NEAR-INTERFACIAL AND INTERFACIAL FRACTURE SIMULATION BY THE EXTENDED FINITE ELEMENT METHOD

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We certify that we have read this dissertation and that, in our opinion, it is satisfactory in scope and quality as a dissertation for the degree of Doctor of Philosophy in Civil Engineering.
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by
Yuhai Yan
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To my parents

and my wife
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Abstract

Design of composite structures in many important industrial applications requires good understanding of the fracture behavior in the vicinity of material interfaces. In this study, near-interfacial and interfacial fractures are modeled by the extended finite element method (XFEM), a numerical technique developed recently to model crack propagation. In the XFEM, a crack, or a discontinuity in displacements, is represented by enriching the nodes around the crack with additional degrees of freedom associated with enrichment interpolation functions. Among the advantages of the XFEM are that no remeshing is needed; the crack path is independent of the finite element mesh; it is applicable to preexisting cracks as well as evolving cracks; and it is numerically robust although extra implementation efforts are needed. In order to deal efficiently with changes in the geometry and mesh topology, the level set method (LSM), an algorithm used to track evolving interfaces, is introduced and combined with the XFEM.

The XFEM is first applied to the simulation of near-interfacial crack propagation in a metal-ceramic layered structure. Experimental evidence indicates that, in a ceramic-metal-ceramic sandwich structure, a near-interfacial crack in the ceramic layer can be drawn to or deflect away from the metal layer depending on the difference in elastic properties across the interface. To model near-interfacial fracture, only the Heaviside functions are used for the XFEM, and the vector LSM method, an
improvement to the original LSM, where the LSM is adapted to the nature of crack propagation problems, is employed for efficient evaluation of the enrichment functions. The crack propagation paths predicted by the XFEM simulation are found to be consistent with the experimental observation. In the simulation of the interfacial fracture, a bi-material plate with a crack on the interface is modeled. In the proposed scheme, the nodes on the crack are enriched with only the Heaviside functions. The stress intensity factor analysis demonstrates that such an enrichment strategy can produce satisfactory results for interfacial fracture problems.
# Table of Contents

Acknowledgements .......................................................... v
Abstract ................................................................. vi
List of Tables .............................................................. ix
List of Figures .............................................................. x
1 Introduction .............................................................. 1
   1.1 Background ....................................................... 1
   1.2 Numerical Simulation of Fracture ............................... 3
      1.2.1 Discrete Inter-element Crack Approach .................... 4
      1.2.2 Smeared Crack Approach .................................... 4
      1.2.3 Embedded Discontinuity Approach .......................... 6
      1.2.4 Extended Finite Element Method ........................... 7
   1.3 Outline of Dissertation ......................................... 9
2 Extended Finite Element Method and Fracture Simulation .......... 10
   2.1 Extended Finite Element Method ............................... 10
      2.1.1 Governing Equations and Weak Form ....................... 10
      2.1.2 XFEM Enrichment Functions ............................... 12
      2.1.3 Heaviside-only Enrichment ............................... 14
      2.1.4 Discrete Equations ....................................... 19
<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.1.5</td>
<td>Numerical Integration</td>
<td>20</td>
</tr>
<tr>
<td>2.2</td>
<td>Vector Level Set Method</td>
<td>22</td>
</tr>
<tr>
<td>2.3</td>
<td>Crack Propagation Simulation</td>
<td>25</td>
</tr>
<tr>
<td>2.4</td>
<td>Stress Intensity Factor Evaluation</td>
<td>27</td>
</tr>
<tr>
<td>3</td>
<td>Near-interfacial Crack Propagation Simulation</td>
<td>33</td>
</tr>
<tr>
<td>3.1</td>
<td>Introduction</td>
<td>33</td>
</tr>
<tr>
<td>3.2</td>
<td>Benchmark Problems</td>
<td>34</td>
</tr>
<tr>
<td>3.2.1</td>
<td>Crack in a Plate Under Uniaxial Loading</td>
<td>34</td>
</tr>
<tr>
<td>3.2.2</td>
<td>Crack Propagation in Double Cantilever Beam</td>
<td>38</td>
</tr>
<tr>
<td>3.3</td>
<td>Near-interfacial Fracture in Layered Structure</td>
<td>42</td>
</tr>
<tr>
<td>3.3.1</td>
<td>Experimental Evidence in Literature</td>
<td>42</td>
</tr>
<tr>
<td>3.3.2</td>
<td>Numerical Model</td>
<td>43</td>
</tr>
<tr>
<td>3.3.3</td>
<td>Results and Discussion</td>
<td>44</td>
</tr>
<tr>
<td>3.4</td>
<td>Concluding Remarks</td>
<td>50</td>
</tr>
<tr>
<td>4</td>
<td>Interfacial Fracture Simulation</td>
<td>52</td>
</tr>
<tr>
<td>4.1</td>
<td>Introduction</td>
<td>52</td>
</tr>
<tr>
<td>4.2</td>
<td>XFEM Interfacial Fracture Simulation</td>
<td>54</td>
</tr>
<tr>
<td>4.3</td>
<td>Stress Intensity Factors of a Bi-material Interfacial Crack and Its Evaluation</td>
<td>56</td>
</tr>
<tr>
<td>4.4</td>
<td>Numerical Examples</td>
<td>59</td>
</tr>
<tr>
<td>4.5</td>
<td>Concluding Remarks</td>
<td>65</td>
</tr>
<tr>
<td>5</td>
<td>Conclusion</td>
<td>66</td>
</tr>
<tr>
<td>A</td>
<td>Near-tip Fields in an Elastic Isotropic Material</td>
<td>68</td>
</tr>
<tr>
<td>A.1</td>
<td>Mode I</td>
<td>68</td>
</tr>
<tr>
<td>A.2</td>
<td>Mode II</td>
<td>69</td>
</tr>
<tr>
<td>B</td>
<td>Near-tip Fields of a Bi-material Interfacial Crack</td>
<td>71</td>
</tr>
</tbody>
</table>
List of Tables

<table>
<thead>
<tr>
<th>Table</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.1</td>
<td>Normalized SIFs for the center crack ($\alpha = 0^\circ$)</td>
<td>37</td>
</tr>
<tr>
<td>3.2</td>
<td>SIFs for the inclined crack</td>
<td>38</td>
</tr>
<tr>
<td>3.3</td>
<td>Material properties and Dundurs' parameters</td>
<td>44</td>
</tr>
<tr>
<td>4.1</td>
<td>SIFs for the interfacial crack ($2a = 40\text{mm}$)</td>
<td>62</td>
</tr>
<tr>
<td>4.2</td>
<td>SIFs for the interfacial crack ($2a = 60\text{mm}$)</td>
<td>62</td>
</tr>
<tr>
<td>4.3</td>
<td>SIFs for the interfacial crack ($2a = 80\text{mm}$)</td>
<td>63</td>
</tr>
</tbody>
</table>
List of Figures

2.1 Domain $\Omega$ of a solid body containing a crack $\Gamma_c$ ................. 11
2.2 Representation of a discontinuity by Heaviside functions ............... 13
2.3 Crack-tip coordinate system ........................................ 14
2.4 Enrichment for the crack-tip element ................................ 16
2.5 Relative position of a crack and a triangular element .................... 18
2.6 Enrichment for the crack-tip element: a general configuration ............ 18
2.7 Partition of elements for Gauss quadrature ................................ 21
2.8 Crack representation in the vector level set method ....................... 23
2.9 Level set update (Ventura et al., 2003) .................................. 24
2.10 Evaluation of the level set function ................................... 25
2.11 $J$ integral .............................................................. 28
2.12 Domain integral ......................................................... 30
2.13 Domain $A$ and weighting function $q$ .................................. 31
3.1 Plate with an inclined crack under uniaxial tension ......................... 35
3.2 Structured mesh: the element arrangement .................................. 36
3.3 Unstructured mesh: the vicinity of the crack ................................ 36
3.4 Double cantilever beam ................................................. 38
3.5 Crack propagation in double cantilever beam .............................. 39
3.6 Effect of element size on simulated crack paths
3.7 Effect of the domain interaction integral radius on simulated crack paths
3.8 Effect of the crack growth increment on simulated crack paths
3.9 Four-point bending specimen
3.10 Mesh for near-interfacial fracture simulation
3.11 Comparison of crack trajectories (1)
3.12 Comparison of crack trajectories (2)
3.13 Comparison of crack trajectories (3)
3.14 Intersection of simulated crack trajectories
3.15 Influence of the mesh density on crack trajectories
4.1 Nodes to be enriched and partition of a crack-tip element
4.2 Nodes to be enriched (Nagashima et al., 2003; Sukumar et al., 2004)
4.3 Representation of a crack in Sukumar and Prévost (2003)
4.4 Local crack-tip coordinates
4.5 Bi-material specimen with an interfacial crack
4.6 Structured mesh for the bi-material fracture simulation
4.7 The influence of mesh density: Error in $K_1$ for the 40 mm crack
Chapter 1
Introduction

1.1 Background

Design of composite structures in many important industrial applications requires good understanding of the fracture behavior near the bi-material interfaces. For example, it has become a widespread practice to strengthen reinforced or prestressed concrete structures with externally bonded fiber-reinforced plastic (FRP) plates. Such a practice has been spurred by FRP’s high stiffness-to-weight ratio, strength-to-weight ratio and durability as compared with other conventional materials (Triantafillou and Antonopoulos, 2000). However, the failure mechanism of the strengthened system has not been understood very well, and has been the subject of extensive experimental and analytical studies in the recent past (Büyüköztürk et al., 2004; Camata et al., 2007; Chen and Teng, 2003; Leung, 2006). Among other failure mechanisms, a greater attention is being paid to debonding failure, that is, debonding of the FRP plate from the concrete substrate.

Debonding is a fracture phenomenon that occurs near or on the interface between two distinct materials. It is closely related to the stress transfer between FRP and the concrete substrate and thereby critically affects the performance of the
strengthening/rehabilitation by FRP. A prevalent failure mode in FRP-strengthened systems, debonding can take place in any one of the constituent materials, which include FRP, concrete substrate and adhesive layer, or at the interfaces between the materials, such as between concrete and adhesive or between FRP and adhesive (Büyüköztürk et al., 2004). Debonding failure tends to be highly brittle and thus must be prevented by adequate design (Büyüköztürk et al., 2004).

Debonding failures, like failures of any other materials or structural components, can be investigated by two different approaches: strength and fracture (Büyüköztürk et al., 2004). A strength approach involves calculating the bond stresses between the FRP and concrete based on elastic material properties. The calculated stresses are then compared with the ultimate bond strength to predict the load level of debonding failure. This approach was taken by, among others, Aprile et al. (2001), Pesic and Pilakoutas (2003) and Wong and Vecchio (2003).

A fracture approach, on the other hand, recognizes debonding as crack propagation, that is, a progressive failure caused by fracture processes. In fact, past experimental studies indicate that the fracture process associated with debonding can be modeled as strain-softening cohesive fracture in concrete (Ali-Ahmad et al., 2006). Computational studies on debonding failure based on such fracture models started to emerge recently, leading to cohesive crack models (or fictitious crack models) by Borg et al. (2002) and Wu and Yin (2003).

Despite the research that has been performed, further investigations have to be carried out to better understand the debonding failure because of the intrinsic complexities of the problem including, among others, the involvement of distinct constituent materials, which inevitably affect the fracture process if the relative elastic properties vary. Since the ultimate behavior has to be considered, it is necessary
to have computational techniques that are applicable for both near-interfacial and interfacial fracture.

This dissertation focuses on application of an emerging numerical method to model the fracture phenomenon near or on the interface between the constituent materials. In the near-interfacial fracture simulation, a sandwich structure under four-point bending is considered. The crack trajectories are reproduced numerically and then compared with the experimental crack paths. In the interfacial fracture simulation, a crack residing on the material interface is modeled and the stress intensity factors are evaluated and compared with a reference solution. The capability of the numerical method to simulate fracture near or on the interface is examined to provide an insight into more complicated fracture phenomena including the debonding in FRP-retrofitted structures.

1.2 Numerical Simulation of Fracture

Many approaches have been developed to numerically simulate fracture in concrete structures, and they can be classified into two categories: discrete and continuum approaches. In the finite-element context, the discrete inter-element approaches can model the crack directly via an inter-element displacement discontinuity, and the continuous part is modeled through classical continuum mechanics. The crack opening can also be represented by introducing discontinuous shape functions into an element so that the crack can grow independently of the mesh topology. This scheme is used by the embedded discontinuity approach and extended finite element method although the concepts behind the two methods are different. In general, for discrete crack approaches additional criteria are needed to determine the direction of the crack propagation.
In the continuum approaches, however, the continuum format is maintained for the entire domain and no inter-element discontinuity is introduced. For instance, in the smeared crack approach cracks are represented by distributing the cracking strain over a certain material volume.

