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Logic and aesthetics in epistemology

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University of Hawaii, 1990

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LOGIC AND AESTHETICS
IN EPISTEMOLOGY

A DISSERTATION SUBMITTED TO THE GRADUATE DIVISION OF THE UNIVERSITY OF HAWAII IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY
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ABSTRACT

The purpose of this dissertation is to present historical evidence in favor of the thesis that many forms of dichotomy appearing in the history of epistemology are related to the duality represented by the mathematical concepts of continuity and discreteness. Parts 1 and 2 give a descriptive and historical account of epistemological dichotomies appearing during the development of mathematics and logic. In part 3, the implications of these dichotomies for general philosophy are explored by means of a collage of analytic, historical, and factual materials, accompanied by brief commentaries, and aimed at establishing the following points:

1. Acceptance of the primordial intuition of continuity distinguishes the constructive model from the structural (digital) model in epistemology.

2. The constructive model emphasizes meaningful thought processes of the reflecting and acting subject, while the structural model emphasizes publicly recognizable and communicable thought processes based upon logic, linguistics, or natural law.

3. An activity which translates between the constructive model and the structural model by means of symbols which include iconic components is to be called "aesthetic".
4. The function of aesthetics is to bring about harmonious and productive relations between individuals and the larger wholes (both natural and social) to which they belong.

The conclusion is reached that, from the point of view of the individual, aesthetics is important for motivation and for the development of cognitive powers and moral sensibilities; from the point of view of the species, aesthetics is important for long-term survival. Therefore, logic and aesthetics are of equal and complementary importance and deserve equal consideration in the study of epistemology.
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PREFACE

In this dissertation, my main philosophical thesis is that the epistemological question stands in need of reformulation on the basis of a complex hypothesis: (1) That there is a connection between kinds of knowledge and modes of acquisition, (2) that there is an ineradicable duality in human thinking which corresponds to a duality in kinds of knowledge, and (3) that this duality is reflected in and intimately connected with the mathematical concepts of continuity and discreteness, which is why the relationship between these concepts is recognized as one of the oldest, if not the oldest, of philosophical problems. It is part of my thesis that the study of human ways of thinking ought not to be regarded as outside the purview of philosophical subject matter; rather, it must fall to philosophers to integrate the legitimate claims of all kinds of knowledge and perforce of all ways of thinking.

My original contribution lies mainly in the suggestion that the many forms of epistemological dichotomy which have appeared in the history of ideas, not only in philosophy but within separate fields in mathematics, the sciences, and the humanities, are all related, either directly or indirectly, with a single underlying duality which can be profitably studied under the form of the mathematical duality between continuity and discreteness.
In both psychology and anthropology, for example, there is an ongoing struggle between those who approach their own discipline as "one of the sciences" and those who approach the same discipline as "one of the humanities." In the fields of art, linguistics, and philosophy itself, there is disagreement between those who consider their discipline as formal, even conventional, and those who contend that this same discipline bears reference to deeper meaning. There are related disagreements as to what methodology is appropriate for each discipline and as to what criteria are to be used in evaluating products or conclusions.

Many may be surprised to learn that a similar battle takes place within the ranks of mathematicians, despite the fact that mathematics is generally conceded to be the most abstract and objective of all disciplines. In this dissertation I have wished to bring out the point that, within the field of mathematics, the manner in which the relationship between continuity and discreteness is resolved determines whether a given thinker falls on the side of what may be called "scientism" or on the side of what is called "humanism." The purely formal approach to mathematics demands that the continuum be thoroughly arithmetized and subjected to analytic method. It is felt that to recognize as equals the geometrical concept of continuity and the arithmetical concept of discreteness is to adulterate the theoretical purity of mathematics and to limit the power of science to penetrate to impersonal laws and so to control or even to alter the forces
of nature. This is because there appears to be an ineradicable connection between the geometrical concept of continuity and syncretistic, humanistic ways of thinking and relating one's self to reality, while, as the brilliant thinkers of the nineteenth century have shown, discreteness can be completely divorced from experience and can be made fully abstract and analytic. But some mathematicians in modern times have contended for the view that continuity is primordial along with discreteness, and they question the disinterestedness, as well as the legitimacy, of motivations which lead to the subordination of continuity to discreteness in mathematical theory.

These considerations seem to me to point toward the possibility that the dichotomies appearing under the names of "sense perception" and "reason," "empiricism" and "rationalism," "psychologism" and "logicism," etc., are one and all rooted in the duality between the geometric concept of continuity and the arithmetical concept of discreteness, and they further suggest that these two concepts must themselves be rooted in a duality which belongs to the nature of human thinking. I have support for this view in the work of certain anthropologists who have attempted to connect ethnographic studies with information theory, and who have concluded that private thought processes have an iconic component which corresponds with the continuous or "analogic" character of many processes in nature, particularly those biological and psychological processes having to do with growth and maturation, while public and social forms of
thought and communication tend to be translations from these continuous processes into digitally structured "yes-no" kinds of statements. For example, the various forms of ritual performance, such as initiation ceremonies and wedding ceremonies, are basic social acts which invariably include both iconic and digital components, and the symbolic nature of which allows translation from one system (the biological-psychological) to another, unlike system (the social-political). Through ritual, continuous processes are translated into public events which make "yes-no" statements, thus setting up sharp dividing lines between "before" and "after," as between childhood and manhood, even though no such dividing lines exist in nature. Hence, the ritual determines in a definite way the economic and political status of the individuals concerned (Rappaport). It is the thesis of some anthropologists, and it is part of my thesis also, that public, digital thought forms are necessarily rooted in, and grow out of, private, iconic, continuous thought forms, while the middle, transitional ground between the two is to be found in mythico-religious, poetic, and artistic forms of expression (Bateson). It is this middle ground which I have called "aesthetics." It is no part of my thesis to suggest that the structures discovered by and reflected in digital thinking are in any way arbitrary, or to detract from the value and power which they confer upon human thinking. Hence my title: Logic and Aesthetics in Epistemology.
It is obvious that the ramifications of such a thesis could not be exhaustively explored in a single dissertation, even if the writer were fully equipped to carry out the exploration. But it has been said that the essence of philosophy lies not in the finding of answers but in the formulation of heuristic questions. Accordingly, I would like to regard my dissertation as a modest contribution toward a reformulation of some very old epistemological questions.

In order to explain the form which the dissertation has taken, I must say something about the conditions under which it came to be written. I came to the study of philosophy with an established and long-term interest in psychology and epistemology. With this background, it was natural that I should be interested in phenomenology. I discovered that a dichotomy existed between the approach to philosophical psychology which had been advocated by Franz Brentano, on the one hand, and that which had been taken by his student, Edmund Husserl, the founder of phenomenology, on the other. I decided to explore this conflict and founded my first dissertation proposal upon that idea.

I noted that Husserl's background, before he came under Brentano's influence, had been in mathematics, and that, when he eventually repudiated Brentano's "radical empiricism," it was in direct response to scathing criticisms received from Gottlob Frege, the early inventor of mathematical logic. I formulated a hypothesis that on one side of the conflict there would be found
a bias in favor of what I then called "mathematical epistemology," and on the other side a bias in favor of "non-mathematical epistemology." After I had written a tentative chapter of my dissertation along these lines, I was asked to explain more precisely the difference between "mathematical" and "non-mathematical" epistemologies.

In response to this request, I set out on an investigation of the history and philosophy of mathematics, and I soon came to believe that I might discover in the developmental history of mathematics itself the key to many forms of epistemological conflict. I wished to write, as a foundation for my main presentation, an exposition which would embody my process of discovery with respect to larger questions arising during the development of mathematics and mathematical logic, and coming to light again and again in non-mathematical contexts throughout the history of ideas.

In Part One and Part Two, therefore, my approach has been descriptive and historical. In Part Three, I have documented the movement of thought which ended in the decisive cutting of linkages between the two forms of thought, and I have described the efforts of Kantian and neo-Kantian thinkers, romanticists and idealists, to repair or compensate for this damage in various ways. I have called attention to aspects of Aristotle's thought which are seldom, if ever, discussed in the literature, such as the presence and the function of the circular syllogism, the identification of the logic of process as "things which are prior
for us," and the fact that the therapeutic use of psychoanalytic techniques is foreshadowed in Aristotle's speculation that intellect and moral will can affect the sensitive-emotional sphere only through the transformation of images. I have suggested that there is a connection between the "story-telling" propensities which are characteristic of human intuition and the geometrical concept of continuity, and I have attempted to link this propensity with the qualitative concept of "neighborhood" which is central to the mathematical discipline of topology.

The originality and the point of the dissertation lie as much in the organization and juxtaposition of materials as in the content. Analytic, historical, and factual materials have been used as a vehicle to convey a synthetic flow of thought, just as an artist may choose to convey an effect through making a collage of given or "found" materials, rather than through drawing or painting on an empty canvas; and the effect of the collage need be no less creative and effective than that of a painting. A unified and coherent flow of thought is conveyed (or I intend it to be conveyed) by means of various ideas and the manner in which they are assembled. This flow of thought constitutes the mainstream of the dissertation and is aimed at building an impression, somewhat after the fashion of poetry or music, of an underground connection among seemingly disparate and disconnected manifestations of thought. The form which the writing has taken is thus itself a part of the message which it seeks to convey. The
working out of the dissertation has itself been an organic part of my personal struggle to integrate two different ways of knowing and communicating—the logical and the aesthetic—in my own thinking and writing. That such integration may be desired, that it cannot be planned or systematically produced, but that it can appear nevertheless in the finished product, is an illustration of what Schelling meant by saying that painstaking conscious efforts become fused with unconscious productivity in aesthetic intuition. It can also be seen as an illustration of the "story-telling" propensities of the human intellect. Readers of the dissertation must judge whether this particular piece of story-telling has achieved a stimulating and heuristic effect.
PART ONE

LOGIC AND MATHEMATICS IN ANCIENT EPISTEMOLOGY
The early development of number theory required the presupposition that there exist discrete and individual units—things of some sort separated by intervals of some sort—which are to be counted. If one conceives of reality as made up of discrete and countable units, a puzzle must arise as to the nature of the units and of the intervals by which they are separated.

In the late nineteenth century, Gottlob Frege remarked that no one has ever been able to give a definition of the property "one." Some have based the concept of one on "inner intuition," but even so, said Frege, there must be criteria for being one, such as being undivided and being isolated.¹

The unexplained notion of "unit" preceded and gave rise to the notions of number and of position in space. The creation of individual unities out of the Unformed first appeared as an archetypal idea in the philosophy of Anaximander, who said that the Infinite (the Apeiron) is the source of all things and that "coming-to-be never fails." Anaximenes, who followed Anaximander, posited air as the primary world-stuff, on the ground that air has
the characteristic of infinity. What the Pythagoreans did was to assimilate air with the void, the boundless and abstract extension, and to identify the Limit with the principle of number.

The early Pythagoreans conceived of numbers themselves as having dimension and occupying space. To see how this came about, one must instruct one's imagination to the effect that numbers are not numerals, rather they are systematic iterations of an identical unit. If the arithmetical "one" be applied to a unit x, the "two" will be conceived as "xx," three as "xxx," and so on. This makes it easier to understand how the Pythagoreans conceived of a logical process whereby numbers became collections of spatial units, such collections of units became lines, such lines became planes, and planes became solid bodies. The geometrical solid was held by the Pythagoreans to consist of the unit-points composing its lines and surfaces; in this way the solid could be thought of as number (made up of units). All of the geometrical elements (points, lines, planes, solids) were understood as ideal in the sense of being imperceptible to the senses, yet they were conceived as actually existing and giving rise to sensible bodies. Aristotle said that the Pythagorean numbers had no existence apart from sensible bodies, but sensible things actually consisted of the numbers present in them.

One might conclude that the early Pythagoreans regarded numbers as material entities, but this would be a reading-in of a later point of view. The Pythagoreans did not distinguish
between the material and the formal elements of reality but explained both by identifying them with number.

The theory that "things are numbers" seems to have arisen from the Pythagorean discovery of numerical formulas for the concordant intervals in music. The expression which the Greeks had for proportionality or ratio was "logos," a term which they used in the sense of something conveying meaning or insight. The close association between the idea of "ratio" and "insight" reflects the Pythagorean notion that ratios express the intrinsic nature of all things.  

The unit-points of Pythagorean theory were not conceived as existing in plurality from the beginning but had to be generated in an orderly way. The first and main pair of opposites was the pair consisting of Limit (Form) and the Unlimited (the Unformed). The imposition of Limit upon the Unlimited seems to have been accomplished through a kind of "breathing-in" of the Unlimited (air) on the part of the principle of Limit. Aristotle wrote:

The Pythagoreans ... held that void exists and that breath enters the heaven itself which as it were inhales; the void distinguishes the nature of things, being a kind of separating and distinguishing factor between terms in a series. This happens primarily in the case of numbers. 

The Pythagorean conception of "unit-points" has been regarded by some historians as the forerunner of the theory of atomism, but the Pythagoreans made numbers the elements of more than just concrete objects; they attributed number formulae, e.g., to justice, soul, mind, opportunity, and marriage. David Furley argues that the
organic and metabolic imagery involved in "breathing-in" and in "generating" precludes an interpretation which would identify Pythagorean cosmogony with atomism. In order to become atoms, the Pythagorean units would have had to undergo a "death."  

The Limit and the Unlimited generated a unit. The unit was said to combine the qualities of Limit and Unlimited. This Pythagorean unit was not itself a number. It was the prototype of oneness, and "One" was not recognized as a number. But "One" was able to give rise to number, and from number, as we have said, came points, lines, planes, solids, and sensible bodies. For the Pythagoreans this progression was both a logical progression and a spatio-temporal process; they did not distinguish the two, for they had not abstracted mathematical thinking from the world.  

The Pythagorean concept of "unit" was close to that of an integral thing, and in Greek number theory there were no real fractions, that is, no fractions of units. For example, one-half never meant "one-half of a unit," but it meant one unit out of two, or some similar ratio between whole numbers.  

Much of Pythagorean arithmetic consisted of studying the results of arranging and re-arranging the "units" in various patterns. What the Pythagoreans studied was the arithmetical form of geometry. Out of this kind of study there came a disturbing discovery.

The Pythagorean theorem, applied to the diagonal of a square, produces such magnitudes as $\sqrt{2}$. Pythagoras himself is credited
with a simple and elegant proof that such a number as $\sqrt{2}$ cannot be resolved into a ratio between whole numbers. This traditional proof appears in Aristotle, and again in Euclid's *Elements*.\(^9\)

Prior to this discovery, all magnitudes had been conceived as commensurable, that is, as resolvable into ratios between whole numbers. As a result of the discovery of incommensurable magnitudes, the Unlimited obtruded itself into the realm of Being. The infinitely divisible continuum had to be accepted as participating in reality along with, or even instead of, unit-points.

In Greek mathematics, there was no means of dealing with irrational numbers within the framework of arithmetic. Their arithmetic involved a theory of proportion which could be applied only to commensurable magnitudes. Also, the proof of several of their geometrical theorems rested on their theory of proportion. The discovery of irrational roots (surds) shattered both the arithmetical and the geometrical methods of the Pythagoreans. One cannot wonder that for a time they ignored their own discovery, and tried to keep it secret from others.

