BOUNDS ON THE NUMBER OF COVERS FOR LATTICES AND RELATED POSETS

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By

Jonathan P. Brown

Thesis Committee:

Ralph S. Freese, Chairperson
Thomas Craven
Martha Crosby
J.B. Nation
Sarah Post

This dissertation is dedicated to the two most inspirational people I have ever known. Tabitha, you are my muse, the light of my life, my wife. Mom, your strength, patience, and love exceed all limits and understanding.

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ABSTRACT

Let L be a lattice of order n. Let $e_{\prec} = e_{\prec}(L)$ be the number of covers in L. It has been long conjectured that

$$\limsup_{n\to\infty} \frac{e_{\prec}}{n^{3/2}} = \left(\frac{1}{2}\right)^{3/2} \approx 0.3536, \tag{0.0.1}$$

where $e_{\prec}$ here is the maximum number of covers in any lattice of size n. More recently it has been conjectured that

$$\max \frac{e_{\prec}}{n^{3/2}} = \frac{35}{64} = 0.546875. \tag{0.0.2}$$

Both of these conjectures stem from studying incidence lattices for projective planes. We prove that Equation (0.0.1) holds for lattices of height less than four. We also show $\limsup_{n\to\infty} \frac{e_{\prec}}{n^{3/2}} = 2\left(\frac{1}{3}\right)^{3/2} \approx 0.3849$ for a specific class of posets, which includes all finite modular lattices. We prove $\limsup_{n\to\infty} \frac{e_{\prec}}{n^{3/2}} < 0.6559$ for a larger class of posets which includes all lattices. This nearly demonstrates Equation (0.0.2). Along the way we will explore the origins of these conjectures. Finally, we pose some open problems as well as some potential leads in proving (0.0.1), a conjecture that is nearly 40 years old [1].

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CHAPTER 1
INTRODUCTION

How many covers can a lattice of order n have? The question is fundamental, yet it has gone unanswered for decades. In pursuing the answer we have generated many results dealing with the combinatorics of lattices and related posets. Although some of our results are stronger than other ones, we present all of them. We do so because we believe that this question fundamentally explores the combinatorics of lattice theory.

The primary focus of this chapter is to introduce or refresh the reader with basic Order Theory, Section 1.1, with a focus on Lattice Theory, Section 1.2. Lastly, there is a lemma from graph theory of great importance to many of our results that we cover in Section 1.3. The original theorem was much stronger, but had a rather involved proof. We present as a lemma a version that suffices for our purposes, as well as a simple and novel proof. The first couple of sections introduce some basic concepts and definitions. Standard references for lattice thoery include [3] or [4].

1.1 Order Theory

We say, $\langle P, \leq \rangle$, is a **partially ordered set** if P is a set with a binary relation, $\leq$, on P, such that for all $a, b, c \in P$ it satisfies the following:

1. The relation is **reflexive**: $a \leq a$,
2. The relation is **antisymmetric**: if $a \leq b$ and $b \leq a$ then $a = b$,
3. The relation is **transitive**: if $a \leq b$ and $b \leq c$ then $a \leq c$.

We call $\langle P, \leq \rangle$ a **poset**. In a poset, if for some $a, b \in P$ we have that $a \leq b$ and there are no elements between them we call (a, b) a **covering pair** and denote this by $a \prec b$. To be more precise, $a \prec b$ if $a \neq b$, $a \leq b$, and whenever $a \leq c \leq b$ then either $a = c$ or $b = c$. Some equivalent phrases are: b **covers** a, a is **covered** by b, or $a \prec b$ is a **cover**.

If you let $V = P$ and $E = \{\{a, b\} \mid a \prec b\}$, then we call $G = \langle V, E \rangle$ the **covering graph** of P. Please note that this is not the conventional definition of a covering graph used by graph theorists, but rather the definition given by Bollobas and Rival [1]. A **Hasse Diagram** is just a drawing of the covering graph of a poset, such that b is drawn physically higher on the page than a whenever $a \prec b$. The figures below demonstrate the Hasse Diagrams for the posets $\{5, 6, 7, 8\}$, Figure 1.1.1, and $\mathcal{P}(\{a, b, c\}) = \{S \mid S \subseteq \{a, b, c\}\}$, Figure 1.1.2, under their natural orders.

Figure 1.1.1

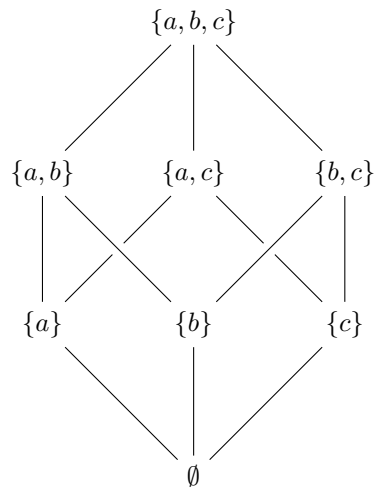

Figure 1.1.2

1.2 Lattice Theory

None of our work deals with general posets, but rather with lattices and other closely related posets. If $\langle L, \leq \rangle$ is a poset such that for any $a, b \in L$ there exists a least upper bound and a greatest lower bound of a and b, then we call this special type of poset a **lattice**. We use $a \wedge b$, pronounced a **meet** b, to represent the element of L that is the greatest lower bound. We use $a \vee b$, pronounced a **join** b, to represent the element of L that is the least upper bound.

An equivalent way to define a lattice is that it is an algebraic structure $\langle L, \wedge, \vee \rangle$ satisfying the following axioms:

1. The Idempotent Laws:

$$a \vee a = a$$

$$a \wedge a = a.$$

2. The Associative Laws:

$$a \vee (b \vee c) = (a \vee b) \vee c$$

$$a \wedge (b \wedge c) = (a \wedge b) \wedge c.$$

3. The Commutative Laws:

$$a \vee b = b \vee a$$

$$a \wedge b = b \wedge a.$$

4. The Absorption Laws:

$$a \vee (a \wedge b) = a$$

$$a \wedge (a \vee b) = a.$$

A poset with all meets and joins defined is a lattice that will follow these axioms. Conversely, an algebra that satisfies all of these axioms is a lattice with its order relation defined by $a \leq b$ if $a \vee b = b$. It should also be noted that there is a symmetry here between join and meet. This is often referred to as duality. If a statement is given and the dual is mentioned, then what is meant is that statement with joins and meets interchanged. The dual of a lattice is a lattice with the order relation reversed, $a \leq' b$ if $b \leq a$.

If we start with a Hasse Diagram, then remove vertices and their connected covers what we end up with still represents a poset. However, if you start with a lattice the remaining object will likely not represent a lattice. We are going to do actions like this in a careful manner so that we either obtain lattices or at least lattice-like posets in our counting arguments. So, we establish a few simple lemmas that tell us when a poset is and isn't a lattice. If a poset has a least element we refer to as 0. If a poset has a greatest element we refer to it as 1.

Lemma 1. *Given lattice L, $a, b, c \in L$, $a \prec c$, and $b \leq c$, then either $a \vee b = c$ or $a \geq b$.*

Proof. Let L, $a, b, c \in L$, $a \prec c$, and $b \leq c$. Let $z = a \vee b$. By definition of join, $a \leq z \leq c$. Hence, either $a = z$ or $c = z$, and we have shown our claim. □

The next lemma is a standard result. A join semilattice is a poset that satisfies axiom 1-3 for join.

Lemma 2. *If a finite join semilattice has a least element, then it is a lattice.*

1.3 Bipartite Graph Lemma

If you take two disjoint antichains, X and Y, from a lattice, L, then there will be no 4-cycles in the covering graph of the poset obtained by restricting the order of L to $X \cup Y$. Furthermore, the covering graph will be bipartite. This is just one of the reasons the combinatorics of bipartite graphs with no 4-cycles are of interest to those studying the combinatorics of lattices. In [8] the authors examine the maximum number of edges in a bipartite graph with no $2k$-cycles. They use the notation $\mathrm{ex}(a, b, C_{2k})$ to represent the maximum number of edges in a bipartite graph with partitions of size a and b with no $2k$-cycles. They find under certain restrictions that

$$\mathrm{ex}(a, b, C_{2k}) \leq \begin{cases} (2k-3)\left[(ab)^{\frac{k+1}{2k}} + a + b\right] & \text{if } k \text{ is odd,} \\ (2k-3)\left[a^{\frac{k+2}{2k}}\, b^{\frac{1}{2}} + a + b\right] & \text{if } k \text{ is even.} \end{cases}$$

We will now present a simple proof for the only case we use in our pursuits, and present some useful variations of the lemma. First, we establish a lemma on convexity.

