THE RESPONSE TO TIDAL FLUCTUATIONS OF A LEAKY AQUIFER SYSTEM

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ANALOG SIMULATION OF TIDAL EFFECTS ON GROUND WATER AQUIFERS

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ABSTRACT

A system of two isotropic and homogeneous infinite aquifers, which are separated by an aquitard and subject to tidal fluctuations along their coastal boundary, has been analyzed and a mathematical model has been developed for the response of this system to tidal changes. The mathematical model consists of equations for the amplitude and phase of the response of both aquifers to a periodic tide. A computer program using the IBM 360 has been written for the evaluation of these equations. Both the mathematical model and the program have been verified by an electric analog model constructed for that purpose.

The mathematical model was evaluated for an aquifer system where both aquifers have the same transmissability (and therefore the same leakage factor) but where the storage of the upper aquifer was 100 times that of the lower aquifer. Tidal periods of 0.5, 1, and 14 days were used. The results indicate that deviations from a response corresponding to the no-leakage case could be from 50 to 100 percent or more for both the amplitude and the phase angle of either aquifer. Also, such deviations were produced by a relatively moderate amount of leakage, i.e., $1/B^2 \geq 0.154 \times 10^{-4}\ ft^{-2}$.

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INTRODUCTION

This report presents a continuation of the general study of the response of coastal aquifers to tidal fluctuations. It deals specifically with the response of an aquifer system consisting of two horizontal and semi-infinite aquifers separated by a semipermeable layer, or aquitard. Both aquifers are of constant thickness, homogeneous and isotropic, and only vertical movement of water through the aquitard is assumed.

Even though this model is highly idealized, it should yield reasonable estimates of the deviation of the amplitude and phase relations from those of similar aquifers but without the presence of leakage. It also offers a means of studying the response of a leaky aquifer under the variation of any of the several parameters involved, *i.e.*, storage, permeability, leakage factor, etc. The model, together with the two non-homogeneous aquifer models developed previously (Williams and Liu, 1971), will help provide a general insight into the interpretation and the use of tidal data for the determination of real aquifer properties which is the ultimate goal of this research.

The Mathematical Model

The two model aquifers will be considered as one-dimensional, homogeneous, and isotropic and their respective constants and variables distinguished between by the use of subscripts: one (1) for the lower aquifer which is necessarily semi-confined and two (2) for the upper aquifer which may be either confined from above or phreatic. The coastline is located at distance L to the right of the origin of coordinates which is in the interior of the aquifer system. A glossary of the terms used and a sketch defining the aquifer system are presented in Appendix A.

A combination of the conservation of mass principle and Darcy's Law together with the assumptions of Dupuit for the two aquifers and of strictly vertical flow in the aquitard leads to the following coupled differential equations

$$\frac{\partial^2 h_1}{\partial x^2} + \frac{h_2 - h_1}{B_1{}^2} = \alpha_1 \frac{\partial h_1}{\partial t} \tag{1a}$$

and

$$\frac{\partial^2 h_2}{\partial x^2} + \frac{h_1 - h_2}{B_2^{\,2}} = \alpha_2 \frac{\partial h_2}{\partial t} \tag{1b}$$

where $\alpha_j = (S/T)_j$ (or $\varepsilon'/K\bar{z}$ for a phreatic aquifer) is the ratio of storage to transmissability, $B^2_j = K_j b_j b'/K'$ is the leakage factor, $h_j = h_j\ (x,t)$ is the piezometric head, and x and t are the independent space and time variables.

Since tidal fluctuations are periodic both h_1 and h_2 may be written as the real part of

$$h_j\ (x,t) = R\ (\zeta_j(x)e^{i\sigma t}), \tag{2}$$

where σ is 2π divided by the tidal period and $\zeta_j(x)$ is the space dependent amplitude of the fluctuation in $h_j(x,t)$.

Elimination of t in equations (1a) and (1b) by use of equation (2) yields the following pair of ordinary differential equations,

$$\zeta_1'' - P_1\zeta_1 = -\frac{\zeta_2}{B_1^{\,2}} \tag{3a}$$

and

$$\zeta_2'' - P_2\zeta_2 = -\frac{\zeta_1}{B_2^{\,2}}, \tag{3b}$$

where

$$P_j = \frac{1}{B_j^{\,2}} + i\sigma\alpha_j \tag{4}$$

Equations (3a) and (3b) may be further reduced by elimination of ζ_2 to give the following fourth order equation:

$$\zeta_1'''' - (P_1 + P_2)\ \zeta_1'' + \left(P_1P_2 - \frac{1}{B_1^{\,2}B_2^{\,2}}\right)\zeta_1 = 0 \tag{5}$$

Equation (5) has a fourth order characteristic equation whose roots are

$$m_1 = p_+ + iq_+ = \rho_+^{\,1/2}\left(\cos\frac{\beta_+}{2} + i\ \sin\frac{\beta_+}{2}\right)$$

$$m_2 = -\ (p_+ + iq_+) = -\ \rho_+^{\,1/2}\left(\cos\frac{\beta_+}{2} + i\ \sin\frac{\beta_+}{2}\right)$$

$$m_3 = p_- + iq_- = \rho_-^{1/2} \left(\cos\frac{\beta_-}{2} + i \sin\frac{\beta_-}{2} \right)$$

$$m_4 = -(p_- + iq_-) = -\rho_-^{1/2} \left(\cos\frac{\beta_-}{2} + i \sin\frac{\beta_-}{2} \right) \qquad (6)$$

where

$$\rho_\pm = 1/2 \quad [\delta_1^2 + \delta_2^2 \pm 2\delta_1\delta_2 \cos(\beta_1 - \beta_2)]^{1/2} ;$$

$$\beta_\pm = \tan^{-1} \left[\frac{\delta_1 \sin\beta_1 \pm \delta_2 \sin\beta_2}{\delta_1 \cos\beta_1 \pm \delta_2 \cos\beta_2} \right] . \qquad (7)$$

and

$$\delta_1 e^{i\beta_1} = P_1 + P_2 ;$$

$$\delta_2 e^{i\beta_2} = \left[(P_1 - P_2)^2 + \frac{4}{B_1^2 B_2^2} \right]^{1/2} . \qquad (8)$$

Hence, the solution to equation (5) can be written as

$$\zeta_1(x) = A_1 e^{m_1 x} + B_1 e^{m_2 x} + C_1 e^{m_3 x} + D_1 e^{m_4 x} \qquad (9)$$

For aquifers which extend infinitely far in the negative x-direction, the constants of integration B_1 and D_1 must be zero. With $\zeta_1(x)$ determined, $\zeta_2(x)$ is also determined by equation (3a).

The boundary condition at $x = L$ requires that the fluctuation in either aquifer coincide with the tide. Thus,

$$\zeta_1(L) = A_1 e^{m_1 L} + C_1 e^{m_3 L} = -i\zeta_0 \qquad (10a)$$

and

$$\zeta_2(L) = B_1^2 \left[(P_1 - m_1^2) A_1 e^{m_1 L} + (P_1 - m_3^2) C_1 e^{m_3 L} \right] = -i\zeta_0 \qquad (10b)$$

where ζ_0 is the amplitude of the tide at the coastline. The equations (10a) and (10b) may be solved for the remaining two constants of integration, *i.e.*,

$$A_1 = A_{1r} + iA_{1i} = \frac{-i\zeta_0}{m_1^2 - m_3^2} (i\sigma\alpha_1 - m_3^2) e^{-m_1 L}$$

$$= \frac{-i\zeta_0}{m_1^2 - m_3^2} A_1 e^{-m_1 L} \qquad (11a)$$

and

$$C_1 = C_{1r} + iC_{1i} = \frac{i\zeta_0}{m_1^2-m_3^2}\ (i\sigma\alpha_1-m_1^2)\ e^{-m_3L} = \frac{i\zeta_0}{m_1^2-m_3^2}\ C_1 e^{-m_3L} \tag{11b}$$

where $A_1 = A_{1r} + i A_{1i}$ and $C_1 = C_{1r} + i C_{1i}$.

Thus, for the aquifer of region 1, the amplitude function becomes

$$\zeta_1\ (x) = \frac{-i\zeta_0}{m_1^2-m_3^2}\left[A_1 e^{m_1(x-L)} - C_1 e^{m_3(x-L)}\right] \tag{12}$$

Finally, substituting equation (12) into equation (2) the expression for the piezometric head in the lower aquifer becomes

$$h_1\ (x,t) = \zeta_0\ \rho_1\ (x)\ \sin(\sigma t + \theta_{p_1}) \tag{13a}$$

where

$$\rho_1\ (x) = \left[\frac{R_1^2(x) + I_1^2(x)}{R_1^2(L) + I_1^2(L)}\right]^{1/2} \tag{13b}$$

and

$$\theta p_1 = \tan^{-1}\left[\frac{R_1(L)I_1(x) - R_1(x)I_1(L)}{R_1(L)R_1(x) + I_1(L)I_1(x)}\right] \tag{13c}$$

with

$$R_1(x) = \left(A_{1r}\cos q_+(x-L) - A_{1i}\sin q_+(x-L)\right) e^{p_+(x-L)}$$

$$+ \left(C_{1r}\cos q_-(x-L) - C_{1i}\sin q_-(x-L)\right) e^{p_-(x-L)} \tag{13d}$$

$$I_1(x) = \left(A_{1r}\sin q_+(x-L) + A_{1i}\cos q_+(x-L)\right) e^{p_+(x-L)}$$

$$+ \left(C_{1r}\sin q_-(x-L) + C_{1i}\cos q_-(x-L)\right) e^{p_-(x-L)} \tag{13e}$$

The amplitude function for piezometric head in the upper aquifer, or region 2, follows from equation (3a), *i.e.*,

$$\zeta_2\ (x) = B_1^2[(P_1-m_1^2)\ A_1 e^{m_1x} + (P_1-m_3^2)\ C_1 e^{m_3x}\ ]$$

which becomes

$$\zeta_2\ (x) = \frac{-i\zeta_0}{m_1^2-m_3^2}\ B_1^2\left[A_2 e^{m_1(x-L)} - C_2 e^{m_3(x-L)}\right] \tag{14}$$

where

$$A_2 = A_{2r} + i\ A_{2i} = (P_1-m_1^2)\ A_1$$

$$C_2 = C_{2r} + i\ C_{2i} = (P_1-m_3^2)\ C_1$$

on application of equations (11a) and (11b). Substitution of equation (14) into equation (3a) and the substitution of that result into equation (2) gives the final expression:

$$h_2(x,t) = \zeta_0 \rho_2(x) \sin(\sigma t + \theta_{\rho_2}) \tag{15a}$$

where*

$$\rho_2(x) = \zeta_0 B_1^2 \left[\frac{R_2^2(x) + I_2^2(x)}{R_1^2(L) + I_1^2(L)}\right]^{1/2} \tag{15b}$$

and

$$\theta_{p_2} = \tan^{-1}\left[\frac{I_2(x)\ R_1(L) - R_2(x)\ I_1(L)}{R_2(x)\ R_1(L) + I_2(x)\ I_1(L)}\right] \tag{15c}$$

with

$$R_2(x) = (A_{2r} \cos q_+(x-L) - A_{2i} \sin q_+(x-L))\ e^{p_+(x-L)}$$
$$+ (C_{2r} \cos q_-(x-L) - C_{2i} \sin q_-(x-L))\ e^{p_-(x-L)} \tag{15d}$$

$$I_2(x) = (A_{2r} \sin q_+(x-L) + A_{2i} \cos q_+(x-L))\ e^{p_+(x-L)}$$
$$+ (C_{2r} \sin q_-(x-L) + C_{2i} \cos q_-(x-L))\ e^{p_-(x-L)} \tag{15e}$$

Two special cases of the above equations, *i.e.*, when the aquitard becomes an aquiclude and when both aquifers are identical, are considered in Appendix B.

