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Maximum principles and Liouville theorems for elliptic partial differential equations

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MAXIMUM PRINCIPLES AND LIOUVILLE THEOREMS
FOR ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS

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ABSTRACT

Part I: Maximum Principles for Elliptic Systems.

Maximum principles and bounds of solutions for weakly coupled second order elliptic systems,
\[ Lu + Cu = 0 \text{ or } f \quad \text{in } D \subset \mathbb{R}^n, \]
are established by using the idea of Liapunov's Second Method. Here
\[ L := \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x) \frac{\partial}{\partial x_i} \]
is a second order real elliptic operator, and the \( m \times 1 \) vectors \( f, u \), and the \( m \times m \) matrix \( C \) all have entries which are complex-valued functions. It is shown that:

If there exists a complex constant matrix \( B > 0 \) such that \( C^*(x)B + BC(x) \leq 0 \) (or \( \leq -E < 0 \), \( E \) is a constant matrix) in \( D \), then for all \( C^2(D) \cap C(\overline{D}) \) solutions \( u \) of \( Lu + Cu = 0 \) (or \( f \)),
\[ \|u\|_{0,D} \leq K_1 \|u\|_{0,\partial D} \quad (\text{or } \|u\|_{0,D} \leq K_1 \|u\|_{0,\partial D} + K_2 \|f\|_{0,D}). \]
Here \( K_1 = \left( \frac{\beta_m}{\beta_1} \right)^{1/2} \) and \( K_2 = \frac{2K_1}{\mu_1} \), where \( \beta_1 \) and \( \beta_m \) are the smallest and biggest eigenvalues of \( B \), respectively, and \( \mu_1 \) is the smallest eigenvalue of \( EB^{-1} \).


Solutions in \( \mathbb{R}^n \) (\( n \geq 2 \)) of the linear, fourth order, variable coefficient, elliptic equation,
\[ L\phi := \sum_{i,j,k,l=1}^{\text{n}} a_{ijkl} \frac{\partial^4 \phi}{\partial x_i \partial x_j \partial x_k \partial x_l} + \sum_{i,j,k=1}^{\text{n}} b_{ijk} \frac{\partial^3 \phi}{\partial x_i \partial x_j \partial x_k} + \sum_{i,j=1}^{\text{n}} c_{ij} \frac{\partial^2 \phi}{\partial x_i \partial x_j} + \sum_{i=1}^{\text{n}} d_i \frac{\partial \phi}{\partial x_i} + e\phi = f, \]

and the related homogeneous equation are investigated. An entire solution is a solution of the equation defined in all \( \mathbb{R}^n \).

Schauder-type a priori estimates are developed for entire solutions with prescribed behavior at infinity. These estimates, of perhaps independent interest themselves, lead to an existence and uniqueness theory for entire solutions, with certain behavior required at infinity, when the operator \( L \) can be separated as \( L = L_2L_1 \). Here \( L_1 \) and \( L_2 \) are two second order elliptic operators.

Under some suitable conditions that guarantee the operator \( L \) approaches the biharmonic operator \( \Delta^2 \) near infinity at a certain rate, it is shown that there exists a unique entire solution in \( \mathbb{R}^n \) (\( n \geq 2 \)) of \( L\phi = f \) (\( f(x) = O(|x|^{-4-\epsilon}) \) near infinity), such that

\[ \phi(x) - \sum_{s=0}^{4-n} \gamma_s \cdot D^s T(x) \to 0 \quad \text{as} \quad x \to \infty, \]

for some constant number \( \gamma_0 \), constant vector \( \gamma_1 \) and constant symmetric matrix \( \gamma_2 \). Here \( T \) is a fundamental solution of the biharmonic equation. Moreover, a one-to-one correspondence between the entire solutions of \( L\phi = f \), with \( \phi(x) - \sum_{s=0}^{4-n} \gamma_s \cdot D^s T(x) = O(|x|^m) \) near infinity, and the biharmonic polynomials of degree no greater than \( m \) is established. When \( f \equiv 0 \), the result is an extension of a Liouville-type theorem.
# TABLE OF CONTENTS

Acknowledgements ........................................ iii
Abstract ......................................................... iv

**Part I. Maximum Principles for Elliptic Systems** ............ 1

1. Introduction ............................................ 2
2. Notation and a Liapunov Stability Theorem ................. 4
3. Maximum Principles ..................................... 6
4. Uniqueness Theorems for Some Boundary Value Problems .... 13
5. Estimate of K ........................................... 17
6. Bounds for Solutions of the Nonhomogeneous Systems ....... 24
7. Applications to Higher Order Equations .................... 28

References for Part I ........................................ 32

**Part II. Schauder Estimates and Existence Theory for Entire Solutions of Linear Fourth Order Elliptic Equations** .. 35

1. Introduction ............................................ 36
2. Norms and a Schauder Estimate for Entire Solutions ........ 38
3. Bounds for Entire Solutions of the Nonhomogeneous Biharmonic Equation ............................. 47
4. An A Priori Bound for the Fourth Order Elliptic Equation .. 63
5. Existence of Solutions ................................... 83

References for Part II ....................................... 94
PART I

MAXIMUM PRINCIPLES FOR ELLIPTIC SYSTEMS
1. INTRODUCTION

Positive definite solutions $B$ of the matrix equation $C^*B + BC = -E$ ($E > 0$) have been successfully used to construct Liapunov functions, and then to prove the stability of some ordinary differential systems $\frac{du}{dx} = Cu$ (cf: [11], [24]). This method usually is called Liapunov's Second Method. In 1974, Chow and Dunninger [2] applied this method to the study of n-metaharmonic functions, and obtained a generalized maximum principle for some classes of n-metaharmonic functions.

In this paper, we transfer the idea of Liapunov's second method to the study of weakly coupled second-order elliptic systems

$$Lu + Cu = 0 \quad \text{or} \quad f \quad \text{in} \quad D \subset \mathbb{R}^n.$$ 

Here

$$L = a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + a_i(x)\frac{\partial}{\partial x_i}, \quad a_{ij} = a_{ji}$$

is a second-order real elliptic operator, and $f = (f_1, \ldots, f_m)^T$, $u = (u_1, \ldots, u_m)^T$, and the $m \times m$ matrix $C$ all have entries which are complex-valued functions.

We establish the following generalized maximum principle for certain classes of the homogeneous system (Theorem 3.1):

If there exists a complex constant matrix $B > 0$ such that

$C^*(x)B + BC(x) \leq 0$ in $D$, then for all $C^2(D) \cap C(\overline{D})$ solutions $u$ of $Lu + Cu = 0$,

$$\|u\|_{0,D} \leq K \|u\|_{0,\partial D}.$$ 

Here $K = \left(\frac{\beta_m}{\beta_1}\right)^{1/2}$, where $\beta_1$ and $\beta_m$ are the smallest and biggest
eigenvalues of $B$, respectively.

We also find a simple sufficient condition for the classical maximum principle ($K = 1$ in the above inequality) holding; this condition is $C^*(x) + C(x) \leq 0$ in $D$ (Theorem 3.3). These results extend the result of Winter and Wong [23] for $C$ being negative semidefinite to a more general form of $C$. Generalized maximum principles for weakly coupled second-order elliptic systems have also been obtained by Dow [3], Hile and Protter [8], Szeptycki [21] and Wasowski [22] under different conditions on the coefficients.

We further show how our maximum principles may be used to prove the uniqueness of various boundary value problems of some classes of elliptic systems over bounded or unbounded domain $D \subset \mathbb{R}^n$. By using a recent result of Hile and Yeh [10], we even obtain uniqueness for a boundary value problem with an exceptional boundary set $\Gamma \subset \partial D$ with the Hausdorff dimension of $\Gamma$ less than $n - 1$.

An estimate of the best possible $K$ in our maximum principle inequality is given when $C$ is a 2 by 2 real matrix. The condition for the classical maximum principle ($K = 1$) holding can be written as

$$\text{Re}(a) \leq 0, \text{Re}(d) \leq 0, |b + \overline{c}|^2 \leq 4\text{Re}(a)\text{Re}(d)$$

when $C = \begin{bmatrix} a(x) & b(x) \\ c(x) & d(x) \end{bmatrix}$ is any 2 by 2 complex-valued matrix function.

We also study the nonhomogeneous system. Miranda [13] has studied the weakly coupled real elliptic system

$$Lu + \sum_{k=1}^{n} B_k \frac{\partial u}{\partial k} + Cu = f \quad \text{in} \quad D \subset \mathbb{R}^n,$$

giving a rather complicated condition which implies the bound for solutions $u$,
When all the matrices $B_k = 0$, $k=1,\ldots,n$, the condition required by Miranda reduces to
\[ \xi^T C \xi \leq -c_0 |\xi|^2 \]
for any $\xi \in \mathbb{R}^m$. We will extend this result to $C$ being a complex matrix function such that
\[ C^* + C \leq -2c_0 I \]  
(Corollary 6.3). Moreover, we prove the following (Theorem 6.1):

If there exist $B > 0$ and $E > 0$ such that $C(x)B + BC(x) \leq -E$ in $D$, then for all $C^2(D) \cap C(\overline{D})$ solutions $u$ of $Lu + Cu = f$,
\[ ||u||_{0,D} \leq K_1 ||u||_{0,\partial D} + K_2 ||f||_{0,D}. \]
Here $K_1 = \left( \frac{\beta_m}{\beta_1} \right)^{1/2}$ and $K_2 = \frac{2}{\mu_1} \left( \frac{\beta_m}{\beta_1} \right)^{1/2}$, where $\mu_1$ is the smallest eigenvalue of $EB^{-1}$.

Results for systems are later used to yield maximum principles and bounds for some higher order elliptic homogeneous and nonhomogeneous equations. Our maximum principles include those of Chow and Dunninger [2], [6] for real metaharmonic functions as a special case. Various maximum principles for higher order elliptic equations were also studied in the papers of Agmon [1], Duffin [4], [5], Fichera [7], Payne [14], Scheaffer and Walter [16], [17] and the books of Miranda [12] and Sperb [19].

2. NOTATION AND A LIAPUNOV STABILITY THEOREM

Unless otherwise stated, all matrices considered in this paper will be over the complex field. Let $X$ be any $m \times n$ matrix. Its transpose, complex conjugate and adjoint will be denoted by $X^T$, $\overline{X}$ and
\( x^* (x^* = x^T) \), respectively. For the sake of brevity, both Hermitian positive definite and real symmetric positive definite matrices will be named positive. Similar abbreviations hold for semipositive, negative and seminegative definite matrices. The notations \( B > 0 \), \( B \geq 0 \), \( B < 0 \) and \( B \leq 0 \) mean that the square matrix \( B \) is positive, semipositive, negative and seminegative, respectively. The norm \( ||*||_{0,D} \) means the sup norm over \( D \); thus for complex-valued vector functions \( u = (u_1, u_2, \cdots, u_m) \),

\[
||u||_{0,D} := \sup_{x \in D} |u(x)| = \sup_{x \in D} (|u_1(x)|^2 + \cdots + |u_m(x)|^2)^{1/2}.
\]

The following well-known result in Liapunov stability theory will be applied several times in this paper. We state this result as a Lemma.

**Liapunov Lemma.** Let \( C \) be an \( m \times m \) complex or real matrix.

(a) Assume that no eigenvalue of \( C \) has positive real part, and moreover that the elementary divisors of \( C \) corresponding to eigenvalues with vanishing real part are linear. Then there exist matrices \( B > 0 \) and \( E \geq 0 \) such that \( C^* B + BC = -E \);

(b) If each eigenvalue of \( C \) has negative real part, then for any \( E > 0 \), there exists a unique \( B > 0 \) such that \( C^* B + BC = -E \).

The proof of this Lemma, both for real and complex versions, can be found in many papers, such as [11], [18] and [24].
3. MAXIMUM PRINCIPLES

Consider a second-order operator,

\begin{equation}
L = a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + a_i(x) \frac{\partial}{\partial x_i}, \quad a_{ij} = a_{ji},
\end{equation}

in a bounded domain \(D\) in \(\mathbb{R}^n\). Here the summation convention is employed. We assume that \(L\) is elliptic in \(D\); i.e., for all \(x \in D\) and all \(y = (y_1, y_2, \ldots, y_n)\) in \(\mathbb{R}^n\setminus\{0\}\) the inequality

\begin{equation}
a_{ij}(x)y_iy_j > 0
\end{equation}

holds. We also suppose that the coefficients \(a_{ij}\) and \(a_i\) are bounded and real-valued functions in \(D\).

Now consider the following weakly coupled second-order elliptic system,

\begin{equation}
Lu_s(x) + c_{sk}(x)u_k(x) = 0, \quad s = 1, 2, \ldots, m, \quad \text{in } D,
\end{equation}

or, in more brief matrix form,

\begin{equation}
Lu(x) + C(x)u(x) = 0 \quad \text{in } D.
\end{equation}

Here \(C(x) = (c_{sk}(x))\) is an \(m \times m\) complex matrix function and \(u\) is a \(C^2\) \(m \times 1\) complex vector function. Associated with (3.3) is the following characteristic equation of \(C\),

\begin{equation}
h(\lambda) = |\lambda I - C| = 0.
\end{equation}

**THEOREM 3.1.** Assume that there exists a constant matrix \(B > 0\) such that

\begin{equation}
C^*(x)B + BC(x) \leq 0, \quad x \in D.
\end{equation}

Then for all \(C^2(D) \cap C(\overline{D})\) solutions \(u\) of (3.3), there exists a positive constant \(K\) such that

\begin{equation}
\|u\|_{0,D} \leq K \|u\|_{0,\partial D}.
\end{equation}
Here \( K = \left( \frac{\beta_m}{\beta_1} \right)^{1/2} \), where \( \beta_1 \) and \( \beta_m \) are the smallest and biggest eigenvalues of \( B \), respectively.

Proof: Define
\[
v = u^* Bu = u \cdot Bu = Bu \cdot u = b_{ks} \overline{u}_ku_s,\]
where \( \"\text{\cdot} \) is the dot product in \( \mathbb{C}^n \) defined by \( x \cdot y = \sum_{k=1}^{n} x_k \overline{y}_k. \)

Then \( v \) is a nonnegative function and,
\[
v, i = \frac{\partial v}{\partial x_i} = b_{ks} \overline{u}_k, u_s + b_{ks} \overline{u}_s, u_k, \]
\[
v, ij = \frac{\partial^2 v}{\partial x_i \partial x_j} = b_{ks} \overline{u}_k, u_s, u_s + b_{ks} \overline{u}_s, u_k, u_k + 2\text{Re}(b_{ks} \overline{u}_k, u_s, u_k), \]
where \( u_k, i = \frac{\partial u_k}{\partial x_i}, u_k, ij = \frac{\partial^2 u_k}{\partial x_i \partial x_j}, \) etc.;
and,
\[
L v = a_{ij} v, ij + a_i v, i = b_{ks} a_{ij} \overline{u}_k, u_s + b_{ks} \overline{u}_s, a_{ij} u_k, u_s + 2b_{ks} a_{ij} \overline{u}_k, u_s, u_k + b_{ks} a_{i} \overline{u}_k, u_s + b_{ks} \overline{u}_s, a_{i} u_k, u_s = b_{ks} \overline{(Lu)} u_s + b_{ks} \overline{u}_s (Lu) + 2a_{ij} u_s, u^* Bu, i.
\]

Thus,
\[
(3.7) \quad L v = -u^* (C^* B + B^* C) u + 2a_{ij} B^{1/2} u, i \cdot B^{1/2} u, j \geq 0, \]
since \( C^* B + B^* C \leq 0 \) and \( a_{ij} v_i \cdot v_j \geq 0 \) for any vectors \( v_1, v_2, \ldots, v_n \). Therefore, by the maximum principle for the elliptic operator \( L \), we have
\[
(3.8) \quad v(x) \leq \max_{y \in \partial D} v(y) \quad \text{for all } x \in D.
\]
Suppose that \( \{\beta_i\}_{i=1}^m \) are the eigenvalues of \( B \) with \( \beta_1 \leq \beta_2 \leq \cdots \leq \beta_m \).

Since \( B > 0 \), we know that \( \beta_1 > 0 \) and
\[
\beta_1 |u(x)|^2 \leq v(x) = u(x)^*Bu(x) \leq \beta_m |u(x)|^2 .
\]

Hence, from (3.8),
\[
|u(x)|^2 \leq \frac{\beta_m}{\beta_1} \max_{y \in \partial D} |u(y)|^2 ,
\]
and consequently, \( \|u\|_{0,D} \leq K \|u\|_{0,\partial D} \), where \( K = \left( \frac{\beta_m}{\beta_1} \right)^{1/2} \).

The Liapunov Lemma yields the following corollary.

**COROLLARY 3.2.** Let \( C(x) = r(x)I + \tilde{C} \) in (3.3), where \( r(x) \leq 0 \) in \( D \) and \( \tilde{C} \) is a constant matrix over the complex field. Assume that none of the eigenvalues of \( \tilde{C} \) has a positive real part, and moreover that the elementary divisors of \( \tilde{C} \) corresponding to eigenvalues with vanishing real part are linear. Then there exists a positive constant \( K \) such that for all \( C^2(D) \cap C(\overline{D}) \) solutions \( u \) of (3.3),
\[
(3.6) \quad \|u\|_{0,D} \leq K \|u\|_{0,\partial D} .
\]

**Proof:** By the Liapunov Lemma in Section 2, there exist matrices \( B > 0 \) and \( E \geq 0 \) such that \( \tilde{C}^*B + BC = -E \leq 0 \). Since \( r \leq 0 \) in \( D \), we get
\[
C^*(x)B + BC(x) = 2r(x)B + \tilde{C}^*B + BC \leq 0 .
\]
Now the result of this corollary follows from Theorem 3.1.

**Remark.** By the Liapunov Lemma, we know that at least one positive definite \( B \), satisfying \( \tilde{C}^*B + BC = -E \leq 0 \), exists if the matrix \( \tilde{C} \) meets the assumption of Corollary 3.2. In fact, if \( \tilde{C} \) satisfies the condition of Corollary 3.2 and has at least one eigenvalue with vanishing real
part, then there is an infinite number of positive definite $B$ such that $C^*B + B^*C = -E \leq 0$ (see [11] and [18]).

Theorem 3.1 and Corollary 3.2 are generalized maximum principles since the value of $K$ in (3.6) may be larger than 1. The best conceivable value of $K$ in (3.6), for any matrix $C$, is $K = 1$, which corresponds to the classical maximum principle.

**THEOREM 3.3.** (The Classical Maximum Principle)

(a) A sufficient condition that

\begin{equation}
(u)_{0,D} \leq \|u\|_{0,\partial D}
\end{equation}

holds, for all $C^2(D) \cap C(\bar{D})$ solutions $u$ of (3.3), is

\begin{equation}
C^* (x) + C(x) \leq 0.
\end{equation}

(b) Assume that the variable matrix $C = C(x)$ in (3.3) is normal (i.e., $C^*(x)C(x) = C(x)C^*(x)$, $x \in D$), and all its eigenvalues have nonpositive real parts for all $x \in D$. Then (3.6) holds for all $C^2(D) \cap C(\bar{D})$ solutions $u$ of (3.3).

Proof: (a) By choosing $B = I$ in Theorem 3.1, (3.6) with $K = 1$ (i.e., (3.6)) follows from the condition (3.9).

(b) Suppose

$$\lambda_1(x), \lambda_2(x), \ldots, \lambda_m(x)$$

are all the eigenvalues of $C(x)$. Since $C(x)$ is normal, there exists a unitary matrix $U(x)$ such that
\[ U^*(x)C(x)U(x) = \begin{bmatrix}
\lambda_1(x) \\
\lambda_2(x) \\
\vdots \\
\lambda_m(x)
\end{bmatrix} . \]

Therefore, by the assumption,

\[ U^*(x)(C^*(x) + C(x))U(x) = \begin{bmatrix}
2\text{Re}\lambda_1(x) \\
\vdots \\
2\text{Re}\lambda_m(x)
\end{bmatrix} =: \Lambda(x) \leq 0. \]

Hence \( C^* + C = UAU^* \leq 0 \); and then (3.6) follows from (a).

Remarks. 1. The condition (3.9) is also a 'necessary' condition for the proof of the classical maximum principle by the method imposed here. In fact, if, in Theorem 3.1, (3.6) holds with \( K = 1 \), then \( \beta_1 = \beta_m \), and so there exists a \( B > 0 \), with an \( m \)-multiple eigenvalue \( \beta > 0 \), such that \( C^*B + BC \leq 0 \); hence \( B = \beta I \), and then \( C^* + C \leq 0 \).

2. Theorem 3.3 contains the result of Winter and Wong [23] for real negative semidefinite \( C = C(x, u, \nu u) \) as a special case; one may view, for given \( u \), \( C(x, u(x), \nu u(x)) \) as a matrix function \( C_1(x) \).

**EXAMPLE 3.4.** For \( n = 2 \), consider

\[ Lu + \begin{bmatrix} a & b \\ c & d \end{bmatrix} u = 0 , \quad a, b, c, d \in \mathbb{R} . \]

The associated characteristic equation,

\[ \lambda^2 - (a+d)\lambda + (ad-bc) = 0 , \]

has roots

\[ \lambda_{\pm} = \frac{(a+d) \pm \sqrt{(a+d)^2 - 4(ad-bc)}}{2} = \frac{(a+d) \pm \sqrt{(a-d)^2 + 4bc}}{2} . \]

Hence, by Corollary 3.2, the inequality (3.6) is valid provided one of
The following conditions is satisfied:

(i) \( a+d < 0 \), \((a-d)^2 + 4bc \leq 0 \);

(ii) \( a+d < 0 \), \((a-d)^2 + 4bc > 0 \), \( ad-bc \geq 0 \);

(iii) \( a+d = 0 \), \( ad-bc > 0 \).

The inequality (3.6) is not valid for the general case, when \( a+d > 0 \) or \( a+d = 0 \), \( ad-bc \leq 0 \). In fact, \( u = \sin x \sin y \begin{bmatrix} 1 \\ -1 \\ -1 \\ -1 \end{bmatrix} \) solves the systems

\[
\Delta u + \begin{bmatrix} 0 & -1 \\ 4 & 4 \end{bmatrix} u = 0
\]

and

\[
\Delta u + \begin{bmatrix} 0 & -1 \\ -4 & 0 \end{bmatrix} u = 0
\]

in \( D = (0,\pi) \times (0,\pi) \) and vanishes on \( \partial D \), but (3.6) does not hold.

Theorem 3.1 gives a sufficient condition for which (3.6) holds. It raises some open questions as to whether Theorem 3.1 can be extended to a more general system (3.3) with weaker restrictions on the matrix \( C \), and as to whether necessary conditions can be determined so that (3.6) holds.

Following from the inequality

(3.7) \( L(u^*Bu) \geq -u^*(C^*B + BC)u + 2a_{ij}B^{1/2}u^1B^{1/2}u^j \geq 0 \),

in the proof of Theorem 3.1, and from Protter and Weinberger’s book [15], are the following two maximum principles for system (3.3).

**Corollary 3.5.** If \( u \in C^2(D) \cap C(\bar{D}) \) is a solution of (3.3), and if \( u^*Bu \) attains a maximum in \( D \) for some positive definite matrix \( B \) such that \( C^*(x)B + BC(x) \) is negative semidefinite in \( D \), then \( u \) is a complex
constant vector in $\Omega$. Moreover, if $C^*(x)B + BC(x)$ is negative definite at some $x \in \Omega$, or, if $C(x)$ is invertible for some $x \in \Omega$, then $u = 0$ in $\Omega$.

Proof: Under the assumption of this corollary, by the proof of Theorem 3.1, inequality (3.7) holds. Thus, by the maximum principle for the second-order elliptic equation (see [15]), $u^*Bu = \text{constant}$.

Hence, from (3.7) again, we have

$$0 = L(u^*Bu) = -u^*(C^*B + BC)u + 2a_{ij}B^{1/2}u^*_i B^{1/2}u^*_j \geq 0 \quad \text{in } \Omega;$$

which implies that $u^*(C^*B + BC)u = 0$ and $a_{ij}B^{1/2}u^*_i B^{1/2}u^*_j = 0$, and then, $B^{1/2}u^*_i = 0$ and $u^*_i = 0$ in $\Omega$, for $1 \leq i \leq n$. Thus $u$ is a complex constant vector in $\Omega$. Moreover, if $C^*(x)B + BC(x) < 0$ at some $x \in \Omega$, then, from $u^*(C^*(x)B + BC(x))u = 0$, we have $u = 0$ in $\Omega$; and if $C(x)$ is invertible for some $x \in \Omega$, then, from the system (3.3), we still have $u = 0$ in $\Omega$.

Note that Corollary 3.5 actually holds even if $\Omega$ is unbounded.

**Corollary 3.6.** Let $u \in C^2(\Omega) \cap C(\overline{\Omega})$ be a solution of (3.3). Suppose that $u^*Bu \leq M$ in $\Omega$ and that $u^*Bu = M$ at a point $P \in \partial \Omega$ for some positive definite $B$ such that $C^*B + BC$ is negative semidefinite.

Here $M$ is a nonnegative constant. Assume that $P$ lies on the boundary of a ball in $\Omega$, and that the outward directional derivative $\frac{\partial u}{\partial v}$ exists at $P$. Then

$$\frac{\partial(u^*Bu)}{\partial v} = 2\text{Re}[u^*B\frac{\partial u}{\partial v}] = 2\text{Re}[(B^{1/2}u)^* \frac{\partial (B^{1/2}u)}{\partial v}] > 0 \quad \text{at } P$$

unless $u$ is a complex constant vector such that $u^*Bu = M$. 

equivalently,
\[ \frac{\partial |B^{1/2}u|}{\partial \nu} > 0 \quad \text{at } P \]
unless \( u \) is constant and \( |B^{1/2}u| = M^{1/2} \).

4. UNIQUENESS THEOREMS FOR SOME BOUNDARY VALUE PROBLEMS

As applications of the maximum principles in section 3, we can prove uniqueness theorems for various boundary value problems.

As an example, consider the first boundary value problem for the elliptic system (3.3). By Theorem 3.1, the problem
\[ (3.3)_N \quad Lu(x) + C(x)u(x) = f(x), \quad x \in D, \]
\[ (4.1) \quad u|_{\partial D} = g(x), \]
where \( C \) satisfies the assumption of Theorem 3.1, has at most one solution.

As a second example, we have uniqueness for the following mixed boundary problem:
\[ (3.3)_N \quad Lu(x) + C(x)u(x) = f(x) \quad \text{in } D \subset \mathbb{R}^n, \]
\[ (4.2) \quad \begin{cases} u(x) = g_1(x) & \text{on } \Gamma_1, \\ \frac{\partial u(x)}{\partial \nu} + \alpha(x)u = g_2(x) & \text{on } \Gamma_2, \end{cases} \]
where \( \nu = \nu(x) \) is a given outward direction on \( \Gamma_2 \), and \( \Gamma_2 = \partial D \setminus \Gamma_1 \).

COROLLARY 4.1. Suppose \( u^1 \) and \( u^2 \) satisfy (3.3)$_N$ and (4.2) in a bounded domain \( D \subset \mathbb{R}^n \) and \( C \) satisfies the assumption of Theorem 3.1. Assume that each point of \( \Gamma_2 \) lies on the boundary of a ball in \( D \). If \( L \) is elliptic as defined by (3.1) and \( \alpha(x) \geq 0 \) on \( \Gamma_2 \), then \( u^1 = u^2 \),
except when $\alpha = 0$, $\Gamma_1$ is vacuous and $C(x)$ is singular for all $x \in D$, in which case $u^1 - u^2$ is a complex constant vector.