Lemma 3. *Suppose $\{a_i\}_{i=1}^n$ is a finite sequence. Then*

$$\sum_{i=1}^{n} a_i^2 \geq \frac{1}{n}\left(\sum_{i=1}^{n} a_i\right)^2$$

Proof. Let$\{a_i\}_{i=1}^n$ be a finite sequence. Note $\sum_{i=1}^n \frac{1}{n} = 1$ and $f(x) = x^2$ is convex. Hence, by convexity

$$\begin{aligned} \frac{1}{n}\sum_{i=1}^{n} a_i^2 &= \sum_{i=1}^{n} \frac{1}{n} a_i^2 \\ &\geq \left(\sum_{i=1}^{n} \frac{1}{n} a_i\right)^2 \\ &= \left(\frac{1}{n}\sum_{i=1}^{n} a_i\right)^2 \\ &= \frac{1}{n^2}\left(\sum_{i=1}^{n} a_i\right)^2. \end{aligned}$$

Multupliying through by n, we show our claim. □

Lemma 4. *Suppose $G = \langle V, E\rangle$ is a graph with bipartition $V = X \dot\cup Y$ that contains no 4-cycles. Let $a = |X|$, $b = |Y|$, and $n = |V| = a + b$. Then*

$$|E| \leq \frac{1}{2}\sqrt{4a^2b - 4ab + b^2} + b/2 \leq a\sqrt{b} + b = a\sqrt{b} + O(n).$$

Proof. Let $d(y)$ be the degree of y and

$$F = \{(x_1, y, x_2) \mid \{x_1, y\}, \{x_2, y\} \in E,\ x_1, x_2 \in X,\ y \in Y, \text{ and } x_1 \neq x_2\}.$$

An immediate consequence of G being 4-cycle free is that, if $(x_1, y_1, x_2), (x_1, y_2, x_2) \in F$ then $y_1 = y_2$. For any ordered pair of distinct elements (x_j, x_k) there can be at most one element (x_j, y, x_k) in F and hence $|F|$ is bound by the number of 2-permutations on X, yielding

$$\begin{aligned}
a(a-1) &\geq |F| \\
&= \sum_{i=1}^{b} d(y_i)(d(y_i) - 1) \\
&= \left(\sum_{i=1}^{b} d(y_i)^2\right) - \left(\sum_{i=1}^{b} d(y_i)\right) \\
&\geq \frac{\left(\sum_{i=1}^{b} d(y_i)\right)^2}{b} - |E| && \text{(Convexity)} \\
&= \frac{|E|^2}{b} - |E|.
\end{aligned}$$

When viewed as a quadratic inequality in terms of $|E|$, we get $|E| \leq \frac{1}{2}\sqrt{4a^2b - 4ab + b^2} + \frac{b}{2}$. Now, since $\sqrt{1+x} \leq 1 + \sqrt{x}$,

$$\begin{aligned}
|E| \leq \frac{1}{2}\sqrt{4a^2b - 4ab + b^2} + b/2 &\leq \frac{1}{2}\sqrt{4a^2b + b^2} + b/2 \\
&= a\sqrt{b}\sqrt{1 + b/4a^2} + b/2 \\
&\leq a\sqrt{b}(1 + \sqrt{b}/2a) + b/2 \\
&= a\sqrt{b} + b \\
&= a\sqrt{b} + O(n)
\end{aligned}$$

Hence, we have shown our claim. $\square$

Here is an immediate consequence.

Lemma 5. *Suppose $G = \langle V, E\rangle$ is a graph with bipartition $V = X \dot\cup Y$ that contains no 4-cycles. Let $a = |X|$, $b = |Y|$, and $n = |V| = a + b$. Then*

$$|E| \leq \min(a,b) \cdot \sqrt{\max(a,b)} + \max(a,b) = \min(a,b) \cdot \sqrt{\max(a,b)} + O(n).$$

This yields the next lemma.

Lemma 6. *Suppose $G = \langle V, E\rangle$ is a graph with bipartition $V = X \dot\cup Y$ that contains no 4-cycles. Let $n = |V|$. Then*

$$|E| \leq \left(\frac{n}{2}\right)^{3/2} + n.$$

Proof. Let $f(x) = \min(x, (n-x)) \cdot \sqrt{\max(x, (n-x))}$, where it is defined. Let us consider to derivative of f. On the left half, $x \in [1, \frac{n}{2})$, the derivative is well defined and positive

$$\begin{aligned}
\frac{df}{dx} &= \sqrt{n-x} - \frac{x}{2\sqrt{n-x}} \\
&> \sqrt{n-x} - \frac{n-x}{2\sqrt{n-x}} \\
&= \frac{\sqrt{n-x}}{2} \\
&> 0.
\end{aligned}$$

On the right half, $x \in (\frac{n}{2}, n]$, the derivative is well defined and negative

$$\begin{aligned}
\frac{df}{dx} &= \frac{n-x}{2\sqrt{x}} - \sqrt{x} \\
&< \frac{x}{2\sqrt{x}} - \sqrt{x} \\
&= -\frac{\sqrt{x}}{2} \\
&< 0.
\end{aligned}$$

Since f is continuous, the maximum must be at $x = \frac{n}{2}$. This means $f(x) \leq \left(\frac{n}{2}\right)^{3/2}$, as desired. By Lemma 6

$$|E| \leq \sqrt{\max(a,b)} + \max(a,b) \leq f\left(\frac{n}{2}\right) + n = \left(\frac{n}{2}\right)^{3/2} + n.$$

□

There is an obvious similarity of Lemma 6 to Equation (0.0.1). This is the foundation of many of our attempts to achieve that conjecture. In some arguments, however, we must leave the square root on b whether or not it was larger. For those arguments we use the following lemma which is proven following a similar optimization argument. Note $\frac{2}{\sqrt{27}} \approx 0.3849$.

Lemma 7. *Suppose $G = \langle V, E \rangle$ is a graph with bipartition $V = X \dot{\cup} Y$ that contains no 4-cycles. Let $a = |X|$, $b = |Y|$, and $n = |V| = a + b$. Then*

$$|E| \leq \frac{2}{\sqrt{27}} n^{3/2} + b \leq \frac{2}{\sqrt{27}} n^{3/2} + n.$$

Proof. By Lemma 4, $|E| \leq a\sqrt{b} + b$. Let $f(x) = x\sqrt{n-x}$, and consider it's derivative over $0 < x < n$.

$$\begin{aligned} \frac{df}{dx} &= \sqrt{n-x} - \frac{x}{2\sqrt{n-x}} \\ &= \frac{1}{2\sqrt{n-x}} \left(2(n-x) - x\right) \\ &= \frac{1}{2\sqrt{n-x}} \left(2n - 3x\right). \end{aligned}$$

This means $\frac{df}{dx} > 0$ for $x < \frac{2}{3}n$, and $\frac{df}{dx} < 0$ for $x > \frac{2}{3}n$. Hence f is maximized at $x = \frac{2}{3}n$ and

$$|E| \leq a\sqrt{b} + b \leq f\left(\frac{2}{3}n\right) + n = \frac{2}{\sqrt{27}} n^{3/2} + n.$$

□

CHAPTER 2
GENERAL POSETS

The primary focus of this dissertation is lattices and related posets. To emphasize the combinatorial differences we briefly present a few simple results for general posets. The first result establishes that $\limsup e_{\prec}$ for posets is order n^2. The second result puts an upper bound on this. Lastly, we tie the two together to conclude that over the variety of posets

$$\frac{1}{4} \leq \limsup_{n\to\infty} \frac{e_{\prec}}{n^2} \leq \frac{1}{2}.$$

We use the notation $e_{\prec}$ or $e_{\prec}(P)$ to denote the number of covers in P. For a finite set, S, of posets, we use $e_{\prec}(S)$ to denote the maximum number of covers for members of S. Notice that the order for general posets is greater than that of lattices.