There was to be from this time onward a growing preoccupation with the problem of Being as continuum. What would be the consequence of conceiving the continuum as infinitely divisible? Or are there such things as infinitesimals (dimensionless and hence indivisible entities)? And what is the relationship between continuity and discreteness? Such questions arose out of the discovery of incommensurable magnitudes.
Heraclitus, a younger contemporary of Pythagoras, had already taken issue with the Pythagorean explanation of Becoming as a logical and spatio-temporal generation of manyness out of unity. He held this to be impossible. He made of Becoming an ultimate eternal principle, and posited as the source of unity a pattern, or an integral complexity, lying beneath the surface of an eternal flux. He called the pattern Logos, here meaning something like "balance of opposites," or something akin to structure or form, to indicate that although there is eternal flux because of the conflict of opposites, the conflict is governed by a structure.

Heraclitus' teaching has always been regarded as obscure and hard to understand. What is Heraclitus' Logos? was a question which no one could answer. It was a principle which is inherent in everything, lies behind change, and structures reality in terms of the conflict and balance of opposites.

Later Heracliteans, the most notable of whom was Cratylus, emphasized the eternal flux at the expense of the Logos. As Guthrie puts it, they inherited from Heraclitus his love of paradox and puzzle without his genius. Guthrie also says that Plato, in his discussion of Heraclitus' philosophy, notably in the Theaetetus, appears to be responding to the views of later Heracliteans, rather than to the teachings of Heraclitus himself.

Heraclitus is noteworthy for his advocacy of the philosophical method of self-search. "He was no man's disciple," wrote Diogenes Laertius. Heraclitus wrote "I searched myself." Even before
Socrates, Heraclitus conceived the clearing ground of the comprehension of the Logos to be the self, says R. A. Frier in Archaic Logic. 13

After many false starts and blunders a man may come upon the Logos within himself. . . . He must seek for himself. . . . Heraclitus claims that it is inherent in all men to know themselves and to think soundly. . . . Evidently there is some connection between the self and thinking of the mind that Heraclitus especially wants to emphasize.

Heraclitus expressed himself as contemptuous of much learning (polymathie) and especially of learning without insight. He seems to have reacted particularly against Pythagoras. "Much learning (polymathie) does not teach sense. Otherwise it would have taught Hesiod and Pythagoras." 15 It seems not fanciful to say that Heraclitus was the first of the romantic philosophers.

Parmenides was born around 510 B.C. He has been described as "a dissident Pythagorean." 16 Unlike Heraclitus, he had been trained in the Pythagorean school of thought, but he, too, objected to the Pythagorean cosmogony. His objections were rooted in the principle that whatever is real can be understood by reason, and if something cannot be understood by reason, then it is not real. It would be a mistake, however, to assume that Parmenides made no appeal to intuitive insight. One must remember that at this stage in the development of philosophic thought, no separation had been made between intuition and reason.

Both Heraclitus and Parmenides began from an intuition of "oneness." Heraclitus conceived of oneness as a characteristic of an
integral complexity or pattern; Parmenides conceived of oneness as the prototype of the logical atom, the Number One. In determining the nature of oneness, Parmenides took his stand upon what we would now call the Laws of Thought. He declared: Either it is, or it is not. There is nothing in between. That it is not is unthinkable. Therefore, it is. Accepting as an original premise that there exists a "One," Parmenides proceeded to demonstrate with logical rigor that if there exists a One, there cannot be more than One, and there can be neither motion nor change (for how could the One move or change if there is neither emptiness nor otherness?)

Parmenides focussed attention upon homogeneity and continuity as attributes of One, which he equated with reality itself. In the writings of Parmenides, the One is considered under the aspect of pure extension; the form of Being is visualized as graphic and geometrical. Cornford explains that Parmenides was denying the void as a nothing which would interrupt the continuity of Being and make it a plurality. In visualizing the "Sphere of Being," Parmenides wished to avoid any variations of density such as might destroy its equilibrium and cause it to break into opposites preying on one another.17

According to the original Pythagorean conception, plane numbers start from three as their ultimate root, the triangle being the most primitive and elementary plane figure. The first solid number is the pyramid, represented by four units. This conception of geometry in its arithmetical form had now to be abandoned.
Parmenides emphasized Being as indivisible and this became an important point for which the Eleatic School, dominated by Parmenides, contended. Zeno of Elea, a disciple of Parmenides, argued that unity and indivisibility necessarily go together. His proof that reality is both one and indivisible is given by Simplicius.

If things are many, the same things must be both finite and infinite in number.

For (a) if things are many, they will be as many as they are, neither more nor less. But if they are as many as they are, they will be finite in number. (b) If things are many, they will be infinite in number. For there will always be others between any of them, and again between these yet others. So things are infinite in number.

But the same things cannot be both finite and infinite in number. Therefore, things are not many.

According to this proof, the notion of plurality is contradictory, since, if reality is divisible at all, it must be infinitely divisible. For it must be a magnitude, and any magnitude is divisible into parts which are still magnitudes and so themselves divisible. But if this is the case, there is nothing that can be called a unit, for anything that one takes as a unit can still be divided and so is not unitary. Zeno is credited with saying that if anyone could explain to him the nature of the unit, he would be able to accept plurality.

The intrusion of the continuum as a mathematical and metaphysical factor to be reckoned with on its own terms gave rise to a crisis in Greek philosophy, and one of the first great consequences of this crisis was that arithmetic and geometry came to be
regarded as separate sciences, while the primacy of number theory
gave way to philosophical preoccupation with measurement, that is,
with geometrical concepts and methods based upon the concept of
Being as continuum. Geometry as the domain of continuous magnitude
and arithmetic as concerned with the discrete set of integers
remained for the Greeks henceforward irreconcilable.

The requirements of Parmenides' metaphysics went hand in hand
with the development of a new science of measurement, for
Parmenides retained the geometrical basis of Pythagorean cosmogony.
It is both important and interesting to realize that Greek
geometry developed as the first postulational-deductive science.
Hermann Weyl writes:

To the Greeks we owe the insight that the structure of space
... is something entirely rational. ... the structure of
space can be exhaustively characterized with the help of a few
exact concepts and in a few statements, the axioms, in such
a manner that all geometrical concepts can be defined in terms
of those basic concepts and every true geometrical statement
follows as a logical consequence from the axioms. Thereby
geometry has become the prototype of a deductive science.

This development had its earliest roots in the writings of the
Eleatic school, and particularly of Parmenides and Zeno.

According to Parmenides, it follows logically from the homo-
genity, continuity, and indivisibility of the One that Being is
immutable, change and becoming are impossible. In the earlier
view everything was naturally endowed with motion—it was
"besouled." In the rational view the world became separated from
soul, and motion and change became unexplainable. Parmenides
was forced to deny their very existence and possibility. Aristotle wrote:

Some removed generation and destruction from the world altogether. Nothing that is, they said, is generated or destroyed, and our conviction to the contrary is illusion. So maintained the school of Melissus and Parmenides.

Zeno proceeded to demonstrate by reductio ad absurdum methods that if Being is made up of separate units, motion is impossible, but if Being is homogeneous, motion is impossible. Therefore, motion is impossible. Zeno's arguments against the reality of motion are given by Aristotle; these are the famous paradoxes of Zeno. Zeno disposed of the possibility that space exists in its own right by saying that everything that exists is in a place, therefore, if place exists, it also exists in a place, and so on ad infinitum.

Views which fly in the face of common sense and all men's experience could be maintained because Parmenides had already taken a crucial step whereby objects of thought became distinguished from objects of experience. He drew a distinction between aestheton and noeton, and said that the latter was real and true, the former unreal. Parmenides also drew a distinction between the eternal and the everlasting; in the eternal, the time sequence has been abolished.

Greek philosophers responded to the challenges of the Eleatics in three separate ways. (1) The pluralists (Empedocles and Anaxagoras) accepted the postulate that being is continuous and
eternal but denied that Being is qualitatively homogeneous, (2) the sophists (e.g., Protagoras and Gorgias) tried to uphold the value of experience against metaphysical rationalism by declaring themselves in favor of subjectivism and relativism, and (3) the atomists (Leucippus and Democritus) tried to effect a reconciliation between rationalism and the facts of experience. 24 Kirk and Raven characterize these responses as follows:

Each of these systems is, in its own way, a deliberate reply to Parmenides. Parmenides seemed, to his contemporaries and immediate successors, to have established once and for all certain canons with which . . . all future cosmologists must somehow comply. Being, in the first place, must not be allowed to spring from Not-Being; anything that was claimed as real must also be ultimate. Again, the void, being sheer non-existence, can find no place in any account of reality. Third, plurality cannot come from an original unity: if there is to be a plurality, it, too, like reality, must be ultimate. And finally, motion must no longer be taken for granted.

For the sophists, "things" had no ultimate reality. The sophists also denied objective reality to purely rational (geometric) entities. Aristotle writes upon this theme:

Indeed the sensible lines are not what the geometers say, because nothing sensible is thus rigorously straight or curved. In fact, circumference does not touch the line (tangent) at a single point but, as Protagoras said, 26 the subject of geometry, along an element of a certain length.

The atomists, Leucippus and Democritus, attempted a synthesis between rationalistic and empiricistic points of view. Their atomic theory combined a belief in the reality of atoms as unchanging elements with a belief in the reality of motion on the part of these elements. Such motion was confined to local motion (i.e., change of place) and was of an external nature only, so that all
change and becoming had now to be explained on the basis of rearrangements in space of elements which themselves remained impenetrable and unchangeable.

The atomists' metaphysics represents a thorough rationalization of Pythagorean metaphysics. Aristotle summarized the atomists' metaphysics as follows:

Leucippus and his associate Democritus say that the full and the empty are the elements, calling the one being and the other non-being (whence they say that being no more is than non-being, because the solid no more is than the empty); and they make these the material causes of things. . . . these philosophers say the differences in the elements are the causes of all other qualities. These differences, they say, are three—shape and order and position.

Leucippus has been credited with the first explicit statement of scientific determinism. He is quoted as saying that "nothing can be produced without a purpose; but everything results from a cause and by reason of necessity."\(^{28}\)

Democritus' evaluation of sense perception differed from that of the Sophists mainly in that he added a metaphysical explanation. Sweet by convention, bitter by convention, cold by convention, hot by convention . . . . In reality, atoms and vacuum.\(^{29}\)

Though sense perception was recognized by Democritus as of a phenomenal character, he wrote at length against Protagoras' theory of the subjectivism of all knowledge and asserted, as did Socrates, that "for all men the same thing is good and true." As a mathematician of stature in his own right, Democritus believed in the existence of the intelligible. We read in a fragment preserved by Sextus Empiricus: "There are two forms of knowledge: a pure or legitimate
one and a shadowy or spurious one. To this latter form belong sight, hearing, taste, smell, and touch. But pure knowledge is completely different. 30

The Eleatics had argued that asserting the divisibility of all beings into parts led to contradictions; hence they had argued that reality could not include plurality, motion, or change. The atomists postulated absolutely indivisible and solid pieces of matter below the level of visibility. Both Eleaticism and atomism had a vital bearing on the development of philosophy, because they provided the basic elements of Western epistemology.
NOTES


3Ibid., p. 13.


5Aristotle Physics 6. 218b22.


9Maziarz and Greenwood, Greek Mathematical Philosophy, pp. 49-50; Aristotle Prior Analytics 1. 23. 4125-30; Euclid Elements, bk. 10.


30 Diels B fragm. 10b and 11, p. 389, quoted by Carruccio in *Mathematics and Logic*, p. 41.
CHAPTER II

THE MATHEMATICAL FOUNDATIONS OF
PLATO'S PHILOSOPHY

Introductory Remarks

After the death of Socrates, Plato made a journey to Southern Italy, and there he held discussions with Pythagoreans and Eleatics which convinced him of the universal value of mathematics. He turned his mind toward the assimilation of mathematical with abstract thinking as it had begun to develop through the investigations of Socrates. The application of Socratic method and the Theory of Universals in the realm of mathematics was aimed at a double effect; on the one hand, mathematical knowledge itself was to become purified and perfected, and on the other hand, all knowledge—moral, physical, political, and psychological—was to become rationalized through a mapping on to the mathematical model. When he returned to Athens, Plato set up the Academy, and it is well known what a prominent place he gave to mathematics in its program of studies.

Plato believed that he could reconstruct the heavens by mathematical methods alone and could determine the principles of harmonics without using the natural relations of sounds to the ear. Thus, in the Republic, he heaped scorn on Glaucon because
"if anyone with back-thrown head should learn something by staring at decorations on a ceiling, you would regard him as contemplating them with the higher reason and not with the eyes," and upon "those whom we just now said we would interrogate about their harmony."

Their method exactly corresponds to that of the astronomer.² For the numbers they seek are those found in these heard concords, but they do not ascend to generalized problems and the consideration which numbers are inherently concordant and which not and why in each case.

But he praised anyone who "by dialectic attempts . . . to find his way to the very essence of each thing and does not desist until he apprehends by thought itself the nature of the good in itself."⁴

Pre-Socratic thinkers had evidently relied to some extent upon intuition and experience in formulating mathematical theories. Plato's insistence on deductive and abstract mathematics actually created the subject as we know it, according to Morris Kline.⁵

Plato paid particular attention to the principles and methods used in mathematics. Certain types of analytical reasoning, especially the method of reductio ad absurdum, were in use before his time, but it was Plato who formalized analytical procedure and pointed out the reciprocal relationship between analysis and synthesis.

