DUAL LINEAR SPACES GENERATED
BY A
NON-DESARGUESIAN CONFIGURATION

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Abstract

A dual linear space is a partial projective plane which contains the intersection of every pair of its lines. Every dual linear space can be extended to a projective plane, usually infinite, by a sequence of one line extensions. Moreover, one may describe necessary conditions for the sequence of one line extensions to terminate after finitely many steps with a finite projective plane. A computer program that attempts to construct a finite projective plane from a given dual linear space by a sequence of one line extension has been written by Dr. Nation. In particular, one would like to extend a dual linear space containing a non-Desarguesian configuration to a finite projective plane of non-prime-power order. This dissertation studies the initial dual linear spaces to be used in this algorithm. The main result is that there are 105 non-isomorphic initial dual linear spaces containing the basic non-Desarguesian configuration.
Contents

Acknowledgements .................................................. iii

Abstract ................................................................. v

List of Tables .......................................................... vii

List of Figures .......................................................... viii

Chapter 1: Introduction .............................................. 1

Chapter 2: Projective Planes ........................................ 3

Chapter 3: Extending Dual Linear Spaces ....................... 25

Chapter 4: Classification of Dual Linear Spaces Generated by a Non-
Desarguesian Configuration .......................................... 39

Chapter 5: Conclusions ............................................... 158

References ............................................................... 160
List of Tables

1  Intersection of lines. ............................................. 40
2  Lines through each point. ........................................ 40
3  Table of inconsistency. ......................................... 44
4  Automorphisms. .................................................. 103
5  Changes in methods corresponding to automorphisms. ....... 104
List of Figures

1  Fano plane. ................................................. 7
2  Desarguesian configuration. ................................. 10
3  Non-Desarguesian configuration. .......................... 39
Chapter 1
Introduction

A dual linear space is a partial projective plane which contains the intersection of every pair of its lines. Every dual linear space can be extended to a projective plane, usually infinite, by a sequence of one line extensions. Moreover, one may describe necessary conditions for the sequence of one line extensions to terminate after finitely many steps with a finite projective plane. A computer program that attempts to construct a finite projective plane from a given dual linear space by a sequence of one line extension has been written by Dr. Nation. In particular, one would like to extend a dual linear space containing a non-Desarguesian configuration to a finite projective plane of non-prime-power order. This dissertation studies the initial dual linear spaces to be used in this algorithm. The main result is that there are 105 non-isomorphic initial dual linear spaces containing the basic non-Desarguesian configuration.

We begin by reviewing some elementary properties of finite projective planes in Chapter 2. We will describe how a projective plane may be constructed from a division ring, an affine plane, a ternary ring, or a double loop. We then introduce Desargues’ theorem and show that a projective plane constructed from a division ring is a Desarguesian plane. The Bruck-Ryser theorem [4] is the classic result on the order of finite projective planes. By this theorem, there is no projective plane of order $n$ when $n$ is congruent to 1 or 2 mod 4 and $n$ is not a sum of two squares. So there is no projective plane of order 6 or 14. In 1989, C. Lam, L. Thiel and S. Swiercz proved that there is no projective plane of order 10 [10]. Hence 12 and 15 are the first two numbers for which we do not
know whether or not there is a projective plane of that order. Likewise, we do not know whether there is a non-Desarguesian plane of a prime order $p$ when $p \geq 11$.

In Chapter 3, we will state the definition of a linear space. Its dual is sometimes called a semiplane. However, the main focus for that Chapter will be to describe and analyze the algorithm for constructing projective planes by extending finite semiplanes.

1. If a finite semiplane can be extended to a projective plane of order $n$, then it can be extended through a sequence of one line extensions.

2. We introduce simple and useful necessary conditions for extendibility to a plane of order $n$.

3. We describe how the computer programs work in general by keeping track of the possible one line extensions that satisfy the conditions.

In Chapter 4, we will prove the main result of this dissertation, that there are 875 different semiplanes generated by a non-Desarguesian configuration, which fall into 105 isomorphism classes. We have checked that all but five of these isomorphism types actually occur in a Hall plane of order 9.

We will summarize the results of our computer search for a projective plane of order 12 in Chapter 5.
Chapter 2
Projective Planes

We begin by reviewing some basic properties of finite projective planes and describing different ways to construct a projective plane in this Chapter. We will use Batten [1], Batten and Beutelspacher [2], Bennett [3], Hall [8], Lam [10], Nation [12], Scherk [13] and Stevenson [14] as the source of the following material.

A projective plane is a structure \( \Pi = (P, L, \leq) \) where \( P \) is a set of points, \( L \) is a set of lines, and \( p \leq l \) means that the point \( p \) is on the line \( l \), satisfying the following axioms:

PP1 Any two distinct points lie on exactly one line.

PP2 Any two distinct lines intersect in exactly one point.

PP3 There exist four points, no three of which are on a line.

If \( p \) and \( q \) are two distinct points in a projective plane, then we let \( p \vee q \) denote the unique line through them. Similarly, \( k \wedge l \) denotes the unique point on both lines \( k \) and \( l \). If two or more points lie on the same line, the points are collinear. If two or more lines pass through the same point, the lines are concurrent. If \( a, b, c, d \) are four points satisfying PP3, then the quadruple \( a, b, c, d \) is called a complete four-point or simply a four-point.

We would like to note here that axiom PP3 is equivalent in the presence of PP1 and PP2 to the following axiom: Every line has at least 3 points and there are at least 2 lines.
If we add 0 and 1 to the lattice representing the incidences between lines and points, we obtain a complemented modular lattice.

If $T$ is a statement about points and lines, then the dual statement of $T$ is the statement obtained by interchanging the words ‘point’ and ‘line’ with suitable adjustments of the words ‘lies on’ and ‘contains’ (or any equivalents). A given statement is true for a projective plane if and only if its dual is true.

**Theorem 1.** Let $n \geq 2$ be an integer. In a projective plane $\Pi$ any one of the following properties implies the rest:

1. One line contains exactly $(n + 1)$ points.
2. One point is on exactly $(n + 1)$ lines.
3. Every line contains exactly $(n + 1)$ points.
4. Every point is on exactly $(n + 1)$ lines.
5. There are exactly $(n^2 + n + 1)$ points in $\Pi$.
6. There are exactly $(n^2 + n + 1)$ lines in $\Pi$.

**Proof.** Let $a, b, c, d$ be four points, no three on a line, whose existence is given by PP3. For our convenience, we call the condition that no three of the points $a, b, c, d$ are on a line ($\star$). PP1 and $(\star)$ imply that there are six distinct lines containing $a, b, c, d$, say, $l_1 = a \lor b$, $l_2 = a \lor c$, $l_3 = a \lor d$, $l_4 = b \lor c$, $l_5 = b \lor d$, and $l_6 = c \lor d$. PP2 and $(\star)$ imply that there are an additional three distinct points $x, y, z$ such that $x = l_1 \land l_5$, $y = l_2 \land l_5$, and $z = l_3 \land l_6$. 


\[ z = l_3 \land l_4. \] Moreover, we may check that there are no further incidences between these seven points and these six lines. Before we continue our proof, we would like to prove two claims.

Claim 1: If \( l \) is a line with exactly \((n + 1)\) points on it, say, \( q_1, q_2, ..., q_{n+1} \), and \( p \) is a point not on \( l \), then \( p \) is on exactly \((n + 1)\) lines.

Proof. Since \( p \) is not on \( l \), we have \( p \lor q_i = l_i, i = 1, ..., n+1 \), by PP1. If \( p \lor q_i = l_i = l_j = p \lor q_j \), then \( p \leq q_i \lor q_j = l \), contrary to our assumption. Hence all the \( l_i \)'s are distinct. Moreover, every line through \( p \) intersects \( l \) and so must be one of the \( l_i \)'s. Therefore there are exactly \((n + 1)\) lines through \( p \). \( \square \)

Claim 2: If \( p \) is a point on exactly \((n + 1)\) lines, say, \( k_1, ..., k_{n+1} \), and \( m \) is a line not through \( p \), then \( m \) has exactly \((n + 1)\) points.

Proof. Claim 2 is the dual of Claim 1. \( \square \)

Now assume property 1. Let \( l \) be a line having exactly \((n + 1)\) points. Note that at least two of the points \( a, b, c, d \) are not on \( l \), say, \( a \) and \( b \). By Claim 1, both of them are on exactly \((n + 1)\) lines, hence it’s clear that property 2 holds. Every line not through \( a \) or not through \( b \) contains exactly \((n + 1)\) points by Claim 2, so every line except possibly \( l_1 \) contains exactly \((n + 1)\) points. Consider \( l_2 \), which does not go through \( b \), and hence has exactly \((n + 1)\) points. Since point \( z \) is not on \( l_2 \), \( z \) is on exactly \((n + 1)\) lines. And \( z \not\in l_1 \) implies that \( l_1 \) must also contain exactly \((n + 1)\) points. Thus property 3 holds. But now for any point \( p \), we may find a line not through it, and so by Claim 1, \( p \) is
on exactly \((n + 1)\) lines, giving property 4. Now let \(p_0\) be a particular point and let \(l_1, \ldots, l_{n+1}\) be the lines through \(p_0\). Each of these lines contains exactly \(n\) points different from \(p_0\), and since \(p_0\) is joined to any point in the plane by one of the \(l_i\)'s, they account for all the points of \(\Pi\), giving a total \(1 + (n + 1)n = n^2 + n + 1\) points in \(\Pi\), proving property 5. Similarly, if \(l_0\) is a line containing points \(p_1, \ldots, p_{n+1}\), each of these points is on exactly \(n\) additional lines, and since each line in the plane intersects \(l_0\) in one of \(p_i\)'s, they account for all the lines in \(\Pi\), giving a total of \(n^2 + n + 1\) lines, proving property 6.

We have shown that property 1 implies the remaining five properties. It is clear that property 3 implies the rest. By duality, we have both property 2 and 4 imply the rest. If property 5 holds, then the line \(l_1 = a \lor b\) contains \((m + 1)\) points for some integer \(m \geq 2\), and so by our earlier argument, \(\Pi\) contains \(m^2 + m + 1\) points. But since both \(m\) and \(n\) are positive integers, \(m^2 + m + 1 = n^2 + n + 1\) implies \(m = n\), whence property 5 implies property 1 and so the remaining ones. Similarly, 6 implies 2 and so the others. Thus all parts of our theorem are proved.

A finite projective plane is said to be of order \(n\) if a line contains exactly \((n + 1)\) points. By Theorem 1, we know that a projective plane of order \(n\) has \(n^2 + n + 1\) points and \(n^2 + n + 1\) lines, each line contains \(n + 1\) points and each point is on \(n + 1\) lines.

Figure 1 gives us the projective plane of order 2, which is also referred to as the Fano plane.

Let \(D\) be an arbitrary division ring. We would like to construct a plane \(\Pi_D = (P, L, \leq)\) as follows. Let \(P = \{[x, y, z] \mid x, y, z \in D\text{ and } x = y = z = 0\text{ is not true}\},\)
Figure 1: Fano plane.

where \([x, y, z] = [x', y', z']\) iff there exists \(r \in D, r \neq 0\), such that \(x = x'r, y = y'r, z = z'r\). Let \(L = \{(a, b, c) \mid a, b, c \in D \text{ and } a = b = c = 0 \text{ is not true}\}\), where \((a, b, c) = (a', b', c')\) iff there exists \(s \in D, s \neq 0\), such that \(a = sa', b = sb', c = sc'.\)

Then define \([x, y, z] \cong (a, b, c)\) iff \(ax + by + cz = 0\).

**Theorem 2.** The plane \(\Pi_D\) is a projective plane.

*Proof.* Let \(p = [x, y, z]\) and \(p' = [x', y', z']\) be distinct points. Since not all of \(x, y, z = 0\), we may assume without loss of generality that \(x \neq 0\). Furthermore, we may assume that \(x = 1\). Since \(p\) and \(p'\) are distinct, \(yx' - y'\) and \(zx' - z'\) cannot be both zero; for otherwise, we would have \(1 \cdot x' = x', yx' = y', zx' = z'\), which implies that \(p = p'\), contrary to our assumption. So we may assume without loss of generality that \(yx' - y' \neq 0\).

First, we want to show that there exists a line containing \(p\) and \(p'\). Let \(L = \langle a, b, c, \rangle\) where \(a = -(by + z), b = (zx' - z')(y' - yx')^{-1}, c = 1\). Now \(ax + by + cz = -(by + z) \cdot 1 + by + 1 \cdot z = 0\). Also \(ax' + by' + cz' = -(by + z)x' + by' + 1 \cdot z' = b(y' - yx') - zx' + z' = (zx' - z')(y' - yx')^{-1}(y' - yx') - zx' + z' = 0\). Thus \(L\) contains \(p\) and \(p'\).
Next, to show uniqueness, suppose that \( L' = (a', b', c') \) also contains \( p \) and \( p' \). If \( c' = 0 \), then \( a' + b'y = 0 \) and \( a'x' + b'y' = 0 \). Thus \(-b'yx' + b'y' = b'(-yx' + y') = 0\). Since \( y' - yx' \neq 0 \), we have \( b' = 0 \). Now \( a' + b'y = 0 \) implies \( a' = 0 \). Thus \( a' = b' = c' = 0 \).

This contradiction establishes that \( c' \neq 0 \); we may assume that \( c' = 1 \). We now have the following equalities:

1. \( a + by + z = 0 \)
2. \( ax' + by' + z' = 0 \)
3. \( a' + b'y + z = 0 \)
4. \( a'x' + b'y' + z' = 0 \)

Equalities (1) and (3) imply that \( a + by = a' + b'y \); (2) and (4) imply that \( ax' + by' = a'x' + b'y' \). So we have:

5. \( a - a' = (b' - b)y \)
6. \( (a - a')x' = (b' - b)y' \)

Now suppose that \( b \neq b' \). Then (5) and (6) imply that \( (b' - b)yx' = (b' - b)y' \), thus \((b' - b)(yx' - y') = 0\). If \( b' - b \neq 0 \), then \( yx' - y' = 0 \), which contradicts our assumption. Hence \( b = b' \). Then equality (5) implies that \( a = a' \), and so we conclude that \( L = L' \).

This proves PP1.

The dual property PP2 may be shown in a similar manner reversing the order of products. Finally the points \([0, 0, 1], [0, 1, 0], [1, 0, 0] \) and \([1, 1, 1] \) form a four-point, thus PP3 holds.
Theorem 2 tells us that projective planes of order $k$ exist for any $k$ a prime power.

**Theorem 3.** Let $p = [x, y, z]$, $p' = [x', y', z']$ and $p'' = [x'', y'', z'']$ be distinct points in $\Pi_D$. Then $p''$ is on the line $p \lor p'$ if and only if $x'' = xr + x's$, $y'' = yr + y's$ and $z'' = zr + z's$ for some $r, s \in D$ where $r, s \neq 0$.

**Proof.** Let $p \lor p' = (a, b, c)$. Note that $ax + by + cz = 0$ and $ax' + by' + cz' = 0$. Thus for $r, s \in D$ and $r, s \neq 0$, $a(xr + x's) + b(yr + y's) + c(zr + z's) = axr + byr + czr + ax's + by's + cz's = (ax + by + cz)r + (ax' + by' + cz')s = 0$. Therefore if $x'' = xr + x's$, $y'' = yr + y's$ and $z'' = zr + z's$, then $p'' \leq p \lor p'$.

Now suppose that $p'' \leq p \lor p'$. Since not all of $x, y, z = 0$, $x, y, z \in D$, we may assume that $x = 1$. Since $p$ and $p'$ are distinct, we know that $yx' - y'$ and $xz' - z'$ cannot be both zero. So without loss of generality, assume $yx' - y' \neq 0$, and thus we may solve the equations for $xr + x's = x''$ and $yr + y's = y''$ for $r$ and $s$. Since we assumed $x = 1$, we have $r = x'' - x's$ and $s = (y' - yx')^{-1}(y'' - yx'')$. If $c = 0$, then $a + by = 0$ and $ax' + by' = 0$. Thus $-byx' + by' = b(y' - yx') = 0$. Since $y' - yx' \neq 0$, we have $b = 0$. Therefore $a = 0$, and we reach a contradiction. Thus $c \neq 0$, and we may assume $c = 1$. Since $p$, $p'$ and $p''$ are collinear, it follows that $a + by + z = 0$, $ax' + by' + z' = 0$ and $ax'' + by'' + z'' = 0$. Thus $z'' = -ax'' - by'' = -a(r + x's) - b(yr + y's) = -ar - ax's - byr - by's = -ar - byr - zr + zr - ax's - by's - z's + z's = (-a - by - z)r -(ax' + by' + z')s + zr + z's = zr + z's$. Clearly, if $r = 0$ or $s = 0$, then $p''$ would not be distinct from $p$ and $p'$. Thus $x'' = xr + x's$, $y'' = yr + y's$, $z'' = zr + z's$, and $r, s \neq 0$.

**Corollary 4.** If $p$, $q$, $r$ are collinear, there exist coordinates $p_i, q_i, i = 1, 2, 3$, of $p, q,$
respectively, such that \( r_i = p_i + q_i \) for given coordinates \( r_i \) of \( r \).

G. Desargues proved the following theorem for the real projective plane in the seventeenth century.

**Theorem 5 (Desargues' Theorem).** If \( x, y, z \) and \( x', y', z' \) are triples of noncollinear points of \( \Pi_R \) such that \( x\vee x', y\vee y', z\vee z' \) intersect in a common point \( o \), then corresponding sides meet in three points \( u, v, w \) which are collinear.

![Figure 2: Desarguesian configuration.](image)

The statement of Desargues' Theorem does not hold in every projective plane \( \Pi \). If Theorem 5 is valid in a projective plane, then it is called a *Desarguesian plane*; otherwise,
it is called a non-Desarguesian plane. Figure 2 shows two triangles in a Desarguesian configuration.

Two triangles $pqr$ and $p'q'r'$ are centrally perspective from point $s$ if and only if the respective vertices are collinear with $s$, that is, $p \lor p'$, $q \lor q'$, and $r \lor r'$ pass through $s$. We say that two triangles are axially perspective from line $L$ if and only if the respective sides of the triangles meet on $L$. Thus in a Desarguesian projective plane, every two centrally perspective triangles are axially perspective.

**Theorem 6.** If $D$ is a division ring, then $\Pi_D$ is a Desarguesian plane.

*Proof.* Let $p$, $q$, $r$ and $p'$, $q'$, $r'$ be vertices of two triangles centrally perspective from a seventh point $s$. Let $p = [p_1, p_2, p_3]$, $q = [q_1, q_2, q_3]$, $r = [r_1, r_2, r_3]$, $s = [s_1, s_2, s_3]$, $p' = [p'_1, p'_2, p'_3]$, $q' = [q'_1, q'_2, q'_3]$ and $r' = [r'_1, r'_2, r'_3]$. Since $s$, $p$ and $p'$ are collinear, by Corollary 4, we may assume that the coordinates of $p$ and $p'$ are such that $p'_i = p_i + s_i$, $i = 1, 2, 3$. Similarly, we may assume that the coordinates of $q$ and $q'$ are such that $q'_i = q_i + s_i$, and that the coordinates of $r$ and $r'$ are such that $r'_i = r_i + s_i$, $i = 1, 2, 3$. Let $a = (p \lor q) \land (p' \lor q')$, $b = (p \lor r) \land (p' \lor r')$, $c = (q \lor r) \land (q' \lor r')$. It's then easily checked that $a = [p_1 - q_1, p_2 - q_2, p_3 - q_3]$, $b = [p_1 - r_1, p_2 - r_2, p_3 - r_3]$ and $c = [q_1 - r_1, q_2 - r_2, q_3 - r_3]$. Therefore $a, b, c$ are collinear by Theorem 3. \[\square\]

The converse of Theorem 6 is also true: if $\Pi$ is a Desarguesian plane, then $\Pi$ is isomorphic to $\Pi_D$ for some division ring $D$. For the proof, see Stevenson [14].

An *affine* plane is an ordered pair $A = (P, L)$ where $P$ is a nonempty set of elements called *points* and $L$ is a nonempty collection of subsets of $P$ called *lines* that have the
following properties:

A1. Any two distinct points of \( A \) are on a unique line.

A2. If \( p \) is a point not on the line \( l \), then there is a unique line \( m \) through \( p \) missing \( l \), i.e., \( l \wedge m = \emptyset \). In this case, we say that \( l \) is parallel to \( m \), written \( l \parallel m \). Note that we consider \( l \parallel l \).

A3. There are at least two points on each line; there are at least two lines.

A pencil of parallel lines is a maximal set of mutually parallel lines, or equivalently, a set consisting of a line, together with all the lines parallel to it.

Suppose \( l \parallel m \), and \( m \parallel h \). If there exists \( p = l \wedge h \), then \( l \) and \( h \) are both lines through \( p \) parallel to \( m \), which contradicts A2. Hence parallelism is an equivalence relation. Each line belongs to one and only one pencil. Two lines \( l \) and \( m \) are parallel if and only if they belong to the same pencil.

We may construct a projective plane from an affine plane by adjoining a common new point to each pencil of parallel lines and gathering the new points together to form an additional line, as described below.

Let \( A = (P, L) \) be an affine plane. For each pencil \( \Phi \) of parallel lines, define a symbol \( P_\Phi \), called a point at infinity. Let \( P' = P \cup \{ P_\Phi \mid \Phi \text{ a pencil of } A \} \). For each \( l \in A \), define \( l' = l \cup \{ P_\Phi \} \), where \( \Phi \) is the unique pencil containing \( l \). Let \( l_\infty = \{ P_\Phi \mid \Phi \text{ a pencil of } A \} \). Finally, let \( L' = \{ l' \mid l \in A \} \cup \{ l_\infty \} \).

**Theorem 7.** If \( A = (P, L) \) is an affine plane, then \( \Pi = (P', L') \) is a projective plane.
Proof. First, we would like to show that Axiom PP1 holds. Fix \( p, q \in \Pi \). If \( p, q \in P \), then there exists a unique line \( l \in L \) containing \( p \) and \( q \). Hence \( l' \) is the unique line in \( \Pi \) containing them. If \( p \in P \), and \( q = P_\Phi \) for some pencil \( \Phi \), then \( p \) is on a unique \( l \in A \) which is in pencil \( \Phi \). Thus \( l' \) is the unique line in \( \Pi \) containing them. If both \( p \) and \( q \) are points at infinity, then \( l_\infty \) is the unique line in \( \Pi \) containing them.

Now fix \( l', m' \in \Pi \). If \( l, m \in L \), then they intersect at either some point \( p \in P \) or some point at infinity. In either case, we have \( l' \) intersects \( m' \) at some unique point in \( \Pi \). If say \( m' = l_\infty \), then \( l' \cap m' = P_\Phi \), where \( P_\Phi \) is the point at infinity on \( l' \). Therefore Axiom PP2 holds.

Since there are at least two points on every \( l \in A \), there are at least three points on every \( l' \in \Pi \). Fix \( l', m' \in \Pi \). Then there are at least two points, say \( p_1 \) and \( p_2 \), that are on \( l' \) but not on \( m' \). Similarly, we can find two points \( q_1 \) and \( q_2 \) that are on \( m' \) but not on \( l' \). Note that no three of the points, \( p_1, p_2, q_1 \) and \( q_2 \), are on a line. Therefore Axiom PP3 holds, and \( \Pi = (P', L') \) is a projective plane. \( \square \)

**Proposition 8.** Let \( A \) be a finite affine plane, that is, \( A \) has finitely many points. Then there exists an integer \( n \geq 2 \) such that every line of \( A \) has exactly \( n \) points.