Theorem 8. *For the family of posets of size n*

$$e_{\prec} \geq \left(\frac{n-1}{2}\right)^2.$$

Proof. If $n = 1$, then the claim is obviously true. Suppose $n > 1$ is even. Consider the poset P of order n such that half of the elements are all minimal, and let the other half of the elements cover every one of the lower elements. Then

$$e_{\prec} \geq e_{\prec}(P) = \frac{n}{2} \cdot \frac{n}{2} > \left(\frac{n-1}{2}\right)^2.$$

If $n > 1$ is odd then let the lower level be size $\frac{n-1}{2}$ and make each element in the upper level of size $\frac{n+1}{2}$ cover every element in the lower level. This makes

$$e_{\prec} \geq e_{\prec}(P) = \frac{n-1}{2} \cdot \frac{n+1}{2} > \left(\frac{n-1}{2}\right)^2.$$

□

Theorem 9. *For the family of posets of size n and height $m - 1 > 0$*

$$e_{\prec} \leq \frac{m-1}{2m} n^2.$$

Proof. Suppose $x_i = |\text{level}_i|$ for $0 \leq i < m$. Then the number of covers between levels i and j can be at most $x_i x_j$. An immediate result is that

$$e_{\prec} \leq \sum_{i=1}^{m-1} \sum_{j=0}^{i-1} x_i x_j.$$

Let $f(\vec{x}) = \sum_{i=1}^{m-1} \sum_{j=0}^{i-1} x_i x_j$ and let us consider its maximum as a real valued function with the domain restrictions $\sum_{i=1}^{m-1} x_i = n$ and $x_i \geq 1$ for all i. If $x_i = \frac{n}{m}$ for all i, then

$$e_{\prec} \leq f(\langle \frac{n}{m}, \frac{n}{m}, \ldots, \frac{n}{m} \rangle) = \frac{m(m-1)}{2} \left(\frac{n}{m}\right)^2 = \frac{m-1}{2m} n^2.$$

Let $S_i = \{j \in \mathbb{Z} \mid -1 < j < m,\, j \neq i\}$. Note

$$\frac{\partial f}{\partial x_i} = \sum_{k \in S_i} x_k > \sum_{k \in S_j} x_k = \frac{\partial f}{\partial x_j}$$

if and only if $x_i < x_j$.

Suppose there is some $x_i < x_j$. Then by the partials and that $\sum_{i=1}^{m-1} x_i = n$, decreasing x_j in favor of x_i will increase f until $x_i = x_j$. Hence, $f(\langle \frac{n}{m}, \frac{n}{m}, \ldots, \frac{n}{m} \rangle) \geq f(\vec{x})$ for any $\vec{x}$ in our domain and we have shown our claim. $\square$

The corollary below follows from the two previous theorems. It is our suspicion that the true supremum is closer to $\frac{1}{4}$.

Corollary 10. *Let $e_{\prec}$ be the maximum number of covers in a poset of size n. Then*

$$\frac{1}{4} \leq \limsup_{n \to \infty} \frac{e_{\prec}}{n^2} \leq \frac{1}{2}.$$

CHAPTER 3
THE ASYMPTOTIC CONJECTURE

In Section 3.1 we prove for lattices of height three or less that Equation (0.0.1) holds. The conjecture that Equation (0.0.1) holds for all lattices is at least 37 years old, since it can be found in [1]. In Section 3.2 we prove that $\limsup_{n\to\infty} \frac{e_\prec}{n^{3/2}} = \frac{2}{\sqrt{27}}$ for a specific family of posets. The difference, $\frac{2}{\sqrt{27}} - \frac{1}{\sqrt{8}} < 0.03135$, between this and Conjecture (0.0.1) is small. Also, this coefficient, $\frac{2}{\sqrt{27}}$, is less than Conjecture (0.0.2), and so with computer verification for small n Conjecture (0.0.2) can be confirmed for the lattices in this family of posets. In Section 3.3 we extend this result to a larger collection of posets.

Let L be a lattice. For $x \in L$, let $h(x)$ be the *height* of x, i.e. the length of a maximally-sized chain from 0 to x. We define the length of a chain involving n elements as $n-1$. We use the term jump cover to mean a cover $x \prec y$ such that $h(y) - h(x) > 1$. Let us also define $\text{level}_i = \{x \in L \,|\, h(x) = i\}$ to be the ith level of L. The height of a lattice is the height of the greatest element.

We use the notation $e_\prec$ or $e_\prec(L)$ to denote the number of covers in L. For a finite set, S, of lattices or posets, we use $e_\prec(S)$ to denote the maximum number of covers for members of S.

3.1 Lattices of Height Less Than Four

Theorem 11. *For lattices of size $n > 2$ and of height less than four,*

$$e_\prec \leq \left(\frac{n-2}{2}\right)^{3/2} + 2(n-2).$$

Proof. If the size is greater than two and the height is less than three, then $e_\prec \leq 2(n-2)$ and there is nothing to show. So, we look at lattices of height three. Let $A = \{x \in L \,|\, 0 \prec x,\, x \not\prec 1\}$ be the set of atoms that are not coatoms, let $C = \{x \in L \,|\, 0 \not\prec x,\, x \prec 1\}$ be set of coatoms that are not atoms, and let $B = \{x \in L \,|\, 0 \prec x,\, x \prec 1\}$ be set of the elements that are both atoms and coatoms. The element 0 will have $|A| + |B|$ many upper covers. The element 1 will have $|C| + |B|$ many covers. The only covers remaining are those between A and C. We apply the Lemma 6 to obtain our estimate

$$e_\prec \leq \left[\left(\frac{|A|+|C|}{2}\right)^{3/2} + |A| + |C|\right] + |A| + |B| + |C| + |B| \leq \left(\frac{n-2}{2}\right)^{3/2} + 2(n-2).$$

□

Using incident lattices for projective planes it is shown in [6] that $\limsup \frac{e_\prec}{n^{3/2}} \geq \left(\frac{1}{2}\right)^{3/2}$. Here and elsewhere $e_\prec$ is $e_\prec(S_n)$, where S_n are the posets of size n under consideration. Combining this with the previous theorem we have an immediate corollary.

Corollary 12. *Over the set of lattices of height less than four,*

$$\limsup_{n\to\infty} \frac{e_\prec}{n^{3/2}} = \left(\frac{1}{2}\right)^{3/2}.$$

3.2 Posets with no Jump Covers

We define a poset to be *cover-bow-tie-free* if there are no distinct elements $x_1, x_2 \prec y_1, y_2$. A poset has *no jump covers* if all covers lie between elements in adjacent levels. In this section we prove the following theorem.

Theorem 13. *For the family of cover-bow-tie-free posets with no jump covers,*

$$\limsup \frac{e_\prec}{n^{3/2}} = 2\left(\frac{1}{3}\right)^{3/2}.$$

Here $e_\prec = e_\prec(S_n)$, where S_n is the set of cover-bow-tie-free posets of order n with no jump covers.

This is very close, $2\left(\frac{1}{3}\right)^{3/2} - \left(\frac{1}{2}\right)^{3/2} \approx 0.0313$, to the Conjecture (0.0.1) for lattices. Although these are not necessarily lattices, we may still use similar notions. Instead of the height, $h(x)$, being the length of a maximally-sized chain from 0 to x, we define the height of an element to be the length a maximally sized chain from any minimal element. We define levels similarly. We begin with an upper bound for $e_\prec$.