1.2.1 Discrete Inter-element Crack Approach

The discrete inter-element crack approach models a crack by means of a separation between element edges upon violating a certain condition of crack initiation. Originated by Ngo and Scordelis (1967), the discrete crack approach was the earliest method used in concrete fracture simulation, and is suitable for those cases for which the crack path is known in advance, such as in mode I fracture, i.e., the opening or tensile mode where the crack surfaces move directly apart. It can even be used for mixed mode fracture if a reasonable guess of the crack path can be made. For the problem in which the crack path is not known in advance, the discrete approach requires remeshing techniques and a continuous change in nodal connectivity. Although such techniques have been implemented in a few finite element codes, a fully general code is not yet available (Bažant and Planas, 1998). This not only limits its usage, but also reduces the computational efficiency significantly.

1.2.2 Smeared Crack Approach

An alternative method to model cracking is the smeared crack approach, in which a cracked solid is dealt with as a continuum. In this approach pioneered by Rashid (1968), the geometric discontinuities induced by cracks are represented by cracking strains distributed over a certain area within the finite element (Rots et al., 1985). Upon crack formation, the local stress-strain relation, which is initially isotropic,
is switched to an orthotropic one that reflects the decrease in the stiffness in the direction orthogonal to the crack. The axes of orthotropy are determined based on the conditions at crack initiation.

In accordance with the method dealing with the crack direction, smeared crack approaches can be categorized into fixed, multi-directional, and rotating crack models, in which the orientation of the crack is kept constant, updated in a stepwise manner and updated continuously, respectively (Rots and Blaauwendraad, 1989). Despite these distinctions, several assumptions are shared by the three models: the decomposition of the total strain into an elastic part and a part due to cracking; the elastic strain is related to the stress by standard equations of linear elasticity; a crack initializes when the principal stress reaches the uniaxial tensile strength; and a traction-separation law relates the crack opening with the residual stress transferred by the crack (Jirések and Zimmermann, 1998).

Unlike the discrete inter-element crack approach, the smeared crack approach has the advantage that the mesh topology is not changed when the crack grows. Furthermore, there are no restrictions with respect to the orientation of the crack. Due to these advantages, the smeared crack approach has come into widespread use (Riggs and Powell, 1986; Crisfield and Wills, 1989) and has been implemented in some commercial finite element codes.

A straightforward use of the traction-separation law, or the strain softening relation, leads, however, to spurious mesh sensitivity in finite-element calculations. In the smeared crack approach, this drawback is resolved by resorting to the crack band model (Bazant and Oh, 1983) that smears out the fracture energy over the area in which the crack localizes. It is also observed that numerical solutions obtained by traditional smeared crack models can become erroneous due to the phenomenon called stress locking, i.e., spurious stress transfer across a widely open crack. For the
fixed crack model, this is mainly due to the fact that the principal strain axes rotate after the crack initiation while the orientation of the crack is fixed, which causes misalignment between the principal axes of strains and stresses. However, locking is also observed for the rotating crack model. Analysis shows that this phenomenon is caused by a poor kinematic representation of the discontinuous displacement field around a macroscopic crack (Jirásek and Zimmermann, 1998). This can be improved by the local enrichment of the kinematic representation of highly localized strains, as is done in the embedded discontinuity approach and extended finite element method.

1.2.3 Embedded Discontinuity Approach

In embedded discontinuity approaches, a crack is modeled by inserting, in the interior of a finite element, a discontinuity in strains or displacements: the enriched degrees of freedom associated with discontinuities can be eliminated by condensation. One speaks of a weak discontinuity if the strain undergoes a discontinuity while a discontinuity in the displacement is referred to as a strong discontinuity. The resulting improvement in the kinematic representation of strain localization has been shown to significantly eliminate stress locking. Unlike the smeared crack approach, one can also avoid spurious mesh dependence of the solutions without introducing a crack-band concept that requires element-dependent mesh-regularization parameters. These advantages and the early pioneering works by Ortiz et al. (1987) and Belytschko et al. (1988) have motivated numerous research studies on this approach in recent years (Armero and Garikipati, 1996; Dvorkin et al., 1990; Jirásek, 2000; Jirásek and Zimmermann, 2001a,b; Larsson and Runesson, 1996; Oliver, 1996a,b; Simo et al., 1993; Wells and Sluys, 2000).
As is done in the smeared crack approach, in the embedded discontinuity model, the strain or displacement is decomposed into a continuous and a discontinuous part due to the opening and sliding of a crack. However, the discontinuous part of the deformation is not smeared over a band or an element: rather, it is represented by a discontinuity path that splits the finite element into two separate zones. By introducing the traction-separation law and traction continuity condition (the stresses in the bulk and the tractions across the crack should satisfy internal equilibrium), the discontinuities can be expressed in terms of the nodal displacements.

The embedded discontinuity approach has recently been further developed by Oliver and Huespe (2004) resulting in a fairly comprehensive concrete fracture analysis procedure that includes an algorithm for tracking multiple cracks.

### 1.2.4 Extended Finite Element Method

Proposed by Belytschko and Black (1999) and Moës et al. (1999), the extended finite element method (XFEM) is a numerical technique for modeling discontinuities, such as crack, voids, etc. In the XFEM special discontinuous functions are added to the finite element approximation under the framework of partition of unity (Melenk and Babuška, 1996). To model cracks, the Heaviside function and linear elastic asymptotic crack-tip displacement fields are used (Moës et al., 1999) although only the Heaviside function is used in other implementations (Zi and Belytschko, 2003). Hence the crack is represented by the additional degrees of freedom for nodes surrounding the crack and therefore the mesh does not have to conform to the crack. As a result, no remeshing techniques are needed for crack propagation modeling.

Both the embedded discontinuity approach and XFEM are discrete intra-element approaches, in which a crack can reside in the interior of a finite element and
grow arbitrarily without regard to the mesh topology. A comparative study of the two methods (Jirasek and Belytschko, 2002) shows that the XFEM has advantages over the embedded discontinuity approach in representing kinematic properties and possesses better numerical robustness.

In this dissertation, the XFEM combined with the level set method (LSM) (Osher and Sethian, 1988) is used to simulate the near-interfacial fracture behavior of layered composite structures and interfacial fracture in bi-material specimens. The XFEM is capable of modeling crack growth without remeshing and possesses advantages over conventional methods as discussed above. The LSM, on the other hand, is capable of handling the motion of an interface, such as a propagating crack. The combined use of the LSM and XFEM can thus greatly simplify the algorithm to model the crack geometry in three dimensions. The LSM is implemented in the current study which is restricted to plane problems so that methodologies can be readily extended in the future to a three-dimensional setting. We use the vector LSM (Ventura et al., 2003), a version of the LSM specifically tailored toward modeling evolving crack geometries, to evaluate XFEM enrichment functions efficiently.

Among the many XFEM formulations available in the literature, the approach proposed by Zi and Belytschko (2003) does not rely on the use of special crack-tip enrichment functions, so that the partition of unity holds in the entire enriched domain. Chessa et al. (2003) report that the accuracy can be impaired when partially enriched elements are created around the crack tip by using the crack-tip enrichment functions.

The XFEM has been applied to a number of problems in fracture mechanics including channel cracking (Huang et al., 2003) and cracks in functionally graded materials (Dolbow and Gosz, 2002; Comi and Mariani, 2007), but few studies have been reported on near-interfacial cracking. The present study is concerned with the
application of the XFEM based on the formulation of Zi and Belytschko (2003) to the simulation of crack growth in layered composite structures, with particular emphasis on the XFEM’s capability in predicting the crack path in near-interfacial fracture.

Bi-material interfacial cracks have also been studied with the XFEM. Sukumar et al. (2004) included in their finite element approximation the crack-tip functions based on near-tip asymptotic displacement fields for the interface of dissimilar materials. Nagashima et al. (2003) applied directly the XFEM formulation for homogeneous materials (Moës et al., 1999) to bi-material problems. In this dissertation, a concise approach based on the XFEM formulation by Chen (2003) and Zi and Belytschko (2003) is employed. Numerical simulations are carried out to investigate its applicability for interfacial fracture problems.

1.3 Outline of Dissertation

The remainder of this dissertation is organized as follows: In Chapter 2 the numerical methods, XFEM and vector LSM, are discussed. The crack propagation criterion used in the research is also presented. Chapter 3 addresses the simulation of near-interfacial fracture propagation. In particular, the crack trajectories in a layered structure are predicted and compared with experimental results. Chapter 4 presents the simulation of interfacial fracture, in which a bi-material plate with a crack along the interface is modeled. The dissertation is concluded with Chapter 5 in which the results are summarized and future work is discussed.
Chapter 2
Extended Finite Element Method and Fracture Simulation

2.1 Extended Finite Element Method

2.1.1 Governing Equations and Weak Form

Consider the domain $\Omega$ with the boundary $\Gamma$, as shown in Figure 2.1. $\Gamma = \Gamma_u + \Gamma_t + \Gamma_c$, where $\Gamma_u$ and $\Gamma_t$ are the subsets of $\Gamma$ where the displacements and tractions, respectively, are prescribed. $\Gamma_c$ represents the crack surface and is assumed to be traction-free. The equilibrium equations and boundary conditions are

$$\nabla \cdot \sigma + b = 0,$$

(2.1)

$$u = \bar{u} \text{ on } \Gamma_u,$$

$$\sigma \cdot n = \bar{t} \text{ on } \Gamma_t,$$

(2.2)

$$\sigma \cdot n = 0 \text{ on } \Gamma_c,$$
where $\sigma$ is the Cauchy stress tensor, $b$ the body force per unit volume, $n$ the unit outward normal vector, $u$ the displacement and $\bar{u}$ and $\bar{t}$ are the prescribed displacements and tractions, respectively.

The stress tensor is related with the strain tensor $\varepsilon$ by elastic constitutive laws

$$\sigma = C : \varepsilon,$$  \hspace{1cm} (2.3)

where $C$ is the Hooke tensor.

The strain is related with the displacement, in the case of small strains and displacements, as

$$\varepsilon = \nabla_s u,$$  \hspace{1cm} (2.4)

where $\nabla_s$ is the symmetric part of the gradient operator.

The weak form of the equilibrium equation 2.1 is

$$\int_{\Omega - \Gamma_c} \varepsilon(u) : C : \varepsilon(\delta u) \, d\Omega - \int_{\Gamma_t} \bar{t} \cdot \delta u \, d\Gamma + \int_{\Omega - \Gamma_o} b \cdot \delta u \, d\Omega,$$  \hspace{1cm} (2.5)
where the displacement \( u \) belongs to the space of trial functions defined by

\[
U = \{ u | u = \bar{u} \text{ on } \Gamma_u; u \text{ is discontinuous on } \Gamma_c \}. \tag{2.6}
\]

Likewise, the space of test functions is given as

\[
\delta U = \{ \delta u | \delta u = 0 \text{ on } \Gamma_u; \delta u \text{ is discontinuous on } \Gamma_c \}. \tag{2.7}
\]

Appropriate regularity suitable for the interpolation functions to be used is assumed for both \( U \) and \( \delta U \). It is shown that the weak form (2.5) is equivalent to the equilibrium equation (2.1) equipped with the boundary conditions and traction-free conditions on the crack surfaces (2.2).

### 2.1.2 XFEM Enrichment Functions

The extended finite element method (XFEM) introduces enrichment functions in the area of interest to model fracture. In the formulation proposed by Moës et al. (1999), two types of enrichment, namely the Heaviside enrichment and crack-tip enrichment, are used to represent, respectively, the displacement jump along the crack and the asymptotic field around the crack tip. With the enrichment functions introduced, the displacement is approximated as

\[
u(x) = \sum_{I \in N_{\text{tot}}} N_I(x) u_I + \sum_{I \in N_{\text{hex}}} N_I(x) (H(x) - H(x_I)) a_I \\
+ \sum_{I \in N_{\text{tip}}} N_I(x) \left( \sum_{J=1}^{4} \Psi_I^J(x) b_I^J \right), \tag{2.8}\]

where \( N_I(x) \) is the standard interpolation function used in conventional finite elements, \( H(x) \) the Heaviside function and \( \Psi_I^J(x) \) the crack-tip enrichment functions.
$u_I$ is the standard displacement degree of freedom whereas $a_I$ and $b_I$ are the degrees of freedom associated with the Heaviside enrichment and crack-tip enrichment, respectively. $N_{\text{tot}}$ is the total set of nodes and $N_{\text{heas}}$ and $N_{\text{tip}}$ are the sets of nodes that need to be enriched by the Heaviside function and crack-tip functions, respectively.

