Period-One Rotating Solution of Parametric Pendulums by Iterative Harmonic Balance

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Hui Zhang

Thesis Committee:

David T. Ma, Chairperson
H. Ronald Riggs
Ian Robertson

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Abstract

In this study, an iterative method based on harmonic balance for the period-one rotation of parametrically excited pendulum is proposed. Based on the characteristics of the period-one rotating orbit, the exact form of the solution is represented using the Fourier series. An iterative harmonic balance process is proposed to estimate the coefficients in the exact solution form. The general formula for each iteration step is presented. The bounds of excitations required for period-one rotations and the convergence of the method are investigated. The method is evaluated using two performance indexes, i.e. system energy error and global residual error. The performance of the proposed method is compared with the existing perturbation method. The numerical results obtained from MATLAB® are used as the baseline of the evaluation.
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1 Introduction

A Planar pendulum is a simple structure, commonly known as a body suspended from a pivot by a rod or cord so that it can swing freely in the vertical plane. If deformations on the rod and body are small enough to be ignored, a pendulum has only one degree of freedom (i.e. angular displacement), and its rotation can be characterized as a rigid-body motion. It is known that the natural frequency of such a pendulum is independent of its material properties.

The dynamics of a planar pendulum is in general governed by a second-order nonlinear differential equation. When the oscillating amplitude of the pendulum is small, the associated equation of motion can be accurately approximated using a second-order linear differential equation. As a result, the oscillating period only depends on the length of the pendulum and the acceleration due to gravity. This property has been used extensively in scientific instruments, such as accelerometers, timekeepers, gravimeters, etc. For large angular displacements, linear approximation is not suitable in describing the pendulums motion due to large errors that arise from ignoring the nonlinear effects. In this situation, nonlinearities must be considered to analyze the dynamics of a pendulum.

The dynamics of a planar pendulum has been extensively studied for a number of years. In general, the external excitations are divided into three elementary categories: torsional, horizontal and vertical vibrations, as summarized in [1]. A torque-driven pendulum may undergo periodic oscillations, symmetric-breaking, or period-doubling bifurcations as investigated in [2]. The latter two cases are known as parametric excitations in that these excitations are a part of the parameters in the dynamical equation. When excited by horizontal vibrations, a pendulum may exhibits periodic oscillations(symmetric or asymmetric), period-doubling bifurcations, running oscillations, or chaos [3, 4]. If a pendulum is excited vertically, periodic oscillations exist when the hanging equilibrium is destabilized [5]. Parametric resonance can be observed when the excitation frequency is equal to twice the natural frequency [6]. This resonance is known as the 2:1 subharmonic resonance. At this resonance, the half-power bandwidth is not only broadened as compared to the linear resonance but
also inversely proportional to the damping level \[7\]. A pendulum can also exhibit continuous rotating motions which symmetrically exist in a clockwise or anticlockwise direction \[8, 9\]. The simplest rotation is the one that makes one complete revolution in each period of excitation. Such a rotation is the so-called period-one rotation. The average angular velocity of a period-one rotation is equal to the excitation frequency. It has been observed that the region of working frequency depends on the damping level and excitation force \[10\]. Within this region, each working frequency corresponds to a continuous, rotating orbit. Under certain conditions, a vertically excited parametric pendulum can exhibit other types of motion, such as chaos (see \[11\]). Motivated by the characteristics of pendulums and their rich dynamics, especially in the nonlinear region, the potential of using pendulum-type rigid-body architecture to enhance the mechanical energy harvesting from ambient vibrations has received attention recently.

Harvesting mechanical energy to provide electricity for low-power applications was originally proposed in late 90’s \[12, 13, 14, 15, 16\]. Numerous schemes and devices have been developed since then for a variety of applications, many of which have been summarized in recent review articles \[17, 18, 19\]. A majority of the devices developed rely on linear mechanical coupling and certain types of electrical damping to harvest energy. Devices are normally designed based on the linear resonant condition with natural frequencies ranging from 20 Hz to a few kilohertz.

When large, flexible structures, such as aerospace, astronautical, and civil engineering structures are of concern, two primary challenges need to be addressed. The vibration energy of such structures is normally concentrated on a spectrum of low frequencies, i.e. a few hertz (see for example \[20, 21\]); the natural frequency and bandwidth of an energy harvesting device must be comparable to those of the structures in order to achieve a reasonable performance. However, to satisfy such a requirement using linear devices has been difficult. On one hand, in a linear device, a low natural frequency is difficult to achieve due to material and geometrical constraints. On the other hand, the bandwidth of a linear device is proportional to its natural frequency and damping level. Low natural frequency yields narrow bandwidth. The bandwidth may be enlarged by increasing the damping level, however, at
the cost of reduced power output [12, 22].

In recent studies, it has been shown that a device with rigid-body architecture may be advantageous, especially in low-frequency applications, over these difficulties [18, 19, 7]. On one hand, a device with pendulum-type architecture does not rely on material deformation to provide mechanical stiffness, it is thus possible to design the device with a low natural frequency, i.e. less than one hertz [23]. On the other hand, the rich dynamical behaviors are also useful for broadening the bandwidth [7, 10]. In particular, it has been shown [10] that the power harvested from the period-one rotation orbit of a pendulum is independent of the natural frequency of the pendulum. Such a feature is advantageous to broadband energy harvesting.

The aim of this thesis is to investigate the period-one rotating solution of a vertically excited parametric pendulum. A method is proposed to obtain an approximation of the period-one rotating solution. The approximation is not only in a stable and explicit form, but also provides excellent performance in predicting period-one rotating orbits.

This thesis is organized as follows: In chapter 2, the background information concerning the vertically excited parametric pendulum, including the method used for the analysis is provided. In this chapter, the equation of motion of the pendulum is firstly introduced. The previous efforts on its dynamics such as harmonic oscillations, subharmonic oscillations, and rotating orbits are then reviewed with a focus on period-one rotating solutions.

Chapter 3 gives a brief description of the harmonic balance used for analyzing nonlinear dynamical system. Harmonic balance is a method developed from frequency domain to calculate the steady state. With the utilization of Fourier transform in the other engineering fields, this method is no longer limited to solve the problems of electrical engineering. The aspects of its application are reviewed in this chapter.

Chapter 4 introduces two approximations for period-one rotations obtained from the perturbation methods. One analytical approximation was obtained from the multi-scale procedure. The first-order solution, developed in [24] is presented along with higher-order
solutions. An alternative solution proposed by Lenci et al. [25] was obtained from the reformulation of the governing equation as an integral equation.

Chapter 5 illustrates a proposed method using an iterative harmonic balance technique for solving period-one rotating solutions. This chapter shows the entire procedure of approximations based on this method from the first to the $n$th ($n \geq 3$) iterations. Two different criteria for evaluating these approximations is also addressed in this chapter. One of them is for system energy, and the other is for the global residual of the governing equation.

Chapter 6 focuses on evaluating the performance of the proposed method. The performance evaluation includes bounds of excitation amplitude, convergence of coefficients, and the error of approximations. In the last one, the approximations obtained from the perturbation methods are also applied for comparison.

Chapter 7 summarizes the research findings and their implications in practical applications. The potential topics of further research are also discussed.
2 Background Information

Pendulum is one of the paradigms used for studying nonlinear dynamics due to the fact that pendulum is a simple physical object. Pendulum can model various kinds of phenomena related to oscillations, bifurcations and chaos. In addition, there are different pendulum systems, such as double pendulums, coupled pendulums, horizontally excited pendulums and vertically excited pendulums. These systems and their variations may be used for modeling other physical phenomena (i.e. the Josephson superconducting unions[26] and charge density waves[27]). In this study, close attention is paid to vertically excited, parametric pendulum. This section focuses on reviewing previous studies.

2.1 Governing Equation

For a pendulum with a point mass of $m$ and massless arm of length of $l$ under a vertical harmonic base excitation (i.e. the displacement $y_0(t)$ of the base is $y_0(t) = A \cos \Omega t$), its dynamical behavior is governed by the following equation.

$$m l \frac{d^2 \theta}{dt^2} + c l \frac{d\theta}{dt} + mg \sin \theta = mA \Omega^2 \cos \Omega t \sin \theta$$

(1)

where $\theta$ is the angular displacement of the pendulum, $c$ is the viscous damping coefficient of the system, and $g$ is the gravitational acceleration.