Theorem 14. *For the family of cover-bow-tie-free posets with no jump covers,*

$$e_\prec \leq 2\left(\frac{n}{3}\right)^{3/2} + 2n.$$

Proof. Let P be a cover-bow-tie-free poset of order n, height $m-1$, and $\chi = \langle |\text{level}_1|, |\text{level}_2|, \ldots, |\text{level}_m| \rangle$. We will use x_i as a real variable in place of the size of level i and let it vary while we look for an upper bound on the number of covers. The variables m and n will remain integer constants. We will make constraints and a utility function from the order theoretic context, but then take a real analysis approach to finding the upper bound for this function. The domain will be a linear equality with only monic terms, the utility

function is continuous, and the partials of the utility function are piecewise continuous. Given a point in the feasible set, this context enables us to use the partials as a guide to move this point to one with a larger output. We first consider points restricted to a very specific subset of the feasible set and show that all of those points will follow a greedy optimization to some point which demonstrates the function is bounded by a certain value. We will then demonstrate that points in other regions go similarly. This makes that value a bound of our utility function. The value is achieved in at least one place in the domain, and hence it is the maximum of the function and a bound on the number of covers.

Set

$$a_i = \begin{cases} x_i\sqrt{x_{i+1}} & \text{if } x_i \leq x_{i+1} \\ \sqrt{x_i}x_{i+1} & \text{if } x_i > x_{i+1}. \end{cases}$$

and define $f(\vec{x}) = \sum_{i=1}^{m-1} a_i$. Since all of the covers lie between adjacent levels, we apply the 4-cycle free graph theorem on each pair of adjacent levels to obtain $e_{\prec} \leq f(\chi) + 2n$. Hence, we will seek to maximize f over $X = \{\vec{x} \in \mathbb{R}^m \mid \sum_{i=1}^m x_i = n, x_i \geq 0 \text{ for all } i\}$. If you add a 0 and a 1 to a cover-bow-tie-free poset of height less than two, then it is clearly a lattice of height less than four. Adding a 0 and a 1 only adds at most $2n$ covers. Since we have already taken care of lattices of height less than four, Theorem 11, we can assume $m \geq 3$. In order to prove this theorem will will demonstrate the claim

$$f(\vec{x}) \leq 2\left(\frac{n}{3}\right)^{3/2}.$$

We will examine the behavior of and bound this function on values, $\vec{x}$, such that $\{x_i\}_{i=1}^m$ is a monotonic increasing sequence. We will then finish by generalizing the argument to values of x that do not have that monotonic property. Note X is convex. Also note that on this set f is continuous and differentiable almost everywhere. Since $\sum_{i=1}^m x_i = n$, we can follow the partials in a simple way to move from a point in the domain to another of greater value.

Suppose $\vec{x} \in X$ such that $\{x_i\}_{i=1}^m$ is monotonic increasing. We will first establish the next statement on our way to prove the claim. Let $k < m-1$ be the first $x_i \neq 0$ and $x_i \neq x_j$ for $i, j \geq k$ on some $\vec{x}$, then $f(\vec{x}) \leq f(\vec{y})$ where $y_k = 0$, or $y_k < x_k$ and $y_i = y_j$ for all $i, j > k$. We will do so in a way that always stays within the monotonic region of the domain.

Inside the monotonic region we see

$$\begin{aligned}
\frac{\partial f}{\partial x_1} &= \sqrt{x_2}, \\
\frac{\partial f}{\partial x_i} &= \frac{x_{i-1}}{2\sqrt{x_i}} + \sqrt{x_{i+1}}, \qquad \text{for } 1 < i < m, \qquad \text{and} \\
\frac{\partial f}{\partial x_m} &= \frac{x_{m-1}}{2\sqrt{x_m}}.
\end{aligned}$$

By monotonicity, $\frac{\partial f}{\partial x_1} = \sqrt{x_2} \leq \sqrt{x_{i+1}} < \frac{\partial f}{\partial x_i}$ for all $1 < i < m$. This means we ought to decrease x_1 in order to increase x_i for $1 < i < m$. However, we do not get this nice comparison of the partials for $i = m$. Hence, if x_m is too small, then it is possible that $x_j = x_k$ for $j, k > 1$ and there is no way to decrease x_1 to 1 without violating monotonicity. Once $x_j = 0$ for some $j < m - 1$, then $\frac{\partial f}{\partial x_{j+1}} = \sqrt{x_{j+2}}$ and $a_j = 0$ for $i < j$. Therefore, the comparison applied to $x_1 = 0$ in a optimal solution may be applied to the first $k < m - 1$ such that $x_i \neq 0$. This demonstrates the statement.

By recursive application of this statement we may conclude that there is some $k < m-1$ such that either $x_i = 0$ for $i < k$ and $x_i = x_j$ for $j > k$ or $x_{m-2} = 0$.

Suppose we are not able to decrease x_k to 0 without violating the monotonicity, we still decrease it as much as we can by transferring its weight to the other components. This makes $a_i = \left(\frac{n-x_k}{m-k}\right)^{3/2}$ for $k < i < m$, $a_i = 0$ for $i < k$, and $a_k = x_k\sqrt{\frac{n-x_k}{m-k}}$. Let $f(x_k) = x_k\sqrt{\frac{n-x_k}{m-k}} + (m-k-1)\left(\frac{n-x_k}{m-k}\right)^{3/2}$ for $0 \leq x_k \leq n$. We may conclude $f(\vec{x}) \leq (m-k-1)\left(\frac{n}{m-k}\right)^{3/2}$ for $m - k \geq 4$, since

$$\begin{aligned}
\frac{df}{dx_k} &= \sqrt{\frac{n-x_k}{m-k}} - \frac{x_k}{2\sqrt{(m-k)(n-x_k)}} - \frac{3(m-k-1)}{2(m-k)^{3/2}}\sqrt{n-x_k} \\
&= \frac{1}{2\sqrt{(m-k)(n-x_k)}}\left(2(n-x_k) - x_k - \frac{3(m-k-1)}{m-k}(n-x_k)\right) \\
&\leq \frac{1}{2\sqrt{(m-k)(n-x_k)}}\left(2(n-x_k) - x_k - 2(n-x_k)\right) \\
&= \frac{-x_k}{2\sqrt{(m-k)(n-x_k)}} \\
&< 0.
\end{aligned}$$

The maximum of $(m-k-1)\left(\frac{n}{m-k}\right)^{3/2}$ for integers m is $2\left(\frac{n}{3}\right)^{3/2}$. If $m-k=3$, then we would have $f(\vec{x}) = x_k\sqrt{\frac{n-x_k}{2}} + \left(\frac{n-x_k}{2}\right)^{3/2}$. Consider

$$\begin{aligned}\frac{\mathrm{d}f}{\mathrm{d}x_k} &= \sqrt{\frac{n-x_k}{2}} - \frac{x_k}{2\sqrt{2(n-x_k)}} - \frac{3}{4}\sqrt{\frac{n-x_k}{2}} \\ &= \frac{1}{4\sqrt{2(n-x_k)}}\left(4(n-x_k) - 2x_k - 3(n-x_k)\right) \\ &= \frac{1}{4\sqrt{2(n-x_k)}}\left(n-3x_k\right).\end{aligned}$$

Since $\frac{\mathrm{d}f}{\mathrm{d}x_k} > 0$ when $x_k < \frac{n}{3}$ and $\frac{\mathrm{d}f}{\mathrm{d}x_k} < 0$ when $x_k > \frac{n}{3}$, we have that $f(\vec{x}) \leq 2\left(\frac{n}{3}\right)^{3/2}$. Either way, if we assume x_k cannot be decreased without violating monotonicity we have $f(\vec{x}) \leq 2\left(\frac{n}{3}\right)^{3/2}$.

Suppose we are able to decrease x_k to 0 without violating monotonicity and $x_i = x_j$ for all $i, j > k$. That means $f(x) = (m-k-1)\left(\frac{n}{m-k}\right)^{3/2}$ and the maximum for this over integers m is $2\left(\frac{n}{3}\right)^{3/2}$. Thus, we have demonstrated the claim in the monotonic region of the domain.