The Heaviside function $H(x)$ is defined as

$$H(x) = \begin{cases} 1 & \text{if } (x - x^*) \cdot n \geq 0 \\ -1 & \text{otherwise} \end{cases}, \quad \text{(2.9)}$$

where $x$ is an integration point, $x^*$ is the closest point to $x$ on the crack and $n$ is the unit outward normal to the crack at $x^*$, namely, the unit normal with a 90° counterclockwise angle with respect to the crack direction (Figure 2.1).

![Figure 2.2: Representation of a discontinuity by Heaviside enrichment functions: one-dimensional case.](image)

The discontinuity due to the Heaviside enrichment functions is illustrated in Figure 2.2 for the one-dimensional case, where the whole domain $\Omega$ is divided into $\Omega_-$ and $\Omega_+$ by the discontinuity at point D. Since only the element connected to node 2 and 3 contains the discontinuity, only those nodes are the members of $N_h$ and need to be
enriched. The resulting enrichment functions along with the standard shape functions are shown in Figure 2.2. A simple calculation reveals that the discontinuity in the displacement is $2N_2(x_D)a_2 + 2N_3(x_D)a_3$. It is clearly seen that the enrichment is local and restricted to the element containing the discontinuity only.

The crack-tip functions $\Psi^J(x)$, $J = 1, \ldots, 4$ are given by

$$\{\Psi^J(x)\} = \left\{ \sqrt{r} \sin \frac{\theta}{2}, \sqrt{r} \cos \frac{\theta}{2}, \sqrt{r} \sin \theta \sin \frac{\theta}{2}, \sqrt{r} \sin \theta \cos \frac{\theta}{2} \right\},$$

(2.10)

where $r$ and $\theta$ are the polar coordinates in the local crack-tip coordinate system as shown in Figure 2.3.

![Crack-tip coordinate system](image)

Figure 2.3: Crack-tip coordinate system

It is noticed that the discontinuity is introduced by both the Heaviside function and the first member of the crack-tip functions ($\sqrt{r} \sin \frac{\theta}{2}$) at $\theta = \pm \pi$.

### 2.1.3 Heaviside-only Enrichment

In the XFEM formulation by Moës et al. (1999) introduced above, two types of enrichment are included in the interpolation function, namely Heaviside and crack-tip enrichment. The latter is introduced to simulate the singular displacement field in the crack-tip elements whereas the former is used for the rest of the enriched elements containing displacement discontinuities. Although this formulation can be
easily implemented for stationary crack modeling, in crack propagation problems
the enrichment type associated with the nodes around the crack tip needs to be
changed as the crack tip advances from one element into another, which may cause
difficulties (Belytschko et al., 2003). Furthermore, the formulation renders the
elements surrounding the crack-tip element partially enriched and it is reported that
the existence of such elements can impair the accuracy of the XFEM approximation
(Chessa et al., 2003). These shortcomings can be overcome by using the approach
proposed by Chen (2003) and Zi and Belytschko (2003) whereby only one type
of enrichment, the Heaviside enrichment, is used to model the whole crack. This
approach has been implemented for linear triangle (CST) and quadratic triangle
(LST) elements. Only the CST is used in this dissertation.

With the Heaviside-only enrichment, the interpolation function can be written as

\[ u(x) = \sum_{I \in N_{\text{tot}}} N_I(x)u_I + \sum_{I \in N_{\text{near}}} N_I(x)(H(x) - H(x_I))a_I \quad (2.11) \]

where \( N_{\text{near}} \) is the set of nodes to be enriched with the Heaviside function and all
other terms have the same meaning as those in equation (2.8). For future reference,
we define the first and second term of Equation (2.11) as \( u_{\text{cont}} \) and \( u_{\text{disc}} \), namely,

\[ u_{\text{cont}}(x) = \sum_{I \in N_{\text{tot}}} N_I(x)u_I, \quad (2.12) \]

\[ u_{\text{disc}}(x) = \sum_{I \in N_{\text{near}}} N_I(x)(H(x) - H(x_I))a_I, \quad (2.13) \]

which implies

\[ u(x) = u_{\text{cont}}(x) + u_{\text{disc}}(x). \quad (2.14) \]
For a CST element, we can express the variables as functions of the area coordinates $\xi = (\xi_1, \xi_2, \xi_3)$. For example, the spatial coordinates $x$ can be represented as
\[
x = \sum_{I=1}^{3} x_I N_I (\xi)
\] (2.15)

For an element that is entirely cut by the crack, all of the nodes are enriched and both $u_{\text{cont}}(x)$ and $u_{\text{dis}}(x)$ can be expressed in terms of the area coordinates straightforwardly.

![Figure 2.4: Enrichment for the crack-tip element](image)

For an element that is partially cut by the crack, additional consideration is necessary to recognize the location of the crack tip within the element, for which the nodes that need to be enriched are determined first. Take the configuration in Figure 2.4, which shows a triangular element that contains the crack tip $T$, as an example. The crack intersects with the edge 1-2 at point $A$. The extension of the crack segment intersects with the edge 3-1 and we enrich only the node other than those on this edge, namely node 2.
The above process produces a virtual triangular element defined by points 1, 2 and 3*, where point 3* is the intersection of the line passing 1 and T with the edge 2-3. The interpolation functions on this virtual element are \( N_i(\xi^*) \), where \( \xi^* \) are the area coordinates of the triangle 123*. The discontinuous part of the shape function can now be written as

\[
\mathbf{u}_{\text{disc}} = \xi_2^*(H(\xi^*) - H(\xi_3^*))\mathbf{a}_2.
\]

(2.16)

In the case depicted in Figure 2.4, \( \xi^* \) are related to \( \xi \) by

\[
\begin{align*}
\xi_1^* &= \xi_1, \\
\xi_2^* &= 1 - \xi_1 - \frac{\xi_3}{\xi_3^*}, \\
\xi_3^* &= \frac{\xi_3}{\xi_3^*},
\end{align*}
\]

(2.17)

where \( \xi_3^* \) is the area coordinate of point 3* in the crack-tip element.

With the crack position in the element varying, there exist six possible configurations in total. All, including the one shown in Figure 2.4, are depicted in Figure 2.5.

In general, suppose the element vertices are denoted as i, j, k and the crack runs through element edge i-j toward edge j-k, as depicted in Figure 2.6. Only node i needs to be enriched and

\[
\begin{align*}
\xi_j^* &= \xi_j, \\
\xi_k^* &= \frac{\xi_k}{\xi_k^*}, \\
\xi_i^* &= 1 - \xi_j - \frac{\xi_k}{\xi_k^*}.
\end{align*}
\]

(2.18)
Figure 2.5: Relative position of a crack and a triangular element: enriched nodes are circled.

Figure 2.6: Enrichment for the crack-tip element: a general configuration
2.1.4 Discrete Equations

Defining $u^e$ and $\delta u^e$ similarly as $u$ and $\delta u$, we write the discrete weak form as

$$
\int_{\Omega^e} \varepsilon(u^e) : C : \varepsilon(\delta u^e) \, d\Omega = \int_{\partial\Omega^e} \bar{t} \cdot \delta u^e \, d\Gamma + \int_{\Omega^e} b \cdot \delta u^e \, d\Omega,
$$

(2.19)

where $\Omega^e$ is the domain of a finite element.

With the enriched shape functions and the discrete weak form, we can derive the discrete equilibrium equation

$$
Kd = f^{\text{ext}},
$$

(2.20)

where $d$ contains the generalized nodal displacements, which includes both the standard nodal displacements and those related to the enriched degrees of freedom:

$$
d = \begin{bmatrix}
u_1 \\ a_1 \\ u_2 \\ a_2 \\ u_3 \\ a_3 
\end{bmatrix}
$$

(2.21)

$K$ is the element stiffness matrix:

$$
K = \int_{\Omega^e} B^T C B \, d\Omega,
$$

(2.22)

where $B$ is the strain-displacement matrix and $C$ the tangent modulus matrix. For an unenriched element,

$$
B = [B_1^0, B_2^0, B_3^0],
$$

(2.23)
and for an enriched element,

\[
B = [B^0_1, B^0_2, B^0_3, B^0_4, B^0_5, B^0_6],
\]

(2.24)

where

\[
B^0_1 = \begin{bmatrix}
N_{I,x} & 0 \\
0 & N_{I,y} \\
N_{I,y} & N_{I,x}
\end{bmatrix},
\]

(2.25)

\[
B^0_2 = \begin{bmatrix}
N_{I,x} (H(x) - H_I) & 0 \\
0 & N_{I,y} (H(x) - H_I) \\
N_{I,y} (H(x) - H_I) & N_{I,x} (H(x) - H_I)
\end{bmatrix}.
\]

(2.26)

In Equation (2.25) and (2.26)

\[
N_1 = \xi_1^*; \quad N_2 = \xi_2^*; \quad N_3 = 1 - \xi_1^* - \xi_2^*,
\]

(2.27)

for crack-tip elements and

\[
N_1 = \xi_1; \quad N_2 = \xi_2; \quad N_3 = 1 - \xi_1 - \xi_2,
\]

(2.28)

for all other elements. \( f^{\text{ext}} \) is the external force:

\[
f^{\text{ext}} = \int_{\delta\Gamma} N^T \mathbf{t} \, d\Gamma + \int_{\Omega^*} N^T \mathbf{b} \, d\Omega.
\]

(2.29)

### 2.1.5 Numerical Integration

For elements cut by a crack, the standard Gauss quadrature may not be adequate due to discontinuities that may exist inside the element. To accurately evaluate
the integrals from the virtual work statement, a new quadrature scheme is adopted. The basic idea is to divide the element domain into subdomains and then apply a quadrature formula to each subdomain. Different division methods exist (Dolbow, 1999) and triangulation is used in this research. When an element is fully cut by the crack into two, the centroid of each subelement is calculated and then the subelement is triangulated by linking its centroid with each of its vertices. The crack-tip element, on the other hand, is first divided into a virtual triangle and the triangle that occupies the rest of the element domain, as described in section 2.1.3. Then the virtual triangle is further cut into three triangles by means of the crack itself and the line joining the crack tip to one of the vertices of the parent element. Figure 2.7 shows how elements completely cut by a crack and a crack-tip element are partitioned for the purpose of Gauss quadrature.

![Figure 2.7: Partition of elements for Gauss quadrature: the nodes with solid circles are enriched with the Heaviside function.](image-url)
2.2 Vector Level Set Method

In the XFEM, the Heaviside function $H(x)$ plays a key role in the enriched displacement field. Although this function can be evaluated directly from Equation (2.9), the level set method (LSM) provides a better alternative, especially in three dimensional cases. Stolarska et al. (2001) first discussed the coupling of the LSM and XFEM and reported that the combination of the two methods is natural and efficient and has a tremendous potential for a wide range of applications.

The LSM is a computational technique developed for modeling a propagating interface over time (Osher and Sethian, 1988). In this method, the interface is described as the zero level set of a function the dimension of which is higher by one than that of the domain where the interface resides. For example, a one-dimensional crack is expressed as the zero level set of a two-dimensional function. Although the use of a function in higher dimensions may require higher computational efforts, the LSM provides many advantages including: no need to change the mesh; straightforwardness in extending to higher dimensions; and ability to naturally handle topology changes (Stolarska et al., 2001).

An improvement to the original LSM, termed as the vector level set method, was later presented (Ventura et al., 2003), where the LSM is adapted to the nature of crack propagation problems. The key idea is to have only the crack tip, instead of the whole crack, evolve during propagation. Such a restriction eliminates the need to solve partial differential equations to determine the evolution of the entire crack. In the vector level set method, the level set function is described in 2D by a 3-tuple: the sign of the level set function and the two components of the closest point projection on the crack. We express them with $H(f(x))$ and $f(x)$. Here $f = x^* - x$ is the vector joining point $x$ and its closest projection on the crack $x^*$. This 3-tuple is stored only
at those nodes that surround the crack. The update of the level set is restricted in the domain where the 3-tuple changes when the crack propagates. In this way, additional computation is trivial. Figure 2.8 shows a part of a crack and the nodes at which the 3-tuples have to be stored.