For simplifying the analysis, Equation (1) can be rewritten in the following non-dimensional equation.

$$\ddot{\theta} + \gamma \dot{\theta} = (p \cos \omega \tau - 1) \sin \theta$$

(2)

where $\dot{()}$ denotes differentiation of the argument with respect to a non-dimensional time variable $\tau$, defined as $\tau = \omega_n t$, where the natural frequency of the pendulum is defined as $\omega_n = \sqrt{g/l}$. The amplitude and frequency of the excitation are normalized such that, $p = A\Omega^2/g$, $\omega = \Omega/\omega_n$, and the damping coefficient is normalized as $\gamma = c/(m\omega_n)$. 
2.2 Literature Review

The dynamics of the parametrically excited pendulum described by Equation (2) has been investigated for many years. Various aspects of pendulum dynamics have been reported, such as oscillations, chaos and purely rotating orbits. Many of the solutions related to these behaviors have been identified by using analytical, numerical and experimental methods.

For oscillations, there exist two equilibrium positions: one of them is for the upside-down equilibrium position (θ = π). Around this equilibrium position, it is known that a parametrically excited pendulum can perform oscillations and even can be stabilized in this inverted position [28] which has been confirmed experimentally. In addition, Acheson [29] pointed out that multiple pendulums can be stabilized in this inverted position, which was also demonstrated experimentally [30]. Later study [31] gave numerically the conditions for keeping stable oscillations of a inverted pendulum. The other equilibrium position is the stable point (θ = 0). Around this equilibrium point, the pendulum can display periodic oscillations located in different resonance zones [9]. The special one is a 2:1 subharmonic oscillation which is located in the resonance zone around ω = 2. Its steady-state solution has been obtained using the multi-scale method [6]. Based on this parametric resonance, Ma et al. [7] designed a pendulum-type device to investigate its potential in harvesting mechanical energy from ambient vibrations. Their results stated that as oppose to the linear resonance, the bandwidth is inversely proportional to the damping level and there exists a optimal damping level for maximizing the power output.

Besides oscillations, the pendulum can undergo chaotic motions. Leven and Koch [32] numerically observed that the parametric pendulum went through apparently chaotic orbits for sufficiently large external excitation. Bishop and Clifford [11] numerically identified zones of chaotic behaviour; these zones were classified as tumbling chaotic motions, oscillating and rotating chaotic motions. The zone of tumbling chaos fills much of the parameter space and is easy to discover both numerically and experimentally, while the zone of oscillating and rotating chaos is very narrow in the phase space.

In addition to the above two motions, the pendulum can also exhibit continuous rotating
motions. Early investigation of purely rotating orbits of a parametrically excited pendulum were reported in the 1980s. McLaughlin [33] observed the rotational orbits in the numerical study of period-doubling bifurcations in both the dissipative and conservative cases. The application of Melnikov method and averaging method to identify the boundaries of sub-harmonic and homoclinic bifurcations indicated the generation of periodic oscillations and rotations [34]. Leven et al. [35] observed the period-one rotations in experimentally studying the periodic and chaotic motion of a parametric pendulum. In a later investigation, a lower bound of the excitation amplitude, required for the period-one rotating orbit, was numerically observed [8]. In later studies [9, 36], it was reported that pure rotations exist only in narrow strips inside the first resonance zone in the parameter space. However, it has been demonstrated in [10] that such narrow strips may become a semi-open region in the parametric space under certain conditions.

The perturbation methods were employed resulting in two approximate solutions for period-one rotations [24, 25]. In [24] an analytical approximation was obtained using the multi-scale procedure and the first-order solution was developed. It was postulated that higher-order solutions may provide better approximation. However, this study shows that using the second order solution may slightly improve the approximation. Solutions with orders higher than two contains non-periodic terms and, thus, are unstable. In [25] an alternative solution was proposed by reformulating the governing equation as an integral equation, which yields an approximate solution in implicit form. Due to the integration of the governing equation, the information about the angular acceleration is lost during calculating the approximate solution. Consequently, the global residual error of this method cannot be controlled.

In this study, a method based on harmonic balance is proposed to develop a stable solution in explicit form. The performance of the proposed method is evaluated using two indexes, i.e. the system energy and global residual error. The numerical result obtained from MATLAB© is used as the baseline of the evaluation. The performance of the proposed method is compared with existing approximate methods developed from the concept of perturbation. The results show that for higher-level excitations, the proposed method
provides noticeably better estimation as compared to the approximations obtained from the perturbation methods.
3 Harmonic Balance

Harmonic balance is an effective method for analyzing nonlinear dynamical systems. The basic idea of the harmonic balance is to assume that the steady state can be represented by Fourier series, and solve the frequency and coefficients from the resulting equations. It is a frequency-domain technique for calculating the steady-state response of nonlinear systems. As electrical signals are naturally processed in frequency domain, harmonic balance is usually used for analyzing electrical circuits (e.g. analog RF and microwave circuits \[37, 38\]). For example, the approach of harmonic balance was utilized to investigate the stability and bifurcations of limit cycles in Chua’s circuit and accurately evaluated fold, flip, and homoclinic bifurcations \[39\].

Besides processing electrical signals, harmonic balance has also been widely applied to other engineering problems \[40, 41, 42, 43, 44\]. For instance, Tesi et al. \[45\] used the harmonic balance technique to analyze period-doubling bifurcations in a large class of continuous-time feedback systems, and the analysis results were applied to control design. Vasegh and Sedigh \[46\] dealt with chaos control in time-delayed systems via time-delayed feedback controllers whose appropriate controller parameters (delay time and feedback gain) were given by the harmonic balance process. Von Groll and Ewins \[47\] demonstrated the application of harmonic balance to rotor/stator contract problems and calculated the steady-state response under periodic excitation. Albertson and Gilbert \[48\] obtained the steady state solutions of a simple one-cylinder cold engine through the harmonic balance method. Gottlieb \[49\] employed harmonic balance method to the analysis of limit cycles of nonlinear jerk equations and indicated that this method was only suitable to solve some special jerk equations which have at least one term in the jerk expression of the type of odd under time-reversal and at least two terms of the type of even under time-reversal. Shen \[43\] identified a series of period-doubling bifurcation points of Mathieu-Duffing oscillator by the incremental harmonic balance method. Liang and Feeny \[50\] applied harmonic balance to identify parameters of a chaotic base-exited double pendulum system. They developed a harmonic balance identification algorithm used for processing the unstable periodic orbits, which were extracted from
recorded experimental response data.

As the period-one rotation orbit is characterized by the fact that the magnitude of the angular velocity of the pendulum fluctuates around the frequency of the excitation, in this study, the Fourier series are used to develop the exact form of the solution. It is shown that there exist infinite number of harmonic terms in the exact solution. Also, due to the generation of beat frequencies, the general form of the coefficients cannot be obtained using harmonic balance directly. An iterative procedure of harmonic balance is proposed to estimate the coefficients in the solution.
4 Perturbation Method

There exist two approximate solutions developed using the perturbation method. One of them is obtained using the multi-scale procedure [24]. In [24], only the first-order solution was provided. It is presented below. In this chapter, higher-order approximations are also included for completeness. The alternative solution was developed by Lenci [25]. It is also presented in this chapter.

4.1 Multi-scale Approximation

It has been demonstrated that for a weakly excited, lightly damped nonlinear system, it is possible to analyze the system dynamics using a non-dimensional, scaling factor, $\sqrt{\epsilon}$, where $\epsilon << 1$. The governing equation (2) can be rewritten in the following form.

\[
\theta'' + \epsilon\tilde{\gamma}\theta' = (\epsilon p \cos \tilde{\omega} \tilde{\tau} - \epsilon) \sin \theta
\]

(3)

where $(\cdot)$ denotes differentiation of the argument with respect to the non-dimensional time variable $\tilde{\tau}$, defined as $\tilde{\tau} = \frac{\tau}{\sqrt{\epsilon}}$, $\tilde{\gamma} = \frac{\gamma}{\sqrt{\epsilon}}$, and $\tilde{\omega} = \sqrt{\epsilon}\omega$.