Proof. If \( l \) is a line of \( A \), we let \( r(l) \) denote the number of points on \( l \). Fix two distinct lines \( l \) and \( m \) of \( A \). If \( p \) is a point not on any of those two lines, then A1 and A2 imply that \( p \) is on \( r(l) + 1 = r(m) + 1 \) lines, so \( r(l) = r(m) \). If there is no such point \( p \), then we claim that \( l \parallel m \). Suppose not, and let \( o = l \cap m \). A3 implies that there exists a point \( a \leq l, a \neq o \). By A1, and since \( l \neq m \), we have \( a \nless m \). Hence there is a line \( k \) through
a parallel to \( m \). Again, by axioms of affine plane, there exists a point \( b \leq k, b \neq a, \) and \( b \not\leq l \). Note that \( b \not\leq m \) as well, since \( k \parallel m \). So we have a point \( b \not\leq l \cup m \), contrary to our assumption. Thus \( l \parallel m \). By A3, we may assume there are two points, \( p_1 \) and \( p_2 \) on \( l \), and two points, \( q_1 \) and \( q_2 \) on \( m \). Let \( k = p_1 \vee q_1, q_2 \not\leq l \cup k \). Hence by earlier argument, \( r(l) = r(k) \). Similarly, \( p_2 \not\leq m \cup k \implies r(m) = r(k) \). So we have \( r(l) = r(m) \) again. Let \( n = r(l) \). Since there are at least two points on each line, \( n \geq 2 \).

If each line of a (finite) affine plane contains exactly \( n \) points, the plane is said to have order \( n \).

**Theorem 9.** Let \( A \) be an affine plane of order \( n \). Then

1. \( A \) has exactly \( n^2 \) points.
2. Each point is on \( n + 1 \) lines.
3. Each pencil contains \( n \) lines.
4. The total number of lines is \( n(n + 1) \).
5. There are \( n + 1 \) pencils of parallel lines.

**Proof.** (i) Fix a line \( l \), and suppose \( p_1, ..., p_n \) are the points on \( l \). Let \( o \) be a point not on \( l \). Let \( m_1 = p_1 \vee o \), and for \( i = 2, ..., n \), let \( m_i \) be the unique line through \( p_i \) parallel to \( m_1 \). Each of the lines \( m_1, ..., m_n \) has \( n \) points, there are \( n \) of these lines, and no pair of lines has a point in common. Therefore there are \( n^2 \) many points contained in \( m_1 \cup ... \cup m_n \). If \( p \) is any point in the plane, then either \( p \in m_1 \), or there is a unique line \( k \) through \( p \)
parallel to $m_1$. In the latter case, $k$ must intersect $l$ at one of the points $p_2, \ldots, p_n$, for otherwise we would have $m_1 \parallel k \parallel l$, with $p_1 = m_1 \cap l$, contradicting A2. Hence $k = m_i$ for some $i = 2, \ldots, n$. Thus every point is in one of the lines $m_1, \ldots, m_n$. Therefore $A$ has exactly $n^2$ points.

(ii) Fix a point $p \in A$ and let $l = \{q_1, \ldots, q_n\}$ be a line not containing $p$. The lines $l_i = q_i \vee p$ and the unique line $l_{n+1}$ through $p$ parallel to $l$ are all distinct. Moreover, if $m$ is any line through $p$, either $m \parallel l$, which implies that $m = l_{n+1}$, or $m \cap l = q_i$ for some $i$, which imples that $m = l_i$ for some $i$. Therefore $p$ is on exactly $n + 1$ lines.

(iii) Fix a line $l \in A$. Suppose $p_1 \leq l$. Let $p_2$ be a point not on $l$. Then consider the line $p_1 \vee p_2 = m = \{p_1, \ldots, p_n\}$. For $i = 2, \ldots, n$, let $l_i$ be the line through $p_i$ parallel to $l$. There are $n - 1$ such lines, and they are all distinct. If $k$ is any line parallel to $l$, since $l$ is the only line containing $p_1$ parallel to $k$, $k \cap m = p_i$, for some $i = 2, \ldots, n$. So $k = l_i$ for some $i$, and consequently, the lines $l_2, \ldots, l_n$ are the only lines parallel to $l$.

(iv) Fix a point $p \in A$, and let $l_1, \ldots, l_{n+1}$ be the lines through $p$. Each of this lines is in a pencil consisting of $n$ lines. So the lines through $p$ together with the lines parallel to them account for $n(n + 1)$ distinct lines. Let $m$ be any line in $A$. If $p \leq m$, then $m$ has been counted as one of the $l_i$'s. If $p \not\leq m$, then there is a unique line $k$ through $p$ parallel to $m$. In this case, $m$ has also been counted as a line in the same pencil as line $k$. Therefore there are $n(n + 1)$ lines in $A$.

(v) Since each of the $n(n + 1)$ lines is in one and only one pencil, and each pencil contains $n$ lines, there are $n + 1$ pencils. □
Note that if we use the construction given just before Theorem 7, we may obtain a projective plane of order \( n \) from an affine plane of the same order. This construction may also be reversed: Given a projective plane, an affine plane is obtained by removing any line and all the points on that line. For the details, see Bennett [3].

Now, following Hall [6], [7], we show how to introduce coordinates in an arbitrary projective plane.

Let \( \Pi \) be a projective plane and let \( X, Y, O, I \) be four points in \( \Pi \) no three on a line. Let \( O = (0,0) \) and \( I = (1,1) \) and \( (X \vee Y) \cap (O \vee I) = (1) \). Choose a set \( R \) which contains 0, 1 and is in one-to-one correspondence with the points of \( O \vee I \setminus \{1\} \), and let \( B = R \setminus \{0,1\} \). For example, if \( \Pi \) has order \( n \), we could take \( B = \{2, \ldots, n-1\} \). Assign the rest of the points on \( O \vee I \) distinct labels \((b, b)\) with \( b \in B \). For any point \( P \) not on \( X \vee Y \) or \( O \vee I \), assign \( P \) the coordinate \((a, b)\) where \((Y \vee P) \cap (O \vee I) = (a, a)\) and \((X \vee P) \cap (O \vee I) = (b, b)\). For any point \( Q \) on \( X \vee Y \) with \( Q \neq Y \), the line \( O \vee Q \) contains a unique point \( (1, m) \) with first coordinate 1; assign \( Q \) the coordinate \((m)\). Finally, let \( Y = (\infty) \).

Let \( l_\infty \) denote the line \( X \vee Y \). Note that \( l_\infty \) contains \((1)\). If \( l \) is any other line through \( Y \) except \( l_\infty \), then \( l \) intersects \( O \vee I \) in some point \((c, c)\). In this case, every point except \( Y \) on \( l \) has coordinates \((c, y)\) for some \( y \), and we let \( l = l_c \). We have labelled all the lines through \( Y \). If \( k \) is a line not through \( Y \), then \( k \cap l_\infty = (m) \) and \( k \cap l_0 = (0, b) \) for some pair \( m, b \). In this case, let \( k = l_{(m,b)} \).

Now note that the lines \( l_{(m,b)} \) have the property that they never contain two points with the same \( x \)-coordinate, since the line through \((c, y)\) and \((c, y')\) is \( l_c \). On the other
hand, for every $c \in R$, $l_{(m,b)} \wedge l_c$ is some standard point $(c,y)$. Thus we can use these lines to define a ternary operation $T(x, m, b)$ on $R$ by

$$T(x, m, b) = y \iff (x, y) \in l_{(m,b)}$$

The ternary operation $T$ defined above satisfies the following:

T1. $T(0, m, c) = T(a, 0, c) = c$.

T2. $T(1, m, 0) = T(m, 1, 0) = m$.

T3. Given $m, b, d$ with $m \neq 0$ there exists exactly one $x$ such that $T(x, m, b) = d$.

T4. Given $a, b, d$ with $a \neq 0$ there exists exactly one $y$ such that $T(a, y, b) = d$.

T5. Given $a, m, d$ there exists exactly one $z$ such that $T(a, m, z) = d$.

T6. Given $m, m', b, b'$ with $m \neq m'$ there exists exactly one $x$ such that $T(x, m, b) = T(x, m', b')$.

T7. Given $a, a', c, c'$ with $a \neq a'$ there exists exactly one pair $m, b$ such that $T(a, m, b) = c$ and $T(a', m, b) = c'$.

A ternary algebra $R = (R, T, 0, 1)$ satisfying T1 - T7 is called a ternary ring. Note that the ternary ring $R$ constructed to coordinatize $\Pi$ depends on the choice of the quadrangle $X, Y, O, I$.

Now given a ternary ring $R = (R, T, 0, 1)$, we may construct a plane $\Pi_T$ as follows. Let $P = \{(x, y) \mid x, y \in R\} \cup \{(x) \mid x \in R\} \cup \{\langle \infty \rangle \}$. Let $L = l_{(m,b)} \cup l_c \cup l_\infty$, where
\( l_{(m,b)} = \{(x,y) \mid y = T(x,m,b), x,y,m,b \in R\} \cup \{(m)\}, \ l_c = \{(x,y) \mid x = c, x,y,c \in R\} \cup \{(\infty)\}, \ \text{and} \ l_{\infty} = \{(m) \mid m \in R\} \cup \{(\infty)\}. \\

**Theorem 10.** \( \Pi_T \) is a projective plane.

*Proof.* First we show that there exists exactly one line passing through two distinct points \( p \) and \( q \).

**Case 1.** Let \( p = (a,b) \) and \( q = (c,d) \). If \( a = c \), then \( p \) and \( q \) lie on line \( l_a \). To show that this line is unique, observe that \( p \) and \( q \) certainly do not occupy \( l_k \) for any \( k, k \neq a \). Also if \( p, q \in l_{(m,k)} \), then \( b = T(a,m,k) = T(c,m,k) = d \), and so \( p = q \), a contradiction.

If \( a \neq c \), then by T7, there exists unique one pair \( m, k \) such that \( T(a,m,k) = b \) and \( T(c,m,k) = d \). Thus \( l_{(m,k)} \) contains both points. Clearly neither \( l_{\infty} \) nor any line \( l_s \) may pass through both points, so the line \( l_{(m,k)} \) is unique.

**Case 2.** Let \( p = (a,b) \) and \( q = (c) \). By T5, there exists exactly one \( k \) such that \( T(a,c,k) = b \). Thus \( p \leq l_{(c,k)} \) and \( q = (c) \leq l_{(c,k)} \). Clearly no other line contains \( q \) and \( q \).

**Case 3.** Let \( p = (a,b) \) and \( q = (\infty) \). Then it’s easy to see that \( l_a \) is the only line containing \( p \) and \( q \).

**Case 4.** Let \( p = (a) \) and \( q = (b) \). Then \( l_{\infty} \) is the only line containing both points.

**Case 5.** Let \( p = (a) \) and \( q = (\infty) \). Then again \( l_{\infty} \) is the unique line passing through both points.

So PP1 is satisfied. A similar argument shows that two lines intersect in exactly one
point. As for PP3, it is easy to verify that (0, 0), (0), (1, 1), and (00) are four point, no three of which are on a line.

**Theorem 11.** If \((R, +, \cdot)\) is a division ring, then \((R, t)\) is a ternary ring where \(t\) is defined by \(t(a, b, c) = ab + c\).

**Proof.** For any \(m, c, a \in R\), we have \(t(0, m, c) = 0m + c = c\), and \(t(a, 0, c) = a0 + c = c\), so \(T1\) holds. For any \(m \in R\), \(t(1, m, 0) = 1m + 0 = m\), and \(t(m, 1, 0) = m1 + 0 = m\), so \(T2\) holds. Given \(m, b, d \in R\), with \(m \neq 0\), let \(x = (d - b)m^{-1}\). Then clearly \(t(x, m, b) = d\). Suppose there exists \(y\) such that \(t(y, m, b) = d\). Then \(xm + b = ym + b\) implies that \(xm = ym\). Multiplying \(m^{-1}\) on both sides, we get \(x = y\). So \(x\) is unique, and \(T3\) holds. Similar arguments show that both \(T4\) and \(T5\) hold. Given \(m, m', b, b'\) with \(m \neq m'\). Then \(m - m' \neq 0\), so \((m - m')^{-1}\) exists. Let \(x = (b' - b)(m - m')^{-1}\). Then \(t(x, m, b) - t(x, m', b') = (b' - b)(m - m')^{-1}m + b - (b' - b)(m - m')^{-1}m' - b' = (b' - b)(m - m')^{-1}(m - m') + b - b' = 0\). So \(t(x, m, b) = t(x, m', b')\). Suppose there exists \(y\) such that \(t(y, m, b) = t(y, m', b')\). Then \(xm + b - xm' - b' = 0 = ym + b - ym' - b'\), which implies that \(x(m - m') = y(m - m')\), hence \((x - y)(m - m') = 0\). Since \(m - m' \neq 0\), we get \(x - y = 0\). So we have \(x = y\), and \(T6\) holds. Given \(a, a', c, c'\) with \(a \neq a'\). Let \(m = (a' - a)^{-1}(c' - c)\) and \(b = c - a(a' - a)^{-1}(c' - c)\). Then it is easy to check that \(t(a, m, b) = c\) and \(t(a', m, b) = c'\). Suppose that \(u, v\) is another pair such that \(t(a, u, v) = c\) and \(t(a', u, v) = c'\). Then we have \(au + v = am + b\) and \(a'u + v = a'm + b\), which implies that \(au + v - am - b = 0 = a'u + v - a'm - b\). So \((a - a')(u - m) = 0\). Since \(a - a' \neq 0\), we have \(u = m\), and \(v = b\) by \(T5\). Therefore \(T7\) holds. \(\square\)
In view of Theorem 11, we may regard the concept of ternary ring as a generalization of the concept of division ring. Now we want to construct ternary rings that are not derived from division rings, so that we can construct non-Desarguesian projective planes.

A loop is an algebra $L = (L, *, e)$ with the following properties.

1. $a * e = a$ and $e * a = a$ for every $a \in L$.
2. Given $a, b \in L$, there exists unique $x \in L$ such that $a * x = b$.
3. Given $c, d \in L$, there exists unique $y \in L$ such that $y * c = d$.

A double loop is an algebra $R = (R, +, \cdot, 0, 1)$ such that

1. $(R, +, 0)$ is a loop,
2. $(R - \{0\}, \cdot, 1)$ is a loop,
3. $x0 = 0$ and $0x = 0$ for every $x \in R$.

**Theorem 12.** A double loop coordinatizes a projective plane if and only if it has the following two properties.

A. Given $m, m', b, b'$ with $m \neq m'$, there is a unique $x \in R$ such that $xm + b = xm' + b'$.

B. Given $a, a', c, c'$ with $a \neq a'$, there is a unique pair $x, y \in R$ such that $ax + y = c$ and $a'x + y = c'$.

Property A reflects that, in an affine plane, two nonparallel lines should intersect in a unique point. Property B says that two points determine a unique line.
Theorem 13. In a finite double loop, property A and property B are equivalent.

For detail of the proof of Theorem 13, see Nation [12].

A quasifield is a double loop \( Q = (Q, +, \cdot, 0, 1) \) such that

1. + is associative,

2. \((a + b)c = ac + bc\) for all \(a, b, c \in Q\),

3. Given \(m, m', c\) with \(m \not= m'\), there is a unique \(x \in Q\) such that \(xm = xm' + c\).

Property 3 in the definition implies that quasifields form a rich class of coordinatizing double loops, and hence a source of non-Desarguesian projective planes. However, the next lemma shows that a projective plane coordinatized by a quasifield will always have prime power order.

Note that property 1 implies that \((Q, +, 0)\) is a group, and property 3 is called the planar property.

Lemma 14. [5] Let \(Q = (Q, +, \cdot, 0, 1)\) be a finite double loop in which addition is associative and the right distributive law holds. Then

1. \((-a)b = -ab\) for all \(a, b \in Q\),

2. \((Q, +, 0)\) is an elementary abelian \(p\)-group, i.e., there is a prime \(p\) such that \(|Q| = p^k\) and every element of \(Q\) has additive order \(p\), and

3. the planar property holds, so \(Q\) is a quasifield.
Proof. By the comment before the lemma, we know that \((Q, +, 0)\) is a group. Fix any \(a, b \in Q\), \(0 = 0b = (-a + a)b = (-a)b + ab\), thus \((-a)b = -ab\).

For any \(m \neq 0\), the map \(\tau_m : x \mapsto xm\) is an automorphism of \((Q, +, 0)\). Moreover, given any \(x, y \in Q\), since \(Q\) is a double loop, there exists unique \(g \in Q\) such that \(xg = y\). So the automorphism group of \((Q, +, 0)\) is transitive, so each row in the multiplication table of a nonzero element is a permutation, which implies that every nonzero element of \(Q\) has the same order. \(Q\) is finite, so some element has a prime order \(p\).

To prove that the additive group \((Q, +, 0)\) is commutative, we would like to prove that the planar property holds first. Let \(m, m' \in Q\) with \(m \neq m'\). Consider the mapping \(\sigma_{m,m'} : x \mapsto -xm' + xm\). Suppose there exist \(x, y \in Q\) such that \(\sigma_{m,m'}(x) = \sigma_{m,m'}(y)\), then \(-xm' + xm = \sigma_{m,m'}(x) = \sigma_{m,m'}(y) = -ym' + ym\), which implies that \(ym' - xm' = ym - xm\), so \((y - x)m' = (y - x)m\). Transitivity of the automorphism group of \((Q, +, 0)\) and \(m \neq m'\) imply that \(y - x = 0\), i.e., \(x = y\). So the map is one-to-one. Finiteness implies that the map is also onto. Therefore the planar property holds, and hence \(Q\) is a quasifield.

Assume there exist elements \(a, b \in Q\) such that \(b + a - b = ac, c \neq 1\). By the planar property, there exists unique element \(d\) such that \(-dc + d = b\), which implies that \(dc + b = d\), and so \(dc + b + a = d + a\). Since \(b + a - b = ac\), we have \(dc + (ac + b) = d + a\). Since addition is associative, we get \((dc + ac) + b = d + a\), and right distributive law gives \((d + a)c + b = d + a\). Thus, \((-d + a)c + (d + a) = b\). The uniqueness of \(d\) then implies \(d + a = d\), or \(a = 0\). Hence \(a \neq 0\) implies \(b + a = a + b\) for all \(b \in Q\), and this clearly holds if \(a = 0\). \(\square\)
Let $F$ be a finite field, and let $Q = F^d$ be a vector space of dimension $d$ over $F$. The first projection embeds $F$ into $Q$, so we can identify $c \in F$ with the vector $(c, 0, \ldots, 0) \in Q$. Suppose that we can find a set of linear transformations $\rho(m), m \in Q$ such that

1. $\rho(0) = O$, the 0 matrix,

2. $\rho(1) = I$, the identity matrix,

3. the first row of $\rho(u)$ is $u$,

4. $\rho(m) - \rho(m')$ is nonsingular whenever $m \neq m'$.

If we define multiplication by $u \circ v = u \rho(v)$, then $Q$ is a quasifield.

For example, let $Q = F^2$, where $F$ is a finite field of order greater than 2. Let $f(x) = x^2 - rx - s$ be an irreducible quadratic over $F$. Define

\[
\rho(c, 0) = \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix}
\]

and for $d \neq 0$,

\[
\rho(c, d) = \begin{pmatrix} c & d \\ -d^{-1}f(c) & -c + r \end{pmatrix}
\]

Then $Q$ is called a Hall quasifield, which coordinatizes a non-Desarguesian plane.

The classic result on the order of finite projective planes is the Bruck-Ryser Theorem [4].

**Theorem 15.** If $n \equiv 1, 2 \mod 4$ and $n$ is not the sum of two squares, then there is no projective plane of order $n$. 

23
We have shown earlier that there is a Desarguesian projective plane of every prime power. The previous quasifield construction yields a non-Desarguesian projective plane of order $q^2$ for any prime power $q \geq 3$. Similar construction of non-Desarguesian projective planes may be found in [5]. Thus far, all known finite projective planes have prime power order. There are infinitely many numbers $n$ of the form given in the Bruck-Ryser Theorem, and 6, 14, 21, 22, 30 are just the first few of them. Lam, Thiel and Swiercz proved that there is no plane of order 10 [10]. Therefore we ask the questions: Is there a projective plane of non-prime-power order? Is there a non-Desarguesian plane of prime power order? In our effort to answer the questions, we have done extensive computer search for a non-Desarguesian plane of order 12, and some search for non-Desarguesian planes of order 11 and 15.
Chapter 3
Extending Dual Linear Spaces

We introduced projective planes, some of the properties and different ways to construct Desarguesian or non-Desarguesian planes in the previous chapter. In our effort to answer the questions raised at the end of that chapter, we will introduce the concept of a linear space [1], [2], [3] and its dual, a semiplane. The computer programs written by Dr. Nation implement the one line extension theory [11]. We will explain how these programs work in general and state the results for some of the tests that we have done.

A linear space is a triple $S = (P, L, \leq)$ consisting of a set $P$ of elements called points, a set $L$ of distinguished subsets of points, called lines, and $p \leq l$ means that the point $p$ is on the line $l$, satisfying the following axioms:

L1 Any two distinct points of $S$ belong to exactly one line of $S$.

L2 Any line of $S$ contains at least two points of $S$.

Note that in a linear space, L1 implies that two distinct lines intersect in at most one point. If a linear space $S$ also has the property that there are three points not on a common line, then $S$ is called a non-trivial linear space.

We define a dual linear space, or semiplane, to be a triple $S' = (P', L', \leq')$ satisfying the dual statements of L1 and L2. Thus in a dual linear space, every pair of lines intersect in exactly one point, and every point lies on at least two lines. The dual to the previous observation implies that two points lie on at most one line in a dual linear space.
Again \( p \lor q \) denotes the line through points \( p \) and \( q \), if there is one. Similarly, \( k \land l \) denotes the point on both lines \( k \) and \( l \), if there is one. As usual, \( p \leq l \) means that point \( p \) is on line \( l \), or equivalently, that line \( l \) passes through point \( p \).

We say that \( S \) is a \textit{finite linear space} if \( S \) has finitely many points. For any point \( p \) in \( S \), we let \( r(p) \) denote the number of lines on \( p \). Similarly, we let \( r(l) \) denote the number of points on the line \( l \). We shall refer to \( r(p) \) and \( r(l) \) as the \textit{rank} of the respective point \( p \) and line \( l \).

**Proposition 16.** Let \( S \) be a linear space. If \( p \) is a point in \( S \) and \( l \) is a line not through \( p \), then \( r(p) \geq r(l) \).

\[ \text{Proof.} \text{ This follows immediately from property L1.} \]

**Proposition 17.** If \( S = (P, L, \leq) \) is a finite linear space, then

\[
\sum_{l \in L} r(l) = \sum_{p \in P} r(p)
\]

\[ \text{Proof.} \text{ The sum on the left counts the number of points on each line, line by line. The sum on the right counts the number of lines through each point, point by point. These are just two different ways of counting point-line incidences. Clearly, they are the same.} \]

**Proposition 18.** If \( S = (P, L, \leq) \) is a finite linear space, then

\[
\sum_{l \in L} r(l)(r(l) - 1) = |P|(|P| - 1)
\]

where \( |P| \) is the number of points.
Proof. Since any two distinct points determine a unique line in $S$, the sum on the left counts the number of ordered pairs of points line by line, which is clearly equal to the right-hand side.