![Figure 2.8: Crack representation in the vector level set method](image)

The algorithm is detailed as follows. Assume the crack is composed by a series of segments, and the crack tip advance segment is $s^n$ with the previous crack segment being $s^{n-1}$. Accordingly, the new and old crack tips are $x_{\text{tip}}^n$ and $x_{\text{tip}}^{n-1}$, respectively.

1. Calculate vector $s^b$ that bisects the angle between $s^{n-1}$ and $s^n$ and the unit vector $s^n_\perp$ which is the counterclockwise normal to $s^n$:

$$s^b = \frac{s^{n-1}}{||s^{n-1}||} + \frac{s^n}{||s^n||}.$$  

(2.30)
2. Determine the set of nodes that need update of the 3-tuple; node $x_I$ needs update of its 3-tuple if the following condition is satisfied:

$$\begin{align*}
(x_I - x_{tip}^n) \cdot s^n &\leq 0, \\
(x_I - x_{tip}^{n-1}) \cdot s^{n-1} &> 0 \text{ or } (x_I - x_{tip}^{n-1}) \cdot s^b > 0.
\end{align*}$$

3. For each node that has been chosen in step 2, update the 3-tuple according to the following equations:

$$H(f(x_I)) = \text{sign} \left[ (x_I - x_{tip}^{n-1}) \cdot s^n \right],$$

$$f(x_I) = \begin{cases} 
-\hat{s}_n \left[ (x_I - x_{tip}^{n-1}) \cdot \hat{s}_n^n \right] & \text{if } (x_I - x_{tip}^{n-1}) \cdot s^n \geq 0 \\
-(x_I - x_{tip}^{n-1}) & \text{if } (x_I - x_{tip}^{n-1}) \cdot s^n < 0
\end{cases}$$
4. Evaluate the level set function (3-tuple) at a generic point $x$. Assume $x_I$ is the closest level set point to $x$, then

$$v_I = f_I + (x_I - x),$$  \hspace{1cm} (2.35)

$$H(x) = H_I \text{sign}(f_I \cdot v_I),$$  \hspace{1cm} (2.36)

$$f(x) = \frac{f_I}{||f_I||} \left( \frac{f_I}{||f_I||} \cdot v_I \right).$$  \hspace{1cm} (2.37)

An illustration is provided in Figure 2.10. The dotted line shown is the tangent line at the projection point of the level set point $x_I$.

![Figure 2.10: Evaluation of the level set function of point $x$: $x_I$ is the closest level set point.](image)

### 2.3 Crack Propagation Simulation

Various crack growth criteria have been proposed to predict crack propagation direction. Well-known ones include maximum hoop stress criterion (Erdogan and Sih, 1963); minimum strain energy criterion (Sih, 1974); maximum energy release rate criterion (Nuismer, 1975); and maximum mode I stress intensity ($K_I$) criterion
(Goldstein and Salganik, 1974). These criteria have been shown to be essentially identical for slightly curved cracks under in-plane loading (McNaney et al., 1994).

Here the maximum hoop stress criterion is used, according to which the crack will propagate in the direction that makes the shear stress equal zero. Since the shear stress near the crack tip is

\[
\sigma_{\theta \theta} = \frac{K_I}{4\sqrt{2}\pi r} \left[ \sin \left( \frac{\theta}{2} \right) + \sin \left( \frac{3\theta}{2} \right) \right] + \frac{K_{II}}{4\sqrt{2}\pi r} \left[ \cos \left( \frac{\theta}{2} \right) + 3 \sin \left( \frac{3\theta}{2} \right) \right],
\]

the crack propagation direction is obtained as

\[
\theta_c = 2 \arctan \frac{1}{4} \left( \frac{K_I}{K_{II}} \pm \sqrt{\left( \frac{K_I}{K_{II}} \right)^2 + 8} \right),
\]

where \( \theta_c \) is the crack propagation angle in the local crack-tip coordinate system and \( K_I \) and \( K_{II} \) are the stress intensity factor of mode I and mode II, respectively. They can be evaluated through domain forms of the interaction integrals once the displacement field is obtained (Yau et al., 1980; Dolbow et al., 2001).

In the present study, the crack grows by a pre-defined length \( \delta a \) in direction \( \theta_c \) when

\[
K_{Iq} \geq K_c,
\]

where \( K_{Iq} \) is the equivalent stress intensity factor calculated by

\[
K_{Iq} = K_I \cos^3 \left( \frac{\theta_c}{2} \right) - \frac{3}{2} K_{II} \cos \left( \frac{\theta_c}{2} \right) \sin(\theta_c),
\]

and \( K_c \) is the fracture toughness of the material under consideration (Dolbow, 1999).
2.4 Stress Intensity Factor Evaluation

Both the stress intensity factors (SIF) and J contour integral are important parameters in fracture mechanics and they can be related to each other for mixed-mode problems in linear elastic materials by

\[ J = \frac{1}{E^*} \left( K_1^2 + K_II^2 \right), \tag{2.42} \]

where

\[ E^* = \begin{cases} E & \text{(plane stress)} \\ E/(1-\nu^2) & \text{(plane strain)} \end{cases} \tag{2.43} \]

and \( K_1 \) and \( K_II \) are stress intensity factors for mode I and II, respectively. \( E \) and \( \nu \) are the Young’s Modulus and Poisson’s ratio. In this dissertation, the stress intensity factors are evaluated using the domain form of interaction integrals (Yau et al., 1980).

The J contour integral is defined as

\[ J = \int \left( \bar{U} dy - \sigma_{ij} n_j \frac{\partial w_i}{\partial x} ds \right), \tag{2.44} \]

where \( \bar{U} = \frac{1}{2} \sigma_{ij} \epsilon_{ij} \) is the stress energy density. The relation \( dy = n_z ds \) (Figure 2.11) reduces Equation (2.44) to

\[ J = \int \left( \frac{1}{2} \sigma_{ij} \epsilon_{ij} \delta_{ik} - \sigma_{ik} \frac{\partial u_i}{\partial x_1} \right) n_k ds. \tag{2.45} \]

For mixed mode crack problems, two states of a cracked body are considered:

- State 1 (actual state): \( \sigma_{ij}^{(1)}, \epsilon_{ij}^{(1)}, u_i^{(1)} \);

- State 2 (auxiliary state): \( \sigma_{ij}^{(2)}, \epsilon_{ij}^{(2)}, u_i^{(2)} \).
The J-integral for the sum of the two states is

\[ J = \int_\Gamma \left[ \frac{1}{2} \left( \sigma_{ij}^1 + \sigma_{ij}^2 \right) \left( \epsilon_{ij}^{(1)} + \epsilon_{ij}^{(2)} \right) \delta_{1k} - \left( \sigma_{ik}^1 + \sigma_{ik}^2 \right) \frac{\partial u_{i}^{(1)}}{\partial x_1} + \frac{\partial u_{i}^{(2)}}{\partial x_1} \right] n_k \, ds. \]  \hspace{1cm} (2.46)

Letting \( W^{(1,2)} = \sigma_{ij}^{(1)} \epsilon_{ij}^{(2)} \) and defining the interaction integral

\[ I^{(1,2)} = \int_\Gamma \left( W^{(1,2)} \delta_{1k} - \sigma_{ik}^{(1)} \frac{\partial u_{i}^{(2)}}{\partial x_1} - \sigma_{ik}^{(2)} \frac{\partial u_{i}^{(1)}}{\partial x_1} \right) n_k \, d\Gamma, \]  \hspace{1cm} (2.47)

we can write Equation (2.46) as

\[ J^{(1+2)} = J^{(1)} + J^{(2)} + I^{(1+2)}. \]  \hspace{1cm} (2.48)
From Equation (2.42), the J-integral for the sum of the two states can also be expressed as

\[
J^{(1+2)} = \frac{1}{E^*} \left[ \left(K_1^{(1+2)}\right)^2 + \left(K_1^{(1+2)}\right)^2\right] \\
= J^{(1)} + J^{(2)} + \frac{2}{E^*} \left(K_1^{(1)}K_1^{(2)} + K_1^{(1)}K_1^{(2)}\right).
\]  

Comparing Equations (2.48) and (2.49), we have

\[
I^{(1,2)} = \frac{2}{E^*} \left(K_1^{(1)}K_1^{(2)} + K_1^{(1)}K_1^{(2)}\right). \tag{2.50}
\]

The stress intensity factors can be calculated by decoupling the mixed mode problem through properly choosing the auxiliary state. To evaluate \(K_1\), we choose the auxiliary state to be the pure mode I asymptotic field \((K_1^{(2)} = 1, K_1^{(2)} = 0)\) so that

\[
I^{(1,2)} = \frac{2}{E^*} K_1^{(1)}, \tag{2.51}
\]

from which

\[
K_1^{(1)} = \frac{E^*}{2} I^{(1,2)}. \tag{2.52}
\]

Similarly, \(K_\Pi\) can be calculated by choosing the pure mode II asymptotic field as the auxiliary state:

\[
K_\Pi^{(1)} = \frac{E^*}{2} I^{(1,2)}. \tag{2.53}
\]

Both the pure mode I and mode II asymptotic fields are listed in Appendix A.

In the finite element computations, the interaction integral \(I^{(1,2)}\) is converted into a domain integral, which can be evaluated more conveniently, by multiplying the
integrand by a sufficiently smooth weighting function \( q(x) \).

\[
q = \begin{cases} 
1 & \text{on } \Gamma \\
0 & \text{on } C_0
\end{cases}
\]  

(2.54)

Figure 2.12: Domain integral: area \( A \) is enclosed by \( \Gamma \), \( C_0 \) and the crack faces \( C_+ \) and \( C_- \).

For such \( q \), the integral \( I^{(1,2)} \) on \( \Gamma \) can be recast as a line integral over the closed contour \( C \) which consists of \( \Gamma \), \( C_0 \), \( C_+ \) and \( C_- \) as shown in Figure 2.12. \( \mathbf{m} \) is the unit outward normal to \( C \). When the crack is parallel to the \( x_1 \) axis, \( m_1 = 0 \) on \( C_+ \cup C_- \), where \( \sigma^{(2)}_{ij} \) vanishes as the auxiliary fields satisfy traction-free conditions and also \( \sigma^{(1)}_{ij} = 0 \) if there is no contact between the crack faces. Thus, we can write the interaction integral in Equation (2.47) as

\[
I^{(1,2)} = \int_c \left[ \sigma^{(1)}_{ij} \frac{\partial u^{(2)}_j}{\partial x_1} + \sigma^{(2)}_{ij} \frac{\partial u^{(1)}_j}{\partial x_1} - W^{(1,2)} \delta_{ij} \right] q m_j \, d\Gamma. 
\]  

(2.55)
Using divergence theorem and passing to the limit as the contour $\Gamma$ is shrunk to the crack tip, we have the interaction integral in domain form as

$$ f^{(1,2)} = \int_A \left[ c_{ij}^{(1)} \frac{\partial^2 u_i^{(2)}}{\partial x_1^2} + c_{ij}^{(2)} \frac{\partial^2 u_i^{(1)}}{\partial x_1^2} - W^{(1,2)} \delta_{ij} \right] \frac{\partial q}{\partial x_j} dA. \quad (2.56) $$

![Figure 2.13: Domain $A$ and weighting function $q$](image)

In this dissertation, domain $A$ is designed to contain all elements which have a node within a ball of radius $r_d$ about the crack tip. $r_d = 2\sqrt{2A_e}$, where $A_e$ is the area of the element. $q = 1$ on all nodes within the ball $r_d$ whereas $q = 0$ on the nodes on the boundary ($\partial \Omega$). Within the elements $q$ can be interpolated using the nodal shape functions, namely,

$$ q = \sum_i N_i q_i \quad (2.57) $$

and

$$ \frac{\partial q}{\partial x_j} = \sum_i \frac{\partial N_i}{\partial x_j} q_i. \quad (2.58) $$

Since the gradient of $q$, instead of $q$ itself, appears in the integral, only the elements with an edge on $\partial \Omega$ will contribute to the evaluation.
For both plane strain and plane stress problems, Equation (2.56) can be further written as

\[
I^{(1,2)} = \int_A \left[ \left( \sigma^{(1)}_{11} \frac{\partial u^{(2)}_1}{\partial x_1} + \sigma^{(1)}_{21} \frac{\partial u^{(2)}_2}{\partial x_1} \right) \frac{\partial q}{\partial x_1} + \left( \sigma^{(1)}_{12} \frac{\partial u^{(2)}_1}{\partial x_1} + \sigma^{(1)}_{22} \frac{\partial u^{(2)}_2}{\partial x_1} \right) \frac{\partial q}{\partial x_2} \\
+ \left( \sigma^{(2)}_{11} \frac{\partial u^{(1)}_1}{\partial x_1} + \sigma^{(2)}_{21} \frac{\partial u^{(1)}_2}{\partial x_1} \right) \frac{\partial q}{\partial x_1} + \left( \sigma^{(2)}_{12} \frac{\partial u^{(1)}_1}{\partial x_1} + \sigma^{(2)}_{22} \frac{\partial u^{(1)}_2}{\partial x_1} \right) \frac{\partial q}{\partial x_2} \right] dA \\
- \left( \sigma^{(2)}_{11} \epsilon^{(1)}_{11} + \sigma^{(2)}_{22} \epsilon^{(1)}_{22} + \sigma^{(2)}_{12} \epsilon^{(1)}_{12} + \sigma^{(2)}_{21} \epsilon^{(1)}_{21} \right) \frac{\partial q}{\partial x_1} dA
\]   

(2.59)
Chapter 3
Near-interfacial Crack Propagation Simulation

3.1 Introduction

In this chapter, the numerical techniques described previously, the XFEM combined with the vector LSM, are employed to simulate the crack propagation in a layered structure. As has been stated, instead of using both the Heaviside enrichment and crack-tip enrichment, the present study follows the XFEM formulation by Chen (2003) and Zi and Belytschko (2003), who employed the Heaviside enrichment only. Dropping crack-tip enrichment functions eliminates the partially enriched elements on which the partition of unity does not hold, thus avoiding the degradation of accuracy in those elements (Chessa et al., 2003). Three-node triangular elements are used exclusively in our implementation for two dimensional situations.