It is assumed that Equation (3) has the general solution with the following form.

\[
\theta = \theta_0 + \epsilon\theta_1 + \epsilon^2\theta_2 + \cdots
\]

(4)

A series of time scales is introduced as $T_0 = \tilde{\tau}$, $T_1 = \epsilon\tilde{\tau}$, $T_2 = \epsilon^2\tilde{\tau}$, ..., which are independent variables. As a result, $\theta(\tilde{\tau}) = \theta(T_0, T_1, T_2, \ldots)$. According to the chain rule, the derivatives of $\theta$ with respect to $\tilde{\tau}$ become

\[
\frac{d}{d\tilde{\tau}} = D_0 + \epsilon D_1 + \epsilon^2 D_2 + \cdots
\]

\[
\frac{d^2}{d\tilde{\tau}^2} = D_0^2 + 2\epsilon D_0 D_1 + \epsilon^2 (2D_0 D_2 + D_1^2) + \cdots
\]

(5)

where $D_n^m = \frac{\partial^n}{\partial T_m^n}$.
Based on the general solution, \( \sin \theta \) can be expanded as

\[
\sin \left( \theta_0 + \epsilon \theta_1 + \epsilon^2 \theta_2 + \epsilon^3 \theta_3 \right) = \\
\sin (\theta_0) \cos (\epsilon \theta_1) \cos (\epsilon^2 \theta_2) \cos (\epsilon^3 \theta_3) - \sin (\theta_0) \sin (\epsilon \theta_1) \cos (\epsilon^2 \theta_2) \sin (\epsilon^3 \theta_3) \\
- \sin (\theta_0) \sin (\epsilon \theta_1) \sin (\epsilon^2 \theta_2) \cos (\epsilon^3 \theta_3) - \sin (\theta_0) \sin (\epsilon \theta_1) \sin (\epsilon^2 \theta_2) \sin (\epsilon^3 \theta_3) \\
+ \cos (\theta_0) \sin (\epsilon \theta_1) \cos (\epsilon^2 \theta_2) \cos (\epsilon^3 \theta_3) - \sin (\theta_0) \sin (\epsilon \theta_1) \sin (\epsilon^2 \theta_2) \sin (\epsilon^3 \theta_3) \\
+ \cos (\theta_0) \cos (\epsilon \theta_1) \sin (\epsilon^2 \theta_2) \cos (\epsilon^3 \theta_3) + \cos (\theta_0) \cos (\epsilon \theta_1) \cos (\epsilon^2 \theta_2) \sin (\epsilon^3 \theta_3)
\]

\[= \sin (\theta_0) \cos (\epsilon \theta_1) \cos (\epsilon^2 \theta_2) \cos (\epsilon^3 \theta_3) - \sin (\theta_0) \sin (\epsilon \theta_1) \cos (\epsilon^2 \theta_2) \sin (\epsilon^3 \theta_3) \\
- \sin (\theta_0) \sin (\epsilon \theta_1) \sin (\epsilon^2 \theta_2) \cos (\epsilon^3 \theta_3) - \sin (\theta_0) \sin (\epsilon \theta_1) \sin (\epsilon^2 \theta_2) \sin (\epsilon^3 \theta_3) \\
+ \cos (\theta_0) \sin (\epsilon \theta_1) \cos (\epsilon^2 \theta_2) \cos (\epsilon^3 \theta_3) - \sin (\theta_0) \sin (\epsilon \theta_1) \sin (\epsilon^2 \theta_2) \sin (\epsilon^3 \theta_3) \\
= \sin \theta_0 + \epsilon \theta_1 \cos \theta_0 + \epsilon^2 \theta_2 \cos \theta_0 + \epsilon^3 \theta_3 \cos \theta_0 - \epsilon^4 \theta_1 \theta_3 \sin \theta_0 - \epsilon^5 \theta_2 \theta_3 \sin \theta_0 - \epsilon^6 \theta_1 \theta_2 \theta_3 \cos \theta_0
\]

Substituting Equations (4), (5) and (6) into Equation (3) and grouping together the terms with the same powers of \( \epsilon \) to equal zero yield

\[\epsilon^0 : D_0^2 \theta_0 = 0 \tag{7}\]

\[\epsilon^1 : D_0^2 \theta_1 = -\hat{\gamma} D_0 \theta_0 - 2D_0 D_1 \theta_0 - \sin \theta_0 + p \cos (\tilde{\omega} T_0) \sin \theta_0 \tag{8}\]

\[\epsilon^2 : D_0^2 \theta_2 = -\hat{\gamma} (D_0 \theta_0 + D_0 \theta_1) - 2D_0 D_2 \theta_0 - 2D_0 D_1 \theta_1 - D_1^2 \theta_0 - \theta_1 \cos \theta_0 + p \theta_1 \cos (\tilde{\omega} T_0) \cos \theta_0 \tag{9}\]

\[\epsilon^3 : D_0^2 \theta_3 = -\hat{\gamma} (D_2 \theta_0 + D_1 \theta_1 + D_0 \theta_2) - 2D_0 D_3 \theta_0 - 2D_1 D_2 \theta_0 - 2D_0 D_2 \theta_1 - 2D_0 D_1 \theta_2 - D_1^2 \theta_0 - \theta_2 \cos \theta_0 + p \theta_2 \cos (\tilde{\omega} T_0) \cos \theta_0 \tag{10}\]

\[\vdots \quad \vdots \quad \vdots \]

The solution of Equation (7) can be solved as

\[\theta_0 = \alpha T_0 + \beta \tag{11}\]

where \( \alpha \) and \( \beta \) are integration constants with respect to \( T_0 \), defined as \( \alpha = \alpha (T_1, T_2, T_3, \ldots) \)
\[ D_{\theta}^2 \theta_1 = -\alpha \ddot{\gamma} - 2D_1 \alpha - \sin (\alpha T_0 + \beta) + \frac{P}{2} \sin (\alpha T_0 - \omega T_0 + \beta) + \frac{P}{2} \sin (\alpha T_0 + \omega T_0 + \beta) \quad (12) \]

On account of the properties of period-one rotating orbits, let \( \alpha = \dot{\omega} \) so that \( \theta_0 \) provides the continuously increasing component in the solution. Thus, higher-order terms of the general solution are expected to be periodic oscillations, which is why the constant terms of high order are required to be zero. The solvable condition of Equation (12) can be derived as

\[ \frac{P}{2} \sin \beta = \dot{\omega} \dot{\gamma} \quad (13) \]

Note that \( \sin \beta \) is bounded in \([-1,1]\). This yields the following critical condition,

\[ p \geq 2\dot{\gamma} \ddot{\omega} \quad (14) \]

Substituting Equation (13) into (12) yields

\[ D_{\theta}^2 \theta_1 = -\sin (\dot{\omega} T_0 + \beta) + \frac{P}{2} \sin (2\dot{\omega} T_0 + \beta) \quad (15) \]

The particular solution of Equation (15) is

\[ \theta_1 = \frac{1}{\dot{\omega}^2} \sin (\dot{\omega} T_0 + \beta) + \frac{-p}{8\dot{\omega}^2} \sin (2\dot{\omega} T_0 + \beta) \quad (16) \]

Next substituting the solutions with respect to \( \theta_0 \) and \( \theta_1 \) into Equation (9) yields

\[
D_{\theta}^2 \theta_2 = -\ddot{\gamma} \left( \frac{1}{\dot{\omega}} \cos (\dot{\omega} T_0 + \beta) + \frac{-p}{4\dot{\omega}} \cos (2\dot{\omega} T_0 + \beta) \right) \\
- \left( \frac{1}{\dot{\omega}^2} \sin (\dot{\omega} T_0 + \beta) + \frac{-p}{8\dot{\omega}^2} \sin (2\dot{\omega} T_0 + \beta) \right) \cos (\dot{\omega} T_0 + \beta) \\
+ p \left( \frac{1}{\dot{\omega}^2} \sin (\dot{\omega} T_0 + \beta) + \frac{-p}{8\dot{\omega}^2} \sin (2\dot{\omega} T_0 + \beta) \right) \cos (\dot{\omega} T_0) \cos (\dot{\omega} T_0 + \beta) \quad (17)
\]
The particular solution of Equation (17) is solved as

\[ \theta_2 = -\frac{p}{16\omega^4} [\sin \omega \tau + 4 \sin (\omega \tau + 2\beta)] + \frac{\gamma}{\omega^3} \cos (\omega \tau + \beta) \]
\[ + \frac{-\gamma p}{16\omega^3} \cos (2\omega \tau + \beta) + \frac{p^2}{128\omega^4} \left[ \sin 2\omega \tau + \left(1 + \frac{16}{p^2}\right) \sin (2\omega \tau + 2\beta) \right] \]
\[ + \frac{-5p}{144\omega^4} \sin (3\omega \tau + 2\beta) + \frac{p^2}{512\omega^4} \sin (4\omega \tau + 2\beta) \]  

(18)