**Proposition 19.** If $S = (P, L, \leq)$ is a finite linear space, then

$$\sum_{p \in P} \tau(p)(\tau(p) - 1) \leq |L|(|L| - 1)$$

where $|L|$ is the number of lines. The equality holds if and only if every pair of lines meet.

Proof. The left-hand side counts the number of intersecting pairs of lines, whereas the right-hand side counts the number of all pairs of lines.

A linear space with $n$ points, where $n \geq 3$, that has one line containing $n - 1$ points and $n - 1$ lines containing two points, is called a near pencil.

The following theorem is the Fundamental Theorem of finite linear spaces. For the proof of the theorem, see [2].

**Theorem 20 (Fundamental Theorem).** Let $S = (P, L, \leq)$ be a finite non-trivial linear space. Then $|L| \geq |P|$. Moreover, equality holds if and only if $S$ is a projective plane or a near-pencil.

Theorem 20 implies that in a finite dual linear space, we have $|P| \geq |L|$.

An embedding of a linear space $S = (P, L, \leq)$ into a linear space $S' = (P', L', \leq')$ is a function $f$ mapping $P$ into $P'$ and $L$ into $L'$ which is one-to-one on both points and lines, and such that $p \leq l$ if and only if $f(p) \leq f(l)$.
We note here that any linear space can be embedded into a linear space containing four points such that no three are collinear.

Marshall Hall proved that a linear space $S$ in which one can find four points, no three collinear, is embeddable in a projective plane using the following construction [6]. Consider all pairs of lines which do not meet, and add a new point to any such pair of lines. Now consider all pairs of points not joined by a line, and add a new line through any such pair. Continuing in this way, we obtain a projective plane of usually infinite order. The question remains: *Can a finite linear space be embedded in a finite projective plane? If so, of what order?*

Let $\Sigma = \langle P, L, \leq \rangle$ be a projective plane. Observe that there is a natural correspondence between subsets of $L$ and semiplanes $\Pi$ contained in $\Sigma$. This is given by the map $K \mapsto \Pi_K$, where $K$ is a subset of $L$ and $\Pi_K = \langle Q, K, \leq^* \rangle$ is a semiplane obtained by taking $Q = \{k \wedge \ell : k, \ell \in K \text{ and } k \neq \ell\}$ and $\leq^*$ the restriction of $\leq$ to $Q \times K$.

Let $\Pi = \langle P_0, L_0, \leq_0 \rangle$ be a semiplane, and let $\Sigma = \langle P, L, \leq \rangle$ be a semiplane properly extending $\Pi$. Let $k \in L - L_0$, and let $M = \{p \in P_0 : p \leq k\}$. Note that $s \lor t$ is undefined in $\Pi$ for all pairs $s, t \in M$. Let $D = \{\ell \in L_0 : \ell \wedge k \notin P_0\}$. For each $\ell \in D$, let $q_\ell = \ell \wedge k$; note that $\ell \neq \ell'$ implies $q_\ell \neq q_{\ell'}$. Define an extension $\Pi' = \langle P_1, L_1, \leq_1 \rangle$ of $\Pi$ as follows.

1. $P_1 = P_0 \cup \{q_\ell : \ell \in D\}$.
2. $L_1 = L_0 \cup \{k\}$.
3. $\leq_1$ is the order induced by $\Sigma$, which consists precisely of the relations holding in $\Pi$ and the new relations $m \leq_1 k$ for all $m \in M$, and $q_\ell \leq_1 \ell$ and $q_\ell \leq_1 k$ for each
\[ \ell \in D. \]

Then \( \Pi' \) is a semiplane with \( \Pi \subseteq \Pi' \subseteq \Sigma \). Moreover, \( s \lor t \) is defined in \( \Pi' \) for all \( s, t \in M \), viz., \( s \lor t = k \).

Now let us describe this construction without reference to \( \Sigma \). Let \( \Pi = \langle P_0, L_0, \leq_0 \rangle \) be a semiplane that is not a projective plane, and let \( M \subseteq P_0 \) be a set of points whose pairwise join is undefined in \( \Pi \). (We allow \( |M| = 0 \) or 1, which can happen in the case of a near pencil, but the important case is when \( |M| \geq 2 \).) Let \( k \) be a symbol for a new line. Let \( D = \{ \ell \in L_0 : p \not\in_0 \ell \text{ for all } p \in M \} \). For each line \( \ell \in D \), introduce a new point \( q_\ell \). Define \( \Pi' = \langle P_1, L_1, \leq_1 \rangle \) by

i. \( P_1 = P_0 \cup \{ q_\ell : \ell \in D \} \).

ii. \( L_1 = L_0 \cup \{ k \} \).

iii. \( \leq_1 \) consists of \( \leq_0 \), the relations \( m \leq_1 k \) for all \( m \in M \), and for each \( \ell \in D \) the relations \( q_\ell \leq_1 \ell \) and \( q_\ell \leq_1 k \).

We will denote the extension \( \Pi' \) obtained thusly by \( \alpha_M(\Pi) \). We refer to the construction as a one line extension, or the one-line extension of \( \Pi \) determined by \( M \).

**Theorem 21.** If \( \Pi \) is a semiplane and \( M \) is a set of points whose pairwise join is undefined in \( \Pi \), then \( \alpha_M(\Pi) \) is a semiplane with \( \Pi \subseteq \alpha_M(\Pi) \). Moreover, let \( \Sigma \) be a semiplane with \( \Pi \subseteq \Sigma \), where \( \Pi = \langle P_0, L_0, \leq_0 \rangle \) and \( \Sigma = \langle P, L, \leq \rangle \). Let \( k \in L - L_0 \), and let \( M = \{ p \in P_0 : p \leq k \} \). Then the subsemiplane \( \Pi_{\langle L_0 \cup \{ k \} \rangle} \) of \( \Sigma \) is isomorphic to \( \alpha_M(\Pi) \).
Let $\Pi$ and $\Sigma$ be two semiplanes. We say that $\Sigma$ properly extends $\Pi$ if $\Pi \subseteq \Sigma$ and $\Sigma$ has at least one line not in $\Pi$.

**Corollary 22.** Let $\Pi$ be a semiplane. Every semiplane properly extending $\Pi$ can be obtained as the union of a (possibly infinite) sequence of one line extensions starting with $\Pi$. Every projective plane that is a minimal extension of $\Pi$ (i.e., does not have a proper subplane containing $\Pi$) can be obtained as the union of a (possibly infinite) sequence of one line extensions with $|M| \geq 2$.

The construction of $\alpha_M(\Pi)$ depends, of course, on the choice of $M$. For example, by always taking $|M| = 2$ we can obtain a sequence whose union is the projective plane freely generated by $\Pi$, as constructed by M. Hall.

Now let us look at some changes in parameters affected by a one line extension $\Pi' = \alpha_M(\Pi)$, with the notation as above.

1. $|P_1| = |P_0| + |D|$.

2. $|L_1| = |L_0| + 1$.

Define the rank $r_\Pi(p)$ of a point $p \in P_0$ by $r_\Pi(p) = |\{ \ell \in L_0 : p \leq \ell \}|$, and define $r_{\Pi'}(p)$, $r_{\Pi}(\ell)$, $r_{\Pi'}(\ell)$ similarly.

3. For $p \in P_0$,

$$r_{\Pi'}(p) = \begin{cases} 
    r_\Pi(p) + 1 & \text{if } p \in M \\
    r_\Pi(p) & \text{if } p \notin M
\end{cases}$$

and $r_{\Pi'}(q_\ell) = 2$ for $\ell \in D$. 
4. For \( \ell \in L_0 \),
\[
\begin{align*}
& r_{\Pi}(\ell) = \\
& \begin{cases} 
& r_{\Pi}(\ell) + 1 & \text{if } \ell \in D \\
& r_{\Pi}(\ell) & \text{if } \ell \notin D 
\end{cases}
\end{align*}
\]

and \( r_{\Pi}(k) = |M| + |D| \).

5. The new joins that are defined in \( \Pi' \) are
\[
\begin{align*}
& s \lor t = k \quad \text{for distinct } s, t \in M, \\
& q_\ell \lor t = k \quad \text{for } \ell \in D \text{ and } t \in M, \\
& q_\ell \lor q_m = k \quad \text{for distinct } \ell, m \in D, \\
& q_\ell \lor u = \ell \quad \text{for } u \leq \ell \in D.
\end{align*}
\]

Finally, given \( \Pi \subseteq \Sigma \), we would like to know when there is a sequence of one line extensions from \( \Pi \) to \( \Sigma \) with \( |M| \geq 2 \) for every step. This cannot be done, for example, if \( \Pi \) is a subplane of the projective plane \( \Sigma \). We need that at each step the new line should intersect the existing lines in at least two existing points, i.e., there are at least two existing points whose join is undefined. The following result gives a sufficient condition for this to occur.

**Theorem 23.** Let \( \Pi \subseteq \Sigma \), where \( \Pi \) is a semiplane and \( \Sigma \) is a projective plane of order \( n \). If \( \Pi \) is not a near pencil or a (sub)plane, then every sequence of one line extensions from \( \Pi \) to \( \Sigma \) has \( |M| \geq 2 \) for every step.

We can use Theorem 21 to find a necessary condition for a semiplane \( \Pi = (P_0, L_0, \leq_0) \) to be extendible to a projective plane of order \( n \). First, we note the obvious necessary
conditions: \(|P_0| \leq n^2 + n + 1, |L_0| \leq n^2 + n + 1, r_{\Pi}(p) \leq n + 1\) for all \(p \in P_0\), and \(r_{\Pi}(\ell) \leq n + 1\) for all \(\ell \in L_0\).

Notice that, by observation 4 above, adding a new point \(q_\ell\) raises the rank of one of the existing lines by exactly one. If \(\Pi\) is to be extended to a plane of order \(n\) by a sequence of one line extensions, then all these line ranks must eventually be raised to \(n + 1\). Thus we must have

\[
\sum_{\ell \in L_0} r_{\Pi}(\ell) + n^2 + n + 1 - |P_0| \geq |L_0|(n + 1).
\]

It is useful to introduce the function \(\rho_n(\Pi) = \sum_{\ell \in L_0} r_{\Pi}(\ell) + n^2 + n + 1 - |P_0| - |L_0|(n + 1)\). The result can then be stated as follows.

**Theorem 24.** If \(\Pi\) is a semiplane that can be extended to a projective plane of order \(n\), then \(\rho_n(\Pi) \geq 0\).

After a one line extension \(\Pi' = \alpha_M(\Pi)\), adding a new line \(k\), we have

\[
\Delta \rho_n = \rho_n(\Pi') - \rho_n(\Pi) = r_{\Pi'}(k) + \left(\sum_{\ell \in L_0} r_{\Pi'}(\ell) - r_{\Pi}(\ell)\right) - |D| - (n + 1)
\]

\[
= r_{\Pi'}(k) + |D| - |D| - (n + 1)
\]

\[
= r_{\Pi'}(k) - (n + 1).
\]

Thus \(\Delta \rho_n\) is nonpositive, i.e., \(\rho_n\) is always decreasing, so long as we keep \(r_{\Pi'}(k) = |M| + |D|\) at most \(n + 1\). For emphasis, we record this.

**Lemma 25.** If \(\Pi\) and \(\Psi\) are semiplanes with \(\Pi \subseteq \Psi\), and \(r_{\Psi}(\ell) \leq n + 1\) for all lines of \(\Psi\), then \(\rho_n(\Pi) \geq \rho_n(\Psi)\), and hence \(\Delta \rho_n \leq 0\).
In practice, we will be trying to extend a relatively small initial semiplane $\Pi$ to a projective plane $\Sigma$ of order $n$ by a sequence of one line extensions. Since we need to keep $\rho_n(\Psi) \geq 0$ for each semiplane $\Psi$ in the sequence of extensions, we want to have $\Delta \rho_n$ small, i.e., heuristically we want $r_{\Pi'}(k) = |M| + |D|$ close to $n + 1$. If $\Pi = \langle P_0, L_0, \leq_0 \rangle$, we may start with $\rho_n(\Pi)$ relatively big, but soon enough it will decrease to a very small number which often is 0, so we can normally expect that $\rho_n(\Pi) \ll n^2 + n + 1 - |L_0|$, which is the number of one line extensions required to go from $\Pi$ to $\Sigma$. Hence most of the one line extensions must have $\Delta \rho_n = 0$. Moreover, since any one line extension raises the rank of an existing point by at most 1, and each new point $q_\ell$ has rank 2, which means it will take at least $n - 1$ extensions to raise the rank of $q_\ell$ to $n + 1$, so no new point $q_\ell$ can be introduced in the last $n - 1$ extensions. So from the conclusion of the $n$th-to-last extension onwards, each line must contain $n + 1$ points. In particular, we must have $r_{\Pi'}(k) = n + 1$, and hence $\Delta \rho_n = 0$, for the last $n$ extensions.

**Note.** The function $\rho_n$ in the condition $\rho_n(\Pi) \geq 0$ depends on $n$, and for small values it may be decreasing in $n$. Thus a semiplane $\Pi$ may be extendible to a plane of order $m < n$ even though $\rho_n(\Pi) < 0$, and indeed this does occur. Let $A_4$ be the affine plane of order 4. Clearly $A_4$ can be extended to the projective plane of order 4, but $\rho_5(A_4) = -25$.

Let $\Sigma = \langle P, L, \leq \rangle$ be a semiplane. The relation $\not\leq$ contained in $P \times L$ induces a Galois connection between the points and lines of $\Sigma$. The corresponding maps are, for
$R \subseteq P$ and $K \subseteq L$,

$$\lambda_{\Sigma}(R) = \{ \ell \in L : r \not\in \ell \text{ for all } r \in R \}$$

$$\pi_{\Sigma}(K) = \{ p \in P : p \not\in k \text{ for all } k \in K \}.$$ 

The usual properties of a Galois connection hold, e.g., $R \subseteq \pi_{\Sigma} \lambda_{\Sigma}(R)$.

For each $R \subseteq P$, define the semiplane $\gamma_{\Sigma}(R) = \Pi(\lambda_{\Sigma}(R))$. Note that if $\gamma_{\Sigma}(R) = (P^*, \lambda_{\Sigma}(R), \leq^*)$, then $p \in P^*$ if and only if $p$ is on at least two lines containing no point of $R$. Since $p \in P - \pi_{\Sigma} \lambda_{\Sigma}(R)$ if and only if $p$ is on at least one line containing no point of $R$, we have $P^* \subseteq P - \pi_{\Sigma} \lambda_{\Sigma}(R) \subseteq P - R$.

**Theorem 26.** If $\Sigma = (P, L, \leq)$ is a semiplane and $R \subseteq P$, then $\gamma_{\Sigma}(R)$ is a semiplane. Moreover, let $\Pi = (P_0, L_0, \leq_0)$ be a semiplane with $\Pi \subseteq \Sigma$. If $R \subseteq \pi_{\Sigma}(L_0)$, i.e., no point of $R$ is on a line of $L_0$, then $\Pi \subseteq \gamma_{\Sigma}(R)$, and there is a sequence of one line extensions refining the sequence $\Pi \subseteq \gamma_{\Sigma}(R) \subseteq \Sigma$.

**Proof.** If $R \subseteq \pi_{\Sigma}(L_0)$, then $\lambda_{\Sigma}(R) \supseteq \lambda_{\Sigma} \pi_{\Sigma}(L_0) \supseteq L_0$. Hence $\Pi_{L_0} \subseteq \Pi(\lambda_{\Sigma}(R)) \subseteq \Pi_L$, so that $\Pi \subseteq \gamma_{\Sigma}(R) \subseteq \Sigma$. By Corollary 22, there is a sequence of one line extensions going from $\Pi$ to $\gamma_{\Sigma}(R)$, which can be extended by a sequence going from $\gamma_{\Sigma}(R)$ to $\Sigma$. \qed

Recall that for a semiplane $\Pi = (P_0, L_0, \leq_0)$ we have defined

$$\rho_n(\Pi) = \sum_{\ell \in L_0} r_{\Pi}(\ell) + n^2 + n + 1 - |P_0| - |L_0|(n + 1).$$

**Theorem 27.** Let $\Pi = (P_0, L_0, \leq_0)$ be a semiplane that can be extended to a projective plane $\Sigma = (P, L, \leq)$ of order $n$. Then $\rho_n(\Pi) = |\pi_{\Sigma}(L_0)|$. 

34
Proof. The lines of $L_0$ contain $|P_0| + \sum_{\ell \in L_0} (n + 1 - r_{\Pi}(\ell))$ points in $\Sigma$. So there are $n^2 + n + 1 + \sum_{\ell \in L_0} r_{\Pi}(\ell) - |P_0| - |L_0|(n + 1) = \rho_n(\Pi)$ points in $P$ that are on no line of $L_0$. \hfill \Box

**Corollary 28.** Let $\Sigma = \langle P, L, \leq \rangle$ be a projective plane of order $n$, and let $R \subseteq P$. If $\Pi$ is a semiplane with $\Pi \subseteq \gamma_\Sigma(R)$, then $\rho_n(\Pi) \geq |R|$. 

Proof. If $\Pi = \langle P_0, L_0, \leq_0 \rangle \subseteq \gamma_\Sigma(R)$, then $L_0 \subseteq \lambda_\Sigma(R)$, whence $\pi_\Sigma(L_0) \supseteq \pi_\Sigma \lambda_\Sigma(R) \supseteq R$. \hfill \Box

Given a projective plane $\Sigma = \langle P, L, \leq \rangle$ and a set of points $R \subseteq P$, we would like to know when $R$ is closed, i.e., when $R = \pi_\Sigma \lambda_\Sigma(R)$. Likewise, we would like to know when a set of lines $K \subseteq L$ is closed, i.e., when $K = \lambda_\Sigma \pi_\Sigma(K)$.

If a set $R$ of points is not too large, then it will be closed.

**Theorem 29.** Let $\Sigma = \langle P, L, \leq \rangle$ be a projective plane of order $n$. If $R \subseteq P$ with $|R| \leq n$, then $\pi_\Sigma \lambda_\Sigma(R) = R$, and hence $\rho_n(\gamma_\Sigma(R)) = |R|$. 

Proof. A point $p \in P$ is in $\pi_\Sigma \lambda_\Sigma(R)$ if and only if every line of $L$ containing $p$ also contains some point $r \in R$. Given a point $p \notin R$, consider (in $\Sigma$) the set of lines $p \lor R = \{ p \lor r : r \in R \}$. If $|p \lor R| = n + 1$ then $p \in \pi_\Sigma \lambda_\Sigma(R)$, while $p \notin \pi_\Sigma \lambda_\Sigma(R)$ whenever $|p \lor R| \leq n$. So if $|R| \leq n$, then $\pi_\Sigma \lambda_\Sigma(R) = R$. \hfill \Box

We have been trying to extend semiplanes containing a non-Desarguesian configuration to projective planes of some finite order using computer programs written by Dr. Nation. All the programs work essentially the same, except possibly accepting
different inputs to suit our testing purposes. The original one basically works in the following way.

1. Read the input of the order of the projective plane and the initial semiplane.

2. Find all choices of $M$ that keep

   (a) $\rho_n \geq 0$,

   (b) $r(k) \leq n + 1$, where $k$ is the new line,

   (c) $r(l) \leq n + 1$ for all line $l$ in the semiplane,

   (d) $r(p) \leq n + 1$ for all point $p$ in the semiplane, and

   (e) the total number of points is no more than $n^2 + n + 1$.

If the number of choices is zero, then go to step (3); otherwise, pick a choice and go to step (4).

3. Remove the last line added, adjust all the parameters. If possible, pick the next choice in line and go to step (4). If there are no more choices, then we have the following two cases:

   (a) If we are not back to the original configuration, repeat step (3).

   (b) Otherwise, all the choices have been tested, and there is no projective plane.

       Output the conclusion that the semiplane cannot be extended to a projective plane of the given order, and stop the program.
4. Add the line, and adjust all the parameters. If the number of lines is $n^2 + n + 1$, then we have obtained a projective plane. Output the plane and stop the program; otherwise go to step (2).

In principle, this is a finite algorithm. However, even for relatively small orders, the number of choices for $M$ at each stage can be large. Hence, it is not practical to run the program straight through as described.

With a Desargues configuration as the initial input, this program does produce the Desarguesian projective planes of order 3, 4, 5 and 7. When we tried to extend a semi-plane containing a non-Desarguesian configuration to a plane of order 9, there are simply too many choices for $M$ at each stage. But with a little help from us, which means that we directed the computer to make certain choices for a few steps, then the program was able to produce a plane of that order. For order 12, there are a lot more choices for $M$ and we cannot provide any special help, so we decided to cut the choices for $M$ at 500 and randomly pick choices for the first 22 lines added. Unless we have made the right choices, the program could still run a significant amount of time. So we also control the number of times that we are willing to try for each sequence, once the count has reached that number and if the number of lines is greater or equal to 48, we print all the choices that we have picked for each stage, and start a new sequence. The number 48 was not chosen at random, but rather we realized, based on all the tests that we have done so far, that the sequence of extensions would be a good candidate if the number of lines is able to reach at least 48. We also did tests for order 11 and 15, but we spent most of our time on order 12.
We also applied the idea of genetic algorithms into our testing. We take a good candidate, change the sequence at one or a few places, or we replace a string of numbers by a string from another good candidate, and then we run the program using the new sequence to see whether or not we are able to obtain another good candidate. This method often produces other semiplanes with more than 48 lines.
Chapter 4
Classification of Dual Linear Spaces Generated by a Non-Desarguesian Configuration

Figure 3: Non-Desarguesian configuration.

Given a non-Desarguesian configuration, there are two basic ways to extend it to a semiplane:

1. make lines go through existing points; or

2. make pairwise nonconcurrent lines meet at new points.
Table 1: Intersection of lines.

<table>
<thead>
<tr>
<th></th>
<th>l₁</th>
<th>l₂</th>
<th>l₃</th>
<th>l₄</th>
<th>l₅</th>
<th>l₆</th>
<th>l₇</th>
<th>l₈</th>
<th>l₉</th>
<th>l₁₀</th>
<th>l₁₁</th>
<th>l₁₂</th>
</tr>
</thead>
<tbody>
<tr>
<td>l₃</td>
<td>p₇</td>
<td>p₇</td>
<td></td>
<td>p₃</td>
<td>p₃</td>
<td>p₆</td>
<td>p₆</td>
<td>p₆</td>
<td>p₆</td>
<td>?</td>
<td>?</td>
<td>?</td>
</tr>
<tr>
<td>l₅</td>
<td>p₁</td>
<td>?</td>
<td>p₃</td>
<td>p₁</td>
<td>p₃</td>
<td>?</td>
<td>p₁₀</td>
<td>p₁₀</td>
<td>p₁₀</td>
<td>p₁₀</td>
<td>p₁₀</td>
<td>p₁₀</td>
</tr>
<tr>
<td>l₇</td>
<td>p₄</td>
<td>p₅</td>
<td>p₉</td>
<td>p₅</td>
<td>p₅</td>
<td>p₉</td>
<td>p₉</td>
<td>p₉</td>
<td>p₉</td>
<td>p₉</td>
<td>p₉</td>
<td>p₉</td>
</tr>
<tr>
<td>l₈</td>
<td>p₄</td>
<td>?</td>
<td>p₆</td>
<td>p₆</td>
<td>p₁₀</td>
<td>p₆</td>
<td>p₁₀</td>
<td>p₁₀</td>
<td>p₁₀</td>
<td>p₁₀</td>
<td>p₁₀</td>
<td>p₁₀</td>
</tr>
</tbody>
</table>

Table 2: Lines through each point.

