \vdots \\
    m_n c_{11} t_1 & m_n c_{12} t_2 & \cdots & c_{nn} t_n
\end{vmatrix} = \begin{vmatrix}
    s_1 f_1^* + m_1 r_1^B \\
    \vdots \\
    s_n f_n^* + m_n r_n^B
\end{vmatrix}.
\]

Then

\[
s_1 s_2 \cdots s_n u_B = s_1 s_2 \cdots s_n \det \begin{bmatrix}
    f_1^* \\
    \vdots \\
    f_n^*
\end{bmatrix} = \det \begin{bmatrix}
    s_1 f_1^* \\
    \vdots \\
    s_n f_n^*
\end{bmatrix} \equiv \Phi \quad (\text{mod } m_1 S_B)
\]

and

\[
\Phi = t_1 t_2 \cdots t_n \Delta \equiv t_1 t_2 \cdots t_n \det \begin{bmatrix}
    e_1^* \\
    \vdots \\
    e_n^*
\end{bmatrix} = t_1 t_2 \cdots t_n u_A \quad (\text{mod } m_1 S_A)
\]

34
so \( s_1s_2 \cdots s_n u_B \equiv t_1t_2 \cdots t_n u_A \pmod{m_1 S} \). Therefore, \( \prod_{i=1}^{n} s_i = \prod_{i=1}^{n} t_i \) in \( \mathbb{Z}/m_1Z \), since \( \theta(u_A), \theta(u_B) \in \mathbb{Z}(m_1, \tau)^{\times} \).

\((\Leftarrow)\) \[ \prod_{i=1}^{n} s_i = \prod_{i=1}^{n} t_i \] in \( \mathbb{Z}(m_1, \tau)^{\times} \) implies that \( \prod_{i=1}^{n} s_i + m_1Z(v + m_1Z) = \prod_{i=1}^{n} t_i + m_1Z(u + m_1Z) \) for some \( u, v \in S^{\times} \cap Z \). By 4.3.1, we can assume without loss of generality that \( s_2 = \cdots = s_n = t_2 = \cdots = t_n = 1, s = s_1, t = t_1, [s] = [t], \) and show that \( A \oplus B + Z^{(1, s)} \approx A \oplus B + Z^{(1, t)} \). Write \( v = \prod q_k^{\beta_k} \) and note that \( sv + m_1Z = tu + m_1Z \). Let \( K = \{ k : q_kB = B \} \); since \( q_kA = A \) for all \( k \not\in K \),

\[ A \oplus B + Z^{(1, s)} \approx A \oplus B + Z^{(1, t)} \]

via \((1, 0) \mapsto (\prod_{k \in K} q_k^{\beta_k}, 0)\) and \((0, 1) \mapsto (0, \prod_{k \in K} q_k^{\beta_k})\). Also, \( \gcd(\prod_{k \in K} q_k^{\beta_k}, m_1) = 1 \) so that

\[ \prod_{k \in K} q_k^{\beta_k} \cdot \prod_{k \in K} q_k^{\beta_k} = A \oplus B + Z^{(1, sv)} \]

Similarly, \( A \oplus B + Z^{(1, t)} \approx A \oplus B + Z^{(1, tu)} \). This completes the proof since \( sv \equiv tu \pmod{m_1 Z} \). \( \square \)

Let \( H = \bigoplus_{i=1}^{n} (A \oplus B + Z^{(1, s_i)}) \) as in (4.3.2), \( n \geq 0 \); we interpret \( n = 0 \) here to mean \( H = 0 \). For \( G \approx A^r \oplus B^s \oplus H, r, s \geq 0 \), we define \( s(G) \) to be \( \prod_{i=1}^{n} s_i \) if \( n > 0 \) and 0 otherwise.

We summarize the classification up to isomorphism in a final theorem.