Then substituting \( \theta_0, \theta_1 \) and \( \theta_2 \) into Equation (10) can solve its particular solution as

\[ \theta_3 = \frac{8p \sin \beta - p^2 \gamma \omega - 32\gamma \omega \tau^2 - 19p\gamma}{128\omega^4} \sin (\omega \tau + \beta) - \frac{11p^2}{256\omega^6} \sin (\omega \tau - \beta) \]
\[ + \frac{1}{16} - \frac{\gamma^2 \omega^2 + \frac{1557}{20736} b^2}{\omega^6} \sin (\omega \tau + \beta) - \frac{p\gamma}{2\omega^5} \sin (\omega \tau + 2\beta) + \frac{p^2}{16\omega^6} \sin (\omega \tau + 3\beta) \]
\[ + \frac{p^2 \gamma}{128\omega^5} \cos (2\omega \tau) - \frac{p^3}{2048\omega^6} \sin (2\omega \tau - \beta) \]
\[ + \frac{20736p\gamma^2 \omega^2 - 13248p - 729p^3}{663552\omega^6} \sin (2\omega \tau + \beta) + \frac{(p^2 + 24) \gamma}{128\omega^5} \cos (2\omega \tau + 2\beta) \]
\[ - \frac{p^3 + 80p}{2048\omega^6} \cos (2\omega \tau + 3\beta) + \frac{269p^2}{82944\omega^6} \cos (3\omega \tau + \beta) - \frac{37p\gamma}{864\omega^5} \cos (3\omega \tau + 2\beta) \]
\[ + \frac{173p^2 + 144}{20736\omega^6} \sin (3\omega \tau + 3\beta) - \frac{5p^3}{32768\omega^6} \sin (4\omega \tau + \beta) + \frac{3p^2 \gamma}{2048\omega^5} \cos (4\omega \tau + 2\beta) \]
\[ - \frac{405p^3 + 8064p}{2654208\omega^6} \sin (4\omega \tau + 3\beta) + \frac{89p^2}{230400\omega^6} \sin (5\omega \tau + 3\beta) - \frac{p^3}{73728\omega^6} \sin (6\omega \tau + \beta) \]  

(19)

Combining the zero-order and the first-order approximations obtains the first-order solution for period-one rotation as

\[ \theta_{m_1} = \omega \tau + \beta + \frac{1}{\omega^2} \sin (\omega \tau + \beta) + \frac{-p}{8\omega^2} \sin (2\omega \tau + \beta) \]

(20)

where \( \theta_{m_1} \) denotes the first-order solution based on multi-scale technique, \( \beta = \pi - \arcsin(2\gamma \omega / p) \).

Next combining \( \theta_{m_1} \) and the second-order approximations yields the second-order solution as

\[ \theta_{m_2} = \theta_{m_1} + \frac{-p}{16\omega^4} [\sin \omega \tau + 4 \sin (\omega \tau + 2\beta)] + \frac{\gamma}{\omega^3} \cos (\omega \tau + \beta) \]
\[ + \frac{-\gamma p}{16\omega^3} \cos (2\omega \tau + \beta) + \frac{p^2}{128\omega^4} \left[ \sin 2\omega \tau + \left(1 + \frac{16}{p^2}\right) \sin (2\omega \tau + 2\beta) \right] \]
\[ + \frac{-5p}{144\omega^4} \sin (3\omega \tau + 2\beta) + \frac{p^2}{512\omega^4} \sin (4\omega \tau + 2\beta) \]  

(21)
where $\theta_{m2}$ denotes the second-order solution based on the multi-scale method.

Note that there is a non-periodic term of order $\tilde{\tau}^2$ in $\theta_3$, solutions with orders equal to three or higher are unstable.

### 4.2 Lenci’s Approximation

Based on the perturbation method, Lenci\cite{25} obtained an alternative approximation for period-one rotating orbits. In\cite{25}, the dynamical equation shown in Equation (2) was firstly reformulated as an integral equation, shown in the following.

\[
\dot{\theta}^2(\tau) - 2 \cos (\theta(\tau)) - \dot{\theta}_i^2 + 2 \cos \theta_i = -2\gamma \int_{\tau_i}^{\tau} \dot{\theta}(r) \frac{d\theta(r)}{dr} dr + 2p \int_{\tau_i}^{\tau} \cos (\omega r) \sin (\theta(r)) \frac{d\theta(r)}{dr} dr
\]  

where $\tau_i$ is the initial time, $\theta_i = \theta(\tau_i)$ and $\dot{\theta}_i = \dot{\theta}(\tau_i)$ are the angular displacement and velocity of the pendulum at the initial time.

Due to $\dot{\theta}(\tau) > 0, \forall \tau$ for period-one rotating orbit, there exists the inverse function $\tau = \tau(\theta)$ of $\theta(\tau)$. So the angular velocity $\dot{\theta}(\tau)$ can also be expressed as the function $\dot{\theta} = \dot{\theta}(\theta)$ (i.e. $\dot{\theta}_i = \dot{\theta}(\theta_i)$). Thus, $\tau$ can be expressed in the form

\[
\tau = \tau_i + \int_{\theta_i}^{\theta} \frac{ds}{\dot{\theta}(s)}
\]  

Based on the above considerations, Equation (22) can be rewritten as the following integral equation.

\[
\dot{\theta}^2(\theta) - 2 \cos (\theta) - \dot{\theta}_i^2 + 2 \cos \theta_i = -2\gamma \int_{\theta_i}^{\theta} \dot{\theta}(s) ds + 2p \int_{\theta_i}^{\theta} \cos \left(\omega \tau_i + \omega \int_{\theta_i}^{\theta} \frac{ds}{\dot{\theta}(s)}\right) \sin z dz
\]  

Because period-one rotating orbits is periodic in space and time, the following relationships can be obtained.

\[
\theta_T = \theta(\tau_i + T) = \theta_i + 2\pi
\]

\[
\dot{\theta}_T = \dot{\theta}(\tau_i + T) = \dot{\theta}_i
\]  

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where \( T = \frac{2\pi}{\omega} \) is the period of period-one rotation, which can also be solved from Equation (23). Combination of the two results can obtain the "synchronicity" equation as

\[
\int_{\theta_i}^{\theta_i+2\pi} \frac{d\theta}{\dot{\theta}(\theta)} = \frac{2\pi}{\omega}
\]  

(26)

Equation (24) can yield the "periodicity" condition as

\[
\gamma \int_{\theta_i}^{\theta_i+2\pi} \dot{\theta}(\theta)d\theta - p \int_{\theta_i}^{\theta_i+2\pi} \cos \left( \omega \tau_i + \omega \int_{\theta_i}^{\theta} \frac{ds}{\dot{\theta}(s)} \right) \sin \theta d\theta = 0
\]  

(27)

For the performance of perturbation analysis, it is assumed that the damping and the excitation are small, \( \gamma = \epsilon \gamma_1 \) and \( p = \epsilon p_1 \), where \( \epsilon \) is a small number. In addition, assume \( \theta_i = 0 \), the approximation for period-one rotation starting from \( \theta = 0 \) at \( \tau = \tau_i \) and ending at \( \theta = 2\pi \) at \( \tau = \tau_i + T \) is presented in the following. According to the previous considerations, Equations (24), (26) and (27) can be rewritten as, respectively,

\[
\dot{\theta}^2(\theta) - 2\cos(\theta) - \dot{\theta}_i^2 + 2 = -2\epsilon \gamma_1 \int_0^{\theta} \dot{\theta}(s)ds + 2\epsilon p_1 \int_0^{\theta} \cos \left( \psi_i + \omega \int_0^{s} \frac{ds}{\dot{\theta}(s)} \right) \sin z dz
\]  

(28)

\[
\int_0^{2\pi} \frac{d\theta}{\dot{\theta}(\theta)} = \frac{2\pi}{\omega}
\]  

(29)

\[
\gamma_1 \int_0^{2\pi} \dot{\theta}(\theta)d\theta - p_1 \int_0^{2\pi} \cos \left( \psi_i + \omega \int_0^{\theta} \frac{ds}{\dot{\theta}(s)} \right) \sin \theta d\theta = 0
\]  

(30)

where \( \psi_i = \omega \tau_i \).