Proof: Define $u = u^1 - u^2$. Then $u$ satisfies

\begin{equation}
Lu + Cu = 0 \quad \text{in } D,
\end{equation}

\begin{equation}
\frac{\partial u}{\partial n} + \alpha u = 0 \quad \text{on } \Gamma_2.
\end{equation}

Choose positive definite $B$, as in the proof of Theorem 3.1, such that $C^*B + BC$ is negative semidefinite, and let $v = u^*Bu$; then (3.7) holds. If $v$ is not a constant, by Corollary 3.5, a nonzero maximum of $v$ must occur at a point $P$ on $\Gamma_2$; and by Corollary 3.6,

$$\text{Re}\left[u^*B\frac{\partial u}{\partial n}\right] > 0 \quad \text{at } P.$$

Hence,

$$\text{Re}\left[u^*B\left(\frac{\partial u}{\partial n} + \alpha u\right)\right] > 0 \quad \text{at } P \in \Gamma_2,$$

which contradicts the boundary condition (4.3). Thus $v = u^*Bu$ must be a constant. From (3.7), by the proof of Corollary 3.5, $u$ must be a complex constant vector. Then, from (3.3) and (4.3), we know that $u = 0$, i.e., $u^1 = u^2$ in $D$, except when $\alpha = 0$, $\Gamma_1$ is vacuous and $C(x)$ is singular for all $x \in D$. $\Box$

Besides having applications to some boundary value problems with bounded domain $D$, the maximum principle we obtained in section 3 can also be used to establish uniqueness theorems for various boundary value problems with unbounded domain $D \subseteq \mathbb{R}^n$. In this case, the point $\omega$ is called the exceptional boundary point, and an appropriate growth restriction on the solution at $\omega$ is required. Moreover, by using a
recent result of Hile and Yeh [10], we even obtain uniqueness for the boundary value problem with an exceptional boundary set \( \Gamma \) such that the Hausdorff dimension of \( \Gamma \) is less than \( n - 1 \).

As a third example of this section, we have uniqueness for the following boundary value problem with unbounded domain \( D \subseteq \mathbb{R}^n \):

\[
\begin{align*}
\mathcal{L}u(x) + C(x)u(x) &= f(x) \quad \text{in } D, \\
\begin{cases}
u(x) = g(x) & \text{on } \partial D, \\
|u(x)| & \text{is bounded in } D.
\end{cases}
\end{align*}
\]

**COROLLARY 4.2.** Let \( D \) be an unbounded domain contained inside a cone, let \( L \) be given in \( D \) by (3.1), with

\[a_i(x) = o(r^{-1}) \quad \text{as } x \to \infty \quad \text{in } D, \quad i = 1, \ldots, n, \quad (r = |x|),\]

and with the uniform ellipticity condition

\[
\delta |y|^2 \leq a_{ij}(x)y_iy_j \leq \Lambda |y|^2, \quad x \in D, \quad y \in \mathbb{R}^n,
\]

holding for some positive constants \( \delta \) and \( \Lambda \). Suppose \( u^1 \) and \( u^2 \) satisfy (3.3) \( _N \), (4.4) in \( D \) and matrix \( C \) satisfies the assumption of Theorem 3.1. Then \( u^1 = u^2 \) in \( D \).

**Proof:** Let \( u = u^1 - u^2 \) and \( v = u^*Bu \), where \( B \), as in Theorem 3.1, is positive definite such that \( C^*(x)B + BC(x) \) is negative semidefinite for all \( x \in D \). Then \( v \) is nonnegative, and the proof of Theorem 3.1 gives

\[
\begin{align*}
\mathcal{L}v &= -u^*(C^* + BC)u + 2a_{ij}B^{1/2}uB^{1/2}u \geq 0 \quad \text{in } D, \\
v(x) &= 0 \quad \text{on } \partial D, \\
v(x) &\text{ is bounded in } D.
\end{align*}
\]

Hence, by the Phragmen-Lindelof Principle ([9], Corollary 2),

---
\[ \lim v(x) = 0, \text{ as } x \to \infty \text{ in } D. \]

Therefore, the maximum principle (Theorem 3.1) implies that \( v = 0 \) in \( D \); i.e., \( u^1 = u^2 \) in \( D \).

**Remark.** When \( L = \Delta \) (Laplace operator) in \((3.3)_N\), the growth restriction of \(|u|\) being bounded in Corollary 4.2 can be replaced by the following weak restriction:

\[
\liminf_{r \to \infty} \frac{\max_{|x| = r} |u(x)|^2}{\log r} = 0 \quad \text{if } n = 2,
\]

or

\[
\liminf_{r \to \infty} \frac{\sup_{|x| = r} |u(x)|^2}{r^{n-2}} = 0 \quad \text{if } n \geq 3
\]

(see [15]).

As a last example, we prove uniqueness for the solution of the following boundary problem without knowing the data on an exceptional boundary set \( \Gamma \subset \partial D \) (the domain \( D \) can be unbounded):

\[
(3.3)_N' \quad \left( \Delta + a_i(x) \frac{\partial}{\partial x_i} \right) u(x) + C(x) u(x) = f(x) \quad \text{in } D,
\]

\[
(4.5) \quad \begin{cases}
  u(x) = g(x) & \text{on } \partial D \setminus \Gamma, \\
  |u(x)| & \text{is bounded in } D.
\end{cases}
\]

**Corollary 4.3.** Suppose \( u^1 \) and \( u^2 \) are two \( C^2 \) solutions of the problem \((3.3)_N'\), (4.5); and assume that \( |a_i(x)|, 1 \leq i \leq n \), are bounded in \( D \) and the matrix \( C(x) \) satisfies the assumption of Theorem 3.1. Let \( \Gamma \) be a subset of \( \partial D \) such that, for each \( y \) on \( \Gamma \), \( D \) is contained on one side of an \((n-1)\)-dimensional hyperplane passing through \( y \). Assume also that the Hausdorff dimension of \( \Gamma \) is less than \( n - 1 \). Then \( u^1 = u^2 \) in \( D \).
Proof: Let $u = u^1 - u^2$ and $v = u^*Bu$ where $B$, as in Theorem 3.1, is positive definite such that $C^*(x)B + BC(x)$ is negative semidefinite over $D$. Then $v$ is nonnegative, and the proof of Theorem 3.1 implies that

$$
\begin{cases}
\left(\Delta + a \frac{\partial}{\partial x_1}\right)v(x) = -u^*(C^*B + BC)u + 2a_{ij}B^{1/2}u_B^{1/2}u \geq 0 \text{ in } D, \\
v(x) = 0 \text{ on } \partial D \setminus \Gamma, \\
v(x) \leq M \text{ in } D, \text{ for some } M > 0.
\end{cases}
$$

Hence, by a result of Hile and Yeh ([10], Corollary 2),

$$
v \equiv 0 \text{ on } \Gamma.
$$

Therefore,

$$
v \equiv 0 \text{ on } \partial D.
$$

Thus, by the maximum principle (if $D$ is bounded) or Phragmen-Lindelof principle (if $D$ is unbounded) for the second order elliptic equation, we have $v \equiv 0$ in $D$. This gives $u^1 = u^2$ in $D$.

5. ESTIMATE OF $K$

In Theorem 3.1, the constant $K = \left(\frac{\beta_m}{\beta_1}\right)^{1/2}$ in (3.6) depends on $C$; in fact, $\beta_1$ and $\beta_m$ are the smallest and largest eigenvalues of $B$, where $B > 0$ is chosen so that

$$
C(x)B + BC(x) \leq 0.
$$

The value of $K$ in (3.6) is important in applications, such as in numerical computation and estimation. The best conceivable value of $K$, for any $C$, is $K = 1$, which corresponds to the classical maximum
principle; and Theorem 3.3 gives a sufficient condition (3.9) to
guarantee the classical maximum principle (K = 1) for solutions of
(3.3). However, for a general matrix C, the best possible value of K
can be larger than 1.

**EXAMPLE 5.1.** For the system

\[ L u(x) + \begin{bmatrix} a(x) & b(x) \\ c(x) & d(x) \end{bmatrix} u(x) = 0, \]

where a, b, c and d are complex-valued functions, by Theorem 3.3 the
classical maximum principle (3.6) holds if

\[ \begin{bmatrix} a & b \\ c & d \end{bmatrix}^* \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 2\text{Re}(a) & b+c \\ b+c & 2\text{Re}(d) \end{bmatrix} \leq 0, \]

which is equivalent to

\[ \text{Re}(a) \leq 0, \text{Re}(d) \leq 0; \text{ and } |b + c|^2 \leq 4\text{Re}(a)\text{Re}(d). \]

The following two examples can be used to compute the best choice
of K = \( \left( \frac{\beta}{\alpha} \right)^{1/2} \) when C = \( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \) is a real constant matrix with
eigenvalues \( \lambda_- \leq \lambda_+ \leq 0 \) (\( \lambda_+ \), \( \lambda_- \) not both zero).

**EXAMPLE 5.2.** Consider

\[ L u + \begin{bmatrix} -1 & \epsilon \\ 0 & -1 \end{bmatrix} u = 0, \quad \epsilon \in \mathbb{R}. \]

In order to determine the best K, we choose a positive definite
B := \( \begin{bmatrix} a & \beta \\ \beta & \gamma \end{bmatrix} \) which minimizes

\[ K = \left( \frac{\beta}{\alpha} \right)^{1/2} = \left( \frac{(\alpha+\gamma) + \sqrt{(\alpha-\gamma)^2 + 4|\beta|^2}}{(\alpha+\gamma) - \sqrt{(\alpha-\gamma)^2 + 4|\beta|^2}} \right)^{1/2}, \]

and satisfies

\[ \begin{bmatrix} -2\alpha & \alpha\epsilon-2\beta \\ \alpha\epsilon-2\beta & \epsilon(\beta+\alpha)-2\gamma \end{bmatrix} \leq 0. \]
Without loss of generality, we can assume that

\[(5.4) \quad \alpha + \gamma = 1. \]

Then the problem can be reduced to the following equivalent problem:

Find \((\alpha, \beta) \in \mathbb{R} \times \mathbb{C}\) which

minimizes \((2\alpha-1)^2 + 4|\beta|^2\),

subject to: \(\alpha > \alpha^2 + |\beta|^2\),

\[\varepsilon(\beta + \overline{\beta}) \leq 2,\]

\[\alpha^2(\varepsilon^2 + 4) + 4|\beta|^2 - 4\alpha \leq 0.\]

Obviously, \(\beta = 0\) is the best. Thus the problem reduces to

minimize \((2\alpha - 1)^2\),

subject to: \(\alpha > \alpha^2\),

\[\alpha^2(\varepsilon^2 + 4) - 4\alpha \leq 0.\]

It is easy to conclude that

\[ (\alpha, \beta) = \begin{cases} \left(\frac{1}{2}, 0\right), & \text{if } |\varepsilon| \leq 2, \\ \left(\frac{4}{4 + \varepsilon^2}, 0\right), & \text{if } |\varepsilon| > 2. \end{cases} \]

Hence by (5.3) and (5.4), the best value of \(K = \left(\frac{\beta_m}{\beta_1}\right)^{1/2}\) for the system (5.2) is

\[ K = \begin{cases} \left\lfloor \frac{1}{|\varepsilon|} \right\rfloor, & \text{if } |\varepsilon| \leq 2, \\ \left\lfloor \frac{2}{|\varepsilon|} \right\rfloor, & \text{if } |\varepsilon| > 2. \end{cases} \]

**EXAMPLE 5.3.** Now we consider the generalized form of (5.2),

\[(5.5) \quad Lu + \left[\begin{array}{cc} -1 & \varepsilon \\ 0 & -\sigma \end{array}\right] u = 0, \quad \sigma \geq 0, \quad \varepsilon \in \mathbb{R}. \]

By going through the same process, we can reduce the problem of minimizing \(K\) in (3.6) to finding

\[ B = \left[\begin{array}{cc} \alpha & \beta \\ \beta & 1-\alpha \end{array}\right] > 0, \]

which
minimizes \((2\alpha-1)^2 + 4|\beta|^2\),

subject to: \(\alpha > 0\),
\[
\alpha - \alpha^2 - |\beta|^2 > 0,
\]
\[
(\sigma-1)(\beta+\bar{\beta})\alpha\varepsilon + 4\sigma(\alpha-\alpha^2) - \alpha^2\varepsilon - (1+\sigma)^2|\beta|^2 \geq 0.
\]

Using Lagrange's method of multipliers and after going through a rather complicated computation, we get

\[
(\alpha, \beta) = \begin{cases}
(\frac{1}{2}, 0), & \varepsilon^2 \leq 4\sigma, \\
((1+\sigma)^2\varepsilon^2 + 2\sqrt{\sigma} (1+\sigma)\sqrt{\varepsilon^2[(\varepsilon^2+1)+(\varepsilon^2-4\sigma)(\sigma-1)\varepsilon^2]}), & \varepsilon^2 > 4\sigma.
\end{cases}
\]

Hence we find that the best choice of \(K\) in (3.6) for the system (5.5) is

\[
K = \begin{cases}
1 & \varepsilon^2 \leq 4\sigma, \\
\frac{1}{\varepsilon^2[(\varepsilon^2+1)+(\varepsilon^2-4\sigma)(\sigma-1)\varepsilon^2]^{1/2}} & \varepsilon^2 > 4\sigma.
\end{cases}
\]

For a general real matrix \(C = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\) with eigenvalues \(\lambda_- \leq \lambda_+ \leq 0\), \(\lambda_- < 0\), by matrix theory (see [20], Chapter 5), there exists an orthogonal matrix \(P := \begin{bmatrix} p & q \\ r & s \end{bmatrix}\) and a real number \(\varepsilon_0\) such that

\[
\begin{bmatrix}
\lambda_- & \varepsilon_0 \\
0 & \lambda_+
\end{bmatrix} = P^T CP = \begin{bmatrix}
p^2a+prc+prb+r^2d & pqa+qrc+spb+sr^2d \\
pqa+pqc+qrb+qsd & q^2a+qsc+qs+q^2s^2d
\end{bmatrix}.
\]

From \(0 = pqa+psc+rqb+rsd\), we have

\[
\varepsilon_0 = pqa+qrc+spb+sr^2d = (ps-rq)(b-c) = \det P^T (b-c) = \pm(b-c).
\]

Let \(B = PB_1P^T\). Then

\[
C^TB + BC = P \begin{bmatrix}
\lambda_- & \varepsilon_0 \\
0 & \lambda_+
\end{bmatrix}^TB_1 + B_1 \begin{bmatrix}
\lambda_- & \varepsilon_0 \\
0 & \lambda_+
\end{bmatrix} P^T \leq 0.
\]
is equivalent to
\[
\begin{bmatrix}
\lambda_- & \epsilon_0 \\
0 & \lambda_+
\end{bmatrix}^T B_1 + B_1 \begin{bmatrix}
\lambda_- & \epsilon_0 \\
0 & \lambda_+
\end{bmatrix} \leq 0 ,
\]
or
\[
\begin{bmatrix}
-1 & -\frac{\epsilon_0}{\lambda_-} \\
0 & -\frac{\lambda_+}{\lambda_-}
\end{bmatrix}^T B_1 + B_1 \begin{bmatrix}
-1 & -\frac{\epsilon_0}{\lambda_-} \\
0 & -\frac{\lambda_+}{\lambda_-}
\end{bmatrix} \leq 0 .
\]
By letting \( \epsilon = -\frac{\epsilon_0}{\lambda_-} = \frac{b-c}{\lambda_-} \) and \( \sigma = \frac{\lambda_+}{\lambda_-} \), from the Example 5.3 we have the following:

**Conclusion:** The best value of \( K = \left( \frac{\beta_m}{\beta_1} \right)^{1/2} \) for \( C = [a \\ b \\ c \\ d] \in \mathbb{R}^{2 \times 2} \) in (3.3) with eigenvalues
\[
\lambda_{\pm} = \frac{(a+d) \pm \sqrt{(a-d)^2 + 4bc}}{2} \leq 0 , \text{ and } (a+d) < 0 ,
\]
is
\[
K^2 = \begin{cases}
1 , & \text{if } (b-c)^2 \leq 4 \det C ; \\
\frac{\epsilon_0^2 \left[ (\lambda_- - \lambda_+)^2 \right] + \left[ -2 \sqrt{\lambda - \lambda_+ (\lambda_- + \lambda_+) (\epsilon_0^2 - 4 \lambda_- \lambda_+) + (\epsilon_0^2 - 4 \lambda_- \lambda_+) \sqrt{\lambda - \lambda_+ (\epsilon_0^2 + (\lambda_- - \lambda_+)^2) \right]}}{\epsilon_0^2 \left[ (\lambda_- - \lambda_+)^2 \right] + \left[ -2 \sqrt{\lambda - \lambda_+ (\lambda_- + \lambda_+) - (\epsilon_0^2 - 4 \lambda_- \lambda_+) \sqrt{\lambda - \lambda_+ (\epsilon_0^2 + (\lambda_- - \lambda_+)^2) \right]}} , & \text{if } (b-c)^2 > 4 \det C .
\end{cases}
\]

**Remarks.** 1. By using
\[
\lambda_- \lambda_+ = \det C ,
\]
\[
\lambda_- + \lambda_+ = \tr C ,
\]
\[
\epsilon_0^2 = (b-c)^2 = a^2 + b^2 + c^2 + d^2 - (a+d)^2 + 2(ad-bc) = (\tr C^T C)^2 - (\tr C)^2 + 2 \det C ,
\]
\[(\lambda^- - \lambda^+)^2 = (\text{tr} C)^2 - 4\text{det} C,\]
we can express \(K\) in terms of the matrix \(C\).

2. The condition for the classical maximum principle \((K = 1)\) is
\[(b-c)^2 \leq 4\text{det} C,\]
and the equivalent condition for it is
\[(b+c)^2 \leq 4ad\] (which is the same as (5.1)),
or
\[(\text{tr} C^T C)^2 \leq (\text{tr} C)^2 + 2\text{det} C.\]

It is clear that not only the normal matrices, but also some other matrices satisfy one of the above conditions.

In case \(C = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{R}^{2 \times 2}\) has two complex conjugate eigenvalues with nonpositive real parts,
\[(5.6) \quad \lambda_{\pm} = \frac{(a+d) \pm \sqrt{-(a-d)^2 - 4bc}}{2} i := \mu \pm \nu i, \quad \left( \mu, \nu \in \mathbb{R}, \mu \leq 0, \nu > 0 \right),\]
the problem of finding the best choice of \(K\) in (3.6) is very complicated. The method we used above for real eigenvalues does not succeed. Here we can only give a formula for an upper bound for the best possible \(K\).

From (5.6), we have
\[2\mu = a + d \leq 0,\]
\[2\nu = \sqrt{-(a-d)^2 - 4bc} > 0.\]
Suppose that \(\xi := \alpha + i\beta \in \mathbb{C}^2\) (when \(\alpha = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}, \beta = \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} \in \mathbb{R}^2\)) is an eigenvector of \(C\) corresponding to the eigenvalue \(\lambda_+ = \mu + \nu i\). Then
\[C(\alpha + i\beta) = (\mu + \nu i)(\alpha + i\beta),\]
\[ C(\alpha-i\beta) = (\mu-i\nu)(\alpha-i\beta), \]

which gives

\[ \begin{align*}
    C\alpha &= \mu\alpha - \nu\beta, \\
    C\beta &= \mu\beta + \nu\alpha,
\end{align*} \]

or, in matrix form

\[ C(\alpha \beta) = (\alpha \beta) \begin{bmatrix} \mu & \nu \\ -\nu & \mu \end{bmatrix}. \]

The general solution for \( (\alpha \beta) \) in the above equation is

\[ (5.7) \quad \alpha = \frac{1}{\nu}(C-\mu I)\beta, \quad \text{for any } \beta = \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} \in \mathbb{R}^2\setminus\{0\}; \]

i.e.,

\[ (5.8) \quad \begin{cases}
    a_1 = \frac{a-\mu \beta_1 + b \beta_2}{\nu}, \\
    a_2 = \frac{c \beta_1 + d - \mu \beta_2}{\nu^2}, \\
\end{cases} \quad \text{for any } \beta_1, \beta_2 \in \mathbb{R}^2\setminus\{0\}. \]

Since \( \nu > 0 \) and \( bc \neq 0 \), we see that \( \alpha \) and \( \beta \) are independent over \( \mathbb{R}^2 \). Hence the matrix \( (\alpha \beta) \) is nonsingular and

\[ C = (\alpha \beta) \begin{bmatrix} \mu & \nu \\ -\nu & \mu \end{bmatrix} (\alpha \beta)^{-1}. \]

Let

\[ B = (\alpha \beta)^{-1}^T B_1 (\alpha \beta)^{-1} > 0; \]

then

\[ C^TB + BC = (\alpha \beta)^{-1}^T \begin{bmatrix} \mu & \nu \\ -\nu & \mu \end{bmatrix} (\alpha \beta)^{-1}^T B_1 (\alpha \beta)^{-1} \]

\[ + (\alpha \beta)^{-1}^T B_1 (\alpha \beta)^{-1} (\alpha \beta) \begin{bmatrix} \mu & \nu \\ -\nu & \mu \end{bmatrix} (\alpha \beta)^{-1} \]

\[ = (\alpha \beta)^{-1}^T \left( \begin{bmatrix} \mu & \nu \\ -\nu & \mu \end{bmatrix} B_1 + B_1 \begin{bmatrix} \mu & \nu \\ -\nu & \mu \end{bmatrix} \right) (\alpha \beta)^{-1}. \]

Hence,

\[ C^TB + BC \leq 0 \quad \text{iff} \]

\[ (5.9) \quad \begin{bmatrix} \mu & \nu \\ -\nu & \mu \end{bmatrix} B_1 + B_1 \begin{bmatrix} \mu & \nu \\ -\nu & \mu \end{bmatrix} \leq 0. \]

The inequality (5.9) holds for \( B_1 = I \), and in this case,

\[ B = (\alpha \beta)^{-1}^T (\alpha \beta)^{-1} = [(\alpha \beta)(\alpha \beta)^T]^{-1} > 0. \]

The characteristic equation for \( (\alpha \beta)(\alpha \beta)^T \) is
\[
\det \left[ \lambda I - (\alpha \beta)(\alpha \beta)^T \right] = \det \left[ \begin{array}{cc}
\lambda - (\alpha_1^2 + \beta_1^2) & -(\alpha_1 \alpha_2 + \beta_1 \beta_2) \\
-(\alpha_1 \alpha_2 + \beta_1 \beta_2) & \lambda - (\alpha_2^2 + \beta_2^2)
\end{array} \right]
\]
\[
= \lambda^2 - \text{tr}(\alpha \beta)(\alpha \beta)^T \lambda + \det(\alpha \beta)(\alpha \beta)^{-1}
\]
\[
= \lambda^2 + (\alpha_1^2 + \alpha_2^2 + \beta_1^2 + \beta_2^2) \lambda + (\alpha_1 \beta_1 - \alpha_2 \beta_2)^2.
\]

Let \( \lambda_1, \lambda_2 \) \((0 < \lambda_1 \leq \lambda_2)\) be the two eigenvalues of \((\alpha \beta)(\alpha \beta)^T\). Then the eigenvalues of \( B \) are \(0 < \lambda_1^{-1} \leq \lambda_2^{-1} \), and
\[
K^2 = \frac{\lambda_1^{-1}}{\lambda_2^{-1}} = \frac{\lambda_2}{\lambda_1} = \frac{(\alpha_1^2 + \alpha_2^2 + \beta_1^2 + \beta_2^2) + \sqrt{\left(\alpha_1^2 + \alpha_2^2 + \beta_1^2 + \beta_2^2\right)^2 - 4(\alpha_1 \beta_1 - \alpha_2 \beta_2)^2}}{(\alpha_1^2 + \alpha_2^2 + \beta_1^2 + \beta_2^2) - \sqrt{\left(\alpha_1^2 + \alpha_2^2 + \beta_1^2 + \beta_2^2\right)^2 - 4(\alpha_1 \beta_1 - \alpha_2 \beta_2)^2}}.
\]

By (5.8), we have
\[
\alpha_1^2 + \alpha_2^2 + \beta_1^2 + \beta_2^2 = \frac{1}{\nu^2} |b-c| (|c\beta_1^2 + (a-d)\beta_1 \beta_2 + b\beta_2^2|),
\]
\[
(\alpha_1 \beta_1 - \alpha_2 \beta_2) = \frac{1}{\nu^2} (-c\beta_1^2 + (a-d)\beta_1 \beta_2 + b\beta_2^2).
\]
Hence, by choosing \(\beta_1, \beta_2 \in \mathbb{R} \setminus \{0\}\) such that
\[-c\beta_1^2 + (a-d)\beta_1 \beta_2 + b\beta_2^2 \neq 0,\]
we obtain
\[
(5.10) \quad K^2 = \frac{1}{\nu^2} \frac{|b-c| + \frac{1}{\nu^2} \sqrt{(b-c)^2 - 4\nu^2}}{\sqrt{(b-c)^2 - 4\nu^2}} = \frac{|b-c| + \sqrt{(b+c)^2 - (a-d)^2}}{|b-c| - \sqrt{(b+c)^2 - (a-d)^2}}.
\]

Since the choice of \(B_1 = I\) in (5.9) may not be the best choice, the formula (5.10) for \(K^2\) is not the best one. From formula (5.10), we obtain \(K = 1\) when \(a = d\) and \(b = -c\).

6. BOUNDS FOR SOLUTIONS OF THE NONHOMOGEneous ELLIPTIC SYSTEM

In section 3 we obtained maximum principles for some homogeneous elliptic systems. Now we extend these results to the nonhomogeneous
elliptic system

\[ (3.3)_N \quad Lu(x) + C(x)u(x) = f(x), \quad x \in D. \]

Under the restriction of \( C \) being a real matrix function such that
\[ \xi^T C \xi \leq -c_0 |\xi|^2 \quad \text{in} \ D, \quad \text{for some} \ c_0 > 0 \quad \text{and for any} \ \xi \in \mathbb{R}^m, \]
Miranda [13] obtained a bound for solutions of the elliptic system \((3.3)_N\):

\[ \|u\|_0,D \leq \|u\|_0,\partial D + c_0^{-1}\|f\|_0,D. \]

In this section, we obtain a similar result for more general complex matrix functions \( C \) in the elliptic system \((3.3)_N\).

**Theorem 6.1.** Suppose, for \( C = C(x) \), there exist two complex constant matrices \( B > 0, E > 0 \) such that \( C^*(x)B + BC(x) \leq -E \). Then for all \( C^2(D) \cap C(\overline{D}) \) solutions \( u \) of \((3.3)_N\), there exist positive constants \( K_1 \) and \( K_2 \) such that

\[ (6.1) \quad \|u\|_0,D \leq K_1 \|u\|_0,\partial D + K_2 \|f\|_0,D. \]

Here \( K_1 = \left( \frac{\beta_m}{\beta_1} \right)^{1/2} \) and \( K_2 = \frac{2 \beta_m^{1/2}}{\mu_1 \beta_1^{1/2}} \), where \( \beta_1 \) and \( \beta_m \) are the smallest and biggest eigenvalues of \( B \), respectively, and \( \mu_1 \) is the smallest eigenvalue of \( EB^{-1} \).

**Proof:** Define \( v = B^{1/2}u \); i.e., \( u = B^{-1/2}v \).

First we will prove that (for \( v \)),

\[ (6.2) \quad \|v\|_0,D \leq \|v\|_0,\partial D + K \|f\|_0,D, \quad K > 0. \]

It is sufficient to show that at an internal relative maximum of \( |v| \)
(or of \( |v|^2 = |B^{1/2}u|^2 = Bu \star u \)),

\[ (6.3) \quad |v(x)| \leq K|f(x)|. \]
We assume $v(x) \neq 0$ at such a maximum; otherwise the inequality is trivial.