<table>
<thead>
<tr>
<th>point</th>
<th>lines</th>
</tr>
</thead>
<tbody>
<tr>
<td>p₁</td>
<td>l₁ l₄ l₅</td>
</tr>
<tr>
<td>p₂</td>
<td>l₂ l₄ l₆</td>
</tr>
<tr>
<td>p₃</td>
<td>l₃ l₅ l₆</td>
</tr>
<tr>
<td>p₄</td>
<td>l₁ l₇ l₈</td>
</tr>
<tr>
<td>p₅</td>
<td>l₁ l₇ lₙ</td>
</tr>
<tr>
<td>p₆</td>
<td>l₃ l₈ lₙ</td>
</tr>
<tr>
<td>p₇</td>
<td>l₁ l₉ l₃</td>
</tr>
<tr>
<td>p₈</td>
<td>l₆ l₉ l₁₀l₁₂</td>
</tr>
<tr>
<td>p₉</td>
<td>l₄ l₇ l₁₀l₁₁</td>
</tr>
<tr>
<td>p₁₀</td>
<td>l₅ l₈ l₁₁l₁₂</td>
</tr>
</tbody>
</table>

Figure 3 illustrates the basic non-Desarguesian configuration, that is, two triangles are centrally perspective from a point but not axially perspective from a line, and establishes our notation.

Based on the non-Desarguesian configuration of Figure 3, Table 1 lists the meets of all the lines. A question mark indicates that the meet is undefined. Table 2 gives the detail of which lines pass through which point. By using these two tables, we can easily
check whether method one or two for our extension would work for certain lines. For instance, from Table 2 we know that $p_1$ is on $l_1$, $l_4$ and $l_5$. So we can only make a line, which does not intersect any of those lines, pass through $p_1$. In other words, we need to see question marks under those lines on some row of Table 1. Now if we check Table 1 carefully, we know that $l_9$ would be our only choice. We would like to note here that no more than three lines can meet at a new point because any set of four lines contains at least one intersecting pair. The following is a complete list of all possible ways to extend the non-Desarguesian configuration of Figure 3. When method one is applied, we group them according to the rank of the points, those with rank 4 and those with rank 3. We further separate the latter ones into three smaller groups depending on whether they are on the first triangle $p_1, p_2, p_3$ or the second triangle $p_4, p_5, p_6$ or neither.

I. Make a line go through a point of rank 4.

1. $l_1 \rightarrow p_8$
2. $l_2 \rightarrow p_{10}$
3. $l_3 \rightarrow p_9$

II. Make a line go through a point of rank 3.

1. Points on the second triangle.
   a. $l_4 \rightarrow p_6$
   b. $l_5 \rightarrow p_5$
   c. $l_6 \rightarrow p_4$
2. Points on the first triangle.
   
a. \( l_7 \rightarrow p_3 \)
b. \( l_8 \rightarrow p_2 \)
c. \( l_9 \rightarrow p_1 \)

3. Point on neither of the triangles.
   
a. \( l_{10} \rightarrow p_7 \)
b. \( l_{11} \rightarrow p_7 \)
c. \( l_{12} \rightarrow p_7 \)

III. Make three pairwise nonconcurrent lines meet at a new point.
   
a. \( l_1, l_8, l_{11} \)
b. \( l_1, l_9, l_{11} \)
c. \( l_2, l_5, l_{10} \)
d. \( l_2, l_9, l_{10} \)
e. \( l_3, l_4, l_{12} \)
f. \( l_3, l_7, l_{12} \)

IV. Make two nonconcurrent lines meet at a new point.

Since in a projective plane any two points lie on exactly one line, and any two lines intersect in exactly one point, some methods from group I, II and III may not be used simultaneously for our extension. For instance, method II cannot go with method IIIc,
for otherwise $p_8 = l_1 \land l_6 = p_4$, a contradiction. Note that method IV can be used at any time of the extension, and it is the only method which can be used repeatedly. The following list shows how meets will be defined if a method is applied.

I1  $l_1 \land l_6 = l_1 \land l_9 = l_1 \land l_{10} = l_1 \land l_{12} = p_8$

I2  $l_2 \land l_5 = l_2 \land l_8 = l_2 \land l_{11} = l_2 \land l_{12} = p_{10}$

I3  $l_3 \land l_4 = l_3 \land l_7 = l_3 \land l_{10} = l_3 \land l_{11} = p_9$

II1a $l_3 \land l_4 = l_4 \land l_8 = l_4 \land l_9 = p_6$

II1b $l_2 \land l_5 = l_5 \land l_7 = l_5 \land l_9 = p_5$

II1c $l_1 \land l_6 = l_6 \land l_7 = l_6 \land l_8 = p_4$

II2a $l_3 \land l_7 = l_5 \land l_7 = l_6 \land l_7 = p_3$

II2b $l_2 \land l_8 = l_4 \land l_8 = l_6 \land l_8 = p_2$

II2c $l_1 \land l_9 = l_4 \land l_9 = l_5 \land l_9 = p_1$

II3a $l_1 \land l_{10} = l_2 \land l_{10} = l_3 \land l_{10} = p_7$

II3b $l_1 \land l_{11} = l_2 \land l_{11} = l_3 \land l_{11} = p_7$

II3c $l_1 \land l_{12} = l_2 \land l_{12} = l_3 \land l_{12} = p_7$

IIia $l_1 \land l_6 \land l_{11} = $ some new point

IIib $l_1 \land l_9 \land l_{11} = $ some new point
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IIIc \( l_2 \wedge l_5 \wedge l_{10} = \text{some new point} \)

IIId \( l_2 \wedge l_8 \wedge l_{10} = \text{some new point} \)

IIle \( l_3 \wedge l_4 \wedge l_{12} = \text{some new point} \)

IIIf \( l_3 \wedge l_7 \wedge l_{12} = \text{some new point} \)

Using the list above, we obtain the table of inconsistency which gives a complete list of all the conflicts among the methods.

**Lemma 30.** Altogether at most six applications of methods from groups I, II1, II2, II3 and III can be made.

**Proof.** If we are to choose seven or more methods from the five groups, not necessarily
from all five, then we are forced to take at least two methods from at least one of the five groups. Note that no more than one method may be chosen from group II3. Also note that only up to three methods may be taken from group III at a time.

Case I: If we take any two methods from group I, then by Table 3, we know that we may possibly pick the remaining one from the same group, one method from group III, one method from group II2, and two methods from group III. But the two methods from group III don’t go together. Therefore, we may take no more than six methods in this case.

For example, if we take I1 and I2, then Table 3 suggests that I3, II1a, II2a, IIIe and IIIf are the possible choices to add. But IIIe and IIIf are inconsistent, hence we may only take up to six methods for this case.

Case II: If we take any two methods from group III1, then we may possibly take the remaining one from the same group, one method from group I, one method from group II3, and three methods from group III.

a. If we take the one from group I, then we may take up to two from group III with or without taking one from group II3.

b. If we take the remaining one from the same group, then we can either take up to three methods from group III, or take one from group II3 together with no more than two from group III.

c. If we do not take the possible one from group I and the remaining one from the same group, then we can take up to a total of three from group II3 and
III.

In any case, we may only take a maximum of six methods.

For example, suppose we take IIIa and IIIb. Then II, IIIc, any one from group II3, IIIa, IIIb, IIId, and IIIf are the possible choices from which we may pick. If we choose II, then we cannot pick IIIc, II3a, II3c, IIIa and IIIb. So now we are left with II3b, IIId and IIIf as our possible choices. Table 3 suggests that II3b has no conflicts with IIId or IIIf. Therefore we may pick all three of them, which gives us a total of six methods. If we choose IIIc, which is the remaining method from the same group, then II and IIIa are eliminated from our list. Since II3a does not go with IIId, II3b does not go with IIIb and II3c does not go with IIIf, we can either take IIIb, IIId and IIIf all at once, or we can take only two of those and one from group II3. Either case gives a total of six methods. Suppose we ignore both II and IIIc. Since no more than three can be chosen from group III, again we are left with only six choices.

Case III: If we take any two methods from group II2, then a similar argument as in Case II will lead to the same conclusion.

Case IV: If we take any two from group III, then only one method from group I, four methods in total from group II1 and II2, one method from group II3, and two methods from group III could possibly be added. But the two methods from group III are inconsistent, and none of them is consistent with either the one from group I or the one from group II3. We may choose up to three of the four methods from group
III and II2 at a time, but if we choose the one method from group I, then only up to two of those methods are possible. Therefore only up to six methods may be taken at a time in this case.

For example, suppose we take IIIa and IIIc, then I3, IIIa, II2a, II2b, II2c, IIIc, IIIf and IIIf are our possible choices to add. Notice that IIIe and IIIf are inconsistent, and both of them are inconsistent with I3 and II3c. Also notice that we may choose II2a, II2b and II2c out of the four methods from group III and II2, but if we choose I3, then we may only take II2b and II2c. Therefore we may take only up to four methods from the list which implies a maximum of six methods for this case.

\begin{proof}
Proposition 31. There are 875 ways to extend the non-Desarguesian configuration of Figure 3 to a semi-plane without adding a new line.

Proof. The proof of this proposition is a straightforward case analysis using our table of inconsistency and the above lemma. The cases are listed in the order of the total number of methods which we pick from group I, III, II2, II3, and III. Note that method IV is the only method which we may use repeatedly, and when we list the case II IIIa, for example, we mean apply methods II and IIIa first to our initial configuration, and then apply method IV as many times as necessary until the meets of every pair of lines are defined. Also if we say pick 2-1 in the proof, we mean pick two methods from one group, and pick the third method from a group that follows in the list. We would like to give a special note here for a few inconsistencies which will be encountered often: no
more than one method can be picked from group II3; IIIa IIIb cannot go together, and
the same for the pairs IIIc IIId and IIIe IIIf. The symbol, \(\rightarrow\), in the proof indicates
inconsistency.

Case I: Pick none from the five groups.

1 IV

Case II: Pick one at a time.

2 II1
3 II2
4 III
5 II1a
6 II1b
7 II1c
8 II2a
9 II2b
10 II2c
11 II3a
12 II3b
13 II3c
14 IIIa
15 IIIb
16 IIIc
17 IIId
Case III: Pick two at a time.

A. Pick two from the same group.

20  I1 I2
21  I1 I3
22  I2 I3
23  IIIa IIIb
24  IIIa IIIc
25  IIIb IIIc
26  II2a II2b
27  II2a II2c
28  II2b II2c
29  IIIa IIIc
30  IIIa IIId
31  IIIa IIIe
32  IIIa IIIf
33  IIIb IIIc
34  IIIb IIId
35  IIIb IIIe
36  IIIb IIIf
37  IIIc IIIe
B. Pick 1-1.

I1 $\leftrightarrow$ IIIc, II2c, II3a, II3c, IIIa, IIIb.

41 I1 II1a
42 I1 II1b
43 I1 II2a
44 I1 II2b
45 I1 II3b
46 I1 IIIc
47 I1 IIId
48 I1 IIIe
49 I1 IIIf

I2 $\leftrightarrow$ IIIb, II2b, II3b, II3c, IIIc, IIId.

50 I2 IIIa
51 I2 II1c
52 I2 II2a
53 I2 II2c
54 I2 II3a
55 I2 IIIa
56 I2 IIIb
I3 →← IIIa, II2a, II3a, II3b, IIIe, IIIf.

IIIa →← II2b, II2c, IIIe.
\[ \text{II}1\text{b} \rightarrow \leftarrow \text{II}2\text{a, II}2\text{c, IIIc.} \]

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\[ \text{II}1\text{c} \rightarrow \leftarrow \text{II}2\text{a, II}2\text{b, IIIa.} \]

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\[ \text{II}2\text{a} \rightarrow \leftarrow \text{IIIf.} \]

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97 & \ II2a \ II3c \\
98 & \ II2a \ IIIa \\
99 & \ II2a \ IIIb \\
100 & \ II2a \ IIIc \\
101 & \ II2a \ IIId \\
102 & \ II2a \ IIIe \\
II2b & \leftrightarrow \ IIId. \\
103 & \ II2b \ IIIa \\
104 & \ II2b \ IIIb \\
105 & \ II2b \ IIIc \\
106 & \ II2b \ IIIa \\
107 & \ II2b \ IIIb \\
108 & \ II2b \ IIIc \\
109 & \ II2b \ IIIe \\
110 & \ II2b \ IIIf \\
II3c & \leftrightarrow \ IIIb. \\
111 & \ II2c \ IIIa \\
112 & \ II2c \ IIIb \\
113 & \ II2c \ IIIc \\
114 & \ II2c \ IIIa \\
115 & \ II2c \ IIIc \\
\end{align*} \]
Case IV: Pick three at a time.

A. Pick three from the same group.

131  I1 I2 I3
B. Pick 2-1.

I1 I2 \rightarrow I1b, I1c, I2b, I2c, I3a, I3b, I3c, I3d, I3e, I3f.

I1 I2 I1a

I1 I2 I2a

I1 I2 I3e

I1 I2 I3f

I1 I3 \rightarrow I1a, I1c, I2a, I2c, I3a, I3b, I3c, I3d, I3e, I3f.

I1 I3 I1b

I1 I3 I2b

I1 I3 I3c

I1 I3 I3d

I2 I3 \rightarrow I1a, I1b, I2a, I2b, I3a, I3b, I3c, I3d, I3e, I3f.
IIa IIb \longrightarrow IIIc, IIIe.

154  IIa IIb IIIa
155  IIa IIb IIIb
156  IIa IIb IIIc
157  IIa IIb IIIa
158  IIa IIb IIIb
159  IIa IIb IIId
160  IIa IIb IIIf

IIa IIc \longrightarrow IIIa, IIIe.

161  IIa IIc IIIa
162  IIa IIc IIIb
163  IIa IIc IIIc
164  IIa IIc IIIb
165  IIa IIc IIIc
166  IIa IIc IIId
167  IIa IIc IIIf

IIb IIc \longrightarrow IIIa, IIIc.

168  IIb IIc IIIa
169  II1b II1c II3b
170  II1b II1c II3c
171  II1b II1c IIIb
172  II1b II1c IIId
173  II1b II1c IIIe
174  II1b II1c IIIf

II2a II2b ←→ IIId, IIIf.
175  II2a II2b II3a
176  II2a II2b II3b
177  II2a II2b II3c
178  II2a II2b IIIa
179  II2a II2b IIIb
180  II2a II2b IIIc
181  II2a II2b IIIe

II2a II2c ←→ IIIb, IIIf.
182  II2a II2c II3a
183  II2a II2c II3b
184  II2a II2c II3c
185  II2a II2c IIIa
186  II2a II2c IIIc
187  II2a II2c IIId
188  II2a II2c IIIe
II2b II2c →← IIIb, IIId.

189 II2b II2c II3a
190 II2b II2c II3b
191 II2b II2c II3c
192 II2b II2c IIIa
193 II2b II2c IIIc
194 II2b II2c IIIe
195 II2b II2c IIIf

C. Pick 1-2.

I1 →← IIIc, II2c, II3a, II3c, IIIa, IIlb.

196 I1 IIIa II1b
197 I1 II2a II2c
198 I1 IIIc IIIe
199 I1 IIIc IIIf
200 I1 IIId IIIe
201 I1 IIId IIIf

I2 →← III1b, II2b, II3b, II3c, IIIc, IIId.

202 I2 IIIa IIIc
203 I2 II2a II2c
204 I2 IIIa IIIe
205 I2 IIIa IIIf
206 I2 IIIb IIIe

58
I3 $\rightarrow \leftarrow$ II1a, II2a, II3a, II3b, IIIe, IIIf.

II1a $\rightarrow \leftarrow$ II2b, II2c, IIIe.

II1b $\rightarrow \leftarrow$ II2a, II2c, IIIc.
IIlc $\rightarrow\leftarrow$ II2a, II2b, IIIa.

II2a $\leftrightarrow$ IIIf.

II2a IIIa IIIc
II2a IIIa IIId
II2a IIIa IIle
II2a IIIb IIIc
II2a IIIb IIId
II2a IIIb IIIe
II2a IIIc IIIe
II2a IIId IIle
II2b $\leftrightarrow$ IIId.

246 II2b IIIa IIIc
247 II2b IIIa IIIe
248 II2b IIIa IIIf
249 II2b IIIb IIIc
250 II2b IIIb IIIe
251 II2b IIIb IIIf
252 II2b IIIc IIIe
253 II2b IIIc IIIf

II2c $\leftrightarrow$ IIIb.

254 II2c IIIa IIIc
255 II2c IIIa IIId
256 II2c IIIa IIIe
257 II2c IIIa IIIf
258 II2c IIIc IIIe
259 II2c IIIc IIIf
260 II2c IIId IIIe
261 II2c IIId IIIf

II3a $\leftrightarrow$ IIIc, IIId.

262 II3a IIIa IIIe
263 II3a IIIa IIIf
264 II3a IIIb IIIe
265 II3a IIIb IIIf

II3b ↔ IIIa, IIIb.

266 II3b IIIc IIIe

267 II3b IIIc IIIf

268 II3b IIId IIIe

269 II3b IIId IIIf

II3c ↔ IIIe, IIIf.

270 II3c IIIa IIIc

271 II3c IIIa IIId

272 II3c IIIb IIIc

273 II3c IIIb IIId

D. Pick 1-1-1.

I1 II1a ↔ II2b, II2c, II3a, II3c, IIIa, IIIb, IIIe.

274 I1 II1a II2a

275 I1 II1a II3b

276 I1 II1a IIIc

277 I1 II1a IIId

278 I1 II1a IIIf

II1b ↔ II2a, II2c, II3a, IIIc, IIIe.

279 I1 II1b II2b

280 I1 II1b II3b

281 I1 II1b IIId
I I IIa → II3a, II3c, IIIa, IIIb, IIIf.

I I IIa II3b

I I IIa IIIc

I I IIa IIId

I I IIa IIIe

I I IIb → II3a, II3c, IIIa, IIIb, IIId.

I I IIb II3b

I I IIb IIIc

I I IIb IIIe

I I IIb IIIf

I II3b II1a, IIIb.

I II3b IIIc

I II3b IIId

I II3b IIIe

I II3b IIIf

II I IIa II2b, II2c, II3b, II3c, IIIa, IIIb, IIIe.

I II1a II2a

I II1a II3a

I II1a IIIa

I II1a IIIb
I2 II1c \rightarrow II2a, II2b, II3b, II3c, IIIa, IIIc, IIId.

I2 II1c II2c

I2 II1c II3a

I2 II1c IIIb

I2 II1c IIIe

I2 II1c IIIf

I2 II2a \rightarrow II3b, II3c, IIIc, IIId, IIIf.

I2 II2a II3a

I2 II2a IIIa

I2 II2a IIIb

I2 II2a IIIe

I2 II2c \rightarrow II3b, II3c, IIIb, IIIc, IIId.

I2 II2c II3a

I2 II2c IIIa

I2 II2c IIIe

I2 II2c IIIf

I2 II3a \rightarrow IIIc, IIId.

I2 II3a IIIa

I2 II3a IIIb

I2 II3a IIIe

I2 II3a IIIf
I3 II1b ↔ II2a, II2c, II3a, II3b, IIIc, IIIe, IIIf.

318 I3 II1b II2b
319 I3 II1b II3c
320 I3 II1b IIIa
321 I3 II1b IIIb
322 I3 II1b IIId

I3 II1c ↔ II2a, II2b, II3a, II3b, IIIa, IIIe, IIIf.

323 I3 II1c II2c
324 I3 II1c II3c
325 I3 II1c IIIb
326 I3 II1c IIIc
327 I3 II1c IIId

I3 II2b ↔ II3a, II3b, IIId, IIIe, IIIf.

328 I3 II2b II3c
329 I3 II2b IIIa
330 I3 II2b IIIb
331 I3 II2b IIIc

I3 II2c ↔ II3a, II3b, IIIb, IIIe, IIIf.

332 I3 II2c II3c
333 I3 II2c IIIa
334 I3 II2c IIIc
335 I3 II2c IIId
I3 II3c $\leftrightarrow$ IIIe, IIIf.

336  I3 II3c IIIa
337  I3 II3c IIIb
338  I3 II3c IIIc
339  I3 II3c IIId

II1a II2a $\leftrightarrow$ IIIe, IIIf.

340  II1a II2a IIIa
341  II1a II2a IIIb
342  II1a II2a IIIc
343  II1a II2a IIIa
344  II1a II2a IIIb
345  II1a II2a IIIc
346  II1a II2a IIId

II1a II3a $\leftrightarrow$ IIIc, IIId, IIIe.

347  II1a II3a IIIa
348  II1a II3a IIIb
349  II1a II3a IIIf

II1a II3b $\leftrightarrow$ IIIa, IIIb, IIIe.

350  II1a II3b IIIc
351  II1a II3b IIId
352  II1a II3b IIIf

II1a II3c $\leftrightarrow$ IIIe, IIIf.
II1b II2b → ← IIIc, IIId.

IIIb II3a

IIIb II3b

IIIb II3c

IIIa II2b IIIa

IIIb II2b IIIb

IIIb II2b IIIe

IIIb II2b IIIf

II1b II3a → ← IIIc, IIId.

IIIb II3a IIIa

IIIb II3a IIIb

IIIb II3a IIIe

IIIb II3a IIIf

II1b II3b → ← IIIa, IIIb, IIIc.

IIIb II3b IIId

IIIb II3b IIIe

IIIb II3b IIIf

II1b II3c → ← IIIc, IIIe, IIIf.

67
371  II1b II3c IIa
372  II1b II3c IIb
373  II1b II3c IIId

II1c II2c ↔ IIa, IIb.
374  II1c II2c II3a
375  II1c II2c II3b
376  II1c II2c II3c
377  II1c II2c IIIc
378  II1c II2c IIId
379  II1c II2c IIIe
380  II1c II2c IIIf

II1c II3a ↔ IIa, IIc, IIId.
381  II1c II3a IIIb
382  II1c II3a IIIe
383  II1c II3a IIIf

II1c II3b ↔ IIa, IIb.
384  II1c II3b IIIc
385  II1c II3b IIId
386  II1c II3b IIIe
387  II1c II3b IIIf

II1c II3c ↔ IIa, IIIe, IIIf.
388  II1c II3c IIIb
II2a II3a →← IIIc, IIId, IIIf.

IIIa, IIIb, IIIf.

II2a II3b →← IIIa, IIIb, IIIf.

II2a II3c →← IIIe, IIIf.

IIIa, IIIb, IIIc, IIId.