As Plato used the term, analysis meant the method of assuming as true the proposition to be proved, and reasoning from it until one arrived at propositions previously established, or at an acknowledged or accepted principle. Reversing the order of the steps...
would give a demonstration of the theorem. Plato had not the use of a formal algebra; it was almost two thousand years after his time that his method led to the invention of analytic geometry.⁶

Through his original insights into the reciprocal relations between analysis and synthesis, Plato developed the concept of rigorous proof. The development of this concept went hand in hand with the development of abstract mathematics. Morris Kline speculates that Plato's interest in mathematics and his admiration for mathematical patterns of thought may have inhibited his interest in the specific structures of deductive logic, since he did not attempt to develop logic as a separate organon.⁷

The logical foundations on which this method rests, known to logicians as the laws of contradiction and the excluded middle, were formulated by Aristotle. . . . In effect, the grand exercise in logic that geometry afforded the Greeks led to the construction and systematization by Aristotle of those laws of thought now accepted.⁸

Plato apparently rejected the Pythagorean notion of a point as well as the Democritean notion of an atom on the ground of their appeal to sense perception. He considered points as geometrical fictions, necessary (presumably) because of the ambiguity of the region between the world of images and the world of pure Ideas. His interest in deductive and abstract mathematics may well have been reinforced by an awareness that the atomist position had run aground on the problem of incommensurable magnitudes. It is Karl Popper's⁹ view that that is why Plato gave a geometrical version of atomism in the *Timaeus*, substituting indivisible planes for indivisible bodies, and doing away with the Void altogether.
Reality and Value: The Problem of Incommensurables

One finds in the Republic and in the Philebus evidence that Plato's interest in abstract mathematics was greatly reinforced by considerations arising out of the mysterious existence of incommensurable magnitudes, and that these considerations afterwards played a central role in the formation of his ontology and epistemology. In the relevant passages (Republic 523b-526, Philebus 23c-26b), the structure of the arguments is as revealing as their content. The following points should be noted:

1. In the Republic, the entire discussion of calculation as the science of pure number begins, not with a discussion of the number series, but with a discussion of comparative qualities of sense perception—bigness and smallness, thickness and thinness, softness and hardness, lightness and heaviness—qualities which must be measured upon a continuum rather than counted, as fingers (for example) are. (523b-524)

2. Emphasis is placed upon the contradiction presented in perception by the "coincidence of opposites" that occurs in sensed qualities, so that perception describes as "one" what thought must understand as "two" and as "distinct." It is in relation to such qualities as pairs of opposites that the mind is forced into an awareness of the distinction which must be made between the "perceptible" and the "intelligible." (Republic 524-524b) This
passage may be further illuminated by comparison with a passage from the Phaedo:

Then we were saying that opposite things come from opposite things; now we are saying that the opposite itself can never become opposite to itself. Then, my friend, we were speaking about objects which possess opposite qualities, and calling them by the name of the latter, but now we are speaking about the qualities themselves. . . . We maintain that the opposites themselves would absolutely refuse to tolerate coming into being from one another. (103b-103e)

3. The question to be put as a result of being forced into reflection by these contradictions is "What in the world is the Great and Small?" and this question is to be asked of the "calculating reason." (Republic 524c) The "Great and Small" is Plato's own peculiar name for the Infinite, or the Unlimited. In the Philebus, it is explicitly stated that the qualities of sense perception belong to the Unlimited. (24-24b) Plato's construction of the Unlimited as a dyad (the Great and Small) is one of the points of departure from Pythagorean thought which are enumerated by Aristotle, the other two being his making the numbers separate from things and his introduction of the Forms. Aristotle also said that "his making the other entity besides the One a dyad was due to the belief that the numbers, except those which were prime, could be neatly produced out of the dyad as out of some plastic material."11

4. The discussion of number which follows the discussion of the continuum (the Great and Small) is presented as a completely parallel case, in which the contemplation of "Unity" in the world of sense perception forces attention because it contains a contradiction: every unit is at once both one and many. This leads to
the question "Whatever then is the One?" The case is extended to all number. (Republic 524d-525b) In the Philebus, Plato identified terms like "equal," "double," and "any term expressing a ratio of one number to another, or of one unit of measurement to another" as "coming under the Limit." (25b) At 25e he spoke of "equal" and "double" and any other "that puts an end to the conflict of opposites with one another, making them well proportioned and harmonious by the introduction of number." To the Limit and the Unlimited, Plato added a third class—the progeny of their mixture, "and you may take me to mean a coming-into-being, resulting from those measures that are achieved with the aid of the Limit." (Philebus 26d) The progeny include not only number (25e) but fair weather and all things beautiful—strength, health, and "a whole host of fair things found in our souls." (26b)

5. In the Republic, the distinction between "numbers attached to visible and tangible bodies" and "pure numbers" is made in terms of numbers which can be cut up or divided into a multiplicity of parts and those which cannot. It is acknowledged that in the sensible realm, numbers appear to be infinitely divisible. (524d-526)

6. The "Great and Small" and the "One" are treated as equal but separate parts of the same science, that science which is to be awarded first place in the list of studies indispensable for the education of the guardians, in that they "beyond anything" compel the soul to employ pure thought, with a view to truth, by
presenting contradictions in the perceptible realm. (Republic 525b-525e)

As in Pythagorean philosophy, the Limit is made equivalent to the One or Unity; it is the active, form-giving element. The Unlimited is made the principle of multitude; it represents a formless numerical matter and is the form-receiving element. 12

In the Sophist, Plato made an equation between "reality" and "wholeness" and again between "wholeness" and "definite number." (245) From this it follows that where definite number cannot be assigned, reality ceases. There is a strict relationship between that which can be defined and definite number. That which contrasts with "definite number" is the indefinite continuum. In the philosophy of Plato it is represented as a dyad and has many species: long and short, broad and narrow, high and low, more and less. The dyad is undetermined, not because it is "nothing," but because it is not anything in particular. This conception is fundamental to the understanding of Plato's whole philosophy and makes comprehensible his notion of an ontological category that lies between Being and non-Being.

Plato's positing of a dyad, his construction of the Infinite out of the Great and Small, is closely related to his novel method of generating the arithmetical continuum. Plato apparently believed that all the real numbers could be generated by a proper combination of the One with the dyad. Though Plato conceived of a generation
of the arithmetical continuum which would account for the irrational numbers, he had not the technical means to bring about the reconciliation of arithmetic and geometry which he visualized.\textsuperscript{13}

A relationship between Plato's method of generating the arithmetical continuum and the process of arriving at specific definitions by the method of Division is suggested by the fact that in both the \textit{Sophist} and the \textit{Statesman} there is repeated emphasis upon bisection of the genus, and particular warnings against taking any smaller fraction of it than we must (\textit{Sophist} 221ff, \textit{Statesman} 262ff), and by the passages in the \textit{Statesman} which make the process of correct definition a matter of finding the due measure which eliminates excess and defect. (284b-284c)

By this view, the essence of a thing will be equated with definite number only in the case of the Forms and will have to be approximated in the realm of sense perception by a process similar to the approximation of a surd. The following passages may be compared for their reflection of a mathematical imagery and color, derived from an implied reference to the interaction of the One with the dyad.

True opinions are a fine thing and do all sorts of good as long as they stay in their place, but they will not stay long. They run away from a man's mind, so they are not worth much until you tether them by working out the reason .... Once they are tied down, they become knowledge, and are stable. That is why knowledge is something more valuable than right opinion. What distinguishes one from the other is the tether. (\textit{Meno} 98)

He is neither mortal nor immortal, for in the space of a day he will be now .... alive and blooming, and now dying, to be
born again, while what he gains will always ebb away as fast. So love is never altogether in or out of need, and stands, moreover, midway between ignorance and wisdom. (Symposium 203e)

... "hotter" never stops where it is but is always going a point further, and the same applies to "colder," whereas definite quantity is something that has stopped going on and is fixed. It follows therefore from what I say that the "hotter" and its opposite with it must be unlimited. (Philebus 24d)

The suggestion is that Plato ultimately conceived of Ideas, Numbers, and all sensible things and experience as being generated by the combination of Limit and Unlimited. The validity of such an interpretation of Plato's philosophy is borne out by Aristotle.

Plato, then, declared himself thus on the points in question; it is evident from what has been said that he has used only two causes, that of the essence and the material cause (for the Forms are the causes of the essence of all other things, and the One is the cause of the essence of the Forms; and it is evident what the underlying matter is, of which the Forms are predicated in the case of sensible things, and the One in the case of the Forms, viz. that this is a dyad, the great and the small. Further, he has assigned the cause of the good and that of evil to the elements, one to each of the two.

Here Aristotle is also suggesting that in Plato's philosophy, as in Pythagorean philosophy, there is a definite relationship between number and value. We have seen that "goodness" is associated with definiteness of structure, while "badness" is associated with lack of definite pattern. J. N. Findlay has proposed that the systematic bringing of badness into the picture was based in part upon an awareness of incommensurable magnitudes.

Plato's Epistemology: Mathematics and Dialectics

In the Republic, the classification "image" occurs at various levels of an epistemological hierarchy. In the realm of the visible,
shadows and reflections are known as images of bodily reality, while in the lower levels of the intelligible realm, bodily reality itself is understood as but an image of mathematical reality. (509d-(510b) The mathematical objects are eternal and unchangeable (527b), but mathematics as science is confined to the lower section of that portion of the divided line which represents the intelligible; in other words, in the intelligible realm the science of mathematics occupies with respect to dialectics a position analogous to that occupied by shadows and reflections, in the realm of the sensible, with respect to physicality. The implication is that mathematical objects, in their turn, present an image or likeness of essential knowledge.

Plato explained that mathematics as a science falls short of essential knowledge on two counts: (1) It makes use of and requires suggestive images arising from the realm which lies below it, and (2) it investigates these images by means of assumptions "from which it proceeds not up to a first principle, but down to a conclusion." (Republic 510b and 511) Here it is being recognized that the Platonic method of analytical regression does not, of itself, lead to any explanatory principle. For this, the method of dialectics will be required, by means of which one "advances from assumptions to a principle that transcends assumption." (510b and 511)

Plato seems to have extended his criticism of mathematics to almost the whole fabric of human knowledge. The concrete sciences
are described as those which have for their object the opinions and desires of men, or the tendance and service, generation and composition, of things that grow and are put together, while the abstract sciences—calculation, geometry, stereometry, astronomy—are merely "dreaming about being." (Republic 533b-533c) None of these deserves to be called "science" in the true and proper sense of the word; "understanding" is the term to be applied to them. (Republic 533d)

In the Republic, it is acknowledged that the procedures of the arts and sciences lead the soul toward the contemplation of what is best. (532c) Essential reality, however, can be grasped only through direct spiritual vision, can be approached only through dialectic. Plato did not fully explain what he meant by dialectic but indicated what it seeks to accomplish.

Tell me, then, what is the nature of this faculty of dialectic? Into what divisions does it fall? What are its ways? (532d-532e)

You will not be able, dear Glaucon, to follow me further, though on my part there will be no lack of good will. And if I could, I would show you, no longer an image and symbol of my meaning, but the very truth as it appears to me. . . . Nothing less than the power of dialectic could reveal this. (533)

No one will maintain . . . that there is any other way of inquiry that attempts systematically and in all cases to determine what each thing really is. . . . is not dialectic the only process of inquiry that advances in this manner, doing away with hypotheses, up to the first principle itself in order to find confirmation there? (533b-533c)

Earlier in the Republic, Plato indicated the kind of knowledge and intellectual power which can be attained by means of dialectic.

By the other section of the intelligible I mean that which the reason itself lays hold of by the power of dialectic,
treating its assumptions not as absolute beginnings but literally as hypotheses... springboards so to speak, to enable it to rise to that which requires no assumption and is the starting point of all, and after attaining to that again taking hold of the first dependencies from it, so to proceed downward to the conclusion, making no use whatever of any object of sense but only of pure ideas, moving on through ideas to ideas and ending with ideas. (511b-511c)

Maziarz and Greenwood interpret this passage to mean that once reason has ascended to the first principle by means of dialectics, it can descend to any particular conclusion of the special sciences without making use of images. Knowledge in this case would be reduced to an algebra—a conception later taken up by Spinoza and Leibniz and in the nineteenth century by the founders of mathematical logic.16

In the Sophist, Plato specified that it is the Forms and the relations among them which one is to grasp through dialectics. (253d-253e) The business of dialectic is to determine the compatibility or incompatibility of the "highest kinds."

Concluding Remarks

Early influences on the development of Plato's thought, together with his profound interest in mathematics, brought him into confrontation with the metaphysical and epistemological dilemmas which had arisen out of the discovery of incommensurable magnitudes. In attempting to solve these dilemmas within the framework of Pythagorean thought, he had to overcome two different forms of aphasia, as represented by the Heraclitean solution on the one hand and the Eleatic solution on the other.
Plato's solution lay in the direction of complete separation between the realm of meanings and the realm of sense perception, with hypostatization of the former, which then became understood as real, while the latter became understood as unreal. His theory ultimately demanded a concept of changeless and unified Being, integrated with a concept of a classificatory hierarchy of unique Forms ranging from specific to generic, all derivable from one or two basic principles in an \textit{a priori} manner.

In the process of working out his solutions to these problems, Plato discovered or invented abstract mathematics, the method of analysis, and the concept of rigorous proof. He conceived of a method of generating the arithmetical continuum in a manner which could account for the irrational numbers and he aimed at a healing of the breach between geometry and arithmetic in Greek thought.

According to one prominent interpretation,\cite{17} based on both the written and unwritten doctrines, the latter derived from the writings of Aristotle and from early historic sources, Plato conceived of the whole of Reality, including the realm of Forms and the realm of instances, as generated by an interaction between the One, or the principle of Limit, and the Indefinite Dyad, his modification of the Pythagorean principle of Unlimitedness. The Forms became identified as numerical formulas or rational patterns; through an identification of the qualities of sense perception with the principle of the Unlimited, the realm of sense perception could be identified with
multitude, imperfection, and irrational number. The imperfection of human knowledge could then be explained on the basis of an ontological category lying between Being and non-Being, with a corresponding epistemological category lying between knowledge and ignorance.

Plato's ontology and epistemology are thus shown to be based on essentially mathematical concepts and imagery. His realm of Forms included also Ideal Acts of Understanding, Reasoning, and Knowledge, not to be confused with individual instantiations of them, but understood rather as complex higher-order ratios, transcendentals par excellence.
NOTES


2 Plato Republic 7. 529-529b, 531b.

3 Plato Republic 7. 531c

4 Plato Republic 7. 532.

5 Maziarz and Greenwood, Greek Mathematical Philosophy, p. 98; Morris Kline, Mathematics in Western Culture (New York: Oxford University Press, 1953), p. 34.


7 Kline, Mathematics in Western Culture, p. 54.

8 Ibid.


10 For the remainder of this chapter, references to Platonic Dialogues will be given in parentheses in the text immediately following the material cited.

11 Aristotle Metaphysics 1. 6. 987b33-35.

12 Maziarz and Greenwood, Greek Mathematical Philosophy, p. 117.

13 Ibid., p. 127.

14 Aristotle Metaphysics 1. 6. 988a7-16.

15 Findlay, Plato: The Written and Unwritten Doctrines, p. 68.

17 Findlay, Plato: *The Written and Unwritten Doctrines*; Maziarz and Greenwood, *Greek Mathematical Philosophy*. 
CHAPTER III

LOGIC IN ANCIENT MATHEMATICAL THOUGHT

The Foundations of Deductive Science
from Plato's Point of View

The method of indirect proof which had been used by Zeno the Eleatic came to be called reductio ad impossibile. It is the earliest form of argument permitting premises of the "If...then..." form. Reductio ad impossibile and reductio ad absurdum were used somewhat interchangeably. The Kneales have suggested that reductio ad absurdum be used to refer to methods of indirect proof which are not strict instances of Zeno's method.¹

Reductio ad absurdum, then, was used before Plato's time, not only in metaphysical argument but also in mathematics. One of the most frequently used forms of indirect proof in mathematics is called the "double reduction." Wishing to prove, for example, that \( A \) is equal to \( B \), a mathematician may proceed by assuming that \( A \) is less than \( B \), and deriving from that assumption a self-contradictory or absurd conclusion. He then makes the alternative assumption that \( A \) is greater than \( B \), and again derives a self-contradictory or absurd conclusion. Since it is axiomatic that \( A \) is less than, equal to, or greater than \( B \), he has proved indirectly by a double reduction that \( A \) is equal to \( B \). Such proofs were used by Hippocrates of Chios.
(circa 470 B.C.), by Eudoxus, Euclid, Apollonius, Archimedes, and many other mathematicians in ancient times. They have continued to be a basic technique and a powerful tool in mathematics. It was not until the twentieth century that the validity of such proofs was questioned. In this century, objections have been raised by the mathematicians who call themselves "intuitionists" to the indiscriminate use of indirect proofs in reasoning about infinite classes.