THE ELECTRIC ANALOG MODEL

The analogous electrical circuit for the leaky aquifer consists of a parallel plate capacitor where one of the plates acts as a conductor and a current is permitted to "leak" into this conducting plate from some external energy source. The differential equation representing this situation has been derived by Karplus (1958) and is

$$\frac{\partial^2 V}{\partial x^2} + \frac{R}{R_\ell} \Delta V = RC \frac{\partial V}{\partial t_e} \tag{16}$$

*Note that $R_2(L) = R_1(L)/B_1^2$ and that $I_2(L) = I_1(L)/B_1^2$

where V is the voltage, R is the resistance per unit of length of the capacitor plate, R_ℓ is the resistance of the media through which the leakage current passes per unit of length along the capacitor plate and ΔV is the voltage with respect to the conducting capacitor plate which drives the leakage current.

Equation (1a) or (1b) can be transformed into an equation similar to (16) using the scale factors defined by Walton and Prickett (1963), *i.e.*,

$$\begin{aligned} q(\mathrm{ft}^3) &= K_1 \mathcal{Q} \text{ (coulombs)} \\ h(\mathrm{ft}) &= K_2 V \text{ (volts)} \\ Q(\mathrm{ft}^3/\mathrm{sec}) &= K_3 i \text{ (amperes)} \\ t(\mathrm{sec}) &= K_4 t_e \text{ (sec)} \end{aligned} \tag{17}$$

where the similarity between Darcy's Law and Ohm's Law and between $q = Qt$ and $\mathcal{Q} = it_e$ requires that

$$K_1 = K_3 K_4 \tag{18a}$$

and

$$RT = K_3/K_2 \tag{18b}$$

respectively. If the unit of length in the aquifer is "a" feet, then by using equation (17), equation (1a) or (1b) can be written as

$$\frac{\partial^2 V}{\partial x^2} + \frac{a^2}{B_1{}^2}(V_0 - V) = \frac{a^2\alpha}{K_4}\frac{\partial V}{\partial t_e}$$

A comparison of this transformed equation with equation (16) defines two more compatibility relations:

$$K_4 = \frac{a^2\alpha}{RC} \tag{18c}$$

and

$$a^2 = B_1{}^2 \frac{R}{R_l}$$

or

$$a = \frac{RT}{R_l(aK')}\, b' = \frac{RT}{(RT)_l}\, b' = b' \tag{18d}$$

Equation (18d) follows from the fact that the product $RT = K_3/K_2$ should remain the same throughout the aquifer system. Hence, in a leaky aquifer the characteristic length "a" is no longer arbitrary but equal to the thickness of the aquitard. The three remaining compatibility equations are unchanged.

To apply these relations, the aquifer system is first defined; then b' is known and "a" is fixed. R and C can be selected for convenience for the lower (or the upper) aquifer and K_4 determined. With K_2 as an arbitrary factor, K_3 and K_1 can then be found by application of equations (18b) and (18a), respectively. Values of R and C for the upper (or the lower) aquifer are given by the application of equations (18b) and (18c) and R_l is determined from equation (18b).

The electric analog model consisted of two arrays of resistors and capacitors and corresponding nodal points in each array were connected by a variable resistor. The circuit diagram and the electrical sizes of the components used are shown in Figure 1. Since the system involved two aquifers of infinite extent, a sufficient number of components had to be incorporated into the analog to produce an undetectable response at the circuit boundary to any signal originating at the coastline. This amounted to extending the circuit to between one and two penetration lengths of the tide, as recommended by Williams and Liu (1971). "Lumped" components were used to simulate the interior two-thirds of the modeled portion of the aquifer system.

EXPERIMENTAL PROCEDURE

The simulated aquifer system was defined by the following values for the pertinent parameters: L = 720 ft, K_1 = 13.3 ft/day, K_2 = 26.6 ft/day, S_1 = 0.002, S_2 = 0.2, b_1 = 100 ft, b_2 = 50 ft, b' = 36 ft, and $B_1^2 = B_2^2 = B^2 = 6.48 \times 10^4$ ft^2 (*i.e.*, K' = 0.739 ft/day). Values of 1000 ohms and 0.01 microfarads were selected for the resistance and capacitance, respectively, for region 1. Consequently, 1000 ohms and 1.0 microfarads were the values for the resistance and capacitance, respectively, determined for region 2. Thus, the time scale factor K_4 from equation (18b) is K_4 = 195 day/sec. Finally, it should be noted that $B^2 = 6.48 \times 10^4$ ft^2 corresponds to R_l = 50 R = 50,000 ohms and the variable resistors in the analog model were set to this value.

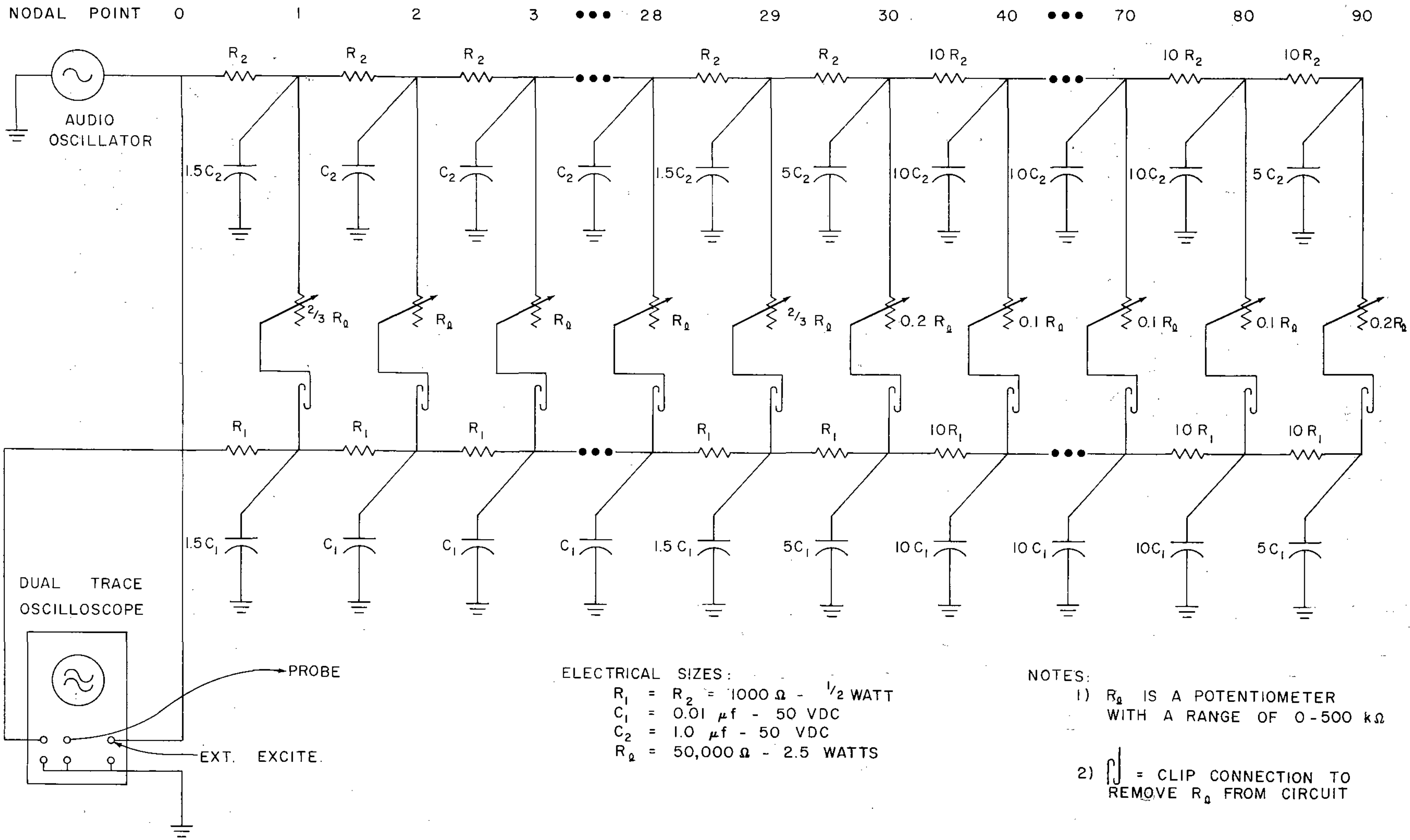


FIGURE 1. CIRCUIT DIAGRAM FOR LEAKY AQUIFER SYSTEM.

Once the variable resistors were adjusted and the harmonic oscillator set to give the desired frequency, the remaining step was to place the oscilloscope probe at each of the nodal points and read and record the amplitude and phase of the response off the oscilloscope screen.

THE RESULTS

The results from both the electric analog and the mathematical model are presented in graphs of the amplitude and the phase angle as functions of the dimensionless position x/L. The amplitudes have been normalized with respect to the tidal amplitude at the coast and the phase angles represent the time required for a particular tidal phase to be observed at a given position x/L.

Figures 2 and 3 show a comparison of the outputs of the electric analog and the mathematical models. The solid curves represent the mathematical model and the individual data points represent the electric analog model. In these two figures, results for several "tidal-periods" are presented. Two of these periods, *i.e.*, 6 hours and 48 hours are not realistic in terms of the components of a real tide, but have been included only to determine the response of the analog model to a set of input frequencies. The abscissa in these two figures includes only the range $0 \leq x/L \leq 1$.

Figures 4 through 9 give results from the mathematical model which include the amplitude and phase angle variations with position for the following values of B^2: 6.48×10^6, 6.48×10^5, 6.48×10^4, $6.48 \times 10^3 ft^2$, and infinity. The limiting case of $B^2 = 0$ has been evaluated by the electric analog model for periods of 0.5 and 1 day and the results included in Figures 4 through 7. (Since the value of $B^2 = 0$ rendered the computer program inoperable, only the electric analog results are available for this case.) Three tidal periods are included: 0.5 day in Figures 4 and 5, 1 day in Figures 6 and 7, and 14 days in Figures 8 and 9. The range of x/L has been doubled for this set of graphs, *i.e.*, $-1 \leq x/L \leq 1$.

The same aquifer system was considered for all analyses and the values for its descriptive parameters are those listed in the previous section.