Incorporated into the XFEM formulation is the vector level set method (Ventura et al., 2003), in which the nature of crack propagation problems is fully exploited for enhanced efficiency. Following the algorithm described in Chapter 2, the level set function of those nodes that surround the crack, namely, the sign of the level set
function and the components of the closest point projection on the crack, is stored and updated accordingly.

In the following, a pilot problem on the computation of stress intensity factors (SIF) for cracks in a plate under uniaxial loading is first considered to verify the implementation. The crack propagation in a double cantilever beam is also modeled and a parametric study is carried out to investigate the factors that may influence the crack propagation direction, such as the mesh density, the domain interaction integral radius and the crack propagation increment. Numerical simulation of near-interfacial crack propagation in a metal-ceramic layered structure follows and the computed crack paths are compared with experimental results. The results are discussed and summarized afterwards.

3.2 Benchmark Problems

3.2.1 Crack in a Plate Under Uniaxial Loading

We use the benchmark problem considered by Huang et al. (2003) to verify the XFEM implementation. In the problem depicted in Figure 3.1, a rectangular plate of height $2h = 3$ and width $2b = 2$ is subjected to a uniaxial tensile stress $\sigma$. The plate contains, at its center, a crack of length $2a = 0.2$, which is inclined at angle $\alpha$ with respect to the horizontal axis. When $\alpha = 0^\circ$, the state of the crack is very close to the mode I as the size of the crack is small compared with the dimensions of the plate. The mode I stress intensity factor $K_I$ at the crack tip is

$$K_I = F \left( \frac{a}{b} \right) \sigma \sqrt{\pi a}$$ (3.1)
where $F(\beta)$ is a dimensionless correction factor depending on the ratio $\beta$. For the case of $\beta = 0.1$, $F(\beta) = 1.006$ (Huang et al., 2003).

Both a structured and an unstructured mesh are used in the XFEM numerical simulation. The structured mesh consists of $200 \times 100$ triangular elements arranged as in Figure 3.2, whereas the unstructured mesh consists of 6032 triangular elements with higher mesh density in the vicinity of the crack (Figure 3.3). In both cases the mesh density is approximately the same around the crack tip.

Mode I and mode II stress intensity factors are computed using domain forms of the interaction integrals (Moës et al., 1999). For the area integral to be evaluated, a domain, conveniently assumed to be a circle with radius $r_d$, needs to be selected with $r_d$ being an arbitrary parameter. The computed SIF values normalized by the
Figure 3.2: Structured mesh: the element arrangement

Figure 3.3: Unstructured mesh: the vicinity of the crack
Table 3.1: Normalized SIFs for the center crack (α = 0°)

<table>
<thead>
<tr>
<th>structured mesh</th>
<th>unstructured mesh</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K_a \theta / \sigma \sqrt{a}$</td>
<td>$K_a \theta / \sigma \sqrt{a}$</td>
</tr>
<tr>
<td>0.245</td>
<td>1.013</td>
</tr>
<tr>
<td>0.490</td>
<td>0.983</td>
</tr>
<tr>
<td>0.735</td>
<td>0.990</td>
</tr>
<tr>
<td>0.980</td>
<td>0.990</td>
</tr>
<tr>
<td>1.225</td>
<td>0.990</td>
</tr>
</tbody>
</table>

Theoretical counterparts are listed in Table 3.1 for a range of $r_d$ used in the domain integral evaluation. It is seen that the SIFs are fairly insensitive to the domain radius, especially when the radius is relatively large, and both the structured and unstructured mesh produce results that are in excellent agreement with theoretical solutions.

When the angle $\alpha$ is greater than 0°, the problem becomes one with a mixed-mode crack and so the mode II ($K_{II}$) as well as mode I ($K_I$) SIFs can be assessed. The theoretical solutions for an infinite domain (Anderson, 2005) are

$$K_I = \sigma \sqrt{\pi a} \cos^2 \alpha, \quad K_{II} = \sigma \sqrt{\pi a} \sin \alpha \cos \alpha. \quad (3.2)$$

Listed in Table 3.2 are the ratios of the calculated to theoretical SIFs for a range of values for $\alpha$. Very good agreements are obtained again. Accurate assessment of SIFs for a mixed-mode crack is particularly relevant for the present study as such information is crucial in determining the crack growth direction as shown in the next example.
Table 3.2: SIF's for the inclined crack

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$K_I$</th>
<th>$K_{II}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>XFEM</td>
<td>theory</td>
</tr>
<tr>
<td>15°</td>
<td>0.527</td>
<td>0.523</td>
</tr>
<tr>
<td>30°</td>
<td>0.421</td>
<td>0.420</td>
</tr>
<tr>
<td>45°</td>
<td>0.278</td>
<td>0.280</td>
</tr>
<tr>
<td>60°</td>
<td>0.136</td>
<td>0.140</td>
</tr>
<tr>
<td>75°</td>
<td>0.039</td>
<td>0.038</td>
</tr>
</tbody>
</table>

3.2.2 Crack Propagation in Double Cantilever Beam

A double cantilever beam with a crack lying slightly above the mid-plane, as shown in Figure 3.4, is simulated. Dimensions of the specimen are $6 \times 2$, and length of the pre-existing crack is 2.

![Double cantilever beam](image)

Figure 3.4: Double cantilever beam

Figure 3.5 shows the crack after 11 steps (crack growth increment $\delta a = 0.1$). Similar results are reported elsewhere (Huang et al., 2003). It is noted that in the simulation once the applied load is big enough to start the crack growth, it
is not necessary to increase the load in the following steps. This can be explained by looking into the equivalent stress intensity factor $K_{eq}$ calculated from Equation (2.41). Suppose, at step $n$, the equivalent stress intensity factor $K_{eq}^n$ under the load $P_n$ is bigger than the fracture toughness $K_c$, satisfying the crack propagation condition, Equation (2.40). Then the crack grows by the prescribed length in the direction of $\theta_c$ and the simulation proceeds to the next step. At step $n + 1$, although the same load $P_{n+1} = P_n$ is applied, the calculated $K_{eq}^{n+1}$ is bigger than $K_{eq}^n$ and consequently is bigger than $K_c$. Hence the crack grows again by the same prescribed length. The process goes on until the desired number of propagation steps is reached.

![Figure 3.5: Crack propagation in double cantilever beam: the mesh is generated by TRIANGLE; the crack is after 11 steps of increment $\delta a = 0.1$.](image)

Influences of some parameters including the mesh, domain interaction integral radius and crack propagation increment are studied. The results are shown in Figure 3.6, 3.7 and 3.8. It is concluded that the crack growth is hardly influenced either by the meshes or by the domain radius. Although the crack propagation increment has the most significant influence, by choosing a suitable value, its influence can be essentially eliminated.
**Figure 3.6:** Effect of element size on simulated crack paths

**Figure 3.7:** Effect of the domain interaction integral radius on simulated crack paths
Figure 3.8: Effect of the crack growth increment on simulated crack paths
3.3 Near-interfacial Fracture in Layered Structure

3.3.1 Experimental Evidence in Literature

To investigate the capability of the XFEM considered in the present study in the simulation of near-interfacial crack propagation in a layered composite structure, we consider fracture in three-layer ceramic-metal-ceramic structures tested by McNaney et al. (1994). A typical specimen subjected to four-point bending is shown in Figure 3.9, where a thin metal layer of thickness $h$ is sandwiched between ceramic materials. The resulting specimen of length $L$ and height $W$ is subjected to a pair of concentrated loads of magnitude $P$ that are distance $d$ apart. A crack of length $a$, initially parallel to a metal-ceramic interface, exists in the ceramic and propagates upward upon the application of the load.

![Figure 3.9: Four-point bending specimen](image)

Experimental as well as theoretical analysis by McNaney et al. (1994) reports that when the crack is situated close to the interface, the crack propagation trajectory is strongly affected by the relative compliance of the two constituent materials for the composite specimen. The crack-tip field of an interfacial crack is affected by Dundurs' parameters $\alpha$ and $\beta$ (Dundurs, 1969), which describe the effect of the difference in
elastic properties of the two constituent materials:

\[
\alpha = \frac{E'_1 - E'_2}{E'_1 + E'_2}, \quad (3.3)
\]

\[
\beta = \frac{\mu_1(\kappa_2 - 1) - \mu_2(\kappa_1 - 1)}{\mu_1(\kappa_2 + 1) + \mu_2(\kappa_1 + 1)}, \quad (3.4)
\]

where, for the plane strain loading condition, \( E'_i = E_i/(1 - \nu_i^2) \) and \( \kappa_i = 3 - 4\nu_i, i = 1, 2 \), and \( E_1, \nu_1 \) and \( \mu_1 \) \((E_2, \nu_2 \text{ and } \mu_2)\) are the Young's modulus, Poisson's ratio and shear modulus of the ceramic (metal), respectively. The study by McNaney et al. (1994) reveals that, for the crack path satisfying the criterion \( K_\Pi = 0 \), the near-interfacial crack is drawn to the interface when \( \beta > 0 \) while the crack is repelled from the interface when \( \beta < 0 \).

### 3.3.2 Numerical Model

XFEM simulations are carried out to numerically replicate the tests that verified such predictions. Two groups of ceramic-metal pairs are considered such that \( \beta > 0 \) for the Alumina/Aluminum pair and \( \beta < 0 \) for the Glass/Copper pair. Elastic properties and the corresponding Dundurs' parameters for the two groups of material pairs are listed in Table 3.3. The dimensions of the four-point bending specimen are \( L = 25 \text{ mm} \) and \( W = 3 \text{ mm} \) for the Alumina/Aluminum specimen and \( L = 30 \text{ mm} \) and \( W = 7.5 \text{ mm} \) for the Glass/Copper specimen. The thickness \( h \) of the metal layer is: 50 ~ 100 \( \mu \text{m} \) and 450 \( \mu \text{m} \) for Aluminum and 125 \( \mu \text{m} \) for Copper. In all cases, the dimension of the specimen in the out-of-plane direction is 3 mm, and so the plane strain loading condition is assumed.

A typical mesh generated by TRIANGLE, a two-dimensional mesh generator and Delaunay triangulator (Shewchuk, 2002), is depicted in Figure 3.10, where a fully
Table 3.3: Material properties and Dundurs’ parameters for two cases of material pairs: Alumina/Aluminum ($\beta > 0$) and Glass/Copper ($\beta < 0$)

<table>
<thead>
<tr>
<th>Material pair 1/2</th>
<th>$E_1$</th>
<th>$\nu_1$</th>
<th>$E_2$</th>
<th>$\nu_2$</th>
<th>$\alpha$</th>
<th>$\beta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Alumina/Aluminum</td>
<td>370 GPa</td>
<td>0.27</td>
<td>71 GPa</td>
<td>0.345</td>
<td>0.666</td>
<td>0.144</td>
</tr>
<tr>
<td>Glass/Copper</td>
<td>74 GPa</td>
<td>0.17</td>
<td>130 GPa</td>
<td>0.343</td>
<td>-0.324</td>
<td>-0.183</td>
</tr>
</tbody>
</table>

developed crack is also shown. A finer mesh density is used in the zone containing the crack as well as the metal layer. The mesh has a symmetric density distribution in the sense that a refinement is provided to both regions that are to the left and right of the metal layer although the region to the right only contains the crack. To improve computational efficiency, meshes with a refined region to the right of the metal layer only have also been used, without any noticeable change in the simulated crack trajectories.