**4.3.3. Theorem.** Fix \( Z \subseteq A, B \subseteq \mathbb{Q} \) and \( m > 1 \) with \( \frac{1}{p} \not\in A, B \) for each prime \( p \mid m \) and \( \tau_A = t(A), \tau_B = t(B) \) incomparable. Set \( \tau = t(S_A S_B) \).
(a) An acd group $G$ with $T_{cr}(G) \subseteq \{\tau_A, \tau_B\}$ and $\exp(G/R(G)) \mid m$ is isomorphic
to

$$A^r \oplus B^s \oplus \bigoplus_{i=1}^{n} \left( A \oplus B + \mathbb{Z}\left(1, \frac{s_i}{m_i}\right) \right)$$

for some $0 \leq n, r, s, m_i, s_i \in \mathbb{Z}$ with $1 < m_1 \mid m_2 \mid \cdots \mid m_n$ and $\gcd(s_i, m_i) = 1$.

(b) A complete set of isomorphism invariants for the acd group $G$ is $\text{rk}_G(\tau_A), \text{rk}_G(\tau_B), \text{IsoCI}(G/R(G))$, and $s(G)$.

(c) If $G$ has no rank-one summands then $\{\text{IsoCI}(H) : H \sim_n G\}$ has the structure of a finite abelian group isomorphic to $(\mathbb{Z}/m_1\mathbb{Z})^\times /\mathbb{Z}(m_1, \tau)^\times$ with multiplication given by

$$\text{IsoCI}\left(\bigoplus_{i=1}^{n} \left( A \oplus B + \mathbb{Z}\left(1, \frac{u_{1i}}{m_i}\right) \right)\right) \ast \text{IsoCI}\left(\bigoplus_{i=1}^{n} \left( A \oplus B + \mathbb{Z}\left(1, \frac{u_{2i}}{m_i}\right) \right)\right) = \text{IsoCI}\left(\bigoplus_{i=1}^{n} \left( A \oplus B + \mathbb{Z}\left(1, \frac{u_{1i}u_{2i}}{m_i}\right) \right)\right).$$

PROOF. (a) and (b) are consequences of 4.2.2 and 4.3.2.

(c) By (a) and 4.3.2, there is a one-to-one correspondence between $\{\text{IsoCI}(H) : H \sim_n G\}$ and $(\mathbb{Z}/m_1\mathbb{Z})^\times /\mathbb{Z}(m_1, \tau)^\times$ given by $\text{IsoCI}(H) \mapsto \prod_{i=1}^{n} u_i$ where $H \approx \bigoplus_{i=1}^{n} \left( A \oplus B + \mathbb{Z}\left(1, \frac{u_{i}}{m_i}\right) \right)$. Hence, $\ast$ is induced by multiplication in $(\mathbb{Z}/m_1\mathbb{Z})^\times /\mathbb{Z}(m_1, \tau)^\times$. \qed

It is shown in [MV1, 5.5] that $\{\text{IsoCI}(H) : H \sim_n G\}$ is a finite abelian group for any acd group $G$; however, this group has been computed explicitly in only two
cases: in the above setting where there are two incomparable types and in the case of acd groups with cyclic regulator quotients [MV2, 2.3].
CHAPTER 5
THE ENDMORPHISM RING

5.1. Structure of the Endomorphism Ring. By 4.3.2 and 4.3.3, up to isomorphism an acd group $G$ with two incomparable critical types looks like

$$A^k \oplus B^l \oplus \left( A \oplus B + \frac{Z(1,s)}{m_1} \right) \oplus \left[ \bigoplus_{i=2}^{n} \left( A \oplus B + \frac{Z(1,1)}{m_i} \right) \right]$$

where $0 \leq k, l, s, n \in Z$, $1 < m_1 | m_2 | \cdots | m_n = \exp(G/R(G))$, $A$ and $B$ are incomparable subgroups of $Q$ containing $Z$ such that $\frac{1}{p} \notin A, B$ for all primes $p | m_n$, and $\gcd(s, m_1) = 1$. Let $G_1 = A \oplus B + \frac{Z(1,s)}{m_1}$, $G_i = A \oplus B + \frac{Z(1,1)}{m_i}$ for $2 \leq i \leq n$, and $H = \bigoplus_{i=1}^{n} G_i$.