Suppose $\{x_i\}$ is monotonic increasing for $i < j$, but $x_j < x_{j-1}$. That is $x_1 \leq x_2 \leq \ldots \leq x_{j-2} \leq x_{j-1} > x_j$. Then either $\frac{\partial f}{\partial x_j} = \sqrt{x_{j-1}} + \sqrt{x_{j+1}}$ or $\frac{\partial f}{\partial x_j} = \sqrt{x_{j-1}} + \frac{x_{j+1}}{2\sqrt{x_j}}$. If $j = 2$, since $x_1 > x_2$

$$\frac{\partial f}{\partial x_1} = \frac{x_2}{2\sqrt{x_1}} < \sqrt{x_1} = \sqrt{x_{j-1}} \leq \frac{\partial f}{\partial x_j}.$$

In that case, we would reduce x_1 in favor of x_2 until $x_1 = x_2$. Although the partials change at $x_1 = x_2$, the function is continuous and so we can move to the boundary as such. If $j > 2$, then we use $x_2 \leq x_{j-1}$ to conclude

$$\frac{\partial f}{\partial x_1} = \sqrt{x_2} \leq \sqrt{x_{j-1}} \leq \frac{\partial f}{\partial x_j}.$$

And so, we should attempt to reduce x_1 to 1, this time in favor of x_j. We continue this process recursively as we did before until $\{x_i\}$ is monotonic increasing. Once we've moved into the monotonic region of the domain we get $f(\vec{x}) \leq 2\left(\frac{n}{3}\right)^{3/2}$. Hence, we have established our claim and since $e_{\prec} \leq f(\chi) + 2n \leq 2\left(\frac{n}{3}\right)^{3/2} + 2n$ we have shown the theorem. □

Now we are ready to complete the proof of Theorem 13.

Proof. Recall the projective plane over the field of order k has k^2+k+1 many points, k^2+k+1 many lines, each line is incident with $k+1$ many points, and each point is incident with $k+1$ many lines. The incident lattice is constructed so that the empty set, the points, the lines, and the whole space are the elements with

containment as the order. This was guaranteed to be a lattice by the definition of this type of geometry. If we take two levels to be the lines and points of a projective plane over a field of order k, and add another level of size k^2+k+1 above such that the relationship between it and the level below is also that of the the relationship between the points and lines in a projective plane over the field of order k we obtain a cover-bow-tie-free poset. The number of elements of this poset is $n = 3(k^2+k+1)$. The number of covers in this poset is $E = 2(k+1)(k^2+k+1)$. An immediate consequence of Theorem 14 is $\limsup_{n\to\infty} \frac{e_{\prec}}{n^{3/2}} \leq 2\left(\frac{1}{3}\right)^{3/2}$. Putting this all together we obtain

$$\begin{aligned} 2\left(\frac{1}{3}\right)^{3/2} &\geq \limsup_{n\to\infty} \frac{e_{\prec}}{n^{3/2}} \\ &\geq \limsup_{n\to\infty} \frac{E}{n^{3/2}} \\ &= \limsup_{k\to\infty} \frac{2(k+1)(k^2+k+1)}{(3(k^2+k+1))^{3/2}} \\ &= 2\left(\frac{1}{3}\right)^{3/2}. \end{aligned}$$

Hence, $\limsup \frac{e_{\prec}}{n^{3/2}} = 2\left(\frac{1}{3}\right)^{3/2}$. □

3.3 Extensions

In this section we present one corollary to Theorem 14. We show that a lattice that does not contain a particular structure can be mapped to a new poset with the same number of covers that is jump cover free and cover-bow-tie-free. The results in the previous section are very strong, and below we show how to extend them to apply to more lattices. We believe that further dealing with this structure is the best path to expanding the previous results to apply to all lattices. We have some promising preliminary results on that matter which are not presented here.

Given a lattice, L, and an element $x \in L$ we define the *depth* of x, $d(x)$, as the length of a maximal-size chain from 1.

Corollary 15. *Given a lattice that does not contain any set of four distinct elements $a, a', c, c' \in L$ such that a and a' are incomparable, $a \prec c$, $a' \prec c'$, with $d(c) = d(c') < d(a \vee a')$, as in Figure 3.3.1,*

$$\frac{e_{\prec}}{n^{3/2}} \leq 2\left(\frac{1}{3}\right)^{2/3}.$$

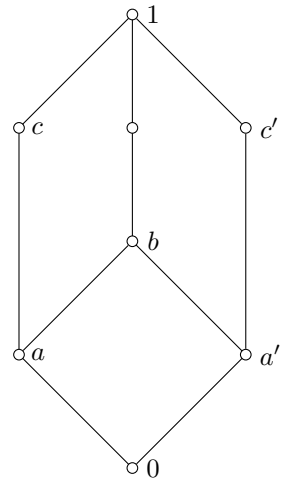

Figure 3.3.1

Proof. Suppose L is a lattice that does not contain any set of four distinct elements $a, a', c, c' \in L$ such that a and a' are incomparable, $a \prec c$, $a' \prec c'$, with $d(c) = d(c') < d(a \vee a')$, as in Figure 3.3.1. Every poset constructed will consist of the same elements at the same depth as L, so we will use the term depth without any direct reference to which poset we are on. We will modify L into a poset P with the same number of covers as L and such that P is cover-bow-tie-free and contains no jump covers. Also the depth function L and P will be the same. We will do this in stages. At the ith stage we will have

$$\{x \in P \mid x \succ_P y,\ d(x) < d(y) - 1,\ d(x) < i \text{ for some } y \in P\} = \emptyset, \tag{IH}$$

$e_{\prec}(P) = e_{\prec}(L)$, $x \leq_P y$ implies $x \leq_L y$ if $d(x), d(y) > i$, and $x \prec_P y$ implies $x \prec_L y$ for all jump covers.

Since a lattice is a cover-bow-tie-free poset such that every coatom $x \in L$ has $d(x) = 1$, the base case is satisfied for $i = 1$ with $P = L$. Suppose it holds for i and P. We must show that there exists a $\langle P', \leq_{P'} \rangle$ with the same number of covers as P such that P' preserves depth and the above properties for $i + 1$. We will sweep through each level with a construction to make each level jump cover free and the overall poset will remain cover-bow-tie-free while retaining most of the structure of the original lattice, including depth and number of covers.

Let $i = \min\{d(x) \mid x \succ_P y \text{ is a jump cover}\}$, and let $a \prec_P c$ be one such cover at minimal depth. Since a is more than one level away from c and $a \prec_P c$, there exists some $b >_P a$ such that $b \not\leq_P c$ and $d(b) = d(c)+1$. We will create a P' by $\leq_{P'} = \leq_P \cup \{(x, c) \mid x \leq_P b\}$. Since $d(b) > d(c)$, P' is still a poset. This destroys the cover $a \prec c$ and adds $b \prec c$. Since all elements with depth less than i do not have any jump covers, (IH),

no covers involving elements above c are destroyed. The only other covers it could affect are those to c from elements below b. Such an element could not be comparable to a, because $a \prec y$. Such an element, in fact, cannot be covered by c because that would make $b \wedge_L c$ undefined in the original lattice. Hence, only one cover is destroyed, one cover is created, and depths are preserved.

We apply this process to all jump covers involving an element with depth i to obtain P'. There cannot be two a and a' below the same b that are both covered by c. If there were, then $b \wedge_L c$ would not be defined in the original lattice, Lemma 1. Hence applying this to all elements with depth i will maintain the same number of covers in the poset. Suppose after applying this process to all elements at depth i that there is a cover bow tie, $b_0, b_1 \prec_{P'} c, c'$. Since P was cover-bow-tie-free and the only covers added are from depth i to depth $i+1$, this is where at least one of the covers involved must lie. Without loss of generality suppose $b_0 \prec_{P'} c$, but $b_0 \not\prec_P c$. That means there exists some $a_0 \in L$ such that $a_0 \prec_L c$, $a_0 <_L b_0$, and $b_0 \not\leq_L c$. By Equation (IH), there are no elements of depth $j < i$ that jump cover any other elements in P'. By the construction, there are also not any elements of depth i that jump cover any other elements in P'. Hence, the entire cover bow tie must lie between elements of depth i and $i+1$. We have three cases; either $b_0 \prec_L c'$, or $a_0 \prec_L c'$, or there exists some $a_0' \in L$ such that $a_0' \prec_L c'$ and $a_0' <_L b$. The third case is excluded by the hypothesis of this theorem. That is, we assumed in the beginning L does not contain incomparable a and a' such that $a \prec_L c$, $a' \prec_L c'$, with $d(c) = d(c') = i$ and $d(a \vee_L a') > i$, as in Figure 3.3.1. In both of the other cases we have $a_0 \leq_L c'$.