II2b II3a →← IIIc, IIId.

II2a II3c, IIIb, IIIe

II2b II3a, IIIb, IIIe

II2a II3c, IIId
II2b II3b IIIc
II2b II3b IIId
II2b II3c → ← IIId, IIIe, IIIf.

II2b II3b IIIa
II2b II3b IIIb
II2b II3b IIIc

II2c II3a → ← IIIb, IIIc, IIId.

II2c II3a IIIa
II2c II3a IIIe
II2c II3a IIIf

II2c II3b → ← IIIa, IIIb.

II2c II3b IIIc
II2c II3b IIId
II2c II3b IIIe
II2c II3b IIIf

II2c II3c → ← IIIb, IIIe, IIIf.

II2c II3c IIIa
II2c II3c IIIc
II2c II3c IIId

Case V: Pick four at a time.

A. Pick 3-1 or 1-3.
Note: I1 I2 I3 don't go with anything.

IIa IIb IIc \rightarrow I1, I2, I3, II2a, II2b, II2c, II3a, II3c, II3e.

421 IIa IIb IIc II3a
422 IIa IIb IIc II3b
423 IIa IIb IIc II3c
424 IIa IIb IIc II3b
425 IIa IIb IIc II3d
426 IIa IIb IIc II3f

II2a II2b II2c \rightarrow I1, I2, I3, II1a, II1b, II1c, II3b, II3d, II3f.

427 II2a II2b II2c II3a
428 II2a II2b II2c II3b
429 II2a II2b II2c II3c
430 II2a II2b II2c IIIa
431 II2a II2b II2c II3c
432 II2a II2b II2c IIIe

IIIa IIIc IIIe \rightarrow I1, I2, I3, II1a, II1b, II1c, II3a, II3b, II3c.

433 II2a IIIa IIIc IIIe
434 II2b IIIa IIIc IIIe
435 II2c IIIa IIIc IIIe

IIIa IIIc IIIf \rightarrow I1, I2, I3, II1b, II1c, II2a, III3a, III3b, III3c.

436 II1a IIIa IIIc IIIf
437 II2b IIIa IIIc IIIf
IIIa IIId IIIe → 11, I2, I3, II1a, II1c, II2b, IIIa, IIIb, IIIc.

IIIb IIIc IIIe → 11, I2, I3, II1a, II1b, II2a, II2c, IIIa, IIIb, IIIc.

IIIb IIIc IIIf → 11, I2, I3, II1a, II1b, II2a, II2c, IIIa, IIIb, IIIc.
454. IIIa IIIb IIId IIIf
455. IIIb IIIb IIId IIIf
456. IIIc IIIb IIId IIIf

B. Pick 2-2.

Note: Any two from group I cannot go with any two from any other group.

II1a II1b → II2a, II2b, II2c, IIIc, IIIe.

457. IIIa IIIb IIIa IIId
458. IIIa IIIb IIIa IIIf
459. IIIa IIIb IIIb IIId
460. IIIa IIIb IIIb IIIf
461. IIIa IIIb IIId IIIf

II1a IIIc → II2a, II2b II2c, IIIa, IIIe.

462. IIIa IIIc IIIb IIIc
463. IIIa IIIc IIIb IIId
464. IIIa IIIc IIIb IIIf
465. IIIa IIIc IIIc IIIf
466. IIIa IIIc IIId IIIf

II1b IIIc → II2a, II2b, II2c, IIIa, IIIc.

467. IIIb IIIc IIIb IIId
468. IIIb IIIc IIIb IIIe
469. IIIb IIIc IIIb IIIf
470. IIIb IIIc IIId IIIe
C. Pick 2-1-1.

II 12 →← II1b, II1c, II2b, II2c, II3a, II3b, II3c, IIIa, IIIb, IIIc, IIId.

II1a →← II2b, II2c, IIIe.

74
Π2a ↔ IIIf.

Π1 I3 ➔← Π1a, Π1c, Π2a, Π2c, Π3a, Π3b, Π3c, Π3a, Π3b, Π3e, Π3f.

Π1b ➔← Π2a, Π2c, Π3c.

Π2b ➔← IIId.

Π2b ➔← IIIc

Π2 I3 ➔← Π1a, Π1b, Π2a, Π2b, Π3a, Π3b, Π3c, Π3d, Π3e, Π3f.

Π1c ➔← Π2a, Π2b, Π3a.

Π2c ➔← IIIb.

Π2 I3 Π2c Π3a

Π1a Π1b ➔← Π2a, Π2b, Π2c, Π3c, Π3e.

Π3a ➔← IIIc, IIId.
II3b $\leftrightarrow$ IIIa, IIIb.

499 \( \text{II}1a \text{ II}1b \text{ II}3b \text{ III}d \)

500 \( \text{II}1a \text{ II}1b \text{ II}3b \text{ III}f \)

II3c $\leftrightarrow$ IIIe, IIIf.

501 \( \text{II}1a \text{ II}1b \text{ II}3c \text{ III}a \)

502 \( \text{II}1a \text{ II}1b \text{ II}3c \text{ III}b \)

503 \( \text{II}1a \text{ II}1b \text{ II}3c \text{ III}d \)

II1a II1c $\leftrightarrow$ II2a, II2b, II2c, IIIa, IIIe.

504 \( \text{II}1a \text{ III}c \text{ III}a \text{ III}b \)

505 \( \text{II}1a \text{ III}c \text{ III}a \text{ III}f \)

506 \( \text{II}1a \text{ III}c \text{ III}b \text{ III}c \)

507 \( \text{II}1a \text{ III}c \text{ III}b \text{ III}d \)

508 \( \text{II}1a \text{ III}c \text{ III}b \text{ III}f \)

509 \( \text{II}1a \text{ III}c \text{ III}c \text{ III}b \)

510 \( \text{II}1a \text{ III}c \text{ III}c \text{ III}c \)

511 \( \text{II}1a \text{ III}c \text{ III}c \text{ III}d \)

II1b II1c $\leftrightarrow$ II2a, II2b, II2c, IIIa, IIIc.

512 \( \text{II}1b \text{ III}c \text{ III}a \text{ III}b \)

513 \( \text{II}1b \text{ III}c \text{ III}a \text{ III}e \)

514 \( \text{II}1b \text{ III}c \text{ III}a \text{ III}f \)

515 \( \text{II}1b \text{ III}c \text{ III}b \text{ III}d \)

516 \( \text{II}1b \text{ III}c \text{ III}b \text{ III}e \)
517 II1b II1c II3b IIIc
518 II1b II1c II3c IIIb
519 II1b II1c II3c IIId

II2a II2b →← IIId, IIIf.
520 II2a II2b II3a IIIa
521 II2a II2b II3a IIIb
522 II2a II2b II3a IIIe
523 II2a II2b II3b IIIc
524 II2a II2b II3b IIIe
525 II2a II2b II3c IIIa
526 II2a II2b II3c IIIb
527 II2a II2b II3c IIIc

II2a II2c →← IIIb, IIIf.
528 II2a II2c II3a IIIa
529 II2a II2c II3a IIIe
530 II2a II2c II3b IIIc
531 II2a II2c II3b IIId
532 II2a II2c II3b IIIe
533 II2a II2c II3c IIIa
534 II2a II2c II3c IIIc
535 II2a II2c II3c IIId

II2b II2c →← IIIb, IIId.
D. Pick 1-2-1.

11 →← II1c, II2c, II3a, II3c, IIIa, IIIb.

12 →← IIIb, II2b, II3b, II3c, IIIc, IIId.
555  I2 II2a II2c IIIe

I3 —— II1a, II2a, II3a, II3b, IIIe, IIIf.

556  I3 II1b II1c IIIc

557  I3 II1b II1c IIIb

558  I3 II1b II1c IIId

559  I3 II2b II2c IIIc

560  I3 II2b II2c IIIa

561  I3 II2b II2c IIIc

E. Pick 1-1-2.

11 —— II1c, II2c, II3a, II3c, IIIa, IIIb.

II1a —— II2b, II2c, IIIe.

562  I1 II1a IIIc IIIf

563  I1 II1a IIId IIIf

II1b —— II2a, II2c, IIIc.

564  I1 II1b IIId IIIe

565  I1 II1b IIId IIIf

II2a —— IIIf.

566  I1 II2a IIIc IIIe

567  I1 II2a IIId IIIe

II2b —— IIId.

568  I1 II2b IIIc IIIe

569  I1 II2b IIIc IIIf

79
$II^3b \leftrightarrow IIIa, IIIb.$

570  $I_1 II^3b II^3c IIIe$
571  $I_1 II^3b II^3c IIIf$
572  $I_1 II^3b IIId IIIe$
573  $I_1 II^3b IIId IIIf$

$I_2 \leftrightarrow II^1b, II^2b, II^3b, II^3c, IIIc, IIId.$

$II^1a \leftrightarrow II^2b, II^2c, IIIe.$

574  $I_2 II^1a IIIa IIIf$
575  $I_2 II^1a IIIb IIIf$

$II^1c \leftrightarrow II^2a, II^2b, IIIa.$

576  $I_2 II^1c IIIb IIIe$
577  $I_2 II^1c IIIb IIIf$

$I_2^2a \leftrightarrow IIIf.$

578  $I_2 II^2a IIIa IIIe$
579  $I_2 II^2a IIIb IIIe$

$I_2^2c \leftrightarrow IIIb.$

580  $I_2 II^2c IIIa IIIe$
581  $I_2 II^2c IIIa IIIf$

$II^3a \leftrightarrow IIIc, IIId.$

582  $I_2 II^3a IIIa IIIe$
583  $I_2 II^3a IIIa IIIf$
584  $I_2 II^3a IIIb IIIe$

80
I2 II3a IIIb IIIf
I3 ← II1a, II2a, II3a, IIIb, IIIe, IIIf.
II1b ← II2a, II2c, IIIc.

I2 II1b IIIa IIId

II1c ← II2a, II2b, IIIa.

I2 II1c IIIb IIIe

II2b ← IIId.

I2 II2b IIIa IIIc

II2c ← IIIb.

I2 II2c IIIa IIIc

II3c ← IIIe, IIIf.

I2 II3c IIIa IIIc

II3c ← IIIb IIIc

II3c ← IIIb IIId

II1a ← II2b II2c IIIe.

II2a ← IIIf. 598 II1a II2a IIIa IIIc

II1a II2a IIIa IIId
111 112 113 114 115
110 111 112 113 114
109 110 111 112 113
108 109 110 111 112
107 108 109 110 111
106 107 108 109 110
105 106 107 108 109
104 105 106 107 108
103 104 105 106 107
102 103 104 105 106
101 102 103 104 105
100 101 102 103 104
99 100 101 102 103
98 99 100 101 102
97 98 99 100 101
96 97 98 99 100
95 96 97 98 99
94 95 96 97 98
93 94 95 96 97
92 93 94 95 96
91 92 93 94 95
90 91 92 93 94
89 90 91 92 93
88 89 90 91 92
87 88 89 90 91
86 87 88 89 90
85 86 87 88 89
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83 84 85 86 87
82 83 84 85 86
81 82 83 84 85
80 81 82 83 84
79 80 81 82 83
78 79 80 81 82
77 78 79 80 81
76 77 78 79 80
75 76 77 78 79
74 75 76 77 78
73 74 75 76 77
72 73 74 75 76
71 72 73 74 75
70 71 72 73 74
69 70 71 72 73
68 69 70 71 72
67 68 69 70 71
66 67 68 69 70
65 66 67 68 69
64 65 66 67 68
63 64 65 66 67
62 63 64 65 66
61 62 63 64 65
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57 58 59 60 61
56 57 58 59 60
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54 55 56 57 58
53 54 55 56 57
52 53 54 55 56
51 52 53 54 55
50 51 52 53 54
49 50 51 52 53
48 49 50 51 52
47 48 49 50 51
46 47 48 49 50
45 46 47 48 49
44 45 46 47 48
43 44 45 46 47
42 43 44 45 46
41 42 43 44 45
40 41 42 43 44
39 40 41 42 43
38 39 40 41 42
37 38 39 40 41
36 37 38 39 40
35 36 37 38 39
34 35 36 37 38
33 34 35 36 37
32 33 34 35 36
31 32 33 34 35
30 31 32 33 34
29 30 31 32 33
28 29 30 31 32
27 28 29 30 31
26 27 28 29 30
25 26 27 28 29
24 25 26 27 28
23 24 25 26 27
22 23 24 25 26
21 22 23 24 25
20 21 22 23 24
19 20 21 22 23
18 19 20 21 22
17 18 19 20 21
16 17 18 19 20
15 16 17 18 19
14 15 16 17 18
13 14 15 16 17
12 13 14 15 16
11 12 13 14 15
10 11 12 13 14
9 10 11 12 13
8 9 10 11 12
7 8 9 10 11
6 7 8 9 10
5 6 7 8 9
4 5 6 7 8
3 4 5 6 7
2 3 4 5 6
1 2 3 4 5
616  IIb  IIc  IIIb  IIIe
617  IIb  IIc  IIIb  IIIf
IIc  \leftrightarrow  IIIa, IIIb.
618  IIb  IIc  IIIb  IIId  IIIe
619  IIb  IIc  IIIb  IIId  IIIf
IIc  \leftrightarrow  IIIe, IIIf.
620  IIb  IIc  IIIa  IIId
621  IIb  IIc  IIIa  IIId
IIc  \leftrightarrow  IIa, IIb, IIIa.
IIc  \leftrightarrow  IIIb
622  IIc  IIc  IIIc  IIIe
623  IIc  IIc  IIIc  IIIf
624  IIc  IIc  IIId  IIIe
625  IIc  IIc  IIId  IIIf
IIc  \leftrightarrow  IIIc, IIId.
626  IIc  IIc  IIIb  IIIe
627  IIc  IIc  IIIb  IIIf
IIc  \leftrightarrow  IIIa, IIIb.
628  IIc  IIc  IIIb  IIIc  IIIe
629  IIc  IIc  IIIb  IIIc  IIIf
630  IIc  IIc  IIIb  IIId  IIIe
631  IIc  IIc  IIIb  IIId  IIIf
$\Pi_3c \leftrightarrow \Pi_1e, \Pi_1f$.

632 $\Pi_1c \Pi_3c \Pi_1b \Pi_3c$
633 $\Pi_1c \Pi_3c \Pi_1b \Pi_1d$
$\Pi_2a \leftrightarrow \Pi_1f$.
$\Pi_3a \leftrightarrow \Pi_1c, \Pi_1d$.

634 $\Pi_2a \Pi_3a \Pi_1a \Pi_1e$
635 $\Pi_2a \Pi_3a \Pi_1b \Pi_1e$
$\Pi_3b \leftrightarrow \Pi_1a, \Pi_1b$.

636 $\Pi_2a \Pi_3b \Pi_1c \Pi_1e$
637 $\Pi_2a \Pi_3b \Pi_1d \Pi_1e$
$\Pi_3c \leftrightarrow \Pi_1e, \Pi_1f$.

638 $\Pi_2a \Pi_3c \Pi_1a \Pi_1c$
639 $\Pi_2a \Pi_3c \Pi_1a \Pi_1d$
640 $\Pi_2a \Pi_3c \Pi_1b \Pi_1c$
641 $\Pi_2a \Pi_3c \Pi_1b \Pi_1d$
$\Pi_2b \leftrightarrow \Pi_1d$.

$\Pi_3a \leftrightarrow \Pi_1c, \Pi_1d$.

642 $\Pi_2b \Pi_3a \Pi_1a \Pi_1e$
643 $\Pi_2b \Pi_3a \Pi_1a \Pi_1f$
644 $\Pi_2b \Pi_3a \Pi_1b \Pi_1e$
645 $\Pi_2b \Pi_3a \Pi_1b \Pi_1f$
$\Pi_3b \leftrightarrow \Pi_1a, \Pi_1b$. 

84
II2b II3b IIIc IIIe
II3c →← IIIe, IIIf.

II2b II3c IIIa IIIc
II2b II3c IIIb IIIc

II2c →← IIIb.

II3a →← IIIc, IIId.

II2c II3a IIIa IIIe
II2c II3a IIIa IIIf

II3b →← IIIa, IIIb.

II2c II3b IIIc IIIe
II2c II3b IIIc IIIf

II2c II3b IIId IIIe
II2c II3b IIId IIIf

II3c →← IIIe, IIIf.

II2c II3c IIIa IIIc
II2c II3c IIIa IIId

F. Pick 1-1-1-1.

II →← II1c, II2c, II3a, II3c IIIa, IIIb.

II1a →← II2b, II2c IIIe.

II2a →← IIIf.
IIIb →← II2a, II2c, IIIc.

II2b →← IIId.

II2 →← II1b, II2b, II3b, IIIc, IIId.

II1a →← II2b, II2c, IIIe.
II2a $\leftrightarrow$ IIIf.

676  I2 II1a II2a II3a
677  I2 II1a II2a IIIa
678  I2 II1a II2a IIIb
679  I2 II1a II3a IIIa
680  I2 II1a II3a IIIb
681  I2 II1a II3a IIIf
682  I2 II2a II3a IIIa
683  I2 II2a II3a IIIb
684  I2 II2a II3a IIIe

II1c $\leftrightarrow$ II2a, II2b, IIIa.

II2c $\leftrightarrow$ IIIb.

685  I2 II1c II2c II3a
686  I2 II1c II2c IIIe
687  I2 II1c II2c IIIf
688  I2 II1c II3a IIIb
689  I2 II1c II3a IIIe
690  I2 II1c II3a IIIf
691  I2 II2c II3a IIIa
692  I2 II2c II3a IIIe
693  I2 II2c II3a IIIf

I3 $\leftrightarrow$ II1a, II2a, II3a, II3b, IIIe, IIIf.
IIb $\rightarrow$ IIa, IIc, IIIc.

IIb $\rightarrow$ IId.

694 I3 II1b II2b II3c
695 I3 II1b II2b IIIa
696 I3 II1b II2b IIIb
697 I3 II1b II3c IIIa
698 I3 II1b II3c IIIb
699 I3 II1b II3c IId
700 I3 II2b II3c IIIa
701 I3 II2b II3c IIIb
702 I3 II2b II3c IIIc

IIc $\rightarrow$ IIa, IIb, IIIa.

IIc $\rightarrow$ IIIb.

703 I3 II1c II2c II3c
704 I3 II1c II2c IIIc
705 I3 II1c II2c IId
706 I3 II1c II3c IIIb
707 I3 II1c II3c IIIc
708 I3 II1c II3c IId
709 I3 II2c II3c IIIa
710 I3 II2c II3c IIIc
711 I3 II2c II3c IId
\( \Pi_{1a} \rightarrow \Pi_{2b}, \Pi_{2c} \Pi_{3e}. \)

\( \Pi_{2a} \leftarrow \Pi_{3f}. \)

712  \( \Pi_{1a} \Pi_{2a} \Pi_{3a} \Pi_{3a} \)
713  \( \Pi_{1a} \Pi_{2a} \Pi_{3a} \Pi_{3b} \)
714  \( \Pi_{1a} \Pi_{2a} \Pi_{3b} \Pi_{3c} \)
715  \( \Pi_{1a} \Pi_{2a} \Pi_{3b} \Pi_{3d} \)
716  \( \Pi_{1a} \Pi_{2a} \Pi_{3c} \Pi_{3a} \)
717  \( \Pi_{1a} \Pi_{2a} \Pi_{3c} \Pi_{3b} \)
718  \( \Pi_{1a} \Pi_{2a} \Pi_{3c} \Pi_{3c} \)
719  \( \Pi_{1a} \Pi_{2a} \Pi_{3c} \Pi_{3d} \)
\( \Pi_{1b} \rightarrow \Pi_{2a}, \Pi_{2c}, \Pi_{3c}. \)

\( \Pi_{2b} \rightarrow \Pi_{3d}. \)

720  \( \Pi_{1b} \Pi_{2b} \Pi_{3a} \Pi_{3a} \)
721  \( \Pi_{1b} \Pi_{2b} \Pi_{3a} \Pi_{3b} \)
722  \( \Pi_{1b} \Pi_{2b} \Pi_{3a} \Pi_{3e} \)
723  \( \Pi_{1b} \Pi_{2b} \Pi_{3a} \Pi_{3f} \)
724  \( \Pi_{1b} \Pi_{2b} \Pi_{3b} \Pi_{3e} \)
725  \( \Pi_{1b} \Pi_{2b} \Pi_{3b} \Pi_{3f} \)
726  \( \Pi_{1b} \Pi_{2b} \Pi_{3c} \Pi_{3a} \)
727  \( \Pi_{1b} \Pi_{2b} \Pi_{3c} \Pi_{3b} \)
\( \Pi_{1c} \rightarrow \Pi_{2a}, \Pi_{2b}, \Pi_{3a}. \)

\( \Pi_{2c} \rightarrow \Pi_{3b}. \)
Case VI: Pick five at a time.

A. Pick 3-2 or 2-3.

Note: II I2 I3 don't go with anything and any two from group I don't go with three from another group.

II1a II1b II1c →← II2a, II2b, II2c, IIIa, IIIc, IIIe.

II2a II2b II2c →← II1a, II1b, II1c, IIIb, IIId, IIIf.

IIa II1b II1c II1a II1c

II2a II2b II2c II1a II1c

II2a II2b II1a II1c

II1a II1b II1c II1a II1c

II2a II2b II1a II1c

II1a II1b II1a II1c

II2a II2b II1a II1c
B. Pick 3-1-1 or 1-3-1 or 1-1-3.

Note: II 12 13 don’t go with anything.

II1a II1b II1c → II, I2, I3, II2a, II2b, II2c, IIIa, IIIc, IIIe.
Note: Pick one from group I or group II, then no three can be picked from group III.

IIa IIb don’t go with IIIe, IIIf, so no three can be picked from group III.

Similarly for IIIb IIIc.

C. Pick 2-2-1.

Note: Any two from group I don’t go with any two from another group. Any two from III don’t go with any two from II.

None for this case.

D. Pick 2-1-2.

Note: Any two from group I don’t go with any two from another group.

IIa IIb IIc IId IIe.

IIa IIb IIc IId IIIa IIIc, IIIf.

IIa IIb IIc IId IIIa IIIc.

IIa IIb IIc IId.

IIa IIb IIc IIIa IIIc.
IIa IIc II3a II3b IIIf
IIa IIc II3b II3c IIIf
IIa IIc II3b IIId IIIf
IIa IIc II3c IIId
IIb IIc ←→ II2a, II2b, II2c, IIIa, IIIc.
IIb IIc II3a IIIb IIIe
IIb IIc II3a IIIb IIIf
IIb IIc II3b IIId IIIe
IIb IIc II3b IIId IIIf
IIb IIc II3c IIIb IIId
II2a II2b ←→ IIId, IIIf.
II2a II2b II3a IIIa IIIe
II2a II2b II3a IIIb IIIe
II2a II2b II3b IIIc IIIe
II2a II2b II3c IIIa IIIc
II2a II2b II3c IIIb IIIc
II2a II2c ←→ IIIb, IIIf.
II2a II2c II3a IIIa IIIe
II2a II2c II3b IIIc IIIe
II2a II2c II3b IIId IIIe
II2a II2c II3c IIIa IIIc
II23 a II2 b II3 c IIIa IIIc
93
E. Pick 1-2-2.