Aside. Note that $f \in \text{End} \, A$ is given by $f(a) = f(1) \cdot a$ so all positive powers of $f(1)$ lie in $A$. Hence, $f(1) \in S_A = \bigcap_{p \neq A} \mathbb{Z}_p$ and it follows that $f(S_A) \subseteq S_A$. Using this fact, it is not hard to show that, in general, if $f \in \text{End}(A^k)$ then $f(S_A^k) \subseteq S_A^k$.

Now, with $G$ as above we have that $\text{End} \, G$ is isomorphic to

$$\begin{bmatrix}
\text{End}(A^k) & 0 & \text{Hom}(H, A^k) \\
0 & \text{End}(B^l) & \text{Hom}(H, B^l) \\
\text{Hom}(A^k, H) & \text{Hom}(B^l, H) & \text{End} \, H
\end{bmatrix}$$

under matrix multiplication (with composition of functions) and addition. In order to explicitly compute $\text{End} \, G$ as a matrix ring, we must compute each of the entries in the above matrix. We will compute $\text{End}(A^k)$, $\text{Hom}(H, A^k)$, $\text{Hom}(A^k, H)$, and $\text{End} \, H$, from which the structure of the other three entries can be inferred.
(1) \( \text{End}(A^k) \approx \text{End}(S_A^k) \approx M_k(S_A) \) where the first (ring) isomorphism is the
restriction map. The second isomorphism is well-known.

(2) \( \text{Hom}(H, A^k) \approx \bigoplus_{i=1}^n \text{Hom}(G_i, A)^k \approx \bigoplus_{i=1}^n m_iS_A^k \) where the first map is the
usual one and the second map is induced by \( \text{Hom}(G_i, A) \approx m_iS_A : f \mapsto f(1, 0), 1 \leq i \leq n. \)

(3) \( \text{Hom}(A^k, H) \approx \text{Hom}(A, H)^k \approx \left[ \bigoplus_{i=1}^n \text{Hom}(A, G_i) \right]^k \approx [S_A^n]^k \) where the first
two maps are the usual ones and the third map is induced by \( \text{Hom}(A, G_i) \approx S_A : f \mapsto f(1, 1, \ldots, 1) \).

(4) \( \text{End} H \) is isomorphic to
\[
\begin{bmatrix}
\text{End} G_1 & \text{Hom}(G_2, G_1) & \cdots & \text{Hom}(G_n, G_1) \\
\text{Hom}(G_1, G_2) & \text{End} G_2 & \cdots & \text{Hom}(G_n, G_2) \\
\vdots & \vdots & \ddots & \vdots \\
\text{Hom}(G_1, G_n) & \text{Hom}(G_2, G_n) & \cdots & \text{End} G_n
\end{bmatrix}.
\]

In order to complete the computation of \( \text{End} G \), we have to compute the entries
of \( \text{End} H \). We will do a little more. We will compute \( \text{Hom}(K, L) \) for arbitrary
indecomposable rank-two acd groups \( K, L \) with the same critical types. We then
apply our results to \( \text{End} H \). First we get some preliminaries out of the way.

5.1.1. Lemma. Let \( A \) be a subgroup of \( \mathbb{Q} \) containing 1 and let \( D \) be a subring
of \( S_A \). If \( 0 \neq m \in \mathbb{Z} \) and \( \frac{1}{p} \notin A \) for all primes \( p \mid m \), then \( mA \cap D = mD \).

Proof. First, \( mD \subseteq mA \cap D \) since \( D \subseteq S_A \subseteq A \). Now suppose that \( a \in A \)
and \( ma \in D \). Write \( a = \frac{u}{v} \) (reduced); then \( p \nmid v \) for all primes \( p \mid m \). Since
\( \gcd(mu, v) = 1, \frac{1}{v} \in D \) whence \( a \in D \). \( \Box \)
Convention. Let $D$ be a subring of $\mathbb{Q}$—recall that such a ring is a PID. For $\alpha, \beta \in D$, let $\gcd(\alpha, \beta)$ denote the least positive integral $\gcd$ in $D$ of $\alpha$ and $\beta$.