At a relative maximum of $|v|^2 = Bu \cdot u$, we have
\[ \frac{\partial}{\partial x_k} |v|^2 = 2\text{Re}(Bu \cdot u)_k = 0 , \]
and the matrix of second derivatives is negative semidefinite; i.e.,
\[ \left[ \frac{\partial^2 |v|^2}{\partial x_i \partial x_k} \right]_{n \times n} = \left[ 2\text{Re}(v_i \cdot v_k + v \cdot v_{ik}) \right]_{n \times n} \leq 0 . \]

Hence,
\[ \text{Re}(Bu \cdot u)_k = \text{Re}(v \cdot v_k) = 0 , \]
\[ \left[ \text{Re}(v_i \cdot v_k + v \cdot v_{ik}) \right]_{n \times n} \leq 0 . \]

Since $(a_{ik}) > 0$, we have that
\[ a_{ik}(v_i \cdot v_k + \text{Re}(v \cdot v_{ik})) \leq 0 . \]

Into this inequality we substitute the system (3.3) and
\[ \text{Re}(Bu \cdot u)_k = 0 \]
to get
\[ 0 \geq a_{ik}v_i \cdot v_k + \text{Re}(Bu \cdot a_{ik}u_{ik}) \]
\[ = a_{ik}v_i \cdot v_k + \text{Re}(Bu(-a_{ik}u_{ik} + Cu + f)) \]
\[ = a_{ik}v_i \cdot v_k - \frac{1}{2}(C^* B + BC)u_{ik}u + \text{Re}(Bu \cdot f) . \]

Therefore,
\[ \rho_m^{1/2} |v| |f| \geq -\text{Re}(Bu \cdot f) \geq a_{ik}v_i \cdot v_k - \frac{1}{2}(C^* B + BC)u_{ik}u \]
\[ \geq \frac{1}{2} Eu \cdot u = \frac{1}{2}(B^{-1/2}EB^{-1/2})v \cdot v . \]

Since $B^{-1/2}EB^{-1/2}$ and $EB^{-1}$ have the same eigenvalues, it follows that
\[ \rho_m^{1/2} |v| |f| \geq \frac{\mu_1}{2} |v|^2 , \]
and then (6.3) holds with
Hence we have proved (6.2).

By using (6.2) and substituting \( v = B^{1/2}u \), we have

\[
\beta_1^{1/2}\|u\|_{0,D} \leq \beta_m^{1/2}\|u\|_{0,D} + K\|f\|_{0,D},
\]

so (6.1) holds with

\[
K_1 = \left(\frac{\beta_m}{\beta_1}\right)^{1/2} \quad \text{and} \quad K_2 = \frac{K}{\beta_1^{1/2}} = \frac{2\beta_m^{1/2}}{\mu_1\beta_1^{1/2}}.
\]

From the Liapunov Lemma, the following corollary is easily obtained:

**COROLLARY 6.2.** Assume that \( C(x) = r(x)I + \tilde{C} \) in (3.3)\(_N\) where \( r \leq 0 \) in \( D \) and each eigenvalue of the complex constant matrix \( \tilde{C} \) has negative real part. Then, for all \( C^2(D) \cap C(\overline{D}) \) solutions \( u \) of (3.3)\(_N\), there exist positive constants \( K_1 \) and \( K_2 \) such that (6.1) holds.

**COROLLARY 6.3.** (a) A sufficient condition that

\[
\|u\|_{0,D} \leq \|u\|_{0,D} + c_0^{-1}\|f\|_{0,D}
\]

holds, for all \( C^2(D) \cap C(\overline{D}) \) solutions \( u \) of (3.3)\(_N\), is

\[
C^*(x) + C(x) \leq -2c_0I < 0, \quad \text{for some} \quad c_0 \in \mathbb{R}.
\]

(b) Suppose that the variable matrix \( C = C(x) \) is normal, and all its eigenvalues \( \{\lambda_i(x)\}_{i=1}^m \) have uniformly negative real parts in \( D \); i.e., \( \text{Re}\lambda_i(x) \leq -c_0 < 0 \) in \( D \), for \( 1 \leq i \leq m \). Then (6.4) holds for all \( C^2(D) \cap C(\overline{D}) \) solutions \( u \) of (3.3)\(_N\).

Proof: (a) By choosing \( B = I \) in Theorem 6.1, the inequality (6.4)
follows from the condition (6.5).

(b) Since $C(x)$ is normal, there exists a unitary matrix $U(x)$ such that

$$U^*(x)[C^*(x) + C(x)]U(x) = \begin{bmatrix} 2\text{Re}\lambda_1(x) & \cdots & \cdots & 2\text{Re}\lambda_m(x) \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & \cdots & \ddots & \vdots \\ \vdots & \cdots & \cdots & -2c_0 I < 0. \end{bmatrix}$$

Hence $C^*(x) + C(x) \leq -2c_0 I$; and then (6.4) follows from (a).

7. APPLICATIONS TO HIGHER ORDER EQUATIONS

Note that the results we obtain are valid for complex systems which may be considered generalizations of some higher order complex equations. For example, consider a 2m-order homogeneous equation of the form

$$(7.1) \quad \mathcal{L}^m u + a_{m-1} \mathcal{L}^{m-1} u + \cdots + a_1 L u + a_0 u = 0,$$

and the nonhomogeneous equation of the same form,

$$(7.1)_N \quad \mathcal{L}u = F.$$ 

Here $a_0, a_1, \cdots, a_{m-1}$ and $F$ are complex-valued functions, $\mathcal{L} := L + r(x)$ where $r \leq 0$ in $D$ and $L$ is the elliptic operator defined by (3.1); and $\mathcal{L}^m = \mathcal{L}(\mathcal{L}^{m-1})$, $\mathcal{L}^0 = I$. Let $u_1 = u$, $u_2 = Lu$, $\cdots$, $u_m = \mathcal{L}^{m-1} u$; then the equations (7.1) and (7.1)_N reduce to the equivalent systems (3.3) and (3.3)_N, respectively, where

$$(7.2) \quad C = rI + \begin{bmatrix} 0 & -1 & \cdots & -1 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & -1 \\ a_0 & a_1 & \cdots & a_{m-1} \end{bmatrix} \quad \text{and} \quad u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix}, \quad f = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ F \end{bmatrix}.$$ 

Associated with each system is the characteristic equation
(7.3) \[ h(\lambda) = (-\lambda)^m + a_{m-1}(-\lambda)^{m-1} + \cdots + a_1(-\lambda) + a_0 = 0. \]

Hence, under the same assumptions on C in (7.2), Theorem 3.1 and Theorem 3.3 hold, but (3.6) changes to

\[
\left( \sum_{j=0}^{m-1} |x_j^i u(x)|^2 \right)^{1/2} \leq K \sup_{x \in \partial D} \left( \sum_{j=0}^{m-1} |x_j^i u(x)|^2 \right)^{1/2}, \quad (K \geq 1); \]

also, under the same assumptions on C in (7.2), Theorem 6.1 holds, but (6.1) changes to

\[
\left( \sum_{j=0}^{m-1} |x_j^i u(x)|^2 \right)^{1/2} \leq K \sup_{x \in \partial D} \left( \sum_{j=0}^{m-1} |x_j^i u(x)|^2 \right)^{1/2} + K_2 \|F\|_{0, \partial D}. \]

In the case of \( m = 2 \) and \( r = 0 \) in \( \text{D} \), the matrix in (7.2) is

\[ C(x) := \begin{bmatrix} 0 & -1 \\ a_0(x) & a_1(x) \end{bmatrix} \]

with eigenvalues \( \lambda_{\pm} = \frac{a_1 \pm \sqrt{a_1^2 - 4a_0}}{2} \).

Hence the requirement for Theorem 3.1 holding is that there exists \( B := \begin{bmatrix} p & s \\ s & q \end{bmatrix} > 0 \) such that

\[
C^*B + BC = \begin{bmatrix} 0 & a_0(x) \\ -1 & a_1(x) \end{bmatrix} \begin{bmatrix} p & s \\ s & q \end{bmatrix} + \begin{bmatrix} p & s \\ s & q \end{bmatrix} \begin{bmatrix} 0 & -1 \\ a_0(x) & a_1(x) \end{bmatrix}
\]

\[
= \begin{bmatrix} 2\text{Re}(sa_0) & qa_0 + sa_1 - p \\ qa_0 + sa_1 - p & 2\text{Re}(qa_1 - s) \end{bmatrix} \leq 0 ,
\]

which is equivalent to the existence of \( p, q \in \mathbb{R}, s \in \mathbb{C} \) such that

\[
(7.5) \begin{cases} 
    p > 0 \\
    pq - |s|^2 > 0 \\
    \text{Re}(sa_0) \leq 0, \quad \text{Re}(qa_1 - s) \leq 0 \\
    4\text{Re}(sa_0) \cdot \text{Re}(qa_1 - s) \geq |qa_0 + sa_1 - p|^2.
\end{cases}
\]

If we choose \( s = 0 \), then

\[
B = \begin{bmatrix} p & 0 \\ 0 & q \end{bmatrix},
\]

\[
K^2 = \frac{\max(p, q)}{\min(p, q)},
\]

and (7.5) becomes
\[
\begin{align*}
\begin{cases}
  p > 0 \\
  q > 0
\end{cases}
\quad \text{and} \quad 
\begin{cases}
  \text{Re} \{a_1(x)\} \leq 0 \\
  a_0 = \frac{p}{q} > 0
\end{cases}
\end{align*}
\]

So for the 4th-order equations,
\[(7.6) \quad \mathcal{L}^2 u(x) + a_1(x) \mathfrak{I} u(x) + a_0 u(x) = 0 \quad \text{in } D,
\]
and
\[(7.6)_N \quad \mathcal{L}^2 u(x) + a_1(x) \mathfrak{I} u(x) + a_0 u(x) = F \quad \text{in } D,
\]
we have

**COROLLARY 7.1.** Assume that \(\text{Re}\{a_1(x)\} \leq 0\) in \(D\) and \(a_0 > 0\), then

(a) for all complex-valued \(C^4(D) \cap C^2(\overline{D})\) solutions \(u\) of (7.6),
\[
\sup_{D}(|u|^2 + |\mathfrak{I} u|^2)^{1/2} \leq \sqrt{\max(a_0, a_0^{-1})} \sup_{\partial D}(|u|^2 + |\mathfrak{I} u|^2)^{1/2}.
\]

(b) if \(r(x) \leq -r_0 < 0\) in \(D\), then for all complex-valued \(C^4(D) \cap C^2(\overline{D})\) solutions \(u\) of (7.6)_N,
\[
\sup_{D}(|u|^2 + |\mathfrak{I} u|^2)^{1/2} \leq \sqrt{\max(a_0, a_0^{-1})} \left[\sup_{\partial D}(|u|^2 + |\mathfrak{I} u|^2)^{1/2} + r_0^{-1} \|F\|_{0, \partial D}\right].
\]

When \(\{a_j\}_{j=0}^{m-1}\) are complex constants, by applying Corollary 3.2 and Corollary 6.2 we have the following corollary for the 2m-order equations (7.1) and (7.1)_N.

**COROLLARY 7.2.** Suppose the roots \(\{\lambda_i\}_{i=1}^m\) of (7.3) satisfy both of the conditions,

(i) \(\text{Re} \lambda_i \leq 0\);

(ii) if \(\text{Re} \lambda_i = 0\), then the corresponding elementary divisor is linear.

Then,

(a) there exists a positive constant \(K\) such that (7.4) holds for
all complex-valued regular solutions of (7.1).

(b) if \( \text{Re} \lambda_i < 0 \), \( 1 \leq i \leq m \); or \( r(x) \leq -r_0 < 0 \) in \( D \), there exist positive constants \( K_1 \) and \( K_2 \) such that (7.4)_N holds for all complex-valued regular solutions of (7.1)_N.

Remark. Corollary 7.2 contains the result of Chow and Dunninger [2], [6] for real metaharmonic functions as a special case. The first part of the Corollary 7.2 was obtained in [2] for the case of \( L = \Delta \) (Laplace operator) and \( a_0, a_1, \ldots, a_{m-1} \) being real constants.
REFERENCES FOR PART I


PART II

SCHAUDER ESTIMATES AND EXISTENCE THEORY FOR
ENTIRE SOLUTIONS OF LINEAR FOURTH ORDER ELLIPTIC EQUATIONS
1. INTRODUCTION

We investigate here solutions in $\mathbb{R}^n$ of the linear, fourth order, elliptic, variable coefficient, nonhomogeneous partial differential equation

$$L\phi := a \cdot D^4 \phi + b \cdot D^3 \phi + c \cdot D^2 \phi + d \cdot D \phi + e \phi$$

$$= \sum_{i,j,k=1}^{n} a_{ijk1} x_i x_j x_k x_1 + \sum_{i,j,k=1}^{n} b_{ijk} x_i x_j x_k + \sum_{i,j=1}^{n} c_{ij} x_i x_j + \sum_{i=1}^{n} d_i x_i + e \phi = f,$$

and the related homogeneous equation,

$$L\phi = 0.$$

Following the prevailing terminology in the literature, we refer to solutions of these equations defined in all of $\mathbb{R}^n$ as entire solutions. We shall develop Schauder-type a priori estimates for entire solutions with prescribed behavior required at infinity. These estimates, of perhaps independent interest themselves, lead to an existence and uniqueness theory for entire solutions with certain behaviour required at infinity whenever the operator $L$ can be separated as

$$L = L_2 L_1.$$

Here

$$L_s := a_s \cdot D^2 + b_s \cdot D + c_s := \sum_{i,j=1}^{n} a_{sij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_{si} \frac{\partial}{\partial x_i} + c_s \quad (s=1,2)$$

are two second order elliptic operators.

The coefficients $a_s$, $b_s$, $c_s$ of $L_s$ ($s = 1, 2$) are assumed to be Holder-continuous in $\mathbb{R}^n \cup \{\omega\}$, and, as $x \to \omega$, the matrices $a_s$ approach the $n \times n$ identity matrix $I$, the vectors $b_s$ approach the $1 \times n$ zero
vector, and the scalars $c_s$ approach zero. Thus $L_s$ ($s = 1, 2$) approach the Laplace operator $\Delta$ and $L = L_2 L_1$ approaches the biharmonic operator $\Delta^2$ near infinity. We also assume the operators $L_s$ are uniformly elliptic in $\mathbb{R}^n$, and $c_s \leq 0$ if $n \geq 3$, $c_s = 0$ if $n = 2$ ($s = 1, 2$). With these assumptions, and for $n \geq 5$, we shall establish a one-to-one correspondence between the entire solutions of (1.1) that are $O(|x|^m)$ near infinity and the polynomial solutions to the biharmonic equation of degree no greater than $m$; when $f = 0$, this result is an extension of a Liouville-type theorem.

For dimension $n \leq 4$ we have similar results, but, like the case of $n = 2$ in [2, 9] for second order equations, there are complications which make the situations in four, three and two dimensions more and more difficult. As Friedman, Begehr and Hile showed in [9, 2], when $n = 2$ is the order of the equation, there need not exist an entire bounded solution of $L_1 \phi = f$, even when $L_1$ is the Laplace operator and $f$ has compact support in $\mathbb{R}^2$. Begehr and Hile also proved that there exists a unique entire solution of $L_1 \phi = f$ with some growth restriction at infinity and the condition $\phi(x) - \gamma \log |x| \to 0$ as $x \to \infty$, for some constant $\gamma$. We obtain a similar result for the fourth order equation (1.1) when $n \leq 4$; our condition at infinity is that $\phi(x) - \sum_{s=0}^{4-n} \gamma_s \delta^s \Gamma(x) \to 0$ as $x \to \infty$, for some constant number $\gamma_0$, constant vector $\gamma_1$ and constant symmetric matrix $\gamma_2$. Here $\Gamma$ is a fundamental solution of the biharmonic equation.

Several authors, for example [2, 5, 9, 10, 13, 15], have investigated existence and/or uniqueness questions for entire
solutions of linear second order elliptic equations and systems. There has also been considerable work on entire solutions of nonlinear and quasilinear second order elliptic equations and systems, as for example in [3, 12, 14, 16, 18–20]. The usual result obtained for a linear or nonlinear second order equation is the traditional statement of Liouville's Theorem, namely, a bounded entire solution is necessarily constant. For the higher order case, Douglis and Nirenberg proved a Liouville-type theorem for certain homogeneous systems with constant coefficients in [4]. Liouville Theorems for polyharmonic and some metaharmonic functions can be found in [1, 6, 8].

Results analogous to the ones presented here for fourth order equations appear to be most closely related to those of Begehr and Hile [2] for second order equations. The Schauder estimates for entire solutions are obtained from analogous Schauder estimates for bounded domains as found in [4].

2. NORMS AND A SCHAUDER ESTIMATE FOR ENTIRE SOLUTIONS

For vectors $v, w$ in $\mathbb{R}^n$ we let $v \cdot w$ denote the dot product of $v$ and $w$, and $|v|$ the Euclidean norm of $v$. For $n \times n$ matrices $A = (a_{ij})$, $B = (b_{ij})$ in $\mathbb{R}^{n \times n}$ we define

$$A \cdot B = \sum_{i,j} a_{ij} b_{ij}, \quad |A| = (A \cdot A)^{1/2} = \left( \sum_{i,j} a_{ij}^2 \right)^{1/2}.$$  

Similarly, for tensors $A = (a_{ijkl})$, $B = (b_{ijk})$ in $\mathbb{R}^{n \times n \times n}$ and $C = (c_{ijkl})$, $D = (d_{ijkl})$ in $\mathbb{R}^{n \times n \times n \times n}$ we define
\[ A \cdot B = \sum_{i,j,k} a_{ijk} b_{ijk}, \quad |A| = \left( \sum_{i,j,k} a_{ijk}^2 \right)^{1/2}, \]
and
\[ C \cdot D = \sum_{i,j,k,l} c_{ijkl} d_{ijkl}, \quad |C| = \left( \sum_{i,j,k,l} c_{ijkl}^2 \right)^{1/2}. \]
(The inequalities \(|A \cdot B| \leq |A||B|\) and \(|C \cdot D| \leq |C||D|\) hold.) We let \( x = (x_1, x_2, \ldots, x_n)^T \) denote a point in \( \mathbb{R}^n \), \( n \geq 2 \), and \( \Omega \) a domain in \( \mathbb{R}^n \). For \( \phi = \phi(x) \), a real valued function defined in \( \Omega \), we let \( D\phi \), \( D^2\phi \), \( D^3\phi \) and \( D^4\phi \) represent the \( n \times 1 \) vector of first derivatives of \( \phi \), the \( n \times n \) matrix of second derivatives of \( \phi \), the \( n \times n \times n \times n \) tensor of third derivatives of \( \phi \), and the \( n \times n \times n \times n \times n \) tensor of fourth derivatives of \( \phi \), respectively. Thus,
\[ D\phi := (\phi_{x_1}, \phi_{x_2}, \ldots, \phi_{x_n})^T, \quad D^2\phi := (\phi_{x_i x_j})_{n \times n}, \]
\[ D^3\phi := (\phi_{x_i x_j x_k})_{n \times n \times n}, \quad D^4\phi := (\phi_{x_i x_j x_k x_l})_{n \times n \times n \times n}. \]
Let \( a = (a_{ijkl})_{n \times n \times n \times n}, b = (b_{ijkl})_{n \times n \times n}, c = (c_{ijkl})_{n \times n}, d = (d_{ij})_{1 \times n}, e \) be functions defined in \( \Omega \), with values in \( \mathbb{R}^{n \times n \times n \times n}, \mathbb{R}^{n \times n \times n}, \mathbb{R}^{n \times n}, \mathbb{R}^n, \mathbb{R} \), respectively. The tensors \( a, b \) and the matrix \( c \) are always assumed to be symmetric; i.e.,
\[ a_{ijkl} = a_{\sigma(i)\sigma(j)\sigma(k)\sigma(l)}, \] for any permutation \( \sigma \) of \( \{i, j, k, l\} \),
\[ b_{ijkl} = b_{ikjl} = b_{jilk} = b_{jkil} = b_{klij}, \text{ and } a_{ij} = a_{ji}. \]
We consider the fourth order linear partial differential operator \( L \), defined by
\[
L\phi := a \cdot D^4\phi + b \cdot D^3\phi + c \cdot D^2\phi + d \cdot D\phi + e\phi
\]
\[ = \sum_{i,j,k,l=1}^n a_{ijkl} \phi_{x_i x_j x_k x_l} + \sum_{i,j,k=1}^n b_{ijkl} \phi_{x_i x_j x_k} + \sum_{i,j=1}^n c_{ijkl} \phi_{x_i x_j} + \sum_{i=1}^n d_{i} \phi_{x_i} + e\phi. \]
Associated with the operator $L$ are the homogeneous equation,
\begin{equation}
L \phi = 0,
\end{equation}
and the nonhomogeneous equation
\begin{equation}
L \phi = f,
\end{equation}
where the right-hand side $f$ is assumed to be a real valued function in $\Omega$.

Using the notation of [11], for points $x, y$ in $\Omega$, we define
\[ d_x := \text{dist}(x, \partial \Omega), \quad d_{x,y} := \min(d_x, d_y), \]
and for $u = u(x)$ a real valued, vector valued, matrix valued, or tensor valued function in $\Omega$ we define the following norms (where $\sigma \in \mathbb{R}, 0 < \alpha \leq 1$):
\begin{align}
|u|^{(\sigma)}_{0,\Omega} &:= \sup_{x \in \Omega} d_x^{\sigma} |u(x)|, \\
|u|^{(\sigma)}_{0,\alpha;\Omega} &:= \sup_{x,y \in \Omega} (d_{x,y})^{\sigma+\alpha} \frac{|u(x) - u(y)|}{|x - y|^\alpha}, \\
|u|^{(\sigma)}_{0,\alpha;\Omega} &:= |u|^{(\sigma)}_{0,\alpha;\Omega} + [u]^{(\sigma)}_{0,\alpha;\Omega}.
\end{align}

By a classical solution of $L \phi = f$, we mean a solution $\phi$ with $\phi, D\phi, D^2\phi, D^3\phi, D^4\phi$ continuous. It is known (see [7], Chapter 9) that if $a, b, c, d, e$ and $f$ are Holder-continuous with exponent $\alpha$ in $\Omega$, $0 < \alpha < 1$, and $\phi$ is a solution of (2.3), then $\phi, D\phi, D^2\phi, D^3\phi$ and $D^4\phi$ are all Holder-continuous with the same exponent $\alpha$ in $\Omega$. So, under the condition that all coefficients and $f$ are Holder-continuous, any solution of (2.3) is a classical solution. Douglis and Nirenberg have obtained a Schauder estimate for certain higher order elliptic systems (Theorem 1, [4]). Applying their result to the single equation (2.3),
we get the following basic Schauder estimate:

**Lemma 2.1 ([4])**. Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n \), \( n \geq 2 \), and let \( \phi \in C^4(\Omega) \), with \( |\phi|^{(0)}_{0,\Omega} \) finite, and with \( \phi \) a classical solution of the nonhomogeneous equation (2.3) in \( \Omega \). Assume \( 0 < \alpha < 1 \), that \( |f|^{(4)}_{0,\alpha;\Omega} \) is also finite, and that for some nonnegative constant \( \Lambda \),

\[
|a|^{(0)}_{0,\alpha;\Omega}, |b|^{(1)}_{0,\alpha;\Omega}, |c|^{(2)}_{0,\alpha;\Omega}, |d|^{(3)}_{0,\alpha;\Omega}, |e|^{(4)}_{0,\alpha;\Omega} \leq \Lambda.
\]

Also assume the symmetric tensor \( a \) satisfies the ellipticity condition

\[
\sum_{i,j,k,l=1}^n a_{ijkl}(x)\xi_i\xi_j\xi_k\xi_l \geq \lambda|\xi|^4, \quad \text{for all } x \in \Omega, \xi \in \mathbb{R}^n,
\]

for some positive constant \( \lambda \). Then for some universal nonnegative constant \( M = M(n, \alpha, \lambda, \Lambda) \) (independent of \( \phi \)) we have the bound

\[
|\phi(x)|^{(0)}_{0,\Omega} + |D\phi|^{(1)}_{0,\Omega} + |D^2\phi|^{(2)}_{0,\Omega} + |D^3\phi|^{(3)}_{0,\Omega} + |D^4\phi|^{(4)}_{0,\alpha;\Omega} \leq M(n, \alpha, \lambda, \Lambda)\{ |\phi|^{(0)}_{0;\Omega} + |f|^{(4)}_{0,\alpha;\Omega} \}.
\]

(Throughout this chapter we denote generic constants by \( M( ) \), where inside the parentheses are listed the entities that determine \( M \).)

We will derive an analogue of Lemma 2.1 for the case \( \Omega = \mathbb{R}^n \).

For this purpose it is useful to define the following norms, where again \( \sigma \in \mathbb{R} \), \( 0 < \alpha \leq 1 \), and now \( u \) is a function defined on all of \( \mathbb{R}^n \).

Let

\[
||u||_\sigma := \sup_{x \in \mathbb{R}^n} (1 + |x|)^{-\sigma}|u(x)|,
\]

\[
||u||_{(\sigma, \alpha)} := \sup_{x, y \in \mathbb{R}^n, 0 < |x - y| \leq (1 + |x|)/2} (1 + |x|)^{-\sigma-\alpha} \frac{|u(x) - u(y)|}{|x - y|^\alpha},
\]

\[
||u||_{\sigma, \alpha} := ||u||_\sigma + ||u||_{(\sigma, \alpha)}.
\]
The norms (2.10)-(2.12) were first defined by H. Begehr and G. N. Hile in [2], and may be viewed as analogous to the norms (2.4)-(2.6), where the distance \( d_x \), vanishing as \( x \) approaches \( \partial \Omega \), has been replaced by the quantity \( (1 + |x|)^{-1} \), which likewise vanishes as \( x \) approaches the boundary point at infinity of \( \mathbb{R}^n \). The condition \( 0 < |x - y| \leq (1 + |x|)/2 \) of (2.11) is a technical convenience, ensuring that \( y \) approaches infinity along with \( x \), and more or less in the same direction.

**DEFINITION 2.2.** We say that a real valued, vector valued, matrix valued, or tensor valued function \( u \) defined on all of \( \mathbb{R}^n \) is in the space

(i) \( B_\sigma \) if and only if \( ||u||_\sigma \) is finite,

(ii) \( B_{\sigma, \alpha} \) if and only if \( ||u||_{\sigma, \alpha} \) is finite.

We see that if \( u \in B_\sigma \), then \( |u(x)| = O(|x|^\sigma) \) as \( x \to \infty \). The space \( B_0 \) consists of bounded functions in \( \mathbb{R}^n \), with \( ||u||_0 \) reducing to the ordinary supremum norm. If \( u \in B_{\sigma, \alpha} \), then the Holder quotient, 

\[ \frac{|u(x) - u(x)|}{|x-y|^\alpha}, \text{ is } O(|x|^{\sigma - \alpha}) \text{ (with the restriction } 0 < |x-y| \leq (1+|x|)/2) \]

In particular, functions in \( B_{\sigma, \alpha} \) are uniformly Holder-continuous with exponent \( \alpha \) on compact subsets of \( \mathbb{R}^n \).

We gather some miscellaneous observations regarding the spaces \( B_\sigma \) and \( B_{\sigma, \alpha} \) into the following lemma. The proof of the lemma can be found in [2].