"Dialectic" originally meant reductio ad impossibile as used by Zeno the Eleatic or reductio ad absurdum as used in verbal argument. It was much used in debate by the Sophists, for example. The rules for dialectic were elaborate. The respondent began by choosing a thesis; the examiner then asked a series of questions, so phrased that the response had to be acceptance or denial of a statement. The purpose was to get the respondent's consent to statements contradictory to his own thesis. If this were successful, the respondent would have an opportunity to fault the premises or the reasoning which had been used, if he could. Such dialectic was the standard form of debate in Plato's Academy. The Platonic Dialogues contain many instances of such discussions.

We know that Plato had developed a method of analysis and synthesis to be used in constructing mathematical proofs, but that in the Republic he had characterized mathematical reasoning as falling short of true science. In the Dialogue Parmenides, Plato made an experiment with metaphysical arguments having conditional statements
as premises. Parmenides offers a series of such arguments as demonstration of the type of logical exercise he recommends to the young Socrates, an exercise which he describes as being of the form used by Zeno the Eleatic, except that the survey is to be extended from the field of visible things to "those objects which are especially apprehended by discourse and can be regarded as Forms." 3

After the Parmenides, Plato made no further attempt to develop deductive argument forms; instead, he returned to the problems of definition. He evidently conceived of the method of Division as developed in the Sophist and in the Statesman as belonging with a new form of dialectic which could lead to positive results at the highest level of abstraction. In his Commentary on the Posterior Analytics, St. Thomas Aquinas advanced the theory that Plato's belief in the Theory of Recollection caused him to regard discursive reasoning as unimportant for the apprehension of true Reality.

He postulated that science in us is not caused by study and exercise, but only that obstacles are removed and man is brought to recall things which he naturally understands, in virtue of an imprint of separated forms. 4

But it is Lynn E. Rose's thesis that the method of Division may have served the purpose of a formal logic for Plato, first, because it is applied exclusively in the realm of Ideas, and second, because it is a kind of a priori showing or revealing. The act of pondering and working out the divisions and relationships among the Ideas provides the needed stimulus which enables us to recollect what we already know, but have forgotten. 5
It has been pointed out in chapter 2 that Plato declared relations among Forms to be the object of dialectic, the highest form of reasoning. This suggests that, insofar as Plato conceived of deductive science at a higher level than that of mathematics, he conceived it as a priori knowledge of structural relations among the Forms. Though Plato had no logic of relations at his command, the dialectic of his mature years, a fusion between the method of Division and the Theory of Recollection, may have served the purpose of such a logic for Plato.

**Systems or Theories of Deduction Developed in Ancient Times**

Aristotle first said that "reasoning is an argument in which, certain things being laid down, something other than these necessarily comes about through them." He then said that arguments start with propositions; propositions are the things "laid down."6 His theory of inference is embodied in the categorical syllogism. The component propositions of the categorical syllogism (its premises and its conclusion) are propositions expressed in simple statements in which a predicational relationship between two terms is asserted through a verb.

The relation between the terms in the premises and in the conclusion of an ideal categorical syllogism is one of inference. The structure of an ideal syllogism is stated by Aristotle as follows:

Whenever three terms are so related to one another that the last is contained in the middle as in a whole, and the middle is contained in, or excluded from the first as in or from a
whole, the extremes must be related by a perfect syllogism. I call that term middle which is itself contained in another and contains another in itself.

Aristotle then said that imperfect syllogisms can all be reduced to perfect syllogisms, and he showed how this is to be done. \(^8\)

Aristotle distinguished between the non-scientific and the scientific syllogism, for he said:

Syllogism should be discussed before demonstration, because syllogism is the more general; demonstration is a sort of syllogism, but not every syllogism is a demonstration.

The distinction between mere syllogism and demonstrative syllogism derives from the epistemic status of the premises; demonstrative syllogism aims at truth, and must be guided by "the real connections of subjects and attributes." \(^10\)

Aristotelian logic is called formal logic because Aristotle recognized—and was the first to recognize—that it is the proper arrangement of the terms in the syllogism which gives it inferential power, i.e., establishes a relationship of entailment between the propositions. He often discussed the structure of the syllogism without reference to the concrete meanings of the terms, using letters such as \(A\), \(B\), and \(C\) to symbolize terms.

Traditionally, the inference schema of the categorical syllogism which later became known as "ideal Barbara" has been formulated as follows:

\[
\begin{align*}
\text{All } B & \text{ is } C; \\
\text{All } A & \text{ is } B; \\
\text{Therefore,} & \\
\text{All } A & \text{ is } C.
\end{align*}
\]
The arguments which were used by Zeno the Eleatic were of another form.

If P, then Q;
If P, then not-Q;
Therefore,
It is impossible that P.

Reflection makes it clear that the symbols P and Q in this latter formulation do not stand for single terms but for entire propositions, and that the relationship between these propositions is not one of inherence. For example, a proposition such as "If it rains, the grass will be wet," is symbolized "If P, then Q."

Zeno was exploring the hypothetical consequences which follow from assuming a certain state of affairs; when he had found that making an assumption led to self-contradictory consequences, he considered that the assumption had been proved false. By the law of the excluded middle, its contradictory had been proved true. Such reasoning is based upon compound propositions, i.e., upon propositions which state relations between propositions, and the method of argument is indirect.

About fifty years after the time of Aristotle, the Stoics proceeded to develop a formal logic in which conditional statements were used as premises. Two things are to be said of such premises: First, they refer to no subject-predicate relation, and second, their validity (as distinguished from truth) depends entirely upon the definitions of the connectives used in formulating them.

Stoic logic came into its own during the nineteenth century, with the advent of mathematical logic.
Epistemology and Logic in Euclid's Elements

The Elements investigates the properties and relations of some mathematical objects according to laws either arising directly from the nature of these objects, stated as axioms, or arrived at by deduction from the axioms. But the underlying logic is not discussed. The methodology with the invention of which Euclid is credited falls into the category of "fishes," which can be studied by "zoologists"—historians, philosophers, and mathematicians who seek to analyze the methodology itself. One such zoologist in ancient times was Proclus.

Proclus was born in Byzantium in the fifth century A.D. While still under twenty years of age, he went to Athens, where the school of Plato was experiencing a revival under Plutarch; he subsequently became teacher as well as student at that school. His Commentary on Euclid is a treasury of information about ancient philosophy. Proclus regarded the methodology of the Elements as representing the distinctive character of all mathematical inquiry, and he wished to refer its underlying logic back to Plato's dialectic.

Proclus said that the Elements contains all the dialectical methods, such as the method of Division for finding kinds, definitions for making statements of essential properties, demonstrations for proceeding from premises to conclusions, and analyses for passing in the reverse direction from conclusions to
principles, and that "the various methods of conversion, both the
simple and the more complex, can be accurately learned in this
treatise."

He specifically credited Plato with the invention of
the method which he called "analysis," but he gave to "synthesis"
its Aristotelian meaning of proving a proposition through an
essential middle term, although, as he pointed out, there are
instances of mathematical proof which do not conform to the ideal
in this respect.

Despite his profession of strict Platonism, Proclus' interpre­
tation of Euclidean methodology represents a synthesis of Platonic
with Aristotelian thought, and there are references throughout the
Commentary to "the inspired Aristotle." But Proclus addressed
several arguments in his Prologue against the Aristotelian doctrine
that mathematical objects are obtained by abstraction from
sense objects. Not surprisingly, he regarded the objects
of mathematics as issuing from the soul itself, the soul having been
reminded of these ideas by sense experiences. Proclus' explanation
of how "the partless realities of Nous" become accessible to human
understanding as extended yet intelligible objects is considered an
original contribution to Platonic philosophy.

According to Proclus, imagination stands in the central
position on the scale of knowing (the divided line of the Republic).
It knows the primary forms, for "the circle in the understanding is
one and simple and unextended, and magnitude itself is without
magnitude there, and figure without shape." But when imagination is
moved to put forth what it knows, drawing its objects out of the undivided center, it expresses them in the medium of division, extension, and figure. Everything it thinks is a picture or a shape of its thought. Therefore, it thinks the geometrical figure as extended. Although this extended figure is free of external matter, it possesses an intelligible matter provided by the imagination itself. Furthermore, the mathematical objects, like the Ideas of which they are images, have a self-moving, substantial quality. They unfold as they are examined, revealing new aspects of their character and their relationships. They carry the understanding with them as they unfold. The entire process is under the guidance of Nous, as the mathematical images derive at every stage directly from the Ideas.

Euclid divided his first principles into definitions, postulates, and axioms. He divided the propositions following from his first principles into "problems" and "theorems."

"Problems," as distinguished from "theorems," are concerned with the construction of figures and various operations upon these constructed figures, as it were in ideal space. Construction, therefore, is an important part of Euclidean methodology. Proclus perceived this as an extension into the sensible world of the all-important functioning of imagination. That Euclid's Elements contains both problems and theorems is evident from his practice of placing at the end of his demonstration sometimes "this is what was to be done," and at other times, "this is what was to be proved."

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In his authoritative work entitled *The Thirteen Books of Euclid's Elements*, Sir Thomas Heath presents a different explanation of the meaning of Euclid's constructions. In the Introduction to volume 1, he discusses the parallelism between Aristotle's stipulations with respect to definitions and the practice which Euclid actually observed in the *Elements*.19

Aristotle was always concerned that a definition should be demonstrably connected with some unique identifiable reality. Except in the case of a few primary things among the first principles of each science, the existence of things defined must always be proved. By "proving the existence of an attribute" Aristotle meant establishing, for example, that some specific number actually has the attribute "odd," or that some specific magnitude actually has the attribute "straight." In the *Metaphysics*, Aristotle said that geometrical constructions "exist potentially" and are discovered by being actualized; the geometer's thinking was described by Aristotle as an actuality from which the potentiality of the geometrical construction first proceeds, "and therefore it is by making constructions that people come to know them."20 In geometry, proving the existence of an attribute takes the form of constructing a figure which possesses the attribute in question. In the *Elements*, Euclid's first three postulates declare the possibility of constructing straight lines and circles; all other things are defined and afterwards constructed and thus proved to exist.
Saccheri gave as an instance the construction of a square in Euclid 1:46. Suppose it to be objected that Euclid had no right to define a square as he does at the beginning of the book, when it was not certain that such a figure exists in nature. Saccheri said that that objection could have force only if Euclid had assumed the aforesaid figure as a given before proving and making the construction. But in fact Euclid never presupposed the existence of the square as defined until after 1:46. It would appear that Euclid's vision was slanted in the direction of an Aristotelian conception of mathematical objects rather than in the direction of understanding them as a priori creations of the (transcendental) imagination.

Euclid derived from Greek philosophy the concept of rigorous proof and rational necessity. These principles were discussed at length by Aristotle in both Prior analytics and Posterior analytics. Aristotle grounded the deductive process on a few principles which were to be stated and granted at the beginning of the reasoning process. In Posterior analytics 1:10 Aristotle discussed the character of these principles. He spoke of basic truths to be used in the demonstrative sciences, some of which are "common truths," such as "take equals from equals and equals remain," and some of which are truths peculiar to the given science. Hypotheses "postulate facts on the being of which depends the being of the fact inferred." This is the pattern which Euclid followed in the Elements.
Of Euclid's methods of reasoning generally speaking, it has been claimed historically that they are syllogistic. To illustrate the untenability of this claim, Ian Mueller presents an analysis of the method used in Proposition 1:1. The Peripatetics had said that the argument in Proposition 1:1 rests upon a categorical syllogism:

Things equal to the same thing are equal to one another.
CA and CB are equal to the same thing.
Therefore,
CA and CB are equal to each other.

But Euclid's actual proof depends upon the relations among three lines and not upon properties of them taken as pairs, and Mueller's analysis shows that no syllogistic reconstruction of his _apodeixis_ is possible. It is unlikely that Euclid regarded his inferences as formal arguments at all. Augustus De Morgan had remarked:

It is not in his system to establish a purely logical inference once for all; accordingly "not-X is always not-Y" is converted into "Y is always X" by one and the same process whenever it is wanted.

In Greek mathematics generally, parallel proofs were often given which could easily have been generalized, from which it appears that Greek geometers trusted their geometrical intuition, and did not rely upon any set of logical principles. Their use of logic was not necessarily based on any rules of inference, but arose out of the fact that natural deduction is an inevitable feature of mathematical activity.
In Euclid's day, his proofs satisfied all requirements of logical and mathematical rigor, but from a modern point of view the situation is otherwise. For one thing, Euclid's definitions do not really define, because Euclid expressed them in terms of things which do not precede them in understanding or in definition, and because he was guilty of logical circularity. Again, Euclid made use not only of his expressed postulates but of many assumptions which he failed to postulate.

The higher standard of rigor demanded and achieved in mathematical proof since the nineteenth century is based upon strict axiomatization, which means that proofs may not make use of any unstated or intuitive principles nor of any undefined terms. An outstanding example of strict axiomatization in the field of geometry is David Hilbert's Foundations of Geometry, first published in 1899.

The need for rigorous axiomatization as a necessary feature of mathematical proof is justified on the ground that intuition might lead one to a false conclusion. Hilbert's contemporary, Felix Klein, demonstrated, as an example, that it is possible to carry out a deduction, making use of Euclid's postulates, which would lead to such false consequences as "all triangles are isosceles." But Ian Mueller argues that while strict axiomatization is certainly necessary in the calculus, where one is working with curves (functions) for which no intuitive picture exists, there is an important difference between the calculus and elementary geometry with respect to the role of intuition. In Euclidean geometry,
conceived as the description of intuitively grasped truth, precautions to avoid falsehoods were unnecessary.  

Concluding Remarks

Greek mathematicians made extensive use of indirect proofs based upon Zeno's methods of argument, as did the sophists, Socrates, and Plato, although the principles upon which these methods are based had not been made explicit. Greek geometers trusted intuition and had little sense of generalization based purely on logical form. Aristotle's syllogistic logic is called formal logic because Aristotle recognized the distinction between syllogism per se and demonstrative syllogism; the former depends merely upon the arrangement of terms, while the latter aims at truth, rather than mere validity, and must be guided by considerations involving reality.

Euclid's reasoning in the Elements is couched in deductive form which resembles modern axiomatic systems in some respects, but analysis of his actual proofs reveals that he, too, relied heavily upon intuition and natural deduction techniques rather than upon formal logic.