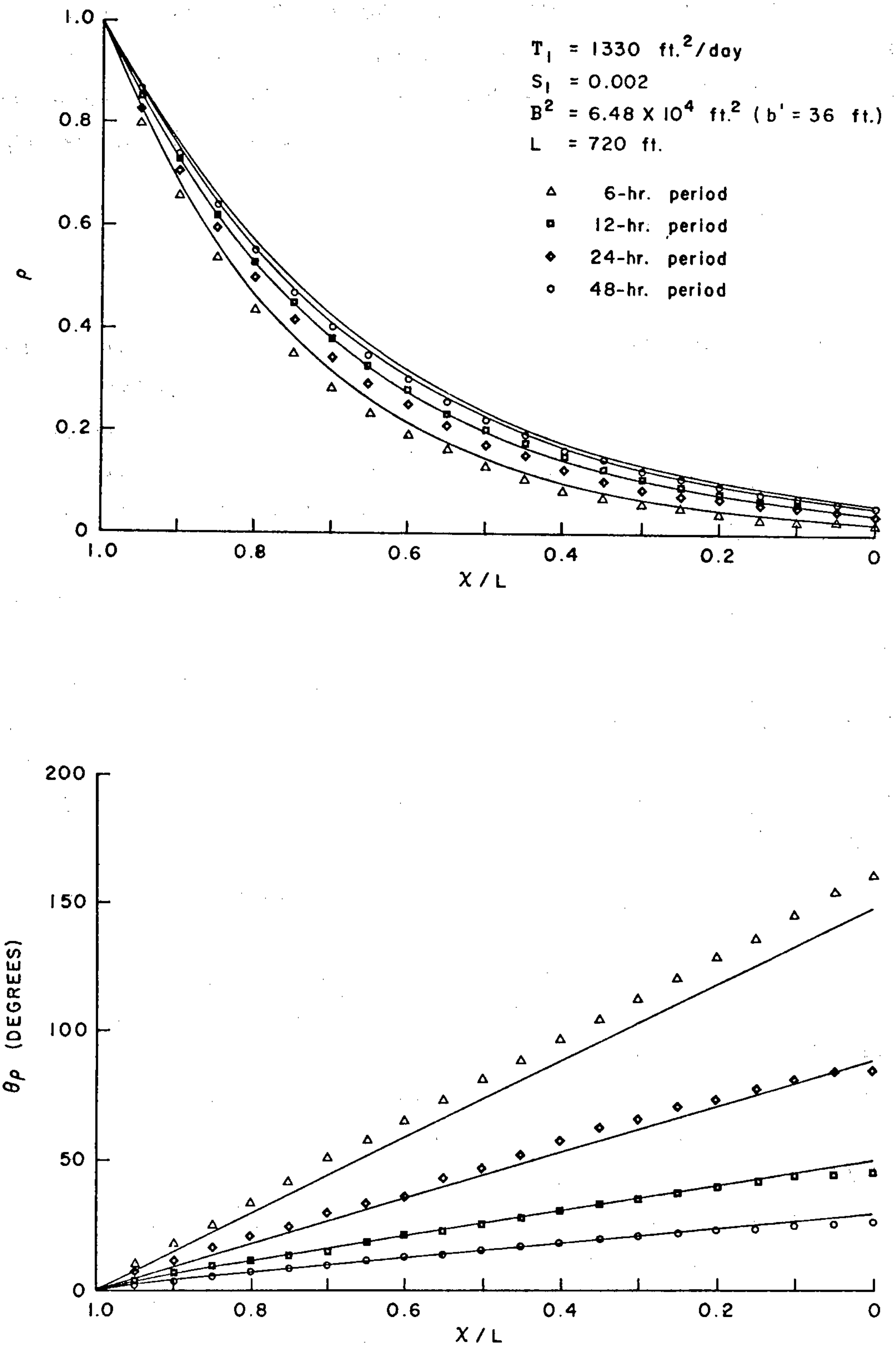
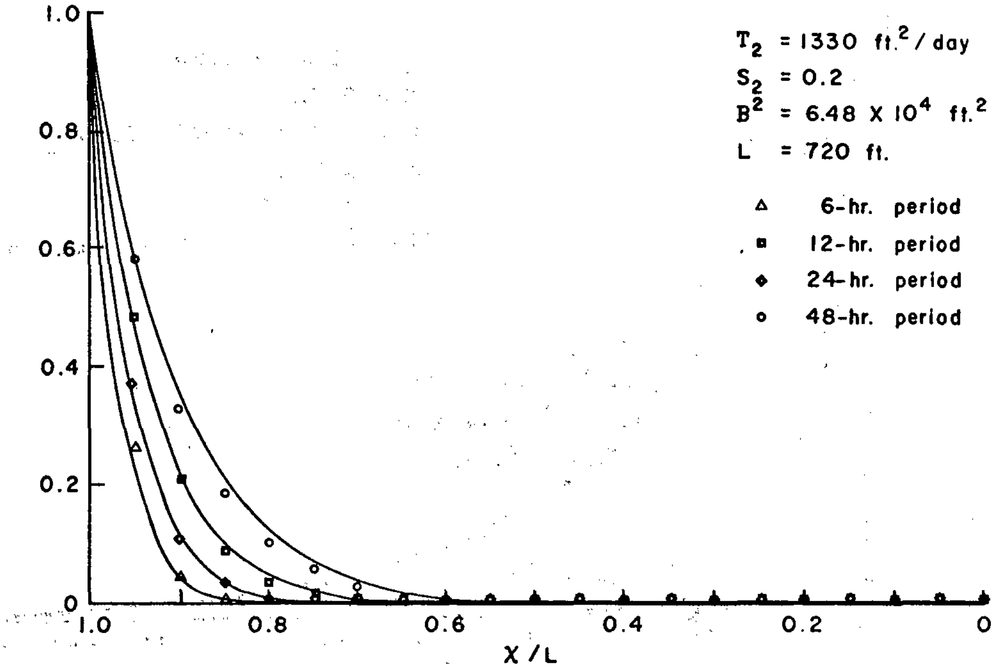


FIGURE 2. AMPLITUDE AND PHASE ANGLE *VS* x/L FOR LEAKY AQUIFER, REGION 1 (CONFINED). JUNE 5, 1972.



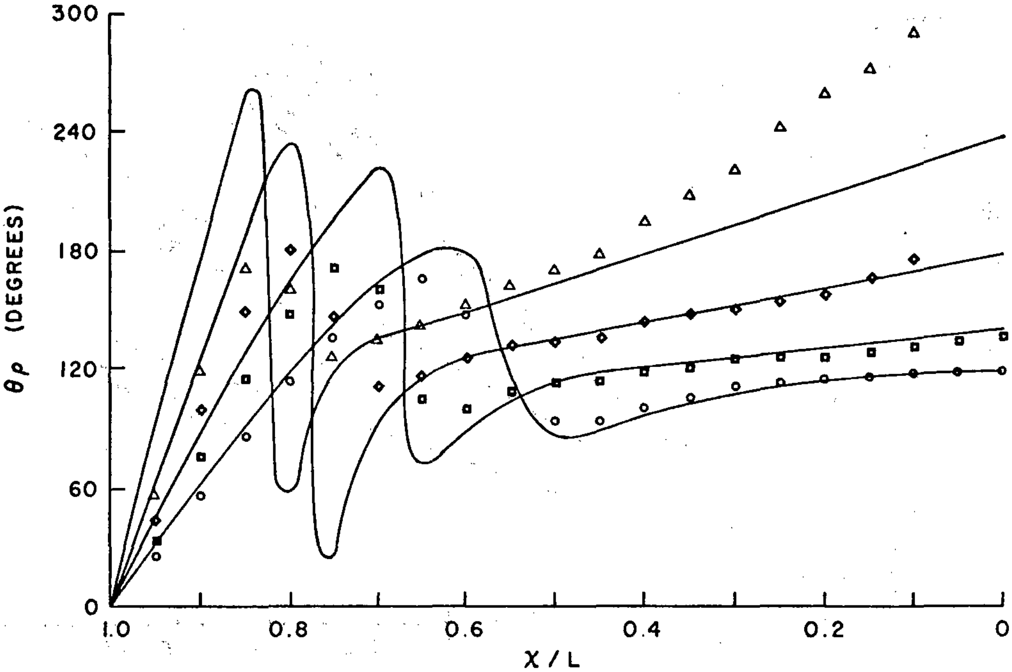


FIGURE 3. AMPLITUDE AND PHASE ANGLE *VS* x/L FOR LEAKY AQUIFER, REGION 2 (PHREATIC). JUNE 6, 1972.

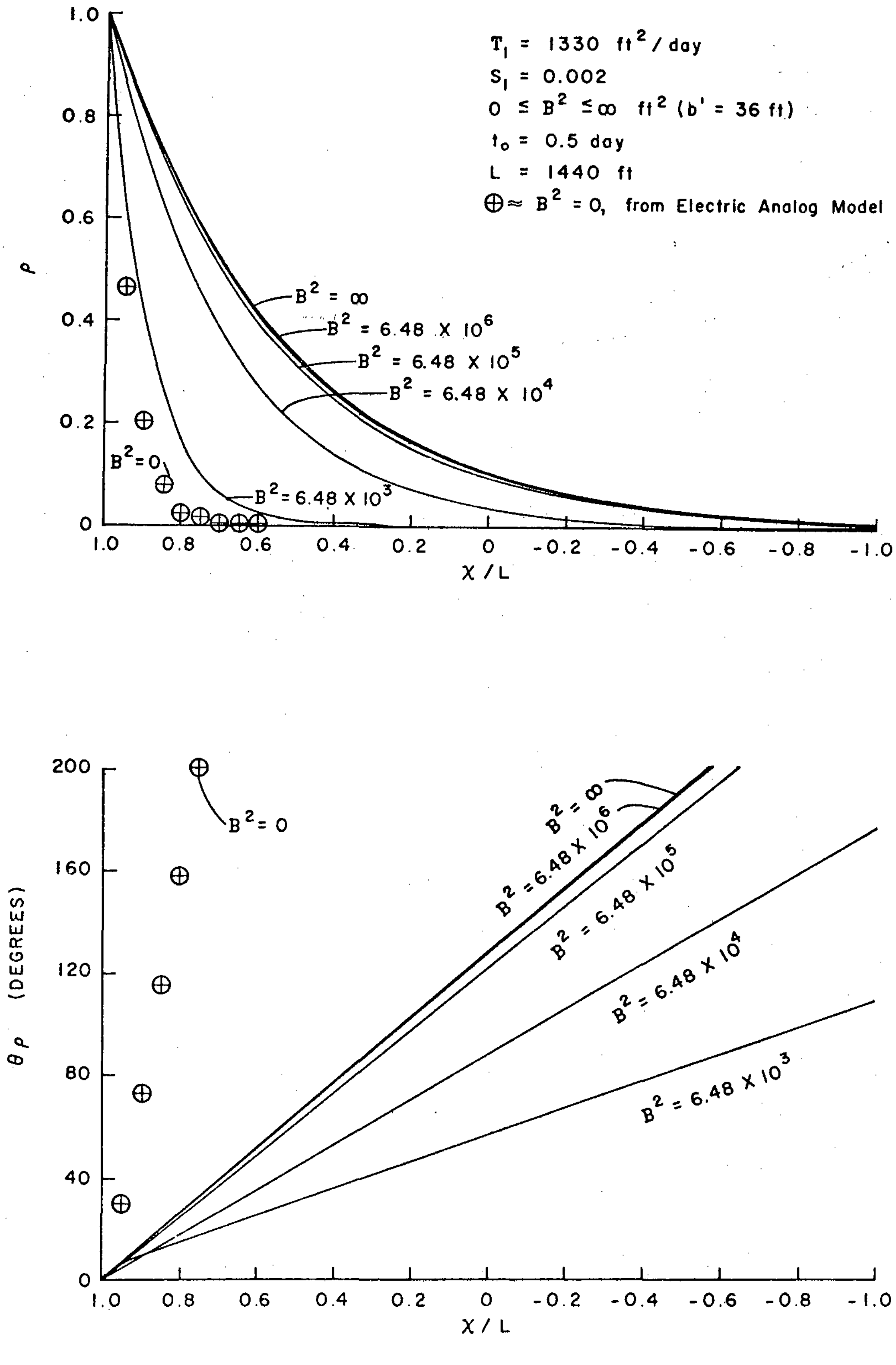


FIGURE 4. AMPLITUDE AND PHASE ANGLE *VS* x/L FOR LEAKY AQUIFER, REGION 1 (CONFINED). AUGUST 7, 1972.