**LEMMA 2.3.** Let \( \sigma, \tau \in \mathbb{R}, 0 < \alpha, \beta \leq 1 \), and let \( u \) and \( v \) denote functions defined on \( \mathbb{R}^n \) (both scalar valued, both vector valued, both
matrix valued, or both tensor valued). Then

(a) \( \sigma \leq \tau \Rightarrow B_\sigma \subset B_\tau \), \( ||u||_\tau \leq ||u||_\sigma \);

(b) \( \sigma \leq \tau, 0 < \beta \leq \alpha \leq 1 \Rightarrow B_{\sigma, \alpha} \subset B_{\tau, \beta} \), \( ||u||_{\tau, \beta} \leq ||u||_{\sigma, \alpha} \);

(c) \( u \in B_\sigma, v \in B_\tau \Rightarrow u \cdot v \in B_{\sigma + \tau} \), \( ||u \cdot v||_{\sigma + \tau} \leq ||u||_\sigma ||v||_\tau \);

(d) \( u \in B_{\sigma, \alpha} , v \in B_{\tau, \alpha} \Rightarrow u \cdot v \in B_{\sigma + \tau, \alpha} \),

\[ ||u \cdot v||_{\sigma + \tau, \alpha} \leq ||u||_{\sigma, \alpha} ||v||_{\tau, \alpha}. \]

Moreover,

(e) if \( \phi \) is a real valued function in \( C^1(\mathbb{R}^n) \), with \( \phi \in B_{\sigma - 1} \),

then \( ||\phi||_{(\sigma, 1)} \) is finite, and in fact

\[ ||\phi||_{(\sigma, 1)} \leq M(\sigma)||\phi||_{\sigma - 1}. \]

Finally,

(f) let \( \{u_m\}^\infty_{m=1} \) be a sequence of functions in \( B_{\sigma, \alpha} \), with all

\( u_m \)'s having values in the same space \( \mathbb{R}, \mathbb{R}^n, \mathbb{R}^{n \times n}, \mathbb{R}^{n \times n \times n} \), or

\( \mathbb{R}^{n \times n \times n \times n} \), and suppose that we have a uniform bound, \( ||u_m||_{\sigma, \alpha} \leq K \) for all \( m \), for some nonnegative constant \( K \).

Then there exists a subsequence \( \{u_m\} \), and a function \( u \) in \( B_{\sigma, \alpha} \), with

also \( ||u||_{\sigma, \alpha} \leq K \), such that \( u_m \rightarrow u \) in the norm of any space

\( B_{\tau} \) or \( B_{\tau, \beta} \) with \( \tau > \sigma, 0 < \beta < \alpha \).

We now present the following modification of Lemma 2.1, where \( \Omega \)
is replaced by \( \mathbb{R}^n \) and the norms (2.4)-(2.6) by newly-defined norms

(2.10)-(2.12).

**THEOREM 2.4** (Schauder Estimate in \( \mathbb{R}^n \)). Let \( \phi \in C^4(\mathbb{R}^n) \), \( n \geq 2 \), with

\( ||\phi||_{\sigma} \) finite for some \( \sigma \in \mathbb{R} \), and with \( \phi \) a classical solution in \( \mathbb{R}^n \) of
the nonhomogeneous equation (2.3). Assume \( 0 < \alpha < 1 \), that \( ||f||_{\sigma - 4, \alpha} \) is
also finite, and that for some nonnegative constant $A$,

\begin{equation}
\|a\|_{0,\alpha} , \|b\|_{-1,\alpha} , \|c\|_{-2,\alpha} , \|d\|_{-3,\alpha} , \|e\|_{-4,\alpha} \leq A.
\end{equation}

Also assume the tensor $a$ is symmetric and satisfies the ellipticity condition

\[ \sum_{i,j,k,l=1}^{n} a_{ijkl}(x) \xi_i \xi_j \xi_k \xi_l \geq \lambda |\xi|^4 \]

for all $x, \xi \in \mathbb{R}^n$, for some positive constant $\lambda$. Then for some nonnegative constant $M = M(n, \alpha, \sigma, \lambda, A)$ we have the bound

\begin{equation}
\| \phi \|_{\sigma, 1} + \| D^2 \phi \|_{\sigma-2, 1} + \| D^3 \phi \|_{\sigma-3, 1} + \| D^4 \phi \|_{\sigma-4, \alpha}
\leq M(n, \alpha, \sigma, \lambda, A) \{ \| \phi \|_{\sigma} + \| f \|_{\sigma-4, \alpha} \}.
\end{equation}

Proof: First, we consider the case $|x| \leq 5$. We apply Lemma 2.1 to the domain

\[ \Omega := \{ z \in \mathbb{R}^n : |z| < 9 \} \]

The restrictions $|x| \leq 5$ and $0 < |x - y| \leq (1 + |x|)/2$ imply that

\begin{equation}
1 \leq d_x, d_y, d_{x,y} \leq 9.
\end{equation}

In order to check that conditions (2.7) hold in $\Omega$, we observe first that

\begin{equation}
\tag{2.16}
|e_{0, \alpha; \Omega}^{(4)}| = \sup_{z \in \Omega} \frac{d^4_x d^4_y}{d_z^4} e(z) |e(z)| \leq 9^4 \sup_{z \in \Omega} (1 + |z|)^4 |e(z)| \leq 9^4 \|e\|_{-4} \leq 9^4 A,
\end{equation}

\begin{equation}
\tag{2.17}
|e_{0, \alpha; \Omega}^{(4)}| = \sup_{z, \xi \in \Omega} \frac{d^4_x d^4_y}{d_z^4} \left( \frac{|e(z) - e(\xi)|}{|z - \xi|^{\alpha}} \right) \leq 9^{4+\alpha} \left\{ \sup_{0 < |z - \xi| \leq (1 + |z|)/2} \left( \frac{|e(z) - e(\xi)|}{|z - \xi|^{\alpha}} \right) + \sup_{|z - \xi| \geq (1 + |z|)/2} \left( \frac{|e(z) - e(\xi)|}{|z - \xi|^{\alpha}} \right) \right\} \leq 9^{4+\alpha} \left\{ \sup_{z, \xi \in \Omega} |e_{\sigma}^{(4)}| (1 + |z|)^{-4-\alpha} + \sup_{z, \xi \in \Omega} \left( \frac{1 + |\xi|}{2} \right)^{-\alpha} |e(z)| \right\} \leq M(\alpha) \{ \|e\|_{-4, \sigma} + \|e\|_{-4} \} = M(\alpha) \|e\|_{-4, \alpha}.
\end{equation}
and hence

\[(2.18)\quad |e|_{0, \alpha; \Omega}^{(4)} \leq M(\alpha) \Lambda.\]

In a similar manner we derive the estimates

\[(2.19)\quad |d|_{0, \alpha; \Omega}^{(3)} \leq M(\alpha) ||d|| -3, \alpha \leq M(\alpha) \Lambda,
|c|_{0, \alpha; \Omega}^{(2)} \leq M(\alpha) ||c|| -2, \alpha \leq M(\alpha) \Lambda,
|b|_{0, \alpha; \Omega}^{(1)} \leq M(\alpha) ||b|| -1, \alpha \leq M(\alpha) \Lambda,
|a|_{0, \alpha; \Omega}^{(0)} \leq M(\alpha) ||a|| 0, \alpha \leq M(\alpha) \Lambda;\]

\[(2.20)\quad |\phi|_{0, \Omega}^{(0)} \leq M(\sigma)||\phi||, \quad |f|_{0, \alpha; \Omega}^{(4)} \leq M(\alpha, \sigma)||f||_{\sigma-4, \alpha} .\]

Therefore Lemma 2.1 applies, and from (2.9), (2.20) we obtain, for \(x, y \in \Omega\), an estimate of the form

\[(2.21)\quad |\phi(x)| + d_x|D\phi(x)| + d_x^2|D^2\phi(x)| + d_x^3|D^3\phi(x)| + d_x^4|D^4\phi(x)|
+ d_x^{4+\alpha} \frac{|D^4\phi(x) - D^4\phi(y)|}{|x - y|^{\alpha}} \leq M(n, \alpha, \sigma, \lambda, \Lambda)\{||\phi|| + ||f||_{\sigma-4, \alpha}\}.\]

But for \(|x| \leq 5, 0 < |x - y| \leq (1 + |x|)/2\), we have (2.15) and also

\[1 \leq 1 + |x| \leq 6, \text{ and therefore we may write (2.21) as}\]

\[(2.22)\quad (1+|x|)^{-\sigma}|\phi(x)| + (1+|x|)^{1-\sigma}|D\phi(x)|
+ (1+|x|)^{2-\sigma}|D^2\phi(x)| + (1+|x|)^{3-\sigma}|D^3\phi(x)|
+ (1+|x|)^{4-\sigma}|D^4\phi(x)| + (1+|x|)^{4-\sigma+\alpha} \frac{|D^4\phi(x) - D^4\phi(y)|}{|x - y|^{\alpha}} \leq M(n, \alpha, \sigma, \lambda, \Lambda)\{||\phi|| + ||f||_{\sigma-4, \alpha}\} .\]

We next consider the case \(|x| \geq 5\). We define

\[(2.23)\quad \rho := (1 + |x|)/6, \quad |x| = 6\rho - 1,\]

and we now apply Lemma 2.1 to the domain

\[\Omega := \{z \in \mathbb{R}^n : \rho < |z| < 10\rho\} .\]

We again check conditions (2.7) in \(\Omega\). For \(z, \zeta \in \Omega\) we have

\[d_z, d_{\zeta}, d_{z, \zeta} \leq 5\rho, \quad \rho \leq 1+|z|, \quad 1+|\zeta| \leq 11\rho.\]
As in (2.16) and (2.17) we derive

\[ |e|_{0;\Omega}^{(4)} \leq 5^4 \sup_{z \in \Omega} (1+|z|)^4 |e(z)| \leq 5^4 \|e\|_{-4} \leq 5^4 \Lambda, \]

\[ [e]_{0,\alpha;\Omega}^{(4)} \leq (5\rho)^{4+\alpha} \left( \sup_{z \in \Omega} (1+|z|)^{-4-\alpha} \right) \]

\[ + 2^{-\alpha} \|e\|_{-4} \sup_{z \in \Omega} (1+|z|)^{-4-\alpha} \]

\[ \leq M(\alpha)\|e\|_{-4,\alpha} \leq M(\alpha)\Lambda. \]

A similar analysis shows that (2.19) continues to hold, but in place of (2.20) we obtain

\[ (2.24) \quad |f|_{0,\alpha;\Omega}^{(4)} = \sup_{z \in \Omega} |f(z)| \leq (11\rho)^\sigma \sup_{z \in \Omega} (1+|z|)^{-\sigma} |f(z)| \leq M(\sigma)\rho^\sigma \|f\|_{-\sigma}, \]

\[ |f|_{0,\alpha;\Omega}^{(4)} \leq M(\alpha)\|f\|_{-4,\alpha} \]

\[ \leq M(\alpha,\sigma)\rho^\sigma \left\{ \sup_{z \in \Omega} (1+|z|)^{4-\sigma} |f(z)| \right\} \]

\[ + \sup_{0<|z-\zeta| \leq (1+|z|)/2} \frac{f(z) - f(\zeta)}{|z - \zeta|^{\alpha}} \]

\[ \leq M(\alpha,\sigma)\rho^\sigma \|f\|_{-\sigma,\alpha} \].

We now apply (2.9) to our particular point \( x \), also using (2.24), to obtain, for \( y \in \Omega \),

\[ (2.25) \quad |\phi(x)| + d_x|D\phi(x)| + d_x^2|D^2\phi(x)| + d_x^3|D^3\phi(x)| \]

\[ + d_x^4|D^4\phi(x)| + d_{x,y}^{4+\alpha} \frac{|D^4\phi(x) - D^4\phi(y)|}{|x - y|^{\alpha}} \]

\[ \leq M(n,\alpha,\sigma,\lambda,\Lambda)\rho^\sigma \{ \|\phi\|_{-\sigma} + \|f\|_{-\sigma,\alpha} \}. \]

Now we have \( 1 + |x| = 6\rho \), and if \( 0 < |x - y| \leq (1 + |x|)/2 \), then \( y \in \Omega \) and \( \rho \leq d_x, d_{x,y} \leq 5\rho \). Thus (2.25) implies (2.22) also for \( |x| \geq 5, 0 < |x - y| \leq (1 + |x|)/2 \).

Upon taking supremums in (2.22), we conclude that
By Lemma 2.3(e) we have $||\phi||_{(\sigma,1)} \leq M(\sigma)||D^2\phi||_{\sigma-1}$, which implies we may add the term $||\phi||_{\sigma,1}$ to the left-hand side of (2.26). Since $D^2\phi \in B_{\sigma-2}$, we may also apply Lemma 2.3(e) to each of the derivatives $u_i$ and conclude that $||D^2\phi||_{(\sigma-1,1)} \leq M(\sigma)||D^2\phi||_{\sigma-2}$. Similarly, since $D^3\phi \in B_{\sigma-3}$ and $D^4\phi \in B_{\sigma-4}$, we also obtain that $||D^2\phi||_{(\sigma-2,1)} \leq M(\sigma)||D^3\phi||_{\sigma-3}$ and $||D^3\phi||_{(\sigma-3,1)} \leq M(\sigma)||D^4\phi||_{\sigma-4}$.

Thus (2.26) implies our desired inequality (2.14).

3. BOUNDS FOR ENTIRE SOLUTIONS OF THE NONHOMOGEOUS BIHARMONIC EQUATION

Before further discussing entire solutions of the general fourth order elliptic equation, it is useful both as motivation and as a mathematical aid to study entire solutions of the nonhomogeneous biharmonic equation,

$$\Delta^2 \phi = f.$$  \hspace{1cm} (3.1)

In $\mathbb{R}^n$, let $\Gamma$ denote the fundamental solution of the biharmonic equation,

$$\Gamma(x) := \begin{cases} 
\frac{1}{2(2-n)(4-n)\omega_n} |x|^{4-n}, & \text{if } n \geq 5 \text{ or } n = 3, \\
- \frac{1}{4\omega_4} \log |x|, & \text{if } n = 4, \\
\frac{1}{4\omega_2} |x|^2(\log |x| - 1), & \text{if } n = 2; 
\end{cases} \hspace{1cm} (3.2)$$

and for a real valued function $f$ on $\mathbb{R}^n$ define the Z-potential of $f$ as
(3.3) \[ Zf(x) := \int_{\mathbb{R}^n} \Gamma(x-y)f(y)dy. \]

Here \( \omega_n \) denotes the surface area of the unit ball in \( \mathbb{R}^n \).

We have the following estimates on the Z-potential.

**Lemma 3.1.** Let \( f \) be a real valued and measurable function on \( \mathbb{R}^n \) (\( n \geq 2 \)), in the space \( B_{-\tau}, \tau > 4 \). Then

(a) for \( x \in \mathbb{R}^n \) the integral \( Zf(x) \) exists and

\[
|Zf(x)| \leq M(n,\tau)\|f\|_{-\tau} \begin{cases} 
(1+|x|)^{4-\tau} & \text{if } n \geq 5, \tau < n, \\
(1+|x|)^{4-n}\log(2+|x|) & \text{if } n \geq 5, \tau = n, \\
(1+|x|)^{4-n} & \text{if } n = 3 \text{ or } \geq 5, \tau > n, \\
\log(2+|x|) & \text{if } n = 4, \\
(1+|x|)^{2}\log(2+|x|) & \text{if } n = 2;
\end{cases}
\]

(b) for \( |x| \geq 1 \) we have,

(i) if \( n = 4, 4 < \tau < 5 \),

\[
|Zf(x) - \gamma_0 \Gamma(x)| = |Zf(x) + \frac{\gamma_0}{4\omega_4} \log|x|| \leq M(\tau)\|f\|_{-\tau} (1+|x|)^{4-\tau}\log(2+|x|);
\]

(ii) if \( n = 3, 4 < \tau < 5 \),

\[
|Zf(x) - \gamma_0 \Gamma(x) + \gamma_1 \Delta \Gamma(x)| = \left| Zf(x) + \frac{\gamma_0}{8\pi} |x| - \frac{\gamma_1}{8\pi|x|} \right| \leq M(\tau)\|f\|_{-\tau} (1+|x|)^{4-\tau};
\]

(iii) if \( n = 2, 4 < \tau < 5 \),

\[
|Zf(x) - \gamma_0 \Gamma(x) + \gamma_1 \Delta \Gamma(x) - \frac{1}{2} \gamma_2 \Delta^2 \Gamma(x)| = \left| Zf(x) - \frac{\gamma_0}{8\pi} |x|^2(\log|x|-1) + \frac{\gamma_1}{8\pi} (2\log|x|-1) \right|
\]
\[- \frac{\gamma_2}{8\pi} \left( \begin{array}{cc}
\log|x| - \frac{1}{2} + \frac{x_1^2}{|x|^2} & \frac{x_1 x_2}{|x|^2} \\
\frac{x_1 x_2}{|x|^2} & \log|x| - \frac{1}{2} + \frac{x_2^2}{|x|^2}
\end{array} \right) \right]
\leq M(\tau)\|f\|_{-\tau} (1+|x|)^{4-\tau} \log(2+|x|);

where

\[
\gamma_0 := \int_{\mathbb{R}^n} f(y) dy, \quad \gamma_1 := \int_{\mathbb{R}^n} yf(y) dy,
\]

(3.8)
\[
\gamma_2 := \int_{\mathbb{R}^n} yy^T f(y) dy = \int_{\mathbb{R}^n} \begin{pmatrix} y_1^2 & y_1 y_2 \\ y_1 y_2 & y_2^2 \end{pmatrix} f(y) dy
\]

are a constant scalar, vector and matrix, respectively.

**Proof:** (a) By applying the estimate

(3.9) \[ |f(y)| \leq \|f\|_{-\tau} (1+|y|)^{-\tau} \]

to the integral (3.3), we obtain an inequality

(3.10) \[ |Zf(x)| \leq \|f\|_{-\tau} \{ I_1(x) + I_2(x) + I_3(x) \}, \]

where we define

(3.11) \[ I_1(x) := \int_{|y-x| \leq (1+|x|)/2} |\Gamma(x-y)| (1+|y|)^{-\tau} dy, \]

(3.12) \[ I_2(x) := \int_{|y-x| \geq 2(1+|x|)} |\Gamma(x-y)| (1+|y|)^{-\tau} dy, \]

(3.13) \[ I_3(x) := \int_{(1+|x|)/2 \leq |y-x| \leq 2(1+|x|)} |\Gamma(x-y)| (1+|y|)^{-\tau} dy. \]

In $I_1(x)$, we observe that

(3.14) \[ |y-x| \leq (1+|x|)/2 \Rightarrow (1+|x|)/2 \leq |y| \leq 3(1+|x|)/2, \]

which gives

(3.15) \[ I_1(x) \leq M(\tau) (1+|x|)^{-\tau} \int_{|z| \leq 1+|x|} |\Gamma(z)| dz \]
\[(1+|x|)^{4-\tau}, \quad \text{if } n = 3 \text{ or } n \geq 5,\]
\[(1+|x|)^{4-\tau} \log(2+|x|), \quad \text{if } n = 2 \text{ or } 4.\]

In \(I_2(x)\),
\[(3.16) \quad |y-x| \geq 2(1+|x|) \implies |x| \leq |x-y|/2 \leq |y|,
\]
and therefore
\[(3.17) \quad I_2(x) \leq M(\tau) \int_{|y-x|\geq 2(1+|x|)} \frac{|\Gamma(x-y)||x-y|^{-\tau} dy}{|y-x|^{1+|x|}} \leq M(\tau) \int_{|z|\geq 1+|x|} |\Gamma(z)||z|^{-\tau} dz \leq M(n,\tau) \begin{cases} (1+|x|)^{4-\tau}, & \text{if } n = 3 \text{ or } n \geq 5, \\ (1+|x|)^{4-\tau} \log(2+|x|), & \text{if } n = 2 \text{ or } 4. \end{cases}\]

In \(I_3(x)\),
\[(3.18) \quad (1+|x|)/2 \leq |y-x| \leq 2(1+|x|) \implies |y| \leq 2+3|x|,
\]
and moreover, if \(n = 3 \text{ or } n \geq 5,
\[(3.19) \quad |\Gamma(x-y)| \leq M(n)|x-y|^{4-n} \leq M(n)(1+|x|)^{4-n};\]
and if \(n = 4,
\[(3.20) \quad -\log 2 \leq \log \frac{1+|x|}{2} \leq \log |x-y| = 4\omega_4 |\Gamma(x-y)| \leq \log 2(1+|x|) \leq 2\log(2+|x|),\]
\[(3.21) \quad |\Gamma(x-y)| \leq \frac{1}{2\omega_4} \log(2+|x|);\]
and if \(n = 2\), applying (3.18) and (3.20),
\[(3.22) \quad |\Gamma(x-y)| \leq \frac{1}{8\pi}|x-y|^2 (\log|x-y| + 1) \leq \frac{1}{\pi}(1+|x|)^2 \log(2+|x|).\]
Therefore, for \(n = 3 \text{ or } n \geq 5,
\[(3.23) \quad I_3(x) \leq M(n)(1+|x|)^{4-n} \int_{|y|\leq 2+3|x|} (1+|y|)^{-\tau} dy \leq M(n,\tau) \begin{cases} (1+|x|)^{4-\tau}, & \text{if } n \geq 5, \tau < n, \\ (1+|x|)^{4-n} \log(2+|x|), & \text{if } n \geq 5, \tau = n, \\ (1+|x|)^{4-n}, & \text{if } n = 3 \text{ or } n \geq 5, \tau > n. \end{cases}\]
For \( n = 4 \),
\[
I_3(x) \leq \frac{1}{2\omega_4} \log(2+|x|) \int \frac{d^2}{|y| \leq 2+|x|} (1+|y|)^{-2} dy \leq M(\tau) \log(2+|x|).
\]

For \( n = 2 \),
\[
I_3(x) \leq \frac{1}{\pi} (1+|x|)^2 \log(2+|x|) \int \frac{d^2}{|y| \leq 2+|x|} (1+|y|)^{-2} dy \leq M(\tau) (1+|x|)^2 \log(2+|x|).
\]

Now combining (3.10), (3.15), (3.17), (3.23)-(3.25), we obtain the required estimate (3.4).

(b) In order to derive (3.5)-(3.7) we write
\[
Zf(x) - \gamma_0 \Gamma(x) = \int_{\mathbb{R}^4} [\Gamma(x-y) - \Gamma(x)] f(y) dy \quad (n=4),
\]
\[
Zf(x) - \gamma_0 \Gamma(x) + \gamma_1 \partial \Gamma(x) = \int_{\mathbb{R}^3} [\Gamma(x-y) - \Gamma(x) + \partial \Gamma(x) \cdot y] f(y) dy \quad (n=3),
\]
\[
Zf(x) - \gamma_0 \Gamma(x) + \gamma_1 \partial \Gamma(x) - \frac{1}{2} \gamma_2 \partial^2 \Gamma(x)
\]
\[
= \int_{\mathbb{R}^2} [\Gamma(x-y) - \Gamma(x) + \partial \Gamma(x) \cdot y - \frac{1}{2} \partial^2 \Gamma(x) \cdot (yy^T)] f(y) dy \quad (n=2),
\]
and apply (3.9) to obtain
\[
\text{ (3.26) } |Zf(x) - \gamma_0 \Gamma(x)| \leq \|f\|_{-\tau} [I_1(x) + I_2(x) + J_0(x) + K_0(x) + N_1(x)] \quad (n=4),
\]
\[
\text{ (3.27) } |Zf(x) - \gamma_0 \Gamma(x) + \gamma_1 \partial \Gamma(x)|
\]
\[
\leq \|f\|_{-\tau} [I_1(x) + I_2(x) + J_0(x) + J_1(x) + K_0(x) + K_1(x) + N_2(x)] \quad (n=3),
\]
\[
\text{ (3.28) } |Zf(x) - \gamma_0 \Gamma(x) + \gamma_1 \partial \Gamma(x) - \gamma_2 \partial^2 \Gamma(x)|
\]
\[
\leq \|f\|_{-\tau} [I_1(x) + I_2(x) + J_0(x) + J_1(x) + J_2(x) + K_0(x) + K_1(x) + K_2(x) + N_3(x)] \quad (n=2),
\]
where \( I_1(x), I_2(x) \) are as defined in (3.11) and (3.12), and we further define
\( J_1(x) := |D^1 \Gamma(x)| \int_{|y-x| \leq (1+|x|)/2} |y|^{i(1+|y|)^{-\tau}} dy, \quad (i = 0, 1, 2), \)

\( K_i(x) := |D^i \Gamma(x)| \int_{|y-x| \geq 2(1+|x|)} |y|^{i(1+|y|)^{-\tau}} dy, \quad (i = 0, 1, 2), \)

\( N_1(x) := \int_{(1+|x|)/2 \leq |y-x| \leq 2(1+|x|)} |\Gamma(x-y) - \Gamma(x)|(1+|y|)^{-\tau} dy, \)

\( N_2(x) := \int_{(1+|x|)/2 \leq |y-x| \leq 2(1+|x|)} |\Gamma(x-y) - \Gamma(x) + y \cdot D\Gamma(x)|(1+|y|)^{-\tau} dy, \)

\( N_3(x) := \int_{(1+|x|)/2 \leq |y-x| \leq 2(1+|x|)} |\Gamma(x-y) - \Gamma(x) + y \cdot D\Gamma(x)| (1+|y|)^{-\tau} dy. \)

By the definition of \( \Gamma(x) \) in (3.2), we have

\[
\text{(3.29)} \quad D\Gamma(x) = \begin{cases} 
- \frac{x}{8\pi|x|} & , \text{if } n = 3, \\
\frac{x}{8\pi} (2\log|x| - 1) & , \text{if } n = 2,
\end{cases}
\]

and

\[
\text{(3.30)} \quad D^2 \Gamma(x) = \frac{1}{4\pi} \begin{pmatrix}
\log|x| - \frac{1}{2} + \frac{x_1^2}{|x|^2} & \frac{x_1x_2}{|x|^2} \\
\frac{x_1x_2}{|x|^2} & \log|x| - \frac{1}{2} + \frac{x_2^2}{|x|^2}
\end{pmatrix}.
\]

In \( J_i(x) \), \( i = 0, 1, 2 \), we have (3.14); thus,

\[
J_i(x) \leq M(\tau) |D^i \Gamma(x)| (1+|x|)^{i-\tau} \int_{|y| \leq 3(1+|x|)/2} dy,
\]

which gives

\[
\text{(3.31)} \quad J_0(x) \leq M(n, \tau) \begin{cases} 
(1+|x|)^{2-\tau} |x|^2 (|\log|x||+1) & , \text{if } n = 2, \\
(1+|x|)^{3-\tau} |x| & , \text{if } n = 3, \\
(1+|x|)^{4-\tau} |\log|x|| & , \text{if } n = 4;
\end{cases}
\]

\[
\text{(3.32)} \quad J_1(x) \leq M(n, \tau) \begin{cases} 
(1+|x|)^{3-\tau} |x|(|\log|x||+1) & , \text{if } n = 2, \\
(1+|x|)^{4-\tau} & , \text{if } n = 3;
\end{cases}
\]
(3.33) \( J_2(x) \leq M(\tau)(1+|x|)^{4-\tau}|\log|x||+1) \), when \( n = 2 \).

Applying (3.16) to \( K_i(x) \), \( i = 0, 1, 2, \), we obtain

\[
K_i(x) \leq |D^i\Gamma(x)| \int_{|y-x| \geq 2(1+|x|)} \left( \frac{|x-y|}{2} \right)^{i-\tau} dy
\]

\[
\leq |D^i\Gamma(x)| \int_{|z| \geq 1+|x|} 2^n |z|^{i-\tau} dz,
\]

which yields

\[
(3.34) \quad K_0(x) \leq M(n,\tau) \begin{cases} (1+|x|)^{2-\tau} |x|^2 \log(2+|x|), & \text{if } n = 2, \\ (1+|x|)^{3-\tau} |x|, & \text{if } n = 3, \\ (1+|x|)^{4-\tau} |\log|x||, & \text{if } n = 4; \end{cases}
\]

(3.35) \( K_1(x) \leq M(n,\tau) \begin{cases} (1+|x|)^{3-\tau} |x| \log(2+|x|), & \text{if } n = 2, \\ (1+|x|)^{4-\tau}, & \text{if } n = 3; \end{cases} \)

(3.36) \( K_2(x) \leq M(\tau)(1+|x|)^{4-\tau} \log(2+|x|), \) when \( n = 2 \).