5.1.2. Lemma. Let $K = A \oplus B + \mathbb{Z}\frac{(1,u)}{m}$ where $1 < m \in \mathbb{Z}$, $\frac{1}{p} \notin A, B$ for all primes $p | m$, and $u \in \mathbb{Z}$ with $\gcd(u, m) = 1$. Let $\alpha \in S_A, \beta \in S_B$, $m' \in \mathbb{Z}$ with $m' | m$ and $\gcd(\alpha, m') = 1$. Let $S = S_A S_B$. Then $\frac{(\alpha, \beta)}{m'} \in K$ if and only if $\alpha \equiv \beta \pmod{m'S}$.

Proof. $(\Rightarrow)$ Suppose that $\frac{(\alpha, \beta)}{m'} \in K$. Write $\frac{(\alpha, \beta)}{m'} = (a, b) + \frac{t(1,u)}{m}$ where $a \in A, b \in B$, and $t \in \mathbb{Z}$. Since $\alpha - m'a = t \frac{m'}{m} \in A$ and $\frac{1}{p} \notin A, B$ for all primes $p | m$, $m't = mt'$ for some $t' \in \mathbb{Z}$. Hence, we have the equations $\alpha - m'a = t'$ and $u\beta - m'b = ut'$. Since $\gcd(u, m') = 1$, there exist $x, y \in \mathbb{Z}$ with $ux + m'y = 1$; hence, $\frac{b}{u} = bx + m'\frac{b}{u}y = bx + (\beta - t')y \in B$. By Lemma 1, $a \in S_A$ and $\frac{b}{u} \in S_B$. Thus, $a - \frac{b}{u} \in S_A + S_B \subseteq S$ and $\alpha - \beta = m'(a - \frac{b}{u}) \in m'S$.

$(\Leftarrow)$ Since $\gcd(\alpha, m') = 1$ and $S_A/m'S_A \approx \mathbb{Z}/m'\mathbb{Z}$, there exist $z_A \in \mathbb{Z}$ with $\alpha \equiv z_A \pmod{m'S_A}$. Since $\alpha \equiv \beta \pmod{m'S}$, $z_A \equiv \beta \pmod{m'S}$ and it follows by 5.1.1 that $z_A \equiv \beta \pmod{m'S_B}$. Hence, $\frac{(\alpha, \beta)}{m'} = \frac{(\alpha - z_A, \beta - z_A)}{m'} + \frac{z_A(1,u)}{m'} \in K$. \qed

The following proposition gives a formula for $\text{Hom}(K, L)$ where $K, L$ are indecomposable rank-two acd groups with the same critical types.

5.1.3. Proposition. Let $K = A \oplus B + \mathbb{Z}\frac{(1,u)}{m}$ and $L = A \oplus B + \mathbb{Z}\frac{(1,v)}{n}$ where
(1) \( \mathbb{Z} \subseteq A, B \subseteq \mathbb{Q} \);

(2) \( \frac{1}{p} \notin A, B \) for all primes \( p \mid mn \);

(3) \( \gcd(u, m) = \gcd(v, n) = 1 \).

Then

\[
\text{Hom}(K, L) = \{ \frac{m}{d}(\gamma, \delta) \in \frac{m}{d}(S_A \times S_B) : \nu \gamma \equiv u \delta \pmod{dS} \}
\]

where \( d = \gcd(m, n) \) and \( S = S_A S_B \); also, \( R(\text{Hom}(K, L)) = mS_A \times mS_B \) if \( S_A, S_B \) are incomparable.

**PROOF.** Let \( f \in \text{Hom}(K, L) \). Write \( f(1, 0) = (\alpha, 0), \alpha \in S_A \), and \( f(0, 1) = (0, \beta), \beta \in S_B \). Write \( m = dm \) and \( n = dn \).

