COST-EFFICIENT PHYSICAL TOPOLOGIES FOR SURVIVABLE
ROUTING OF DATA NETWORK RINGS
IN WDM-BASED NETWORKS

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To my parents.
Acknowledgments

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ABSTRACT

Wavelength Division Multiplexing (WDM) is a promising technology that can construct backbone networks to comply with its exponentially increasing bandwidth requirements. An important part of this network’s cost is the total length of the fiber-links needed to create the network. A second significant cost is the bandwidth cost that reflects how efficiently the network’s topology facilitates its use of bandwidth. A network model is used where nodes of the network are points on the unit square, and the costs of fiber-links between nodes are the Euclidean distances between the nodes. Fiber-link and bandwidth costs are defined with respect to this model. Two network topologies are investigated that provide both low fiber-link and bandwidth costs. These two network topologies can support survivable data rings, although their methods for routing lightpaths differ.
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Chapter 1

Introduction

Wavelength Division Multiplexing (WDM) is the present choice for constructing backbone networks to comply with its exponentially increasing bandwidth requirements. A WDM network consists of a set of nodes and the fiber-links connecting them. This network supports lightpaths, which are end-to-end optical connections. A lightpath occupies a wavelength-channel per fiber-link along its path. The nodes contain optical cross-connects (OXC s) that cross-connect wavelength-channels to form lightpaths.

Lightpaths are utilized as optical connections (or links) for additional networks, such as IP or SONET networks. Figure 1.1 [1] illustrates an example, where lightpaths in a WDM network are links between routers of an IP network. This creates a two-layer network hierarchy with the WDM network at the lower optical layer, and the IP or SONET networks at the higher electronic layer. We refer to the IP or SONET network as a data network, and the electronic layer as the data layer. Often, in such a hierarchical paradigm, the data network is said to be overlaid on the WDM network, while the topology of the data network is referred to as the virtual topology.

We propose a scenario where the WDM network is part of an Optical Transport Service Provider (OT-SP), which leases lightpaths to organizations or corporations that set up their own data networks. These networks are enterprise networks or IP service providers (IP-SP). We assume that the data network topologies are ring networks, which
we call \textit{data rings}. This is a reasonable assumption since ring networks use a minimal number of links and are resilient to failures. Specifically, rings remain connected after any single link or node fault. SONET and Resilient Packet Ring (RPR) networks \cite{2} are ring oriented due, in part, to these reasons. We assume that the sizes of the rings are limited to some constant $K$ number of nodes, which is sensible because large rings are not bandwidth-efficient.

![Figure 1.1: An IP network overlaid on a WDM network.](image)

Although a data ring is fault tolerant to single failures, a fiber-link or OXC can carry multiple lightpaths. Hence, a single failure on the WDM network (e.g., fiber-link or OXC failure) can cause multiple failures on the data ring, leaving it disconnected. For instance, in Figure 1.1, the failure of fiber-link (1,5) disconnects the IP router at node 1 from the rest of the data ring.
In this thesis, we consider WDM network topologies that support survivable data rings for any single fiber-link failure. These topology layouts have minimal costs. We now define these costs and the network model. The WDM network is assumed to exist in a geographical region modeled by the unit square. A node resides on a point in the square. It is also presumed that the nodes are randomly located, independent of each other, and uniformly distributed over the square. The cost of constructing a fiber-link between two points is the Euclidean distance between them, which we refer to as the fiber-link cost. The fiber-link cost of the network is the sum of all its fiber-link costs. Another cost we consider is the cost to set up a lightpath. A lightpath's cost is the sum of the costs of the fiber-links along its path. A data ring's cost is the sum of its lightpath costs.

A fiber-link's cost reflects the actual costs to set up a fiber-link. This includes the cost of the fiber cable, which often includes many fibers. It also entails the cost to manually layout the cable and the cost of any initial equipment along its path, such as amplifiers. All of these costs are approximately proportional to the length of the link. Therefore, a WDM network's fiber-link's cost is proportional to the initial cost of setting up the network.

A lightpath's cost denotes the amount of bandwidth it consumes. The cost of the bandwidth of a fiber-link is proportional to the fiber-link's cost. It is desirable that the network topology facilitates bandwidth efficiently. We define the bandwidth cost as the average cost of a data ring of size $K$, where the nodes are randomly chosen. The bandwidth costs reflects how the network topology facilitates the use of bandwidth.
Next, we consider two WDM network topologies, shown in Figure 1.2, which minimize these costs. The first topology in Figure 1.2a is a ring network. A WDM ring can support survivable data rings of size $K$ as follows. The WDM ring network has two node disjoint paths between any pair of nodes. A lightpath can use these two paths so that one can carry the lightpath’s signal on one path, and the other is used as back up if the first fails. The path that carries the signal under normal conditions is called the working path, and the back up is called the protection path. When a lightpath is implemented with a working and protection path, it is called a protected lightpath. Such a protected lightpath can survive any single fiber-link or node failure. The disadvantage of a protected lightpath is that it uses more bandwidth than an unprotected lightpath. This is due to the bandwidth for an additional protection path, and the additional constraints of disjoint paths, making them longer and consuming more bandwidth.

![Ring Topology](image)

![Double Hub Topology](image)

**Figure 1.2:** Two common topologies that seek to minimize costs.
A survivable data ring can be implemented on a WDM ring by having all but one of its lightpaths protected. Thus, when a fiber-link or node fails, the data ring will have at most one lightpath failing. Since this leaves the data ring connected, it is survivable.

Next, we survey the typical costs of a WDM ring network by first inspecting its fiber-link cost. It was shown in [3] that layouts of the ring topology have a fiber-link cost of $O(\sqrt{N})$. To show that this is efficient, note that in [4,5], the cost of a minimum spanning tree is $\Theta(\sqrt{N})$ with high probability (close to 1). The minimum spanning tree is a connected network with the least cost, and thus, it is also a lower bound on the cost of the WDM network. A ring has a fiber-link cost of $O(\sqrt{N})$, and therefore, it is efficient.

The disadvantage of the WDM ring is that it has a high average lightpath cost. For a WDM ring network with $N$ nodes, a typical lightpath, routed along a shortest hop path, must traverse approximately $N/4$ fiber-links. This is a fourth of the circumference of the WDM ring network. Since an efficiently constructed WDM ring network has fiber-link costs of $\Theta(\sqrt{N})$, an average lightpath has cost $\Theta(\sqrt{N})$. However, this is quite high compared to the following lower bound. Note that the Euclidean distance between two points is a lower bound on the lightpath cost between these points. Also note that the average Euclidean distance between two random points on the unit square is $\Theta(1)$. This is a lower bound on the average lightpath cost and is significantly less than the $\Theta(\sqrt{N})$ average lightpath cost for a WDM ring network. The bound implies that the bandwidth cost has a lower bound of $\Theta(K)$ because a data ring has $K$ lightpaths. For a WDM ring network, the bandwidth cost is $\Theta(K\sqrt{N})$, which is much higher.
The second network topology we consider is a multi-hub network, as shown in Figure 1.2b. This topology can support survivable data rings of size $K$ when all the lightpaths of the data ring are unprotected. Such network topologies that support survivable data rings for any single fiber-link failure are examined in [6,11]. However, the authors considered networks with a minimum number of links rather than minimum cost. They assumed links have the same cost, independent of their length. This models cases when the equipment at the end points of a link dominate costs. A tight upper bound is given on the number of fiber-links in an $N$ node WDM topology needed to support survivable data rings of size $K$ for any single fiber-link failure. Necessary conditions were also presented for the WDM network topology for data rings of size $K$. They described several topologies that meet these upper bounds. For instance, a double hub topology will support data rings of size $K = N - 2$ for $N$ even and $K \leq N - 3$ for $N$ odd. A modified double hub topology is a double hub with a single link joining the two hubs. An architecture of this type, containing $N$ nodes, can support all logical rings of size $N - 1$ in a survivable manner. Furthermore, a three-hub structure with connecting links between hubs supports data rings of any size.

Figure 1.2b depicts a double hub network, where two nodes are designated as hubs, and the other nodes directly connect to them. The double hub networks have an average lightpath cost of $O(1)$, since a lightpath requires only two hops to connect two nodes in a network. This is comparable to the lower bound $\Theta(1)$, i.e., the average Euclidean distance between two nodes. Hence, it implies that the bandwidth cost for the double hub network is $O(K)$, which is the lower bound.
However, as we demonstrate next, a double hub network has higher fiber-link costs. Consider a double hub, and let $N$ denote the number of its nodes. Of these $N$ nodes, $N - 2$ are non-hub nodes that each have two fiber-links connecting them to the hubs. Thus, there are $2(N - 2)$ fiber-links. In addition, on average, a fiber-link in the network has cost $\Theta(1)$. Therefore, a double hub network has $O(N)$ fiber-link costs, which is considerably higher than the $\Theta(\sqrt{N})$ cost of a minimum spanning tree.