Suppose that $a_0 \prec_L c$, $a_0 <_L b_0$, and $b_0 \not\leq_L c$ and $a_0 \leq_L c'$. That means $c \wedge_L c' = a_0$. Hence $b_1 \not\prec_L c, c'$. That means must exist some $a_1 <_L b_1$ such that $a_1 \prec_L c$ or some $a_1' <_L b_1$ such that $a_1' \prec_L c'$. By the hypothesis of this theorem, there cannot be both. That would make either $a_1 = c \wedge_L c'$ or $a_1' = c \wedge_L c'$. Either way a contradiction with $c \wedge_L c' = a_0$. Thus, the new poset P' is cover-bow-tie-free. Since it inherits all covers from P except for the new ones added between elements of depth i and those of depth $i+1$ while destroying only those involving elements with depth i, P' satisfies (IH) for $i+1$ and we shown our claim. $\square$

CHAPTER 4
GENERAL LATTICES

Here we present our results for a family of posets that contains all lattices. We do not require that a poset have a join and meet for all elements. We only require that if there is a common upper bound for a pair of elements then there is a least upper bound, and similarly for meet. We refer to this property as *bow tie free*. This name is from the bow tie like shape in the Hasse Diagram that would be made by two elements and their two incomparable upper bounds. If we only require that the Hasse diagram be free of such bow ties involving only covers, i.e. $x, y \prec w, v$ all distinct elements, then we call that property *cover-bow-tie-free*.

We present some results that are defeated by others since they highlight the combinatorial structure of lattices, perhaps the techniques may be of interest, and they might even inspire a solution to Equation (0.0.1). Section 4.1 will start off Chapter 4 with the first significant advancement [7] towards proving Equation (0.0.2). We then introduce in Section 4.2 what we refer to as the Width Theorem from [6] as well as a minor improvement. In Section 4.3 we show the progression of our best results for general lattices that culminates in nearly, within 0.11, proving the coefficient in Equation (0.0.2).

4.1 First Improvement

In [7] they prove a theorem similar to Theorem 16. This was a significant advancement towards the conjectured supremum, (0.0.1), compared to the coefficient of 3 first proven by [1]. We present the following result, because the conclusion is stronger, we demonstrate it applies to a larger family of posets, and we show some details that are left out of the original article.

We will use the following statements about convex optimization. The text [2] is a good reference, however these statements are spread through out it. See [10] for a clear explanation of convex optimization.

Suppose we have a convex function, $f(\vec{x}) : \mathbb{R}^n \to \mathbb{R}$, to be minimized over the variable $\vec{x}$ subject to the convex inequality constraints $g_i(\vec{x}) \leq 0$. Define the Lagrangian function

$$L(\vec{x}, \lambda_0, \ldots, \lambda_m) = \lambda_0 f(\vec{x}) + \lambda_1 g_1(\vec{x}) + \ldots + \lambda_m g_m(\vec{x}).$$

For each point $\vec{x}$ in the domain that minimizes f, there exist real numbers $\lambda_0, \ldots, \lambda_m$ called Lagrange multipliers, that satisfy these conditions simultaneously:

1. x minimizes $L(\vec{y}, \lambda_0, \lambda_1, \ldots, \lambda_m)$ over all $\vec{y}$ in the domain of f,

2. $\lambda_0 \geq 0$, $\lambda_1 \geq 0$, …, $\lambda_m \geq 0$, with at least one $\lambda_k > 0$.

3. $\lambda_1 g_1(\vec{x}) = 0$, …, $\lambda_m g_m(\vec{x}) = 0$ (complementary slackness).

Conversely, if some $\vec{x}$ satisfies (1)-(3) for scalars $\lambda_0, \ldots, \lambda_m$ with $\lambda_0 = 1$, then $\vec{x}$ is certain to minimize f.

Theorem 16. *Given a cover-bow-tie-free poset of size n, then*

$$e_{\prec} \leq \sqrt{\frac{8}{9}} n^{3/2} - \sqrt{\frac{8}{9}}.$$

Proof. Let C be the cover set of a cover-bow-tie-free poset, L, of size n. Let $\{0, 1, 2, \ldots n-1\}$ be the elements of L. Since there exists a linear extension of any poset, [9], we may assume without loss of generality that the numerical order on L is a linear extension of the order on L. Note that the elements are numbered by a linear extension and so 1 in general does not represent the greatest element. Let $X_j = \{i \mid i \prec j\}$ and $x_j = |X_j|$. Since L is cover-bow-tie-free, there are no distinct i, j, k, l such that $i, j \prec k, l$. Hence $|X_k \cap X_l| \leq 1$ for $k \neq l$.

Let $Y_k = X_k \cup \{k\}$, and consider two element subsets of these Y_k. If $k \neq l$, then Y_k and Y_l share no common two element subset. Let $i, j \in Y_k$ be distinct. Suppose $i, j \in X_k$. We have already established that both i and j cannot exist in some X_l for $k \neq l$. If $i, j \in Y_l$ for some $k \neq l$, then without loss of generality $j = l$. That means $i \prec j \prec k$ would violate $i \prec k$. Instead suppose, without loss of generality, that $i \in X_k$ and $j = k$. It cannot be that $i, j \in Y_l$ for some $k \neq l$, since $i \prec j \prec l$ would violate the definition of $i \prec l$. Hence, the two element subsets are distinct. On the left we count the number of two element subsets of all Y_i for $i \leq k$. On the right we use the fact that the elements are ordered by a linear extension to bound this count by the number two element subsets of $\{0, 1, \ldots, k-1\}$. Since there are no lower covers of 0, we will start our count at 1 in the following the estimates for any $k = 1, 2, \ldots, n-1$

$$\begin{aligned} \sum_{i=1}^{k} \binom{x_i + 1}{2} &\leq \binom{k+1}{2} \\ \sum_{i=1}^{k} \left(x_i^2 + x_i\right) &\leq k^2 + k \\ \sum_{i=1}^{k} x_i^2 &\leq k^2 + k. \end{aligned} \tag{4.1.1}$$

We will now treat each x_k as a non-negative real variable and find the maximum of $f(\vec{x}) = \sum_{i=1}^{n-1} x_i$ subject to the $n-1$ constraints of (4.1.1) for $0 < k < n$. This is a continuous function on a compact set,

so the extrema do exist. The reader may wonder if using Inequality (4.1.1) instead of the stricter inequality directly above it might be less optimal. However, with the techniques employed in this proof the later inequality produces the better result.

Each of the domain constraints are convex, and so the intersection of them is also convex. We are going use some typical techniques refered to as convex optimization. Since $-f$ is a convex function, a solution to this maximization problem has to also be a solution of the Lagrangian Multiplier minimization problem of

$$\hat{f}(\vec{x}) = -\lambda_0 f(\vec{x}) + \lambda_1 \left(x_1^2 - 2\right) + \ldots + \lambda_{n-1} \left(\sum_{i=1}^{n-1} x_i^2 - \left(n^2 - n\right)\right),$$

where $\lambda_k \geq 0$ are scalars and $\lambda_k \left(\sum_{i=1}^{k} x_i^2 - \left(k^2 + k\right)\right) = 0$ for all k, and $\lambda_k > 0$ for some k. Furthermore, if we find a $\vec{x}$ that minimizes $\hat{f}$ under these conditions with $\lambda_0 = 1$, then $\vec{x}$ will minimize $-f$ and hence maximize f. Since this new $\hat{f}$ is also convex, if we find a local minimum then it will be a global minimum. We will demonstrate $x_k = \sqrt{2k}$ for all k is one such solution by using the first derivative test followed by calculating the eigenvalues of the Hessian.