Based on the perturbation method, the general solution of Equations (28), (29) and (30) is assumed to be

\[
\psi_i = \psi_{i0} + \epsilon \psi_{i1} + \ldots
\]

\[
\dot{\psi}_i = \dot{\psi}_{i0} + \epsilon \dot{\psi}_{i1} + \ldots
\]

\[
\dot{\theta}(\theta) = \dot{\theta}_0(\theta) + \epsilon \dot{\theta}_1(\theta) + \ldots
\]  

(31)

Substituting Equation (31) into Equations (28) and equating the coefficients with the same
powers of $\epsilon$ to zero can obtain

$$
e^0 : \dot{\theta}_0(\theta) = \sqrt{2 \cos \theta + \dot{\theta}_0^2 - 2} \tag{32}$$

$$
e^1 : \dot{\theta}_0(\theta)\dot{\theta}_1(\theta) = \dot{\theta}_{i0} \dot{\theta}_{i1} - \gamma_1 \int_0^\theta \dot{\theta}_0(s)ds + p_1 \int_0^\theta \cos \left( \psi_{i0} + \omega \int_0^z \frac{ds}{\dot{\theta}_0(s)} \right) \sin zdz \tag{33}$$

$$
e^2 : \dot{\theta}_0(\theta)\dot{\theta}_2(\theta) = \dot{\theta}_{i0} \dot{\theta}_{i2} \frac{\dot{\theta}_1^2(\theta)}{2} - \frac{\dot{\theta}_1^2(\theta)}{2} - \gamma_1 \int_0^\theta \dot{\theta}_1(s)ds + p_1 \int_0^\theta \left( \psi_{i1} - \omega \int_0^z \frac{\dot{\theta}_1(s)}{\dot{\theta}_0^2(s)} ds \right) \sin \left( \psi_{i0} + \omega \int_0^z \frac{ds}{\dot{\theta}_0(s)} \right) \sin zdz \tag{34}$$

\[\vdots\]

where $\dot{\theta}_{i0}, \dot{\theta}_{i1}, \ldots$, and $\psi_{i0}, \psi_{i1}, \ldots$, can be determined by the synchronicity

$$
e^0 : \int_0^{2\pi} \frac{d\theta}{\dot{\theta}_0(\theta)} = \frac{2\pi}{\omega} \tag{35}$$

$$
e^1 : \int_0^{2\pi} \frac{\dot{\theta}_1(\theta)}{\dot{\theta}_0^2(\theta)} d\theta = 0 \tag{36}$$

$$
e^2 : \int_0^{2\pi} \frac{\dot{\theta}_2(\theta)}{\dot{\theta}_0^2(\theta)} d\theta = \int_0^{2\pi} \frac{\dot{\theta}_1^2(\theta)}{\dot{\theta}_0^2(\theta)} d\theta \tag{37}$$

\[\vdots\]

and periodicity conditions

$$
e^1 : \gamma_1 \int_0^{2\pi} \dot{\theta}_0(\theta)d\theta - p_1 \int_0^{2\pi} \cos \left( \psi_{i0} + \omega \int_0^\theta \frac{ds}{\dot{\theta}_0(s)} \right) \sin \theta d\theta = 0 \tag{38}$$

$$
e^2 : \gamma_1 \int_0^{2\pi} \dot{\theta}_1(\theta)d\theta - p_1 \int_0^{2\pi} \left( \psi_{i1} - \omega \int_0^\theta \frac{\dot{\theta}_1(s)}{\dot{\theta}_0^2(s)} ds \right) \sin \left( \psi_{i0} + \omega \int_0^\theta \frac{ds}{\dot{\theta}_0(s)} \right) \sin \theta d\theta = 0 \tag{39}$$

\[\vdots\]

From Equations \((32)\) and \((35)\), the first-order solution can be solved as

$$
\dot{\theta}_{L1}(\theta) = \sqrt{2 \cos \theta + \dot{\theta}_0^2 - 2} \tag{40}
$$
where $\theta \in [0, 2\pi]$, $\dot{\theta}_{L1}$ denotes the first-order solution, and $\dot{\theta}_0$ is a constant determined by

$$
\int_0^{2\pi} \frac{d\theta}{\sqrt{2 \cos \theta + \dot{\theta}_0^2 - 2}} = \frac{2\pi}{\omega} \tag{41}
$$

Next by the first periodicity condition (38), the following form is obtained

$$
\sin \psi_i = \frac{\gamma_1}{p_1} \frac{\int_0^\pi \dot{\theta}_0(\theta) d\theta}{\int_0^\pi \sin \left( \int_0^\theta \frac{ds}{\dot{\theta}_0(s)} \right) \sin \theta d\theta} \approx -\frac{\gamma}{p} \frac{2(1 + \omega^2 + \omega^4)}{(\omega + \omega^3)} \tag{42}
$$

Since $\sin \psi_i$ is bounded in $[-1,1]$, Equation (42) yields the necessary condition for the existence of the solution as

$$
p > 2\gamma \frac{(1 + \omega^2 + \omega^4)}{(\omega + \omega^3)} \tag{43}
$$

From Equation (33), it can solve

$$
\dot{\theta}_1(\theta) = \frac{\dot{\theta}_0 \dot{\theta}_1 - \gamma_1 \int_0^\theta \dot{\theta}_{L1}(s) ds}{\dot{\theta}_{L1}(\theta)} + \frac{p_1 \int_0^\theta \cos \left( \psi_i + \omega \int_0^z \frac{ds}{\dot{\theta}_{L1}(s)} \right) \sin(z) dz}{\dot{\theta}_{L1}(\theta)} \tag{44}
$$

where $\dot{\theta}_1$ is a constant, solved by Equation (36) as

$$
\dot{\theta}_1 = \gamma_1 \frac{\int_0^{2\pi} 1/\dot{\theta}_{L1}^2(\theta) \left[ \int_0^\theta \dot{\theta}_{L1}(s) ds \right] d\theta}{\dot{\theta}_{L1} \int_0^{2\pi} 1/\dot{\theta}_{L1}^2(\theta) \left[ \int_0^\theta \dot{\theta}_{L1}(s) ds \right] d\theta} - p_1 \frac{\int_0^{2\pi} 1/\dot{\theta}_{L1}^2(\theta) \left[ \int_0^\theta \cos \left( \psi_i + \omega \int_0^z \frac{ds}{\dot{\theta}_{L1}(s)} \right) \sin(z) dz \right] d\theta}{\dot{\theta}_{L1} \int_0^{2\pi} 1/\dot{\theta}_{L1}^2(\theta) \left[ \int_0^\theta \dot{\theta}_{L1}(s) ds \right] d\theta} \tag{45}
$$

Based on the previous results, the second-order solution can be obtained as

$$
\dot{\theta}_{L2}(\theta) = \dot{\theta}_{L1}(\theta) + \frac{\dot{\theta}_0 \left( e \hat{\theta}_1 \right) - \gamma \int_0^\theta \dot{\theta}_{L1}(s) ds}{\dot{\theta}_{L1}(\theta)} + \frac{p \int_0^\theta \cos \left( \psi_i + \omega \int_0^z \frac{ds}{\dot{\theta}_{L1}(s)} \right) \sin(z) dz}{\dot{\theta}_{L1}(\theta)} \tag{46}
$$

where $\dot{\theta}_{L2}$ denotes the second-order solution. Following the same procedure, higher-order solutions can also be obtained.
5 Iterative harmonic balance

Period-one rotation is a type of continuous, rotatory response of a parametrically excited
pendulum (Eq. (2)), the angular velocity of which is equal to the frequency of the excitation
with a small periodic perturbation with zero mean over one period \[\text{[8, 24]}\]. Using the
non-dimensional time variable \(\tau\) defined previously, the angular velocity can be written as

\[
\dot{\theta}(\tau) = \omega + \epsilon(\tau)
\]

(47)

where \(\epsilon(\tau)\) is a periodic function with small amplitude, which satisfies

\[
|\epsilon(\tau)| << \omega
\]

(48)

and

\[
\int_{\tau_0}^{\tau_0 + 2\pi/\omega} \epsilon \, d\tau = 0
\]

(49)

Due to the periodicity of the perturbation, using the Fourier series \(\epsilon(\tau)\) can be represented
as

\[
\epsilon(\tau) = \sum_{k=1}^{\infty} \epsilon_k \cos (k\omega \tau + \varphi_k)
\]

(50)

where coefficients \(\epsilon_k\) and \(\varphi_k\) are the amplitude and phase of the \(k\)th harmonics of frequency
\(k\omega\), respectively.

Using Equations (47) and (50), the angular displacement can be readily obtained as

\[
\theta(\tau) = \omega \tau + \varphi_0 + \epsilon_\theta
\]

(51)

where \(\varphi_0\) is the initial phase of \(\theta\), and

\[
\epsilon_\theta = \sum_{k=1}^{\infty} C_k \sin (k\omega \tau + \varphi_k)
\]

(52)
in which \( C_k = \epsilon_k / (k\omega) \), thus

\[
|\epsilon_\theta| < \frac{|\epsilon|}{\omega} \ll 1 \quad (53)
\]

Note that based on the definition of the period-one rotation, Equation (51) represents the exact form of the solution. However, due to the infinite number of coefficients to be determined, i.e. \( \epsilon_k, \varphi_k, \varphi_0 \), and the parametrical term in the governing equation that generates beat frequencies, it is impossible to determine exactly the parameters by direct substitution and harmonic balance. In this study, an iterative method is proposed to approximately determine these parameters.