I1 →← II1c, II2c, II3a, II3c, IIIa, IIIb.

II1a II1b →← II2a, II2b, II2c IIIc, IIIe.

II2a II2b →← IIId, IIIf.

II2a II2c →← IIIb, IIIf.

II2a II2c →← IIIb, IIIf.

II2a II2c →← IIIa IIIe

I2 →← II1b, II2b, II3b, II3c, IIIc, IIId.

II1a II1c →← II2a, II2b, II2c, IIIa, IIIe.

II2a II2c →← IIIb, IIIf.

II2a II2c →← IIIa IIIe

I3 →← II1a, II2a, II3a, II3b, IIIe, IIIf.

II1b II1c →← II2a, II2b, II2c, IIIa, IIIc.

IIIb II1c II1b IIId
II2b II2c $\leftrightarrow$ IIIb, IIId.

801  I3 II2b II2c IIIa IIIc

F. Pick 2-1-1-1.

I1 I2 $\leftrightarrow$ II1b, II1c, II2a, II2b, II2c, II3a, II3b, II3c, IIIa, IIIb, IIIc, IIId.

II1a $\leftrightarrow$ II2b, II2c, IIIe.

II2a $\leftrightarrow$ IIIf.

So nothing works with I1 I2. Similarly, nothing works with I1 I3 and I2 I3.

Note: Any two from group II1 don’t go with anything from group II2.

None for this case.

G. Pick 1-2-1-1.

I1 $\leftrightarrow$ II1c, II2a, II2b, II2c, IIIa, IIIb.

II1a II1b $\leftrightarrow$ II2a, II2b, II2c, IIIc, IIIe.

802  I1 II1a II1b II3b IIId
803  I1 II1a II1b II3b IIIf

II2a II2b $\leftrightarrow$ IIId, IIIf.

804  I1 II2a II2b II3b IIIc
805  I1 II2a II2b II3b IIIe

I2 $\leftrightarrow$ II1b, II2b, II3b, II3c, IIIc, IIId.

II1a II1c $\leftrightarrow$ II2a, II2b, II2c, IIIa, IIIe.

806  I2 II1a II1c II3a IIIb
807  I2 II1a II1c II3a IIIf
II2a II2c → II1b, IIIf.

808  I2 II2a II2c II3a II1a

809  I2 II2a II2c II3a IIIe

I3 → II1a, II2a, II3a, II3b, IIIe, IIIf.

II1b II1c → II2a, II2b, II2c, IIIa, IIIc.

810  I3 II1b II1c II3c IIIb

811  I3 II1b II1c II3c IIId

II2b II2c → IIIb, IIId.

812  I3 II2b II2c II3c IIIa

813  I3 II2b II2c II3c IIIc

Note: Any one from group III doesn't go with any two from group II2.

H. Pick 1-1-2-1.

Note: Any two from group II2 don't go with anything from III1.

None for this case.

I. Pick 1-1-1-2.

II → II1c, II2c, II3a, II3c, II1a, II1b.

II1a → II2b, II2c, IIIe.

II2a → IIIf.

II3b → IIIa, IIIb.

814  I1 II1a II3b IIIc IIIf

815  I1 II1a II3b IIId IIIf

96
$\Pi_1 b \leftrightarrow \Pi_1 a, \Pi_2 c, \Pi_3 c.$

$\Pi_2 b \leftrightarrow \Pi_3 d.$

816 $I_1 \Pi_1 b \Pi_3 b \Pi_3 d \Pi_3 e$
817 $I_1 \Pi_1 b \Pi_3 b \Pi_3 d \Pi_3 f$
818 $I_1 \Pi_2 a \Pi_3 b \Pi_3 c \Pi_3 e$
819 $I_1 \Pi_2 a \Pi_3 b \Pi_3 d \Pi_3 e$
820 $I_1 \Pi_2 b \Pi_3 b \Pi_3 c \Pi_3 e$
821 $I_1 \Pi_2 b \Pi_3 b \Pi_3 c \Pi_3 f$

$I_2 \leftrightarrow \Pi_1 b, \Pi_2 b, \Pi_3 b, \Pi_3 c \Pi_3 c, \Pi_3 d.$

$\Pi_1 a \leftrightarrow \Pi_2 b, \Pi_2 c, \Pi_3 e.$

$\Pi_2 a \leftrightarrow \Pi_3 f.$

$\Pi_3 a \leftrightarrow \Pi_3 c, \Pi_3 d.$

822 $I_2 \Pi_1 a \Pi_3 a \Pi_3 a \Pi_3 f$
823 $I_2 \Pi_1 a \Pi_3 a \Pi_3 b \Pi_3 f$

$\Pi_1 c \leftrightarrow \Pi_2 a, \Pi_2 b, \Pi_3 a.$

$\Pi_2 c \leftrightarrow \Pi_3 b.$

824 $I_2 \Pi_1 c \Pi_3 a \Pi_3 b \Pi_3 e$
825 $I_2 \Pi_1 c \Pi_3 a \Pi_3 b \Pi_3 f$
826 $I_2 \Pi_2 a \Pi_3 a \Pi_3 a \Pi_3 e$
827 $I_2 \Pi_2 a \Pi_3 a \Pi_3 b \Pi_3 e$
828 $I_2 \Pi_2 c \Pi_3 a \Pi_3 a \Pi_3 e$
829 $I_2 \Pi_2 c \Pi_3 a \Pi_3 a \Pi_3 f$
13 →← II1a, II2a, II3a, II3b, IIIe, IIIf.

II1b →← II2a, II2c, IIIc.

II2b →← IIId.

830  I3 II1b II3c IIIa IIId

831  I3 II1b II3c IIIb IIId

II1c →← II2a, II2b, IIIa.

II2c →← IIIb.

832  I3 II1c II3c IIIb IIIc

833  I3 II1c II3c IIIb IIId

834  I3 II2b II3c IIIa IIIc

835  I3 II2b II3c IIIb IIIc

836  I3 II2c II3c IIIa IIIc

837  I3 II2c II3c IIIa IIId

II1a →← II2b, II2c, IIIe.

II2a →← IIIf.

II3a →← IIIc, IIId.

II3b →← IIIa, IIIb.

II3c →← IIIe, IIIf.

838  II1a II2a II3c IIIa IIIc

839  II1a II2a II3c IIIa IIId

840  II1a II2a II3c IIIb IIIc

841  II1a II2a II3c IIIb IIId
$\Pi_1b \leftrightarrow \Pi_2a, \Pi_2c, \Pi_3c.$

$\Pi_2b \leftrightarrow \Pi_3d.$

842 $\Pi_1b \Pi_2b \Pi_3a \Pi_3a \Pi_3e$

843 $\Pi_1b \Pi_2b \Pi_3a \Pi_3a \Pi_3f$

844 $\Pi_1b \Pi_2b \Pi_3a \Pi_3b \Pi_3e$

845 $\Pi_1b \Pi_2b \Pi_3a \Pi_3b \Pi_3f$

$\Pi_1c \leftrightarrow \Pi_2a, \Pi_2b, \Pi_3a.$

$\Pi_2c \leftrightarrow \Pi_3b.$

846 $\Pi_1c \Pi_2c \Pi_3b \Pi_3c \Pi_3e$

847 $\Pi_1c \Pi_2c \Pi_3b \Pi_3c \Pi_3f$

848 $\Pi_1c \Pi_2c \Pi_3b \Pi_3d \Pi_3e$

849 $\Pi_1c \Pi_2c \Pi_3b \Pi_3d \Pi_3f$

J. Pick 1-1-1-1-1-

$\Pi_1 \leftrightarrow \Pi_1c, \Pi_2c, \Pi_3a, \Pi_3c, \Pi_3a, \Pi_3b.$

$\Pi_1a \leftrightarrow \Pi_2b, \Pi_2c, \Pi_3e.$

$\Pi_2a \leftrightarrow \Pi_3f.$

850 $\Pi_1 \Pi_1a \Pi_2a \Pi_3b \Pi_3c$

851 $\Pi_1 \Pi_1a \Pi_2a \Pi_3b \Pi_3d$

$\Pi_1b \leftrightarrow \Pi_2a, \Pi_2c, \Pi_3c.$

$\Pi_2b \leftrightarrow \Pi_3d.$

852 $\Pi_1 \Pi_1b \Pi_2b \Pi_3b \Pi_3e$

853 $\Pi_1 \Pi_1b \Pi_2b \Pi_3b \Pi_3f$
Case VII: Pick six at a time.

A. Any combination with 3 from one group.

Note: I 12 13 don’t go with anything. II1a II1b II1c don’t go with anything.
from group I and group II. II2a II2b II2c don’t go with anything from group I and group III.

B. Any combination with 2 from one group.

Note: Any two from group I may possibly go with one from group III, one from group II and one from group III, which will only make five in total. So we cannot pick two from group I.

Note: Any two from group III don’t go with anything from group II2, and vice versa.

870 1 I1 I1a I1b I1b IIIb IIIc IIId IIIf
871 1 I2 I2a I2b I2c IIIa IIIc IIIe
872 I2 I1a I1c IIIa IIIb IIIe
873 I2 I2a I2c IIIa IIIc IIIe
874 I3 I1b I1c IIIc IIIb IIId
875 I3 I2b I2c IIIc IIIa IIIc
Theorem 32. Any automorphism of the non-Desarguesian configuration of Figure 3 fixes $p_7$.

Proof. Since $p_7$ has rank 3, it could probably be mapped only to one of the points from $p_1$ to $p_7$. Note that every point on the lines passing through $p_7$ has rank 3, but there always exists a point of rank 4 on at least one of the lines passing through any of the points from $p_1$ to $p_6$. Therefore, any automorphism of the non-Desarguesian configuration of Figure 3 must map $p_7$ to $p_7$.

Corollary 33. There are 12 automorphisms of the non-Desarguesian configuration of Figure 3.

Proof. Since any automorphism will fix $p_7$, we may only map $l_1$, $l_2$, and $l_3$ into themselves. This implies that we may map $p_1$ to itself, $p_2$, $p_3$, $p_4$, $p_5$, or $p_6$. And under each of the cases mentioned, there are two choices to map $p_2$ to. Therefore we obtain 12 automorphisms. Table 4 gives all the details.

Among the 875 extensions, some of them may be isomorphic. Table 5 shows how the methods change when automorphisms are applied.

Theorem 34. We can extend the non-Desarguesian configuration of Figure 3 to 105 semiplanes up to isomorphism without adding a new line.

When we apply the automorphisms to the semiplanes listed in Theorem 31, we obtain a list of all 105 isomorphism classes, which is shown at the end of this Chapter.
Table 4: Automorphisms.

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Table 5: Changes in methods corresponding to automorphisms.

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In summary, one semiplane, with 37 points, forms one class by itself, 6 semiplanes with 35 points form one class, 9 semiplanes with 34 points form 2 classes, 15 semiplanes with 33 points form 3 classes, 42 semiplanes with 32 points form 4 classes, 47 semiplanes with 31 points form 7 classes, 75 semiplanes with 30 points form 9 classes, 111 semiplanes with 29 points form 12 classes, 95 semiplanes with 28 points form 11 classes, 126 semiplanes with 27 points form 15 classes, 120 semiplanes with 26 points form 12 classes, 79 semiplanes with 25 points form 9 classes, 66 semiplanes with 24 points form 9 classes, 45 semiplanes with 23 points form 5 classes, 26 semiplanes with 22 points form 3 classes, 6 semiplanes with 21 points form one class and 6 semiplanes with 20 points
form one class. In the list, the parenthetical number(s) following a semiplane indicate(s) the automorphism(s) which will map the first semiplane in the class to that semiplane. Note that semiplanes with the same number of points may not have the same \( \rho \) value. Also note that a semiplane with fewer points does not necessarily have a smaller \( \rho \) value than a semiplane with more points.

Class 1. Semiplane with 37 points and \( \rho_{12} = 51 \).

(a) IV

Class 2. Semiplanes with 33 points and \( \rho_{12} = 48 \).

(a) l1,IV

(b) l2,IV (3,4,9,10)

(c) l3,IV (5,6,11,12)

Class 3. Semiplanes with 34 points and \( \rho_{12} = 49 \).

(a) II1a,IV

(b) II1b,IV (2,5)

(c) II1c,IV (4,6)

(d) II2a,IV (7,9)

(e) II2b,IV (8,11)

(f) II2c,IV (10,12)
Class 4. Semiplanes with 34 points and $\rho_{12} = 49$.

(a) II3a,IV

(b) II3b,IV (3,5,9,11)

(c) II3c,IV (2,4,8,10)

Class 5. Semiplanes with 35 points and $\rho_{12} = 50$.

(a) IIIa,IV

(b) IIIb,IV (7,8)

(c) IIIc,IV (3,4)

(d) IIId,IV (9,10)

(e) IIIe,IV (5,6)

(f) IIIf,IV (11,12)

Class 6. Semiplanes with 29 points and $\rho_{12} = 45$.

(a) I1,I2,IV

(b) I1,I3,IV (2,5,8,11)

(c) I2,I3,IV (4,6,10,12)

Class 7. Semiplanes with 31 points and $\rho_{12} = 47$.

(a) II1a,II1b,IV

(b) II1a,II1c,IV (3,4)
Class 8. Semiplanes with 33 points and $\rho_{12} = 49$.

(a) IIIa,IIIc,IV
(b) IIIa,IIIe,IV (2,5)
(c) IIIb,IIId,IV (7,9)
(d) IIIb,IIIf,IV (8,11)
(e) IIIc,IIIe,IV (4,6)
(f) IIId,IIIf,IV (10,12)

Class 9. Semiplanes with 33 points and $\rho_{12} = 49$.

(a) IIIa,IIId,IV
(b) IIIa,IIIf,IV (2,11)
(c) IIIb,IIIc,IV (3,7)
(d) IIIb,IIIe,IV (5,8)
(e) IIIc,IIIf,IV (4,12)
(f) IIId,IIIf,IV (6,10)
Class 10. Semiplanes with 30 points and $\rho_{12} = 46$.

(a) I,II1a,IV
(b) I,II1b,IV (2)
(c) I,II2a,IV (7)
(d) I,II2b,IV (8)
(e) I2,II1a,IV (3)
(f) I2,II1c,IV (4)
(g) I2,II2a,IV (9)
(h) I2,II2c,IV (10)
(i) I3,II1b,IV (5)
(j) I3,II1c,IV (6)
(k) I3,II2b,IV (11)
(l) I3,II2c,IV (12)

Class 11. Semiplanes with 30 points and $\rho_{12} = 46$.

(a) I1,II3b,IV
(b) I2,II3a,IV (3,4,9,10)
(c) I3,II3c,IV (5,6,11,12)

Class 12. Semiplanes with 31 points and $\rho_{12} = 47$. 

108
(a) II,IIIc,IV

(b) II,IIId,IV (7)

(c) II,IIIe,IV (2)

(d) II,IIIf,IV (8)

(e) I2,IIIa,IV (3)

(f) I2,IIIb,IV (9)

(g) I2,IIIe,IV (4)

(h) I2,IIIf,IV (10)

(i) I3,IIIa,IV (5)

(j) I3,IIIb,IV (11)

(k) I3,IIIc,IV (6)

(l) I3,IIId,IV (12)

Class 13. Semiplanes with 31 points and $\rho_{12} = 47$.

(a) II1a,II2a,IV

(b) II1b,II2b,IV (2,5,8,11)

(c) II1c,II2c,IV (4,6,10,12)

Class 14. Semiplanes with 31 points and $\rho_{12} = 47$.

(a) II1a,II3a,IV

(b) II1a,II3b,IV (3)
(c) II1b, II3b, IV (5)
(d) II1b, II3c, IV (2)
(e) II1c, II3a, IV (6)
(f) II1c, II3c, IV (4)
(g) II2a, II3a, IV (7)
(h) II2a, II3b, IV (9)
(i) II2b, II3a, IV (11)
(j) II2b, II3c, IV (8)
(k) II2c, II3a, IV (12)
(l) II2c, II3c, IV (10)

Class 15. Semiplanes with 31 points and $\rho_{12} = 47$.

(a) IIIa, IIIc, IV
(b) III1b, IIIa, IV (2,5)
(c) III1c, IIIb, IV (4,6)
(d) III2a, IIIc, IV (7,9)
(e) III2b, IIIa, IV (8,11)
(f) III2c, IIIb, IV (10,12)

Class 16. Semiplanes with 32 points and $\rho_{12} = 48$.

(a) III1a, IIIa, IV
Class 17. Semiplanes with 32 points and $\rho_{12} = 48.$
(i) II2b,IIIa,IV (8)

(j) II2b,IIIe,IV (11)

(k) II2c,IIIc,IV (10)

(l) II2c,IIIe,IV (12)

Class 18. Semiplanes with 32 points and \( \rho_{12} = 48 \).

(a) II1a,IIIf,IV

(b) II1b,IIId,IV (2,5)

(c) II1c,IIIb,IV (4,6)

(d) II2a,IIIf,IV (7,9)

(e) II2b,IIIc,IV (8,11)

(f) II2c,IIIa,IV (10,12)

Class 19. Semiplanes with 32 points and \( \rho_{12} = 48 \).

(a) II3a,IIIa,IV

(b) II3a,IIIb,IV (7)

(c) II3a,IIIe,IV (6)

(d) II3a,IIIf,IV (12)

(e) II3b,IIIc,IV (3)

(f) II3b,IIId,IV (9)

(g) II3b,IIIe,IV (5)
(h) II3b,III,f,IV (11)
(i) II3c,II1a,IV (2)
(j) II3c,II1b,IV (8)
(k) II3c,II1c,IV (4)
(l) II3c,II1d,IV (10)

Class 20. Semiplane with 25 points and $\rho_{12} = 42$.

(a) II1,II2,II3,IV

Class 21. Semiplanes with 28 points and $\rho_{12} = 45$.

(a) II1a,II1b,II1c,IV

(b) II2a,II2b,II2c,IV (7,8,9,10,11,12)

Class 22. Semiplanes with 31 points and $\rho_{12} = 48$.

(a) IIIa,IIIc,IIIe,IV

(b) IIIb,III1d,III1f,IV (7,8,9,10,11,12)

Class 23. Semiplanes with 31 points and $\rho_{12} = 48$.

(a) IIIa,IIIc,IIIe,IV

(b) IIIa,III1d,III1e,IV (2,5)

(c) IIIa,III1d,III1f,IV (10,12)

(d) IIIb,IIIc,IIIe,IV (4,6)
(e) IIIb, IIIc, IIIf, IV (8,11)
(f) IIIb, IIId, IIIe, IV (7,9)

Class 24. Semiplanes with 26 points and $\rho_{12} = 43$.

(a) I1, I2, II1a, IV
(b) I1, I2, II2a, IV (7,9)
(c) I1, I3, IIIb, IV (2,5)
(d) I1, I3, II2b, IV (8,11)
(e) I2, I3, IIIc, IV (4,6)
(f) I2, I3, II2c, IV (10,12)

Class 25. Semiplanes with 27 points and $\rho_{12} = 44$.

(a) I1, I2, IIIe, IV
(b) I1, I2, IIIf, IV (7,9)
(c) I1, I3, IIIc, IV (2,5)
(d) I1, I3, IIId, IV (8,11)
(e) I2, I3, IIIa, IV (4,6)
(f) I2, I3, IIIb, IV (10,12)

Class 26. Semiplanes with 28 points and $\rho_{12} = 45$.

(a) II1a, II1b, II3a, IV
Class 27. Semiplanes with 28 points and $\rho_{12} = 45$.

(a) $\Pi_1a,\Pi_1b,\Pi_3c,IV$

(b) $\Pi_1a,\Pi_1b,\Pi_3b,IV$ (2)

(c) $\Pi_1a,\Pi_1c,\Pi_3b,IV$ (3)

(d) $\Pi_1a,\Pi_1c,\Pi_3c,IV$ (4)

(e) $\Pi_1b,\Pi_1c,\Pi_3a,IV$ (6)

(f) $\Pi_1b,\Pi_1c,\Pi_3b,IV$ (5)

(g) $\Pi_2a,\Pi_2b,\Pi_3a,IV$ (7)

(h) $\Pi_2a,\Pi_2b,\Pi_3c,IV$ (8)

(i) $\Pi_2a,\Pi_2c,\Pi_3b,IV$ (9)

(j) $\Pi_2a,\Pi_2c,\Pi_3c,IV$ (10)

(k) $\Pi_2b,\Pi_2c,\Pi_3a,IV$ (12)

(l) $\Pi_2b,\Pi_2c,\Pi_3b,IV$ (11)

Class 28. Semiplanes with 29 points and $\rho_{12} = 46$.

(a) $\Pi_1a,\Pi_1b,\Pi_3b,IV$

(b) $\Pi_1a,\Pi_1c,\Pi_3a,IV$ (3,4)

(c) $\Pi_1b,\Pi_1c,\Pi_3c,IV$ (5,6)

(d) $\Pi_2a,\Pi_2b,\Pi_3b,IV$ (7,8)

(e) $\Pi_2a,\Pi_2c,\Pi_3a,IV$ (9,10)

(f) $\Pi_2b,\Pi_2c,\Pi_3c,IV$ (11,12)
(a) II1a,II1b,IIIa,IV
(b) II1a,II1c,IIIc,IV (3,4)
(c) II1b,II1c,IIIe,IV (5,6)
(d) II2a,II2b,IIIb,IV (7,8)
(e) II2a,II2c,IIId,IV (9,10)
(f) II2b,II2c,IIIIf,IV (11,12)

Class 29. Semiplanes with 29 points and $\rho_{12} = 46$.

(a) II1a,II1b,IIIb,IV
(b) II1a,II1c,IIIId,IV (3,4)
(c) II1b,II1c,IIIIf,IV (5,6)
(d) II2a,II2b,IIIA,IV (7,8)
(e) II2a,II2c,IIIC,IV (9,10)
(f) II2b,II2c,IIIFe,IV (11,12)

Class 30. Semiplanes with 29 points and $\rho_{12} = 46$.

(a) II1a,II1b,IIIId,IV
(b) II1a,II1b,IIIIf,IV (2)
(c) II1a,II1c,IIIb,IV (3)
(d) II1a,II1c,IIIIf,IV (4)
(e) II1b,II1c,IIIb,IV (5)
Class 31. Semiplanes with 27 points and $\rho_{12} = 44$.

(a) II,IIIa,II1b,IV
(b) II,II2a,II2b,IV (7,8)
(c) II1a,IIIa,IV (3,4)
(d) II2a,II2c,IV (9,10)
(e) II2b,II2c,IV (11,12)

Class 32. Semiplanes with 29 points and $\rho_{12} = 46$.

(a) II1c,IIIe,IV
(b) II1d,IIIf,IV (7,8)
(c) II1a,IIle,IV (3,4)
(d) II1b,IIIf,IV (9,10)
(e) I3, IIIa, IIIc, IV (5,6)
(f) I3, IIIb, IIId, IV (11,12)

Class 33. Semiplanes with 29 and $\rho_{12} = 46$.

(a) I1, IIIc, IIIf, IV
(b) I1, IIIe, IIId, IV (2,7)
(c) I2, IIIa, IIIf, IV (3,10)
(d) I2, IIIb, IIIe, IV (4,9)
(e) I3, IIIa, IIId, IV (5,12)
(f) I3, IIIb, IIIc, IV (6,11)

Class 34. Semiplanes with 30 points and $\rho_{12} = 45$.