To estimate \( N_1(x) \) \( (n = 4) \), we write, considering \( \Gamma \) as a function of \(|x| \) rather than \( x \),

\[
\Gamma(x-y) - \Gamma(x) = -\frac{1}{4\omega^4} t^{-1}(|x-y| - |x|),
\]

where \( t \) is some positive number between \(|x| \) and \(|x-y| \). Since (3.18) holds in \( N_1(x) \), we have

(3.37) \(|x| \geq |x|/2, |x-y| \geq |x|/2 \Rightarrow t, |x-sy| \geq |x|/2 \) for \( s \in [0,1] \),

and hence

\[
|\Gamma(x-y) - \Gamma(x)| \leq M|x|^{-1}|y| \quad (n = 4).
\]

Substituting into \( N_1(x) \), again noting (3.18), we have

\[
N_1(x) \leq M|x|^{-1} \int_{|y| \leq 2+3|x|} |y|(1+|y|)^{-\tau} dy \quad (n = 4),
\]

which gives

(3.38) \( N_1(x) \leq M(\tau)|x|^{-1}(1+|x|)^{5-\tau}, \) if \( 4 < \tau < 5, \) \( n = 4. \)
We substitute (3.15), (3.17), (3.31), (3.34), (3.38) into (3.26), requiring that $|x| \geq 1$, and obtain the desired estimate (3.5).

To estimate $N_2(x)$ ($n = 3$), and $N_3(x)$ ($n = 2$), we define

$$f(t) := \Gamma(x-ty), \ t \in \mathbb{R}.$$  

By Taylor’s Theorem,

$$f(1) - f(0) - f'(0) = \frac{1}{2} f''(s), \text{ for some } s \in [0,1],$$

and

$$f(1) - f(0) - f'(0) - \frac{1}{2} f''(0) = \frac{1}{6} f'''(s), \text{ for some } s \in [0,1],$$

and therefore, we have

$$\Gamma(x-y) - \Gamma(x) + \frac{1}{6\pi} \{ |x-y| - |x| + \frac{x \cdot y}{|x|} \} = \frac{1}{16\pi} \left\{ \frac{|y|^2}{|x-y|} - \frac{|(x-sy) \cdot y|^2}{|x-sy|^3} \right\} \quad (n = 3),$$

and

$$\Gamma(x-y) - \Gamma(x) + \frac{1}{6\pi} \frac{1}{2} \frac{D^2 \Gamma(x) \cdot yy^T}{|x-y|} = \frac{1}{8\pi} \left\{ |x-y|^2 ( \log |x-y| - 1 ) - |x|^2 ( \log |x| - 1 ) + (2 \log |x| - 1)x \cdot y \right\} \left\{ \log |x| - \frac{1}{2} + \frac{x^2}{|x|^2} \right\} \right. \nonumber$$

$$\left. \left\{ \frac{x_1 x_2}{|x|^2} - \frac{x_1^2}{|x|^2} \log |x| - \frac{1}{2} + \frac{x_2^2}{|x|^2} \right\} \right) \ast (yy^T) \right\} \right\} \left( \frac{1}{24\pi} \right) \left\{ \frac{2}{|x-y|^4} \left[ \frac{|(x-sy) \cdot y|^3}{|x-sy|^3} - 3 |y|^2 \frac{(x-sy) \cdot y}{|x-sy|^2} \right] \right\} \quad (n = 2).$$

Since (3.18) and (3.37) hold in $N_2(x)$, by substituting (3.39) into $N_2(x)$, we obtain

$$N_2(x) \leq M|x|^{-1} \int_{|y| \leq 2+3|x|} |y|^2 (1+|y|)^{-\gamma} \, dy \quad (n = 3),$$

which gives

$$N_2(x) \leq M(\gamma) |x|^{-1} (1+|x|)^{5-\gamma} \quad \text{if } 4 < \gamma < 5, \ n = 3.$$
We substitute (3.15), (3.17), (3.31), (3.32), (3.34), (3.35), (3.41) into (3.27), requiring that \( |x| \geq 1 \), and obtain the desired estimate (3.6).

Now substituting (3.40) into \( N_3(x) \), again noting (3.18) and (3.37), we have

\[
N_3(x) \approx M|x|^{-1} \int_{|y| \leq 2+3|x|} |y|^3(1+|y|)^{-\tau} \, dy \quad (n = 2),
\]

which gives

(3.42) \( N_3(x) \leq M(\tau)|x|^{-1}(1+|x|)^{5-\tau} \), if \( 4 < \tau < 5 \), \( n = 2 \).

Finally, we substitute (3.15), (3.17), (3.31)-(3.36), (3.42) into (3.28), requiring that \( |x| \geq 1 \), and obtain the desired estimate (3.7).

From the definition of \( \Gamma(x) \) in (3.2), it is easy to derive the following estimates for the first, second, third and fourth derivatives of \( \Gamma \):

\[
\left| \frac{\partial}{\partial x_1} \Gamma(x) \right| \leq \begin{cases} \frac{|x|^{3-n}}{2(n-2)\omega_n}, & \text{if } n \geq 3, \\ \frac{1}{4\omega_2}|x|(2|\log|x||+1), & \text{if } n = 2; \end{cases}
\]

\[
\left| \frac{\partial^2}{\partial x_1 \partial x_j} \Gamma(x) \right| \leq \begin{cases} -\frac{1}{\omega_n}|x|^{2-n}, & \text{if } n \geq 3, \\ -\frac{1}{4\omega_2}(2|\log|x||+3), & \text{if } n = 2; \end{cases}
\]

\[
\left| \frac{\partial^3}{\partial x_1 \partial x_j \partial x_k} \Gamma(x) \right| \leq \frac{n+3}{2\omega_n}|x|^{1-n}, \quad \text{if } n \geq 2;
\]

\[
\left| \frac{\partial^4}{\partial x_1 \partial x_j \partial x_k \partial x_1} \Gamma(x) \right| \leq \frac{n^2+8n+3}{2\omega_n}|x|^{-n}, \quad \text{if } n \geq 2.
\]

By using the proofs of [11, Lemma 4.1, Lemma 4.2], the following
lemma concerning the Z-potential on bounded domains can be proved from
the above estimates.

**Lemma 3.2.** Let f be bounded and locally Hölder continuous (with
exponent $0 < \alpha < 1$) in a bounded domain $\Omega$. Then

$$(3.43) \quad z(x) := \int_{\Omega} \Gamma(x-y)f(y)dy$$

is in $C^4(\Omega)$; and for any $x \in \Omega$,

$$\frac{\partial z(x)}{\partial x_i} = \int_{\Omega} \frac{\partial \Gamma(x-y)}{\partial x_i} f(y)dy, \quad \frac{\partial^2 z(x)}{\partial x_i \partial x_j} = \int_{\Omega} \frac{\partial^2 \Gamma(x-y)}{\partial x_i \partial x_j} f(y)dy,$$

$$(3.44) \quad \frac{\partial^3 z(x)}{\partial x_i \partial x_j \partial x_k} = \int_{\Omega} \frac{\partial^3 \Gamma(x-y)}{\partial x_i \partial x_j \partial x_k} f(y)dy,$$

$$\frac{\partial^4 z(x)}{\partial x_i \partial x_j \partial x_k \partial x_l} = \int_{\Omega_0} \frac{\partial^4 \Gamma(x-y)}{\partial x_i \partial x_j \partial x_k \partial x_l} [f(y)-f(x)]dy$$

$$- f(x) \int_{\partial \Omega_0} \frac{\partial^2 \Gamma(x-y)}{\partial x_i \partial x_j \partial x_k} \nu_1(y)dS_y.$$

Here $i, j, k, l = 1, 2, \ldots, n$, $\nu = (\nu_1, \nu_2, \ldots, \nu_n)$ is the outward
normal direction, $\Omega_0$ is any domain containing $\Omega$ for which the
divergence theorem holds, and $f$ is extended to vanish outside $\Omega$.

From Lemma 3.2 we have, for any $x \in \Omega$,

$$\Delta z(x) = \int_{\Omega} \Delta \Gamma(x-y)f(y)dy$$

$$(3.45)$$

$$= \begin{cases} \frac{1}{(2-n)\omega_n} \int_{\Omega} \frac{1}{|x-y|^{n-2}} f(y)dy, & \text{if } n \geq 3, \\ - \frac{1}{\omega_2} \int_{\Omega} \log|x-y|f(y)dy, & \text{if } n = 2, \end{cases}$$

which is the well known Newtonian potential of $f$ on the domain $\Omega$.

Therefore, we also have, for $x \in \Omega$,

$$\Delta^2 z(x) = \Delta(\Delta z(x)) = f(x).$$

(3.46)
LEMMA 3.3. Let $f$ be real valued on $\mathbb{R}^n$ and in some space $B_{r,\alpha}$, with $\tau > 4$, $0 < \alpha < 1$. Then $Zf \in C^4(\mathbb{R}^n)$, with

$$\Delta(Zf) = Sf,$$

and

$$\Delta^2(Zf) = \Delta(Sf) = f$$

in $\mathbb{R}^n$, where

$$Sf(x) := \begin{cases} \frac{1}{(2-n)\omega_n} \int_{\mathbb{R}^n} |x-y|^{2-n}f(y)dy, & \text{if } n \geq 3, \\ \frac{1}{2\pi} \int_{\mathbb{R}^2} \log|x-y|f(x)dy & \text{if } n = 2 \end{cases}$$

is the well known Newtonian potential of $f$ on $\mathbb{R}^n$. Moreover,

(a) if $n \geq 5$, then $Zf$ vanishes at infinity;

(b) if $n = 4$, then $Zf + \frac{\gamma_0}{4\omega_4} \log|x|$ vanishes at infinity;

(c) if $n = 3$, then $Zf + \frac{\gamma_0}{8\pi}|x| - \frac{\gamma_1}{8\pi|x|} \cdot x$ vanishes at infinity;

(d) if $n = 2$, then $Zf + \frac{1}{8\pi} \left[ -\gamma_0 |x|^2(\log|x|-1) + (2\log|x|-1)x \cdot \gamma_1 \\
- (\log|x| - \frac{1}{2}) \text{tr}(\gamma_2) - \frac{x \cdot \gamma_2 x}{|x|^2} \right]$ vanishes at infinity.

Remark Being aware of the fact that $D^i\Gamma(x)$ vanishes at infinity when $n \geq 5-i$, $(i = 0,1,2)$, we can write (a)-(d) of Lemma 3.3 in the following uniform way: for $n \geq 2$,

$$Zf - \gamma_0 \Gamma + \gamma_1 \cdot D\Gamma - \frac{1}{2} \gamma_2 \cdot D^2\Gamma \text{ vanishes at infinity.}$$

Proof of Lemma 3.3: By choosing $\tau$ smaller if necessary, we may prescribe $4 < \tau < n$ if $n \geq 5$, and $4 < \tau < 5$ if $n = 2, 3, 4$. Then (a)-(d) follow from Lemma 3.1.
In order to discuss differentiability of $Zf$, we let $\rho$ denote some large radius and form the decomposition

$$
Zf(x) = \int_{|y| \leq \rho} \Gamma(x-y)f(y)dy + \int_{|y| \geq \rho} \Gamma(x-y)f(y)dy 
= z_\rho(x) + w_\rho(x).
$$

Lemma 3.2 gives that $z_\rho(x)$, the first integral of (3.51), is $C^4$ in the region $\Omega_\rho = \{x \in \mathbb{R}^n : |x| < \rho\}$, with (3.45), (3.46) valid if we replace $z(x)$ and $\Omega$ by $z_\rho(x)$ and $\Omega_\rho$. It is clear that $w_\rho(x)$, the second integral of (3.51), is $C^\infty$ in $\Omega_\rho$ with

$$
|\Delta w_\rho(x)| \leq M(n)||f||_{-\tau} \left\{ \begin{align*}
&\int_{|y| \geq \rho} |x-y|^{2-n}(1+|y|)^{-\tau}dy, \quad \text{if } n \geq 3, \\
&\int_{|y| \geq \rho} |\log|x-y|||1+|y||^{-\tau}dy, \quad \text{if } n = 2,
\end{align*} \right.
$$

and

$$
\Delta^2 w_\rho(x) = 0 \quad \text{in } \Omega_\rho.
$$

Since $\rho$ is arbitrary, $Zf$ is $C^4$ in $\mathbb{R}^n$. From (3.52), it follows easily that $\Delta w_\rho(x) \to 0$, as $\rho \to \infty$, for any (fixed) $x \in \mathbb{R}^n$. By letting $\rho \to \infty$ in (3.52)-(3.53), and also in (3.45)-(3.46) with $z(x)$ and $\Omega$ replaced by $z_\rho(x)$ and $\Omega_\rho$, we obtain (3.47) and (3.48). □

Douglis and Nirenberg have proved a Liouville-type theorem for certain homogeneous elliptic systems with constant coefficients and homogeneous order (Theorem 3, [4]). Applying their result to a single equation of fourth order, we obtain the following:

**Lemma 3.4.** (Liouville-Type Theorem) Let $\phi$ be a $C^4$ solution of

$$a \cdot D^4 \phi = 0 \quad \text{in } \mathbb{R}^n,$$

where $a$ is a constant matrix.
where $a$ is a constant tensor in $\mathbb{R}^{n\times n\times n\times n}$ and satisfies the elliptic condition (2.8), and assume that, for some nonnegative integer $m$, 
$$\phi(x) = o(|x|^m)$$
at infinity.
Then $\phi \equiv 0$ if $m = 0$, and $\phi$ is a polynomial of degree $\leq m-1$ if $m > 0$.
Particularly, the result holds for entire solutions of the biharmonic equation (in which case, $D^4 = \Delta^2$).

The following basic result for the nonhomogeneous biharmonic equation will serve as our model for the general fourth order elliptic equation.

**Theorem 3.5.** Let $f$ be real valued on $\mathbb{R}^n$ and in some space $B_{-\tau, \alpha}$ with $\tau > 4$, $0 < \alpha < 1$. Then there exists exactly one entire $C^4$ solution $\phi$ of the equation $\Delta^2 \phi = f$, namely the solution $\phi = Zf$, such that

(a) when $n \geq 5$, $\phi$ vanishes at infinity; if moreover $4 < \tau < n$,
then for this solution we have the bound

\[
||\phi||_{4, \tau, 1} + ||D\phi||_{3, \tau, 1} + ||D^2\phi||_{2, \tau, 1} + ||D^3\phi||_{1, \tau, 1} + ||D^4\phi||_{-\tau, \alpha} \leq M(n, \alpha, \tau)||f||_{-\tau, \alpha};
\]

(b) when $n = 4$, $\phi(x) - \gamma_0 \Gamma(x)$ vanishes at infinity for some constant $\gamma_0$; moreover, for this solution $\phi$ and any positive constant $\epsilon$ we have the bound

\[
||\phi||_{\epsilon, 1} + ||D\phi||_{\epsilon-1, 1} + ||D^2\phi||_{\epsilon-2, 1} + ||D^3\phi||_{\epsilon-3, 1} + ||D^4\phi||_{\epsilon-4, \alpha} \leq M(\alpha, \tau, \epsilon)||f||_{-\tau, \alpha};
\]

(c) when $n = 3$, $\phi(x) - \gamma_0 \Gamma(x) + \gamma_1 \nabla \Gamma(x)$ vanishes at infinity for some constant $\gamma_0$ and constant vector $\gamma_1$; moreover, for this solution we have the bound
\begin{align}
(3.56) & \quad \|\phi\|_{1,1} + \|D\phi\|_{0,1} + \|D^2\phi\|_{-1,1} + \|D^3\phi\|_{-2,1} + \|D^4\phi\|_{-3,1} \\
& \quad \leq M(\alpha, \tau)\|f\|_{-\tau,\alpha} ;
\end{align}

(d) when $n = 2$, $\phi(x) - \gamma_0 \Gamma(x) + \gamma_1 \cdot D\Gamma(x) - \frac{1}{2} \gamma_2 \cdot D^2\Gamma(x)$ vanishes at infinity for some constant $\gamma_0$, constant vector $\gamma_1$ and constant symmetric matrix $\gamma_2$; moreover, for this solution $\phi$ and any positive constant $\varepsilon$ we have the bound

\begin{align}
(3.57) & \quad \|\phi\|_{\varepsilon+2,1} + \|D\phi\|_{\varepsilon+1,1} + \|D^2\phi\|_{\varepsilon,1} + \|D^3\phi\|_{\varepsilon-1,1} + \|D^4\phi\|_{\varepsilon-2,1} \\
& \quad \leq M(\alpha, \tau, \varepsilon)\|f\|_{-\tau,\alpha} .
\end{align}

The uniquely determined constant $\gamma_0$, constant vector $\gamma_1$ and constant matrix $\gamma_2$ in (b), (c) and (d) are defined by (3.8).

**Proof:** (a) By Lemma 3.4 for biharmonic functions (also see [6, 8]), uniqueness of the solution is clear since the difference of two solutions would be biharmonic and vanish at infinity, thereby being identically zero. Lemma 3.3 states that $\phi = Z\Gamma$ is a solution vanishing at infinity. In order to derive the bound on $\phi$, we apply (2.14) of Theorem 2.4 to the special case of the biharmonic operator, with $\sigma = 4 - \tau$, obtaining

\begin{align}
\|\phi\|_{4-\tau,1} + \|D\phi\|_{3-\tau,1} + \|D^2\phi\|_{2-\tau,1} + \|D^3\phi\|_{1-\tau,1} + \|D^4\phi\|_{0-\tau,1} \\
& \quad \leq M(n, \alpha, \tau)\{\|\phi\|_{4-\tau} + \|f\|_{-\tau,\alpha}\} .
\end{align}

But Lemma 3.1 implies, for $4 < \tau < n$,

\begin{align}
\|\phi\|_{4-\tau} \leq M(n, \tau)\|f\|_{-\tau} ,
\end{align}

which, upon substitution into the previous inequality, yields (3.54).

(b) For uniqueness, suppose there were two $C^4$ solutions $\phi_1$, $\phi_2$ and real constants $\gamma_{01}$, $\gamma_{02}$, with $\phi_1(x) - \gamma_{01} \Gamma(x)$ vanishing at
infinity, i = 1, 2. Setting \( \phi = \phi_1 - \phi_2, \gamma_0 = \gamma_{01} - \gamma_{02} \), we have 
\( \Delta^2 \phi = 0 \), and \( \phi(x) + \frac{\gamma_0}{4\omega} \log|x| \) vanishes at infinity. By Lemma 3.4, 
\( \phi = \text{constant} \), and therefore \( \gamma_0 = 0 \) and \( \phi = 0 \).

Again, Lemma 3.3 states that \( \phi = Zf \) is a solution of the problem, with \( \gamma_0 \) defined by (3.8). To obtain the bound on \( \phi \), we observe that since 
\[
\log(2 + |x|) \leq M(\varepsilon)(1 + |x|)^{\varepsilon},
\]
Lemma 3.1 (a) implies the inequality 
\[
||\phi||_{\varepsilon} = ||Zf||_{\varepsilon} \leq M(\tau, \varepsilon)||f||_{-\tau}.
\]
We also have, since \( \varepsilon - 4 > -\tau \), 
\[
||f||_{\varepsilon-4,\alpha} \leq ||f||_{-\tau,\alpha}.
\]
We substitute these inequalities into the right-hand side of (2.14), as applied to the biharmonic equation with \( \sigma = \varepsilon \), and obtain (3.55).

(c) For uniqueness, we need to prove that the only solution of 
\( \Delta^2 \phi = 0 \) on \( \mathbb{R}^3 \), with 
\[
\phi(x) + \frac{\gamma_0}{8\pi}|x| - \frac{\gamma_1 \cdot x}{8\pi|x|}
\] vanishing at infinity for some \( \gamma_0 \in \mathbb{R}, \gamma_1 \in \mathbb{R}^3 \), is \( \phi = 0 \), in which case \( \gamma_0 = 0, \gamma_1 = 0 \).

By Lemma 3.4, we have \( \phi(x) = q \cdot x + p, \) a polynomial with degree \( \leq 1 \). If \( \alpha \neq 0 \), we choose a unit vector \( \vec{x} \) such that \( q \cdot \vec{x} = 0 \); then setting \( x = k\vec{x} \) in (3.59) and letting \( k \to \infty \), we conclude that \( \gamma_0 = 0 \), and then \( q = 0 \). Therefore, we must have \( \phi = p, \gamma_1 = 0 \) and 
\[
p - \frac{\gamma_1 \cdot x}{8\pi|x|} \to 0 \text{ as } x \to \infty.
\]
Now, if \( \gamma_1 \neq 0 \), by letting \( x = k\gamma_1, k \to \infty \), and then \( x = -k\gamma_1, k \to \infty \) in (3.60), we get \( p = \frac{1}{8\pi}|\gamma_1| = -\frac{1}{8\pi}|\gamma_1| \). Thus we have proved that
\( \phi = 0, \gamma_0 = 0 \) and \( \gamma_1 = 0 \).

Again, Lemma 3.3 states that \( \phi = Zf \) is a solution of the problem, with \( \gamma_0 \) and \( \gamma_1 \) defined by (3.8). The bound (3.56) follows from (2.14) of Theorem 2.4 (with \( \sigma = 1 \)), Lemma 3.1 (a) and Lemma 2.3 (b).

(d) For uniqueness, we must show that the only solution of

\[
\Delta^2 \phi = 0 \quad \text{on} \quad \mathbb{R}^2,
\]

\[
(3.61) \quad 8\pi \phi(x) - \gamma_0 |x|^2 \left( \log |x| - 1 \right) + (2 \log |x| - 1) \gamma_1
\]

\[= \gamma_2 \cdot \left( \begin{array}{cc}
\log |x| - \frac{1}{2} + \frac{x_1^2}{|x|^2} & \frac{x_1 x_2}{|x|^2} \\
\frac{x_1 x_2}{|x|^2} & \log |x| - \frac{1}{2} + \frac{x_2^2}{|x|^2}
\end{array} \right) \rightarrow 0,
\]

as \( x \rightarrow \infty \), for some \( \gamma_0 \in \mathbb{R}, \gamma_1 \in \mathbb{R}^2 \), symmetric \( \gamma_2 \in \mathbb{R}^{2 \times 2} \), is \( \phi = 0 \), in which case \( \gamma_0 = 0, \gamma_1 = 0 \) and \( \gamma_2 = 0 \).

By Lemma 3.4, \( \phi \) is a polynomial of degree \( \leq 2 \). Therefore, from (3.61), we must have \( \gamma_0 = 0 \). Thus \( \phi = o(|x|^2) \) at infinity, and again Lemma 3.4 implies that \( \phi \) is a polynomial with degree \( \leq 1 \). Hence, again from (3.61) (with now \( \gamma_0 = 0 \), \( \gamma_1 = 0 \). Applying Lemma 3.4 again, we conclude that \( \phi \equiv p, \) a constant, and (3.61) becomes

\[
(3.62) \quad 8\pi p - \left( \log |x| - \frac{1}{2} + \frac{x_1^2}{|x|^2} \right) \gamma_{11} - \left( \log |x| - \frac{1}{2} + \frac{x_2^2}{|x|^2} \right) \gamma_{22}
\]

\[- \frac{2x_1 x_2}{|x|^2} \gamma_{12} \rightarrow 0, \quad \text{as} \quad x \rightarrow \infty,
\]

where \[
\begin{pmatrix}
\gamma_{11} & \gamma_{12} \\
\gamma_{12} & \gamma_{22}
\end{pmatrix} := \gamma_2
\]

From (3.62), we must have

\[
(3.63) \quad \gamma_{11} + \gamma_{22} = 0.
\]
Now by letting \( x = (x_1, 0)^T \to \infty \) and then \( x = (0, x_2)^T \to \infty \) in (3.61), we get

\[
8 \pi \beta - \frac{1}{2} \gamma_{11} + \frac{1}{2} \gamma_{22} = 0, \tag{3.64}
\]
and

\[
8 \pi \beta + \frac{1}{2} \gamma_{11} - \frac{1}{2} \gamma_{22} = 0. \tag{3.65}
\]
Solving the linear system (3.63)-(3.65), we obtain \( \phi = p = 0, \gamma_{11} = \gamma_{22} = 0; \) then, from (3.62), we also have \( \gamma_{12} = 0. \) Thus we have proved that \( \phi = 0, \gamma_0 = 0, \gamma_1 = 0 \) and \( \gamma_2 = 0. \)

Again, Lemma 3.3 states that \( \phi = Zf \) is a solution of the problem, with \( \gamma_0, \gamma_1 \) and \( \gamma_2 \) defined by (3.8). The bound (3.57) follows from (2.14) (with \( \sigma = \epsilon + 2 \)), (3.58), Lemma 3.1 (a) and Lemma 2.3 (b).

4. AN A PRIORI BOUND FOR THE FOURTH ORDER ELLIPTIC EQUATION

We consider the nonhomogeneous equation,

\[
L \phi := a \cdot D \phi + b \cdot D^3 \phi + c \cdot D^2 \phi + d \cdot D \phi + e \phi = f, \tag{4.1}
\]
and derive an \textit{a priori} bound for entire solutions analogous to Theorem 3.4. The conditions on the coefficients will guarantee that the operator L approaches the biharmonic operator near infinity at a certain rate; and the condition of the separability of L (= \( \mathbf{L}_2 \mathbf{L}_1 \)) will be used to prove the uniqueness of the solution. Let \( J \) denote the \( n \times n \times n \times n \) tensor \((\delta_{ij} \delta_{kl})\) with \( J \cdot D^4 = \Delta^2. \) We require:

\textbf{Condition F} (i) The operator L can be separated as \( L = \mathbf{L}_2 \mathbf{L}_1, \) where

\[
L_s = a_s \cdot D^2 + b_s \cdot D + c_s \quad (s = 1, 2),
\]
and \( a_s, b_s, c_s \) are matrix, vector, scalar valued functions on \( \mathbb{R}^n, \)
respectively, with \( a_1, b_1 \) and \( c_1 \) being \( C^2 \) functions and
\[
(4.2) \quad c_s = 0 \quad \text{for} \quad n \geq 3, \quad \text{and} \quad c_s = 0 \quad \text{for} \quad n = 2 \quad (s = 1, 2).
\]
For these coefficients, there exist constants \( \delta, \alpha, \Lambda \), with
\( \delta > \max\{0, 4-n\} \), \( 0 < \alpha < 1 \), \( \Lambda \geq 0 \), such that
\[
(4.3) \quad \|a_s - 1\|_{-\delta, \alpha'}, \|b_s\|_{-1-\delta, \alpha'}, \|c_s\|_{-2-\delta, \alpha'}
\]
\[
\|D^s a_1\|_{-s-\delta, \alpha'}, \|D^s b_1\|_{-1-s-\delta, \alpha'}, \|D^s c_1\|_{-2-s-\delta, \alpha} \leq \Lambda, \quad s = 1, 2.
\]
\[\text{(ii) The matrices} \quad a_s, \quad s = 1, 2, \quad \text{are symmetric, and for some constant} \quad \lambda, \lambda > 0, \]
\[a_s(x)\xi \cdot \xi \geq \lambda|\xi|^2, \quad \text{for any} \quad x, \xi \in \mathbb{R}^n, \quad s = 1, 2.\]

**Remark** From \( L = L_2 L_1 \), i.e., from
\[
(4.4) \quad a \cdot D^4 + b \cdot D^3 + c \cdot D^2 + d \cdot D + e = (a_2 \cdot D^2 + b_2 \cdot D + c_2) (a_1 \cdot D^2 + b_1 \cdot D + c_1),
\]
and from Lemma 2.3, it follows from condition (4.3) that
\[
(4.3)' \quad \|a - J\|_{-\delta, \alpha'}, \|b\|_{-1-\delta, \alpha'}, \|c\|_{-2-\delta, \alpha'}, \|d\|_{-3-\delta, \alpha'}, \|e\|_{-4-\delta, \alpha} \leq M(\Lambda).
\]
So, under Condition F, we can use both (4.3) and (4.3)'.

**Theorem 4.1.** Suppose Condition F holds, that \( f \in B_{-\tau, \alpha} \) for some \( \tau \), \( \tau > 4 \), and that \( \alpha \) is the same as in Condition F. Let \( \phi \) be an entire \( C^4 \) solution of (4.1).