Though cast in the form of an axiomatic deductive science, the first of its kind, Euclid's Elements in fact presents a picture of intuitively grasped truth. In modern times, attempts to recast this description in stricter deductive form brought about a radical revolution in theoretical understanding of the nature of space,
as well as in philosophical understanding of reasoning itself, particularly with respect to the role of intuition.
NOTES


3. Parmenides 135d-e.


6. Topics 1. 1. 100b25-27.


12. Ibid., p. 57

13. Ibid., p. 54.


15. Ibid., pp. 10-13.

17Proclus: A Commentary on Euclid's Elements, pp. 40-44.

18Morrow, Introduction to Proclus, pp. xxxix-xl.


22Posterior analytics 1. 10. 76a40-b39.


A deductive science presupposes an underlying logic which need not be explicitly stated as part of the body of knowledge constituting that science. Therefore, a deductive science includes and presupposes another system, a system or theory of deduction. The logical basis for such a system is called "natural deduction." It consists of valid rules of inference which are presumed to be closely related to those employed in natural reasoning processes.

As we have seen in the previous chapter, Euclidean method begins with fundamental assumptions which are accepted on the basis of their plausibility; these assumptions are called postulates. From them certain conclusions are derived, but the underlying logic is never made explicit, and Euclid may not have been consciously aware of any rules of inference. His system appears to have been based upon natural reasoning processes combined with intuition. Yet his method became the model for exact reasoning, i.e., for deductive science.
In modern times, theories of deduction have been classified as either "natural" or "axiomatic," according to whether explicitly stated axioms or inexplicit rules of inference play the more prominent role. By axiomatizing a theory of deduction, one can reduce the number of natural rules of inference which will be employed. An axiomatic system is constructed by what is called "axiomatic method." The construction proceeds as follows: First, a small number of expressions is selected to function as primitive terms. Their sense is given by the way they are used in the context of the axioms. All other expressions of the system are defined by means of these primitive expressions. Second, certain sentences are accepted in the system without demonstration. These are axioms (postulates, basic theorems, primitive statements). All other statements of the theory are deduced from the axioms. The formalization of such a deductive system exhibits its structure independently of the meaning of any terms other than the logical constants—copulas, connectives, quantifiers, and qualifiers.¹

A formal deductive system is still not a science of logic. The underlying logic of a deductive science is merely an abstract system which will lack rigor unless the notion of logical derivation itself is precisely defined. In a logistic system, in which the subject matter is deduction itself, the logical terms must also be replaced by uninterpreted symbols. In such a system, the language used, consisting entirely of
uninterpreted symbols, is called an "object language," while
the language in which its characteristics are described and
discussed is called a "metalanguage." The criteria for
distinguishing significant from non-significant symbols in an
object language are purely formal, that is, they are syntactical
and not semantical.² The study of deduction itself is concerned
with logical syntax (the relations between signs and signs) to
the exclusion of semantics (the relations between signs and objects
or events).

With each greater degree of formalization, a greater degree
of generality is obtained. When the deductive system, or the
underlying logic, of any science has been formalized, its empty
structure is exhibited; the science from which the formalization
began then becomes only one model of the system, and there may be
many other models of the same deductive system.

In this connection, it is instructive to examine the
exposition of Stephen Cole Kleene in his Introduction to
Metamathematics. Kleene contrasts genetic or constructive methods
of "introducing a system of objects into mathematics" with the
axiomatic or postulational method. As an example of the former,
he adduces the inductive definition of the natural numbers, where
the natural numbers are conceived as being generated or constructed
by an orderly, rule-determined procedure. In the axiomatic method,
on the other hand, propositions are laid down at the outset as "the
assumption of conditions on a system S of objects."³
Kleene explains what is meant by a "system of objects" as follows:

By a system $S$ of objects we mean a (non-empty) set or class or domain $D$ . . . of objects among which are established certain relationships. . . .

When the objects of the system are known only through the relationships of the system, the system is abstract. What is established in this case is the structure of the system, and what the objects are, in any respects other than how they fit into the structure, is left unspecified.

Then any further specification of what the objects are gives a representation or model of the abstract system . . . . These objects are not necessarily more concrete, as they may be chosen from some other abstract system . . . .

Two representations of the same abstract system are (simply) isomorphic. i.e. can be put into 1-1 correspondence preserving the relationships.

An analytic proposition is one which finds its justification in belonging to such a purely formal system, the first principles of which have simply been postulated without any reference to "reality."

Euclid did not formulate his axiomatic system according to these principles; he thought of his axioms as expressing fundamental properties of real space. Axiomatic method in this sense is called informal or material axiomatics. In this older sense of axiomatic method, the objects of the system are supposed to be known prior to the axioms, and to determine them, whereas in the later or formal axiomatic method, the axioms are supposed to be given first, and to determine the conditions for the existence of the objects. In the former case, postulational method is combined with a synthetic approach, while in the latter case, the method is analytic.
E. T. Bell says of the age of Descartes, Fermat, Newton and Leibniz that if the fundamental difference between this "second great age" of mathematics and its first great age can be suggested in a word, it may be said that the spirit of the first age was synthetic, while that of the second age was analytic. Euclid's *Elements* presents us, in general, with a synthetic approach to deductive science, in that Euclid sought, and was bound by, an ability to find in construction, aided by intuition, an interpretation of his deductive system that would assure him of its self-consistency—i.e., would assure him that the application of its rules could not lead him into self-contradiction. On the other hand, despite the fact that Euclid's is a synthetic metric geometry, his postulational method, the apparent intent of which is strict deduction from explicitly stated assumptions that have been recognized as such, is now conceived as analytic, and as a method of great importance in algebra as well as in geometry. But the word "analytic" now implies an avoidance of any appeal to objects of perception or intuition for establishing the truth of a proposition or the validity of an inference. Insofar as intuition and experience are ruled out of account in mathematics, it becomes necessary to justify mathematical propositions on the basis of logical principles alone, and in the philosophy of Gottlob Frege, the possibility of such justification, in the case of any proposition, makes the proposition analytic, no matter how it may have been arrived at in the first place.
The contrast between ancient methods of mathematical reasoning and modern method is well brought out by a comparison of Eudoxus' techniques with the modern concept of limit, for the limit concept is at the foundation of almost all mathematical analysis, and it is further remarkable in that it could hardly have been developed without the aid of geometrical intuitions, yet in the end it seemed to contradict them.7

In chapter 1 of the present work, the content of the term "number" as used by the Pythagoreans was discussed. In their conception, the concept of number is equated with that of "integer." The Pythagoreans had no such thing as a generalized concept of number that might have been extended to include fractional parts of a unit, still less to include irrationals. Therefore, number could not be brought into strict relationship with continuity.

For the Pythagoreans, two quantities were to each other as the ratio of the whole numbers (the number of units) contained in each. They did not conceive of lengths of lines or of areas of surfaces as such. Rather, they would ask: What is the ratio of the length of two lines? What is the ratio of the areas of two circles? Ratios were ascertained by a method of superimposing, i.e., of applying one length to another, or one plane surface to another.

The method of application of areas enabled the Greeks to compare a figure bounded by straight lines with another such figure, and to ascertain that the one figure was greater than,
equivalent to, or less than the other. But this method could not be used to compare a figure bounded by straight lines with one which was curvilinear. Therefore, Antiphon the Sophist and Bryson proposed a method of inscribing a regular polygon within a circle and doubling the number of its sides successively, again and again, so that the polygon approached more and more nearly to coinciding with the circumference of the circle.  

Eudoxus then developed this method into a more rigorous type of argument. The axiom upon which he based his argument is given in Euclid 10.1. It reads as follows:

Proposition 1. Two unequal magnitudes being set out, if from the greater there be subtracted a magnitude greater than its half, and from that which is left a magnitude greater than its half, and if this process be repeated continually, there will be left some magnitude which will be less than the lesser magnitude set out.

As this process is continued, the difference between the area of the polygon and that of the circle can be made as small as one pleases. The method implies that space is an unbroken continuity that is infinitely divisible. If the inscribed polygon were actually to become identical with the circle, this would put an end to the process of subdividing the sides, and space could no longer be held to be infinitely divisible. Therefore, Greek mathematicians never considered the process as being literally carried out an infinite number of times. They always conceived that there would be a quantity left over, however small. The process never contradicted their intuition of the nature of space, nor
did it contradict the evidence of sensory perception. In order to make the argument rigorous, Eudoxus introduced a _reductio ad absurdum_ procedure, by means of which it could be proved that, if the ratio between the area of the polygon and that of the circle did not, at some point, reach equality, this would contradict the principle that had been stated in the axiom.¹⁰

This theory later came to be inaccurately called "the method of exhaustion." It is in some ways similar to the method employed in the calculus, involving the mathematical concept of _limit_, but there is a marked difference in epistemological viewpoint. It is this difference which is of interest here.

Explanation of the limit concept in simple terms is not easy, but an approach can be made through a discussion of the derivative, treating it, up to a point, as an intuitive concept.

There are many instances where it is desirable to determine the value of a curvilinear figure at a given point on the curve. For example, one may wish to discover the velocity of an object in motion at a given instant in time. "Instantaneous velocity" implies that the time interval has become zero. Such a time interval does not exist in nature, and such a velocity is not a velocity at all in the ordinary sense of the word. Yet instantaneous velocity is treated as a mathematical entity.

One may discover average velocity by dividing the length of a time interval into the distance covered by a motion during this time interval. But division by zero is a mathematical
impossibility, so that this ordinary method will not serve to
determine instantaneous velocity. It is here that the concept of
limit must be introduced. One considers successive values of
a quotient as the distance and time intervals grow smaller and
smaller. An infinite sequence of values can be produced in this
way, that is, a sequence which can be continued forever without
reaching an end. Such a sequence may appear to be approaching
nearer and nearer to some fixed value. In such a case, the fixed
value which it appears to be approaching is called the limit of
the sequence, or alternatively, the derivative of the function.
In the above example, the derivative of the function gives the
instantaneous velocity of a moving particle at a given instant
in time. 

The limit concept may be used to describe any case where,
as $x$ approaches a certain value, $y$ approaches or tends toward
another value. For example, it can be found by straight substi-
tution in the equation

$$y = -2x^2 + 8x - 4$$

that when $x$ is 3, $y$ is 2. This can be expressed by saying that
as $x$ comes closer and closer to the value 3, $y$ (or function of $x$
written $f(x)$) comes closer and closer to the value 2. Stated
more generally: If a function $f$ is defined for values of $x$ about
the number $a$, and if as $x$ tends toward $a$, the values of $f(x)$ get
closer to some specific number $L$, we can say that the limit of
$f(x)$ as $x$ approaches $a$ is $L$. 11
The usefulness of such a general statement becomes apparent in cases like that of instantaneous velocity described above, where the value of a function at a point cannot be determined by ordinary algebraic operations. The limit concept, then, must be defined by extending the terms of ordinary processes in algebra so as to include infinite processes.

In a purely logical definition of the limit concept, intuitive notions such as those implied by the word "approach" must be abandoned. One does not imagine the infinite sequence as coming to an end; one simply regards the limit as a number which possesses certain properties which are stated in its definition.

As Russell explains it, it used to be supposed that "limit" was a quantitative notion. But in fact the notion of "limit" is a notion involving order or sequence rather than quantity. For example, the cardinal number Aleph Null is thought of as the limit of the whole series of numbers 1,2,3, . . . , yet from a quantitative point of view, finite numbers get no closer to Aleph Null as they grow larger; what makes Aleph Null the limit of the finite numbers is just the fact that it comes immediately after them in the series.

The logical concept of limit demands acceptance of the concept of an actual infinite sequence, whereas the method of exhaustion relies upon the concept of the infinite sequence as mere potentiality. All four of Zeno's paradoxes of motion, involving
the continuity of space and time, result from an assumption that all of the terms of an infinite series must be separately visualized in spatial intuition. Since this cannot be done, Zeno argued that either motion does not exist or there is no plurality of elements. The solution to Zeno's paradox of the flying arrow directly involves the concept of the actual infinite.

The argument in this paradox, as also that in the stade, is met by the assumption that the distance and time intervals contain an infinite number of subdivisions. Mathematical analysis has shown that the conception of an infinite class is not self-contradictory, and that the difficulties here, as also in the case of the first two paradoxes, are those of conceiving intuitively the nature of the continuum and of infinite aggregates.49


Concluding Remarks

Certain characteristics of the limit concept must be underlined. Where the ancient method of exhaustion involved intuitional concepts only, the concept of limit demands an extrapolation beyond the possibilities of sense experience and spatial intuition, and even beyond comprehension. It must be regarded as a purely verbal concept,16 and since it can be justified only on the basis of its definition, it is an analytic concept.

In order for continuity to be represented analytically, the number concept had to be generalized to include irrational numbers, and there had to be an algebraic representation of abstract terms.
that could be conceived as taking on an infinite succession of
different values. The logical definition of the limit concept
therefore required (1) an algebraic mode of expression, (2) the
concepts of variability and functionality, and (3) a generalized
concept of number, including a definition of the real numbers.
Until these requirements had been met, the so-called
arithmetization of continuity could not be accomplished.
NOTES


4Ibid., pp. 24-25.


6Frege, Foundations of Arithmetic, pp. 3e-4e.

7In discussing this contrast, I am heavily indebted to Carl Boyer, Concepts of the Calculus.

8Boyer, Concepts of the Calculus, pp. 32-33.


10Boyer, Concepts of the Calculus, pp. 33-36.

11Ibid., pp. 6-8.


13Ibid.

14Maziarz and Greenwood, Greek Mathematical Philosophy, pp. 60-61.

15Boyer, Concepts of the Calculus, p. 25.

16Ibid., pp. 3-6.
PART TWO

LOGIC AND AESTHETICS IN MODERN MATHEMATICS
Arabic algebra was mastered in Western Europe during the Renaissance. During the period of transition from the Renaissance to the modern world, Arabic algebra became transformed into symbolic logic.

One prominent figure in the effecting of this transformation was the Frenchman, Francois Viète. Viète realized that the "unknown" need not be a number or a geometrical line but might be conceived as anything at all, since algebra reasons, not about concrete individual things, but about types or species. He distinguished between logistica speciosa (algebra) and logistica numerosa (arithmetic). He also noted that in problems involving the "cosa," or "unknown," the reasoning used was similar to that which the ancients had called "analysis." That is, in algebra one begins with an assumption and reasons from this assumption to a necessary conclusion. It must then be shown that the reasoning process can be followed out in reverse. Since this type of reasoning is typical of algebra, and since it is the type of reasoning which the ancients had regarded as analysis, Viète called algebra "the analytic art."
Symbolic algebra reached full maturity in Descartes' *La géométrie*. This is the earliest mathematical text which makes use of a notation which a present-day student of algebra can understand. Descartes was the first to use letters near the beginning of the alphabet to designate parameters and letters near the end to designate unknown quantities. He also institutionalized the use of the Germanic symbols + and - and a method of exponential notation like that in present-day use.¹

Today we read $x^2$ without seeing in our mind's eye an actual square, but this was not true at the time of Descartes (1596-1650). Parameters and unknowns were still thought of as line segments, rather than as numbers. From Descartes' own point of view, the fundamental purpose of *La géométrie* was to furnish a geometrical interpretation of algebraic operations, which he had characterized in the Discours as "confused and obscure." He sought not only to give meaning to the operations of algebra through geometric interpretations, but also to free geometry from its too-great dependence upon diagrams.²

It was Descartes, and not Euclid, who explicitly advocated the use of intuition in preference to formal logic in mathematical proofs.