We examined two network topologies and their fiber-link and bandwidth costs. The WDM ring network has efficient fiber-link costs but high bandwidth costs, while the double hub WDM network has efficient bandwidth costs but high fiber-link costs. In this thesis, we present two WDM network topologies that are efficient in both costs. They are named the tree-of-rings (TOR) and the tree-of-multi-hubs (TOMH), and we examine them in Chapter 2. Both topologies support data rings that can survive a single fiber-link failure.

Figure 1.3 compares their fiber-link and bandwidth costs and the lower bounds. The costs for the TOR can be found in Section 2.1.1 as Theorems 2.1 and 2.2, and the costs for the TOMH can be found in Section 2.2.1 as Theorems 2.4 and 2.5. The parameter $H$ is for the TOR topology and it may be chosen to be constant. It will be explained in Chapter 2. Chapter 3 concludes the thesis by discussing directions of future research. For the remainder of this section, we survey related work.

In [7], the authors considered networks that minimized total bandwidth costs, where the bandwidth cost of a link is proportional to the physical length of the link. They show that a single hub network topology (i.e., a star topology) can produce near optimal costs [7]. A star network is non-blocking and its cost is at most twice the minimum cost...
when the integer source capacity equals the integer valued sink capacity for all nodes. They also show that with non-blocking switches, any tree-structured network that is non-blocking for point-to-point channels is also non-blocking for multicast channels. The authors also consider other topologies and characteristics. However, their topologies are not survivable with single failures.

<table>
<thead>
<tr>
<th>Fiber-link Cost</th>
<th>Ring</th>
<th>Double Hub</th>
<th>TOR</th>
<th>TOMH</th>
<th>Lower Bounds</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bandwidth Cost</td>
<td>$0(\sqrt{N})$</td>
<td>$0(N)$</td>
<td>$0(\sqrt{H}\sqrt{N})$</td>
<td>$0(\sqrt{K}\sqrt{N})$</td>
<td>$0(\sqrt{N})$</td>
</tr>
<tr>
<td>Cost</td>
<td>$0(K\sqrt{N})$</td>
<td>$0(K)$</td>
<td>$0(K\cdot H)$</td>
<td>$0(K)$</td>
<td>$0(K)$</td>
</tr>
</tbody>
</table>

Figure 1.3: Comparison of fiber-link costs and bandwidth costs.

Finally, we consider the Euclidean minimum spanning tree and traveling salesman tour problems as they contain similar aspects to the WDM network topology layouts we examine. For these problems, the nodes are points located on a unit square. A link is created between any pair of nodes, and the cost of the link is the Euclidean distance between the two nodes. The Euclidean minimum spanning tree and traveling salesman tour problems are to find the minimum cost spanning tree and the minimum cost traveling salesman tour, respectively. As mentioned earlier, [3,4,5] have results when the nodes are placed randomly and independently, and are uniformly distributed on the square. Additional results on the probabilistic Euclidean traveling salesman problem are given in [8,9]. In [9], a number of simple solutions were compared, including nearest neighbor and spacefilling curve methods, and in [8], the Euclidean traveling salesman problem is proved as NP-Complete. In [10], asymptotic convergence results for the
probabilistic Euclidean minimum spanning tree and traveling salesman problems were given.
Chapter 2

Two Network Topologies

We present the tree-of-rings (TOR) network topology in Section 2.1, the tree-of-multi-hubs (TOMH) network topology in Section 2.2, and a comparison of the two in Section 2.3. The TOR topology supports two node-disjoint paths between any pair of nodes. Therefore, between any pair of nodes, we can find a working and protection path that use different fiber-links and nodes. Then, between any pair of nodes, we can set up a protected lightpath, which can survive a single fiber-link or node failure. Since the TOR has enough connectivity to support protected lightpaths, it can support survivable data rings. In particular, a data ring can be implemented by having all its lightpaths protected except possibly one. Then, any single fiber-link or node failure in the WDM ring will result in at most a single link failure in the data ring. This will leave the data ring connected.

The connectivity of the TOMH ensures that a data ring can be implemented using only unprotected lightpaths, so that it survives any single fiber-link failure. Since the lightpaths are unprotected, they must not have any common fiber-links. Thus, any fiber-link failure will cause at most one lightpath to fail. A shortcoming of the TOMH is its high fiber-link costs when the data ring size $K$ becomes large.
2.1 Tree-of-Rings (TOR)

To create the topology, we first assign nodes to regions within the unit square. First, we describe these regions, and then illustrate how to form the topology with respect to these same regions.

The unit square can be divided into four equal sized sub-squares as shown in Figure 2.1. These sub-squares can also be divided into four equal sized sub-squares and so forth. The unit square is designated as the square at depth $0$, while its four sub-squares are squares at depth $1$. In general, for integer $d \geq 0$, a square at depth $d$ has four sub-squares at depth $d+1$. A square is called the parent of its four sub-squares while the sub-squares are known as its children. Note that each child square has only one parent. This parent-child relationship forms a tree of squares (TOS) with the unit square as the root of the tree. The TOS is comprised of an infinite number of sub-squares because the sub-squares can become infinitesimally small.

![Figure 2.1: Unit square with four sub-squares.](image)

The TOR is organized with respect to the TOS as follows. A square in the TOS can have a set of nodes assigned to it, and these nodes must reside in the square. The size of this set is at most a pre-defined integer value, $H$. We assume that $H$ is at least 4.
If the size of the set of nodes is greater than 2, then these nodes are connected by fiber-links to form a ring. This set is called a ring of a square (ROS), while single-node sets are noted as degenerate rings. The fiber-links of the ROS are called ring links. Each ROS has a depth that is the depth of its square. Squares are known as occupied if they have nodes assigned to it, and unoccupied, otherwise. Each node is assigned to only one of the squares.

If two ROSs \( R_1 \) and \( R_2 \) exist, and the square of \( R_1 \) is the parent of the square of \( R_2 \), then \( R_1 \) is the parent of \( R_2 \), while \( R_2 \) is the child of \( R_1 \).

Nodes are assigned to squares such that if a child square is occupied then so is its parent. Hence, the unit square contains an ROS. An additional constraint is that each parent ROS has at least three nodes. These rings are arranged into a tree-of-rings with the ROS of the unit square as the root.

Finally, in a TOR, links connect a child ROS to its parent. Such links are in addition to the ring links creating the rings. Two distinct links, known as tree-links connect a child ROS to its parent. The links should have distinct nodes unless the child ROS consists of only one node. In this case, the links have a common node at the child ROS but distinct nodes at the parent ROS. The tree-links are called the tree-links for the child square.

Figure 2.2 illustrates an algorithm that creates a TOR. It assigns nodes to squares, where, initially, all nodes are assigned to the unit square. At all times, nodes are assigned to squares where they are physically located. If a square \( S \) has at most \( H \) nodes then the nodes become an ROS for the square. If a square \( S \) has more than \( H \) nodes, then the nodes are reassigned to the children squares of \( S \) such that (i) \( S \) has at most \( H \) nodes but
has at least three nodes, (ii) no square has exactly two nodes (note that two nodes make it impossible to form a ring), and (iii) nodes are assigned to squares where they are physically located.

Algorithm BUILD-TOR

Input: A set of nodes and their locations in the unit square. It is assumed that there are at least 3 nodes.

Output: A TOR.

Label all squares in the TOS unoccupied. (Note that this can be done implicitly, requiring no computation.)

Assign all nodes to the unit square (eventually the root of the TOS) and label the unit square occupied but incomplete

while there is an occupied but incomplete square S in the TOS do

begin

Let M denote the nodes assigned to S.

if $|M| > H$ nodes then do

begin

Reassign all the nodes of S to its children squares so that the nodes are assigned to squares they are physically located.

If there are any children squares with exactly two nodes then reassign exactly one of their nodes back to S.

Comment: No children square has two nodes. In addition, S has at most four nodes. The next while-loop insures that S will have either 3 or 4 nodes. It also insures there are no children squares have two nodes.

while $|M| < 3$ do

begin

Find a child square S* with a node.

if S* has three nodes then reassign two nodes back to S.
else reassign one node back to S.

end

Label all children squares of S that have nodes as occupied but incomplete.

end

Label S complete.

Let the assigned nodes $M$ of S be an ROS for S and include the ROS into the TOR.

end

Connect the ROSs with ring links.

Connect children ROSs with their parents with tree-links.

Figure 2.2: Algorithm for Tree-of-Rings.
Figure 2.3 illustrates how the algorithm BUILD-TOR forms ROSs at depths 0 and 1 when $H = 4$. The ring links at depth 1 are shorter than the ring links at depth 0 because of the decreasing dimensions of their sub-squares. Figure 2.4 exhibits a complete TOR topology, while Figure 2.5 shows how these rings form a tree structure. The set of dashed lines in Figure 2.3 correspond to the bold faced lines in Figure 2.5, and these indicate the tree-links between squares at depths 0 and 1. The rings at depths 0 and 1 in Figure 2.4 coincide with the outlined squares in the tree in Figure 2.5.