Then \( \frac{(1, u)}{m} \mapsto \frac{(\alpha, u\beta)}{m} \in L \); with a little work, it follows that \( \tilde{m} \mid \gcd(\alpha, \beta) \). Write \( \alpha = \tilde{m}\alpha' \) and \( \beta = \tilde{m}\beta' \). Choose \( \beta'' \in S_B \) with \( v\beta'' \equiv u\beta' \pmod{dS_B} \). Then \( \frac{(\alpha', u\beta'')}{d} \in L \). By 5.1.2, \( \alpha' \equiv \beta'' \pmod{dS} \); thus, \( \nu \alpha' \equiv u\beta' \pmod{dS} \). Therefore, \( f \mapsto (\alpha, \beta) \) is a well-defined monomorphism. It is easy to check that this map is surjective. The computation of \( R(\text{Hom}(K, L)) \) is straightforward. \( \square \)

Now, back to computing \( \text{End}_K \). Using 5.1.3, we get

(a) \( \text{End}G_i = \{(\alpha, \beta) \in S_A \times S_B : \alpha \equiv \beta \pmod{m_iS}\} \);

(b) \( \text{Hom}(G_1, G_i) = \{(\alpha, \beta) \in S_A \times S_B : \alpha \equiv s\beta \pmod{m_1S}\} \) if \( 1 < i \);

(c) \( \text{Hom}(G_i, G_1) = \{(\frac{m_i}{m_1}(\alpha, \beta) \in \frac{m_i}{m_1}(S_A \times S_B) : s\alpha \equiv \beta \pmod{m_1S}\} \) if \( 1 < i \);

(d) \( \text{Hom}(G_i, G_j) = \{(\alpha, \beta) \in S_A \times S_B : \alpha \equiv \beta \pmod{m_iS}\} \) if \( 1 < i < j \);

(e) \( \text{Hom}(G_i, G_j) = \{(\frac{m_i}{m_j}(\alpha, \beta) \in \frac{m_i}{m_j}(S_A \times S_B) : \alpha \equiv \beta \pmod{m_jS}\} \) if \( 1 < j < i \).
This completes the computation of the structure of $\text{End } G$.

5.2. Nearly Isomorphic Groups with Non-Isomorphic Endomorphism Rings. The following example shows that nearly isomorphic groups need not have isomorphic endomorphism rings. Since isomorphic groups have isomorphic endomorphism rings, the example indicates that there may be a pertinent notion of morphism lying somewhere between near isomorphism and isomorphism.

5.2.1. Example. Let $A = 11^{-\infty}\mathbb{Z} = S_A$, $B = 31^{-\infty}\mathbb{Z} = S_B$, $S = AB = (11 \cdot 31)^{-\infty}\mathbb{Z}$. Let
\[ G_1 = A \oplus B + \mathbb{Z}^{(1,1)}_5, \]
\[ G_2 = H_2 = A \oplus B + \mathbb{Z}^{(1,1)}_{25}, \]
\[ H_1 = A \oplus B + \mathbb{Z}^{(1,2)}_5. \]

Let $G = G_1 \oplus G_2$ and $H = H_1 \oplus H_2$. Note that $G \sim_n H$.

Set $E_G = \text{End } G$ and $E_H = \text{End } H$. Identify $E_G, E_H$ with their matrix representations as before with multiplication inherited from $\text{M}_2(\mathbb{Q} \times \mathbb{Q})$:

\[
E_G = \begin{bmatrix}
\text{End } G_1 & \text{Hom}(G_2, G_1) \\
\text{Hom}(G_1, G_2) & \text{End } G_2
\end{bmatrix},
E_H = \begin{bmatrix}
\text{End } H_1 & \text{Hom}(H_2, H_1) \\
\text{Hom}(H_1, H_2) & \text{End } H_2
\end{bmatrix}
\]

where

$\text{End } G_1 = \text{End } H_1 = \{(\alpha, \beta) \in A \times B : \alpha \equiv \beta \pmod{55}\},$

$\text{End } G_2 = \text{End } H_2 = \{(\alpha, \beta) \in A \times B : \alpha \equiv \beta \pmod{255}\},$