Let $\vec{y} = \langle \sqrt{2}, \sqrt{4}, \ldots, \sqrt{2(n-1)} \rangle$ and $\lambda_0 = 1$. If $\sum_{i=1}^{k-1} x_i^2 - \left(k^2 - k\right) = 0$ and $\sum_{i=1}^{k} x_i^2 - \left(k^2 + k\right) = 0$ for all k, then $x_k = \sqrt{k^2 + k - (k^2 - k)} = \sqrt{2k}$ for all k and vice versa. Hence $\lambda_k \left(\sum_{i=1}^{k} x_i^2 - \left(k^2 + k\right)\right) = 0$ for all k when $\vec{x} = \vec{y}$. For each k, $\frac{\partial \hat{f}}{\partial x_k} = -1 + 2x_k \sum_{i=k}^{n-1} \lambda_k$. We find the scalar values for each λ_k that makes $\vec{y}$ a critical point by solving the system

$$\begin{aligned}
0 &= -1 + 2\lambda_1 x_1 + 2\lambda_2 x_1 + \ldots + 2\lambda_{n-1} x_1 \\
&\vdots \\
0 &= -1 + 2\lambda_{n-3} x_{n-3} + 2\lambda_{n-2} x_{n-3} + 2\lambda_{n-1} x_{n-3} \\
0 &= -1 + 2\lambda_{n-2} x_{n-2} + 2\lambda_{n-1} x_{n-2} \\
0 &= -1 + 2\lambda_{n-1} x_{n-1},
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
\lambda_1 &= \frac{1}{2x_1} - \lambda_2 - \lambda_3 - \ldots - \lambda_{n-1} = \frac{1}{2x_1} - \frac{1}{2x_2} \\
&\vdots \\
\lambda_{n-3} &= \frac{1}{2x_{n-3}} - \lambda_{n-2} - \lambda_{n-1} = \frac{1}{2x_{n-3}} - \left(\frac{1}{2x_{n-2}} - \frac{1}{2x_{n-1}}\right) - \frac{1}{2x_{n-1}} = \frac{1}{2x_{n-3}} - \frac{1}{2x_{n-2}}
\end{aligned}$$

$$\lambda_{n-2} = \frac{1}{2x_{n-2}} - \lambda_{n-1} = \frac{1}{2x_{n-2}} - \frac{1}{2x_{n-1}}$$
$$\lambda_{n-1} = \frac{1}{2x_{n-1}}.$$

Under this set of scalars the first derivative test identifies y as a critical point.

Let us now consider the Hessian of $\hat{f}$ with this set of scalars. We have $\frac{\partial \hat{f}}{\partial x_k \partial x_j} = 0$ when $j \neq k$ and $\frac{\partial \hat{f}}{\partial^2 x_k} = 2\sum_{i=k}^{n-1} \lambda_i$ for any k. Hence, the Hessian is diagonal and the eigenvalues are $2\sum_{i=k}^{n-1} \lambda_i$ for $0 < k < n$. Since the sequence of x_k is increasing, each $\lambda_k = \frac{1}{2x_k} - \frac{1}{2x_{k+1}} > 0$. Therefore each eigenvalue of the Hessian is positive and so $f(\vec{y})$ is an upper bound for $e_{\prec}$. Lastly, we can estimate the sum a of an increasing series such as $\{y_k\}$ with its integral to prove our claim

$$\begin{aligned} e_{\prec} &\leq f(\vec{y}) \\ &= \sqrt{2} + \sqrt{4} + \ldots + \sqrt{2(n-1)} \\ &\leq \int_1^n \sqrt{2x}\, dx \\ &= \left[\frac{(2x)^{3/2}}{3}\right]_1^n \\ &= \sqrt{\frac{8}{9}} n^{3/2} - \sqrt{\frac{8}{9}}. \end{aligned}$$

□

4.2 Width Theorem

Given a poset, P, an antichain is any $A \subseteq P$ such that $i \not< j$ for all $i, j \in A$. We define the width of a poset, P, to be size of the largest antichain in P. Trying to find a good bound on the number of covers in a lattice proves itself difficult to do directly. The best results we have found for general lattices comes from breaking the problem into cases: when the width is small and when the width is large. The starting point of this is what we refer to as the Width Theorem, [6]. Note we state it as it was originally, but the proof in the text demonstrates the conclusion holds for all cover-bow-tie-free posets.

Theorem 17 (Width Theorem)**.** *Given a lattice of size n and width at most k, $e_{\prec} \leq n\sqrt{k} + \frac{n}{2}$.*

By combining this style of argument with that in [7], we obtain a minor improvement on this result in terms of the coefficient on $n^{3/2}$ but the $O(n)$ term gets a little worse.

Theorem 18. *Given a cover-bow-tie-free poset of size n of width at most cn, then*

$$e_{\prec} \leq \sqrt{c - \frac{1}{n}}\left(1 - c\left(1 - \sqrt{\frac{8}{9}}\right)\right) n^{3/2} + n < \sqrt{c}\left(1 - c\left(1 - \sqrt{\frac{8}{9}}\right)\right) n^{3/2} + n.$$

Proof. Suppose L is a cover-bow-tie-free poset of size n and width at most cn. As in the proof of Theorem 16 we assume the elements of L are $0, 1, \ldots, n-1$ ordered by a linear extension, let $X_j = \{i \mid i \prec j\}$, and set $x_j = |X_j|$. Instead of counting two element subsets, we are going to count "forks" this time. We define the set of forks below l as $F_l = \{(i, j, k) \mid i, k \prec j \leq l\}$. Note that each ordered pair (i, k) of distinct elements of uniquely determine the fork (i, j, k). This means $|F_l| = \sum_{i=1}^{l} x_i(x_i - 1)$. The expressions for x_i are less easy to work with in the inequalities we will generate, so instead we let $y_i = x_i - 1$. Once we find $\max \sum_{i}^{n-1} y_i = \delta$, then we will know $e_{\prec} \leq \delta + n$. For now assume that $cn \in \mathbb{Z}$. Since L is cover-bow-tie-free we know that each distinct pair $i, k \in L$ determine at most one fork, and hence

$$\begin{aligned} \sum_{i=1}^{l} x_i(x_i - 1) &\leq l(l-1) \\ \sum_{i=1}^{l} (y_i + 1)y_i &\leq l(l-1) \\ \sum_{i=1}^{l} y_i^2 &\leq l^2 - l. \end{aligned} \tag{4.2.1}$$

By a calculation similar to one in Theorem 16, $g(\vec{y}) \leq \sqrt{\frac{8}{9}}(cn-1)^{3/2}$ where $g(\vec{y}) = \sum_{i=1}^{cn} y_i$. Note that as in the last theorem using the tighter inequality here does not provide overall better results. We attribute this again to the estimation technique used later.

For $m > cn$ we find a better result. We will still use the calculation $|F_m| = \sum_{i=1}^{m}(y_i + 1)y_i$. However, instead of simply bounding $|F_m|$ by the number of two permutations on m, we will use the width. Since the width is at most cn, the poset can be partitioned into at most cn chains, [5]. If (i, j, k) is a fork, we assert that there cannot be another fork (i, j, l) where k and l were in the same chain. Without loss of generality suppose $k < l$. If (i, j, l) were a fork, then $k < l \prec j$ witnesses $k \not\prec j$. Therefore the two lower elements have to come from different chains, of which there are at most cn. There are m many ways to pick an $i \in L$ such that $i < m$ to be the first lower element of a fork and $cn - 1$ many ways to pick the second lower element of a fork, and so

$$\sum_{i=1}^{m} (y_i + 1)y_i \leq m(cn - 1)$$

$$\sum_{i=1}^{m} y_i^2 \leq m(cn-1). \tag{4.2.2}$$

We seek to maximize $f(\vec{y}) \leq g(\vec{y}) + h(\vec{y})$, where $h(\vec{y}) = \sum_{i=cn}^{n-1} y_i$. We've already established that under cn of the constraints of the form (4.2.1) $g(\vec{y}) \leq \sqrt{\frac{8}{9}}(cn-1)^{3/2}$, and so adding the rest of the constraints would not increase that sum. By a simple calculation $f(\vec{y})$ subject only to (4.2.2) type restraints is maximized when $y_i = \sqrt{cn-1}$ for all i. Since f is a simple sum and $y_i = \sqrt{cn-1}$ for all i in a maximal solution under just some of the constraints we know $h(\vec{y}) \leq \sqrt{cn-1}(n-cn)$. We do have to concern ourselves with the case that $cn \notin \mathbb{N}$. However, since we included $i = cn$ in both g and h, there is no problem there. Therefore,

$$\begin{aligned}
e_{\prec} &= f(\vec{y}) + n \\
&\leq \sqrt{\frac{8}{9}}(cn-1)^{3/2} + \sqrt{cn-1}(n-cn) + n \\
&= \sqrt{c - \frac{1}{n}}\sqrt{\frac{8}{9}}cn^{3/2} + \sqrt{c - \frac{1}{n}}(1-c)n^{3/2} + n \\
&= \sqrt{c - \frac{1}{n}}\left(1 - c\left(1 - \sqrt{\frac{8}{9}}\right)\right)n^{3/2} + n \\
&< \sqrt{c}\left(1 - c\left(1 - \sqrt{\frac{8}{9}}\right)\right)n^{3/2} + n.
\end{aligned}$$

□

Both of these results give a bound for cover-bow-tie-free posets, more specifically lattices, with a width less than a given cn. Next we present a bound for such posets with a width larger than a given cn. It is by combining these two types of results that we achieve our best results for general lattices in the next section.