Figure 2.3: Formation of rings at depths 0 and 1.
Figure 2.4: Complete Tree-of-Rings topology.

Figure 2.5: Tree structure of Tree-of-Rings topology.
2.1.1 Properties of TOR

We now prove the following two theorems.

**Theorem 2.1.** The TOR of BUILD-TOR has the fiber-cost of \( O(\sqrt{H} \sqrt{N}) \).

**Theorem 2.2.** Consider a TOR of BUILD-TOR. Between any two nodes, two node-disjoint paths exist such that each path has length at most \( O(H) \).

**Theorem 2.3.** The bandwidth cost of a tree-of-rings is \( O(K \cdot H) \).

We use the following Lemmas 2.1-2.4 to prove Theorem 2.1.

**Lemma 2.1.** The cost of a ring with \( H \) nodes in a square at depth \( d \) is at most \( H \sqrt{2} \cdot 2^{-d} \).

**Proof.** The length of a side of the square is \( 2^{-d} \). The distance between any two points in this square is at most \( \sqrt{2} \cdot 2^{-d} \), which is an upper bound on the cost of a ring link. A ring has at most \( H \) links, so the lemma is implied. ■

**Lemma 2.2.** Consider an occupied square at depth \( d \). The cost of the square's tree-links (i.e., the tree-links between its nodes and its parent's nodes) is at most \( \sqrt{2} \cdot 2^{-d+2} \).

**Proof.** A tree-link connects a node in a square with a node in the square's parent. A square's parent has depth \( d - 1 \), and thus, the square's parent has a dimension of \( 2^{-d+1} \). The cost of such a link is \( \sqrt{2} \cdot 2^{-d+1} \). Since two tree-links connect a child ring to its parent, the lemma is implied. ■

Let the fiber-link cost of an occupied square be defined as the sum of the fiber-link costs of the ring and tree-links for the square.

**Lemma 2.3.** The fiber-link cost of an occupied square with \( H \) nodes at depth \( d \) is at most \( (H + 4) \sqrt{2} \cdot 2^{-d} \).
Proof. The cost is the sum of the ring and tree-link costs. Lemmas 2.1 and 2.2 provide upper bounds on these sums. The lemma is implied by these upper bounds.

Lemma 2.4. The upper bound on the fiber-link cost of the tree-of-rings is $O(\sqrt{H\sqrt{N}})$.

Proof. Consider an arbitrary node $u$. Let $c_{\text{ring}}(u)$ be the fiber-link cost of its ROS, and let $p(u)$ be the number of nodes in its ROS. Let $c(u)$ be defined as

$$c(u) = \frac{c_{\text{ring}}(u)}{p(u)}.$$

Let the upper bound in Lemma 2.3 be denoted by $f$, and let $d(u)$ represent the depth of the ROS. Then,

$$c(u) \leq \frac{f}{p(u)} \leq \left(1 + \frac{4}{p(u)}\right) \cdot 2^{-d(u)} \leq 5\sqrt{2} \cdot 2^{-d(u)}$$

where the last inequality is implied since $\left(1 + \frac{4}{p(u)}\right)$ is maximized when $p(u) = 1$. Then, the fiber-link cost of the TOR is

$$\sum_u c(u) \leq \sum_u 5\sqrt{2} \cdot 2^{-d(u)}$$

For the remainder of the proof, we will determine an upper bound on $\sum_u 5\sqrt{2} \cdot 2^{-d(u)}$.

For arbitrary $\delta$, let $n_\delta$ denote the number of nodes at depth $\delta$. Then,

$$\sum_u 5\sqrt{2} \cdot 2^{-d(u)} = 5\sqrt{2} \sum_\delta n_\delta \cdot 2^{-\delta}.$$
Note that the values \( \{n_\delta\} \) have the following two constraints. First, \( \sum n_\delta = N \), the total number of nodes. Second, for each \( \delta \), there are \( 4^\delta \) squares at depth \( \delta \), and each square can be occupied by at most \( H \) nodes. Thus, \( n_\delta \leq H \cdot 4^\delta \). Then, an upper bound to \( \sum_{\delta=0}^5 5\sqrt{2} \cdot 2^{-d(\nu)} \) is the solution to the following optimization problem:

\[
\max 5\sqrt{2} \sum_{\delta} n_\delta \cdot 2^{-\delta} \\
\text{subject to } \sum_{\delta} n_\delta = N
\]

for each \( \delta \geq 0 \), \( n_\delta \leq H \cdot 4^\delta \).

The problem is optimized when \( n_\delta \) is maximized for small values of \( \delta \). In other words, a solution to the problem is

\[
n_\delta = \begin{cases} 
H \cdot 4^\delta & \text{if } \delta < \delta^* \\
0 & \text{if } \delta > \delta^*,
\end{cases}
\]

where \( \delta^* \) is the largest value that satisfies \( \sum_{\delta=0}^\delta H \cdot 4^\delta \leq N \). We proceed to find an upper bound to \( \delta^* \). Note that \( \sum_{\delta=0}^\delta H \cdot 4^\delta \leq N \) is equivalent to \( \frac{4^{\delta^*} - 1}{4 - 1} \leq \frac{N}{H} \). This is simplified to

\[
4^{\delta^*} \leq 3 \frac{N}{H} + 1 \leq 3 \left(\frac{N}{H}\right) + 1 \leq 4 \left(\frac{N}{H}\right)
\]

Then, \( \delta^* \leq \log_4\left(\frac{N}{H}\right) + 1 \).

Next, notice that \( n_\delta = 0 \) if \( \delta > \delta^* \). Therefore, an upper bound on the optimization problem is
\[
5\sqrt{2} \sum_{\delta=0}^{\delta^*} (H \cdot 4^\delta \cdot (2^{-\delta})) = 5\sqrt{2} \cdot H \sum_{\delta=0}^{\delta^*} 2^\delta
\]
\[
= 5\sqrt{2} \cdot H (2^{\delta^*+1} - 1)
\]
\[
\leq 5\sqrt{2} \cdot H (2^{\log_4(N)+2} - 1)
\]
\[
\leq 5\sqrt{2} \cdot H (4 \cdot \sqrt[\log_4(N)]{-1})
\]

where in the second to the last inequality, we applied the upper bound \(\delta^* \leq \log_4 N + 1\).

Thus, the solution of the optimization problem is \(O(\sqrt{H \cdot \sqrt{N}})\), and the lemma is implied.

Lemma 2.4 implies Theorem 2.1.

We now describe a routing algorithm, ROUTE-TOR, for lightpaths in a TOR.

The algorithm routes two node-disjoint paths between any pair of nodes \((u,v)\) as follows.

Let \(S(u)\) and \(S(v)\) denote the squares of \(u\) and \(v\), respectively. Let \(d(u)\) and \(d(v)\) denote the depths of \(S(u)\) and \(S(v)\), respectively. Note that \(S(u)\) and \(S(v)\) have common ancestor squares. Let \(S^*\) denote their common ancestor with the largest depth, and denote its depth by \(d^*\). Figure 2.6 illustrates such a scenario that identifies the depths of the nodes but not the nodes themselves. Note that \(S^*\) could be \(S(u)\) if \(S(u)\) is an ancestor of \(S(v)\), or vice versa. The algorithm will find a pair of node-disjoint paths from node \(u\) to the ring in \(S^*\), and another pair of node-disjoint paths from \(v\) to the ring in \(S^*\). Then, the paths are connected in the ring in \(S^*\). The algorithm ROUTE-TOR will find the node-disjoint paths from \(u\) (and \(v\)) to the ring in \(S^*\) as follows.

First, it finds two node-disjoint paths from node \(u\), through its ROS, to the tree-links leading to the parent \(S'(u)\) of \(S(u)\). The paths continue through the tree-links to the ROS of \(S'(u)\). The paths are routed through the ROS until it reaches the tree-links leading to the parent of \(S'(u)\). The paths continue through the tree-links and ROSs until
they reach the ROS in the common ancestor, $S^*$. Similar paths from $v$ to the common ancestor are found. The paths from $u$ and $v$ are connected in the ROS of $S^*$ to form a pair of node-disjoint paths.

![Diagram](image)

**Figure 2.6:** Path between a pair of nodes.

**Theorem 2.2.** The total length of the pair of node-disjoint paths routed by ROUTE-TOR is $O(H)$.

**Proof.** Consider the paths between two arbitrary nodes $u$ and $v$. Let $d(u)$ and $d(v)$ represent the depths of the square of $u$ and $v$, respectively. Let $d^*$ denote the largest depth of a common ancestor of the squares of $u$ and $v$.