(a) For \( n \geq 5 \), assume that \( \phi(x) \) vanishes at infinity. Then
\[
(4.5) \quad \|\phi\|_0,1 + \|D\phi\|_{-1,1} + \|D^2\phi\|_{-2,1} + \|D^3\phi\|_{-3,1} + \|D^4\phi\|_{-4,\alpha}
\]
\[
\leq M(n, \alpha, \delta, \tau, \lambda, \Lambda)\|f\|_{-\tau, \alpha'}.
\]
Moreover, for any \( \varepsilon \) such that, \( \varepsilon \leq \min\{\delta, \tau-4\}, \)
\[
(4.6) \quad \|\Delta^2 \phi\|_{-4-\varepsilon, \alpha} \leq M(\alpha, \delta, \tau, \lambda, \Lambda)\|f\|_{-\tau, \alpha'}
\]
and when also \( 0 < \varepsilon < 1 \) and \( |x| \geq 1, \)
\[
(4.7) \quad |\phi(x)| \leq M(\alpha, \delta, \tau, \varepsilon, \lambda, \Lambda)\|f\|_{-\tau, \alpha}(1+|x|)^{-\varepsilon}.
\]
(b) For $n = 4$, assume that $\phi(x) - \gamma_0 \log |x|$ vanishes at infinity for some constant $\gamma_0$. Then

\[
|\gamma_0| \leq M(\alpha, \delta, \tau, \lambda, \Lambda)\|f\|_{-\tau, \alpha},
\]
and for any $\epsilon > 0$,

\[
||\phi||_{\epsilon, 1} + ||D\phi||_{\epsilon-1, 1} + ||D^2\phi||_{\epsilon-2, 1} + ||D^3\phi||_{\epsilon-3, 1} + ||D^4\phi||_{\epsilon-4, \alpha} \leq M(\alpha, \delta, \tau, \epsilon, \lambda, \Lambda)\|f\|_{-\tau, \alpha}.
\]

If, moreover, $\epsilon < \delta$, $\epsilon \leq \tau - 4$, then

\[
||\Delta^2\phi||_{-4-\epsilon, \alpha} \leq M(\alpha, \delta, \tau, \epsilon, \lambda, \Lambda)\|f\|_{-\tau, \alpha},
\]
and if also $0 < \epsilon < 1$ and $|x| \geq 1$, then

\[
|\phi(x) - \gamma_0 \log |x|| \leq M(\alpha, \delta, \tau, \epsilon, \lambda, \Lambda)\|f\|_{-\tau, \alpha}(1+|x|)^{-\epsilon}\log(2+|x|).
\]

(c) For $n = 3$, assume that $\phi(x) - \gamma_0 |x| + \frac{\gamma_1 x}{|x|}$ vanishes at infinity for some $\gamma_0 \in \mathbb{R}$ and $\gamma_1 \in \mathbb{R}$. Then

\[
|\gamma_0|, |\gamma_1| \leq M(\alpha, \delta, \tau, \lambda, \Lambda)\|f\|_{-\tau, \alpha},
\]

\[
||\phi||_{1, 1} + ||D\phi||_{0, 1} + ||D^2\phi||_{-1, 1} + ||D^3\phi||_{-2, 1} + ||D^4\phi||_{-3, \alpha} \leq M(\alpha, \delta, \tau, \lambda, \Lambda)\|f\|_{-\tau, \alpha}.
\]

Moreover, for any $\epsilon$ such that $\epsilon \leq \min \{\delta-1, \tau-4\}$,

\[
||\Delta^2\phi||_{-4-\epsilon, \alpha} \leq M(\alpha, \delta, \tau, \lambda, \Lambda)\|f\|_{-\tau, \alpha},
\]
and when also $0 < \epsilon < 1$ and $|x| \geq 1$,

\[
|\phi(x) - \gamma_0 |x| + \gamma_1 \frac{x}{|x|}| \leq M(\alpha, \delta, \tau, \epsilon, \lambda, \Lambda)\|f\|_{-\tau, \alpha}(1+|x|)^{-\epsilon}.
\]

(d) For $n = 2$, assume that

\[
\phi(x) - \gamma_0 |x|^2(\log |x|-1) + (2\log |x|-1)\gamma_1 \cdot x
\]

\[
\left[\begin{array}{cc}
\log |x| - \frac{1}{2} & \frac{x_1}{|x|^2} \\
\frac{x_1}{|x|^2} & \frac{x_2}{|x|^2}
\end{array}\right]
\]

\[
- \gamma_2 \left[\begin{array}{cc}
\log |x| - \frac{1}{2} & \frac{x_1 x_2}{|x|^2} \\
\frac{x_1 x_2}{|x|^2} & \log |x| - \frac{1}{2} & \frac{x_2}{|x|^2}
\end{array}\right]
\]
vanishes at infinity for some constant \( \gamma_0 \), constant vector \( \gamma_1 \) and constant matrix \( \gamma_2 \). Then

\[
|\gamma_0|, |\gamma_1|, |\gamma_2| \leq M(\alpha, \delta, \tau, \lambda, \Delta)||f||_{-\tau, \alpha},
\]

and for any \( \epsilon > 0 \),

\[
||\phi||_{\epsilon+2,1} + ||D\phi||_{\epsilon+1,1} + ||D^2\phi||_{\epsilon,1} + ||D^3\phi||_{\epsilon-1,1} + ||D^4\phi||_{\epsilon-2,\alpha} \leq M(\alpha, \delta, \tau, \epsilon, \lambda, \Delta)||f||_{-\tau, \alpha}.
\]

If, moreover, \( \epsilon < \delta-2 \), \( \epsilon \leq \tau-4 \), then

\[
||\Delta^2\phi||_{-4-\epsilon, \alpha} \leq M(\alpha, \delta, \tau, \epsilon, \lambda, \Delta)||f||_{-\tau, \alpha},
\]

and if also \( 0 < \epsilon < 1 \) and \( |x| \geq 1 \), then

\[
|\phi(x) - \gamma_0|x|^2(\log |x|-1) + (2\log |x|-1)\gamma_1 x|
- \gamma_2 \left( \begin{array}{c}
\frac{x_1}{|x|^2} \\
\frac{x_2}{|x|^2}
\end{array} \right)
\leq M(\alpha, \delta, \tau, \epsilon, \lambda, \Delta)||f||_{-\tau, \alpha}(1+|x|)^{-7\log(2+|x|)}).
\]

Proof: (a) First assume that (4.5) holds for all \( C^4 \) solutions \( \phi \) such that \( \phi \) vanishes at infinity. We rewrite (4.1) as

\[
\Delta^2\phi = g,
\]

where

\[
g := -(a-J)\ast D\phi - b\ast D\phi - c\ast D\phi - d\ast D\phi - e\phi + f.
\]

If \( \epsilon \) is any fixed real number with \( \epsilon \leq \min{\delta, \tau-4} \), then from (4.20),

(4.21), Lemma 2.3 (b) and (c), Condition F (with (4.3)'), and (4.5) we have the bound

\[
||\Delta^2\phi||_{-4-\epsilon, \alpha} \leq ||a-J||_{-\epsilon, \alpha}||D^4\phi||_{-4, \alpha} + ||b||_{-1-\epsilon, \alpha}||D^3\phi||_{-3, \alpha} + ||c||_{-2-\epsilon, \alpha}||D^2\phi||_{-2, \alpha} + ||d||_{-3-\epsilon, \alpha}||D\phi||_{-1, \alpha}
\]
\[ + \|e\|_{-4, \epsilon, a} + \|f\|_{-4, \epsilon, a} \]
\[ \leq M(\Lambda) (\|b^4\phi\|_{-4, a} + \|D^3\phi\|_{-3, a} + \|D^2\phi\|_{-2, a}) \]
\[ + \|D\phi\|_{-1, a} + \|\phi\|_{0, a} + \|f\|_{-\tau, a} \]
\[ \leq M(\alpha, \delta, \tau, \lambda, \Lambda) \|f\|_{-\tau, a}, \]

which is (4.6). When also \(0 < \epsilon < 1\) and \(|x| \geq 1\), the inequality (4.7) is a consequence of Lemma 3.1 (a) and (4.22).

Therefore, in order to finish the proof of (a) it is sufficient to show that (4.5) holds. We choose \(\epsilon\) sufficiently small that
\[ 0 < \epsilon < \min \{1, \delta/2, \tau-4\}. \]

Observe that (4.3) implies
\[ \|a_s\|_{0, a} \leq \|a_s - I\|_{0, a} + \|I\|_{0, a} \leq \|a_s - I\|_{-\delta, a} + \|I\|_{0} \leq \Lambda + \sqrt{n}, \]
\[ \|b_s\|_{-1, a} \leq \|b_s - I\|_{-1-\delta, a} \leq \Lambda, \quad \|c_s\|_{-2, a} \leq \|c_s - I\|_{-2-\delta, a} \leq \Lambda, \]
\[ \|D^3 a_s\|_{-s, a} \leq \|D^3 a_s - I\|_{-s-\delta, a} \leq \Lambda, \]
\[ \|D^3 b_s\|_{-s, a} \leq \|D^3 b_s - I\|_{-s-\delta, a} \leq \Lambda, \]
\[ \|D^3 c_s\|_{-s, a} \leq \|D^3 c_s - I\|_{-s-\delta, a} \leq \Lambda \quad (s = 1, 2). \]

Hence, from (4.4) and Lemma 2.3, we have (2.13) with \(A\) replaced by \(M(\Lambda)\). Thus Theorem 2.4 can be applied with \(\sigma = 0\) to get the estimate
\[ \|\phi\|_{0, 1} + \|D\phi\|_{-1, 1} + \|D^2\phi\|_{-2, 1} + \|D^3\phi\|_{-3, 1} + \|D^4\phi\|_{-4, 1} \]
\[ \leq M(n, \alpha, \tau, \lambda, \Lambda) \{\|\phi\|_{0} + \|f\|_{-\tau, a}\}. \]

Therefore it is sufficient to derive, instead of (4.5), the more conservative estimate
\[ \|\phi\|_{0} \leq M(n, \alpha, \delta, \tau, \lambda, \Lambda) \|f\|_{-\tau, a}. \]

Suppose that (4.26) is false. Then there exist sequences \(\{a_{s}^m\}, \{b_m\}, \{c_m\}, \{f_m\}, \{\phi_m\}, s = 1, 2; m = 1, 2, 3, \ldots\), such that all \(a_{s}^m, b_m, c_m\) satisfy Condition F with the same constants \(\delta, \alpha, \lambda, \Lambda, \ldots\).
such that each $f_m$ is in $B_{-\tau, \alpha}$, each $\phi_m$ is of class $C^4$ and vanishes at infinity, with

\begin{equation}
(4.27) \quad a_m \cdot D^4 \phi_m + b_m \cdot D^3 \phi_m + c_m \cdot D^2 \phi_m + d_m \cdot D \phi_m + e_m \phi_m = f_m
\end{equation}

and with

\begin{equation}
(4.28) \quad \|\phi_m\|_0 = 1, \quad \|f_m\|_{-\tau, \alpha} \leq \frac{1}{m}.
\end{equation}

Then inequalities (4.24), (4.25) are satisfied by the members of these sequences; and in particular, for $s = 1, 2$, the sequences

$\{a_{sm} - I\}, \{b_{sm}\}, \{c_{sm}\}, \{D^s a_{1m}\}, \{D^s b_{1m}\}, \{D^s c_{1m}\}, \{f_m\}, \{\phi_m\}, \{D\phi_m\}, \{D^2 \phi_m\}, \{D^3 \phi_m\}, \{D^4 \phi_m\}$ $(s = 1, 2)$

are all uniformly bounded with respect to the appropriate norms in the spaces $B_{-\delta, \alpha'}, B_{-1-\delta, \alpha'}$ $B_{-2-\delta, \alpha'}, B_{-3-\delta, \alpha'}, B_{-2-5-\delta, \alpha'}, B_{-1-1', \alpha'}, B_{-2, 1'}$, $B_{-3}, B_{-4, \alpha'}$ respectively. By Lemma 2.3 (f), there exist functions $a_{s-I}, b_{s}, c_{s}, f, \phi$ together with $D^s a_{1}, D^s b_{1}, D^s c_{1}$ $(s = 1, 2), D\phi, D^2 \phi, D^3 \phi, D^4 \phi$ all contained in the same spaces as the corresponding sequences such that (noting (4.23)), after passing to subsequences (preserving (4.28)), we have with respect to the norms of the spaces involved,

\begin{equation}
(4.29) \quad (a_{sm} - I) \rightarrow (a_{s} - I) \text{ in } B_{-\varepsilon}, \quad \text{and hence } a_{sm} \rightarrow a_{s} \text{ in } B_{-\varepsilon},
\end{equation}

\begin{equation}
\begin{split}
b_{sm} \rightarrow b_{s} \text{ in } B_{-1-\varepsilon}, \quad c_{sm} \rightarrow c_{s} \text{ in } B_{-2-\varepsilon}, \\
D^s a_{1m} \rightarrow D^s a_{1} \text{ in } B_{-s-\varepsilon}, \quad D^s b_{1m} \rightarrow D^s b_{1} \text{ in } B_{-1-s-\varepsilon}, \\
D^s c_{1m} \rightarrow D^s c_{1} \text{ in } B_{-2-s-\varepsilon} \quad (s = 1, 2), \\
f_m \rightarrow f \text{ in } B_{-4}, \quad \phi_m \rightarrow \phi \text{ in } B_{\varepsilon},
\end{split}
\end{equation}

\begin{equation}
\begin{split}
D\phi_m \rightarrow D\phi \text{ in } B_{\varepsilon-1}, \quad D^2 \phi_m \rightarrow D^2 \phi \text{ in } B_{\varepsilon-2}, \\
D^3 \phi_m \rightarrow D^3 \phi \text{ in } B_{\varepsilon-3}, \quad D^4 \phi_m \rightarrow D^4 \phi \text{ in } B_{\varepsilon-4};
\end{split}
\end{equation}
and, from Lemma 2.3, we also have

\[ (4.30) \quad \psi := a_1 \cdot D^2 \phi + b_1 \cdot D \phi + c_1 \phi \in B_{-2}. \]

From (4.28) we obtain

\[ \|f\|_{-4} = \lim \|f_m\|_{-4} \leq \lim \|f_m\|_{-7} \leq \lim (1/m) = 0. \]

Therefore, upon passing to the limit in (4.27) we find that \( \phi \) is an entire \( C^4 \) solution of the homogeneous equation

\[ (4.31) \quad a \cdot D^4 \phi + b \cdot D^3 \phi + c \cdot D^2 \phi + d \cdot D \phi + e \phi = 0. \]

Moreover, it is clear that (4.2) and (ii) of Condition F on \( c_s \) and \( a_s \) are fulfilled. Since (4.30) implies that \( \psi = a_1 \cdot D^2 \phi + b_1 \cdot D \phi + c_1 \phi \)
vanishes at infinity, the maximum and minimum principles (applied to arbitrarily large spheres in \( \mathbb{R}^n \)) for second-order elliptic equations ([11], [17]) may be applied to solutions \( \psi \) of the equation (4.31) to obtain

\[ (4.32) \quad a_1 \cdot D^2 \phi + b_1 \cdot D \phi + c_1 \phi \equiv 0 \quad \text{in } \mathbb{R}^n. \]

We need to show that \( \phi \equiv 0 \), but first we have to investigate the behavior of each \( \phi_m \) near infinity. We again rewrite (4.27) as

\[ (4.33) \quad \Delta^2 \phi_m = g_m, \]

where

\[ (4.34) \quad g_m := -(a_m - J) \cdot D^4 \phi_m - b_m \cdot D^3 \phi_m + c_m \cdot D^2 \phi_m + d_m \cdot D \phi_m + e_m \phi_m + f_m. \]

Then, from Lemma 3.1, Condition F (with (4.3)') and (4.28) for \( a_m, b_m, c_m, d_m, e_m, \) and \( \phi_m, f_m \), we have the estimate

\[ (4.35) \quad \|g_m\|_{-4,\epsilon, \alpha} \leq \|a_m - J\|_{-2, \epsilon, \alpha} \|D^4 \phi_m\|_{\epsilon - 4, \alpha} + \|b_m\|_{-1 - 2, \epsilon, \alpha} \|D^3 \phi_m\|_{\epsilon - 3, \alpha} + \|c_m\|_{-2 - 2, \epsilon, \alpha} \|D^2 \phi_m\|_{\epsilon - 2, \alpha} + \|d_m\|_{-3 - 2, \epsilon, \alpha} \|D \phi_m\|_{\epsilon - 1, \alpha} + \|e_m\|_{-4 - 2, \epsilon, \alpha} \|\phi_m\|_{\epsilon, \alpha} + \|f_m\|_{-4 - \epsilon, \alpha}. \]
\[ \leq M(A)\{M(\alpha, \epsilon, \lambda, A) \cdot 2\} + 1. \]

Thus \( g_m \in B_{-4-\epsilon, \alpha} \) and then Theorem 3.5 (a) and Lemma 3.1 (a) imply
\[ (4.36) \quad \phi = Zg_m, \quad |\phi_m(x)| \leq M(n, \epsilon)\|g_m\|_{-4-\epsilon}(1+|x|)^{-\epsilon}. \]

By substituting (4.35) and letting \( m \to \infty \), we see that \( \phi \) vanishes at infinity; and then by applying the maximum and minimum principles (to (4.32)), we conclude \( \phi \equiv 0 \). Hence (4.30) yields \( \phi_m \to 0, D\phi_m \to 0, \)
\( D^2\phi_m \to 0, D^3\phi_m \to 0, D^4\phi_m \to 0 \) in the spaces \( B_{\epsilon}, B_{\epsilon-1}, B_{\epsilon-2}, B_{\epsilon-3}, B_{\epsilon-4} \), respectively. Again from (4.34), (4.3)', and (4.28) for \( a_m', b_m', c_m', d_m', e_m \) and \( \phi_m, f_m \), we observe that
\[ (4.37) \quad \|g_m\|_{-4-\epsilon} \leq \|a_m-J\|_{-2\epsilon}\|D^4\phi_m\|_{\epsilon-4} + \|b_m\|_{-1-2\epsilon}\|D^3\phi_m\|_{\epsilon-3} \]
\[ + \|c_m\|_{-2-2\epsilon}\|D^2\phi_m\|_{\epsilon-2} + \|d_m\|_{-3-2\epsilon}\|D\phi_m\|_{\epsilon-1} \]
\[ + \|e_m\|_{-4-2\epsilon}\||\phi_m||_{\epsilon} + \|f_m\|_{-\tau} \]
\[ \leq M(A)\{\|b^4\phi_m\|_{\epsilon-4} + \|b^3\phi_m\|_{\epsilon-3} + \|D^2\phi_m\|_{\epsilon-2} \]
\[ + \|D^\phi_m\|_{\epsilon-1} + ||\phi_m||_{\epsilon}\} + \frac{1}{m}. \]

Hence \( \|g_m\|_{-\tau} \to 0 \) as \( m \to \infty \). But then from (4.36), Lemma 3.1 (a) we obtain
\[ \|\phi_m\|_{\epsilon} \leq M(n, \epsilon)\|g_m\|_{-4-\epsilon} \to 0 \] as \( x \to \infty \),
which contradicts the assumption (4.28).

(b) First assume that (4.9) holds for all \( \epsilon > 0 \) and all \( C^4 \) solutions \( \phi \) such that \( \phi(x) - \gamma_0 \log|x| \) vanishes at infinity for some constant \( \gamma_0 \). If \( \epsilon < \delta, \epsilon \leq \tau-4 \), then from (4.1), Condition F (with (4.3)'), and (4.9) (with \( \epsilon \) replaced by \( \delta-\epsilon \)) we obtain the bound
\[ (4.38) \quad \|D^2\phi\|_{-4-\epsilon, \alpha} \]
\[ \leq \|a-J\|_{-\delta, \alpha}\|b^4\phi\|_{\delta-4, \alpha} + \|b\|_{-1-\delta, \alpha}\|D^3\phi\|_{\delta-3, \alpha} \]
which is (4.10). Moreover, by Theorem 3.5 (b),

\[ \phi = Z(\Delta^2\phi), \quad \gamma_0 = -\frac{1}{4\omega^4} \int_{\mathbb{R}^4} (\Delta^2\phi)(y)dy, \]

which gives the bound

\[ |\gamma_0| \leq \frac{1}{4\omega^4} \int_{\mathbb{R}^4} |\Delta^2\phi|_{-4-\epsilon} (1+|y|)^{-4-\epsilon} dy \leq M(\epsilon)||\Delta^2\phi||_{-4-\epsilon}. \]

We take \( \epsilon = \min\{\delta/2, \tau - 4\} \) (thereby removing dependence on \( \epsilon \)), and combine (4.40) with (4.38) to obtain (4.8). When also \( 0 < \epsilon < 1 \) and \( |x| \geq 1 \), the inequality (4.11) is a consequence of Lemma 3.1 (b) (with \( \tau \) replaced by \( 4 + \epsilon \)) and (4.38).

Therefore, in order to complete the proof of (b) it is sufficient to show that (4.7) holds for all \( \epsilon > 0 \). First we observe that the left-hand side of (4.7) increases as \( \epsilon \) decreases. Thus we may take \( \epsilon \) smaller, if necessary, so that

\[ 0 < \epsilon < \min\{1, \delta/2\}, \quad \epsilon \leq \tau - 4. \]

The inequalities (4.24) hold as when \( n \geq 5 \), and if \( \phi(x) - \gamma_0 \log|x| \) vanishes at infinity for some \( \gamma_0 \), then \( \phi \in B_\epsilon \) for all \( \epsilon > 0 \). Thus we may apply Theorem 2.4 (with \( \sigma = \epsilon \)) and conclude, analogously to (4.25),

\[ ||\phi||_{\epsilon,1} + ||D\phi||_{\epsilon-1,1} + ||D^2\phi||_{\epsilon-2,1} + ||D^3\phi||_{\epsilon-3,1} + ||D^4\phi||_{\epsilon-4,}\]

\[ \leq M(\alpha, \epsilon, \lambda, \Lambda) (||\phi||_\epsilon + ||f||_{-\tau,}\epsilon). \]
Therefore it is sufficient to derive the estimate

\[(4.43) \quad \|\phi\|_{\epsilon} \leq M(\alpha, \delta, \tau, \epsilon, \lambda, \Lambda) \|f\|_{-\tau, \alpha}.\]

Assuming that \((4.43)\) is false, we obtain sequences \(\{a_{sm}\}, \{b_{sm}\}, \{c_{sm}\}, \{f_m\}, \{\phi_m\}, \{\gamma_{om}\}\), \(s = 1, 2; m = 1, 2, 3, \ldots\), such that all \(a_{sm}, b_{sm}, c_{sm}\) satisfy Condition F, such that each \(f_m\) is in \(B_{-\tau, \alpha}\), each \(\phi_m\) is of class \(C^4\) and a solution of \((4.27)\), with \(\phi_m(x) - \gamma_{om} \log |x|\) vanishing at infinity, and with

\[(4.44) \quad \|\phi_m\|_{\epsilon} = 1, \quad \|f_m\|_{-\tau, \alpha} \leq \frac{1}{m}.\]

We substitute \((4.44)\) into \((4.42)\) (applied to all pairs \(\{\phi_m, f_m\}\)) and obtain

\[(4.45) \quad \|\phi_m\|_{\epsilon, 1} + \|D\phi_m\|_{\epsilon-1, 1} + \|D^2\phi_m\|_{\epsilon-2, 1} + \|D^3\phi_m\|_{\epsilon-3, 1} + \|D^4\phi_m\|_{\epsilon-4, \alpha} \leq M(\alpha, \delta, \tau, \epsilon, \lambda, \Lambda)(2).\]

Since \(\delta > \delta - \epsilon > \epsilon > 0\) (see \((4.41)\)) we may pass to subsequences so that

\begin{align*}
a_{sm} & \to a_{s-1} \text{ in } B_{\epsilon-\delta}, \quad b_{sm} \to b_{s} \text{ in } B_{\epsilon-\delta-1}, \\
c_{sm} & \to c_{s} \text{ in } B_{\epsilon-\delta-2}, \quad D^s a_{1m} \to D^s a_{1} \text{ in } B_{\epsilon-\delta-s}, \\
D^s b_{1m} & \to D^s b_{1} \text{ in } B_{\epsilon-\delta-s-1}, \quad D^s c_{1m} \to D^s c_{1} \text{ in } B_{\epsilon-\delta-s-2} \quad (s=1,2), \\
f_m & \to 0 \text{ in } B_{-4}, \quad \phi_m \to \phi \text{ in } B_{\delta-\epsilon}, \\
D\phi_m & \to D\phi \text{ in } B_{\delta-\epsilon-1}, \quad D^2\phi_m \to D^2\phi \text{ in } B_{\delta-\epsilon-2}, \\
D^3\phi_m & \to D^3\phi \text{ in } B_{\delta-\epsilon-3}, \quad D^4\phi_m \to D^4\phi \text{ in } B_{\delta-\epsilon-4}. \\
\end{align*}

Then \(\phi\) is an entire \(C^4\) solution of the equation \((4.31)\) and, from Lemma 2.3 \((f)\) and \((c)\),

\[(4.47) \quad \psi := a_1 \cdot D^2\phi + b_1 \cdot D\phi + c_1 \phi \in B_{\epsilon-2}.\]

We will conclude that \(\phi = 0\), but first we must investigate more carefully the behaviour of each \(\phi_m\) near infinity. We again have
(4.33), with \( g_m \) defined by (4.34). From (4.34), (4.44), (4.45) we have the estimate (4.35). Thus \( g_m \in B_{-\epsilon, \alpha} \), and Theorem 3.5 (b) implies

\[
\phi_m = Zg_m, \quad \gamma_{0m} = -\frac{1}{4\omega_4} \int_{\mathbb{R}^4} g_m(y) dy.
\]

By making estimates as in (4.40), with (4.35) being applied, we see that the sequence \( \{\gamma_{0m}\} \) is a bounded sequence of real numbers. On passing again to subsequences, we may assume \( \gamma_{0m} \to \gamma_0 \) for some real number \( \gamma_0 \). Lemma 3.1 (b) implies that, for \( |x| \geq 1 \),

\[
|\phi_m(x) - \gamma_{0m} \log|x|| \leq M(\epsilon)\|g_m\|_{-\epsilon}(1+|x|)^{-\epsilon} \log(2+|x|).
\]

By substituting (4.35) and letting \( m \to \infty \), we obtain for \( |x| \geq 1 \),

\[
|\phi(x) - \gamma_0 \log|x|| \leq M(\alpha, \epsilon, \lambda, \Lambda)(1+|x|)^{-\epsilon} \log(2+|x|),
\]

and hence

\[
\phi(x) - \gamma_0 \log|x| \text{ vanishes at infinity, for some } \gamma_0 \in \mathbb{R}.
\]

Since (4.47) implies that \( \psi \) vanishes at infinity, the maximum and minimum principles for second order elliptic equations may be applied to solutions \( \psi \) of the equation (4.31) to obtain

\[
\psi = a_1 \cdot D^2\phi + b_1 \cdot D\phi + c_1 \phi = 0 \text{ in } \mathbb{R}^4.
\]

Since \( \phi \in B_{\epsilon} \) is a solution of (4.49), a modified theorem of Begehr and Hile ([2], Theorem 5.3 with \( B_m \) replaced by \( B_{m+\epsilon} \), \( \epsilon < \min\{1, \delta\} \)) may be applied to solutions of the equation (4.49) to get

\[
\phi(x) - p \to 0 \text{ as } x \to \infty \text{ for some constant } p;
\]

but then (4.40) implies that \( \gamma_0 = p = 0 \), and thus \( \phi \) vanishes at infinity. Applying the maximum and minimum principles again, we conclude that \( \phi \equiv 0 \). Hence (4.46) yields \( \phi_m \to 0 \), \( D\phi_m \to 0 \), \( D^2\phi_m \to 0 \), \( D^3\phi_m \to 0 \), \( D^4\phi_m \to 0 \) in the spaces \( B_{\delta, -\epsilon}, B_{\delta, -\epsilon-1}, B_{\delta, -\epsilon-2}, B_{\delta, -\epsilon-3} \).
B_{\delta-\varepsilon-4}, respectively. Again from (4.34), (4.44), we have an estimate
\begin{equation}
\|g_m\|_{-4-\varepsilon} \leq \|a_m-J\|\|D^2 \phi_m\|_{\delta-\varepsilon-4} + \|b_m\|_{-1-\delta}\|D^3 \phi_m\|_{\delta-\varepsilon-3}
+ \|c_m\|_{-2-\delta}\|D^2 \phi_m\|_{\delta-\varepsilon-2} + \|d_m\|_{-3-\delta}\|D^3 \phi_m\|_{\delta-\varepsilon-1}
+ \|e_m\|_{-4-\delta}\|\phi_m\|_{\delta-\varepsilon} + \|f_m\|_{-4-\varepsilon}
\leq M(\Lambda)\|D^4 \phi_m\|_{\delta-\varepsilon-4} + \|D^3 \phi_m\|_{\delta-\varepsilon-3} + \|D^2 \phi_m\|_{\delta-\varepsilon-2}
+ \|D\phi_m\|_{\delta-\varepsilon-1} + \|\phi_m\|_{\delta-\varepsilon} + \frac{1}{m}.
\end{equation}
Hence \(\|g_m\|_{-4-\varepsilon} \to 0\) as \(m \to \infty\). Finally, from Lemma 3.1 (a) we have
for \(x \in \mathbb{R}^4\),
\[|\phi_m(x)| \leq M(\varepsilon)\|g_m\|_{-4-\varepsilon} \log(2+|x|),\]
which implies that \(\|\phi_m\|_\varepsilon \to 0\) as \(m \to \infty\), contradicting (4.44).