$\text{Hom}(G_1, G_2) = \text{End } G_1,$

$\text{Hom}(G_2, G_1) = \{5(\alpha, \beta) \in 5A \times 5B : \alpha \equiv \beta \pmod{55}\},$
\[
\text{Hom}(H_1, H_2) = \{(\alpha, \beta) \in A \times B : \alpha \equiv 2\beta \pmod{5S}\}, \text{ and}
\]
\[
\text{Hom}(H_2, H_1) = \{5(\alpha, \beta) \in 5A \times 5B : 2\alpha \equiv \beta \pmod{5S}\}.
\]

Suppose by way of contradiction that there is a ring monomorphism \( \theta : E_G \to E_H \) such that the induced map \( \bar{\theta} : E_G/R(E_G) \to E_H/R(E_H) \) is an isomorphism.

We will use the following projection maps:

\[
\Pi_A : M_2(A \times B) \to M_2(A), \quad \Pi_B : M_2(A \times B) \to M_2(B).
\]

We proceed in 5 steps toward a contradiction.

(1) \[
R = \text{R}(E_G) = \text{R}(E_H) = \begin{bmatrix}
5A \times 5B & 25A \times 25B \\
5A \times 5B & 25A \times 25B
\end{bmatrix}
\]

by 5.1.3.

(2) For \( X = \begin{bmatrix}
(0,0) & (0,0) \\
(1,1) & (0,0)
\end{bmatrix}, \quad Y = \begin{bmatrix}
(0,0) & (5,5) \\
(0,0) & (0,0)
\end{bmatrix} \in E_G
\]

we have that

\[
XY = \begin{bmatrix}
(0,0) & (0,0) \\
(0,0) & (5,5)
\end{bmatrix} \quad \text{and} \quad YX = \begin{bmatrix}
(5,5) & (0,0) \\
(0,0) & (0,0)
\end{bmatrix}
\]

so

\[
\frac{1}{5}XY, \frac{1}{5}YX \text{ are idempotents. Hence, } \frac{1}{5}\theta(XY), \frac{1}{5}\theta(YX) \text{ are idempotents since } \theta \text{ is a ring homomorphism.}
\]
(3) It is easy to check that nilpotent elements of $M_2(\mathbb{Q})$ have one of the following forms:

$$
\begin{bmatrix} 0 & 0 \\ t & 0 \end{bmatrix}, \begin{bmatrix} r & s \\ -r^2/s & -r \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}
$$

where $s, t \neq 0$. Note that $\theta(X), \theta(Y)$ are non-zero nilpotent elements since $\theta$ is a ring monomorphism and $X^2 = Y^2 = 0$.

Furthermore, idempotent elements of $M_2(\mathbb{Q})$ have one of the following forms:

$$
\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1/2 & s \\ 1/4s & 1/2 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1/2 & s \\ 1/4s & 1/2 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} r & s \\ r(1-r)/s & 1-r \end{bmatrix}
$$

where $r, s, t \neq 0$.

(4) $\theta(X) + R$ and $\theta(Y) + R$ have order 5 in $E_H/R$ so that, in view of the nilpotent forms given in (3), $\theta(X)$ must have the form 

$$
\theta(X) = \begin{bmatrix} (5a, 5b) & (5\tilde{a}, 5\tilde{b}) \\ -(a', b') & -(5a, 5b) \end{bmatrix}
$$

and $\theta(Y)$ must have the form

$$
\theta(Y) = \begin{bmatrix} (5c, 5d) & (5\gamma, 5\delta) \\ -(\tilde{c}, \tilde{d}) & -(5c, 5d) \end{bmatrix}
$$

where $a, \tilde{a}, a', c, \gamma, \tilde{c} \in A$ and $b, \tilde{b}, b', d, \delta, \tilde{d} \in B$ with $a \equiv b \pmod{5S}$, $2\tilde{a} \equiv \tilde{b} \pmod{5S}$, $a' \equiv 2b' \pmod{5S}$, $c \equiv d \pmod{5S}$, $2\gamma \equiv \delta \pmod{5S}$, $\tilde{c} \equiv 2\tilde{d} \pmod{5S}$.