(a) II1a, IIIa, IIIc, IV
(b) II1b, IIIa, IIIe, IV (2,5)
(c) II1c, IIIc, IIIe, IV (4,6)
(d) II2a, IIIb, IIId, IV (7,9)
(e) II2b, IIIb, IIIf, IV (8,11)
(f) II2c, IIId, IIIf, IV (10,12)

Class 35. Semiplanes with 30 points and $\rho_{12} = 47$.

(a) II1a, IIIa, IIId, IV
Class 36. Semiplanes with 30 points and $\rho_{12} = 47$.

(a) IIIa,IIIa,IIIa,IV

(b) IIIa,IIIb,IIIc,IV (3)

(c) IIIb,IIIa,IIIf,IV (2)

(d) IIIb,IIIb,IIIe,IV (5)

(e) IIIc,IIIc,IIIf,IV (4)

(f) IIIc,IIIe,IIIe,IV (6)

(g) III2a,IIIa,IIIa,IV (9)

(h) III2a,IIIb,IIIc,IV (7)

(i) III2b,IIIa,IIIf,IV (11)

(j) III2b,IIIb,IIIe,IV (8)

(k) III2b,IIIc,IIIe,IV (12)

(l) III2c,IIIe,IIIe,IV (10)
(h) II2a,IIId,IIIe,IV (9)
(i) II2b,IIIb,IIIc,IV (8)
(j) II2b,IIIc,IIIf,IV (11)
(k) II2c,IIId,IIIf,IV (10)
(l) II2c,IIId,IIIf,IV (12)

Class 37. Semiplanes with 30 points and $\rho_{12} = 47$.

(a) II1a,IIIb,IIIId,IV
(b) II1b,IIIb,IIIIf,IV (2,5)
(c) II1c,IIIId,IIIIf,IV (4,6)
(d) II2a,IIId,IIIc,IV (7,9)
(e) II2b,IIId,IIIe,IV (8,11)
(f) II2c,IIIc,IIIe,IV (10,12)

Class 38. Semiplanes with 30 points and $\rho_{12} = 47$.

(a) IIIa,IIIb,IIIIf,IV
(b) IIIa,IIIId,IIIIf,IV (3)
(c) IIIb,IIIb,IIIId,IV (2)
(d) IIIb,IIIId,IIIIf,IV (5)
(e) IIIc,IIIb,IIIId,IV (4)
(f) IIIc,IIIb,IIIIf,IV (6)
Class 39. Semiplanes with 30 points and $\rho_{12} = 47$.

(a) II3a,IIIa,IIIe,IV

(b) II3a,IIIb,IIIIf,IV (7,12)

(c) II3b,IIIc,IIIe,IV (3,5)

(d) II3b,IIIId,IIIIf,IV (9,11)

(e) II3c,IIIa,IIIc,IV (2,4)

(f) II3c,IIIb,IIIId,IV (8,10)

Class 40. Semiplanes with 30 points and $\rho_{12} = 47$.

(a) II3a,IIIa,IIIIf,IV

(b) II3a,IIIb,IIIe,IV (6,7)

(c) II3b,IIIc,IIIIf,IV (3,11)

(d) II3b,IIIId,IIIe,IV (5,9)

(e) II3c,IIIa,IIIId,IV (2,10)
Class 41. Semiplanes with 27 points and $\rho_{12} = 44$.

(a) $I_1, II_1a, II_2a, IV$

(b) $I_1, II_1b, II_2b, IV (2,8)$

(c) $I_2, II_1a, II_2a, IV (3,9)$

(d) $I_2, II_1c, II_2c, IV (4,10)$

(e) $I_3, II_1b, II_2b, IV (5,11)$

(f) $I_3, II_1c, II_2c, IV (6,12)$

Class 42. Semiplanes with 27 points and $\rho_{12} = 44$.

(a) $I_1, II_1a, II_3b, IV$

(b) $I_1, II_1b, II_3b, IV (2)$

(c) $I_1, II_2a, II_3b, IV (7)$

(d) $I_1, II_2b, II_3b, IV (8)$

(e) $I_2, II_1a, II_3a, IV (3)$

(f) $I_2, II_1c, II_3a, IV (4)$

(g) $I_2, II_2a, II_3a, IV (9)$

(h) $I_2, II_2c, II_3a, IV (10)$

(i) $I_3, II_1b, II_3c, IV (5)$

(j) $I_3, II_1c, II_3c, IV (6)$
(k) I3,II2b,II3c,IV (11)
(l) I3,II2c,II3c,IV (12)

Class 43. Semiplanes with 28 points and $\rho_{12} = 45$.

(a) I1,II1a,IIIc,IV
(b) I1,II1b,IIIe,IV (2)
(c) I1,II2a,IIId,IV (7)
(d) I1,II2b,IIIf,IV (8)
(e) I2,II1a,IIIa,IV (3)
(f) I2,II1c,IIIe,IV (4)
(g) I2,II2a,IIIb,IV (9)
(h) I2,II2c,IIIf,IV (10)
(i) I3,II1b,IIIa,IV (5)
(j) I3,II1c,IIIc,IV (6)
(k) I3,II2b,IIIb,IV (11)
(l) I3,II2c,IIId,IV (12)

Class 44. Semiplanes with 28 points and $\rho_{12} = 45$.

(a) I1,II1a,IIId,IV
(b) I1,II1b,IIIf,IV (2)
(c) I1,II2a,IIIc,IV (7)
Class 45. Semiplanes with 28 points and $\rho_{12} = 45$.

(a) 11,11a,11f,IV

(b) 11,11b,11d,IV (2)

(c) 11,112a,11e,IV (7)

(d) 11,112b,11c,IV (8)

(e) 12,111a,111b,IV (3)

(f) 12,111c,111f,IV (4)

(g) 12,112a,111a,IV (9)

(h) 12,112c,111e,IV (10)

(i) 13,111b,111b,IV (5)

(j) 13,111c,111d,IV (6)

(k) 13,112b,111a,IV (11)

(l) 13,112c,111c,IV (12)
Class 46. Semiplanes with 28 points and $\rho_{12} = 45$. 

(a) I1,II3b,IIIc,IV 
(b) I1,II3b,IIIc,IV (7) 
(c) I1,II3b,IIIc,IV (2) 
(d) I1,II3b,IIIc,IV (8) 
(e) I2,II3a,IIIa,IV (3) 
(f) I2,II3a,IIIb,IV (9) 
(g) I2,II3a,IIIc,IV (4) 
(h) I2,II3a,IIIc,IV (10) 
(i) I3,II3c,IIIa,IV (5) 
(j) I3,II3c,IIIa,IV (11) 
(k) I3,II3c,IIIa,IV (6) 
(l) I3,II3c,IIIa,IV (12) 

Class 47. Semiplanes with 28 points and $\rho_{12} = 45$. 

(a) II1a,II2a,II3a,IV 
(b) II1a,II2a,II3b,IV (3,9)
(c) II1b, II2b, II3b, IV (5,11)
(d) II1b, II2b, II3c, IV (2,8)
(e) II1c, II2c, II3a, IV (6,12)
(f) II1c, II2c, II3c, IV (4,10)

Class 48. Semiplanes with 28 points and $\rho_{12} = 45$.

(a) II1a, II2a, II3c, IV
(b) II1b, II2b, II3a, IV (2,5,8,11)
(c) II1c, II2c, II3b, IV (4,6,10,12)

Class 49. Semiplanes with 29 points and $\rho_{12} = 46$.

(a) II1a, II2a, IIIa, IV
(b) II1a, II2a, IIIb, IV (7)
(c) II1a, II2a, IIIc, IV (3)
(d) II1a, II2a, IIId, IV (9)
(e) II1b, II2b, IIIa, IV (2)
(f) II1b, II2b, IIIb, IV (8)
(g) II1b, II2b, IIIe, IV (5)
(h) II1b, II2b, IIIf, IV (11)
(i) II1c, II2c, IIIc, IV (4)
(j) II1c, II2c, IIId, IV (10)
(k) II1c,II2c,IIIe,IV (6)
(l) II1c,II2c,IIIf,IV (12)

Class 50. Semiplanes with 29 points and $\rho_{12} = 46$.

(a) II1a,II3a,IIia,IV
(b) II1a,II3b,IIic,IV (3)
(c) II1b,II3b,IIIe,IV (5)
(d) II1b,II3c,IIia,IV (2)
(e) II1c,II3a,IIIe,IV (6)
(f) II1c,II3c,IIic,IV (4)
(g) II2a,II3a,IIib,IV (7)
(h) II2a,II3b,IIId,IV (9)
(i) II2b,II3b,IIIf,IV (11)
(j) II2b,II3c,IIIb,IV (8)
(k) II2c,II3a,IIIf,IV (12)
(l) II2c,II3c,IIId,IV (10)

Class 51. Semiplanes with 29 points and $\rho_{12} = 46$.

(a) II1a,II3a,IIib,IV
(b) II1a,II3b,IIIid,IV (3)
(c) II1b,II3b,IIIf,IV (5)
Class 52. Semiplanes with 29 points and $\rho_{12} = 46$.

(a) $\Pi_1a,\Pi_3a,\Pi_3f,\Pi_4$

(b) $\Pi_1a,\Pi_3b,\Pi_3f,\Pi_4$ (3)

(c) $\Pi_1b,\Pi_3b,\Pi_3d,\Pi_4$ (5)

(d) $\Pi_1b,\Pi_3c,\Pi_3d,\Pi_4$ (2)

(e) $\Pi_1c,\Pi_3a,\Pi_3b,\Pi_4$ (6)

(f) $\Pi_1c,\Pi_3c,\Pi_3b,\Pi_4$ (4)

(g) $\Pi_2a,\Pi_3a,\Pi_3e,\Pi_4$ (7)

(h) $\Pi_2a,\Pi_3b,\Pi_3e,\Pi_4$ (9)

(i) $\Pi_2b,\Pi_3b,\Pi_3c,\Pi_4$ (11)

(j) $\Pi_2b,\Pi_3c,\Pi_3a,\Pi_4$ (8)

(k) $\Pi_2c,\Pi_3a,\Pi_3e,\Pi_4$ (12)

(l) $\Pi_2c,\Pi_3c,\Pi_3c,\Pi_4$ (10)
(j) II2b,II3c,IIIc,IV (8)

(k) II2c,II3a,IIIa,IV (12)

(l) II2c,II3c,IIIa,IV (10)

Class 53. Semiplanes with 29 points and \( \rho_{12} = 46 \).

(a) II1a,II3c,IIIa,IV

(b) II1a,II3c,IIIc,IV (3)

(c) II1b,II3a,IIIa,IV (2)

(d) II1b,II3a,IIIe,IV (5)

(e) II1c,II3b,IIIC,IV (4)

(f) II1c,II3b,IIIe,IV (6)

(g) II2a,II3c,IIIb,IV (7)

(h) II2a,II3c,IIId,IV (9)

(i) II2b,II3a,IIIb,IV (8)

(j) II2b,II3a,IIIf,IV (11)

(k) II2c,II3b,IIId,IV (10)

(l) II2c,II3b,IIIf,IV (12)

Class 54. Semiplanes with 29 points and \( \rho_{12} = 46 \).

(a) II1a,II3c,IIIb,IV

(b) II1a,II3c,IIId,IV (3)
(c) II1b,II3a,IIIb,IV (2)
(d) II1b,II3a,IIIf,IV (5)
(e) II1c,II3b,IIId,IV (4)
(f) II1c,II3b,IIIf,IV (6)
(g) II2a,II3c,IIIa,IV (7)
(h) II2a,II3c,IIIc,IV (9)
(i) II2b,II3a,IIIa,IV (8)
(j) II2b,II3a,IIIe,IV (11)
(k) II2c,II3b,IIIc,IV (10)
(l) II2c,II3b,IIIe,IV (12)

Class 55. Semiplanes with 25 points and $\rho_{12} = 47$.

(a) IIIa,II1b,II1c,II3a,IV
(b) IIIa,II1b,II1c,II3b,IV (3,5)
(c) IIIa,II1b,II1c,II3c,IV (2,4)
(d) II2a,II2b,II2c,II3a,IV (7,12)
(e) II2a,II2b,II2c,II3b,IV (9,11)
(f) II2a,II2b,II2c,II3c,IV (8,10)

Class 56. Semiplanes with 26 points and $\rho_{12} = 44$.

(a) IIIa,II1b,II1c,IIIb,IV
(b) II1a,II1b,II1c,IIId,IV (3,4)
(c) II1a,II1b,II1c,IIIf,IV (5,6)
(d) II2a,II2b,II2c,IIIa,IV (7,8)
(e) II2a,II2b,II2c,IIIc,IV (9,10)
(f) II2a,II2b,II2c,IIIe,IV (11,12)

Class 57. Semiplanes with 28 points and $\rho_{12} = 46$.

(a) II1a,IIIa,IIIc,IIIf,IV
(b) II1b,IIIa,IIId,IIIe,IV (2,5)
(c) II1c,IIIb,IIIc,IIIe,IV (4,6)
(d) II2a,IIIb,IIId,IIIe,IV (7,9)
(e) II2b,IIIb,IIIc,IIIf,IV (8,11)
(f) II2c,IIIa,IIId,IIIf,IV (10,12)

Class 58. Semiplanes with 28 points and $\rho_{12} = 46$.

(a) II1a,IIIa,IIId,IIIf,IV
(b) II1a,IIIb,IIIc,IIIf,IV (3)
(c) II1b,IIIa,IIId,IIIf,IV (2)
(d) IIIb,IIIb,IIId,IIIe,IV (5)
(e) II1c,IIIb,IIIe,IIIf,IV (4)
(f) IIIc,IIIb,IIId,IIIe,IV (6)

131
(g) II2a,IIIa,IIIId,IIIe,IV (9)
(h) II2a,IIIb,IIIc,IIIe,IV (7)
(i) II2b,IIIa,IIIc,IIIIf,IV (11)
(j) II2b,IIIb,IIIc,IIIe,IV (8)
(k) II2c,IIIa,IIIc,IIIIf,IV (12)
(l) II2c,IIIa,IIIId,IIIe,IV (10)

Class 59. Semiplanes with 28 points and $\rho_{12} = 46$.

(a) I11a,II1b,IIIId,IIIIf,IV
(b) II1b,IIIb,IIIId,IIIIf,IV (2,5)
(c) I11c,IIIb,IIIId,IIIIf,IV (4,6)
(d) II2a,IIIa,IIIc,IIIe,IV (7,9)
(e) II2b,IIIa,IIIc,IIIe,IV (8,11)
(f) II2c,IIIa,IIIc,IIIe,IV (10,12)

Class 60. Semiplanes with 27 points and $\rho_{12} = 45$.

(a) I11a,II1b,IIIa,IIIId,IV
(b) I11a,II1b,IIIa,IIIIf,IV (2)
(c) I11a,II1c,II1b,IIIc,IV (3)
(d) I11a,II1c,IIIc,IIIIf,IV (4)
(e) I11b,II1c,IIIb,IIIe,IV (5)
Class 61. Semiplanes with 27 points and $\rho_{12} = 45$.

(a) II1a,II1b,II1c,IIIb,IV
(b) II1a,II1b,II1c,IIIf,IV (2)
(c) II1a,II1c,IIIb,IIId,IV (3)
(d) II1a,II1c,IIIb,IIIf,IV (4)
(e) II1b,II1c,IIIb,IIIf,IV (5)
(f) II1b,II1c,IIIb,IIIf,IV (6)
(g) II2a,II2b,IIIa,IIIc,IV (7)
(h) II2a,II2b,IIIa,IIIe,IV (8)
(i) II2a,II2c,IIIa,IIIc,IV (9)
(j) II2a,II2c,IIIa,IIIe,IV (10)
(k) II2b,II2c,IIIa,IIIe,IV (11)
(l) II2b,II2c,IIIf,IV (12)
Class 62. Semiplanes with 27 points and $\rho_{12} = 45$.

(a) II1a,II1b,IIId,IIIf,IV

(b) II1a,II1c,IIlb,IIIf,IV (3,4)

(c) II1b,II1c,IIlb,IIId,IV (5,6)

(d) II2a,II2b,IIlc,IIIe,IV (7,8)

(e) II2a,II2c,IIIa,IIIe,IV (9,10)

(f) II2b,II2c,IIIa,IIIc,IV (11,12)

Class 63. Semiplanes with 23 points and $\rho_{12} = 41$.

(a) II1,II2,II1a,II2a,IV

(b) II1,II3,II1b,II2b,IV (2,5,8,11)

(c) II2,II3,II1c,II2c,IV (4,6,10,12)

Class 64. Semiplanes with 24 points and $\rho_{12} = 42$.

(a) II1,II2,II1a,IIIf,IV

(b) II1,II3,II1b,IIId,IV (2,5)

(c) II2,II3,II1c,IIlb,IV (4,6)

(d) II1,II2a,IIle,IV (7,9)

(e) II1,II2b,IIlc,IV (8,11)
Class 65. Semiplanes with 26 points and $\rho_{12} = 44$.

(a) II1a,II1b,II3a,IIIa,IV

(b) II1a,II1b,II3c,IIIa,IV (2)

(c) II1a,II1c,II3b,IIIc,IV (3)

(d) II1a,II1c,II3c,IIIc,IV (4)

(e) II1b,II1c,II3a,IIIe,IV (6)

(f) II1b,II1c,II3b,IIIe,IV (5)

(g) II2a,II2b,II3a,IIIb,IV (7)

(h) II2a,II2b,II3c,IIIb,IV (8)

(i) II2a,II2c,II3b,IIId,IV (9)

(j) II2a,II2c,II3c,IIId,IV (10)

(k) II2b,II2c,II3a,IIIf,IV (12)

(l) II2b,II2c,II3b,IIIf,IV (11)

Class 66. Semiplanes with 26 points and $\rho_{12} = 44$.

(a) II1a,II1b,II3a,IIIb,IV

(b) II1a,II1b,II3c,IIIb,IV (2)

(c) II1a,II1c,II3b,IIId,IV (3)

(d) II1a,II1c,II3c,IIId,IV (4)
Class 67. Semiplanes with 26 points and $\rho_{12} = 44$. 

(a) II1a,II1b,II3a,IIIb,IV

(b) II1a,II1b,II3c,IIIc,IV (2)

(c) II1a,II1c,II3b,IIIc,IV (3)

(d) II1a,II1c,II3c,IIIb,IV (4)

(e) II1b,II1c,II3a,IIIb,IV (6)

(f) II1b,II1c,II3b,IIIc,IV (5)

(g) II2a,II2b,II3a,IIIe,IV (7)

(h) II2a,II2b,II3c,IIIc,IV (8)

(i) II2a,II2c,II3b,IIIc,IV (9)

(j) II2a,II2c,II3c,IIIa,IV (10)

(k) II2b,II2c,II3a,IIIe,IV (12)

(l) II2b,II2c,II3b,IIIe,IV (11)
(k) II2b,II2c,II3a,IIIa,IV (12)

(l) II2b,II2c,II3b,IIIc,IV (11)

Class 68. Semiplanes with 26 points and $\rho_{12} = 44$.

(a) II1a,II1b,II3b,IIId,IV

(b) II1a,II1b,II3b,IIIf,IV (2)

(c) II1a,II1c,II3a,IIId,IV (3)

(d) II1a,II1c,II3a,IIIf,IV (4)

(e) II1b,II1c,II3c,IIId,IV (5)

(f) II1b,II1c,II3c,IIIc,IV (6)

(g) II2a,II2b,II3b,IIIC,IV (7)

(h) II2a,II2b,II3b,IIIe,IV (8)

(i) II2a,II2c,II3a,IIIA,IV (9)

(j) II2a,II2c,II3a,IIIe,IV (10)

(k) II2b,II2c,II3c,IIIA,IV (11)

(l) II2b,II2c,II3c,IIIC,IV (12)

Class 69. Semiplanes with 24 points and $\rho_{12} = 42$.

(a) I1,II1a,II1b,II3b,IV

(b) I2,II1a,II1c,II3a,IV (3,4)

(c) I3,II1b,II1c,II3c,IV (5,6)
Class 70. Semiplanes with 25 points and \( \rho_{12} = 43 \).

(a) I,IIa,II1b,IIIc,IV
(b) I,II1a,II1b,IIIc,IV (2)
(c) I,II1a,II1b,IIId,IV (3)
(d) I,II1a,II1b,IIId,IV (4)
(e) I,II1a,II1b,IIId,IV (5)
(f) I,II1a,II1b,IIId,IV (6)
(g) I,II2a,II2b,IIIc,IV (7)
(h) I,II2a,II2b,IIIc,IV (8)
(i) I,II2a,II2b,IIIc,IV (9)
(j) I,II2a,II2b,IIIc,IV (10)
(k) I,II2a,II2b,IIIc,IV (11)
(l) I,II2a,II2b,IIIc,IV (12)

Class 71. Semiplanes with 26 points and \( \rho_{12} = 44 \).

(a) I,II1a,IIIc,IIIc,IV
(b) I,II1b,IIIc,IIIc,IV (2)
Class 72. Semiplanes with 26 points and $\rho_{12} = 44$.

(a) II1a,IIId,IIIb,IV (3)
(b) II1b,IIId,IIIf,IV (2)
(c) II1a,IIId,IIIb,IV (3)
(d) II1c,IIId,IIIf,IV (4)
(e) II1b,IIId,IIIb,IV (5)
(f) II1c,IIId,IIIc,IV (6)
(g) II2a,IIId,IIIe,IV (7)
(h) II2b,IIId,IIIe,IV (8)
(i) II2a,IIId,IIIe,IV (9)
(j) II2c,IIId,IIIf,IV (10)
(k) II2b,IIId,IIIc,IV (11)
(l) II2c,IIId,IIIa,IV (12)
Class 73. Semiplanes with 26 points and $\rho_{12} = 44$.

(a) 11,II3b,IIIc,IIIe,IV
(b) 11,II3b,IIIc,IIIe,IV (7,8)
(c) 12,II3a,IIIa,IIIe,IV (3,4)
(d) 12,II3a,IIIa,IIIe,IV (9,10)
(e) 13,II3c,IIIa,IIIc,IV (5,6)
(f) 13,II3c,IIIa,IIIc,IV (11,12)

Class 74. Semiplanes with 26 points and $\rho_{12} = 44$.

(a) 11,II3b,IIIc,IIIe,IV
(b) 11,II3b,IIIc,IIIe,IV (2,7)
(c) 12,II3a,IIIa,IIIe,IV (3,10)
(d) 12,II3a,IIIa,IIIe,IV (4,9)
(e) 13,II3c,IIIa,IIIc,IV (5,12)
(f) 13,II3c,IIIa,IIIc,IV (6,11)
Class 75. Semiplanes with 27 points and $\rho_{12} = 45$.

(a) II1a,II2a,IIIa,IIIc,IV

(b) II1a,II2a,IIIb,IIId,IV (7,9)

(c) II1b,II2b,IIIa,IIIe,IV (2,5)

(d) II1b,II2b,IIIb,IIIe,IV (8,11)

(e) II1c,II2c,IIIc,IIIe,IV (4,6)

(f) II1c,II2c,IIId,IIIe,IV (10,12)

Class 76. Semiplanes with 27 points and $\rho_{12} = 45$.