## 5.1 Iterative Method

Substituting Equation (51) into the right-hand side of Equation (2) and expanding \( \sin \theta \) at \( \epsilon_\theta = 0 \) yield

\[
\ddot{\theta} + \gamma \dot{\theta} = (p \cos \omega \tau - 1) \left[ \sin (\omega \tau + \varphi_0) + \cos (\omega \tau + \varphi_0) \cdot \epsilon_\theta + \cdots \right] \quad (54)
\]

In this study, the following iterative process is proposed for the estimation of the parameters in Equation (51):

**Step 1:** The iteration starts with the zero-order approximation of \( \sin \theta \approx \sin(\omega \tau + \varphi_0) \).

Equation (54) can be rewritten as

\[
\ddot{\theta} + \gamma \dot{\theta} = (p \cos \omega \tau - 1) \sin (\omega \tau + \varphi_0) \quad (55)
\]

An approximate solution \( \theta_1 \) can be obtained and using Equations (51) and (52).

**Step 2:** Using the approximated parameters in Equation (54) and truncating the series at desired order, a modified equation of motion can be obtained and solved in combination of Equations (51) and (52).

\[
\ddot{\theta} + \gamma \dot{\theta} = \cdots \cdots
\]
Step \( n \): Using the solution from the \((n-1)\)th step and follow the same process in Step 2, the solution for the \( n \)th iteration can be obtained.

5.1.1 First Iteration

The solution of Equation (55) is

\[
\theta_1 = \frac{p\tau}{2\gamma} \sin \varphi_{01} + \varphi_{01} + \sum_{k=1}^{2} C_{1k} \sin (k\omega \tau + \varphi_{1k}) \tag{56}
\]

where

\[
\varphi_{01} = \pi - \arcsin \left( \frac{2\gamma\omega}{p} \right) \tag{57}
\]

and

\[
C_{1k} = \sqrt{A_{1k}^2 + B_{1k}^2}, \quad \varphi_{1k} = \arctan \frac{B_{1k}}{A_{1k}} \tag{58}
\]

in which

\[
A_{1k} = \frac{R_{k1} + \frac{\gamma}{k\omega} R_{k2}}{(k\omega)^2 + \gamma^2}, \quad B_{1k} = \frac{\frac{\gamma}{k\omega} R_{k1} - R_{k2}}{(k\omega)^2 + \gamma^2}
\]

\[
R_{11} = \cos \varphi_{01}, \quad R_{12} = -\sin \varphi_{01}
\]

\[
R_{21} = -\frac{p}{2} \cos \varphi_{01}, \quad R_{22} = \frac{p}{2} \sin \varphi_{01}
\]

Note that \( \sin \varphi_{01} \) is bounded in \([-1,1]\). Consequently, the necessary condition for the solution (56) to exist is

\[
p \geq 2\gamma\omega \tag{59}
\]

which gives a lower bound on the normalized excitation amplitude, \( p \), the same as that obtained in [24] and [25].

As in the period-one rotating orbits of a parametrically excited pendulum, the angular velocity must never change its sign, i.e. the orbit must be a pure, continuous rotating. Thus, for \( \theta_1 \) to be a valid approximation, the following necessary condition must be satisfied.

\[
|\theta_1| \geq \omega - \sum_{k=1}^{2} k\omega C_{1k} > 0 \tag{60}
\]
which gives an upper bound on the normalized excitation amplitude, \( p \), as

\[
p < 4\omega^2 \sqrt{1 + \left( \frac{\gamma}{2\omega} \right)^2} \left( 1 - \frac{1}{\omega^2 \sqrt{1 + (\gamma/\omega)^2}} \right)
\]  

(61)

5.1.2 Second Iteration

In the second iteration, using the result of the first iteration and considering the first-order approximation, the following modified governing equation is obtained.

\[
\ddot{\theta} + \gamma \dot{\theta} = (p \cos \omega \tau - 1) \left[ \sin (\omega \tau + \varphi_{02}) + \cos (\omega \tau + \varphi_{02}) \sum_{k=1}^{2} C_{1k} \sin (k \omega \tau + \varphi_{1k}) \right]
\]

(62)

which is solved as

\[
\theta_2 = \frac{\sqrt{K_1^2 + K_2^2}}{2\gamma} \tau \sin (\varphi_{02} + \psi) + \varphi_{02} + \sum_{k=1}^{4} C_{2k} \sin (i \omega \tau + \varphi_{2k})
\]

(63)

where

\[
\varphi_{02} = \pi - \arcsin \left( \frac{2\gamma \omega}{\sqrt{K_1^2 + K_2^2}} \right) - \psi
\]

(64)

and

\[
K_1 = p + A_{11} - \frac{p A_{12}}{2}, \quad K_2 = -B_{11} + \frac{p B_{12}}{2}
\]

\[
\psi = \arctan \frac{K_2}{K_1}
\]

\[
C_{2k} = \sqrt{A_{2k}^2 + B_{2k}^2}, \quad \varphi_{2k} = \arctan \frac{B_{2k}}{A_{2k}}
\]

(65)

\[
k = 1, 2, 3, 4
\]

in which

\[
A_{2k} = \frac{R_{k1} + \frac{\gamma}{k \omega} R_{k2}}{(k \omega)^2 + \gamma^2}, \quad B_{2k} = \frac{\frac{\gamma}{k \omega} R_{k1} - R_{k2}}{(k \omega)^2 + \gamma^2}
\]

and

\[
R_{11} = \cos \varphi_{02} - \frac{p C_{11}}{4} \cos(\varphi_{11} + \varphi_{02}) + \frac{C_{12}}{2} \cos(\varphi_{12} - \varphi_{02})
\]

\[
R_{12} = -\sin \varphi_{02} + \frac{p C_{11}}{4} \sin(\varphi_{11} + \varphi_{02}) - \frac{C_{12}}{2} \sin(\varphi_{12} - \varphi_{02}) + \frac{p C_{11}}{2} \sin(\varphi_{11} - \varphi_{02})
\]
\[ R_{21} = -\frac{p}{2} \cos \varphi_{02} + \frac{C_{11}}{2} \cos(\varphi_{11} + \varphi_{02}) - \frac{pC_{12}}{2} \cos \varphi_{12} \cos \varphi_{02} \]

\[ R_{22} = \frac{p}{2} \sin \varphi_{02} - \frac{C_{11}}{2} \sin(\varphi_{11} + \varphi_{02}) + \frac{pC_{12}}{2} \sin \varphi_{12} \cos \varphi_{02} \]

\[ R_{31} = -\frac{pC_{11}}{4} \cos(\varphi_{11} + \varphi_{02}) + \frac{C_{12}}{2} \cos(\varphi_{12} + \varphi_{02}) \]

\[ R_{32} = \frac{pC_{11}}{4} \sin(\varphi_{11} + \varphi_{02}) - \frac{C_{12}}{2} \sin(\varphi_{12} + \varphi_{02}) \]

\[ R_{41} = -\frac{pC_{12}}{4} \cos(\varphi_{12} + \varphi_{02}) \]

\[ R_{42} = \frac{pC_{12}}{4} \sin(\varphi_{12} + \varphi_{02}) \]

As \( \sin(\varphi_{02} + \psi) \) is bounded in \([-1, 1]\), the necessary condition for the solution to exist is

\[ \frac{2\gamma \omega}{\sqrt{K_1^2 + K_2^2}} \leq 1 \quad (66) \]

Also, as in the first iteration, the condition \(|\dot{\theta}_2| \geq \omega - \sum_{k=1}^{4} k\omega C_{2k} > 0\) must be satisfied to guarantee a continuous rotating orbit, which can be simplified as

\[ 1 > \sum_{k=1}^{4} kC_{2k} \quad (67) \]

5.1.3 \( n \)th Iteration (\( n > 3 \))

In the \( n \)th iteration, the modified governing equation is

\[ \ddot{\theta} + \gamma \dot{\theta} = (p \cos \omega \tau - 1) \left[ \sin(\omega \tau + \varphi_{0n}) + \cos(\omega \tau + \varphi_{0n}) \cdot \epsilon_{\theta_{n-1}} \right] \quad (68) \]

where \( \epsilon_{\theta_{n-1}} = \sum_{k=1}^{2(n-1)} C_{(n-1)k} \sin(k\omega \tau + \varphi_{(n-1)k}) \) is obtained from the \((n-1)\)th iteration.