The ROUTE-TOR algorithm finds disjoint paths from $u$ and $v$ to the common ancestor square at depth $d^*$. The routes traverse rings and tree-links along the way, using distinct fiber-links. For a square of depth $\delta$, a ring link and tree-link has length at most $\sqrt{2} \cdot 2^{-\delta}$ and $\sqrt{2} \cdot 2^{-\delta+1}$, respectively. Since the square has at most $H$ ring-links and two tree-links, its total length of ring and tree-links is at most $(H + 4)\sqrt{2} \cdot 2^{-\delta}$. Then, the routes from $u$ and $v$ to the common ancestor square have total length at most
Since routes are joined in the ring of the common ancestor square, the total length of the paths of the ROUTE-TOR is

\[ \sum_{\delta=\delta' + 1}^{d(u)} (H + 4)\sqrt{2} \cdot 2^{-\delta} + \sum_{\delta=\delta' + 1}^{d(v)} (H + 4)\sqrt{2} \cdot 2^{-\delta}. \]

Since \( L_2 \) and \( L_3 \) are both 0 (1), the entire expression is \( O(H) \), and the theorem is implied.

**Theorem 2.3.** The bandwidth cost of a tree-of-rings is \( O(K \cdot H) \).

**Proof.** A data ring of size \( K \) may be set up by having \( K - 1 \) of its links be protected lightpaths, and the last link as an unprotected lightpath. From Theorem 2.2, there are routes for both types of lightpaths that have cost \( O(H) \). The bandwidth cost is the average cost of a data ring of size \( K \), and thus, the total bandwidth cost for the data ring is \( O(K \cdot H) \).

2.1.2 Discussion of Results

The fiber-link cost of a TOR arises from a scenario where the \( N \) nodes are randomly located, independent of each other, and uniformly distributed over the unit square. The results in Section 2.1.1 do not consider the specific placement of these nodes. Two unique cases arise when the nodes are in particular positions. For instance, consider the case when the \( N \) nodes are systematically and uniformly distributed in the unit square. Figure 2.7 shows that creating rings with specific nodes from each square and sub-square can correlate to the formation of a complete tree. This scenario produces
the constraints for the maximization problem in Lemma 2.4. The maximum number of squares exists at each depth; namely, for each \( \delta \), there are \( 4^\delta \) squares at depth \( \delta \).

![Figure 2.7: A complete tree.](image)

In a contrasting scenario, all \( N \) nodes are clustered within a small area of the unit square. Figure 2.8 examines the tree structure of a TOR when this occurs. The tree has only one branch with \( \left\lfloor N/H \right\rfloor \) rings, and the lack of rings at low depths provides for a low fiber-link cost. For instance, the cost of a ring at depth 4 is only a fraction of the cost of a ring at depth 1. As the ring's depth approaches \( \infty \), the ring's fiber-link cost approaches 0. If the TORs depicted in Figures 2.7 and 2.8 contain the same number of rings, then their actual fiber-link costs will differ due to the unequal number of rings at specific depths in each topology. However, the upper bound is the equivalent in both scenarios.

When calculating the bandwidth cost of the TOR, we assume that the traffic flow between any two points is uniform and that a lightpath can easily traverse the tree. However, a bottleneck occurs at the root of the tree because many lightpaths use the ring at depth 0 as part of a route from one node to another. The root square is always a common ancestor of two squares although it is not necessarily the common ancestor with
the largest depth. For instance, it is the common ancestor for two nodes in two different sub-squares at depth 1, and thus, the lightpaths must travel through the root square.

Figure 2.8: A tree with one branch.

When comparing a TOR with a ring, it is evident that their bandwidth costs differ. These differences are due to the two topologies' average lightpath cost. In a ring, an average lightpath cost is \( \Theta(\sqrt{N}) \), while in a TOR, the length of a pair of node-disjoint paths is \( O(H) \). The bandwidth costs of a ring and TOR are equal when \( H = \sqrt{N} \).

Although the upper bound fiber-link costs of a TOR and ring are not equivalent, a closer inspection indicates that the order of these costs are the same. When creating a TOR using the algorithm BUILD-TOR, \( H \), the maximum number of nodes in a square, is a constant value. Thus, it does not affect the order of the fiber-link cost.
An examination of the lower bound costs with those of the TOR shows that the order of these costs are equivalent. Since $H$ is a constant value, both costs of a TOR reduce to the lower bounds of $0(\sqrt{N})$ for the fiber-link cost and $0(K)$ for the bandwidth cost. Smaller constant values of $H$ produce lower cost topologies, as it affects the actual costs but not the order of the bounds. The TOR is a cost-efficient topology that supports survivable data rings if all but one lightpath is protected with both a working and protection path.

2.2 Tree-of-Multi-Hubs (TOMH)

We will describe the TOMH topology. The topology can support survivable data rings of size at most $K$ using only unprotected lightpaths. However, the data rings can only survive single fiber-link failures and not necessarily single node failures. This differs from a TOR, which requires that almost all lightpaths of data rings are protected, and the data rings can survive both single fiber-link and node failures.

The first part of this section will describe how to create a physical topology of the TOMH for a collection of nodes in the unit square. Next, we present its fiber-link costs by using multiple lemmas. Then, we describe the routing algorithm for lightpaths of an arbitrary survivable data ring of size at most $K$. This algorithm routes the lightpaths such that they do not share any fiber-links, and this ensures that any fiber-link failure will disconnect at most one lightpath. Thus, the data ring will survive any single fiber-link failure.

The properties and construction for a TOR still hold here with a few notable modifications. The physical layout of the TOMH uses the TOS, defined in Section 2.1.
As with the TOR, we assign nodes to squares, and these nodes must physically reside in the square. In addition, each node is assigned to exactly one square.

If a square has nodes assigned to it, it is referred to as *occupied*; otherwise, it is *unoccupied*. If a square is occupied, then so is its parent. These occupied squares, as in the TOR, form a tree, which we call the tree of occupied squares (TOOS). The number of nodes in an occupied square must be at least a parameter $J$, where $J = 3K + 4$. We assume that $K$ is at least 3. The value $J$ ensures that the square has a sufficient number of nodes such that their interconnections can facilitate lightpath routing. In particular, it must facilitate the routing of lightpaths for a data ring so that the lightpaths do not share fiber-links. There is also an upper bound on the number of nodes in an occupied square, and we describe this shortly.

In the TOOS, each square contains nodes that are connected to nodes of its parent and children squares. The exceptions are the root square, which does not have a parent, and leaf squares, which do not have a child. The fiber-links that implement such a connection are referred to as *transit fiber-links*. $K$ transit fiber-links exist between a square and its parent. Nodes within a square are connected to one another by fiber-links, referred to as *local fiber-links*.

The nodes in a square that are incident to transit fiber-links are referred to as *transit nodes*. They are the nodes through which lightpaths pass from one square to another. A transit node has fiber-links either from the parent or children squares. If the fiber-links are from the parent, then the node is a *transit-p* node; and if the fiber-links are from the children, then the node is a *transit-c* node. There exists $K$ transit-p nodes, each with exactly one transit fiber-link, and there exists $2K$ transit-c nodes, each with at most
two transit fiber-links. Note that it is possible for each transit-c node to have at most two fiber-links because the $2K$ transit-c nodes of a square are incident to at most $4K$ transit fiber-links from its children squares.

Besides the $3K$ total transit nodes, four distinguished nodes also exist in a square. Two of them are referred to as transit-hubs, while the remaining two are called local-hubs. Along with the local fiber-links, these four nodes connect all nodes within a square.

A square contains two types of local fiber-links, known as local-t (for local-transit) and local-s (for local-square). The local-t fiber-links connect the $3K$ transit nodes to the two transit-hubs, such that each transit node has a local-t fiber-link to each transit-hub. This forms a double hub topology with $6K$ local-t fiber-links. We refer to this as the transit-hub topology of the square. The local-s fiber-links connect the two local-hubs to all other nodes of the square. This also forms a double hub topology that we call the local-hub topology of the square. Note that if a square is assigned $M$ total nodes, then the number of local-s fiber-links is $2(M - 2)$. Figure 2.9 illustrates an example of a square, its node, and its fiber-links.

The combination of the transit-hub topology and the local-hub topology of a square forms a set of multi-hubs. We refer to this as the multi-hub of a square (MHOS). As with the TOR, each MHOS has a depth that is the depth of its square. If two MHOSs $MH_1$ and $MH_2$ exist, and the square of $MH_1$ is the parent of the square of $MH_2$, then $MH_1$ is the parent of $MH_2$, while $MH_2$ is the child of $MH_1$. Note that the parent-child relationships between the MHOSs form a tree. This completes the description of the tree-of-multi-hubs (TOMH) topology.
Figure 2.9: A square with nodes and links in a Tree-of-Multi-Hubs.