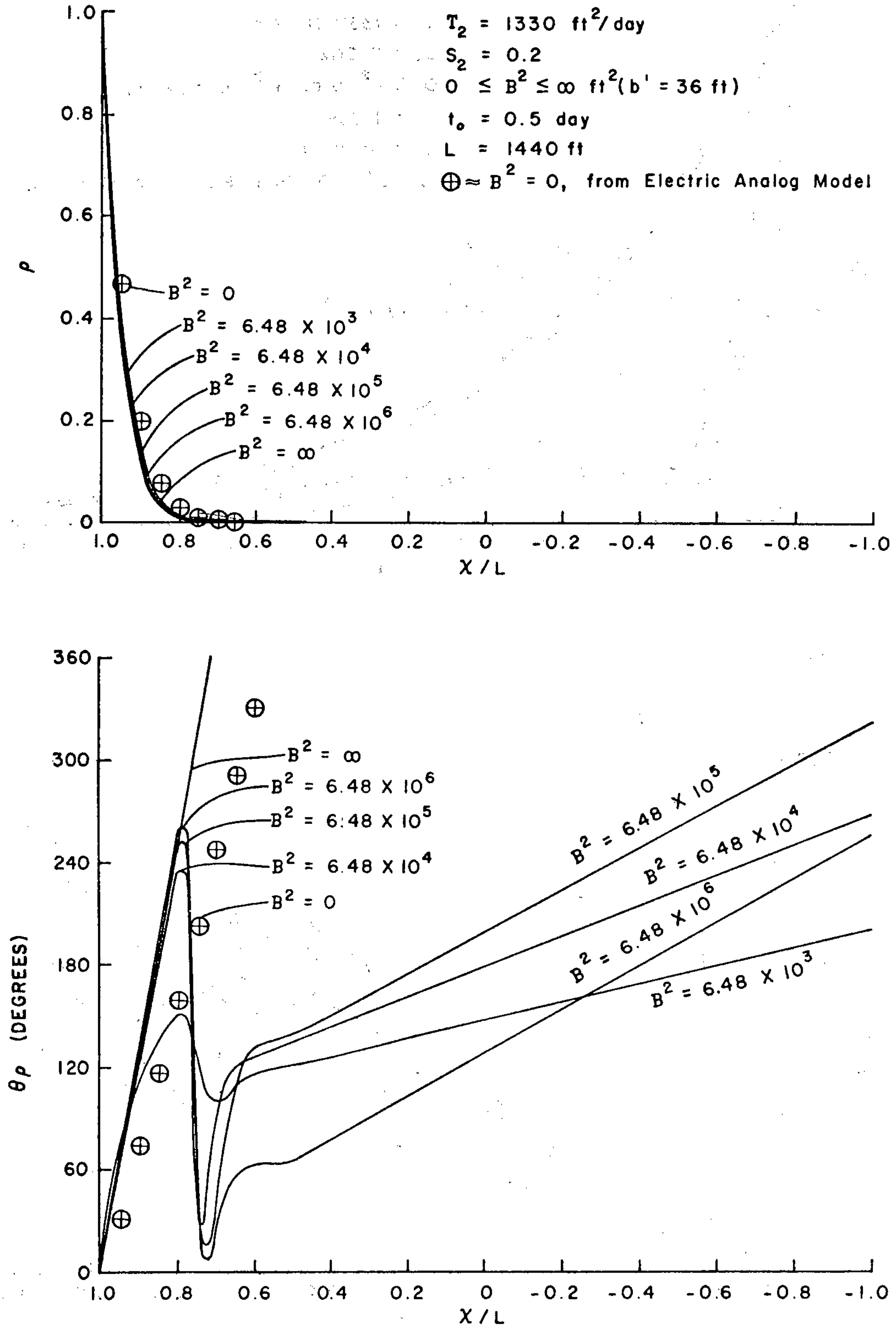


FIGURE 5. AMPLITUDE AND PHASE ANGLE *VS* x/L FOR LINEAR AQUIFER, REGION 2 (PHREATIC). AUGUST 7, 1972.

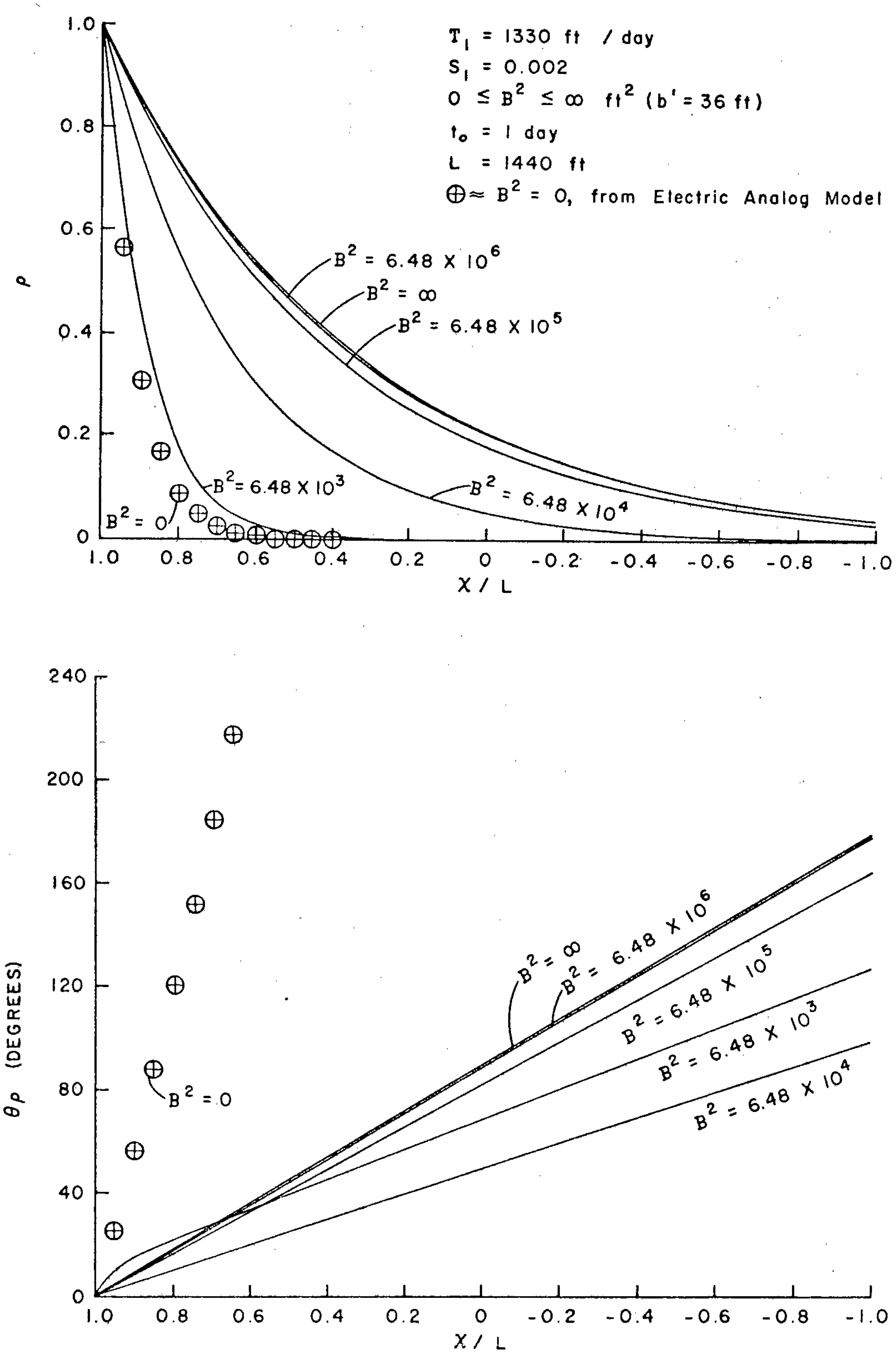
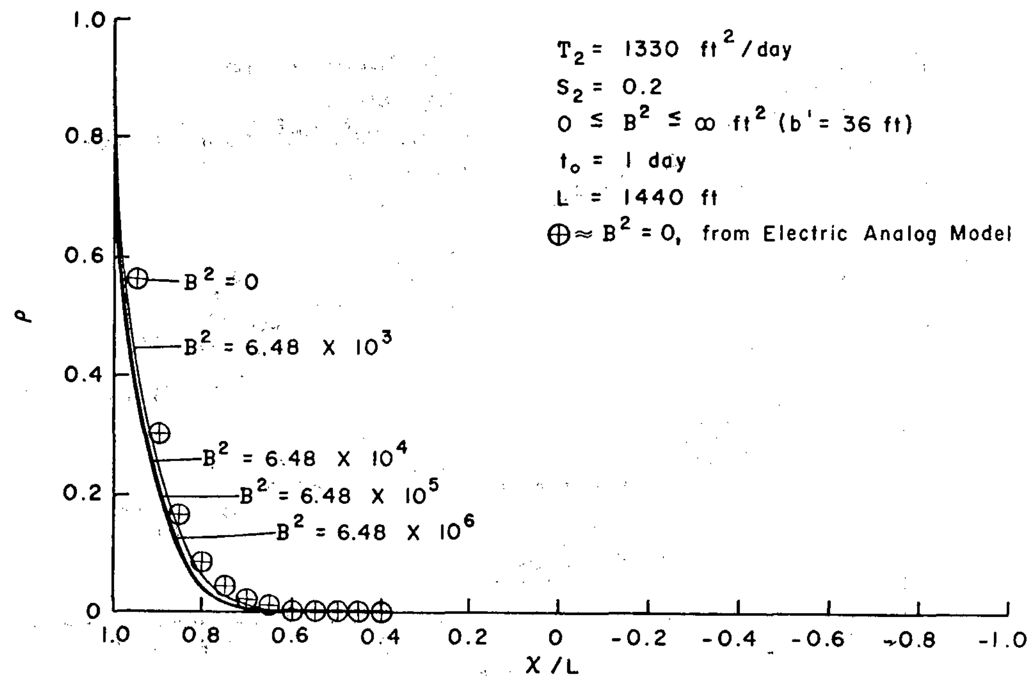


FIGURE 6. AMPLITUDE AND PHASE ANGLE *VS* x/L FOR LEAKY AQUIFER, REGION 1 (CONFINED). AUGUST 7, 1972.



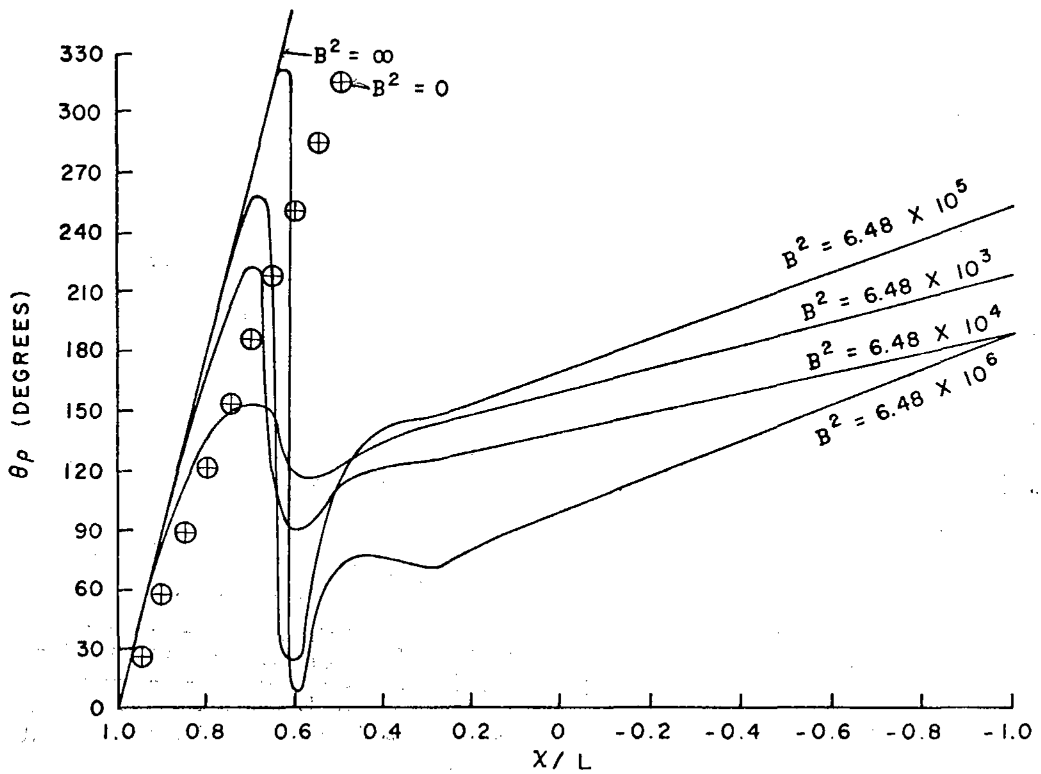


FIGURE 7. AMPLITUDE AND PHASE ANGLE *VS* x/L FOR LEAKY AQUIFER, REGION 2 (PHREATIC). AUGUST 7, 1972.