Theorem 19. *Given a cover-bow-tie-free poset of size n with width at least cn,*

$$e_{\prec} \leq \left((1-c)\sqrt{c} + \sqrt{\frac{8}{9}}(1-c)^{3/2}\right)n^{3/2} + 2n, \qquad \text{if } c \geq \frac{1}{3} \text{ and}$$

$$e_{\prec} \leq \left(\sqrt{\frac{4}{27}} + \sqrt{\frac{8}{9}}(1-c)^{3/2}\right)n^{3/2} + 2n, \qquad \text{always.}$$

Proof. Suppose L is a cover-bow-tie-free poset of size n with width at least cn. This means we can find a maximally sized antichain, $A \subseteq L$, of size $rn \geq cn$. Let $T = \{x \in L \mid x > a \text{ for some } a \in A\}$ and $B = \{x \in L \mid x < a \text{ for some } a \in A\}$. This creates a tripartition of L. Let E_A be the number of covers from A to $L \setminus A$ and $E_{L\setminus A}$ be the number of covers in L not involving A. Note $e_{\prec} = E_A + E_{L\setminus A}$.

Let us first consider when $c \geq \frac{1}{3}$. Since L is cover-bow-tie-free, given any two distinct $a_1, a_2 \in A$, there can be at most one $t \in T$ such that $a_1, a_2 \prec t$. By the construction of T there cannot be any $t \in T$ such that $t \prec a_1, a_2$. Hence the set $A \dot{\cup} T$ with the collection of covers between A and T create a bipartite graph with no 4-cycles. This means we may apply Lemma 4 to conclude that there are at most $|T|\sqrt{rn} + rn$ many covers between A and T. Similarly there are at most $|B|\sqrt{rn} + rn$ many covers between A and B. Therefore $E_A \leq (|T| + |B|)\sqrt{rn} + 2rn = (1-r)\sqrt{r}n^{3/2} + 2rn$. The derivative $\frac{d}{dx}\left((1-x)\sqrt{x}n^{3/2}\right) = (1-3x)\frac{n^{3/2}}{2\sqrt{x}}$ is negative for $x > \frac{1}{3}$ and hence

$$E_A \leq (1-r)\sqrt{r}n^{3/2} + 2rn \leq (1-c)\sqrt{c}n^{3/2} + 2n.$$

Let us now consider the poset $L \setminus A$ with an order determined by the cover set inherited by L. Since L was cover-bow-tie-free, so is $L \setminus A$. We estimate $E_{L \setminus A}$ by applying Theorem 16,

$$\begin{aligned} E_{L\setminus A} &\leq \sqrt{\frac{8}{9}}\left(|T \cup B|\right)^{3/2} \\ &= \sqrt{\frac{8}{9}}(1-r)^{3/2}n^{3/2} \\ &\leq \sqrt{\frac{8}{9}}(1-c)^{3/2}n^{3/2}. \end{aligned}$$

Therefore, we obtain $e_\prec = E_A + E_{L\setminus A} \leq \left((1-c)\sqrt{c} + \sqrt{\frac{8}{9}}(1-c)^{3/2}\right)n^{3/2} + 2n$ when $c \geq \frac{1}{3}$.

We look again at the derivative $\frac{d}{dx}\left((1-x)\sqrt{x}n^{3/2}\right) = (1-3x)\frac{n^{3/2}}{2\sqrt{x}}$. It is negative when $x > \frac{1}{3}$ and positive when $x < \frac{1}{3}$. Since the function is continuous on the domain $(0,1)$, we have that the max is $\left(1-\frac{1}{3}\right)\sqrt{\frac{1}{3}}n^{3/2} = \sqrt{\frac{4}{27}}n^{3/2}$. Hence, $e_\prec \leq \left(\sqrt{\frac{4}{27}} + \sqrt{\frac{8}{9}}(1-c)^{3/2}\right)n^{3/2} + 2n$. □

4.3 Best Results for General Lattices

Here we will show our best results for general lattices. There is only one theorem in this section, however we have at least one better result that is not presented and a couple of preliminary better results as well.

Theorem 20. *Given a cover-bow-tie-free poset L of size n*

$$e_\prec \leq \sqrt{\frac{1}{6}\left(4 - 10\sqrt[3]{\frac{2}{3\sqrt{69}-11}} + 2^{2/3}\sqrt[3]{3\sqrt{69}-11}\right)}n^{3/2} + 2n \approx 0.65587n^{3/2} + O(n).$$

Proof. Let $\alpha = \frac{1}{6}\left(4 - 10\sqrt[3]{\frac{2}{-11+3\sqrt{69}}} + 2^{2/3}\sqrt[3]{-11+3\sqrt{69}}\right)$. Note $\alpha \approx 0.43016$. We will prove this theorem by induction. Clearly for $n = 1$ this is true. Suppose it is true for all $n < N$. We must show it is true for N. Let L be a cover-bow-tie-free poset of size N and let cN be its width. Either $c \leq \alpha$ or $c > \alpha$.

Suppose $c \leq \alpha$. By the Width Theorem, $e_{\prec} \leq \sqrt{\alpha}n^{3/2} + \frac{n}{2}$.

Suppose $c > \alpha$. Let A, B, and T be defined as in Theorem 19. Since $c > \alpha > \frac{1}{3}$, we apply the first argument in that theorem to bound the number of covers involving elements from A by $(1-\alpha)\sqrt{\alpha}N^{3/2}+2\alpha N$. We define $P = T \cup B$ by the cover set $C_P = C_L \setminus \{(a,b) \mid a \in A \text{ or } b \in A\}$. The poset P inherits being cover bow tie free from L. Since $|P| < N$, we can apply the inductive hypothesis to P

$$\begin{aligned} e_{\prec} &\leq (1-\alpha)\sqrt{\alpha}N^{3/2} + 2\alpha N + \sqrt{\alpha}|P|^{3/2} + 2|P| \\ &\leq (1-\alpha)\sqrt{\alpha}N^{3/2} + 2\alpha N + \sqrt{\alpha}(N-\alpha N)^{3/2} + 2(N-\alpha N) \\ &\leq \left((1-\alpha)\sqrt{\alpha} + \sqrt{\alpha}(1-\alpha)^{3/2}\right)N^{3/2} + 2N. \end{aligned}$$

Cardano's Equation gives us that α is the solution to $x^3 - 2x^2 + 3x - 1 = 0$. Observe,

$$\begin{aligned} \alpha^3 - 2\alpha^2 + 3\alpha - 1 &= 0 \\ \alpha^2 &= 1 - 3\alpha + 3\alpha^2 - \alpha^3 \\ \alpha^2 &= (1-\alpha)^3 \\ \alpha &= (1-\alpha)^{3/2} && (\alpha > 0) \\ 1 &= (1-\alpha) + (1-\alpha)^{3/2} \\ \sqrt{\alpha} &= (1-\alpha)\sqrt{\alpha} + \sqrt{\alpha}(1-\alpha)^{3/2}. \end{aligned}$$

Therefore, $e_{\prec} \leq \sqrt{\alpha}N^{3/2} + 2N$ and we have proven our theorem by cases. □

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