Figure 2.10 displays the tree structure of such a topology. Here, each rectangle containing $M$ total nodes represents a square in the TOS, and each is partitioned to identify the various types of nodes. As stated earlier, a lower bound exists on the number of nodes in an occupied square. If the number of nodes in a square, $M$, is greater than $J$, the remaining $M - J$ nodes are referred to as extra nodes.
We now discuss how to design a TOMH topology network from a collection of nodes in the unit square. First, we assign nodes to the squares in the TOS. Recall that the assignment must satisfy the following conditions: (i) nodes are only assigned to squares where they are physically located; (ii) if a square is occupied with nodes, then it has at least \( J \) nodes; and (iii) if a square is occupied, then so is its parent. We impose a fourth condition, which is that if a square is occupied, then it has at most \( J^* = 5J - 4 \) nodes. This leads to an upper bound in the fiber-link costs for the topology.

The BUILD-TOMH algorithm presented in Figure 2.11 accomplishes an assignment of nodes to squares that satisfies all four conditions. It is similar to the BUILD-TOR algorithm discussed in Section 2.1. It begins by first assigning nodes to squares. Initially, all nodes are assigned to the unit square, and at all times, nodes are assigned to squares where they are physically located. After nodes are assigned to
squares, $J$ of the nodes within squares are designated as transit, transit-hub, and local-hub nodes. The specification is arbitrary so that $K$ nodes are transit-p, $2K$ nodes are transit-c, two nodes are transit-hubs, and two nodes are local-hubs. This designation is always possible because the algorithm guarantees at least $J$ nodes in an occupied square.

The rest of the nodes of the square are reassigned to the square's children so that the nodes are physically in the children. If a child square is assigned less than $J$ nodes, then the nodes are reassigned back to the parent. These are referred to as extra nodes for the parent. Note that the parent could possess as many as $4(J-1)$ extra nodes from its four children. Thus, a square will have at most $J + 4(J-1)$ nodes, which is equal to $J^*$. 

The next step is to connect nodes between parent and child squares with transit fiber-links. Recall that $K$ transit fiber-links exist between nodes of a parent square and a child square. Such a fiber-link is incident to a transit-c node in the parent square and a transit-p node in the child square. In addition, (i) each transit-p node is incident to one transit fiber-link, and (ii) a transit-c node is incident to at most two transit fiber-links. We can always set up the transit fiber-links so that the second restriction is satisfied because for a parent square, its children squares have a total of at most $4K$ transit-p nodes. This implies that there are at most $4K$ transit fiber-links for the $2K$ transit-c nodes of a parent square. Thus, each transit-c node can be connected so that it has at most two incident transit fiber-links. Finally, in each square, local fiber-links form the transit-hub and local-hub topologies.

Figure 2.12 illustrates the result of the algorithm BUILD-TOMH when $K = 3$. MHOSs are illustrated at depths 0 and 1, and as the depth increases, the length between
two nodes in a multi-hub decreases. Figure 2.12 is also the complete TOMH topology, and Figure 2.13 shows how this architecture forms a tree structure.

Algorithm BUILD-TOMH

**Input:** A set of nodes and their locations in the unit square. It is assumed that there are at least $3K+4$ nodes.

**Output:** A TOMH.

Label all squares in the TOS *unoccupied*. (Note that this can be done implicitly, requiring no computation.)

Assign all nodes to the unit square (eventually the root of the TOS) and label the unit square *occupied* but *incomplete*

while there is an occupied but incomplete square $S$ in the TOS do

begin

Let $M$ denote the nodes assigned to $S$.

if $|M| > 3K+4$ nodes then do

begin

Designate $3K+4$ nodes in $M$ as follows: $2K$ nodes are transit-c nodes, $K$ nodes are transit-p nodes, two nodes are transit-hubs, and two nodes are local-hubs.

Reassign all the other nodes of $S$ to its children squares so that the nodes are assigned to squares they are physically located in.

If there are any children squares with less than $3K+4$ nodes then reassign all of their nodes back to $S$. They are designated as extra nodes.

Comment: No children square has less than $3K+4$ nodes. In addition, $S$ has at least $3K+4$ nodes and at most $5(3K+3)+1$ nodes. The next while-loop insures that $S$ will have at least $3K+4$ nodes. It also insures there are no children squares have less than $3K+4$ nodes.

Label all children squares of $S$ that have nodes as occupied but incomplete.

end

Label $S$ *complete*.

Let the assigned nodes $M$ of $S$ be an MHOS for $S$ and include the MHOS into the TOMH.

end

For each MHOS,

Connect children MHOSs with their parents using transit fiber-links (connect transit-p nodes with transit-c nodes of parent).

Connect the $3K$ transit nodes with the 2 transit-hubs using local-t links.

Connect all nodes with the 2 local-hubs using local-s links.

Figure 2.11: Algorithm for Tree-of-Multi-Hubs.
Figure 2.12: Complete Tree-of-Multi-Hubs topology.

Figure 2.13: Tree structure of Tree-of-Multi-Hubs topology.
2.2.1 Properties of TOMH

We now prove the following theorems and lemmas that examine the fiber-link and bandwidth costs of a TOMH. These indicate that the fiber-link cost is $O(\sqrt{K}\sqrt{N})$, while the bandwidth cost is $O(K)$.

Theorem 2.4. The TOMH of BUILD-TOMH has the fiber cost of $O(\sqrt{K}\sqrt{N})$.

Theorem 2.5. The total length of a path routed by ROUTE-TOMH is $O(1)$.

Theorem 2.6. The bandwidth cost of a tree-of-multi-hubs is $O(K)$.

We use Lemmas 2.5-2.8 to prove Theorem 2.4.

Lemma 2.5. The cost of the local fiber-links of an MHOS with $M$ nodes at depth $d$ is at most $\sqrt{2}(2(M-2)+6K)\cdot 2^{-d}$.

Proof. The length of a side of the square is $2^{-d}$. The distance between any two points in this square is at most $\sqrt{2} \cdot 2^{-d}$, which is an upper bound on the cost of a local fiber-link. Now, let $M$ denote the number of nodes in the MHOS. The local fiber-links of the MHOS are either in the square’s transit-hub topology or the local-hub topology. The transit-hub topology has $2\cdot 3K$ fiber-links, and the local-hub topology has $2(M-2)$ fiber-links. The total number of local fiber-links is $2(M-2)+2\cdot 3K$. Since each fiber-link has cost at most $\sqrt{2} \cdot 2^{-d}$, the lemma is implied.

For an MHOS, we consider its transit fiber-links as the ones between its transit-p nodes to its parent’s transit-c nodes.

Lemma 2.6. Consider an MHOS $P$ at depth $d$. The cost of its transit-links is at most $K\sqrt{2} \cdot 2^{-d+1}$.
Proof. MHOS $P$ has at most $K$ transit fiber-links. Each such fiber-link connects a node in $P$ with a node in its parent. The parent has depth $d - 1$, and thus, it has a dimension of $2^{-d+1}$. The cost of such a fiber-link is $\sqrt{2} \cdot 2^{-d+1}$. Thus, the total cost of $P$'s transit fiber-links is at most $K \sqrt{2} \cdot 2^{-d+1}$, and the lemma is implied. □

Let the fiber-link cost of an MHOS be defined as the sum of its local and transit fiber-links.

Lemma 2.7. The fiber-link cost of an MHOS with $M$ nodes at depth $d$ is at most

$$(2(M - 2) + 8K)\sqrt{2} \cdot 2^{-d}.$$  

Proof. This is implied by Lemmas 2.5 and 2.6. □

Lemma 2.8. Assume $N \geq J^*$. The upper bound on the fiber-link cost of the tree-of-multi-hubs is $O(\sqrt{K} \sqrt{N})$.

Proof. Consider an arbitrary node $u$. Let $c_{\text{multi-hub}}(u)$ be the fiber-link cost of its MHOS, and let $p(u)$ be the number of nodes that occupy it. Let $c(u)$ be defined as

$$c(u) = \frac{c_{\text{multi-hub}}(u)}{p(u)}.$$  

Let the upper bound in Lemma 2.7 be denoted by $f$, and let $d(u)$ represent the depth of the square. Then,

$$c(u) = \frac{c_{\text{multi-hub}}(u)}{p(u)} \leq \frac{f}{p(u)} \leq \frac{f}{p(u)} \cdot \frac{2(p(u) - 2) + 8K}{p(u)} \sqrt{2} \cdot 2^{-d(u)},$$

$$\leq \sqrt{2} \cdot 2^{-d(u)} + \frac{8K - 4}{p(u)} 2^{-d(u)} \leq \sqrt{2} \cdot 2^{-d(u)} + \frac{8K - 4}{J} 2^{-d(u)}.$$
where the last inequality is implied since \( \frac{8K-4}{p(u)} \) is maximized when \( p(u) \) is minimized.