(a) II1a,II2a,IIIa,IIId,IV

(b) II1a,II2a,IIIb,IIIc,IV (3,7)

(c) II1b,II2b,IIIa,IIIf,IV (2,11)

(d) II1b,II2b,IIIb,IIIe,IV (5,8)

(e) II1c,II2c,IIIc,IIIe,IV (4,12)

(f) II1c,II2c,IIId,IIIe,IV (6,10)

Class 77. Semiplanes with 27 points and $\rho_{12} = 45$.

(a) II1a,II3a,IIIa,IIIe,IV

(b) II1a,II3b,IIIc,IIIe,IV (3)

(c) II1b,II3b,IIId,IIIe,IV (5)
(d) II1b,II3c,IIIa,IIIId,IV (2)
(e) II1c,II3a,IIIb,IIIe,IV (6)
(f) II1c,II3c,IIIb,IIIc,IV (4)
(g) II2a,II3a,IIIb,IIIe,IV (7)
(h) II2a,II3b,IIIId,IIIe,IV (9)
(i) II2b,II3b,IIIc,IIIe,IV (11)
(j) II2b,II3b,IIIc,IIIf,IV (8)
(k) II2c,II3a,IIIa,IIIf,IV (12)
(l) II2c,II3c,IIIa,IIIId,IV (10)

Class 78. Semiplanes with 27 points and $\rho_{12} = 45$.

(a) II1a,II3a,IIIb,IIIf,IV
(b) II1a,II3b,IIIId,IIIf,IV (3)
(c) II1b,II3b,IIIId,IIIf,IV (5)
(d) II1b,II3c,IIIb,IIIId,IV (2)
(e) II1c,II3a,IIIb,IIIf,IV (6)
(f) II1c,II3c,IIIb,IIIId,IV (4)
(g) II2a,II3a,IIIa,IIIe,IV (7)
(h) II2a,II3b,IIIc,IIIe,IV (9)
(i) II2b,II3b,IIIc,IIIe,IV (11)
Class 79. Semiplanes with 27 points and $\rho_{12} = 45$.

(a) II1a,II3c,IIIa,IIIc,IV
(b) II1b,II3a,IIIa,IIIe,IV (2,5)
(c) II1c,II3b,IIIc,IIIe,IV (4,6)
(d) II2a,II3c,IIIb,IIId,IV (7,9)
(e) II2b,II3a,IIIb,IIIf,IV (8,11)
(f) II2c,II3b,IIId,IIIf,IV (10,12)

Class 80. Semiplanes with 27 points and $\rho_{12} = 45$.

(a) II1a,II3c,IIIa,IIIc,IV
(b) II1a,II3c,IIIb,IIIc,IV (3)
(c) II1b,II3a,IIIa,IIIf,IV (2)
(d) II1b,II3a,IIIb,IIIe,IV (5)
(e) II1c,II3b,IIIc,IIIf,IV (4)
(f) II1c,II3b,IIIId,IIIe,IV (6)
(g) II2a,II3c,IIIa,IIIId,IV (9)
(h) II2a,II3c,IIIb,IIIc,IV (7)
(i) II2b,II3a,IIIa,IIIf,IV (11)
(j) II2b,II3a,IIIb,IIIe,IV (8)
(k) II2c,II3b,IIIc,IIIf,IV (12)
(l) II2c,II3b,IIIId,IIIe,IV (10)

Class 81. Semiplanes with 27 points and $\rho_{12} = 45$.

(a) IIla,II3c,IIIb,IIId,IV
(b) II1b,II3a,IIIb,IIIf,IV (2,5)
(c) II1c,II3b,IIIId,IIIf,IV (4,6)
(d) II2a,II3c,IIIa,IIIc,IV (7,9)
(e) II2b,II3a,IIIa,IIIe,IV (8,11)
(f) II2c,II3b,IIIc,IIIe,IV (10,12)

Class 82. Semiplanes with 24 points and $\rho_{12} = 42$.

(a) I1,II1a,II2a,II3b,IV
(b) I1,II1b,II2b,II3b,IV (2,8)
(c) I2,II1a,II2a,II3a,IV (3,9)
(d) I2,II1c,II2c,II3a,IV (4,10)
(e) I3,II1b,II2b,II3c,IV (5,11)
(f) I3,II1c,II2c,II3c,IV (6,12)

144
Class 83. Semiplanes with 25 points and $\rho_{12} = 43$.

(a) I1,II1a,II2a,IIIc,IV

(b) I1,II1a,II2a,IIIId,IV (7)

(c) I1,II1b,II2b,IIId,IV (2)

(d) I1,II1b,II2b,IIIff,IV (8)

(e) I2,II1a,II2a,IIIa,IV (3)

(f) I2,II1a,II2a,IIIb,IV (9)

(g) I2,II1c,II2c,IIId,IV (4)

(h) I2,II1c,II2c,IIIff,IV (10)

(i) I3,II1b,II2b,IIIa,IV (5)

(j) I3,II1b,II2b,IIIa,IV (11)

(k) I3,II1c,II2c,IIId,IV (6)

(l) I3,II1c,II2c,IIIId,IV (12)

Class 84. Semiplanes with 25 points and $\rho_{12} = 43$.

(a) I1,II1a,II3b,IIIc,IV

(b) I1,II1b,II3b,IIId,IV (2)

(c) I1,II2a,II3b,IIId,IV (7)

(d) I1,II2b,II3b,IIIff,IV (8)

(e) I2,II1a,II3a,IIIa,IV (3)
Class 85. Semiplanes with 25 points and $\rho_{12} = 43$. 

(a) I,II1a,II3b,IIIId,IV 
(b) I,II1b,II3b,IIIIf,IV (2) 
(c) I,II2a,II3b,IIIc,IV (7) 
(d) I,II2b,II3b,IIIe,IV (8) 
(e) I,II1a,II3a,IIIb,IV (3) 
(f) I,II1c,II3a,IIIIf,IV (4) 
(g) I,II2a,II3a,IIIa,IV (9) 
(h) I,II2c,II3a,IIIe,IV (10) 
(i) I,II1b,II3c,IIIb,IV (5) 
(j) I,II1c,II3c,IIIId,IV (6) 
(k) I,II2b,II3c,IIIa,IV (11)
Class 86. Semiplanes with 25 points and $\rho_{12} = 43$.

(a) I1,II1a,II3b,IIIf,IV
(b) I1,II1b,II3b,IIId,IV (2)
(c) I1,II2a,II3b,IIle,IV (7)
(d) I1,II2b,II3b,IIlc,IV (8)
(e) I2,II1a,II3a,IIIf,IV (3)
(f) I2,II1c,II3a,IIIb,IV (4)
(g) I2,II2a,II3a,IIle,IV (9)
(h) I2,II2c,II3a,IIla,IV (10)
(i) I3,II1b,II3c,IIId,IV (5)
(j) I3,II1c,II3c,IIlb,IV (6)
(k) I3,II2b,II3c,IIlc,IV (11)
(l) I3,II2c,II3c,IIla,IV (12)

Class 87. Semiplanes with 26 points and $\rho_{12} = 44$.

(a) II1a,II2a,II3a,IIla,IV
(b) II1a,II2a,II3a,IIlb,IV (7)
(c) II1a,II2a,II3b,IIlc,IV (3)
(d) II1a,II2a,II3b,IIld,IV (9)
(e) II\textsubscript{1}b, II\textsubscript{2}b, II\textsubscript{3}b, III\textsubscript{e}, IV (5)

(f) II\textsubscript{1}b, II\textsubscript{2}b, II\textsubscript{3}b, III\textsubscript{f}, IV (11)

(g) II\textsubscript{1}b, II\textsubscript{2}b, II\textsubscript{3}c, III\textsubscript{a}, IV (2)

(h) II\textsubscript{1}b, II\textsubscript{2}b, II\textsubscript{3}c, III\textsubscript{b}, IV (8)

(i) II\textsubscript{1}c, II\textsubscript{2}c, II\textsubscript{3}a, III\textsubscript{e}, IV (6)

(j) II\textsubscript{1}c, II\textsubscript{2}c, II\textsubscript{3}a, III\textsubscript{f}, IV (12)

(k) II\textsubscript{1}c, II\textsubscript{2}c, II\textsubscript{3}c, III\textsubscript{c}, IV (4)

(l) II\textsubscript{1}c, II\textsubscript{2}c, II\textsubscript{3}c, III\textsubscript{d}, IV (10)

Class 88. Semiplanes with 26 points and $\rho_{12} = 44$.

(a) II\textsubscript{1}a, II\textsubscript{2}a, II\textsubscript{3}c, III\textsubscript{a}, IV

(b) II\textsubscript{1}a, II\textsubscript{2}a, II\textsubscript{3}c, III\textsubscript{b}, IV (7)

(c) II\textsubscript{1}a, II\textsubscript{2}a, II\textsubscript{3}c, III\textsubscript{c}, IV (3)

(d) II\textsubscript{1}a, II\textsubscript{2}a, II\textsubscript{3}c, III\textsubscript{d}, IV (9)

(e) II\textsubscript{1}b, II\textsubscript{2}b, II\textsubscript{3}a, III\textsubscript{a}, IV (2)

(f) II\textsubscript{1}b, II\textsubscript{2}b, II\textsubscript{3}a, III\textsubscript{b}, IV (8)

(g) II\textsubscript{1}b, II\textsubscript{2}b, II\textsubscript{3}a, III\textsubscript{e}, IV (5)

(h) II\textsubscript{1}b, II\textsubscript{2}b, II\textsubscript{3}a, III\textsubscript{f}, IV (11)

(i) II\textsubscript{1}c, II\textsubscript{2}c, II\textsubscript{3}b, III\textsubscript{c}, IV (4)

(j) II\textsubscript{1}c, II\textsubscript{2}c, II\textsubscript{3}b, III\textsubscript{d}, IV (10)

148
(k) \( \Pi_{1c},\Pi_{2c},\Pi_{3b},\Pi_{1e},\Pi_{4} \) (6)

(l) \( \Pi_{1c},\Pi_{2c},\Pi_{3b},\Pi_{1f},\Pi_{4} \) (12)

Class 89. Semiplanes with 24 points and \( \rho_{12} = 43 \).

(a) \( \Pi_{1a},\Pi_{1b},\Pi_{1c},\Pi_{1d},\Pi_{1d},\Pi_{4} \)

(b) \( \Pi_{1a},\Pi_{1b},\Pi_{1c},\Pi_{1b},\Pi_{1f},\Pi_{4} \) (2,5)

(c) \( \Pi_{1a},\Pi_{1b},\Pi_{1c},\Pi_{1d},\Pi_{1f},\Pi_{4} \) (4,6)

(d) \( \Pi_{2a},\Pi_{2b},\Pi_{2c},\Pi_{3a},\Pi_{3e},\Pi_{4} \) (7,9)

(e) \( \Pi_{2a},\Pi_{2b},\Pi_{2c},\Pi_{3a},\Pi_{3e},\Pi_{4} \) (8,11)

(f) \( \Pi_{2a},\Pi_{2b},\Pi_{2c},\Pi_{3e},\Pi_{4} \) (10,12)

Class 90. Semiplanes with 25 points and \( \rho_{12} = 44 \).

(a) \( \Pi_{1a},\Pi_{1b},\Pi_{1c},\Pi_{1d},\Pi_{3f},\Pi_{4} \)

(b) \( \Pi_{1a},\Pi_{1c},\Pi_{1b},\Pi_{1c},\Pi_{1f},\Pi_{4} \) (3,4)

(c) \( \Pi_{1b},\Pi_{1c},\Pi_{1b},\Pi_{1d},\Pi_{1e},\Pi_{4} \) (5,6)

(d) \( \Pi_{2a},\Pi_{2b},\Pi_{2c},\Pi_{3c},\Pi_{3e},\Pi_{4} \) (7,8)

(e) \( \Pi_{2a},\Pi_{2c},\Pi_{3a},\Pi_{3d},\Pi_{3e},\Pi_{4} \) (9,10)

(f) \( \Pi_{2b},\Pi_{2c},\Pi_{3a},\Pi_{3c},\Pi_{3f},\Pi_{4} \) (11,12)

Class 91. Semiplanes with 25 points and \( \rho_{12} = 44 \).

(a) \( \Pi_{1a},\Pi_{1b},\Pi_{1c},\Pi_{1d},\Pi_{3f},\Pi_{4} \)
(b) II1a,II1c,IIIb,IIIId,IIIIf,IV (3,4)
(c) IIIb,II1c,IIIb,IIIId,IIIIf,IV (5,6)
(d) II2a,II2b,IIIa,IIIc,IIIe,IV (7,8)
(e) II2a,II2c,IIIa,IIIc,IIIe,IV (9,10)
(f) II2b,II2c,IIIa,IIIc,IIIe,IV (11,12)

Class 92. Semiplanes with 23 points and \( \rho_{12} = 42 \).

(a) II1a,II1b,IIIc,II3a,IIIb,IV
(b) II1a,II1b,IIIc,II3a,IIIIf,IV (6)
(c) II1a,II1b,IIIc,II3b,IIIId,IV (3)
(d) II1a,II1b,IIIc,II3b,IIIIf,IV (5)
(e) II1a,II1b,IIIc,II3c,IIIf,IV (2)
(f) II1a,II1b,IIIc,II3c,IIIId,IV (4)
(g) II2a,II2b,II2c,II3a,IIIa,IV (7)
(h) II2a,II2b,II2c,II3a,IIIe,IV (12)
(i) II2a,II2b,II2c,II3b,IIIc,IV (9)
(j) II2a,II2b,II2c,II3b,IIIe,IV (11)
(k) II2a,II2b,II2c,II3c,IIIa,IV (8)
(l) II2a,II2b,II2c,II3c,IIIc,IV (10)

Class 93. Semiplanes with 24 points and \( \rho_{12} = 43 \).
(a) II1a, II1b, II3a, IIIa, IIIf, IV
(b) II1a, II1b, II3c, IIIa, IIId, IV (2)
(c) II1a, II1c, II3b, IIIc, IIIf, IV (3)
(d) II1a, II1c, II3c, IIIb, IIIc, IV (4)
(e) II1b, II1c, II3a, IIIb, IIIe, IV (6)
(f) II1b, II1c, II3b, IIId, IIIe, IV (5)
(g) II2a, II2b, II3a, IIIb, IIIe, IV (7)
(h) II2a, II2b, II3c, IIIb, IIIc, IV (8)
(i) II2a, II2c, II3b, IIId, IIIe, IV (9)
(j) II2a, II2c, II3c, IIIa, IIId, IV (10)
(k) II2b, II2c, II3a, IIIa, IIIf, IV (12)
(l) II2b, II2c, II3b, IIIc, IIIf, IV (11)

Class 94. Semiplanes with 24 points and ρ₁₂ = 43.
(g) II2a,II2b,II3a,IIIa,IIIe,IV (7)
(h) II2a,II2b,II3c,IIIa,IIIc,IV (8)
(i) II2a,II2c,II3b,IIIc,IIIe,IV (9)
(j) II2a,II2c,II3c,IIIa,IIIc,IV (10)
(k) II2b,II2c,II3a,IIIa,IIIe,IV (12)
(l) II2b,II2c,II3b,IIIc,IIIe,IV (11)

Class 95. Semiplanes with 24 points and $\rho_{12} = 43$.

(a) I11a,II1b,II3b,IIId,IIIf,IV
(b) I11a,II1c,II3a,IIIb,IIIf,IV (3,4)
(c) II1b,II1c,II3c,IIib,IIId,IV (5,6)
(d) II2a,II2b,II3b,IIIc,IIIe,IV (7,8)
(e) II2a,II2c,II3a,IIIa,IIIe,IV (9,10)
(f) II2b,II2c,II3c,IIIa,IIIc,IV (11,12)

Class 96. Semiplanes with 23 points and $\rho_{12} = 42$.

(a) I1,II1a,IIIb,IIId,IIIf,IV
(b) I2,II1a,IIIc,IIIb,IIIf,IV (3,4)
(c) I3,II1b,II1c,IIIb,IIId,IV (5,6)
(d) I1,II2a,II2b,IIIc,IIIe,IV (7,8)
(e) I2,II2a,II2c,IIIa,IIIe,IV (9,10)
Class 97. Semiplanes with 22 points and $\rho_{12} = 41$.

(a) II,IIa,IIb,II3b,IIIc,IV
(b) II,IIa,IIb,II3b,IIIf,IV
(c) I2,IIa,II1c,II3a,IIIb,IV
(d) I2,IIa,II1c,II3a,IIIf,IV
(e) I3,II1b,II1c,II3c,IIIb,IV
(f) I3,II1b,II1c,II3c,IIIc,IV
(g) II,II2a,II2b,II3b,IIIc,IV
(h) I1,II2a,II2b,II3b,IIIe,IV
(i) I2,II2a,II2c,II3a,IIIa,IV
(j) I2,II2a,II2c,II3a,IIIe,IV
(k) I3,II2b,II2c,II3c,IIIa,IV
(l) I3,II2b,II2c,II3c,IIIc,IV

Class 98. Semiplanes with 23 points and $\rho_{12} = 42$.

(a) I1,II1a,II3b,IIIc,IIIf,IV
(b) I1,II1b,II3b,IIIf,II1e,IV
(c) I1,II2a,II3b,IIId,II1e,IV
(d) I1,II2b,II3b,IIIc,IIIf,IV
Class 99. Semiplanes with 23 points and $\rho_{12} = 43$. 

(a) 11,II1a,II3b,IIId,IIIf,IV 

(b) 11,II1b,II3b,IIId,IIIf,IV (2) 

(c) 11,II2a,II3b,IIIc,IIIe,IV (7) 

(d) 11,II2b,II3b,IIIc,IIIe,IV (8) 

(e) 12,II1a,II3a,IIIb,IIIf,IV (3) 

(f) 12,II1b,II3a,IIIb,IIIf,IV (4) 

(g) 12,II2a,II3a,IIIb,IIIe,IV (9) 

(h) 12,II2c,II3a,IIIa,IIIe,IV (10) 

(i) 13,II1b,II3c,IIIa,IIId,IV (5) 

(j) 13,II1c,II3c,IIIa,IIId,IV (6) 

(k) 13,II2b,II3c,IIIb,IIIc,IV (11) 

(l) 13,II2c,II3c,IIIa,IIId,IV (12)
(k) I3,II2b,II3c,IIIa,IIIc,IV (11)

(l) I3,II2c,II3c,IIIa,IIIc,IV (12)

Class 100. Semiplanes with 24 points and $\rho_{12} = 43$.

(a) II1a,II2a,II3c,IIIa,IIIc,IV

(b) II1a,II2a,II3c,IIIb,IIIc,IV (7,9)

(c) II1b,II2b,II3a,IIIa,IIIc,IV (2,5)

(d) II1b,II2b,II3a,IIIb,IIIc,IV (8,11)

(e) II1c,II2c,II3b,IIIc,IIIe,IV (4,6)

(f) II1c,II2c,II3b,IIIc,IIIe,IV (10,12)

Class 101. Semiplanes with 24 points and $\rho_{12} = 43$.

(a) II1a,II2a,II3c,IIIa,IIIc,IV

(b) II1a,II2a,II3c,IIIb,IIIc,IV (3,7)

(c) II1b,II2b,II3a,IIIa,IIIe,IV (2,11)

(d) II1b,II2b,II3a,IIIb,IIIe,IV (5,8)

(e) II1c,II2c,II3b,IIIc,IIIe,IV (4,12)

(f) II1c,II2c,II3b,IIIc,IIIe,IV (6,10)

Class 102. Semiplanes with 22 points and $\rho_{12} = 41$.

(a) II1a,II2a,II3b,IIIc,IV

155
Class 103. Semiplanes with 22 points and $\rho_{12} = 42$.

(a) $\Pi_1a, \Pi_1b, \Pi_1c, \Pi_3b, \Pi_3d, \Pi_3f, IV$

(b) $\Pi_2a, \Pi_2b, \Pi_2c, \Pi_3a, \Pi_3c, \Pi_3e, IV$ (7-12)

Class 104. Semiplanes with 21 points and $\rho_{12} = 41$.

(a) $\Pi_1a, \Pi_1b, \Pi_1c, \Pi_3a, \Pi_3b, \Pi_3c, \Pi_3f, IV$

(b) $\Pi_2a, \Pi_2b, \Pi_2c, \Pi_3a, \Pi_3c, \Pi_3e, IV$ (7,12)
Class 105. Semiplanes with 20 points and $\rho_{12} = 40$.

(a) $I_1, I_1a, I_1b, I_3b, IIId, IIIf, IV$

(b) $I_1, I_1b, I_2b, I_3b, IIIc, IIIe, IV$ (7,8)

(c) $I_2, I_1a, I_1b, I_3a, IIIb, IIIf, IV$ (3,4)

(d) $I_2, I_2a, I_2b, I_3a, IIIa, IIIe, IV$ (9,10)

(e) $I_3, I_1b, I_1c, I_3c, IIIb, IIId, IV$ (5,6)

(f) $I_3, I_2b, I_2c, I_3c, IIIa, IIIc, IV$ (11,12)
Chapter 5
Conclusions

If a finite semiplane can be extended to a projective plane of order $n$, then it can be extended through a sequence of one line extensions. Dr. Nation observed that it was necessary to keep $\rho_n \geq 0$ and this provides a good criterion for restricting the choices for extensions. The algorithm for one line extensions was implemented to computer programs which can be used to attempt to construct non-Desarguesian projective planes.

The main result of this dissertation is that there are 105 non-isomorphic semiplanes generated by a non-Desarguesian configuration, which serve as the initial configuration to the programs. So far we have done more than a hundred thousand hours of computer search. In all cases, $\rho_n$ decreases to 0 in just a few steps. At the beginning of our computer search, we focused mainly on extending semiplane 105 to a plane of order 12, since it is a minimal case. But after testing all the 105 possibilities, we decided to shift our focus to semiplanes 41, 42 and 86 which seem to produce more good candidates than the others. However, we should properly test our semiplanes more thoroughly in the future. We did work on semiplane 98 for a while, but once we found that it cannot be extended to a Hall plane of order 9 and also its dual, we stopped using that configuration as initial input.

From all the tests that we have done so far on extending non-Desarguesian semiplanes to a plane of order 12, we have more than fourteen hundred cases in which we were able to extend the semiplanes from 12 lines to 49 lines, and more than thirty cases to 50 lines, and one case so far to 51 lines. We did some search on order 11 and 15 as well.
In the construction of all the lower order planes, we observed that once the number of lines exceeded roughly one third of the total required, the extension to a projective plane was completed immediately with only one choice for the extension in all remaining steps. While we do not understand the exact nature of this phenomenon (and hope to work on it in the future), it is clear that we are not necessarily as far from a plane of order 12 as it may appear.

We also checked that there are no embeddings among the 105 isomorphism types. All of them can be extended to a Hall plane of order 9, except five of them, namely semiplanes 89, 90, 95, 96 and 98. These five types cannot be extended to the dual of that Hall plane either, together with two additional types, semiplanes 93 and 99.

For other results on projective plane of order 12, see Hall and Wilkinson [9] and Suetake [15].
References


[12] Nation, J.B. Notes on finite linear projective planes,