For our mind is so constituted by nature that general propositions are formed out of the knowledge of particulars. One must contemplate a concrete object, yet the object to be contemplated is not material. It must be at one and the same
time a particular and the representative of an essence. Descartes wrote in the Fifth Meditation:

For example, when I imagine a triangle, although there may nowhere in the world be such a figure outside my thought, or ever have been, there is nevertheless in this figure a certain determinate nature, form, or essence, which is immutable and eternal, which I have not invented, and which in no wise depends on my mind, as appears from the fact that diverse properties of that triangle can be demonstrated, viz. that its three angles are equal to two right angles, that the greatest side is subtended by the greatest angle, and the like, which now, whether I wish it or do not wish it, I recognise very clearly as pertaining to it, although I never thought of the matter at all when I imagined a triangle for the first time.

Descartes' intuitionism contains the germ of Kantian intuitionism. In the Critique of Pure Reason, Kant stated that the employment of pure knowledge in geometry depends upon the condition that objects be given to us in intuition. The result of demonstration, then, is determined not by axioms but by the conditions of construction, which implies that the theorems of Euclidean geometry could all be discovered through intuition alone. This is Kant's synthetic a priori.

Descartes distinguished two types of curves. The first type he called "geometrical curves." These included the line, the circle, and the conics (ellipse, parabola, and hyperbola), and other curves whose properties can be discovered by the use of ordinary algebraic operations. The second type, which Descartes called "mechanical curves," were those whose properties cannot be discovered without the aid of infinitesimal methods or of the concept of limit. Descartes thought of his geometrical curves
as those which could be exactly described, while his mechanical curves he believed could be described only inexactly.\textsuperscript{6} 

Descartes was acquainted with infinitesimal methods and had used them in the discussion of scientific experiments, but his mathematical methods were purely algebraic. Yet a large number of works devoted to infinitesimal methods appeared in print both before and after the publication of the \textit{Discours de la M\'ethode}. With the exception of Fermat's, these works were based largely upon synthetic geometry. One of the most influential books of the early modern period was published in 1635 by Cavalieri, a disciple of Galileo. The argument of this book, entitled \textit{Geometria indivisibilibus continuorum}, was that an area can be thought of as made up of indivisibles, and that a solid volume can be thought of as composed of areas that are quasi-atomic volumes. This type of reasoning goes back to Archimedes, indeed back to Democritus, who had conceived of the pyramid as made up of infinitely thin slabs. Despite what he calls the "specious plausibility" of the method of indivisibles, Carl Boyer considers that some of Cavalieri's work amounted to analytical geometry and calculus, before either of these subjects had been invented.\textsuperscript{7}

In England, John Wallis (1616-1703) sought to free arithmetic and algebra from geometrical representations. In 1655 Wallis published his \textit{Arithmetica infinitorum}. In this book he associated the infinitely many indivisibles in Cavalieri's geometric figures with numerical values. This demanded the use of what are now
called the "real numbers" (the concept of real number includes that of "rational number" and also that of "irrational number"). No such numbers had as yet been defined, but this did not worry Wallis. His work was based upon what he called "interpolation" and "induction." Interpolation involved a principle of continuity according to which a rule which held for certain known numerical values was to hold for all values which occurred intermediately between them. Induction meant the use of analogy, as in scientific induction. By these principles Wallis was able to assume that the properties of the infinitely many numbers which include the irrationals are the same as the properties of finite numbers. Wallis' work displays a certain looseness of thought, of a type which was prevalent among mathematicians at that time. It was also immensely creative. Even though in modern mathematics the concept of the infinite is very different from that of Wallis, the calculus could not have been invented without such efforts as he was willing to make. This illustrates the fact that at certain points in the history of mathematics, it is necessary to disregard the demands of logical rigor in order to progress.

Sir Isaac Newton (1642-1727) acknowledged that he was led to his first discoveries in analysis by the *Arithmetica infinititorum* of Wallis, and his generalized conception of number is very close to that of Wallis. Wallis believed that arithmetic is independent of geometry and that geometrical concepts should be replaced by numerical ones wherever this is possible. Such arithmetization
scandalized the English philosopher Thomas Hobbes (1588-1679), as did the use of algebra in mathematics generally. Hobbes protested violently against "the whole herd of them who apply their algebra to geometry," and called Wallis' work "a scab of symbols." Hobbes adhered to the view that mathematics is an idealization of sense perception and not a branch of abstract formal logic. The influence of Hobbes may have helped to lead mathematicians away from a purely abstract view of mathematics. Both Newton and Leibniz (1646-1716) tried to explain their work in terms of notions based on the evidence of sense perceptions in relation to the generation of magnitude rather than upon the logical conception of number.

In discussing the epistemological orientation of Descartes, Newton, Leibniz, and most of their successors during the eighteenth century, E. T. Bell explains that, in the context of the analytic practice of the period, "intuitionism" refers to "an unreasoning faith in the validity of what the senses report . . . concerning motion and geometrical diagrams," and he adds that, in this sense of the word, "Newton was the greatest of the intuitionists."

Newton's first calculus was abstracted from intuitive ideas of motion, based upon the "infinitely short" path traced by a point in an "infinitely short" time. This path he called the "momentum." When the momentum was divided by the infinitely short time, the result was what Newton called the "fluxion."
1676, Newton changed this description of the calculus to one involving what he called "prime and ultimate ratios."¹¹

Through a combination of methods involving the concepts of the infinitely small and of instantaneous velocity with the use of algebraic methods applied to geometric configurations, Newton (and a little later, Leibniz) invented the calculus. From the point of view of epistemology, the application of algebraic methods immediately gave rise to an enormous increase in generality and abstractness. For example, thanks to these methods, Newton was able to classify all plane cubic curves into seventy-eight species, and to exhibit all of these except six, while only sixty-seven years before, geometers had been working laboriously with the conic sections and a few other curves by the methods of Apollonius, and had not begun to imagine the curves which Newton discovered.

E. T. Bell writes:

Now here at last was the universal solvent for all the intractabilities of classical geometry and astronomy . . . . Difficulties that would have baffled Archimedes were easily overcome by men not worthy to strew the sand in which he traced his diagrams. Leibniz did not exaggerate when (1691) he boasted that "My new calculus [and Newton's] . . . offers truths by a kind of analysis, and without any effort of the imagination—which often succeeds only by accident."

Among the highly significant implications of algebraic methods, let us mention three:

1. The rules of algebra may be applied to the equation of a curve, producing by purely algebraic means new equations which
express properties and relations of the curve in question, which
could not have been discovered otherwise.

2. The equation of a curve expresses in abstract form
a relation which holds between coordinates of every point on the
curve. The relation is expressed symbolically as a relation
between variables $x$ and $y$. The relation holds between $x$ and $y$
when these represent irrational numbers as well as when they
represent rational numbers. The possibility of working algebraic­
ically with relations between irrational numbers is opened up,
and the concept of continuity (which includes the concept of
continuous motion and change as well as the intuitive
characteristics of space) becomes accessible to algebraic treatment.

3. Since for each curve there is a definite equation, and
for each equation there is a definite curve, hitherto unknown
curves can be discovered by first discovering their equations.

Despite the immense power of the instrument created by
Newton and Leibniz, its foundations remained open to criticism.
The undefined concepts involved were to prove a source of
distraction, not only to mathematicians of the eighteenth century,
but also to at least one eighteenth-century non-mathematician.

Bishop George Berkeley (1685-1753) published a tract in 1734
under the title _The Analyst_, with subtitle "Or a Discourse
Addressed to an Infidel Mathematician. Wherein It Is Examined
Whether the Object, Principles, and Inferences of the Modern
Analysis Are More Distinctly Conceived, or More Evidently Deduced, than Religious Mysteries and Points of Faith." Berkeley wrote that there are first, second, third, etc., differences considered in the calculus in an infinite progression toward nothing. He continued:

That men who have been conversant only about clear points should with difficulty admit obscure ones might not seem altogether unaccountable. But he who can digest a second or third fluxion, a second or third difference need not, methinks, be squeamish about any point in divinity.

Berkeley denied the possibility of an "instantaneous velocity" in which distance and time increments have vanished. He expressed it thus:

And what are these fluxions? The velocities of evanescent increments. And what are these same evanescent increments? They are neither finite quantities, nor quantities infinitely small, nor yet nothing. May we not call them the ghosts of departed quantities?

Berkeley pointed out that in finding fluxions or ratios of differentials, mathematicians must first assume that increments are given to the variables, and then take the increments away by assuming them to be zero. Thus "by virtue of a twofold mistake you arrive, though not at science, yet at the truth."

According to Bell, Bishop Berkeley's was one of the ablest critiques by a non-mathematician in any period. As a philosopher, he turned the tables on mathematicians by convicting them of changing their hypothesis in the middle of an argument, since the analysts must first assume that there are increments, and then that there are no increments. Neither Berkeley nor his criticisms were taken seriously at the time; yet, in enumerating the net gains to
mathematics during the eighteenth century, Bell lists as one of the gains that "Berkeley disposed of fluxions and of prime and ultimate ratios." 17

In the context of mathematical analysis as it developed during the eighteenth century, "formalism" means the untrammelled manipulation of formulas to achieve new formulas (and new curves and new intuitions derived from them) without any attempt to develop a substratum of postulates and definitions as a basis for this reasoning process. Where Newton was the greatest of the intuitionists, the greatest of the formalists in this sense of the word was Leonhard Euler (1707-1783). It was Euler who developed the differential calculus and the method of fluxions into an independent branch of mathematics, the study of infinite processes, which has ever since been known as "analysis." Analytic treatment of trigonometric functions was first established by Euler. For example, the sine became henceforth a number or a ratio, the ordinate of a point, or the number defined by an infinite series, instead of being thought of as a line segment. Other mathematical relations were treated in the same way by Euler and became, under the title "Euler identities," familiar tools of mathematical analysis.

As a result of Euler's work, the name and idea of "function" became fundamental in mathematics. For Euler, a function simply meant a set of formal statements which could be manipulated and
transformed into one another by any device which would serve.

Bell writes:

If the end ever justifies the means in analysis, Euler was justified. He sought beautiful formulas, and he found them in overwhelming abundance. But obviously the calculus could not continue indefinitely on the primrose path so happily followed by this boldest and most successful formalist in history.  

Euler was not alone in relying on formalist methods in this manner. On the contrary, he was eminently representative of the era. Richard Courant says in What Is Mathematics? that during the seventeenth and eighteenth centuries the Greek ideal of precise reasoning simply disappeared.

In a veritable orgy of intuitive guesswork, of cogent reasoning interwoven with nonsensical mysticism, with a blind confidence in the superhuman power of the formal procedure, they [the mathematicians of the seventeenth and eighteenth centuries, the "new pioneers"] conquered a mathematical world of immense riches.

Concluding Remarks

Symbolic algebra was perfected by Descartes in 1637, but Descartes intended his algebraic method to be used in conjunction with geometric diagrams. It was John Wallis who first introduced the idea of arithmetizing the geometric continuum, and he accomplished this largely by a bold leap of imaginative thought, using indeterminate concepts of analogy between the properties of ordinary numbers and those of irrationals.

Through a combination of Wallis' methods with symbolic algebra, and with intuitive ideas of motion and diagrams, Newton and Leibniz invented the calculus. Subsequently, by unrestricted
manipulation of symbolic formulas, and by inventing the analytic concept of a function, their successors in the eighteenth century expanded the calculus without regard for logical rigor, thus creating an independent branch of mathematics dealing with infinite processes and called "analysis." Despite the fact that it had no rational foundation, the instrument thus created was far more powerful than any that had been known previously in mathematics.

Descartes based his mathematical reasoning upon a priori intuition without recourse to logic, and Kant's synthetic a priori developed out of Descartes' intuitionism. But Leibniz regarded mathematical truth as analytic, in the sense of being independent of intuition. He wished to establish mathematics upon a purely logical basis and to show that all mathematical truths are tautological.

As inventor of strict deductive reasoning in mathematics, Euclid had made use of postulational method, but his axiomatic system was nevertheless based upon a "postulate of reality," and he relied upon construction and intuition as guarantors of consistency. His methodology may be described as synthetic, in that its emphasis was upon interpretation of his formalism in terms of some independent reality.

During the nineteenth century, symbolic algebra was to be combined with a formalized version of Euclid's postulational method. The new technique was to lead to concentration upon abstract generalized relations at the expense of interest in the nature
and metaphysical status of such mathematical entities as points, lines, numbers, etc., which can be known through intuition. In modern method, a postulational method is constructed without regard to interpretation; the logical relations are established within the system. The reasoning within such a system depends solely upon its definitions and postulates, and its validity depends upon the internal consistency of its logical relations. This methodology gives a new meaning to the word "analytic."
NOTES

3Descartes, Philosophical Works, 2:38.
4Ibid., 1:180.
8Ibid., pp. 175-76, 178.
9Ibid., p. 178.
10Bell, Development of Mathematics, p. 262.
11Ibid., pp. 135-36.
12Ibid., p. 118.
14Ibid.
15Ibid.
16Bell, Development of Mathematics, p. 264.
17 Ibid., pp. 264, 268.
18 Ibid., p. 266.
CHAPTER VI
SYMBOLIC LOGIC AS AN EXACT SCIENCE

Introductory Remarks

A new attitude was in the process of formation during the first half of the nineteenth century. An increasingly formal approach to mathematics through the use of uninterpreted systems of symbols was rapidly coming into fashion. The new attitude is represented in the logico-mathematical writings of George Boole (1815-1864), who espoused a far more general view of mathematics than had been current. He wrote:

We might justly assign it as the definitive character of a true calculus, that it is a method resting upon the employment of Symbols, whose laws of combination are known and general, and whose results admit of a consistent interpretation.

This is a clear expression of the view that the essential character of mathematics is not so much its content as its form, and that if any topic is presented in such a way that it consists of symbols and precise rules of operation upon those symbols, subject only to the requirements of inner consistency, that topic is part of mathematics.

Concurrently with this new attitude towards mathematics, a new attitude toward logic was developing also, an attitude which was to culminate in what is now called "mathematical logic." In the
first stages of the history of logic, logical theorems had been derived from ordinary language. In the mathematical stage, a purely formal system is constructed, and an interpretation is only later looked for, whether in ordinary speech or elsewhere.

The new attitude toward logic is well represented in Boole's "logical calculus," which not only lent itself to mechanization but led to the construction of the first electronic computer at Harvard University in 1947.\(^2\)

The new understanding of logic and mathematics demanded a return to the use of Euclid's postulational method; not, however, to his informal or material axiomatics but to a purely formal version of it. To arrive at this stage, mathematics had to evolve through a number of preliminary stages.