Since \( p(u) \) must be at least \( J \), the inequality is implied. Thus,

\[
c(u) \leq \sqrt{2} \cdot 2^{-d(u)} + \frac{8K-4}{J} \cdot 2^{-d(u)} \leq \sqrt{2} \cdot 2^{-d(u)} + \frac{8K-4}{3K+4} \cdot 2^{-d(u)} \leq \sqrt{2} \cdot 2^{-d(u)} + \frac{8}{3} \cdot 2^{-d(u)} \leq (\sqrt{2} + \frac{8}{3})2^{-d(u)}
\]

where the last inequality is due to the fact that \( \frac{8K-4}{3K+4} \) is an increasing function. For simplicity, let \( g = (\sqrt{2} + \frac{8}{3}) \). Then, \( c(u) \leq g \cdot 2^{-d(u)} \). Then, the fiber-link cost of the TOMH is

\[
\sum_u c(u) \leq \sum_u g \cdot 2^{-d(u)}
\]

For the remainder of the proof, we will determine an upper bound on the summation.

For arbitrary \( \delta \), let \( n_{\delta} \) denote the number of nodes at depth \( \delta \). Then,

\[
\sum_u g \cdot 2^{-d(u)} = g \sum_\delta n_{\delta} \cdot 2^{-\delta}.
\]

Note that the values \( \{n_{\delta}\} \) have the following two constraints. First, \( \sum_\delta n_{\delta} = N \), the total number of nodes. Second, for each \( \delta \), there are \( 4^\delta \) squares at depth \( \delta \), and each square can be occupied by at most \( J^* \) nodes. Thus, \( n_{\delta} \leq J^* \cdot 4^{\delta} \). Then, an upper bound to

\[
\sum_u g \cdot 2^{-d(u)} \text{ is the solution to the following optimization problem:}
\]
\[ \max g \sum_{\delta} n_\delta \cdot 2^{-\delta} \]

subject to \( \sum_{\delta} n_\delta = N \)

for each \( \delta \geq 0, n_\delta \leq J^* \cdot 4^\delta \)

The problem is optimized when \( n_\delta \) is maximized for small values of \( \delta \). In other words, a solution to the problem is

\[
n_\delta = \begin{cases} 
J^* \cdot 4^\delta & \text{if } \delta < \delta^* \\
0 & \text{if } \delta \geq \delta^*
\end{cases}
\]

where \( \delta^* \) is the largest value that satisfies \( \sum_{\delta=0}^{\delta^*} J^* \cdot 4^\delta \leq N \). Using the arguments from the proof of Lemma 2.4, we have \( \delta^* \leq 3(\nicefrac{1}{J^*}) + 1 \). Then, \( \delta^* \leq 3(\nicefrac{1}{J^*} + 1) \leq 4(\nicefrac{1}{J^*}) \). This implies \( \delta^* \leq \log_4 \left( \frac{1}{J^*} \right) + 1 \).

Using the arguments in the proof of Lemma 2.4, we can conclude that an upper bound on the optimization problem is

\[
g \sum_{\delta=0}^{\delta^*} (J^* \cdot 4^\delta) \cdot 2^{-\delta} = g \cdot J^* \sum_{\delta=0}^{\delta^*} 2^\delta = g \cdot J^* (2^{\delta^*+1} - 1) \leq g \cdot J^* \cdot 2^{\delta^*+1}
\]

Then, applying the upper bound \( \delta^* \leq \log_4 \left( \frac{1}{J^*} \right) + 1 \), we have that the upper bound on the solution of the optimization problem is

\[
g \cdot J^* \cdot 2^{\delta^*+1} \leq g \cdot J^* \cdot 2^{\log_4 \left( \frac{1}{J^*} \right) + 1} \leq g \cdot 4 \cdot J^* \cdot \sqrt{\frac{1}{J^*}} \leq g \cdot 4 \cdot \sqrt{J^*} \cdot \sqrt{N}
\]
This is an upper bound on the fiber-link cost of the TOMH. Since $J^*$ is $O(K)$, the
lemma is implied.

Lemma 2.8 implies Theorem 2.4.

We now describe a routing algorithm, ROUTE-TOMH, for an arbitrary data ring
of size $k \leq K$. In this data ring, $k$ lightpaths are routed. The objective of the algorithm is
to find a route for a lightpath beginning from its two end nodes. For instance, consider a
lightpath that has end nodes denoted by $u$ and $v$. Let $S(u)$ and $S(v)$ denote the squares of $u$
and $v$, respectively. Also, let $d(u)$ and $d(v)$ denote the depths of $S(u)$ and $S(v)$,
respectively. Note that $S(u)$ and $S(v)$ have common ancestor squares. Let $S^*$ denote their
common ancestor with the largest depth, and denote its depth by $d^*$. Figure 2.6 in
Section 2.1.1 illustrated such a scenario. Note that $S^*$ could be $S(u)$ if $S(u)$ is an ancestor
of $S(v)$, or vice versa. The algorithm will find a path from node $u$ through the TOS until
it reaches the nodes of $S^*$. Another path from $v$ to the square in $S^*$ is found similarly.
These paths are parts of the route of the lightpath and are referred to as partial routes.
The partial routes are then joined at $S^*$ to form the complete route for the lightpath.

The partial route from node $u$ is computed by starting from node $u$ and routing on
the local fiber-links of $S(u)$ until reaching a transit-p node. The partial route continues on
a transit fiber-link to the nodes of the parent square of $S(u)$. We denote the parent square
of $S(u)$ by $S'(u)$. The route follows local fiber-links of $S'(u)$ until it reaches a transit-p
node. It continues up the TOS, through transit fiber-links and squares, until reaching the
common ancestor square, $S^*$. A similar partial route from $v$ to the common ancestor is
found.
The algorithm ROUTE-TOMH must ensure that the \( k \) lightpath routes traverse disjoint fiber-links. This is accomplished, in part, by using the results of the following lemma. The lemma applies for double hub network topologies such as the transit-hub and local-hub topologies. Recall that a double hub topology has two nodes designated as \textit{hubs}, and that all other nodes have links to each of them. The lemma shows that it is possible to find link-disjoint routes under certain conditions. It uses the following definitions. First, denote each route by its pair of end nodes, e.g., \((x,y)\), and refer to the pair as a \textit{virtual link}. Note that the nodes and virtual links form a topology we call a \textit{virtual topology}.

**Lemma 2.9.** Consider a double hub topology network with \( n \) nodes and the problem of finding link-disjoint routes between \( k \) pairs of nodes. Let the pairs of nodes be virtual links. Suppose the virtual topology satisfies the following conditions: (i) each node is incident to at most two virtual links; (ii) the virtual topology is either a single ring or a collection of disjoint paths; and (iii) \( 2k \leq n - 3 \). Then, we can find link-disjoint paths between the pairs of nodes.

**Proof.** The first condition implies that the virtual topology is composed of disjoint paths and rings. The second condition states that there are two cases for the virtual topology: it is either a single ring, or it is composed of only disjoint paths. In the first case, since the number of nodes in the ring is \( k \leq n - 3 \), we can apply Theorem 4.1 of [11], which shows that we can find link-disjoint paths between the pair of nodes.

In the second case, the virtual topology is a collection of disjoint paths. Let the number of paths be denoted by \( m \), and the let the paths be denoted by \( p(0), p(1), \ldots, p(m-l) \). Let the end nodes of path \( p(i) \) be denoted by \((u(i),v(i))\). Let us also define \( m \) dummy
links, where dummy link $i$ is $(v(i), u(i+1 \mod m))$. Notice that the paths and the dummy links form a single ring. There are $k$ links in the paths and $m$ dummy links. Thus, the single ring has $k + m$ links. Again, we can apply Theorem 4.1 [11] (similar results can be found in [6]) to this ring because the ring has size $k + m \leq 2k \leq n - 3$. Since we can find link-disjoint paths for the virtual and dummy links, we can find link-disjoint paths for the virtual links and their end nodes.

To ensure that the routes are disjoint, the algorithm ROUTE-TOMH routes multiple lightpaths together. It begins from the MHOSs without children MHOSs, i.e., the leaves of the TOMH. From these leaf MHOSs, partial routes of lightpaths from data ring nodes are computed. The partial routes are continued through transit fiber-links to parent MHOSs, up through the TOMH. Partial routes are computed together through the MHOSs, one MHOS at a time. At each MHOS, partial routes may continue through to the parent MHOS or may join to complete a lightpath route. In addition, nodes of the data ring that reside in the MHOS will begin their partial routes.

We now proceed to describe the algorithm in detail. It is a recursive algorithm that computes partial routes through MHOSs, one at a time. A MHOS that has its partial routes computed is referred to as routed. Otherwise, it is unrouted. The algorithm initially labels all MHOSs as unrouted. If an unrouted MHOS $P$ has no unrouted children MHOSs, then the algorithm will compute partial routes through $P$ as follows. For $P$, note that its children are routed but its parent is unrouted. Its children may continue their partial routes through it. The partial routes will arrive through the transit fiber-links and the transit-c nodes of $P$. Since each transit-c node is incident to at most two transit fiber-links, they have at most two partial routes from children of $P$. 