The solution of Equation (68) can be obtained as

\[ \theta_n = \frac{\sqrt{K_1^2 + K_2^2}}{2\gamma} \tau \sin(\varphi_{0n} + \psi) + \varphi_{0n} + \sum_{k=1}^{2n} C_{nk} \sin(i\omega \tau + \varphi_{nk}) \quad (69) \]

where

\[ \varphi_{0n} = \pi - \arcsin \left( \frac{2\gamma \omega}{\sqrt{K_1^2 + K_2^2}} \right) - \psi \quad (70) \]
and

\[ K_1 = p + A_{n-1} - \frac{pA_{n-1}^2}{2}, \quad K_2 = -B_{n-1} + \frac{pB_{n-1}^2}{2} \]

\[ \psi = \arctan \frac{K_2}{K_1} \]

\[ C_{nk} = \sqrt{A_{nk}^2 + B_{nk}^2}, \quad \varphi_{nk} = \arctan \frac{B_{nk}}{A_{nk}} \]

\[ k = 1, 2, 3, 4, ..., 2n \]

in which

\[ A_{nk} = \frac{R_{k1} + \frac{\gamma}{k\omega} R_{k2}}{(k\omega)^2 + \gamma^2}, \quad B_{nk} = \frac{\gamma}{k\omega} R_{k1} - R_{k2} \]

and

\[ R_{11} = \cos \varphi_{0n} - \frac{pC_{n-1}1}{4} \cos(\varphi_{(n-1)1} + \varphi_{0n}) + \frac{C_{(n-1)2}}{2} \cos(\varphi_{(n-1)2} - \varphi_{0n}) \]

\[ - \frac{pC_{(n-1)3}}{4} \cos(\varphi_{(n-1)3} - \varphi_{0n}) \]

\[ R_{12} = -\sin \varphi_{0n} + \frac{pC_{n-1}1}{4} \sin(\varphi_{(n-1)1} + \varphi_{0n}) - \frac{C_{(n-1)2}}{2} \sin(\varphi_{(n-1)2} - \varphi_{0n}) \]

\[ + \frac{pC_{(n-1)11}}{2} \sin(\varphi_{(n-1)1} - \varphi_{0n}) + \frac{pC_{(n-1)3}}{4} \sin(\varphi_{(n-1)3} - \varphi_{0n}) \]

\[ R_{21} = -\frac{p}{2} \sin \varphi_{0n} + \frac{C_{n-1}1}{2} \cos(\varphi_{(n-1)1} + \varphi_{0n}) - \frac{C_{(n-1)2}}{2} \cos(\varphi_{(n-1)2} \cos \varphi_{0n}) \]

\[ + \frac{C_{(n-1)3}}{2} \cos(\varphi_{(n-1)3} - \varphi_{0n}) - \frac{pC_{(n-1)4}}{4} \cos(\varphi_{(n-1)4} - \varphi_{0n}) \]

\[ R_{22} = \frac{p}{2} \sin \varphi_{0n} - \frac{C_{n-1}1}{2} \sin(\varphi_{(n-1)1} + \varphi_{0n}) + \frac{pC_{(n-1)2}}{2} \sin(\varphi_{(n-1)2} \cos \varphi_{0n}) \]

\[ - \frac{C_{(n-1)3}}{2} \sin(\varphi_{(n-1)3} - \varphi_{0n}) + \frac{pC_{(n-1)4}}{4} \sin(\varphi_{(n-1)4} - \varphi_{0n}) \]

\[ R_{m1} = -\frac{pC_{(n-1)(m-2)}}{4} \cos(\varphi_{(n-1)(m-2)} + \varphi_{0n}) + \frac{C_{(n-1)(m-1)}}{2} \cos(\varphi_{(n-1)(m-1)} + \varphi_{0n}) \]

\[ - \frac{pC_{(n-1)m}}{2} \cos(\varphi_{(n-1)m} \cos \varphi_{0n}) + \frac{C_{(n-1)(m+1)}}{2} \cos(\varphi_{(n-1)(m+1)} - \varphi_{0n}) \]

\[ - \frac{pC_{(n-1)(m+2)}}{4} \cos(\varphi_{(n-1)(m+2)} - \varphi_{0n}) \]

\[ R_{m2} = \frac{pC_{(n-1)(m-2)}}{4} \sin(\varphi_{(n-1)(m-2)} + \varphi_{0n}) - \frac{C_{(n-1)(m-1)}}{2} \sin(\varphi_{(n-1)(m-1)} + \varphi_{0n}) \]

\[ + \frac{pC_{(n-1)m}}{2} \sin(\varphi_{(n-1)m} \cos \varphi_{0n}) - \frac{C_{(n-1)(m+1)}}{2} \sin(\varphi_{(n-1)(m+1)} - \varphi_{0n}) \]

\[ + \frac{pC_{(n-1)(m+2)}}{4} \sin(\varphi_{(n-1)(m+2)} - \varphi_{0n}) \]

\[ m = 3, 4, ..., 2n \]

and \( C_{(n-1)k} = 0 \) where \( k > 2(n-1) \)
As \( \sin (\varphi_0 + \psi) \) is bounded in \([-1,1]\), the necessary condition for the solution to exist is

\[
\frac{2\gamma \omega}{\sqrt{K^2_1 + K^2_2}} \leq 1 \quad (72)
\]

Similarly, the necessary condition for the solution to be continuous, rotatory is \(|\dot{\theta}_n| \geq \omega - \sum_{k=1}^{2n} k\omega C_{nk} > 0\), which can be simplified as

\[
1 > \sum_{k=1}^{2n} kC_{nk} \quad (73)
\]

5.2 Error Analysis

In this study, the performance of the proposed method is evaluated using two different indexes, which are: (1) from a physical perspective, system energy, and (2) from a mathematical perspective, global residual of the governing equation.

The instantaneous energy of the pendulum, \( E \) can be written as

\[
E = 1 - \cos \theta + \frac{\dot{\theta}^2}{2} \quad (74)
\]

The energy given by the approximate solution \((\theta_a, \dot{\theta}_a)\) is

\[
E_a = 1 - \cos \theta_a + \frac{\dot{\theta}_a^2}{2} \quad (75)
\]

The relative error, denoted by \( \Delta E \), between \( E_a \) and \( E \) is defined as

\[
\Delta E = \frac{E_a - E}{E} \times 100\% \quad (76)
\]

Since the true analytical solution of the system equation cannot be obtained, the numerical solution, \((\theta_{\text{num}}, \dot{\theta}_{\text{num}})\) obtained using the ODE45 solver of MATLAB\(^\circledast\) is used to calculate
an estimate of the system energy $E_{num}$, which gives the error definition of

$$
\Delta E = \frac{E_a - E_{num}}{E_{num}} \times 100\%
$$

(77)

In this study, the root-mean-square value of the relative error in energy in one period is used as the performance index $\eta$, which can be written as

$$
\eta = \sqrt{\frac{1}{2\pi} \int_{\theta_0}^{\theta_0 + 2\pi} \Delta E^2 d\theta}
$$

(78)

The other criterion used in this study is the root-mean-square value of the global residual error in one period $R_e$, which is defined as

$$
R_e = \sqrt{\frac{1}{T} \int_{\tau_0}^{\tau_0 + T} \left[ \dot{\theta}_a + \gamma \dot{\theta}_a - (p \cos \omega \tau - 1) \sin \theta_a \right]^2 d\tau}
$$

(79)
6 Performance Evaluation

In order to evaluate the performance of the proposed method, this study examined bounds of excitation amplitude, convergence of coefficients, and the error of approximations in this chapter. The results from both theoretical analysis and numerical simulations are demonstrated for comparison. The numerical solution was obtained by numerically solving Equation (2) with the ODE45 solver of MATLAB®.

6.1 Bounds of Excitation Amplitude

For a given excitation frequency, the period-one rotating orbit only exists in a stable zone in the parameter space, with a related energy level. As can be seen from Equation (74), the energy level of the orbit can be characterized by the square of the rotating speed \( \omega \). In order to maintain a stable orbit, it is necessary to maintain a balance and be matchable at rate between the energy provided by the excitation and the energy dissipated by the damping. In other words, the energy of the excitation is bounded in a specific range \([9, 10, 25]\). The lower bound prescribes the minimal energy level required to stabilize the orbit. Beyond the upper bound, the dynamics will leave the zone of period-one rotation.