**Developing the Logic of Relations**

With the development of symbolic logic, mathematicians and logicians perceived that there is a fundamental difference between properties (the predicates in a subject-predicate proposition) and relations. For example, such a relation as "less than" cannot be expressed in a subject-predicate proposition. But "less than" is a highly-important ordering relation in mathematics. Mathematics cannot be based upon subject-predicate logic, but when the logic of relations was finally developed, an intimate connection between number theory and logic could be perceived.
Augustus De Morgan (1806-1871) was one of the earliest investigators of the logic of relations. His contributions to logic are embodied in a book entitled *Formal Logic*, first published in 1847, and in a series of five essays contributed to the *Transactions of the Cambridge Philosophical Society* between 1847 and 1862. The collection includes De Morgan's most original contribution to logic, an essay entitled "On the Syllogism: IV, and On the Logic of Relations."

In this 1859 essay, De Morgan said that the forms of algebra, "having been born and educated in arithmetic, have left their parent and set up for themselves," developing into a calculus in which the symbols need not be magnitudes at all but may be simply directions how to operate. Algebra is distinguished from arithmetic by its gradual approach to the following theorem: Every pair of opposite relations is indistinguishable from every other pair in the instruments of operation required.

The example of algebra brings before the mind the idea of an abstract copula—a mode of joining two terms in logic by a formal symbol which carries no meaning but obeys the laws which are necessary to make inference follow. In those forms of thought which have no relation to magnitude, no such symbol had been created. The verb *is* was made to do the work of such a symbol. Furthermore, the conditions for the use of the copula in inference had never been defined.
De Morgan recognized that the relations expressed by "is" are a special type of relation which he called "onymatic." They are distinguished by the fact that they have all to do with naming and are all relations of whole and part. Such relations are one case, and one case only, of composition of relations. De Morgan showed the necessity for an analysis of the laws of thought connected with the notion of relation as such. 6

The supreme law of syllogism in three terms, the law which governs every possible case, and to which every variety of expression must be brought before inference can be made, is this: "Any relation of X to Y compounded with any relation of Y to Z gives a relation of X to Z."

Recognition of this general principle of inference enabled De Morgan to bring all theories of deduction under a single rubric. By agreeing that the word "syllogism" shall mean a composition of any two relations into one, we can open the field of logic, said De Morgan, and he concluded:

"It is to algebra that we must look for the most habitual use of logical forms. . . . so soon as the syllogism is considered under the aspect of combination of relations, it becomes clear that there is more of syllogism, and more of its variety, in algebra than in any other subject whatever, . . . And here the general idea of relation emerges, and for the first time in the history of knowledge, the notions of relation and relation of relation are symbolized."

On June 18, 1870, Charles S. Peirce sailed from New York, carrying with him copies of an article of his own entitled "Description of a Notation for the Logic of Relatives, Resulting from an Amplification of the Conceptions of Boole's Calculus of Logic." In London, he delivered a copy to De Morgan's residence.
But De Morgan was in a decline, and he died in March of the following year.

Peirce began the development of his system of notation by stating that certain principles of analogy must be formulated, in order that there may be a wider definition of old symbols when they are applied to new subjects. Accordingly, he developed general definitions of the algebraic signs. Peirce's definitions of the algebraic signs were intended to be free of dependence upon interpretation; at the same time, he had to make certain that they would allow the symbols to be interpreted as numbers, and that, when determinate numbers were substituted for the letters which are operated upon, the signs could still be interpreted according to their ordinary definitions and the equations would still hold good. "It is necessary that arithmetical algebra should be included under the notation as a special case of it," he wrote. This meant that the general conception of each sign must be such that it could accommodate the ordinary arithmetical conception and also the general logical interpretation.

In a paper of 1883, Peirce introduced several important innovations to his logic of relative terms. He invented signs for quantifiers—signs to indicate that logical addition or multiplication is to take place over all the objects of a certain range. To these signs he added relational signs with the use of subscripts, as in the compound sign "\(1_i j\)," which means that the
person "i" is a lover of the person "j." He thus created a more adequate symbolism for the logic of relations.

Peirce called his invention a General Algebra of Logic, but recent custom has reserved this title for the part of logical theory which can be presented without the use of quantifiers. This includes the logic of propositions studied by the Stoics and the calculus of classes or attributes formulated by Boole. The honor of being the first to conceive of a comprehensive logic must go to Frege rather than to Peirce, since Frege's *Begriffschrift* was published in 1879, but Peirce had never heard of Frege when he published his 1883 paper, and his system is identical in syntax with the system now in use.

The Epistemology of Structure in Modern Mathematics

"Mathematics," said Carl Friedrich Gauss (1777-1855) in 1831, "is concerned with the enumeration and comparison of relations." Bertrand Russell (1872-1970) later explained that given any relation, we can make a map of it, and that what is revealed by the map is the "structure" of the relation. The structure does not depend upon the particular terms that make up the field of the relation, so that two relations have the same structure when the same map will do for both.

If it is possible to establish a one-one correspondence between the postulates of two systems such that the correlated postulates have the same structure, these systems also are said
to have the same structure, that is, they are abstractly identical. It is then sufficient to develop the mathematics of one system in order to have the mathematics of all systems sharing the same structure. The postulates represent abstract symbolic formulations of laws and conditions, and if the logical consequences of the postulates are developed, the body of theorems so obtained will be valid for all the analogous interpretations of the structure.

By the application of structural reasoning, "objectivity" becomes equated with those aspects of a structure which remain invariant under every transformation, and abstract structure is recognized as the sole subject matter of both logic and mathematics. This generalization of the use of symbols is regarded as a leading characteristic of mathematics.

In logic, the object of complete symbolization is to get rid of ambiguities of ordinary language by providing an ideography in which signs represent ideas and relations between them without the intervention of words and to manage so that all logical conclusions are drawn by means of rules of transformation, like those of algebra.

Similar reasoning may be applied in science. Russell characterizes the nature of structural reasoning in science in these words:

Given some statement in a language of which we know the grammar and the syntax, but not the vocabulary, what are the possible meanings of such a statement, and what are the meanings of the unknown words that would make it true? The reason that this question is important is that it represents, much more nearly than might be supposed, the state of our knowledge of nature... We know much more
about the form of nature than about the matter. Accordingly, what we really know when we enunciate a law of nature is only that there is probably some interpretation of our terms which will make the law approximately true.

In physical science particularly, what matters is not the intrinsic nature of the terms but the logical nature of their interrelations. This means that the study of continuous processes in nature, i.e., processes involving motion, space, and time, must also be reduced to the study of logical relations.

The Epistemology of Structure:
Continuity and Infinity

Before the concept of continuity could be provided with a logic, it was necessary that continuity should be brought into strict relationship with discreteness, since only discreteness is subject to logical analysis. The first requirement was as adequate definition of "real number."

The concept of real number is very different from that of the series of integers known to the Greeks. Indeed, the modern concept of number in general is different, not only from that of the Greeks, but from that which was formerly held by all mathematicians. A positive integer such as +2 was once thought to be identical with the number 2, and it was thought that a fraction such as 2/1 might also be identified with the number 2. Again, an irrational number such as \(\sqrt{2}\) was supposed to find a place in the series of fractions, being greater than some of them, and less than others. But it has now been understood that +2 and -2
are relations, and so are ratios whose denominator is 1. These are not identical with integers. Similarly, numbers which can be greater or less that \( \sqrt{2} \) must not be identified as belonging with the series of fractions. Rather, we must acknowledge a new kind of number, called "real number." Rational real numbers correspond to fractions, but they are not the same as fractions. Irrational real numbers correspond to no fractions, and they cannot be distinctly known by intuition at all. They must be analyzed on the basis of linguistic concepts and logical procedures. Real numbers can only be understood as intellectual structures. These distinctions became clear in the late nineteenth century.\(^{15}\)

Galileo had illustrated the difference between continuity and discreteness by the difference between a liquid and a finely powdered dust, and both Galileo and Leibniz had tried to explain continuity as the perfect density of points in a line—that is, in terms of the proposition that between any two points in a line there is always another point. But the inadequacy of this explanation was made apparent by the consideration that the same claim can be made of the rational numbers, although these do not constitute a continuity.

In the early years of the nineteenth century, Bernhard Bolzano (1781-1848), a Czechoslovakian priest of whom Abraham Fraenkel says that he was the greatest logician between Leibniz and the development of modern logic,\(^{16}\) proposed that the
The continuum of real numbers can be correlated, one to one, with the infinite aggregation of points in a line. He went on to show that there are just as many points in a line segment one inch long as there are in a line segment two inches long, and that there are just as many real numbers between 0 and 1 as there are between 0 and 2. Bolzano called these relations "paradoxes of the infinite."\textsuperscript{17} Such paradoxical properties belong only to the infinite considered as an actual, completed, or absolute magnitude, but in nineteenth-century mathematics the infinite was, for the most part, still appearing as potential infinity.

J. W. R. Dedekind (1831-1916) became fascinated with the question: What is the nature of continuity? He answered this question through the observation that when a line is divided and its points thereby fall into two classes, such that every point in one class is to the left of every point in the other class, there must be then precisely one point, and one point only, which produces the division. This is not the case with the ordered system of fractions, and it marks a difference between continuity and that system.

Dedekind's discovery can be explained as follows: We can divide all fractions into two classes such that all the terms in one class are less than all the terms in the other. Suppose them to be divided into all fractions whose squares are less than 2, and all fractions whose squares are greater than 2. Between these two classes, where $\sqrt{2}$ ought to be, there is nothing.
There are indeed fractions which approach nearer and nearer to having their squares equal to 2, but there is no maximum to the fractions whose square is less than 2, and no minimum to those whose square is greater than 2.\textsuperscript{18}

Dedekind concluded that the property of compactness combined with the property of being divisible by exactly one point distinguishes the points in a line from the rational numbers. It must represent the essence of continuity and is the reason why the points in a line form a continuity, while the rational numbers do not. If one assumes as an axiom Bolzano's insight that the points in a line can be put into one-one correspondence with the real numbers, it becomes possible to say that for every division of the rational numbers into two classes, such that any number of the first is less than every number of the second, there is one and only one real number producing the Schnitt, or "Dedekind cut."

Later, Russell suggested a refinement of Dedekind's definition, according to which a real number is defined as a whole segment of rational numbers.\textsuperscript{19} Russell defined a real number as the lower class of the corresponding cut in the series of fractions. For example, $\sqrt{2}$ is defined as the class or the property of those fractions whose square is less than 2. The rational real number $1/3$ is defined as the class of all fractions smaller than itself. In other words, where Dedekind had called the cutting element the number, Russell took
one of the two classes created by the cut (the lower class) and called that class the number. Now an irrational real number is a segment of the series of ratios which has no boundary, while a rational real number is a segment of the series of ratios which does have a boundary.

On the basis of this definition, the entire arithmetic of the real numbers can be developed. It is to be noted that the rational real number 1/3, for example, comes after the infinite series of fractions which defines it, since it is defined by all fractions less than itself. In this definition, the essential character of a real number is not its magnitude but its position in the ordered series of real numbers. It is to be noted that the fractions less than 1/3 are infinite in number; therefore, 1/3, like all the real numbers, is the name of an infinite class of fractions.

Georg Cantor (1845-1918) had an unprecedented mathematical imagination that had been stimulated by his interest in philosophy, and by his becoming intrigued with the arguments of the medieval theologians concerning continuity and the infinite. Cantor designated infinite classes or infinite aggregates "sets," and he asserted that not all infinite sets are equal, just as not all ordinary numbers are equal. He showed that the rational fractions, as well as the set of all perfect squares and other infinite sets, do indeed have the same Mächtigkeit, or "power," but he showed by a reductio ad absurdum procedure that the like is not true of the set of all real numbers. These cannot be put into one-one
correspondence with the positive integers, i.e., they are not countable. Cantor expressed this by saying that the set of all real numbers is non-denumerable, and that it has a higher power than that of the positive integers.

Following up his discoveries, Cantor set out to generalize the number system, even beyond infinity. Treating the power of an infinite set as a "transfinite number," Cantor built up an entire transfinite arithmetic. He gave a name to the smallest transfinite "number," which is the power of the set of all positive integers, and of all other denumerable infinite sets. He gave a different name to the transfinite "number" which belongs to the continuum. He then proceeded to extrapolate to still higher transfinite numbers, to sets of sets of sets, sets of sets of sets of sets, etc. Cantor found ways to name the ordinal series of infinite numbers and also the cardinal transfinite numbers, and he worked out combinatorial rules for transfinite numbers.

Before Cantor's time, mathematicians had become accustomed to dealing with the infinitely large and the infinitely small as something in the process of becoming, as potentiality. But the sort of infinite that had been traditional in mathematics is quite different from the Cantorian infinite.

In the thirteenth century, Scholastic philosophers, becoming familiar with the work of Aristotle and particularly with his discussions in the Physics concerning the infinite, the infinitesimal, and continuity, had begun to study these topics and
to speculate upon their metaphysical significance. Petrus Hispanus (Pope John XXI) had recognized two kinds of infinity, a categorematic infinity in which all terms are actually realized, and a syncategorematic infinity in which the notion of potentiality is present. Thomas Bradwardine, the greatest English mathematician of the fourteenth century, had explained the difference by saying that the categorematic infinity is a quantity without end, whereas the syncategorematic infinity is a quantity which is not so great but that it can be made greater. 20

Between 1873 and 1897, Cantor introduced into philosophy and mathematics the theory of the actual (categorematic) infinity. The actual infinite is represented, for example, when the totality of numbers (1, 2, 3, . . .) or the totality of points in a line is regarded as complete, already existing, present all at once. The method of reasoning which depends upon the assumption of such an infinite totality, given in advance of the reasoning process, is peculiar to analysis and set theory.

It appeared that the actual infinite, which had been rejected by Aristotle, could be used to provide a logical (structural) foundation for arithmetic and analysis. Gottlob Frege (1848-1925) showed that the ordinary cardinal numbers can be defined as sets of finite sets. For example, two is the set of all couples, three the set of all triads, etc. In determining such sets, Frege made use of the relation of one-one correspondence between the elements of sets, because it involves an operation more
primitive than, and logically prior to, the operation of counting, and because it saves the definition from circularity. The number of the set of all such sets is the infinite number which Cantor had called "Aleph Null."

Concluding Remarks

Mathematics is distinguished from all other sciences by the fact that in mathematics the concept of infinity plays a central role. The concept of infinity is closely associated with the concept of continuity. In order to provide a rational foundation for the concept of continuity, the intuitive (geometric) notion of continuity had to be translated into a concept of discreteness. Such a concept could only be based upon the real numbers, which include both rational and irrational numbers.