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It is possible that a transit-c node is also one of the \( k \) nodes of the data ring. We say that a transit-c node:

- has an *overload of two* if it is a data ring node and it has two partial routes from descendants
- has an *overload of one* if
  - it is a data ring node and has one partial route from descendants, or
  - it is not a data ring node and has two partial routes from descendants
- is *free* if it is not a data ring node and has no partial routes from descendants.

The routing algorithm will forward partial routes from overloaded nodes to free nodes such that each free node will receive at most one partial route. In particular, if a transit-c node has an overload of two, then its two partial routes are forwarded. If a transit-c node has an overload of one, then one of its partial routes are forwarded.

The forwarding of the partial routes are done over the transit-hub topology for \( P \). Figure 2.14 illustrates how a transit-c node with an overload of two forwards two partial routes to two free transit-c nodes. The two dashed lines indicate the forwarding of the partial routes, while the two solid lines signify the two partial routes from the descendant MHOSs. The open circles represent free transit-c nodes. Figure 2.15 examines the scenario when a transit-c node has an overload of one because it is a data ring node that has one partial route from a descendant. The dashed lines indicate the forwarding of a partial route that utilizes the transit-hub topology.
Figure 2.14: Routing when a Transit-C node has an overload of two.

Figure 2.15: First type of routing when a Transit-C node has an overload of one.
Figure 2.16 illustrates a second type of routing when a transit-c node has an overload of one. In this case, the node is not a data ring node, but it has two partial routes from descendants. The dashed lines indicate the routing within a square.

![Diagram of routing](image)

Figure 2.16: Second type of routing when a Transit-C node has an overload of one.

The transit-hub topology will also carry another type of partial route forwarding, which is from some or all of the transit-p nodes. These transit-p nodes are part of the \( k \) nodes of the data ring, and we will refer to these nodes as *transit-p ring nodes*. Each of these nodes terminates two lightpaths. The partial routes of the two lightpaths of a transit-p ring node begin at the node and are forwarded to free transit-c nodes.

Recall that each free transit-c node will have at most one partial route forwarded to it, either from an overloaded transit-c node or from a transit-p ring node. Figure 2.17 illustrates this as the two lightpaths originating at a transit-p ring node terminate at two
different free transit-c nodes. The following lemma implies that there exist a sufficient number of free transit-c nodes.

Lemma 2.10. For an MHOS, there is a forwarding of overloaded transit-c nodes and transit-p ring nodes to free transit-c nodes so that the free nodes each have at most one forwarding.

Proof. Let $m$ denote the transit-c nodes that are also nodes of the data ring. We will refer to these nodes as transit-c ring nodes, and the other transit-c ring nodes as transit-c non-ring nodes. Next note that the forwardings correspond to partial routes of lightpaths. Since there are $k$ lightpaths, there are at most $2k$ partial routes. However, we must exclude $2m$ partial routes that begin from the $m$ transit-c ring nodes because they do not contribute to forwardings. Thus, there are at most $2(k - m)$ forwardings.
There are a total of $2k - m$ transit-c non-ring nodes. Since $2(k - m) \leq 2k - m$, there are at least as many transit-c non-ring nodes as forwardings. Thus, there is a sufficient number of free nodes to imply the lemma.

A node pair $(u, v)$ can denote each forwarding, where node $u$ is the origin and node $v$ is the destination of the forwarded partial route. We consider the node pairs as virtual links for the transit-hub topology. Note that each node in this topology is incident to at most two virtual links. In addition, each virtual link is incident to a free transit-c node, and such a node is incident to at most one virtual link. Therefore, the virtual links form node-disjoint paths. We can apply Lemma 2.9 to conclude that the forwarding of partial routes can be accomplished over the transit-hub topology by using fiber-link-disjoint paths. The path computations can be done using the algorithms in [6,11].

After the partial routes are forwarded, the results are the following three scenarios.

- Each transit-c node is
  - a data ring node with no partial routes from children MHOSs, or
  - not a data ring node and has at most one partial route.

We refer to the nodes that satisfy the first condition as type $A$, and the nodes that satisfy the second condition as type $B$.

- Each transit-p ring node has the partial routes of its lightpaths forwarded away.

There may exist other nodes in $P$, which are not transit nodes, that are part of the $k$ nodes of a data ring. The nodes are also called type $A$.

In summary, from a type $A$ node, there exist partial routes for its two lightpaths because it is a data ring node, while from a type $B$ node, there exists one partial route that
originated from another node. These partial routes are then forwarded through the local-hub topology in one of two ways: (i) they continue beyond $P$ by advancing up the TOMH, or (ii) they join with another partial route within $P$ to complete a route for a lightpath. The first type of partial route is known as transit, while the second type is termed local.

The transit partial routes are forwarded to distinct transit-$p$ nodes using the local-$s$ fiber-links of the local-hub topology of $P$. It is always possible to find distinct transit-$p$ nodes because there are $K$ transit-$p$ nodes and at most $K$ lightpaths in the data ring, which implies at most $K$ partial routes whose forwarding is required. A forwarding corresponds to a pair of nodes, where one node is the origin of the partial route, and the other is its destination node. This pair is now considered a virtual link for the local-hub topology.

Nodes with local partial routes are paired such that the joining of partial routes of a pair completes a lightpath's route. These local partial routes are joined together by utilizing a path through the local-$s$ links of the local-hub topology. These pairs of nodes also form virtual links for the local-hub topology.

We apply Lemma 2.9 to the local-hub topology and virtual links, but first, we discuss why we can apply the conditions of the lemma. Note that the nodes have at most two virtual links. These virtual links form node-disjoint paths and cycles. If a cycle is formed, then all nodes of the cycle must have two virtual links and the nodes are of type $A$. This implies that all nodes of the cycle are part of the data ring, which in turn implies that the virtual topology corresponds to the entire data ring. Therefore, the virtual topology is either a single cycle that corresponds to the entire data ring, or it is a collection of node-disjoint paths. Thus, the lemma is applicable. Then, the partial routes
are forwarded using fiber-link-disjoint paths over the local-hub topology. This can be accomplished using the algorithms in [6,11].

Once the forwarding over the local-hub topology is complete, then any partial routing that are at the transit-p nodes and must go to the parent MHOS of $P$ are forwarded up the transit-fiber links. Then, the partial routing through the MHOS $P$ is complete. It is relabeled routed, and the routing algorithm ROUTE-TOMH considers another unrouted MHOS.

**Theorem 2.5.** The total length of a path routed by ROUTE-TOMH is $O(1)$.

**Proof.** Consider the paths between two arbitrary nodes $u$ and $v$. Let $d(u)$ and $d(v)$ represent the depths of the MHOS of $u$ and $v$, respectively. Let $d^*$ denote the longest depth of a common ancestor of the squares of $u$ and $v$.

The ROUTE-TOMH algorithm finds a path from $u$ and $v$ to the common ancestor MHOS at depth $d^*$. The routes traverse MHOSs and transit fiber-links along the way.

For an MHOS of depth $\delta$, a local fiber-link and transit fiber-link has length at most $\sqrt{2} \cdot 2^{-\delta}$ and $\sqrt{2} \cdot 2^{-\delta+1}$, respectively. Due to the transit-hub and local-hub topologies, a lightpath employs at most four local fiber-links between a pair of nodes in an MHOS because in a transit-hub topology and a local-hub topology, a lightpath uses at most two hops. The total length of the fiber-links within an MHOS and the transit fiber-link to reach the parent is at most $(4 + 1)\sqrt{2} \cdot 2^{-\delta} = 5\sqrt{2} \cdot 2^{-\delta}$. Then, the routes from $u$ and $v$ to the common ancestor square have total length at most

$$\sum_{\delta = \delta^* + 1}^{d(u)} 5\sqrt{2} \cdot 2^{-\delta} + \sum_{\delta = \delta^* + 1}^{d(v)} 5\sqrt{2} \cdot 2^{-\delta}.$$

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Since routes are joined in the common ancestor MHOS, the total length of a path of the ROUTE-TOMH is

$$4\sqrt{2} \cdot 2^{-\delta} + \sum_{\delta = \delta^* + 1}^{d(u)} 5\sqrt{2} \cdot 2^{-\delta} + \sum_{\delta = \delta^* + 1}^{d(v)} 5\sqrt{2} \cdot 2^{-\delta}.$$  

Since $\sum_{\delta = \delta^* + 1}^{d(u)} 2^{-\delta}$ and $\sum_{\delta = \delta^* + 1}^{d(v)} 2^{-\delta}$ are both $O(1)$, the entire expression is $O(1)$, and the theorem is implied.

**Theorem 2.6.** The bandwidth cost of a tree-of-multi-hubs is $O(K)$.

**Proof.** A data ring of size $K$ may be set up by having all links as unprotected lightpaths. Each fiber-link is used at most once during the implementation of a data ring, and thus, one fiber-link failure causes at most one lightpath failure. From Theorem 2.5, routes for lightpaths have cost $O(1)$. The bandwidth cost is the average cost of a data ring of size $K$, and thus, the total bandwidth cost for the data ring is $O(K)$.