(c) Again we first assume that (4.13) holds for all \(C^4\) solutions \(\phi\) such that \(\phi(x) - y_0|x| + \gamma_1 \frac{x}{|x|}\) vanishes at infinity for some constant \(\gamma_0\) and constant vector \(\gamma_1\). If \(\varepsilon\) is any fixed real number with \(\varepsilon \leq \min(\delta-1, \tau-4)\), then from (4.1), Condition F (with (4.3)'), and (4.11) we obtain the bound
\begin{equation}
\|\Delta^2 \phi\|_{-4-\varepsilon,\alpha} \leq \|a-J\|_{-1-\varepsilon,\alpha}\|D^2 \phi\|_{-3,\alpha} + \|b\|_{-2-\varepsilon,\alpha}\|D^3 \phi\|_{-2,\alpha}
+ \|c\|_{-3-\varepsilon,\alpha}\|D^2 \phi\|_{-1,\alpha} + \|d\|_{-4-\varepsilon,\alpha}\|D\phi\|_{0,\alpha}
+ \|e\|_{-5-\varepsilon,\alpha}\|\phi\|_{1,\alpha} + \|f\|_{-4-\varepsilon,\alpha}
\leq M(\Lambda)\|D^4 \phi\|_{-4,\alpha} + \|D^3 \phi\|_{-3,\alpha} + \|D^2 \phi\|_{-2,\alpha}
+ \|D\phi\|_{0,\alpha} + \|\phi\|_{1,\alpha} + \|f\|_{-\tau,\alpha}
\leq M(\alpha,\delta,\tau,\lambda,\Lambda)\|f\|_{-\tau,\alpha'},
\end{equation}
which is (4.14). Moreover, by Theorem 3.5 (c),
\[
\phi = Z(\Delta^2 \phi), \quad \gamma_0 = -\frac{1}{8\pi} \int_{\mathbb{R}^3} (\Delta^2 \phi)(y)dy, \quad \gamma_1 = -\frac{1}{8\pi} \int_{\mathbb{R}^3} \gamma(\Delta^2 \phi)(y)dy.
\]
Thus, by (4.51), we have the bound
\[
|\gamma_s| \leq \frac{1}{8\pi} \int_{\mathbb{R}^3} |y|^{3(1+|y|)^{-4-\varepsilon}} \|\Delta^{2}\phi\|_{-4-\varepsilon} dy \leq M(\varepsilon) \|\Delta^{2}\phi\|_{-4-\varepsilon} \leq M(\varepsilon) \|\Delta^{2}\phi\|_{-4-\varepsilon, \alpha} \leq M(\alpha, \delta, \tau, \varepsilon, \lambda, \Lambda) \|f\|_{-\tau, \alpha} (s = 0, 1).
\]
Choose \( \varepsilon = \min \{\delta - 1, \tau - 4\} \). Then \( M \) in (4.52) is independent of \( \varepsilon \), and (4.12) holds. When also \( 0 < \varepsilon < 1 \) and \( |x| \geq 1 \), the inequality (4.15) is a consequence of Lemma 3.1 (b) and (4.51).

Therefore, in order to finish the proof of (c) it is sufficient to show that (4.13) holds. We choose \( \varepsilon \) sufficiently small that
\[
(4.53) \quad 0 < \varepsilon < \min \{1, (\delta - 1)/2, \tau - 4\}.
\]
Since inequalities (4.24) hold as when \( n \geq 5 \), Theorem 2.4 (with \( \sigma = 1 \)) may be applied to obtain an estimate
\[
(4.54) \quad \|\phi\|_{1,1} + \|D\phi\|_{0,1} + \|D^{2}\phi\|_{-1,1} + \|D^{3}\phi\|_{-2,1} + \|D^{4}\phi\|_{-3,1} \leq M(\alpha, \delta, \tau, \lambda, \Lambda) (\|\phi\|_{1} + \|f\|_{-\tau, \alpha}).
\]
Hence it is sufficient to derive the estimate
\[
(4.55) \quad \|\phi\|_{1} \leq M(\alpha, \delta, \tau, \lambda, \Lambda) \|f\|_{-\tau, \alpha}.
\]
Suppose that (4.55) is false. Then there exist sequences \( \{a_{sm}\} \), \( \{b_{sm}\} \), \( \{c_{sm}\} \), \( \{f_{m}\} \), \( \{\phi_{m}\} \), \( \{\gamma_{0m}\} \), \( \{\gamma_{1m}\} \), \( s = 1, 2; m = 1, 2, 3, \ldots \),
such that all \( a_{sm}, b_{sm}, c_{sm} \) satisfy Condition F, each \( f_{m} \) is in \( B_{-\tau, \alpha} \), and each \( \phi_{m} \) is of class \( C^{4} \) and a solution of (4.27), with
\[
(4.56) \quad \|\phi_{m}\|_{1} = 1, \quad \|f_{m}\|_{-\tau, \alpha} \leq \frac{1}{m}.
\]
Substituting (4.56) into (4.54) (applied to all pairs \( \{\phi_{m}, f_{m}\} \)), we get
\[
(4.57) \quad \|\phi_{m}\|_{1,1} + \|D\phi_{m}\|_{0,1} + \|D^{2}\phi_{m}\|_{-1,1} + \|D^{3}\phi_{m}\|_{-2,1} + \|D^{4}\phi_{m}\|_{-3,1} \leq M(\alpha, \delta, \tau, \lambda, \Lambda) (2).
\]
By (4.53), we may pass to subsequences so that
(4.58) \[ \begin{align*}
  a_{sm} & \to a_s - I \text{ in } B_{-1-\varepsilon}, \quad b_{sm} \to b_s \text{ in } B_{-2-\varepsilon}, \\
  c_{sm} & \to c_s \text{ in } B_{-3-\varepsilon}, \quad D_s^a a_{1m} \to D_s^a a_1 \text{ in } B_{-1-s-\varepsilon}, \\
  D_s^b b_{1m} & \to D_s^b b_1 \text{ in } B_{-2-s-\varepsilon}, \quad D_s^c c_{1m} \to D_s^c c_1 \text{ in } B_{-3-s-\varepsilon}, \\
  f_m & \to 0 \text{ in } B_{-4}, \quad \phi_m \to \phi \text{ in } B_{\varepsilon+1}, \\
  D_s^\phi \phi_m & \to D_s^\phi \phi \text{ in } B_{\varepsilon}, \quad D_s^2 \phi_m \to D_s^2 \phi \text{ in } B_{\varepsilon-1}, \\
  D^3_s \phi_m & \to D^3_s \phi \text{ in } B_{\varepsilon-2}, \quad D^4_s \phi_m \to D^4_s \phi \text{ in } B_{\varepsilon-3} \quad (s = 1, 2). 
\end{align*} \]

Therefore, \( \phi \) is a \( C^4 \) solution of the equation (4.31), with

(4.59) \[ \psi = a_1 \cdot D^2 \phi + b_1 \cdot D \phi + c_1 \phi \in B_{-1}. \]

We need to show that \( \phi = 0 \). We again have (4.33) and (4.34), and from (4.34), (4.56), (4.57), we also have, analogously to (4.51), the estimate

(4.60) \[ |||s_m|||_{-4-\varepsilon, \alpha} \leq M(A) \{ |||D_s^4 \phi_m|||_{-3, \alpha} + |||D_s^3 \phi_m|||_{-2, \alpha} + |||D_s^2 \phi_m|||_{-1, \alpha} \]
\[ + |||D_s \phi_m|||_{0, \alpha} + |||\phi_m|||_{1, \alpha} \} + |||f_m|||_{-\tau, \alpha} \]
\[ \leq M(A) \{ M(\alpha, \delta, \tau, \lambda, A) \cdot 2 \} + 1. \]

Thus \( g_m \in B_{-4-\varepsilon, \alpha} \), and then Theorem 3.5 (c) implies

(4.61) \[ \phi_m = Z g_m, \quad \gamma_{om} = -\frac{1}{8\pi} \int_{\mathbb{R}^3} g_m(y) dy, \quad \gamma_{1m} = -\frac{1}{8\pi} \int_{\mathbb{R}^3} y g_m(y) dy. \]

By making estimates as in (4.52) (for \( g_{sm} \) and \( g_m \)), with (4.60) being applied, we find that the sequences \( \{\gamma_{om}\}, \{\gamma_{1m}\} \) are bounded sequences of real numbers and real vectors, respectively. On passing again to subsequences, we may assume \( \gamma_{om} \to \gamma_0 \) and \( \gamma_{1m} \to \gamma_1 \) for some \( \gamma_0 \in \mathbb{R} \) and \( \gamma_1 \in \mathbb{R}^3 \). Lemma 3.1 (b) implies that, for \( |x| \geq 1 \),

\[ \left| \phi_m(x) - \gamma_{om} |x| + \gamma_{1m} \frac{x}{|x|} \right| \leq M(\varepsilon) ||g_m||_{-4-\varepsilon}(1+|x|)^{-\varepsilon}. \]

By substituting (4.60) and letting \( m \to \infty \), we see that

(4.62) \[ \phi(x) - \gamma_0 |x| + \gamma_1 \frac{x}{|x|} \text{ vanishes at infinity,} \]

for some \( \gamma_0 \in \mathbb{R}, \gamma_1 \in \mathbb{R}^3 \).
Since, from (4.59), \( \psi \) vanishes at infinity, the maximum and minimum principles for solutions \( \psi \) of the second order equation (4.31) imply that
\[
\psi = a_1 \partial^2 \phi + b_1 \partial \phi + c_1 \phi \equiv 0 \quad \text{in } \mathbb{R}^3.
\]
Again, since \( \phi \in B_1 \) is a solution of (4.63), Theorem 5.3 of [2] states that
\[
(4.64) \quad \phi(x) - q \cdot x - p \to 0 \quad \text{as } x \to \infty \quad \text{for some } p \in \mathbb{R} \text{ and } q \in \mathbb{R}^3.
\]
Then, (4.62) and (4.64) imply that
\[
(4.65) \quad q \cdot x + p - \gamma_0 |x| + \gamma_1 \frac{x}{|x|} \text{ vanishes at infinity},
\]
for some \( p, \gamma_0 \in \mathbb{R} \) and some \( q, \gamma_1 \in \mathbb{R}^3 \). A proof analogous to that of Theorem 3.4 (c) may be applied to ensure that \( p = \gamma_0 = 0 \) and \( q = \gamma_1 = 0 \).
Hence \( \phi \) vanishes at infinity, and then \( \phi = 0 \) follows from the maximum and minimum principles (for the equation (4.63)).

We now have from (4.58) that \( \phi_m \to 0, D\phi_m \to 0, D^2\phi_m \to 0, D^3\phi_m \to 0, D^4\phi_m \to 0 \) in the spaces \( B_{\varepsilon+1}, B_{\varepsilon}, B_{\varepsilon-1}, B_{\varepsilon-2}, B_{\varepsilon-3} \), respectively. Again from (4.34), (4.56), we have
\[
\|g_m\|_{-4-\varepsilon} \leq \|a_m-J\|_{-1-2\varepsilon} \|D^4\phi_m\|_{-3} + \|b_m\|_{-3-2\varepsilon} \|D^3\phi_m\|_{-2} + \|c_m\|_{-3-2\varepsilon} \|D^2\phi_m\|_{-1} + \|d_m\|_{-4-2\varepsilon} \|D\phi_m\|_{\varepsilon} + \|e_m\|_{-4-\varepsilon} \|\phi_m\|_{\varepsilon+1} + \|f\|_{-4-\varepsilon} \leq M(A) [\|D^4\phi_m\|_{-3} + \|D^3\phi_m\|_{-2} + \|D^2\phi_m\|_{-1} + \|D\phi_m\|_{\varepsilon} + \|\phi_m\|_{\varepsilon+1}] + \frac{1}{m}.
\]
Hence \( \|g_m\|_{-4-\varepsilon} \to 0 \) as \( x \to \infty \); and then, from Lemma 3.1 (a),
\[
\|\phi_m\|_{1} \leq M(\varepsilon) \|g_m\|_{-4-\varepsilon} \to 0 \quad \text{as } m \to \infty
\]
which contradicts (4.56).

(d) As in the proof of (b), we first assume that (4.17) holds for
all $\epsilon > 0$ and all $C^4$ solutions $\phi$ such that

\begin{equation}
\phi(x) - \gamma_0 |x|^2 (\log |x|-1) + (2 \log |x|-1) \gamma_1 \cdot x
- \gamma_2 \left( \log|x| - \frac{1}{2} + \frac{x_1^2}{|x|^2} + \frac{x_1 x_2}{|x|^2} \log|x| - \frac{1}{2} + \frac{x_2^2}{|x|^2} \right) \longrightarrow 0,
\end{equation}

as $x \rightarrow \infty$, for some $\gamma_0 \in \mathbb{R}$, $\gamma_1 \in \mathbb{R}^2$, symmetric $\gamma_2 \in \mathbb{R}^{2 \times 2}$. If $0 < \epsilon < 1$, $\epsilon < \delta - 2$, $\epsilon \leq \tau - 4$, then from (4.1), Condition F, and (4.17) (with $\epsilon$ replaced by $\delta - 2 - \epsilon$) we obtain the estimate (4.38) which is (4.18).

Hence Theorem 3.5 (d) implies that

\begin{align*}
\gamma_0 &= \frac{1}{8\pi} \int_{\mathbb{R}^2} \Delta^2 \phi(y) \, dy, \\
\gamma_1 &= \frac{1}{8\pi} \int_{\mathbb{R}^2} y \Delta^2 \phi(y) \, dy, \\
\gamma_2 &= \frac{1}{8\pi} \int_{\mathbb{R}^2} y y^T \Delta^2 \phi(y) \, dy.
\end{align*}

Thus, using (4.18), we have

\begin{equation}
|\gamma_s| \leq \frac{1}{8\pi} \int_{\mathbb{R}^2} |y|^6 (1 + |y|)^{-4 - \epsilon} \|\Delta^2 \phi\|_{-4 - \epsilon} \, dy
\leq M(\epsilon) \|\Delta^2 \phi\|_{-4 - \epsilon} \leq M(\alpha, \delta, \tau, \epsilon, \lambda, \Lambda) \|f\|_{-\tau, \alpha} \quad (s = 0, 1, 2).
\end{equation}

Choose $\epsilon = \min \{(\delta - 2)/2, \tau - 4\}$; then this bound is independent of $\epsilon$ and becomes (4.16). Again, when also $0 < \epsilon < 1$ and $|x| \geq 1$, the inequality (4.19) is a consequence of Lemma 3.1 (b) and (4.18).

Therefore, it is sufficient to prove that (4.17) holds for all $\epsilon > 0$. Without lose of generality, we may assume, analogously to (4.41),

\begin{equation}
0 < \epsilon < \min \{1, (\delta - 2)/2\}, \quad \epsilon \leq \tau - 4.
\end{equation}

The inequalities (4.24) still hold, and if $\phi$ satisfies (4.66), then $\phi \in B_{\epsilon+2}$ for all $\epsilon > 0$. Thus we may apply Theorem 2.4 (with $\sigma = 2 + \epsilon$)
and obtain

\[ (4.69) \quad \|\phi\|_{\epsilon+2,1} + \|D\phi\|_{\epsilon+1,1} + \|D^2\phi\|_{\epsilon,1} + \|D^3\phi\|_{\epsilon-1,1} + \|D^4\phi\|_{\epsilon-2,\alpha} \leq M(\alpha, \delta, \tau, \epsilon, \lambda, \Lambda)(\|\phi\|_{\epsilon+2} + \|\epsilon\|_{-\tau, \alpha}). \]

It is, therefore, sufficient to derive instead of (4.17) the estimate

\[ (4.70) \quad \|\phi\|_{\epsilon+2} \leq M(\alpha, \delta, \tau, \epsilon, \lambda, \Lambda)\|f\|_{-\tau, \alpha}. \]

Suppose that (4.70) is false. Then there exist sequences \( \{a_{sm}\}, \{b_{sm}\}, \{c_{sm}\}, \{f_m\}, \{\phi_m\}, \{\gamma_{0m}\}, \{\gamma_{1m}\}, \{\gamma_{2m}\} \), \( s = 1, 2 \); such that all

\( a_{sm}, b_{sm}, c_{sm} \) satisfy Conditon F (hence each \( c_{sm} = 0 \) in this case), each \( f_m \) is in \( B_{-\tau, \alpha'} \) each \( \phi_m \) is in \( B_{\epsilon+2} \) and a solution of (4.27) (with \( c_{sm} = 0 \)), with (4.66) holding for all \( \{\phi_m, \gamma_{0m}, \gamma_{1m}, \gamma_{2m}\} \) and with

\[ (4.71) \quad \|\phi\|_{\epsilon+2} = 1, \quad \|f_m\|_{-\tau, \alpha} \leq \frac{1}{m}. \]

Therefore, by substituting (4.71) into (4.69) (applied to all pairs \( \{\phi_m, f_m\} \)), we have

\[ (4.72) \quad \|\phi_m\|_{\epsilon+2,1} + \|D\phi_m\|_{\epsilon+1,1} + \|D^2\phi_m\|_{\epsilon,1} + \|D^3\phi_m\|_{\epsilon-1,1} + \|D^4\phi_m\|_{\epsilon-2,\alpha} \leq M(\alpha, \delta, \tau, \epsilon, \lambda, \Lambda)(2). \]

Since now \( \delta > \delta - \epsilon > \epsilon + 2 \) (see (4.68)) we may pass to subsequences to ensure that (4.46) holds. Then \( \phi \) is a \( C^4 \) solution of the equation

\[ (4.31) \quad \text{with } c_1 = c_2 = 0, \ e = 0; \] and

\[ (4.73) \quad \psi(x) - \Delta \phi(x) = a_1(x) \cdot D^2 \phi(x) + b_1(x) \cdot D\phi(x) - \Delta \phi(x) \to 0 \text{ as } x \to \infty. \]

Since \( \Delta^2 \phi = - (a-J) \cdot D^4 \phi - b \cdot D^3 \phi - c \cdot D^2 \phi - d \cdot D\phi \in B_{\epsilon-\delta-2} \subset B_{-\epsilon}, \) by

Theorem 3.5 (d) and Lemma 3.3, we have

\[ \phi = Z(\Delta^2 \phi) \quad \text{and} \quad \Delta \phi = \Delta(Z(\Delta^2 \phi)) = S(\Delta^2 \phi); \]

and then, by [2, Lemma 3.2], there is a real constant \( \gamma, \)

\[ \gamma = \frac{1}{2\pi} \int_{\mathbb{R}^2} (\Delta^2 \phi)(x) dx, \] such that \( \Delta \phi(x) - \gamma \log |x| \to 0 \) as \( x \to \infty. \)

Hence, from (4.73), we have
\[ \psi(x) - \gamma \log|x| \to 0 \text{ as } x \to \infty \text{ for some } \gamma \in \mathbb{R}. \]

But since \( \psi \) solves the second order elliptic equation,
\[ a_2 \cdot D^2 \psi + b_2 \cdot D\psi = 0, \]
the maximum and minimum of \( \psi \) on any circle about the origin must occur on the boundary of this circle. We conclude that we must have \( \gamma = 0 \), \( \psi = 0 \).

Now we need to prove that \( \phi = 0 \). We again have (4.33), (4.34); and from (4.34), (4.68), Condition F (with (4.3)'), (4.72), (4.71), we also have, analogously to (4.35),
\begin{equation}
(4.74) \quad \|g_m\|_{-4-\epsilon, \alpha} \leq \|a_m - J\|_{-2-2\epsilon, \alpha} \|D^4 \phi_m\|_{\epsilon-2, \alpha} + \|b_m\|_{-3-2\epsilon, \alpha} \|D^2 \phi_m\|_{\epsilon-1, \alpha} \\
+ \|c_m\|_{-4-2\epsilon, \alpha} \|D^2 \phi_m\|_{\epsilon, \alpha} + \|d_m\|_{-5-2\epsilon, \alpha} \|D\phi\|_{\epsilon+1, \alpha} \\
+ \|e_m\|_{-6-2\epsilon, \alpha} \|\phi_m\|_{\epsilon+2, \alpha} + \|f_m\|_{-4-\epsilon} \\
\leq M(A)\{M(\alpha, \delta, r, \epsilon, \lambda, A) + 2\} + 1.
\end{equation}

Thus \( g_m \in B_{-4-\epsilon, \alpha} \), and Theorem 3.5 (d) implies
\[ \phi_m = Zg_m, \quad \gamma_{0m} = \frac{1}{8\pi} \int_{\mathbb{R}^2} g_m(y)dy, \]
\[ \gamma_{1m} = \frac{1}{8\pi} \int_{\mathbb{R}^2} yg_m(y)dy, \quad \gamma_{2m} = \frac{1}{8\pi} \int_{\mathbb{R}^2} yy^T g_m(y)dy. \]

By making estimates as in (4.67), with (4.74) being applied, we find that the sequences \( \{\gamma_{0m}\}, \{\gamma_{1m}\}, \{\gamma_{2m}\} \) are bounded sequences of real numbers, vectors, symmetric matrices, respectively. On passing again to subsequences, we may assume \( \gamma_{0m} \to \gamma_0, \gamma_{1m} \to \gamma_1, \gamma_{2m} \to \gamma_2 \) for some \( \gamma_0 \in \mathbb{R}, \gamma_1 \in \mathbb{R}^2, \) symmetric \( \gamma_2 \in \mathbb{R}^{2 \times 2} \). From Lemma 3.1 (b) we have, for \( |x| \geq 1, \)
By substituting (4.74) and letting $m \to \infty$, we see that (4.66) holds as $x \to \infty$.

Since $\psi(x)$ equals zero for all $x \in \mathbb{R}^2$, we have

\begin{equation}
(4.76) \quad a_1 \cdot D^2 \phi + b_1 \cdot D \phi = 0 \quad \text{in} \ \mathbb{R}^2.
\end{equation}

Again, since $\phi \in B_{\epsilon+2}$ is a solution of (4.76), the same modified theorem of [2] implies that

\begin{equation}
(4.77) \quad \phi(x) - p_2(x) - \gamma \log|x| \to 0 \quad \text{as} \quad x \to \infty,
\end{equation}

for some $\gamma \in \mathbb{R}$ and some harmonic polynomial

\begin{equation}
(4.78) \quad p_2(x) = r_{11} x_1^2 + 2 r_{12} x_1 x_2 + r_{22} x_2^2 + q_1 x_1 + q_2 x_2 + p.
\end{equation}

Then (4.66) and (4.77) imply that (4.66) holds with $\phi(x)$ replaced by $p_2(x) + \gamma \log|x|$. Since the growth rates of $x_1^2, x_1 x_2, x_2^2, x_1, x_2, \log|x|$, $|x|^2 \log|x|-1$ and $(2 \log|x|-1)x_1$ and $(2 \log|x|-1)x_2$ are variant as $x \to \infty$ from different directions, it is easy to show, from (4.66), (4.77) and (4.78), that

\begin{equation}
(4.79) \quad r_{11} = r_{12} = r_{22} = q_1 = q_2 = \gamma_0 = 0, \quad \gamma_1 = 0, \quad \gamma = \gamma_1 + \gamma_2,
\end{equation}

and

\begin{equation}
(4.80) \quad p = \left( \frac{x_1^2}{|x|^2} - \frac{1}{2} \right) \gamma_{11} - \left( \frac{x_2^2}{|x|^2} - \frac{1}{2} \right) \gamma_{22} - 2 \gamma_{12} \frac{x_1 x_2}{|x|^2} \to 0,
\end{equation}
as \( x \to \infty \), where \( \begin{pmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{pmatrix} := \gamma_2 \).

Now by letting \( x = (x_1, 0)^T \to \infty \) and \( x = (x_2, 0)^T \to \infty \) in (4.80), we have

\[ p \pm \frac{1}{2}(\gamma_{11} - \gamma_{12}) = 0, \]

and then combining with \( \gamma = \gamma_{11} + \gamma_{22} \), we get

\[ (4.81) \gamma_{11} = \gamma_{22} = \gamma/2, \quad p = 0. \]

Thus, from (4.80) (with (4.81) applied), we also have \( \gamma_{12} = 0 \).

Hence, from (4.78), (4.79), (4.81), we have \( p_2 = 0 \); and then, from (4.77), we see that \( \phi(x) - \gamma \log |x| \) vanishes at infinity. But since \( \phi \) solves the second order elliptic equation (4.76), the maximum and minimum of \( \phi \) on any circle about the origin must occur on the boundary of this circle. We conclude that \( \gamma = 0 \) (hence \( \gamma_{11} = \gamma_{22} = 0 \)), \( \phi = 0 \). Therefore, (4.46) yields \( D^s \phi_m \to 0 \) in \( B_{\delta-\epsilon-s} \), \( s = 0, 1, 2, 3, 4 \). Again from (4.34) and (4.71) we have estimate (4.50). Hence

\[ \|g_m\|_{-4-\epsilon} \to 0 \text{ as } x \to \infty. \]

Finally, from Lemma 3.1 (a),

\[ |\phi_m(x)| \leq M(\epsilon)\|g_m\|_{-4-\epsilon}(1+|x|)^{2\log(2+|x|)}, \]

which implies that \( \|\phi_m\|_{\epsilon+2} \to 0 \) as \( x \to \infty \), contradicting (4.71).