The crack growth direction is determined by the maximum hoop stress criterion (Erdogan and Sih, 1963), which gives the angle of the crack growth

$$\theta_c = 2 \arctan \frac{1}{4} \left( \frac{K_I}{K_{II}} \pm \sqrt{\left( \frac{K_I}{K_{II}} \right)^2 + 8} \right)$$

(3.5)

with respect to the local coordinate system for which $\theta_c = 0$ corresponds to the direction of the existing crack. The maximum hoop stress criterion is known to lead to the crack trajectory such that $K_{II} = 0$ (Sukumar and Prévost, 2003).

3.3.3 Results and Discussion

Comparisons of the crack path trajectories simulated by the XFEM with the experimental results obtained by McNaney et al. (1994) are shown in Figure 3.11 and 3.12 (Alumina/Aluminum; $h = 50 - 100 \, \mu m$ and $h = 450 \, \mu m$, respectively) and
3.13 (Glass/Copper; $h = 125 \, \mu m$). In Figure 3.11 two groups of XFEM results, one for $h = 50 \, \mu m$ and the other for $h = 100 \, \mu m$, are plotted since the experimental results are from specimens for which $h$ was varied from 50 to 100 $\mu m$ and the actual $h$ for each curve could not be discerned from McNaney et al. (1994). A range of values
for the initial distance of the crack from the interface, namely, $Y/h$ at $a/W = 0$ (to be referred to as $Y_0/h$), are considered to study its influence on the resulting crack trajectory.

![Graph](image)

Figure 3.11: Comparison of crack trajectories: Alumina/Aluminum ($\beta > 0$), $h = 50 - 100 \, \mu m$

In all cases, XFEM simulations correctly capture the important feature of near-interfacial fracture: the crack is drawn to the interface for $\beta > 0$ (Figure 3.11 and 3.12) whereas the crack is deflected away from the interface for $\beta < 0$ (Figure 3.13).

It is also accurately simulated that the effect of the sign of $\beta$ on crack trajectories becomes less obvious as $Y_0/h$ increases: Figure 3.12 shows that for $Y_0/h \leq 1$ the crack is drawn to and ultimately reaches the interface whereas the crack path is less affected by the presence of the metal layer as $Y_0/h$ increases, staying in the ceramic layer until the specimen breaks into two ($a/W = 1$).

Although qualitative features of the experiment are captured very well by the XFEM, some discrepancies exist as far as exact crack trajectories are concerned. For the purpose of such comparisons, it is first noted that one-to-one comparison of the
Figure 3.12: Comparison of crack trajectories: Alumina/Aluminum ($\beta > 0$), $h = 450 \mu m$

Figure 3.13: Comparison of crack trajectories: Glass/Copper ($\beta < 0$), $h = 125 \mu m$

XFEM and experimental results, as alluded to in Figure 3.11, 3.12 and 3.13, may not be justifiable in some cases. This is so because $Y_0/h$ for those cases cannot be clearly
identified from the drawings provided in McNaney et al. (1994). Nevertheless, care was exercised in the XFEM simulation to ensure that $Y_0/h$ would be comparable for simulations and experiments.

For the Alumina/Aluminum specimen with $h = 450 \mu m$ (Figure 3.12), the simulated crack trajectories are in excellent agreement with the test results for a wide range of $Y_0/h$. On the other hand, the degree of agreement for $h = 50 - 100 \mu m$ depends on $Y_0/h$ (Figure 3.11). For each value of $Y_0/h$ the two trajectories obtained by the XFEM, one for $h = 50 \mu m$ and the other for $100 \mu m$, provide an envelope, within which a trajectory corresponding to the experimental one exists. It is noted that, in the case of $Y_0/h \approx 2$ and 3, the simulated trajectories for $h = 50 \mu m$ are relatively close to the experimental ones while the crack path for $h = 100 \mu m$ more closely follows the experimental result in the case of $Y_0/h \approx 6$. The envelope for $Y_0/h \approx 4$, however, does not embrace the test result with comparable accuracy. It is also observed that the experimental curves for $Y_0/h \approx 3$ and 4 intersect with each other, which may be a clear sign that different $h$'s were used for those cases. Intersecting crack trajectories can also be simulated by the XFEM as illustrated in Figure 3.14, albeit with a different combination of $Y_0/h$ and $h$.

In the case of the Glass/Copper specimen (Figure 3.13), the simulated trajectories are quite close to the experimental ones especially when the crack is still in its early stage of growth. Discrepancies become larger as the crack is fully developed with an increased rate of departure from the interface. In contrast, the simulated paths are seen to be deflected away from the interface with approximately a constant slope. However, test results involve intersecting curves, which suggests that the accelerated departure of the cracks in some cases may be misrepresenting the true behavior.

The influence of the mesh density on crack trajectories is also investigated by considering three increasingly refined meshes for the zone where the crack resides.
Figure 3.14: Intersection of simulated crack trajectories: Alumina/Aluminum ($\beta > 0$), $h = 50 \, \mu m$ and $200 \, \mu m$.

Figure 3.15: Influence of the mesh density on crack trajectories: Alumina/Aluminum ($\beta > 0$), $h = 100 \, \mu m$. The area of elements in the zone where the crack resides is 200 $\mu m^2$, 50 $\mu m^2$ and 12.5 $\mu m^2$ for mesh 1, 2 and 3, respectively.
The area of the elements for the coarsest mesh is 200 \( \mu m^2 \), and the area is reduced by a factor of four from one level of refinement to the next. In the case of the Alumina/Aluminum specimen with \( h = 100 \mu m \), the results shown in Figure 3.15 indicate that the mesh density has relatively insignificant influences. For some values of \( Y_0/h \), the cracks due to a finer mesh are drawn to the interface slightly faster, being closer to the experimental results. The foregoing discussion indicates that overall comparisons of computational and experimental results are deemed very good. Nevertheless, it is expected that improvements can be attained by introducing additional ingredients into the numerical model. First, the simulation entails complete breakage of specimens \( (a/W = 1) \), for which consideration of large deformation may be necessary. By the same token, plasticity of the metal may have affected the displacement field around the crack. While pure brittle cracking was assumed in the present study for a crack initiation criterion, use of a proper cohesive crack model is expected to more closely capture the fracture behavior of Alumina as evidenced by experiments (Llorca and Steinbrech, 1991). Use of higher-order elements that allows for a better kinematic representation of the crack geometry as well as the finite-element displacement field may enhance the comparison further.

3.4 Concluding Remarks

An XFEM relying on Heaviside enrichment only has been applied to the analysis of near-interfacial fracture in ceramic/metal/ceramic layered structures. The method accurately captures the important aspect of the fracture behavior that the crack growth trajectory is either drawn to or repelled from the ceramic/metal interface depending on the relative elastic properties of the constituent materials. The crack paths observed in experiments are also closely reproduced by XFEM simulations.
It is expected that some observed discrepancies in crack paths may be reduced by considering inelastic behavior of the structure as well as higher-order kinematic descriptions in the XFEM formulations. Incorporation of such additional modeling capabilities are the subject of future studies.
In this chapter, the extended finite element method (XFEM) is applied to the simulation of the bi-material interfacial fracture. Only the Heaviside functions are used for the XFEM with the nodes on the crack only enriched. The stress intensity factor analysis demonstrates that such an enrichment strategy can produce satisfactory results for interfacial fracture problems.

4.1 Introduction

Fracture along the interface of two distinct materials has been of great practical importance. In civil engineering, for example, it has become a widespread practice to use externally-bonded fiber-reinforced plastic (FRP) composites in the form of a plate or sheet to strengthen reinforced or prestressed concrete structures. One crucial concern in this type of application is the integrity of the interface between the concrete substrate and the FRP plate/sheet since the debonding that may occur along the interface is brittle and can lead to failure of the whole strengthening system. Various studies have been carried out both experimentally and analytically. Pan and Leung (2007) designed a novel experimental set-up to investigate the pulling/peeling
effects associated with the debonding along the FRP-concrete interface, proposing a four-parameter analytical model to determine the strain distribution near the crack as well as the ultimate failure load. An energy-based tri-layer model is developed by Au and Büyükoztürk (2006) to characterize the debonding between the interface of FRP/epoxy or epoxy/concrete. In their model, the classical beam theory with the linear-elastic assumption was used to derive the interface fracture energy release rate, which allowed for the evaluation of the interface fracture toughness. The debonding failure along the FRP/concrete interface between two adjacent flexural cracks is studied in Chen et al. (2007), where explicit expressions based on a bond-slip model are provided to predict the ultimate load and the effective bond length. Different interfacial stress-based models were also developed by researchers and a review of their studies can be found in Smith and Teng (2002).

This chapter focuses on the numerical simulation of interfacial fracture. Although, for such purposes, the finite element method (FEM) and the boundary element method (BEM) (Miyazaki et al., 1993; Červenka et al., 1998; Ikeda and Sun, 2001) have been widely used, recently, the extended finite element method (XFEM) (Belytschko and Black, 1999; Moës et al., 1999) has been used due to its enhanced capability for representing kinematic property and its numerical robustness. The XFEM is a local partition of unity finite element method and has been used to model discontinuities. To model cracks in a homogeneous and isotropic material, Moës et al. (1999) introduced an XFEM interpolation enriched by the Heaviside function and linear elastic asymptotic crack-tip displacement fields of a homogeneous crack. Nagashima et al. (2003), in their attempt to model a bi-material crack, adopted, without changes, the enrichment functions used by Moës et al. (1999). Sukumar et al. (2004) proposed a more comprehensive crack-tip enrichment determined from the elastostatic near-tip stress and displacement fields of a bi-material interfacial
crack, choosing the crack-tip enrichment functions deliberately so that they can span
the asymptotic displacement fields of an interfacial crack along two distinct materials.
In this chapter, the Heaviside-only enrichment approach described in Chapter 2 is
employed to model interfacial cracks. The strategy for the selection of nodes that
need to be enriched is presented accordingly. Numerical examples are also presented
to illustrate the accuracy of the proposed approach.

4.2 XFEM Interfacial Fracture Simulation

Figure 4.1: Nodes to be enriched (circled) and partition of a crack-tip element (dotted
lines)

The present study follows the XFEM formulation proposed by Chen (2003) and
Zi and Belytschko (2003) employing only the Heaviside enrichment. To model a
crack on the interface between two distinct materials, the nodes on the crack only are
enriched. Since the interfacial crack overlaps the element edges, no element partition
is employed except for the crack-tip elements. Figure 4.1 shows the nodes to be enriched and the partition of a crack-tip element for a typical structured mesh.

The approach proposed herein is different from the ones followed by Nagashima et al. (2003) and Sukumar et al. (2004), who employed, albeit different, the crack-tip enrichment as well as the Heaviside enrichment. Accordingly, the nodal enrichment scheme in their methods differs from the one proposed here in that the nodes surrounding the crack tips are enriched with the crack-tip functions along with those on the crack enriched with the Heaviside function, as illustrated in Figure 4.2. It is noted that the approach employed herein is straightforward and easier to implement.

Figure 4.2: Nodes to be enriched (Nagashima et al., 2003; Sukumar et al., 2004): circled nodes are enriched with the Heaviside function whereas squared nodes with the crack-tip functions.

The proposed approach is also different from the method mentioned by Sukumar and Prévost (2003), who also studied the XFEM with only the Heaviside enrichment. In their implementation, if the crack is terminated between two nodes of an element edge, it would be modeled as if the crack were extended to the closest node ahead of the crack tip. As such, the position of the crack tip would be represented inaccurately (Figure 4.3). Although it is possible to adjust the mesh so that the crack tips coincide with nodes, or to use a finer mesh so that the actual crack is indistinguishable from
the modeled one, such measures would require additional computational burden and greater care. In the method proposed in this dissertation, the crack-tip element by Chen (2003) and Zi and Belytschko (2003) allows for an accurate representation of the crack tip location.