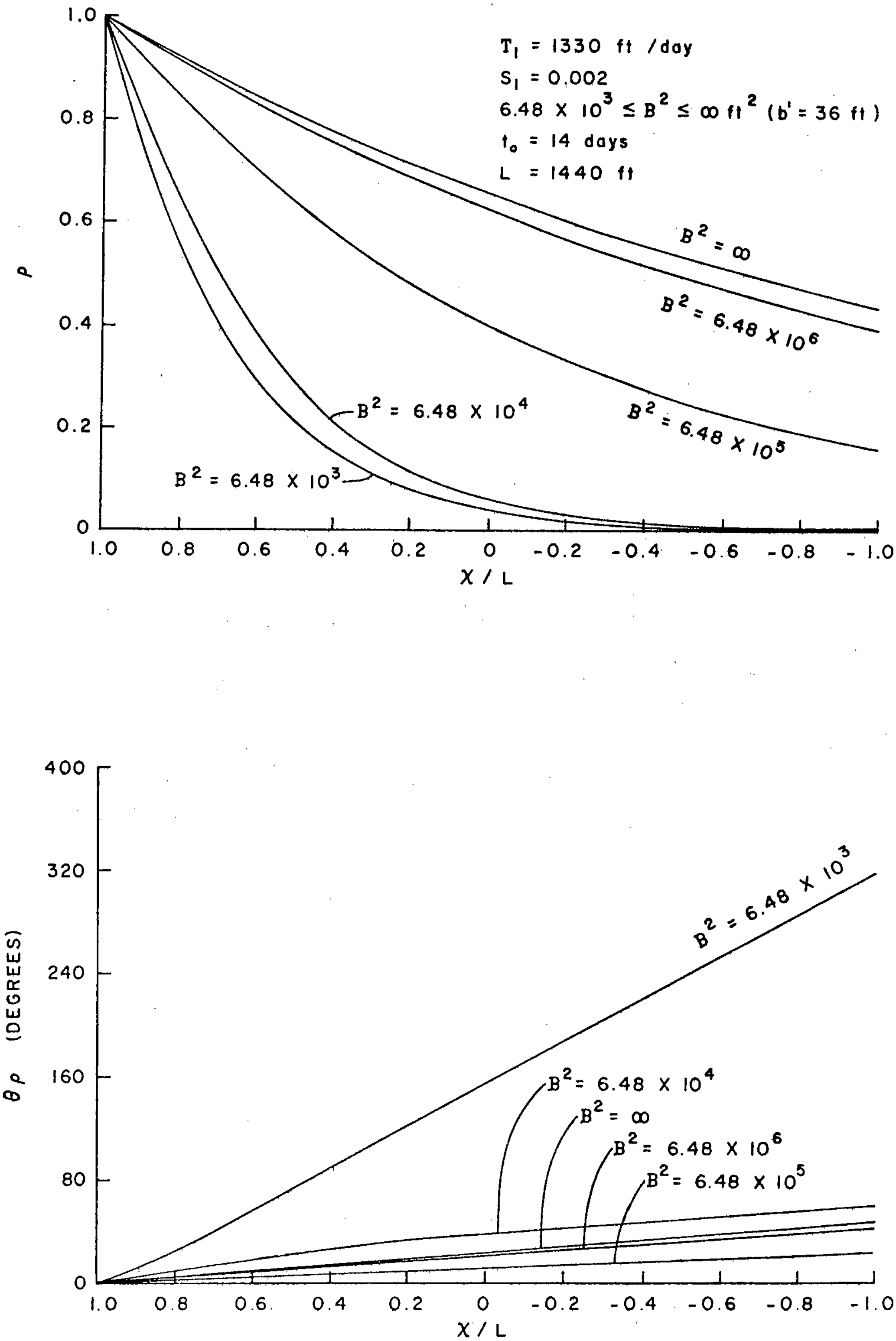


FIGURE 8. AMPLITUDE AND PHASE ANGLE *VS* x/L FOR LEAKY AQUIFER, REGION 1 (CONFINED). AUGUST 11, 1972.

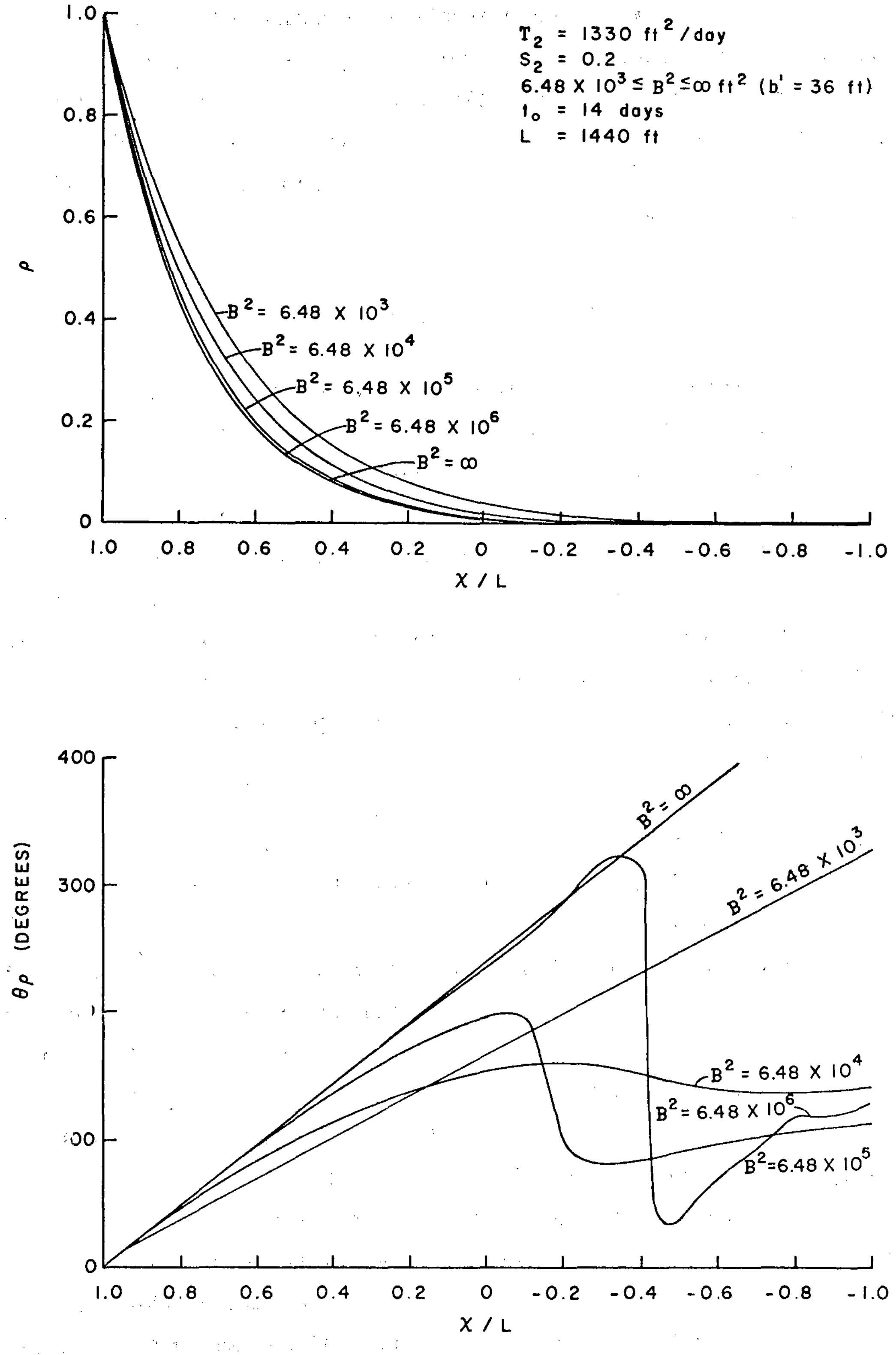


FIGURE 9. AMPLITUDE AND PHASE ANGLE *VS* x/L FOR LEAKY AQUIFER, REGION 2 (PHREATIC). AUGUST 11, 1972.

The IBM 360 computer was used to evaluate equations 13b and 13c and 15b and 15c for the given aquifer properties at predetermined values of x/L. These computed values of amplitude and phase angle provided the points through which the solid line curves in Figures 2 through 9 are drawn. The program and the printout for t_0 = 0.5 day and B^2 = 6.48 x 10^4 ft^2 are presented in Appendix C.

DISCUSSION OF THE RESULTS

The values selected for the several parameters of the system are considered to be reasonably representative of field conditions in some areas of Hawaii. It is to be noted that a somewhat special situation is considered here, since the transmissabilities of both regions are equal and, consequently, the leakage factors for both regions are also equal. However, as previously indicated, the storage coefficient for region 2 is one hundred times that of region 1. This arrangement--equal transmissabilities (and leakage factors) but storage coefficients differing by several orders of magnitude--is considered to be approximated in some local aquifer systems.

The primary purpose of the electric analog model was to provide a check on the mathematical model and the computer program. The results in Figures 2 and 3 show sufficiently good agreement between the two outputs to verify the model and the program.

Good agreement exists for all periods tested with the exception of the phase angles in region 2, where there is generally poor agreement for the 6-hour tidal period. For this period, the penetration length λ* in region 2 is 145 ft and resulted in a ratio of penetration length to grid spacing of approximately 4, which is considerably smaller than 50 as recommended by Williams and Liu (1971). Note that for the 48-hour period, this ratio is 11.3 and the agreement between the two models is good. Also, the electric analog results consistently deviate from those of the mathematical model in the neighborhood of the local maximum and minimum points for all tidal periods. This is to be expected since the change in the phase angle from maximum to minimum occurs within one grid spacing or

*See Appendix A

less.

For region 1, penetration lengths are 10 times greater than in region 2, hence, $\lambda/a \geq 40$. However, the overall circuit length now becomes a factor and may be partially responsible for the measured phase angles being somewhat greater than those predicted by the model.

It should be observed that errors resulting from the finite difference approach to the problem are proportional to the fourth and higher derivatives of the piezometric head. Equation (5) shows that the fourth derivative (as well as higher derivatives) is not zero and that it is composed of two parts: the first part is $(m_1^2 + m_3^2)$ times the second derivative of the piezometric head and the second part is equal to the constant $(m_1^2 \cdot m_3^2)$*. Hence, the errors in measurements on either region depend upon the properties of both regions.

In addition to providing a means of checking the computer program and the mathematical models, Figures 2 and 3 also indicate that, as expected, the damping of the piezometric surface increases as the tidal period decreases. The variation of phase angle position remains linear for region 1, but for region 2, a rather abrupt decrease followed by an increase at a lesser rate develops in the curves. For the 14-day period, this feature is not present in the first 720 feet from the coastline but it does appear in the second 720 feet (Figure 9) where the pressures in the two regions are sufficiently out of phase.