Theorem 4.1 can be restated as the following uniform way:

**Theorem 4.1'**. Suppose Condition F holds, that \( f \in B_{-r, \alpha} \) for some \( r \), \( r > 4 \), and that \( \alpha \) is the same as in Condition F. Assume that \( \phi \) is an entire \( C^4 \) solution of (4.1) in \( \mathbb{R}^n \) (\( n \geq 2 \)) such that \( \phi(x) - \sum_{s=0}^{4-n} \gamma_s D^s T(x) \) vanishes at infinity for some constants \( \gamma_s \) of the same dimensions as \( D^s T \). Then
\begin{equation}
|\gamma_s| \leq M(\alpha, \delta, \tau, \lambda, \Lambda) ||f||_{-\tau, \alpha}
\end{equation}

and for any \( \varepsilon > 0 \) (\( \varepsilon \) may equal zero whenever \( n \) is odd or \( n \geq 4+1 \)),

\begin{equation}
\sum_{s=0}^{4-1} ||D^s \phi||_{\max(0,4-n)+\varepsilon-1,1} + ||D^4 \phi||_{\max(0,4-n)+\varepsilon-4,1}
\leq M(\alpha, \delta, \tau, \varepsilon, \lambda, \Lambda) ||f||_{-\tau, \alpha}.
\end{equation}

If, moreover, \( \varepsilon < \delta - \max(0,4-n) \), \( \varepsilon \leq \tau - 4 \), then

\begin{equation}
||A^{4/2} \phi||_{-4-\varepsilon,1} \leq M(\alpha, \delta, \tau, \varepsilon, \lambda, \Lambda) ||f||_{-\tau, \alpha};
\end{equation}

and if also \( 0 < \varepsilon < 1 \) and \( |x| \geq 1 \), then

\begin{equation}
\left| \phi(x) - \sum_{s=0}^{4-n} \gamma_s \ast D^s \Gamma(x) \right|
\leq M(\alpha, \delta, \tau, \varepsilon, \lambda, \Lambda) ||f||_{-\tau, \alpha}(1+|x|)^{-\varepsilon}, \text{ if } n \text{ is odd or } n \geq 4+1,
\leq M(\alpha, \delta, \tau, \varepsilon, \lambda, \Lambda) ||f||_{-\tau, \alpha}(1+|x|)^{-\varepsilon} \log(2+|x|), \text{ if } n \text{ is even and } n \leq 4.
\end{equation}

5. EXISTENCE OF SOLUTIONS

We now demonstrate the existence of entire solutions of the nonhomogenous equation,

\begin{equation}
L \phi := a \ast D^4 \phi + b \ast D^3 \phi + c \ast D^2 \phi + d \ast D \phi + e \phi
= \sum_{s=0}^{4} (a_s \ast D^s \phi + b_s \ast D \phi + c_s) = f,
\end{equation}

with certain prescribed behaviour at infinity. Again we assume Condition F on the coefficients of \( L \).

**Theorem 5.1.** Suppose Condition F holds on the coefficients of \( L \) and that \( f \in B_{-\tau, \alpha} \) for some \( \tau > 4 \).

(a) For \( n \geq 5 \), there exists a unique entire \( C^4 \) solution \( \phi \) of (5.1) such that \( \phi \) vanishes at infinity. Moreover, for this solution \( \phi \) the bound (4.5)-(4.7) of Theorem 4.1 (a) hold.
(b) For \( n = 4 \), there exists a unique entire \( C^4 \) solution \( \phi \) of (5.1) such that \( \phi(x) - \gamma_0 \log|x| \) vanishes at infinity for some constant \( \gamma_0 \). Moreover, for this solution \( \phi \) the bounds (4.8)-(4.11) of Theorem 4.1 (b) hold.

(c) For \( n = 3 \), there exists a unique entire \( C^4 \) solution \( \phi \) of (5.1) such that \( \phi(x) - \gamma_0 |x| + \gamma_1 \cdot \frac{x}{|x|} \) vanishes at infinity for some constant \( \gamma_0 \) and constant vector \( \gamma_1 \). Moreover, for this solution \( \phi \) the bounds (4.12)-(4.15) of Theorem 4.1 (c) hold.

(d) For \( n = 2 \), there exists a unique entire \( C^4 \) solution \( \phi \) of (5.1) such that

\[
\phi(x) - 2 \log|x| + (2 \log|x| - 1) \gamma_1 \cdot x
\]

as \( x \to \infty \), for some constant \( \gamma_0 \), constant vector \( \gamma_1 \) and constant symmetric matrix \( \gamma_2 \). Moreover, for this solution \( \phi \) the bounds (4.16)-(4.19) of Theorem 4.1 (d) hold.

Proof: In all cases (a)-(d), the uniqueness of the solution and bounds (4.5)-(4.17) are consequences of Theorem 4.1. Therefore, we only need to establish existence of a solution.

(a) Let \( \varepsilon \) be a fixed number such that

\[
0 < \varepsilon < 1, \quad \varepsilon \leq \min \{ \delta/2, \tau-4 \}.
\]

We consider a Banach space \( B \) defined by

\[
B := \{ \phi \in C^4(\mathbb{R}^n) : \Delta^2 \phi \in B_{-4-\varepsilon, \alpha}, \phi(x) \text{ vanishes at infinity} \}.
\]
with norm

\[(5.3) \quad ||\phi||_* := ||\Delta^2 \phi||_{-4-\epsilon, \alpha}^*
\]

By Theorem 4.1 (a), applied to \( L = \Delta^2 \), for any \( \phi \) in \( B \) we have

\[(5.4) \quad ||\phi||_{0,1} + ||D\phi||_{-1,1} + ||D^2\phi||_{-2,1} + ||D^3\phi||_{-3,1} + ||D^4\phi||_{-4,\alpha}
\leq M(n,\alpha,\epsilon)||\Delta^2 \phi||_{-4-\epsilon, \alpha}^*
\]

and for \(|x| \geq 1,\n\]

\[(5.5) \quad |\phi(x)| \leq M(n,\alpha,\epsilon)||\Delta^2 \phi||_{-4-\epsilon, \alpha}^*(1+|x|)^{-\epsilon}.
\]

From (5.4)-(5.5), it is easy to verify that \( B \) is indeed a Banach space (see the similar proof in (b)).

We also consider two families of linear operators defined by

\[(5.6) \quad P_t := (1-t)\Delta^2 + tL_2 \Delta, \quad 0 \leq t \leq 1,
\]

and

\[(5.7) \quad \Sigma_t := (1-t)L_2 \Delta + tL_2 L_1, \quad 0 \leq t \leq 1.
\]

Setting

\[(5.8) \quad a_{st} := (1-t)I + ta_s, \quad b_{st} := tb_s, \quad c_{st} := tc_s \quad (s = 1, 2),
\]

we have

\[(5.9) \quad P_t \phi = (a_{2t} \cdot D^2 + b_{2t} \cdot D + c_{2t})\Delta \phi
\]

and

\[(5.10) \quad \Sigma_t \phi = L_2(a_{1t} \cdot D^2 \phi + b_{1t} \cdot D\phi + c_{1t} \cdot D\phi).
\]

We first note that, for \( 0 \leq t \leq 1, \)

\[(5.10) \quad ||a_t||_{-1-\delta, \alpha} = t||a_t||_{-1-\delta, \alpha} \leq \Lambda, \quad ||b_t||_{-1-\delta, \alpha} \leq \Lambda, \quad ||c_t||_{-2-\delta, \alpha} \leq \Lambda,
\]

\[(5.10) \quad ||D^s a_t||_{-s-\delta, \alpha} \quad ||D^s b_t||_{-1-s-\delta, \alpha} \quad ||D^s c_t||_{-2-s-\delta, \alpha} \quad (s = 1, 2),
\]

and then, from (5.5), Lemma 2.3, (4.3)' and (5.4), if \( \phi \in B, \)
(5.11) \[ \|\Sigma_t \phi\|_{-4-\epsilon, \alpha} \leq \Lambda \|\phi\|_{0, \alpha} + \|D\phi\|_{-1, \alpha} + \|D\phi\|_{-2, \alpha} \]
\[ + \|D\phi\|_{-3, \alpha} + \|D\phi\|_{-4, \alpha} + \|\Delta^2 \phi\|_{-4-\epsilon, \alpha} \]
\[ \leq M(n, \alpha, \epsilon, \Lambda) \|\phi\|_\ast. \]

Similarly, if \( \phi \in \mathcal{B} \),
\[ \|P_t \phi\|_{-4-\epsilon, \alpha} \leq M(n, \alpha, \epsilon, \Lambda) \|\phi\|_\ast. \]

Hence, both \( P_t \) and \( \Sigma_t \), \( 0 \leq t \leq 1 \), map \( \mathcal{B} \) into \( \mathcal{B}_{-4-\epsilon, \alpha} \). We also get from Condition F that
\[ c_{st} = t c_s \leq 0 \text{ if } n \geq 3, \text{ and } c_{st} = 0 \text{ if } n = 2 \text{ (s = 1, 2); } \]
and, for any \( x, \xi \in \mathbb{R}^n \),
\[ a_{st}(x)\xi \cdot \xi = (1-t)|\xi|^2 + t a_s(x)\xi \cdot \xi \]
\[ \geq (1-t)|\xi|^2 + t \lambda |\xi|^2 \geq \min\{1, \lambda\} |\xi|^2 \text{ (s = 1, 2).} \]

Therefore, Theorem 4.1 can be applied, and for all \( \phi \in \mathcal{B} \) and \( 0 \leq t \leq 1 \) we have estimates
\[ \|\phi\|_\ast \leq M(n, \alpha, \delta, \tau, \lambda, \Lambda) \|P_t \phi\|_{-\tau, \alpha} \]
and
\[ \|\phi\|_\ast \leq M(n, \alpha, \delta, \tau, \lambda, \Lambda) \|\Sigma_t \phi\|_{-\tau, \alpha}. \]

According to Theorem 5.2 in [11], the Schauder continuation method applies. Theorem 3.5 (a) and Theorem 4.1 (a) assert that the operator \( P_0 = \Delta^2 \) maps \( \mathcal{B} \) onto \( \mathcal{B}_{-4-\epsilon, \alpha} \). Hence, from (5.15), any \( P_t \), \( 0 \leq t \leq 1 \), does the same and, in particular, \( P_1 = L_2 \Delta = \Sigma_0 \) maps \( \mathcal{B} \) onto \( \mathcal{B}_{-4-\epsilon, \alpha} \).

Then, from (5.16), applying the Schauder continuation method again, we conclude that \( L_1 L_1 = L \) maps \( \mathcal{B} \) onto \( \mathcal{B}_{-4-\epsilon, \alpha} \). Since \( \mathcal{B}_{-\tau, \alpha} \subseteq \mathcal{B}_{-4-\epsilon, \alpha} \), the proof of (a) is complete.

(b) Let \( \epsilon \) be chosen and fixed so that
\[ 0 < \epsilon < 1, \quad \epsilon \leq \min\{\delta/2, \tau-4\}. \]
We define $\mathcal{B}$ now as the Banach space
\[(5.17) \quad \mathcal{B} := \{ \phi \in C^4(\mathbb{R}^4) : \Delta^2 \phi \in B_{-4-\epsilon, \alpha}, \phi(x) - \gamma_0 \log|x| \text{ vanishes at infinity for some real constant } \gamma_0 \}, \]
with the norm (5.3). By Theorem 4.1 (b), applied to $L = \Delta^2$, for any $\phi$ in $\mathcal{B}$ with corresponding $\gamma_0$ we have
\[(5.18) \quad |\gamma_0| \leq M(\alpha, \epsilon) \|\Delta^2 \phi\|_{-4-\epsilon, \alpha}, \]
\[(5.19) \quad \|\phi\|_{\epsilon, 1} + \|D\phi\|_{\epsilon-1, 1} + \|D^2\phi\|_{\epsilon-2, 1} + \|D^3\phi\|_{\epsilon-3, 1} + \|D^4\phi\|_{\epsilon-4, \alpha} \leq M(\alpha, \epsilon) \|\Delta^2 \phi\|_{-4-\epsilon, \alpha}, \]
and for $|x| \geq 1$,
\[(5.20) \quad |\phi(x) - \gamma_0 \log|x|| \leq M(\alpha, \epsilon) \|\Delta^2 \phi\|_{-4-\epsilon, \alpha}(1+|x|)^{-\epsilon \log(2+|x|)}. \]
(These results also follow from Theorem 3.5 and Lemma 3.1.) In order to verify that $\mathcal{B}$ is a Banach space, we consider a sequence $\{\phi_m\}$, Cauchy in the norm (5.3) of $\mathcal{B}$, with corresponding associated constants $\{\gamma_{0m}\}$. Inequalities (5.18)-(5.20) show that $\{\phi_m\}$ is also Cauchy in the norm determined by the left hand side of (5.19), that the sequence $\{\gamma_{0m}\}$ is a Cauchy sequence of real numbers, and that the limits $\phi$ and $\gamma_0$ of $\{\phi_m\}$ and $\{\gamma_{0m}\}$ satisfy an inequality similar to (5.20), thereby implying that $\phi(x) - \gamma_0 \log|x|$ vanishes at infinity. Hence $\phi \in \mathcal{B}$, and we conclude that $\mathcal{B}$ is indeed a Banach space with respect to the norm (5.3).

We again consider the families of operators $\mathcal{P}_t$ and $\mathcal{I}_t$, $0 \leq t \leq 1$, defined by (5.6)-(5.9), with (5.10), (5.13), (5.14) also being valid. Then, from (5.10), Lemma 2.3 and (5.19), we have, analogously to (5.11)-(5.12),
This shows that both $P_t$ and $T_t$, $0 \leq t \leq 1$, map $B$ into $B_{-4-\varepsilon, \alpha}$. Then Theorem 4.1 (b) (with $\tau = 4 + \varepsilon$) gives the bound, for all $\phi \in B$,

$$
\|\phi\| \leq M(\alpha, \delta, \varepsilon, \lambda, \Lambda) \cdot \min\{\|P_t \phi\|_{-4-\varepsilon, \alpha}, \|T_t \phi\|_{-4-\varepsilon, \alpha}\}.
$$

Theorem 3.4 (b) and Theorem 4.1 (b) assert that $P_0 = \Delta^2$ maps $B$ onto $B_{-4-\varepsilon, \alpha}$. Hence so does $P_1 = L_2 \Delta = I_0$, and therefore so does $I_1 = L_2 L_1 = L$. Since $B_{-\tau, \alpha} \subseteq B_{-4-\varepsilon, \alpha}$ the proof of (b) is complete.

(c) Let $\varepsilon$ be a fixed number such that

$$
0 < \varepsilon < 1, \quad \varepsilon < \min\{(\delta - 1)/2, \tau - 4\}.
$$

Define $B$ as the Banach space

$$
B := \{\phi \in C^4(\mathbb{R}^3) : \Delta^2 \phi \in B_{-4-\varepsilon, \alpha}, \phi(x) - \gamma_0 |x| + \gamma_1 x^T |x| \text{ vanishes at infinity for some constant } \gamma_0 \text{ and constant vector } \gamma_1\}.
$$

with the norm (5.3). The rest of the proof is similar to the proofs that we presented in (a) and (b).

(d) Let $\varepsilon$ be chosen and fixed so that

$$
0 < \varepsilon < 1, \quad \varepsilon < \min\{(\delta - 2)/2, \tau - 4\}.
$$

Define $B$ now as the Banach space

$$
B := \{\phi \in C^4(\mathbb{R}^2) : \Delta^2 \phi \in B_{-4-\varepsilon, \alpha} \text{ and (5.2) holds as } x \to \infty \text{ for some } \gamma_0 \in \mathbb{R}, \gamma_1 \in \mathbb{R}^2, \text{ symmetric } \gamma_2 \in \mathbb{R}^{2 \times 2}\}
$$

with the norm (5.3). Again, the rest of the proof can be obtained by following the procedure of (a) or (b).

In order to discuss the existence of entire solutions of (5.1) with polynomial growth at infinity, we require the following
modification of Condition F.

**Condition MF** The coefficients of L satisfy Condition F, and moreover, there exists a nonnegative integer m with \( m' := m - \max\{0, 4-n\} \)
such that

\[
\begin{align*}
||a||_{m'-\delta, \alpha'} ||b||_{m'-1-\delta, \alpha'} ||c||_{m'-2-\delta, \alpha'} \|D^8 a_1\|_{m'-s-\delta, \alpha'} \\
\|D^8 b_1\|_{m'-1-s-\delta, \alpha'} \|D^8 c_1\|_{m'-2-s-\delta, \alpha'} \leq \Lambda \quad (s = 1, 2).
\end{align*}
\]

From (5.1) and Lemma 2.3, we see that (5.23) implies the following condition on the coefficients a, b, c, d and e:

\[
\begin{align*}
(5.23)' ||a - J||_{m'-\delta, \alpha'} ||b||_{m'-1-\delta, \alpha'} ||c||_{m'-2-\delta, \alpha'} \\
\|d||_{m'-3-\delta, \alpha'} \|e||_{m'-4-\delta, \alpha'} \leq M(A).
\end{align*}
\]

After replacing Condition F by the stronger Condition MF, we will have the following extension of Theorem 5.1.

**THEOREM 5.2.** Suppose Condition MF holds on the coefficients of L, and that \( f \in B_{-\tau, \alpha} \) for some \( \tau > 4 \). Then:

(i) For any biharmonic polynomial \( p \), with (degree \( p \)) \( \leq m \), there exists a unique entire \( C^4 \) solution \( \phi \) of (5.1) such that

(a) if \( n \geq 5 \), \( \phi(x) - p(x) \to 0 \) as \( x \to \infty \);

(b) if \( n = 4 \), \( \phi(x) - p(x) - \gamma_0 \log|x| \to 0 \) as \( x \to \infty \) for some constant \( \gamma_0 \in \mathbb{R} \);

(c) if \( n = 3 \), \( \phi(x) - p(x) - \gamma_0 \log|x| + \gamma_1 \frac{x}{|x|} \to 0 \) as \( x \to \infty \) for some \( \gamma_0 \in \mathbb{R} \) and \( \gamma_1 \in \mathbb{R}^3 \);
(d) if \( n = 2 \), \( \phi(x) - p(x) - \gamma_0 |x|^2 (\log |x| - 1) + (2 \log |x| - 1) \gamma_1 x \\
- \gamma_2 \left( \log |x| - \frac{1}{2} + \frac{x_1^2}{|x|^2} \right) - \frac{x_1 x_2}{|x|^2} \log |x| - \frac{1}{2} + \frac{x_2^2}{|x|^2} \right) \rightarrow 0

as \( x \rightarrow \infty \) for some \( \gamma_0 \in \mathbb{R}, \gamma_1, \gamma_2 \in \mathbb{R}^2 \), symmetric \( \gamma_2 \in \mathbb{R}^{2 \times 2} \).

(ii) If \( \phi \) is an entire \( C^4 \) solution of (5.1), with \( \phi \in B_{\infty, \sigma} \) and
\[ 0 \leq \sigma < \min \{1, \delta - \max(0, 4-n)\}, \]
then there exists a unique biharmonic polynomial \( p \), with (degree \( p \)) \( \leq m \), such that (a)-(d) above hold.

Proof: (i) The proof of the uniqueness of \( \phi \) again follows from the a priori bounds of Theorem 4.1. By Lemma 2.3 (e), we have \( p \in B_{m, 1}, Dp \in B_{m-1, 1}, D^2 p \in B_{m-2, 1}, D^3 p \in B_{m-3, 1}, D^4 p \in B_{m-4, 1} \). Writing
\[ Lp = \Delta^2 p + (L - \Delta^2)p = (a-J)\Delta^4 p + b\Delta^3 p + c\Delta^2 p + d\Delta p + ep, \]
and applying Condition MF (with (5.23)' and (4.3)' being applied), we have, for \( 0 < \varepsilon < \delta - \max(0, 4-n) \),
\begin{align*}
\|Lp\|_{-4-\varepsilon, \alpha} &\leq \|a-J\|_{-m-\varepsilon, \alpha} \|D^4 p\|_{m-4, \alpha} + \|b\|_{-m-1-\varepsilon, \alpha} \|D^3 p\|_{m-3, \alpha} + \|c\|_{-m-2-\varepsilon, \alpha} \|D^2 p\|_{m-2, \alpha} + \|d\|_{-m-3-\varepsilon, \alpha} \|Dp\|_{m-1, \alpha} \\
&\quad + \|e\|_{-m-4-\varepsilon, \alpha} \|p\|_{m, \alpha} \\
&\leq M(A)[\|D^4 p\|_{m-4, 1} + \|D^3 p\|_{m-3, 1} + \|D^2 p\|_{m-2, 1} \\
&\quad + \|Dp\|_{m-1, 1} + \|p\|_{m, 1}],
\end{align*}
and hence \( Lp \in B_{-4-\varepsilon, \alpha} \). By Theorem 5.1, there exists a unique entire \( C^4 \) solution \( \Phi \) of the equation
\[ L\Phi = f - Lp \]
such that if \( n \geq 5 \), \( \Phi(x) \rightarrow 0 \) as \( x \rightarrow \infty \); if \( n = 4 \), \( \Phi(x) - \gamma_0 \log |x| \rightarrow \)
0 as \( x \to \infty \) for some \( \gamma_0 \in \mathbb{R} \); if \( n = 3 \), \( \Phi(x) - \gamma_0 |x| + \gamma_1 * \frac{x}{|x|} \to 0 \) as \( x \to \infty \) for some \( \gamma_0 \in \mathbb{R} \) and \( \gamma_1 \in \mathbb{R}^3 \); and if \( n = 2 \),

\[
(5.25) \quad \Phi(x) - \gamma_0 |x|^2 (\log |x| - 1) + (2 \log |x| - 1) \gamma_1 \cdot x
\]

as \( x \to \infty \) for some \( \gamma_0 \in \mathbb{R} \), \( \gamma_1 \in \mathbb{R}^2 \) and symmetric \( \gamma_2 \in \mathbb{R}^{2 \times 2} \). Setting \( \phi := \Phi + p \), we obtain the desired solution \( \phi \).

(iii) The proof of the uniqueness of \( p \) follows from the a priori bounds of Theorem 3.5 applied to \( \Delta^2 \).

In order to show existence of \( p \), we observe that if \( \phi \) is a \( C^4 \) solution of (5.1), with \( \phi \in B_{m+\sigma} \), then Theorem 2.4 implies that

\[
\phi \in B_{m+\sigma,1} \quad \text{and} \quad D\phi \in B_{m+\sigma-1,1}, \quad D^2\phi \in B_{m+\sigma-2,1}, \quad D^3\phi \in B_{m+\sigma-3,1}, \quad \text{and} \quad D^4\phi \in B_{m+\sigma-4,1}.
\]

We also have

\[
\Delta^2 \phi = L\phi - (L-\Delta^2)\phi = f - (a-J) \cdot D^4\phi - b \cdot D^3\phi - c \cdot D^2\phi - d \cdot D\phi - e\phi.
\]

Choose \( \varepsilon \) so that \( 0 < \varepsilon < \tau-4 \) and \( \varepsilon \leq \min \{ 1, \delta-\max(0,4-n) \} - \sigma \). Then, from Lemma 2.3 and Condition MF,

\[
||\Delta^2 \phi||_{-4-\varepsilon,\alpha} \leq ||f||_{-4-\varepsilon,\alpha} + ||a-J||_{-\varepsilon-\sigma,\alpha} ||D^4\phi||_{m+\sigma-4,\alpha} + ||b||_{-\varepsilon-\sigma,\alpha} ||D^3\phi||_{m+\sigma-3,\alpha} + ||c||_{-\varepsilon-\sigma,\alpha} ||D^2\phi||_{m+\sigma-2,\alpha} + ||d||_{-\varepsilon-\sigma,\alpha} ||D\phi||_{m+\sigma-1,\alpha} + ||e||_{-\varepsilon-\sigma,\alpha} ||\phi||_{m+\sigma,\alpha}
\]

\[
\leq ||f||_{-\tau,\alpha} + M(\Lambda) \sum_{s=0}^{4} ||D^s\phi||_{m+\sigma-s,\alpha},
\]

which implies \( \Delta^2 \phi \in B_{-4-\varepsilon,\alpha} \). Therefore, by Lemma 3.3 the potential \( Z(\Delta^2 \phi) \) is \( C^4 \) in \( \mathbb{R}^n \), with \( \Delta^2(Z(\Delta^2 \phi)) = \Delta^2 \phi \). Moreover, (a)-(d) of Lemma
3.3 are satisfied by $Z(\Delta^2 \phi)$. The function

\begin{equation}
(5.26) \quad p := \phi - Z(\Delta^2 \phi)
\end{equation}

is biharmonic in $\mathbb{R}^n$. If $n \geq 4$, then, since $\phi \in B_{m+\sigma}$, $Z(\Delta^2 \phi) \in B_\sigma$.

Lemma 3.4 asserts that $p$ is a polynomial with (degree $p$) $\leq m$. The above reasoning applies for the cases of $n = 3$, $m \geq 1$ and $n = 2$, $m \geq 2$ since $Z(\Delta^2 \phi)$ is in $B_1$ and $B_{2+\sigma}$, respectively. For $n = 3$, $m = 0$, we have $\phi \in B_\sigma$, with $Z(\Delta^2 \phi) - \gamma_0 |x| + \gamma_1 \cdot \frac{x}{|x|}$ vanishing at infinity, and then Lemma 3.4 implies $p(x) = a \cdot x + p_0$, a polynomial with degree $\leq 1$.

But then

$$\phi(x) - q \cdot x - p_0 - \gamma_0 |x| + \gamma_1 \cdot \frac{x}{|x|} \to 0 \quad \text{as} \quad x \to \infty,$$

with $\phi \in B_\sigma$, $\sigma < 1$. It follows that $\gamma_0 = 0$, $q = 0$, and $p(x) = p_0$ is a polynomial of degree zero. For $n = 2$, $m = 0$ or $1$, we have $\phi \in B_{m+\sigma} \subset B_{1+\sigma}$ and $(5.25)$ holds with $\Phi$ replaced by $Z(\Delta^2 \phi)$. Then Lemma 3.4 implies that

$$p(x) = r_{11} x_1^2 + 2r_{12} x_1 x_2 + r_{22} x_2^2 + q_1 x_1 + q_2 x_2 + p_0,$$

a polynomial of degree $\leq 2$. Hence, from $(5.26)$, $(5.25)$ for $Z(\Delta^2 \phi)$,

$$\phi(x) - r_{11} x_1^2 - 2r_{12} x_1 x_2 - r_{22} x_2^2 - q_1 x_1 - q_2 x_2 - p_0 - \gamma_0 |x|^2 (\log |x| - 1)$$

$$+ (2\log |x| - 1) \gamma_1 \cdot x - \gamma_2 \cdot \begin{pmatrix} \log |x| - \frac{1}{2} + \frac{x_1^2}{|x|^2} \\ \frac{x_1 x_2}{|x|^2} \\ \frac{x_1 x_2}{|x|^2} \end{pmatrix}$$

$$\to 0 \quad \text{as} \quad x \to \infty.$$

If $m = 1$, then $\phi \in B_{1+\sigma}$, and hence $\gamma_0 = r_{11} = r_{12} = r_{22} = 0$, and

$$p(x) = q_1 x_1 + q_2 x_2 + p_0$$

is a polynomial of degree $\leq 1$; if $m = 0$, then $\phi \in B_\sigma$, and it follows that $\gamma_1 = 0$, $q_1 = q_2 = 0$, and hence $p(x) = p_0$. 

a polynomial of degree zero. Finally, writing $\phi - p = Z(\Delta^2 \phi)$ gives (a)-(d).

Remark. Theorem 5.1 and 5.2 can also be restated in a uniform way as we did for Theorem 4.1. For example, the statements (a)-(d) of Theorem 5.2 can be written in the following uniform way:

\begin{equation}
\phi(x) - p(x) - \sum_{s=0}^{4-n} \gamma_s \cdot D^s \Gamma(x) \to 0 \quad \text{as } x \to 00
\end{equation}

for some constant $\gamma_s$ of the same dimensions as $D^s \Gamma$.

For the special case of the homogeneous equation,

\begin{equation}
\tag{5.28} L \phi = L_2 L_1 \phi = 0,
\end{equation}

Theorem 5.2 implies the following:

**Corollary 5.3.** Suppose Condition MF holds on the coefficients of $L$. Then the space of entire $C^4$ solutions of (5.28) which are in $B_m$ forms a finite dimensional vector space, of dimension the same as that of the space of biharmonic polynomials of degree no greater than $m$. There is a one-to-one correspondence between the members of these two spaces, determined by (5.27) or, equivalently, by statements (a)-(d) of Theorem 5.2.
REFERENCES FOR PART II