Figure 1 shows the relationship between the bounds and the excitation frequency. Systems with different damping levels are considered, such as \( \gamma = 0.005, 0.01, 0.05, \) and 5. The results from both theoretical analysis (shown in dotted and dashed lines) and extensive numerical simulations (shown in solid lines with markers) are presented for comparison. The approximate bounds can be obtained from each iteration of the proposed method. Here, only results obtained from the first and second iterations are summarized, shown in Figure 1 (a) and (b), respectively. It is seen that the lower bound can be predicted by Equations (59) and (66) with satisfactory accuracy. At lower frequencies, using only one iteration underestimates the lower bound, while using two iterations slightly overestimates. As the frequency increases, the discrepancy diminishes. The upper bound is quadratically proportional to the non-dimensional frequency, \( \omega \). For systems with low levels of damping, i.e.
\[ \gamma \ll 1, \] the upper bound is almost kept in the same value, which is orders of magnitude higher than the lower bound, indicating a relatively large stable zone of period-one rotating orbit. Comparing numerical and analytical results shows that the prediction of first iteration, Equation (61) overestimates the upper bound in all cases considered, summarized in Figure 1 (a), while the prediction of second iteration, Equation (67) is closer to the upper bound, shown in Figure 1 (b). Combining the lower with the upper bound indicates that

![Figure 1: Bounds of excitation amplitude, \( p \), (a) First iteration and (b) Second iteration. Dotted: Theoretical lower bound; Dashed: Theoretical upper bound; Solid line with markers: Numerically calculated (\( \triangle \):lower bound, \( \triangledown \): upper bound)](image-url)
the predicted zone from first iteration is larger than the actual stable zone, while the zone predicted by the second iteration is smaller.

Figure 2: Bounds of excitation amplitude, \( p \). Dotted: Theoretical lower bound; Dashed: Theoretical upper bound; Solid line with markers: Numerically calculated (\( \triangle \):lower bound, \( \triangledown \): upper bound)

The relationship between the predictions of bounds and the iterative number are presented in Figure 2. The damping level of the systems was fixed at \( \gamma = 0.05 \). The iterative step were considered up to 5. The numerical results are also shown for comparison. As can be seen from Figure 2, the prediction of lower bound becomes larger than the actual with the increase of iterative step at the lower frequency (i.e. \( \omega < 5 \)), while the predictions of lower bound obtained from all iterations converge to the actual bound when the frequency increases. The upper bounds obtained from all iterations overestimate the bound. However, the predicted upper bound is closer to the numerical calculation and approaches a limit when the iterative step increases.

6.2 Convergence of coefficients

The effect of number of iterations on the estimation of the coefficients was also evaluated by exciting the system with different excitations. The number of iterations considered was
from 1 to 16. The damping level of the system, $\gamma$ was fixed at 0.01. The non-dimensional frequency of the excitation was fixed at $\omega = 3$ for four different cases: $p = 0.2$, $p = 0.5$, $p = 1$, or $p = 10$, where $p = 0.2$ is close to the lower limit, $p = 10$ is close to the upper limit. The coefficients evaluated include $C_k$, $\varphi_0$, and $\varphi_k$. The results are summarized in Figures 3, 4 and 5 respectively.

Figure 3: Convergence of coefficients $C_k$, $k = 1, 2, \cdots, 8$ for $\gamma = 0.01$ and $\omega = 3$ and (a) $p = 0.2$, (b) $p = 0.5$, (c) $p = 1$ and (d) $p = 10$.

Figure 3 shows the relationship between the steps of iteration and the convergence of coefficients, $C_k$. It was found that the coefficients, $C_k$, for higher order harmonics are orders of magnitude small than those for the lower order ones, thus only the results for $C_1$ to $C_{16}$ are shown. It is seen that all the coefficients converge quickly – within two or three iterations after they are introduced when more iterations are performed. Also, for all the cases considered, only one or two iterations are needed as the coefficients generated from three or more iterations, i.e. $C_5$ and higher, are insignificant.

Figures 4 and 5 respectively illustrate the effect of iterative number on the convergence...
Figure 4: Convergence of coefficient $\varphi_0$ for the system with damping level of $\gamma = 0.01$ at $\omega = 3$ with the excitations of $p = 0.2, 0.5, 1, \text{ or } 10$ of the phases, $\varphi_0$ and $\varphi_k$ under different excitation levels. Due to the insignificance of the amplitudes of higher order harmonics, e.g. $C_5$ and higher, only the results for $\varphi_1$ to $\varphi_4$ are presented, as shown in Figure 5. For the excitations located away from the lower bound, e.g. $p = 0.5, 1, \text{ and } 10$, it is evident that all the phases can reach convergence with one or two iterations after they are introduced when more iterations are carried out. When the excitation is located nearby the lower limit, however, the phases have tendencies towards the convergence with one or two iterations after they are generated, but the tendency of their convergence becomes divergence with the increase of the iterations, i.e. after the sixth iteration. The results seem to indicate that the increase of iterations, after two iterations, is not beneficial to improve the evolution of the approximate solution, especially in the case of that the excitation amplitude is located nearby the lower bound.

### 6.3 Error Evaluation

The error of the proposed method was analyzed under different levels of excitation. The non-dimensional excitation frequency was fixed at $\omega = 3$, and four levels of excitation, i.e. $p = 0.1$, $0.5$, $1$, or $10$. The error evaluation was performed to assess the accuracy and convergence of the solution obtained through the proposed method.
Figure 5: Convergence of coefficient $\phi_k, k = 1, 2, 3, 4$ for the system with damping level of $\gamma = 0.01$ at $\omega = 3$ with the excitations of $p = 0.2, 0.5, 1, \text{ or } 10$: (a) $\phi_1$, (b) $\phi_2$, (c) $\phi_3$, (d) $\phi_4$.

$p = 0.5, p = 1, \text{ and } p = 10,$ were respectively considered. For each case, the damping level, $\gamma$ was swept from nearly zero to near the critic value, as show in Equation (59). On account of the effect of iterative number on the coefficients, only two iterations are considered in this section. For comparison, the results of the approximations obtained from the perturbation methods are also presented.

The root-mean-square values of the relative error in system energy defined in (78), $\eta$, of the cases considered are shown in Figure 6. Under low-level excitations, e.g. $p = 0.1$ (Fig. 6a), the errors of the first-order approximations obtained from the perturbation methods, i.e. Equations (20) and (40), are very similar to that from the first iteration in the proposed method and they are almost independent on the level of damping. The errors of the second-order approximations from the perturbation methods vary significantly with the damping level, and yet are noticeably lower than those from the first-order approximations as well as those from the first and second iteration in the proposed method. It is noted that in the
proposed method, the error obtained from the second iteration is noticeably higher than that from the first iteration. For higher excitation levels, i.e. $p = 0.5, 1$ (Fig. 6), the allowed damping level increases. The errors obtained with the first-order approximation in [25] increase noticeably while the errors of the multi-scale method and those of the second-order approximations in [25] remain at an almost similar level. The error from the first iteration in the proposed method is similar to that from the first-order multi-scale method, however, compared to the first-order multi-scale method, the proposed method performs slightly better at higher damping levels. The second-order approximations from the perturbation method as well as that from the second iteration in the proposed method perform significantly better than their first-order counterparts, while the error from the proposed method is the lowest. For the excitation with $p = 10$, the proposed method generally performs better than the perturbation methods, especially at higher damping levels, Fig. 6d.

The performance of the proposed method evaluated using the global residual error is summarized in Fig. 7. The method proposed in [25] generates the largest error, while the...
second-order multi-scale method surpasses the others at low-level excitations, e.g. \( p = 0.1 \) (Fig. 7a) and as the level of excitation increases, the performance of the proposed method becomes better as compared to the others (Fig. 7b,c,d).

Figures 7 demonstrate the performance of the methods considered using the phase portrait over one period. The results confirm the observations from the two evaluation criteria used in this study.
Figure 8: Phase portrait over one period (a) $\gamma = 0.01$ and (b) $\gamma = 0.016$ at $p = 0.1$ and $\omega = 3$
Figure 9: Phase portrait over one period (a) $\gamma = 0.01$ and (b) $\gamma = 0.06$ at $p = 0.5$ and $\omega = 3$
Figure 10: Phase portrait over one period (a) $\gamma = 0.1$ and (b) $\gamma = 0.16$ at $p = 1$ and $\omega = 3$
Figure 11: Phase portrait over one period (a) $\gamma = 0.1$ and (b) $\gamma = 1.6$ at $p = 10$ and $\omega = 3$
7 Conclusions

In this study, an approximate analytical solution based on iterative harmonic balance for period-one rotation of parametric pendulums has been proposed. The general formula for each iteration step has been presented. The numerical solutions obtained from MATLAB have been used as the baseline for performance evaluation. The proposed method has been validated in comparison with the perturbation methods using two performance indexes, i.e. error in system energy and the global residual error. The second-order multi-scale method has also been developed in this study for completeness. It has been found that two iterations are sufficient in proposed method. For the cases of higher excitation and damping levels, the proposed method has been found to perform the best as compared to the approximations obtained from the perturbation method. Due to its simplicity in development, the proposed method may be modified to develop solutions for other rotating orbits of parametrically excited pendulums.
References