Figures 4 through 9 show the influence on the amplitude and phase relations of variations in B^2 as well as with those of the tidal period. The general trend for all three periods represented is an increase in the rate of amplitude decay in region 1 with a decrease in B^2. The amplitudes for $B^2 \leq 6.48 \times 10^4$ ft^2 (*i.e.*, $K' \geq 0.739$ ft/day) are substantially smaller than those for $B^2 = \infty$ (*i.e.*, $K' = 0$). At the same time amplitudes in region 2 increase compared with the amplitude for $B = \infty$. This behavior is the result of the pressure change propagating more quickly (by a factor of 10 for the aquifers studied here) in region 1 than in region 2. Thus, a volume of water moves vertically through the aquitard as region 2 provides additional storage for region 1, causing the amplitude curve for region 1 to decay more rapidly than if no leakage were present. At the

*See Appendix B

same time, this volume of water causes the amplitude in region 2 to decay at a lesser rate than it would if no aquitard were present. The fact that the net decrease in the amplitudes of region 1 is so much larger than the increases produced in region 2 is the result of having a storage coefficient one hundred times greater in region 2 than in region 1.

The phase angle relations for region 1 remain essentially linear except for $B^2 = 6.48 \times 10^3$ ft^2 (*i.e.*, K' = 7.39 ft/day). There is an apparent anomaly in the relative positions of the curves representing the several values of B^2 which is most noticeable in Figure 8 and to a lesser degree in Figures 4 and 6. This can be explained in the following way. The value of $B^2 = \infty$ represents the limiting case of two separate infinite aquifers, each with a phase angle which increases linearly with distance from the coastline. The other limiting case occurs for $B^2 = 0$, implying that the region separating the two aquifers offers no resistance to flow. In the latter limiting condition, the aquifer system would behave as a single aquifer and the phase relation would essentially be determined by the greater storage of region 2. This greater storage causes the phase angle of region 1 to increase with respect to distance from the coastline at a greater rate than it otherwise would. Confirmation of this fact is provided by the data points representing $B^2 = 0$ plotted in Figures 4, 5, 6, and 7. For values of B^2 that are relatively small but not zero, this effect on the phase angle is still quite noticeable for the longer period tides (See Figure 8 where t = 14 days), since the responses of each aquifer remain essentially in phase for some distance from the coastline. However, as the tidal period decreases, the responses of the two aquifers experience an increasing phase difference at any given distance from the coastline. As a result of this greater phase difference the aquifer of region 1 uses a decreased portion of the storage available in region 2. Thus, the phase lag with respect to the coast in region 1 shows a subsequent decrease as shown in Figures 4 and 6.

The most striking departure from the linear phase angle relation of the infinitely long aquifer occurs in region 2 as shown in Figures 5, 7, and 9. Here the phase angles follow closely the linear change for the single aquifer for some distance from the coast and then decrease rapidly (almost discontinuously for the larger values of B^2) after which they increase again. In several instances, a second set of maximum and minimum points are present, but not to such extreme positions as the initial set.

This sudden decrease of the phase angle in region 2 is caused by water moving through the aquitard under a pressure gradient produced by the more rapidly propagating tidal response in region 1. Hence, at that distance from the coastline where significant flow through the aquitard develops, the phase of region 1 is imposed on that of region 2 with the resultant phase in the upper aquifer showing a decrease. For small values of B^2 this process repeats itself at a greater distance from the coastline, producing a second set of maximum and minimum points (see Figure 7).

It should be pointed out that the tidal periods used here are not those of a real tide. However, any of the components of the semi-diurnal, diurnal, or lunar fortnightly tides are adequately approximated by the .5, 0.5, 1, and 14 day periods, respectively.

CONCLUSIONS

From the results presented in Figures 2 through 9 and the preceding discussion, it is evident that leakage can significantly influence tidal response data from either of the two aquifers.

For the aquifer with the greatest storage, the deviations in the phase angle are the most significant. However, these deviations take place at a point where the amplitudes have decayed to 5 percent or less of their initial value and phase angle measurements at such small amplitudes may not be usable because of a lack of accuracy. The amplitudes seem to be influenced less than the phase angles. However, for the range of leakage values studied, they may be larger than those for no leakage by as much as 100 percent for the 14-day period, 20 percent for the 1-day period and less than 20 percent for the 0.5-day period.

For the aquifer with the least storage both the amplitude and the phase angle values can be substantially different from those when no leakage is present. The phase angles can be either larger or smaller than those when no leakage occurs. The amplitudes are smaller than those when there is no leakage and may be so by as much as an order of magnitude or more for any of the three tidal periods considered. In general, the error for both the phase angles and the amplitudes is smaller for larger values of B^2.

In view of these results, it is apparent that aquifer properties based on tidal data may be completely erroneous if the data came from an aquifer system where leakage was present as described above. Also, a relatively moderate amount of leakage ($1/B^2 \geq 0.154 \times 10^{-4}$) can produce errors of 100 percent or more for an aquifer system similar to the one tested.

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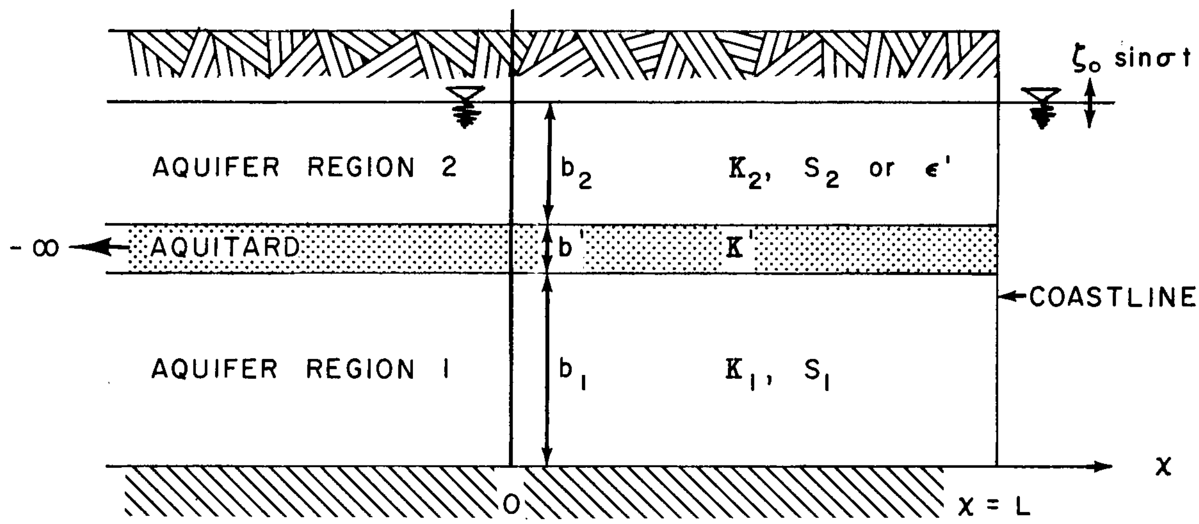
APPENDICES

APPENDIX A. LIST OF SYMBOLS.

a	characteristic length, ft
A_1, A_2	complex constants
b_1, b_2; b'	thickness of aquifer; of aquitard
B_j, B_1, B_2	the square root of the leakage factor, ft
C_1, C_2; C	complex constant; capacitance per unit of length, farads per grid spacing
e	base of natural logarithms
h_j, h_1, h_2	piezometric head, ft.
i	$\sqrt{-1}$; current, amperes
I_1, I_2	imaginary part of complex function for piezometric head
$K_1, \ldots K_4$; K'	coefficient of permeability of aquifer, ft/day or scale factors for electric analog model; coefficient of permeability of aquitard ft/day
L	distance to coast from origin of coordinates
$m_1, \ldots, m_4$	roots to fourth order characteristic equation
p_+ , p_-	real parts of the roots to the characteristic equation
P_j, P_1, P_2	complex coefficients of basic differential equation
Q	discharge, ft^3/sec
q_+, q_-; q	imaginary parts of the roots to the characteristic equation; volume of water, ft^3
R_1, R_2; R, R_ℓ	real part of complex function for piezometric head; resistance per unit length, ohms/grid spacing
S	aquifer storage
t, t_e; t_0	time, seconds, or days; tidal period, days
T_j, T_1, T_2	transmissivity, ft^2/day or gal/day - ft
V, V_0	electrical potential, volts
x, z	space variables, ft or "a" feet units
A, B, C, D	constants of integration

$\mathcal{Q}$	quantity of charge, coulombs
R	real part of
$\beta_{\pm}$; β_1, β_2	argument of complex roots to characteristic equations; argument of complex constant $(P_1 + P_2)^2 \, [(P_1 - P_2)^2 + 4/\beta_1{}^2\beta_2{}^2]^{1/2}$
α_j, α_1, α_2	coefficient combining aquifer properties and tidal frequency, ft^{-1}
δ_1, δ_2	moduli of complex coefficients in the fourth order differential equations
ε'	effective porosity
ζ_0; ζ_j, ζ_1, ζ_2	amplitude of the tide at the coast; amplitude of piezometric surface
θ_p	phase angle, degrees
λ	penetration length - is the distance from the coastline to the point where fluctuations in the piezometric head are in phase with the tide, *i.e.*, $(4T\, t_0/S)^{1/2}$
$\rho_{\pm}$; ρ_1, ρ_2	modulus of complex roots to characteristic equation; dimensionless amplitude functions for region 1 and 2, respectively
σ	angular frequency of tide, radians/sec



NOTE: REGION 2 MAY BE EITHER CONFINED FROM ABOVE OR PHREATIC. IN THE LATTER CASE THE STORAGE EQUALS THE EFFECTIVE POROSITY, ϵ'.

FIGURE A-1. DEFINING SKETCH FOR AQUIFER SYSTEM

APPENDIX B

Several special cases may be investigated by considering the characteristic equation of (differential) equation (5). This characteristic equation is

$$m^4 - (P_1 + P_2)\, m^2 + (P_1P_2 - 1/B_1^2B_2^2) = 0$$

and m_1, m_2, m_3, and m_4 are the four roots given in equation (6). From the theory of equations

$$m_1^2 + m_3^2 = P_1 + P_2$$

$$m_1^2 \cdot m_3^2 = P_1P_2 - 1/(B_1B_2)^2$$

$$m_1 = -m_2;\ m_3 = -m_4$$

and therefore

$$m_1^2 = \frac{(P_1 + P_2) + \sqrt{(P_1 - P_2)^2 + 4/B_1^2B_2^2}}{2}$$

$$m_3^2 = \frac{(P_1 + P_2) - \sqrt{(P_1 - P_2)^2 + 4/B_1^2B_2^2}}{2}$$

The first case is that in which the aquitard becomes an aquiclude *i.e.*, K' approaches zero. Then $1/B_1^2 = 1/B_2^2 = 0$ and

$$m_1^2 = i\sigma\alpha_1 \ ; \quad m_3^2 = i\sigma\alpha_2$$

The constants of integration in region 1 given by equations (11a) and (11b) become

$$\frac{-i\zeta_0}{m_1^2-m_3^2}\, A_1e^{-m_1L} = \frac{-i\zeta_0}{m_1^2-m_3^2}\,(i\sigma\alpha_1-m_3^2)\ e^{-m_1L} = -i\zeta_0e^{\sqrt{i\sigma\alpha_1}\,L}$$

$$\frac{i\zeta_0}{m_1^2-m_3^2}\, C_1e^{-m_3L} = \frac{i\zeta_0}{m_1^2-m_3^2}\,(i\sigma\alpha_1-m_1^2)\ e^{-m_3L} = 0$$

Therefore, equation (12) reduces to

$$\zeta_1\ (x) = -i\zeta_0e^{\sqrt{i\sigma\alpha_1}\,(x-L)} ,$$

which is the correct amplitude function for a single, one-dimensional aquifer of infinite extent in region 1.