### 2.2.2 Discussion of Results

An analysis of the properties of a double hub and the TOMH indicates that their fiber-link costs are dissimilar, while their bandwidth costs remain the same. The bandwidth cost of a TOMH is equivalent to the bandwidth cost of a double hub, which is also the lower bound. This similarity arises because in each case, the average lightpath cost is $O(1)$. Their fiber-link costs are different as double hub’s cost is $O(N)$, while the TOMH’s is $O(\sqrt{K}\sqrt{N})$. The TOMH is more cost-efficient because of its lower degree of $N$. However, if $K = N$, both the double hub and TOMH have a fiber-link cost of $O(N)$.
Although the fiber-link costs of the lower bound and the TOMH appear different, a closer inspection indicates that the order of these costs are the same. When creating a TOMH using the algorithm BUILD-TOMH, the size of the data ring, $K$, remains a constant. Thus, $K$ only affects the actual fiber-link cost and not its upper bound.

Minimizing the number of nodes in each multi-hub corresponds to decreasing the fiber-link cost in a TOMH. As seen earlier, a high number of nodes at lower depths induce a higher cost than a lesser number of nodes at these same depths. Therefore, it is advantageous to assign a small amount of nodes to each multi-hub in the tree. It is also beneficial if these nodes are clustered within a small area, such that the lengths of their links are relatively short. These two conditions ensure that the fiber-link cost in this topology is minimal.

The theoretical circumstances that minimize the fiber-link cost do not consider the connectivity needed to ensure that the topology can support a data ring of at most size $K$. Since the objective of creating a TOMH is to support survivable data rings such that all links are implemented as unprotected lightpaths, each multi-hub is constructed from at least $3K+4$ nodes. Similarly, the $K$ transit fiber-links also ensure that a connected network remains after a single failure. These constraints directly oppose the concept that each multi-hub encompasses a smallest number of nodes that minimize the fiber-link cost. Hence, a trade-off exists between the connectivity of the network and a low fiber-link cost. The importance of connectivity justifies a higher cost. Note that forming multi-hubs of at least size $J$ in areas that contain clusters of nodes alleviates the high fiber-link cost, while protecting the connectivity of the network.
2.3 Comparison of TOR and TOMH

Although the fiber-link costs of the TOR and TOMH appear different, they both are equal to the lower bounds. The values \( H \), in the TOR, and \( K \), in the TOMH, are constant numerical values, and thus do not contribute to the order of the cost. However, the actual costs of the two topologies differ even if each is comprised of the same nodes. For instance, consider the case of \( N \) nodes in a unit square, and assume that each ROS has at most \( X \) nodes, while each MHOS also has at most \( X \) nodes. Assuming that the formation of rings and multi-hubs is random, the TOMH will have a greater cost because it contains more physical links than a TOR. A ring containing \( X \) nodes is formed with \( X \) ring links, whereas according to Lemma 2.5, a multi-hub of \( X \) nodes in our topology is comprised of \( 2(X - 2) + 6K \) links. A TOMH also includes \( K \) distinct links between MHOSs, while a TOR only employs two such links. At any given depth, the maximum length of a ring link equals the maximum length of a hub link, and the length of the links connecting children squares to parents are the same in both topologies. The formation of the two topologies indicates that a TOMH contains a greater number of physical links at lower depths than a TOR, and thus has a higher fiber-link cost. The fiber-link costs are only upper bound costs, and forming ROSs and MHOSs within clusters of nodes reduces the actual costs for each topology. In addition, the two topologies have similar fiber-link costs if specific values of \( H \) for the TOR and \( K \) for the TOMH are chosen.

The bandwidth costs are not equal in the two topologies, but each equals the order of the lower bound. A lightpath may need to traverse a large fraction of a ROS before arriving at a node with a tree-link to either its parent or child, and thus, the length of a pair of node-disjoint paths routed by ROUTE-TOR is \( O(H) \) as shown in Theorem 2.2.
Due to a multi-hub’s connectivity, a lightpath traverses at most four links in a MHOS to find a path between two nodes. The routing of a lightpath between two nodes in a transit-hub topology accounts for at most two hops, while the routing in a local-hub topology also justifies at most two hops. Theorem 2.5 shows that in a TOMH, the total path length between a pair of nodes is $O(1)$. Since the maximum path lengths in the two topologies are not equal, the bandwidth costs for the TOR and the TOMH are also different. The actual bandwidth cost of a TOR is greater than both the TOMH and the lower bounds, but the constant value $H$ does not affect the order of the bounds. Thus, the order of the bandwidth costs of both the TOR and the TOMH is equivalent to the lower bound.

The TOR supports survivable lightpaths as it finds two disjoint paths between any two nodes, but it does not guarantee the full support of data rings. Since we do not assume that a single fiber-link carries at most one lightpath, a single fiber-link failure can cause multiple lightpath failures. This then causes the failure of the data ring. However, this data ring can survive if every lightpath except one has both a working and protection lightpath.

Due to the connectivity of the TOMH, the lightpaths do not require protection, but they still ensure that a data ring will not fail. The lightpath routes traverse disjoint fiber-links. Thus, any single fiber-link failure results in at most one lightpath failure. A topology that ensures that a data ring will not fail requires a greater number of fiber-links than a topology that does not guarantee the survivability of the data ring. Hence, a trade-off exists between the fiber-link cost and the connectivity of a network.
Chapter 3

Conclusion

In this thesis, we considered fiber networks that support a data network, which assumes the topology of a ring. A ring's characteristics ensure that a single node or link failure will still leave a connected network. Obtaining efficient fiber-link and bandwidth costs are important aspects when constructing a fiber network. The fiber-link costs refer to the initial set up of WDM links, while the bandwidth costs pertain to the costs of the lightpaths traversing through the data network. In the first chapter, we outlined the scope of the problem and considered research papers that seek to minimize the two costs. We also examined mathematical papers that contain a relevance to the network problem.

The second chapter explored two hierarchical network topologies that achieve efficient fiber-link and bandwidth costs. The first is a TOR that has a fiber-link cost of $O(\sqrt{H}\sqrt{N})$ and a bandwidth cost of $O(H \cdot K)$, while the second, the TOMH has a fiber-link cost of $O(\sqrt{K}\sqrt{N})$ but whose bandwidth cost is $O(K)$. To find these upper bounds, we solved maximization problems that assumed certain values of $p(u)$, the number of nodes in a ROS and in a MHOS. For a TOR, $p(u) = 1$, but for a TOMH, $p(u) = J$. These extreme cases only provide upper bound values, and thus, depending on the specific formation of the rings and multi-hubs, the actual costs may vary. The order of the fiber-link and bandwidth costs for both topologies compare favorably with the lower bounds, and thus both the TOR and TOMH are cost-efficient.
A trade-off exists between the connectivity of the network and a low fiber-link cost. An efficient cost occurs when only a few nodes are at small depths of the tree, but we must allow for enough nodes in each ring or multi-hub to guarantee that the minimum connectivity still exists. An intelligent method of forming rings and multi-hubs is to create them within clusters of nodes, such that the lengths of their interconnections are minimal.

Throughout this thesis, the topology of a ring represents the data network because of its 2-connectiveness and of its simplicity. However, it is worthy to develop physical topologies that can withstand a single fiber-link failure for any arbitrary 2-connected logical topologies.

Another assumption throughout is that the nodes within a unit square are randomly located, independent, and uniformly distributed, yet, in reality, nodes are clustered in certain regions. For instance, assume that we want to connect every home in Utah using the TOR. To do this, we designate each home as a node, and we let the unit square represent the state of Utah. Nodes are clustered in highly populated areas of the state, such as Salt Lake City and Provo, while no nodes are present in areas of treacherous terrain, such as the Rocky Mountains. Assume that a lightpath wants to travel from node (home) \( a \) in Salt Lake City to another, node \( b \), that is 50 yards away. Due to the clustering of nodes in this area, and depending on the maximum size of a ring, nodes \( a \) and \( b \) are not necessarily in the same ROS, and thus, a lightpath must traverse up and down the TOR to connect these two nodes. However, the distance between these nodes is so small that it is probably more efficient to directly connect them. Thus, it is
worthwhile to consider arbitrarily placed nodes in a unit square to determine how to improve the algorithms of the two topologies.

Throughout this thesis, we assume that the unit square contains \( N \) points and that the value \( N \) is relatively large. We do this to justify the need to create new topologies and to generalize our cost results. If, however, only 10 nodes occupy a unit square, it is more appropriate to utilize a simple topology, such as the ring or double hub.

In Chapter 2, we propose algorithms to create both the TOR and TOMH, and we illustrate the construction of these topologies. We do not run simulations to examine the different configurations of rings and multi-hubs in each square. Nonetheless, it is useful to write a program for these algorithms to examine the actual average fiber-link costs and to inspect the average running time.
Bibliography