![Figure 4.3: Representation of a crack in Sukumar and Prévost (2003): the crack shown in (a) is modeled as if the crack were extended to the closest nodes as shown in (b).](image)

4.3 Stress Intensity Factors of a Bi-material Interfacial Crack and Its Evaluation

SIFs of an Interfacial Crack

Different from the homogeneous case, the stress intensity factors of an interfacial crack do not have the simple physical interpretation and the symmetric and skew-symmetric
loadings are intermixed in the definition of $K_I$ and $K_{II}$ (Rice and Sih, 1965). Two distinct SIF definitions have been proposed, both of which are defined through the complex stresses along the $x$-axis of the local crack-tip coordinates system around the interfacial crack tip (Figure 4.4). Erdogan (1965) defined the SIFs as

$$\left(\sigma_{yy} + i\sigma_{xy}\right)_{\theta=0} = \frac{K_1 + iK_{II}}{\sqrt{2\pi r}} \left(\frac{r}{l_k}\right)^{ik}, \quad (4.1)$$

whereas Hutchinson et al. (1987) gave the definition of the SIFs for an interfacial crack as

$$\left(\sigma_{yy} + i\sigma_{xy}\right)_{\theta=0} = \frac{K_1 + iK_{II}}{\sqrt{2\pi r}} r^{ik}, \quad (4.2)$$

where $K_I$ and $K_{II}$ are the SIFs of an interfacial crack for mode I and II, respectively, $i$ is the imaginary unit, $l_k$ is a reference length and $\epsilon$ reflects the material difference, which is given by

$$\epsilon = \ln \left(\frac{\frac{\kappa_1}{\mu_1} + \frac{1}{\mu_2}}{\frac{\kappa_2}{\mu_2} + \frac{1}{\mu_1}}\right), \quad (4.3)$$

Figure 4.4: Local crack-tip coordinates
where
\[ \kappa_i = \begin{cases} 
\frac{3 - \mu_i}{1 + \nu_i} & \text{(plane stress)} \\
3 - 4\nu_i & \text{(plane strain)} 
\end{cases} \] (4.4)

and \( \mu_i \) and \( \nu_i \) are the shear modulus and Poisson's ratio of material \( i, i = 1, 2 \). For consistency, the material of the upper part in a domain that consists of two distinct materials is referred to as material 1 and the lower part as material 2 (Figure 4.4). In this chapter, the definition of Erdogan (1965) is adopted.

Evaluation of SIFs of an Interfacial Crack

The domain integral method (Yau et al., 1980) is used to evaluate the stress intensity factors. In this method, the stress intensity factors \( K_I \) and \( K_{II} \) of a bi-material interfacial crack and interaction energy integral \( I \) are related to each other by

\[ I = 2\beta(K_I^{aux} + K_{II}^{aux}). \] (4.5)

As for a homogeneous material, two states of a cracked body are involved: one is the state under consideration and the other is the auxiliary state. In this implementation, the solution of elastostatic near-tip stress and displacement fields of a bi-material interface crack (Sun and Jih, 1987) is used for the auxiliary state. \( K_I^{aux} \) and \( K_{II}^{aux} \) are mode I and mode II stress intensity factors for the auxiliary states and \( \beta \) is given by

\[ \beta = \frac{\kappa_1 + 1 + \kappa_2 + 1}{\mu_1 + \mu_2}. \] (4.6)

The interaction integral \( I \) can be obtained using Equation (2.59) in Chapter 2.

As is done for a homogeneous material, by selecting a proper auxiliary state, we can compute \( K_I \) and \( K_{II} \) of the state under consideration. For instance, choosing an
auxiliary state such that $K_{I}^{\text{aux}} = 1$ and $K_{II}^{\text{aux}} = 0$ and denote the corresponding $I$ as $I_{K_{I}^{\text{aux}}=1}$, we have, from Equation (4.5),

$$K_{I} = \frac{I_{K_{I}^{\text{aux}}=1}}{2\beta}. \quad (4.7)$$

The corresponding auxiliary state can be computed by letting $K_{I}^{\text{aux}} = 1$ and $K_{II}^{\text{aux}} = 0$ in the near-tip stress and displacement fields given by Sun and Jih (1987). Similarly, by choosing $K_{I}^{\text{aux}} = 0$ and $K_{II}^{\text{aux}} = 1$, we obtain

$$K_{II} = \frac{I_{K_{II}^{\text{aux}}=1}}{2\beta}. \quad (4.8)$$

### 4.4 Numerical Examples

To verify the proposed approach, we consider an interfacial crack of length $2a$ in a rectangular plate subjected to a tensile load in the plane stress loading condition (Figure 4.5). The plate is $4W$ long and $2W$ wide and consists of two parts with material properties $E_{1}$, $\nu_{1}$ and $E_{2}$, $\nu_{2}$, where $E_{i}$ and $\nu_{i}$ are the Young's modulus and Poisson's ratio, respectively, for material $i$. Appropriate displacement boundary conditions are applied to eliminate the rigid body movements.

In the simulation, $W$ is set to 50 mm, the Young's modulus for lower material $E_{2}$ is 205.8 GPa whereas $E_{1}$ varies from 205.8 GPa to 20580.0 GPa so that the ratio of $E_{1}$ to $E_{2}$ is between 1 to 100. Both $\nu_{1}$ and $\nu_{2}$ are set to 0.3. Different crack lengths ($2a = 40, 60, 80$ mm) are considered. Structured meshes as shown in Figure 4.6 are used for the simulation.
Figure 4.5: Bi-material specimen with an interfacial crack subjected to a tensile load
Figure 4.6: Structured mesh for the bi-material fracture simulation: (a) entire mesh; (b) mesh in the vicinity of the crack tip
Table 4.1: SIFs for the interfacial crack (2a = 40mm)

<table>
<thead>
<tr>
<th>$E_1/E_2$</th>
<th>$K_I$</th>
<th>$K_{II}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>XFEM</td>
<td>reference</td>
</tr>
<tr>
<td>1</td>
<td>1.109</td>
<td>1.111</td>
</tr>
<tr>
<td>2</td>
<td>1.097</td>
<td>1.104</td>
</tr>
<tr>
<td>3</td>
<td>1.086</td>
<td>1.096</td>
</tr>
<tr>
<td>4</td>
<td>1.078</td>
<td>1.089</td>
</tr>
<tr>
<td>10</td>
<td>1.054</td>
<td>1.066</td>
</tr>
<tr>
<td>100</td>
<td>1.027</td>
<td>1.039</td>
</tr>
</tbody>
</table>

Table 4.2: SIFs for the interfacial crack (2a = 60mm)

<table>
<thead>
<tr>
<th>$E_1/E_2$</th>
<th>$K_I$</th>
<th>$K_{II}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>XFEM</td>
<td>reference</td>
</tr>
<tr>
<td>1</td>
<td>1.301</td>
<td>1.305</td>
</tr>
<tr>
<td>2</td>
<td>1.285</td>
<td>1.296</td>
</tr>
<tr>
<td>3</td>
<td>1.270</td>
<td>1.284</td>
</tr>
<tr>
<td>4</td>
<td>1.259</td>
<td>1.273</td>
</tr>
<tr>
<td>10</td>
<td>1.224</td>
<td>1.239</td>
</tr>
<tr>
<td>100</td>
<td>1.186</td>
<td>1.201</td>
</tr>
</tbody>
</table>

The stress intensity factors $K_I$ and $K_{II}$ are calculated using the domain integral method and compared with the reference solutions obtained by the boundary element method (Miyazaki et al., 1993).

Listed in Table 4.1, 4.2 and 4.3 are the stress intensity factors computed from the XFEM simulation and the reference solutions as well as the ratio between them when the crack length is 40, 60 and 80 mm, respectively. It can be seen that, in all cases, good accuracy is obtained throughout the range of $E_1/E_2$ considered.

Three meshes consisting of 10,000, 40,000 and 160,000 triangular elements are used to investigate the effects of the mesh density. The relative errors are evaluated
Table 4.3: SIFs for the interfacial crack ($2a = 80\text{mm}$)

<table>
<thead>
<tr>
<th>$E_1/E_2$</th>
<th>$K_1$</th>
<th>$K_\text{II}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>XFEM</td>
<td>reference</td>
</tr>
<tr>
<td>1</td>
<td>1.803</td>
<td>1.818</td>
</tr>
<tr>
<td>2</td>
<td>1.775</td>
<td>1.799</td>
</tr>
<tr>
<td>3</td>
<td>1.747</td>
<td>1.774</td>
</tr>
<tr>
<td>4</td>
<td>1.725</td>
<td>1.752</td>
</tr>
<tr>
<td>10</td>
<td>1.659</td>
<td>1.686</td>
</tr>
<tr>
<td>100</td>
<td>1.587</td>
<td>1.612</td>
</tr>
</tbody>
</table>

by

$$\text{Error} = \left| \frac{K_{\text{XFEM}} - K_{\text{ref}}}{K_{\text{ref}}} \right| \times 100\% \quad (4.9)$$

and shown in Figure 4.7 in the case of $K_1$ for a 40 mm crack. It is obvious that the computed stress intensity factors get closer to the reference values consistently when the mesh is refined. Similar conclusions can be made for the cracks with different lengths. This trend can also be observed for $K_\text{II}$ in the majority of the cases considered. However, a refinement does not necessarily lead to a greater accuracy when $E_1/E_2$ is relatively small. This may be caused by the fact that $K_\text{II}$, different from $K_\text{II}$, is small in those cases.
Figure 4.7: The influence of mesh density: Error in $K_I$ for the 40 mm crack
4.5 Concluding Remarks

An XFEM relying on only the Heaviside enrichment function has been applied to bi-material interfacial fracture simulation. A nodal enrichment scheme accommodating the enrichment function selected has been proposed. The stress intensity factors evaluated by the domain interaction integral are in good agreement with reference solutions, demonstrating that the proposed approach can be used for modeling interfacial fracture.
Chapter 5
Conclusion

In this dissertation, the extended finite element method combined with the level set method has been used for the simulation of fracture near or on material interfaces.

In the near-interfacial fracture simulation, the XFEM is applied, for the first time, to the crack propagation modeling in a layered ceramic/metal/ceramic structure. The study shows that the simulated crack trajectories based on the maximum hoop stress criterion are fairly consistent with the one observed in the experiments, confirming the influence of the Dundurs' parameters on near-interfacial crack propagation.

In the interfacial fracture simulation, an XFEM enrichment approach different from the ones used in other researches is proposed. Its application to the simulation of the crack on bi-material interfaces illustrates that the proposed enrichment scheme based on Heaviside functions only is simple and effective.

The modeling capability demonstrated in this study indicates that the XFEM is an effective numerical simulation tool for modeling arbitrary crack growth and may adequately tackle more involved problems in near-interfacial and interfacial fracture including debonding in FRP-strengthened concrete structures. It is expected that detailed information from simulation results can provide an insight into the debonding mechanism of the system, leading potentially to an improved design approach to avoid
debonding failure. For this purpose, it may be necessary to introduce other features into the simulation. For example, it is of interest to include a cohesive crack model taking account of the quasi-brittleness of the concrete material since the debonding may occur in the concrete substrate. With the presence of reinforcement in concrete, the interaction between the reinforcement and concrete should also be modeled, especially if the crack propagates into the area surrounding the reinforcement. It may also be desirable to further extend the implementation to three dimensions in the case of problems that involve more complicated geometries and three-dimensional effects. In the Copper/Glass sandwich experiments, cracks starting from different distance from the interface intersect with each other, which may indicate that the materials used are not as homogeneous as they are in the simulation. This could be an even more important concern when concrete material is considered and it would be helpful to have a reliable numerical technique employed to reflect the material inhomogeneity.
Appendix A

Near-tip Fields in an Elastic Isotropic Material

A.1 Mode I

Displacement

\[ u_1 = \frac{K_1}{2\mu} \sqrt{\frac{\tau}{2\pi}} \cos\left(\frac{\theta}{2}\right) \left[ \kappa - 1 + 2\sin^2\left(\frac{\theta}{2}\right) \right] \]  
(A.1)

\[ u_2 = \frac{K_1}{2\mu} \sqrt{\frac{\tau}{2\pi}} \sin\left(\frac{\theta}{2}\right) \left[ \kappa + 1 - 2\cos^2\left(\frac{\theta}{2}\right) \right] \]  
(A.2)