The constants of integration for region 2 can be calculated in a similar fashion, *i.e.*,

$$\frac{B_1{}^2 A_2}{m_1{}^2-m_3{}^2} = \frac{A_1}{m_1{}^2-m_3{}^2} + B_1{}^2(i\sigma\alpha_1-m_1{}^2)\frac{A_1}{m_1{}^2-m_3{}^2} =$$

$$1 + \lim_{K'\to 0} B_1{}^2(i\sigma\alpha_1-m_1{}^2) = 0$$

$$\frac{B_1{}^2 C_2}{m_1{}^2-m_3{}^2} = \frac{C_1}{m_1{}^2-m_3{}^2} + B_1{}^2(i\sigma\alpha_1-m_3{}^2)\frac{C_1}{m_1{}^2-m_3{}^2} =$$

$$0 + \lim_{K'\to 0} B_1{}^2(i\sigma\alpha_1-m_1{}^2) = -1$$

where

$$\lim_{K'\to 0} [B_1{}^2(i\sigma\alpha_1-m_1{}^2)] = \lim_{K'\to 0} \frac{i\sigma\alpha_1-m_1{}^2}{P_1-i\sigma\alpha_1} = -1$$

since $m_1{}^2 \to P_1$ as $K' \to 0$. Equation (14) becomes

$$\zeta_2(x) = i\zeta_0 e^{\sqrt{i\sigma\alpha_2}\,(x-L)}$$

which is the correct amplitude function for a single, one-dimensional aquifer of infinite extent in region 2.

The second case is that in which both aquifers are identical. Then

$$P_1 = P_2 = P, \ \alpha_1 = \alpha_2 = \alpha$$

and

$$m_1{}^2 = P + 1/B^2 \ ; \ m_3{}^2 = P - 1/B^2$$

This gives, for region 1

$$\frac{A_1}{m_1{}^2-m_3{}^2} = \frac{i\sigma\alpha - P + 1/B^2}{2/B^2} = 0,$$

$$\frac{C_1}{m_1{}^2-m_3{}^2} = \frac{i\sigma\alpha - P - 1/B^2}{2/B^2} = -1$$

and

$$\zeta_1(x) = -i\zeta_0 e^{\sqrt{i\sigma\alpha}\,(x-L)} .$$

Similarly, for region 2

$$\frac{B_1{}^2(P_1-m_1{}^2)\ A_1}{m_1{}^2-m_3{}^2} = -\frac{B^2}{2} A_1 = 0,$$

$$\frac{B_1{}^2(P_1-m_3{}^2)\ C_1}{m_1{}^2-m_3{}^2} = -\frac{C_1}{m_1{}^2-m_3{}^2} = -1$$

and

$$\zeta_2(x) = -i\zeta_0 e^{\sqrt{i\sigma\alpha}\,(x-L)} .$$

APPENDIX C

Computer program and print-out for mathematical model of leaky aquifer system:

$t_0 = 0.5$ day

$B^2 = 6.48 \times 10^4$ ft^2

$T_1 = T_2 = 1330$ ft^2/day

$S_1 = 0.002$

$S_2 = 0.2$

```
C     LEAKY AQUIFER - REGION 1 (CCNFINED)
C     AL = AQUIFER LENGTH (FEET)
C     PERIOD = TIDAL PERIOD (HOURS)
C     B = AQUIFER THICKNESS (FEET)
C     CK = HYDRAULIC CCNDUCTIVITY (FT./DAY)
C     S = STORAGE COEFFICIENT
C     BSQRT = LEAKAGE FACTCR (FT.**2)
      DOUBLE PRECISICN CK3,PERIOD,AL,CK1,CK2,B1,B2,S1,S2,T1,T2,B3,SIGMA,
     1        ALPHA1,ALPHA2,CB1,CB2,B,A,RHC1,Z,BD1,ZP1,ZP2,RHO2P,RHC2,ZP,
     2        BD2P,BD2,BM1,BM2,BM3,BM,RHOP,RHCM,PP,PM,QP,QM,A1R,A1I,C1R,
     3        C1I,TENX,X,PPXL,PMXL,QPXL,QMXL,AIX,AIL,ARX,ARL,RI1,RI2,RIL,
     4        BP1,BP2,BP3,BP,RHO,TAM,THETA,DEGREE,CP,CM,D,BSQRT
   10 READ(5,1,END=30)AL,B1,B2,B3,S1,S2,CK1,CK2,CK3,PERICD
    1 FORMAT(F7.0,7F6.0,2F10.0)
      BSQRT = (CK1*B1*B3)/CK3
      WRITE(6,100)AL,PERIOD,B1,CK1,B2,CK2,B3,CK3,S1,S2,BSCRT
  100 FORMAT(1H1,'AL = ',F7.1,' FT',5X,'PERIOD = ',F12.3,' HRS'/1H0,
     1       'B1 = ',F7.1,' FT',5X,'CK1 = ',F12.3,' FT/DAY'/1H0,'B2 = ',
     2       F7.1,' FT',5X,'CK2 = ',F12.3,' FT/DAY'/1H0,'B3 = ',F7.1,
     3       ' FT',5X,'CK3 = ',F12.6,' FT/DAY'/1H0,'S1 = ',F7.5,8X,
     4       'S2 = ',F7.5/1H0,'BSQRT = ',1PD11.5,' FT**2')
      PERIOD = PERICD*3600.0
      CK1 = CK1/86400.0
      CK2 = CK2/86400.0
      CK3 = CK3/86400.0
      T1 = CK1*B1
      T2 = CK2*B2
      SIGMA = 6.2831853/PERICD
      ALPHA1 = S1/T1
      ALPHA2 = S2/T2
      WRITE(6,4)
    4 FORMAT(//1H0,'LOCATION',6X,'AMPLITUDE',9X,'PHASE ANGLE'/)
      CB1 = DSQRT((CK1*B1*B3)/CK3)
      CB2 = DSQRT((CK2*B2*B3)/CK3)
      B = SIGMA*(ALPHA1+ALPHA2)
      A = (1.0/(CB1**2))+(1.0/(CB2**2))
      RHC1 = DSQRT(A**2+B**2)
      Z = B/A
      BD1 = DATAN(Z)
      IF (B .GT. 0.0 .AND. A .LT. 0.0) BD1 = BD1+3.1415927
      IF (B .LT. 0.0 .AND. A .LT. 0.0) BD1 = BD1+3.1415927
      IF (B .LT. 0.0 .AND. A .GT. 0.0) BD1 = BD1+6.2831853
      CP = (1.0/(CB1**2))+(1.0/(CB2**2))
      CM = (1.0/(CB1**2))-(1.0/(CB2**2))
      D = SIGMA*(ALPHA1-ALPHA2)
      ZP1 = 2.0*CM*D
      ZP2 = CP**2-D**2
      RHO2P = DSQRT(ZP2**2+ZP1**2)
```

```
      RHO2 = DSQRT(RHO2P)
      ZP = ZP1/ZP2
      BD2P = DATAN(ZP)
      IF (ZP1 .GT. 0.0 .AND. ZP2 .LT. 0.0) BD2P = BD2P+3.1415927
      IF (ZP1 .LT. 0.0 .AND. ZP2 .LT. 0.0) BD2P = BD2P+3.1415927
      IF (ZP1 .LT. 0.0 .AND. ZP2 .GT. 0.0) BD2P = BD2P+6.2831853
      BD2 = BD2P/2.0
      BP1 = RHO1*DSIN(BD1)+RHO2*DSIN(BD2)
      BP2 = RHO1*DCOS(BD1)+RHO2*DCOS(BD2)
      BP3 = BP1/BP2
      BP = DATAN(BP3)
      IF (BP1 .GT. 0.0 .AND. BP2 .LT. 0.0) BP = BP+3.1415927
      IF (BP1 .LT. 0.0 .AND. BP2 .LT. 0.0) BP = BP+3.1415927
      IF (BP1 .LT. 0.0 .AND. BP2 .GT. 0.0) BP = BP+6.2831853
      BM1 = RHO1*DSIN(BD1) - RHO2*DSIN(BD2)
      BM2 = RHO1*DCOS(BD1) - RHO2*DCOS(BD2)
      BM3 = BM1/BM2
      BM = DATAN(BM3)
      IF (BM1 .GT. 0.0 .AND. BM2 .LT. 0.0) BM = BM+3.1415927
      IF (BM1 .LT. 0.0 .AND. BM2 .LT. 0.0) BM = BM+3.1415927
      IF (BM1 .LT. 0.0 .AND. BM2 .GT. 0.0) BM = BM+6.2831853
      RHOP = DSQRT(RHO1**2+RHO2**2+2.0*RHO1*RHO2*DCOS(BD1-BD2))/2.0
      RHOM = DSQRT(RHO1**2+RHO2**2-2.0*RHO1*RHO2*DCOS(BD1-BD2))/2.0
      PP = (DSQRT(RHOP))*DCOS(BP/2.0)
      PM = (DSQRT(RHOM))*DCOS(BM/2.0)
      QP = (DSQRT(RHOP))*DSIN(BP/2.0)
      QM = (DSQRT(RHOM))*DSIN(BM/2.0)
      A1R = QM**2-PM**2
      A1I = SIGMA*ALPHA1-2.0*PM*QM
      C1R = (-1.0)*(QP**2-PP**2)
      C1I = (-1.0)*(SIGMA*ALPHA1-2.0*PP*QP)
      TENX = 20.0
   20 X = TENX/20.0
      PPXL = PP*(X-1.0)*AL
      PMXL = PM*(X-1.0)*AL
      QPXL = QP*(X-1.0)*AL
      QMXL = QM*(X-1.0)*AL
      AIX = (A1R*DSIN(QPXL)+A1I*DCOS(QPXL))*DEXP(PPXL)
     6     +(C1R*DSIN(QMXL)+C1I*DCOS(QMXL))*DEXP(PMXL)
      AIL = A1I+C1I
      ARX = (A1R*DCOS(QPXL)-A1I*DSIN(QPXL))*DEXP(PPXL)
     7     +(C1R*DCOS(QMXL)-C1I*DSIN(QMXL))*DEXP(PMXL)
      ARL = A1R+C1R
      RI1 = AIL*AIX + ARL*ARX
      RI2 = -AIL*ARX + ARL*AIX
      RHO = DSQRT((ARX**2+AIX**2)/(ARL**2+AIL**2))
      TAM = RI2/RI1
      THETA = DATAN(TAM)
```

```
      IF (RI2 .GT. 0.0 .AND. RI1 .LT. 0.0) THETA = THETA-3.1415927
      IF (RI2 .LT. 0.0 .AND. RI1 .LT. 0.0) THETA = THETA-3.1415927
      DEGREE = THETA*360.0/6.2831853
      WRITE (6,3) X, RHO, DEGREE
    3 FORMAT(1H0,D9.3,2X,D15.6,3X,D16.6)
      IF (X .LE. 0.0) GO TO 10
      TENX = TENX-1.0
      GO TO 20
   30 CALL EXIT
      END
```

AL =	720.0 FT	PERIOD =	12.000 HRS
B1 =	100.0 FT	CK1 =	13.300 FT/DAY
B2 =	50.0 FT	CK2 =	26.600 FT/DAY
B3 =	36.0 FT	CK3 =	0.738900 FT/DAY