Then the $(2,2)$ entry of $\frac{1}{5} \theta(XY)$ is

$$
-\frac{1}{5} \left[ (5a'\gamma, 5b'\delta) - (25ac, 25bd) \right] = -(a'\gamma - 5ac, b'\delta - 5bd)
$$
so \( 5 \nmid \gamma, \delta, a', b' \) since the order of \( \frac{1}{5}\theta(XY) + R \) is 25. In view of the nilpotent forms given in (3), \( 5 \nmid a', b' \) implies that \( 5 \mid \tilde{a}, \tilde{b} \). Similarly, \( 5 \mid \tilde{c}, \tilde{d} \). Thus, \( \theta(X) \) must have the form

\[
\theta(X) = \begin{bmatrix}
(5a, 5b) & (25a, 25\beta) \\
-(a', b') & -(5a, 5b)
\end{bmatrix}
\]

and \( \theta(Y) \) must have the form

\[
\theta(Y) = \begin{bmatrix}
(5c, 5d) & (5\gamma, 5\delta) \\
-(5c', 5d') & -(5c, 5d)
\end{bmatrix}.
\]

(5) Multiplying out, we get that

\[
\frac{1}{5}\theta(XY) = \begin{bmatrix}
(5ac - 25ac', 5bd - 25\beta d') & (5a\gamma - 25ac, 5b\delta - 25\beta d) \\
-(a'c - 5ac', b'd - 5bd') & -(a'\gamma - 5ac, b'\delta - 5bd)
\end{bmatrix} \in E_H.
\]

If \( \Pi_A(\frac{1}{5}\theta(XY)) = 0_{M_2(A)} \), then \( 5 \mid (b'\delta - 5bd) \) since \( a'\gamma - 5ac \equiv b'\delta - 5bd \) (mod 255); but this contradicts the fact that the order of \( \frac{1}{5}\theta(XY) + R \) is 25. Also, \( \Pi_A(\frac{1}{5}\theta(XY)) \neq 1_{M_2(A)} \) since 5 divides the (1,1) entry of \( \frac{1}{5}\theta(XY) \). Similarly, we get that \( \Pi_B(\frac{1}{5}\theta(XY)) \neq 0_{M_2(B)}, 1_{M_2(B)} \). Referring to the idempotent forms given in (3), we get the equations

\[
1 - 5ac + 25ac' = -a'\gamma + 5ac \quad \text{and} \quad 1 - 5bd + 25\beta d' = -b'\delta + 5bd.
\]

Referring to the nilpotent forms given in (3), the form of \( \theta(X) \) and \( \theta(Y) \) given in (4), and the above equations, we see that \( a = 0 \) if and only if \( \alpha = 0 \); also, \( c = 0 \).
if and only if \( c' = 0 \). If \( a = 0 \) or \( c = 0 \) then \(-a'\gamma = 1\) in which case \( \gamma \in S^\times \); using the fact that \( S = (11 \cdot 31)^{-\infty} \mathbb{Z} \), it is easy to show that \( \gamma \equiv \pm 1 \pmod{5S} \).

If \( a, c \neq 0 \) then \( a' = a^2/\alpha \in S \) and \( c' = c^2/\gamma \in S \) so that \( 1 - 5ac + 25\alpha c^2/\gamma = -\frac{a^2}{\alpha} \gamma + 5ac \); thus, \( a'\alpha \gamma^2 - 10ac\alpha \gamma + 25\alpha^2 c'\gamma = -\alpha \gamma \neq 0 \) so \( a'\gamma - 10ac + 25\alpha c' = -1 \).