Stress

\[ \sigma_{11} = \frac{K_1}{\sqrt{2\pi\tau}} \cos\left(\frac{\theta}{2}\right) \left[ 1 - \sin\left(\frac{\theta}{2}\right) \sin\left(\frac{3\theta}{2}\right) \right] \]  
(A.3)

\[ \sigma_{22} = \frac{K_1}{\sqrt{2\pi\tau}} \cos\left(\frac{\theta}{2}\right) \left[ 1 + \sin\left(\frac{\theta}{2}\right) \sin\left(\frac{3\theta}{2}\right) \right] \]  
(A.4)

\[ \sigma_{12} = \frac{K_1}{\sqrt{2\pi\tau}} \cos\left(\frac{\theta}{2}\right) \sin\left(\frac{\theta}{2}\right) \cos\left(\frac{3\theta}{2}\right) \]  
(A.5)
Derivative of displacement

\[
\frac{\partial u_1}{\partial x_1} = \frac{K_1}{2\mu} \sqrt{\frac{1}{2\pi r}} \cos \left( \frac{\theta}{2} \right) \left\{ \frac{1}{2} \left[ \kappa - 1 + 2\sin^2 \left( \frac{\theta}{2} \right) \right] - \sin^2 \theta \right\} \tag{A.6}
\]

\[
\frac{\partial u_2}{\partial x_1} = \frac{K_1}{2\mu} \sqrt{\frac{1}{2\pi r}} \sin \left( \frac{\theta}{2} \right) \left\{ -\frac{1}{2} \left[ \kappa + 1 - 2\cos^2 \left( \frac{\theta}{2} \right) \right] - \sin^2 \theta \right\} \tag{A.7}
\]

In all equations, \( \mu \) is the shear modulus. \( \kappa \) is calculated by

\[
\kappa = \begin{cases} 
3 - 4\nu & \text{(plane strain)} \\
3 - \frac{\nu}{1 + \nu} & \text{(plane stress)}
\end{cases} \tag{A.8}
\]

**A.2 Mode II**

Displacement

\[
u_1 = \frac{K_\Pi}{2\mu} \sqrt{\frac{\tau}{2\pi r}} \sin \left( \frac{\theta}{2} \right) \left[ \kappa + 1 + 2\cos^2 \left( \frac{\theta}{2} \right) \right] \tag{A.9}
\]

\[
u_2 = -\frac{K_\Pi}{2\mu} \sqrt{\frac{\tau}{2\pi r}} \cos \left( \frac{\theta}{2} \right) \left[ \kappa - 1 - 2\sin^2 \left( \frac{\theta}{2} \right) \right] \tag{A.10}
\]

Stress

\[
\sigma_{11} = -\frac{K_\Pi}{\sqrt{2\pi r}} \sin \left( \frac{\theta}{2} \right) \left[ 2 + \cos \left( \frac{\theta}{2} \right) \cos \left( \frac{3\theta}{2} \right) \right] \tag{A.11}
\]

\[
\sigma_{22} = \frac{K_\Pi}{\sqrt{2\pi r}} \sin \left( \frac{\theta}{2} \right) \cos \left( \frac{\theta}{2} \right) \cos \left( \frac{3\theta}{2} \right) \tag{A.12}
\]

\[
\sigma_{12} = \frac{K_\Pi}{\sqrt{2\pi r}} \cos \left( \frac{\theta}{2} \right) \left[ 1 - \sin \left( \frac{\theta}{2} \right) \sin \left( \frac{3\theta}{2} \right) \right] \tag{A.13}
\]
Derivative of displacement

\[
\frac{\partial u_1}{\partial x_1} = \frac{K_{II}}{2\mu} \sqrt{\frac{1}{2\pi r}} \sin \left( \frac{\theta}{2} \right) \left\{ - \frac{1}{2} \left[ \kappa + 1 + 2 \cos^2 \left( \frac{\theta}{2} \right) \right] + \sin^2 \theta \right\} \tag{A.14}
\]

\[
\frac{\partial u_1}{\partial x_1} = \frac{K_{II}}{2\mu} \sqrt{\frac{1}{2\pi r}} \cos \left( \frac{\theta}{2} \right) \left\{ - \frac{1}{2} \left[ \kappa - 1 - 2 \sin^2 \left( \frac{\theta}{2} \right) \right] - \sin^2 \theta \right\} \tag{A.15}
\]

As in mode I, \( \mu \) is the shear modulus. \( \kappa \) can be calculated by Equation (A.8).
Appendix B

Elastostatic Near-tip Fields of a Bi-material Interfacial Crack

Displacement

\[
(u_1)_j = M_2 \left\{ K_1 \left[ \kappa_j \omega_j h_{11} - \frac{1}{\omega_j} h_{12} + \omega_j h_{13} \right] + K_{II} \left[ \kappa_j \omega_j h_{21} - \frac{1}{\omega_j} h_{22} + \omega_j h_{23} \right] \right\} 
\]

\[
(u_2)_j = M_2 \left\{ K_1 \left[ \kappa_j \omega_j h_{21} - \frac{1}{\omega_j} h_{22} - \omega_j h_{23} \right] + K_{II} \left[ -\kappa_j \omega_j h_{11} + \frac{1}{\omega_j} h_{12} + \omega_j h_{13} \right] \right\} 
\]

Stress

\[
(\sigma_{11})_j = M_1 \left\{ K_1 \left[ \omega_j f_{11} - \frac{1}{\omega_j} \cos(\theta - \Theta) \right] + K_{II} \left[ \omega_j f_{II} - \frac{1}{\omega_j} \sin(\theta - \Theta) \right] \right\} 
\]

\[
(\sigma_{22})_j = M_1 \left\{ K_1 \left[ \omega_j f_{22} + \frac{1}{\omega_j} \cos(\theta - \Theta) \right] + K_{II} \left[ \omega_j f_{II} + \frac{1}{\omega_j} \sin(\theta - \Theta) \right] \right\} 
\]

\[
(\sigma_{12})_j = M_1 \left\{ K_1 \left[ \omega_j f_{12} - \frac{1}{\omega_j} \sin(\theta - \Theta) \right] + K_{II} \left[ \omega_j f_{II} + \frac{1}{\omega_j} \cos(\theta - \Theta) \right] \right\} 
\]

where

\[
M_1 = \frac{1}{2\sqrt{2\pi r \cosh(\pi \alpha)}} 
\]
\[ M_2 = \frac{\sqrt{2\pi r}}{4\pi \mu_j \cosh (\pi \alpha)} \]  
(B.7)

\[ \alpha = \frac{1}{2\pi} \ln \left( \frac{\mu_1 + 1}{\mu_2 + 1} \right) \]  
(B.8)

\[ \Theta = \alpha \ln \left( \frac{r}{2\alpha} \right) + \frac{\theta}{2} \]  
(B.9)

\[ \kappa_j = \begin{cases} 
3 - 4\nu_j & \text{(plane strain)} \\
3 - \nu_j & \frac{1}{1 + \nu_j} & \text{(plane stress)} 
\end{cases}, \quad (j = 1, 2) \]  
(B.10)

\[ \mu_j = \text{shear modulus} \]  
(B.11)

\[ \nu_j = \text{Poisson's ratio} \]  
(B.12)

\[ \omega_1 = e^{-\alpha(\pi - \theta)} \]  
(B.13)

\[ \omega_2 = e^{\alpha(\pi + \theta)} \]  
(B.14)

\[ f_{11}^I = 3 \cos \Theta + 2\alpha \sin \theta \cos(\theta + \Theta) - \sin \theta \sin(\theta + \Theta) \]  
(B.15)

\[ f_{11}^H = -3 \cos \Theta - 2\alpha \sin \theta \cos(\theta + \Theta) - \sin \theta \cos(\theta + \Theta) \]  
(B.16)

\[ f_{22}^I = \cos \Theta - 2\alpha \sin \theta \cos(\theta + \Theta) + \sin \theta \sin(\theta + \Theta) \]  
(B.17)

\[ f_{22}^H = -\sin \Theta + 2\alpha \sin \theta \sin(\theta + \Theta) + \sin \theta \cos(\theta + \Theta) \]  
(B.18)

\[ f_{12}^I = 2\alpha \sin \theta \sin(\theta + \Theta) + \cos \theta \sin(\theta + \Theta) \]  
(B.19)

\[ f_{12}^H = 2\alpha \sin \theta \cos(\theta + \Theta) + \cos \theta \cos(\theta + \Theta) \]  
(B.20)

\[ h_{11} = [\cos(\theta - \Theta) - 2\alpha \sin(\theta - \Theta)]/(1 + 4\alpha^2) \]  
(B.21)

\[ h_{12} = [\cos \Theta + 2\alpha \sin \Theta]/(1 + 4\alpha^2) \]  
(B.22)
\[ h_{13} = \sin \theta \sin \Theta \]  
(B.23)

\[ h_{21} = \left[ \sin(\theta - \Theta) + 2\alpha \cos(\theta - \Theta) \right]/(1 + 4\alpha^2) \]  
(B.24)

\[ h_{12} = [-\sin \Theta + 2\alpha \cos \Theta]/(1 + 4\alpha^2) \]  
(B.25)

\[ h_{23} = \sin \theta \cos \Theta \]  
(B.26)

Derivative of displacement

\[ \frac{\partial (u_1)}{\partial x_1} = H \left\{ K_1 \left[ g_{11} \omega_j - \frac{1}{\omega_j} \cos(\theta - \Theta) \right] + K_{11} \left[ g_{11}^H \omega_j + \frac{1}{\omega_j} \sin(\theta - \Theta) \right] \right\} \]  
(B.27)

\[ \frac{\partial (u_2)}{\partial x_2} = H \left\{ K_2 \left[ g_{21} \omega_j - \frac{1}{\omega_j} \sin(\theta - \Theta) \right] + K_{21} \left[ g_{21}^H \omega_j + \frac{1}{\omega_j} \cos(\theta - \Theta) \right] \right\} \]  
(B.28)

\[ \frac{\partial (u_2)}{\partial x_1} = H \left\{ K_1 \left[ g_{21}^H \omega_j - \frac{1}{\omega_j} \sin(\theta - \Theta) \right] + K_{12} \left[ g_{12}^H \omega_j + \frac{1}{\omega_j} \cos(\theta - \Theta) \right] \right\} \]  
(B.29)

\[ \frac{\partial (u_2)}{\partial x_2} = H \left\{ K_1 \left[ g_{21} \omega_j + \frac{1}{\omega_j} \cos(\theta - \Theta) \right] + K_{21} \left[ g_{21}^H \omega_j + \frac{1}{\omega_j} \sin(\theta - \Theta) \right] \right\} \]  
(B.30)

where

\[ H = \frac{1}{4\mu_j \sqrt{(2\pi r)} \cosh(\alpha r)} \]  
(B.31)

\[ \Theta = \alpha \ln \left( \frac{r}{2\alpha} \right) + \frac{\theta}{2} \]  
(B.32)

\[ \omega_1 = e^{-\alpha(\pi - \theta)} \]  
(B.33)

\[ \omega_2 = e^{\alpha(\pi + \theta)} \]  
(B.34)

\[ g_{11}^I = \kappa_j \cos \Theta + 2\alpha \sin \theta \cos(\theta + \Theta) - \sin \theta \sin(\theta + \Theta) \]  
(B.35)

\[ g_{11}^H = -\kappa_j \sin \Theta - 2\alpha \sin \theta \sin(\theta + \Theta) - \sin \theta \cos(\theta + \Theta) \]  
(B.36)

\[ g_{12} = \kappa_j \sin \Theta + 2\alpha \sin \theta \sin(\theta + \Theta) - \sin \theta \cos(\theta + \Theta) + 2 \cos \theta \sin(\theta + \Theta) \]  
(B.37)
\[ g_{12}^{\Pi} = \kappa_j \cos \Theta + 2\alpha \sin \theta \cos(\theta - \Theta) + \sin \theta \sin(\theta + \Theta) + 2 \cos \theta \cos(\theta + \Theta) \quad (B.38) \]

\[ g_{21}^I = -\kappa_j \sin \Theta + 2\alpha \sin \theta \sin(\theta + \Theta) + \sin \theta \cos(\theta + \Theta) \quad (B.39) \]

\[ g_{21}^{\Pi} = -\kappa_j \cos \Theta + 2\alpha \sin \theta \cos(\theta + \Theta) - \sin \theta \sin(\theta + \Theta) \quad (B.40) \]

\[ g_{22}^I = \kappa_j \cos \Theta - 2\alpha \sin \theta \cos(\theta + \Theta) - \sin \theta \sin(\theta + \Theta) - 2 \cos \theta \cos(\theta + \Theta) \quad (B.41) \]

\[ g_{22}^{\Pi} = -\kappa_j \sin \Theta + 2\alpha \sin \theta \sin(\theta + \Theta) - \sin \theta \cos(\theta + \Theta) + 2 \cos \theta \sin(\theta + \Theta) \quad (B.42) \]
References