S1 = 0.00200   S2 = 0.20000

BSQRT = 6.47990D 04 FT**2

LOCATION	AMPLITUDE	PHASE ANGLE
0.100D 01	0.100000D 01	0.0
0.950D 00	0.853804D 00	-0.476665D 01
0.900D 00	0.725786D 00	-0.928643D 01
0.850D 00	0.617290D 00	-0.136732D 02
0.800D 00	0.525473D 00	-0.180433D 02
0.750D 00	0.447434D 00	-0.224266D 02
0.700D 00	0.380977D 00	-0.268178D 02
0.650D 00	0.324374D 00	-0.312100D 02
0.600D 00	0.276177D 00	-0.356016D 02
0.550D 00	0.235141D 00	-0.399927D 02
0.500D 00	0.200203D 00	-0.443838D 02
0.450D 00	0.170457D 00	-0.487749D 02
0.400D 00	0.145130D 00	-0.531660D 02
0.350D 00	0.123566D 00	-0.575571D 02
0.300D 00	0.105206D 00	-0.619482D 02
0.250D 00	0.895745D-01	-0.663393D 02
0.200D 00	0.762653D-01	-0.707304D 02
0.150D 00	0.649336D-01	-0.751215D 02
0.100D 00	0.552857D-01	-0.795126D 02
0.500D-01	0.470712D-01	-0.839037D 02
0.0	0.400772D-01	-0.882949D 02

```
C       LEAKY AQUIFER - REGICN 2 (PHREATIC)
C       AL = AQUIFER LENGTH (FEET)
C       PERIOD = TIDAL PERIOD (HOURS)
C       B = AQUIFER THICKNESS (FEET)
C       CK = HYDRAULIC CCNDUCTIVITY (FT./DAY)
C       S = STCRAGE CCEFFICIENT
C       BSQRT = LEAKAGE FACTCR (FT.**2)
        DOUBLE PRECISICN CK3,PERIOD,AL,CK1,CK2,B1,B2,S1,S2,T1,T2,B3,SIGMA,
     1        ALPHA1,ALPHA2,CB1,CB2,B,A,RHO1,Z,BD1,ZP1,ZP2,RHO2P,RHO2,ZP,
     2        BD2P,BD2,BM1,BM2,BM3,BM,RHOP,RHOM,PP,PM,QP,QM,A1R,A1I,C1R,
     3        C1I,TENX,X,PPXL,PMXL,QPXL,QMXL,AIX,AIL,ARX,ARL,RI1,RI2,RIL,
     4        BP1,BP2,BP3,BP,RHO,TAM,THETA,DEGREE,CP,CM,D,XX,YY,BSQRT
   10 READ(5,1,END=30)AL,B1,B2,B3,S1,S2,CK1,CK2,CK3,PERIOD
    1 FORMAT(F7.0,7F6.0,2F10.0)
      BSQRT = (CK2*B2*B3)/CK3
      WRITE(6,100)AL,PERIOD,B1,CK1,B2,CK2,B3,CK3,S1,S2,BSQRT
  100 FORMAT(1H1,'AL = ',F7.1,' FT',5X,'PERIOD = ',F12.3,' HRS'/1H0,
     1       'B1 = ',F7.1,' FT',5X,'CK1 = ',F12.3,' FT/DAY'/1H0,'B2 = ',
     2       F7.1,' FT',5X,'CK2 = ',F12.3,' FT/DAY'/1H0,'B3 = ',F7.1,
     3       ' FT',5X,'CK3 = ',F12.6,' FT/DAY'/1H0,'S1 = ',F7.5,8X,
     4       'S2 = ',F7.5/1H0,'BSQRT = ',1PD11.5,' FT**2')
      PERIOD = PERIOD*3600.0
      CK1 = CK1/86400.0
      CK2 = CK2/86400.0
      CK3 = CK3/86400.0
      T1 = CK1*B1
      T2 = CK2*B2
      SIGMA = 6.2831853/PERIOD
      ALPHA1 = S1/T1
      ALPHA2 = S2/T2
      WRITE(6,4)
    4 FORMAT(//1H0,'LCCATICN',6X,'AMPLITUDE',9X,'PHASE ANGLE'/)
      CB1 = DSQRT((CK1*B1*B3)/CK3)
      CB2 = DSQRT((CK2*B2*B3)/CK3)
      B = SIGMA*(ALPHA1+ALPHA2)
      A = (1.0/(CB1**2))+(1.0/(CB2**2))
      RHO1 = DSQRT(A**2+B**2)
      Z = B/A
      BD1 = DATAN(Z)
      IF (B .GT. 0.0 .AND. A .LT. 0.0) BD1 = BD1+3.1415927
      IF (B .LT. 0.0 .AND. A .LT. 0.0) BD1 = BD1+3.1415927
      IF (B .LT. 0.0 .AND. A .GT. 0.0) BD1 = BD1+6.2831853
      CP = (1.0/(CB2**2))+(1.0/(CB1**2))
      CM = (1.0/(CB2**2))-(1.0/(CB1**2))
      D = SIGMA*(ALPHA2-ALPHA1)
      ZP1 = 2.0*CM*D
      ZP2 = CP**2-D**2
      RHO2P = DSQRT(ZP2**2+ZP1**2)
```

```
      RHC2 = DSQRT(RHO2P)
      ZP = ZP1/ZP2
      BD2P = DATAN(ZP)
      IF (ZP1 .GT. 0.0 .AND. ZP2 .LT. 0.0) BD2P = BD2P+3.1415927
      IF (ZP1 .LT. 0.0 .AND. ZP2 .LT. 0.0) BD2P = BD2P+3.1415927
      IF (ZP1 .LT. 0.0 .AND. ZP2 .GT. 0.0) BD2P = BD2P+6.2831853
      BD2 = BD2P/2.0
      BP1 = RHO1*DSIN(BD1)+RHO2*DSIN(BD2)
      BP2 = RHO1*DCOS(BD1)+RHO2*DCOS(BD2)
      BP3 = BP1/BP2
      BP = DATAN(BP3)
      IF (BP1 .GT. 0.0 .AND. BP2 .LT. 0.0) BP = BP+3.1415927
      IF (BP1 .LT. 0.0 .AND. BP2 .LT. 0.0) BP = BP+3.1415927
      IF (BP1 .LT. 0.0 .AND. BP2 .GT. 0.0) BP = BP+6.2831853
      BM1 = RHO1*DSIN(BD1) - RHO2*DSIN(BD2)
      BM2 = RHO1*DCOS(BD1) - RHO2*DCOS(BD2)
      BM3 = BM1/BM2
      BM = DATAN(BM3)
      IF (BM1 .GT. 0.0 .AND. BM2 .LT. 0.0) BM = BM+3.1415927
      IF (BM1 .LT. 0.0 .AND. BM2 .LT. 0.0) BM = BM+3.1415927
      IF (BM1 .LT. 0.0 .AND. BM2 .GT. 0.0) BM = BM+6.2831853
      RHOP = DSQRT(RHO1**2+RHO2**2+2.0*RHO1*RHO2*DCOS(BD1-BD2))/2.0
      RHOM = DSQRT(RHO1**2+RHO2**2-2.0*RHO1*RHO2*DCOS(BD1-BD2))/2.0
      PP = (DSQRT(RHOP))*DCOS(BP/2.0)
      PM = (DSQRT(RHOM))*DCOS(BM/2.0)
      QP = (DSQRT(RHOP))*DSIN(BP/2.0)
      QM = (DSQRT(RHOM))*DSIN(BM/2.0)
      A1R = (QM**2-PM**2)*((1.0/CB1**2)-PP**2+QP**2)-((SIGMA*ALPHA1-
     6      2.0*PM*QM)*(SIGMA*ALPHA1-2.0*PP*QP))
      A1I = ((1.0/CB1**2)-PP**2+QP**2)*(SIGMA*ALPHA1-2.0*PM*QM)+
     7      ((SIGMA*ALPHA1-2.0*PP*QP)*(QM**2-PM**2))
      C1R = (PP**2-QP**2)*((1.0/CB1**2)-PM**2+QM**2)+((SIGMA*ALPHA1-
     8      2.0*PP*QP)*(SIGMA*ALPHA1-2.0*PM*QM))
      C1I = (-1.0)*(((1.0/CB1**2)-PM**2+QM**2)*(SIGMA*ALPHA1-
     9      2.0*PP*QP)+((QP**2-PP**2)*(SIGMA*ALPHA1-2.0*PM*QM)))
      TENX = 20.0
   20 X = TENX/20.0
      PPXL = PP*(X-1.0)*AL
      PMXL = PM*(X-1.0)*AL
      QPXL = QP*(X-1.0)*AL
      QMXL = QM*(X-1.0)*AL
      AIX = (A1R*DSIN(QPXL)+A1I*DCOS(QPXL))*DEXP(PPXL)
     6    +(C1R*DSIN(QMXL)+C1I*DCOS(QMXL))*DEXP(PMXL)
      AIL = A1I+C1I
      ARX = (A1R*DCOS(QPXL)-A1I*DSIN(QPXL))*DEXP(PPXL)
     7    +(C1R*DCOS(QMXL)-C1I*DSIN(QMXL))*DEXP(PMXL)
      ARL = A1R+C1R
      XX = (PP**2-QP**2)-(PM**2-QM**2)
```

```
      YY = 2.0*(PP*CP-PM*QM)
      RI2 = AIX*XX-ARX*YY
      RI1 = ARX*XX+AIX*YY
      RHO = (CB1**2)*(DSQRT((ARX**2+AIX**2)/(XX**2+YY**2)))
      TAM = RI2/RI1
      THETA = DATAN(TAM)
      IF (RI2 .LT. 0.0 .AND. RI1 .LT. 0.0) THETA = THETA-3.1415927
      IF (RI2 .GT. 0.0 .AND. RI1 .LT. 0.0) THETA = THETA-3.1415927
      DEGREE = THETA*360.0/6.2831853
      WRITE (6,3) X, RHO, DEGREE
    3 FORMAT(1H0,D9.3,2X,D15.6,3X,D16.6)
      IF (X .LE. 0.0) GO TO 10
      TENX = TENX-1.0
      GO TO 20
   30 CALL EXIT
      END
```

AL = 720.0 FT PERIOD = 12.000 HRS

B1 = 100.0 FT CK1 = 13.300 FT/DAY

B2 = 50.0 FT CK2 = 26.600 FT/DAY

B3 = 36.0 FT CK3 = 0.738900 FT/DAY

S1 = 0.00200 S2 = 0.20000

BSQRT = 6.47990D 04 FT**2

LOCATION	AMPLITUDE	PHASE ANGLE
0.100D 01	0.100000D 01	0.380230D-15
0.950D 00	0.335209D 00	-0.633064D 02
0.900D 00	0.113762D 00	-0.124458D 03
0.850D 00	0.364419D-01	-0.180945D 03
0.800D 00	0.861273D-02	-0.234964D 03
0.750D 00	0.151285D-02	-0.272327D 02
0.700D 00	0.321596D-02	-0.936265D 02
0.650D 00	0.301094D-02	-0.115992D 03
0.600D 00	0.240910D-02	-0.126548D 03
0.550D 00	0.194984D-02	-0.131166D 03
0.500D 00	0.164077D-02	-0.134610D 03
0.450D 00	0.140147D-02	-0.138589D 03
0.400D 00	0.119684D-02	-0.142959D 03
0.350D 00	0.101975D-02	-0.147403D 03
0.300D 00	0.868094D-03	-0.151818D 03
0.250D 00	0.738983D-03	-0.156211D 03
0.200D 00	0.629155D-03	-0.160600D 03
0.150D 00	0.535678D-03	-0.164989D 03
0.100D 00	0.456090D-03	-0.169380D 03
0.500D-01	0.388324D-03	-0.173772D 03
0.0	0.330626D-03	-0.178163D 03