Write \( a' = r_1^2 \bar{a}, \gamma = r_2^2 \bar{\gamma}, \alpha = r_3^2 \bar{\alpha}, \) and \( c' = r_4^2 \bar{c} \) where \( \bar{a}, \bar{\gamma}, \bar{\alpha}, \bar{c} \) are square-free integers and \( r_i \in S \). Then \( a = \bar{a} \) and \( c = \bar{c} \) since \( a^2 = a'\alpha = r_1^2 \bar{a}r_3^2 \bar{\alpha} \) and \( c^2 = r_4^2 \bar{c}r_2^2 \bar{\gamma} \). Therefore, \( a = \pm r_1r_3 \bar{a}, c = \pm r_2r_4 \bar{c}, \) and \( r_1^2 \bar{a}r_2^2 \bar{c} \pm 10r_1r_2r_3r_4 \bar{a} \bar{c} + 25r_2^2 \bar{c}r_4^2 \bar{\gamma} = -1 \); that is, \( \bar{a}c(r_1r_2 \pm 5r_3r_4)^2 = -1 \). It follows that \( \bar{c} \in S^\times \) and \( \gamma = r_2^2 \bar{c} \equiv \pm 1 \pmod{5S} \) since \( 5 \mid \gamma \).

Similarly, \( \delta \equiv \pm 1 \pmod{5S} \). This gives the desired contradiction because \( 2\gamma \equiv \delta \pmod{5S} \). □

Note that our example shows a little more than the title of this section states. It shows that there is no ring monomorphism \( E_G \rightarrow E_H \) such that the induced map \( E_G/\text{R}(E_G) \rightarrow E_H/\text{R}(E_H) \) is an isomorphism.

On the other hand, in many cases two such nearly isomorphic groups have the property that there exists a monomorphism between the endomorphism rings inducing an isomorphism of the regulator quotients:

**5.2.2. Example.** Let \( A, B \) be rank-one groups with \( Z \subseteq A, B \subseteq \mathbb{Q} \) and \( \frac{1}{p} \notin A, B \) for all primes \( p \mid m_2 \) where \( m_1 \mid m_2 \in \mathbb{Z} \) and \( S_A, S_B \) are incomparable. Let

\[
G_1 = A \oplus B + \mathbb{Z}\frac{(1,1)}{m_1},
\]
\[H_1 = A \oplus B + \mathbb{Z}^{(1,s^2)}_{m_1},\]
\[G_2 = H_2 = A \oplus B + \mathbb{Z}^{(1,1)}_{m_2}\]

where \(\gcd(s, m_2) = 1\). Let \(m = m_1\) and \(r = \frac{m_2}{m_1}\). Let \(G = G_1 \oplus G_2\) and \(H = H_1 \oplus H_2\).

Choose \(v \in \mathbb{Z}\) such that \(vs \equiv 1 \pmod{m_2\mathbb{Z}}\) and \(\gcd(v, rms) = 1\). Choose \(w \in \mathbb{Z}\) such that \(vs + wm = 1\); note that \(r \mid w\). Choose \(k', l' \in \mathbb{Z}\) such that \(k'v + l'rms = 1\) and set \(k = wk', l = wl'\). Let \(a = s^2 + smk, b = s^2 - rm^2s^4v, c = d = vl\); let \(\alpha = s^2, \beta = s^4, \gamma = -v^2, \text{ and } \delta = -1\).

Let
\[
X = \begin{bmatrix}
(0,0) & (0,0) \\
(1,1) & (0,0)
\end{bmatrix}
\text{ and } Y = \begin{bmatrix}
(0,0) & (r,r) \\
(0,0) & (0,0)
\end{bmatrix}.
\]

Now, \(S_A, S_B\) are incomparable and \(E_G\) is the purification of the subring generated by \(\Pi_A(m_1 X), \Pi_B(m_1 X), \Pi_A(m_1 Y), \text{ and } \Pi_B(m_1 Y)\), so the correspondence
\[
X \mapsto \begin{bmatrix}
rm(a,b) & r^2m^2(\alpha, \beta) \\
-(a^2/\alpha, b^2/\beta) & -rm(a,b)
\end{bmatrix}
\text{ and } Y \mapsto \begin{bmatrix}
rm(c,d) & r(\gamma, \delta) \\
-rm^2(c^2/\gamma, d^2/\delta) & -rm(c,d)
\end{bmatrix}
\]

uniquely determines a ring monomorphism \(\theta: E_G \to \mathbb{Q}E_H\). It then suffices to check that \(\theta(E_G) + R(E_H) = E_H\). \(\Box\)
REFERENCES