In the nineteenth century, the real numbers were defined as infinite sets of rational numbers. This definition required a method of reasoning based upon the actual, rather than the merely potential, infinite. The concept and the theory of the actual infinite were introduced and developed by Georg Cantor. Through the subsequent efforts of Peano, Frege, Russell, and Whitehead, an expanded deductive system was created, based upon Stoic logic and the logic of relations, and including axioms of Cantor's set theory. By means of this expanded system, infinity and continuity could be defined as abstract logical structures.
The ancient Greek mathematicians had also invented methods for dealing with the problems of infinity and continuity. They had used synthetic metric geometry, *reductio ad absurdum* procedures, and a concept of infinity as potentiality to provide a logical basis for their intuitions of space as an unbroken continuity. By contrast, in modern mathematics, algebraic-analytic methods, based upon the concept of the actual infinite, were invented in order to extrapolate beyond the possibilities of intuition to generalized abstract structures, accessible only to pure reason.

First, analysis was arithmetized, i.e., the continuum was understood in terms of real numbers. Then, the real numbers were defined in terms of integers. Finally, numbers themselves were seen to disappear when the propositions which seemed to contain them were fully written out. Mathematics thus became, at least arguably, reducible to logic, though not to logic in its traditional form. As Russell said, all of this generalizing had produced "a set of new deductive systems, in which traditional arithmetic is at once dissolved and enlarged," and in which traditional logic was also dissolved and enlarged, as in the work of Frege and in *Principia Mathematica*.

In *Principia Mathematica*, the calculus of relations was expanded so that ordinary arithmetic, the theory of measurement, and Cantor's set theory could all alike be treated from the standpoint of abstract relations, or structure. Thus, by
a transformation of traditional (Aristotelian) logic into the form of Stoic logic, by generalizing this form to that of a function, and by the use of set theory and the logic of relations, arithmetic and analysis were finally reduced to a formal theory of abstract relations. This rationalizing and formalizing of arithmetic and analysis was achieved through a long evolution, at the end of which geometrical intuitions of space were replaced by analytic definitions of the number concept.

This history affords an example of how mathematical concepts are formed by emancipation of ideas from the sense data and primary intuitions out of which these ideas have been born. It illustrates the movement away from the synthetic, intuitive spirit and toward the analytic spirit that is typical of modern mathematics and modern science, and of much of modern philosophy as well.

For example, it was Russell's belief that a great deal of speculation in traditional philosophy might have been avoided if the importance of structure had been realized. He remarks, in connection with the Kantian philosophy, that every proposition having a communicable significance must be true either of both the phenomenal world and its objective counterpart, or of neither, so that the only difference between them must lie in "just that essence of individuality which always eludes words and baffles
description, but which for that very reason is irrelevant to science."²⁴
NOTES


3 De Morgan, On the Syllogism, p. 50.

4 Ibid., p. 23.

5 Ibid., pp. 50-51.

6 Ibid., pp. 81, 96, 213-15.

7 Ibid., p. 228.

8 Ibid., p. 241.


10 Kneale and Kneale, The Development of Logic, pp. 432-34.

11 Ibid., p. 432.


14 Ibid., p. 55.

15 Ibid., pp. 63-65, 71-72.


24 Ibid., p. 61.
CHAPTER VII

EPISTEMOLOGICAL PLATONISM IN
MATHEMATICS AND SCIENCE

The complete formalization of mathematics was subject to criticism from its beginnings. For example, in 1841 A. A. Cournot protested that some concepts of the understanding exist independently of the definitions which may be given them, and he argued that such concepts should not be subjected to logical definition. J. J. Sylvester (1814-1897) also objected to formalizing tendencies, on the ground that the objective of pure mathematics is an "unfolding of the laws of human intelligence," and he recommended "continually renewed introspection of that inner world of thought of which the phenomena are as varied and require as close attention as those of the outer physical world."2

At the second international congress of mathematicians, held in Paris in 1900, Henri Poincaré (1854-1912) delivered a paper in which he compared the roles of logic and intuition in mathematics. After this time, mathematicians and philosophers interested in the foundations of mathematics became divided more or less into three camps: the logicists, headed by Russell, the formalists, headed by Hilbert, and the intuitionists, headed in a loose manner by Poincaré.
at first, and later headed in a much more definite way by L. E. J. Brouwer (1881-1966).

Controversies arose from disagreements as to the relationships among mathematics and logic, epistemology and psychology. Mathematicians and philosophers have approached the question from two points of view. From one point of view, the principles and procedures developed in connection with mathematical practices point the way to the development of epistemological theories, and the adequacy of the latter must be measured by their power to integrate the former. From another point of view, epistemology takes precedence over mathematics and should be used as a critical tool for evaluating the legitimacy of mathematical principles and procedures.

There are those who subscribe to the point of view that logic (hence mathematics) takes precedence over psychology and epistemology. But there are also those who subscribe to the point of view that psychology and epistemology (hence mathematics) take precedence over logic. The first group may be called the platonists (or the realists), and the second group the intuitionists. Paul Bernays speaks of "a tendency to view the objects and relations of a theory as cut off from all links with the reflecting subject." This tendency he calls "platonism" because it is prominent in the philosophy of Plato. By emphasizing the lack of connecting links between the objects and relations of a theory and the reflecting and acting subject, Bernays gives to the word
"platonism" an epistemological, rather than a merely ontological, content. I shall speak of the tendency he describes as "epistemological platonism." Of intuitionism, on the other hand, Bernays says that its characteristic general feature is not that of being founded on pure intuition, as is generally supposed, but that of being founded on "the relation of the reflecting and acting subject to the whole development of science." 4

Speaking loosely, those who subscribed to epistemological platonism were the followers of Russell and Hilbert, and those who subscribed to intuitionism were the followers of Poincaré and Brouwer. The lines may become blurred, however, and a given author may not fit into one school only. For instance, Hilbert shared some views with the intuitionists, although these were of minimal importance in the development of his formalism, and Russell sometimes used language which suggests a conceptualist, rather than a platonist, point of view. It also appears that the philosophical motivations for subscribing to one point of view or the other may vary.

Logicism was founded by Frege, although its chief proponent was Russell. The view that Frege's strong anti-psychologism was connected with a protest against German idealism, and was a reaction similar to the reaction of Moore and Russell to British idealism thirty years later, may not reflect the actual situation. Already by the time of Hegel's death in 1831, idealism had ceased to be a power in German thought, and had been supplanted by a new ideology, scientific naturalism. Scientific naturalism gave birth
to physiological psychologism, and "psychologism" became identified with the thesis that the laws of logic and mathematics are simply empirical, natural laws.\(^5\)

Frege used the determinacy and invariability of the laws of pure thought as his criteria for distinguishing between the logical and the psychological, saying:

All these phases of consciousness [i.e., sensations, mental pictures, and the like "psychological" phenomena] are characteristically fluctuating and indefinite, in strong contrast to the definiteness and fixity of the concepts of mathematics.

Since Kant is credited with an absolutist view of logic and mathematics equal to Frege's own, it is reasonable to suppose that Frege had in mind empirical or physiological psychologism, the attempt to give a naturalistic foundation to logic and mathematics, rather than Kant's transcendental psychologism.

For Russell, however, the notion of psychologism was more closely associated with idealism, and his main interest lay in achieving a final refutation of British idealism. Russell claimed that the object of judgment can form no part of the mental presentation at all, not even with respect to such imaginary objects as a "golden mountain."\(^7\)

The definition of the realist position as formulated by Russell states that "every presentation and every belief must have an object other than itself, and extra mental," and that "truth and falsehood apply not to beliefs, but to their objects." This position admittedly created a difficulty in the matter of false
propositions. At the time, Russell inclined toward the view that some propositions are true and some false, just as some roses are red and some white. Belief, then, is simply a certain attitude toward propositions. It is called "knowledge" when the propositions are true, and "error" when they are false. Of course, this leaves our preference for truth "a mere unaccountable prejudice." At first, Russell tried to find a way out of this difficulty by speculating that our preference for truth is based upon an ethical ground. 8

The central tenets of logicism may be briefly summed up as follows: Mathematics is a purely formal, hypothetico-deductive system, which is reducible to logic. Logic and mathematics have no substantial or empirical subject matter; they are independent of the actual world and have nothing to do with individuals. At the same time, this science is not an arbitrary, purely postulational one, for its ultimate premises are the truths of reason—the rules of deduction themselves. The truth of these premises is independent of the existence of the universe, and they must be unconditionally true, otherwise the system of deduction would be invalid. 9

Since the only truths of mathematics are logical truths, and these must be unconditionally true, Russell claimed that the possibility of mathematical knowledge refutes both traditional empiricism and Kantian idealism, since it shows us on the one hand that human knowledge is not wholly deduced from facts of sense, and on the other hand that a priori knowledge cannot be explained in a subjective,
psychological manner. Russell argued that the ultimate premises upon which both logic and mathematics are founded cannot be derived from sense experience only, since they allow us to make assertions, not only about cases which we have been able to observe, but about all actual or possible cases. On the other hand, these general and a priori truths must have the same objectivity and the same independence of the mind that the particular facts of the physical world possess. For if the truths of logic expressed only psychological facts, we could not use them legitimately to deduce a fact from other facts.10

Later, under the influence of Wittgenstein, Russell's realism, as here expressed, was supplanted by the belief that operations of logic concern only rules for manipulating expressions. This did not mean that the logically true statement was in any way contingent; it meant that logical and hence mathematical truths were recognized as analytic, or linguistic.

An analytic, or logically necessary, proposition is one which is true in virtue of its form alone. Such propositions are "true" in a different sense than that in which factual propositions are true. "All mathematical proof consists merely in saying in other words part or the whole of what is said in the premises," Russell wrote.11 Carl Hempel elaborated: Since mathematical as well as logical reasoning is a conceptual technique of making explicit what is implicitly contained in a set of premises, the system of mathematics is "a vast and ingenious conceptual structure without
empirical content and yet an indispensable and powerful theoretical instrument for the scientific understanding and mastery of the world of our experience.\textsuperscript{12}

The system of mathematics referred to by Russell and Hempel of course includes set theory, which is based on the concept of the actual infinite. Despite its apparent compatibility with reason, the conceptual structure based on the idea of the actual infinite displays, from the point of view of the reflecting subject, a high degree of what might be called "uncanniness," so much so that even Cantor wrote at one point, "I see it, but I don't believe it." As one example, consider \textit{consecutiveness}, a quality exhibited by ordinary progressions. Most infinite series are said not to have this quality, for it is stipulated in the definition of continuity that the infinite series is compact, that is, that between any two members of such a series there are always other members. Since these infinite intermediaries are thought of as actually in existence, not merely as potential, no two members of such a series are consecutive.

The set of points in any line segment can be placed in one-one correspondence with the set of points in an infinitely extended line, or with the set of points in any area, in any volume, or in all of three-dimensional space. Thus, in relation to infinite sets, extension begins to lose its meaning altogether. There are infinite sets, for instance the set of all sets of sets of real numbers, which cannot even be put into one-one correspondence with all the
points in space. Such sets could not be identical with the output of any physical or "mechanical" process, even if one could accept the notion of such a process being an infinite one. No physical structure could serve as a model for such sets.

In set theory, the discourse is about infinite sets or (considered as logic) about infinite classes. In pure logic and pure mathematics, no elements of sets except other sets need ever be considered, since the propositions do not concern individuals at all. The elements of such collections could not be known to us except by definition. This means that a property is named which is the defining characteristic of every one of the infinite elements of the set, common to them and to no others; and it is supposed that, presented with any element from the universe of discourse, we will be able to say whether it does or does not belong to the set in question. But set theory also demands (as in the axiom of choice, or multiplicative axiom) the use of arbitrary collections whose members are selected without reference to a defining characteristic. The uncanniness (my choice of word) in the idea of such an arbitrary collection is described by Benacerraf and Putnam:

If we say that . . . the members of a "collection" need not be proximate in space and time, need not be "similar" in any respect, and so on, then we are left with the notion of something like a random listing. . . . And if we say that the members of a "collection" (a) need not be objects, numbers, and so on but may themselves be "collections," and (b) need not even be capable of being listed (or for that matter, named in language), even by a random device working through an infinity of time, then what notion are we supposed to form at all?"
It is therefore not surprising that questions were raised both with regard to the inclusion of set theory as part of logic, and with regard to the claim that set theory, as well as elementary logic, is analytic. Such doubts were reinforced by the discovery of the antinomies or paradoxes of set theory.

In Frege's (and Russell's) definition of cardinal number, it is declared that numbers do not apply to individuals at all, but only to concepts, and the same is true of "existence," at least when used in its strictly logical sense. But the development of number theory and set theory demanded consideration, not merely of concepts, but of their extensions—i.e., of classes of things to which concepts apply, or to which properties belong.

In 1889, Cantor had proved that there can be no greatest cardinal number. An infinite set is an infinite number; by Cantor's proof, no matter how great such a number may be, the number of its subsets will be greater still. Hence there can be no set of all sets, no greatest cardinal. That this embodies a contradiction when applied to classes or extensions of properties is shown by the fact that there are certain logical properties, such as "\(x = x\)," which belong to all entities. The cardinal number of entities representing the extension of such a property must be the greatest of cardinal numbers. But there is no such cardinal number. Hence "Cantor's paradox."

Paul Halmos explains this antinomy as showing that "Whatever \(A\) may be, there exists something (call it \(B\)) that does not belong to
A, or in other words, that "nothing contains everything," or that "there is no universe." Here "universe" may be taken to mean universe of discourse," that is, the set that contains all the objects entering into a given discussion. The paradox shows that no logical theory and no language can be formulated which will apply with logical consistency to all the objects referred to in the theory or the language.

In 1901, Russell attempted to apply Cantor's argument that there is no greatest cardinal to "the class of all the things there are." In so doing, he discovered the well-known paradox present in the notion of "the class of all classes which are not members of themselves." This property does not define a class, said Russell, for if it did, and if this class were not a member of itself, then it would be a member of itself, and vice versa.

Other paradoxes had been discovered before Russell's. Ten or twelve paradoxes, several of which are variations of the others, have been discussed in the literature. These paradoxes have since been divided into the logical or mathematical paradoxes, and the linguistic or semantical paradoxes. Russell's is of the first type. The second type is illustrated by the case of someone stating "I am lying," a statement which cannot consistently be judged to be either true or false.

In the course of thinking about ordinary events in the realm of the finite, the antinomies need pose no problem. But at the high levels of abstraction and generalization which are involved
in set theory, it had been discovered that application of apparently correct logical procedures lead one into self-contradiction. Gödel remarked that Russell had brought to light "the amazing fact that our logical intuitions (i.e., intuitions concerning such notions as truth, concept, being, class, etc.) are self-contradictory."

Russell devised a number of different solutions to the problem of the paradoxes. Each of them led to difficulties, either because they did not succeed in avoiding the paradoxes, or because they demanded a serious curtailment of the forms of reasoning used in analysis and set theory. The most successful and lasting of the ways devised by Russell was his "Theory of Logical Types." In its "simple" form this theory aimed to guarantee that we cannot meaningfully say of any property either that it belongs to itself or that it does not. The simple theory of types has been accepted by most proponents of modern logic as legitimate and necessary.

But the acceptance of type theory also created a new problem in the derivation of mathematics from logic. This derivation required that there be an infinite number of objects in the universe of discourse, in order that the inductive numbers (for example) may form an infinite progression, in which no number will be equal to its successor. On the basis of Cantor's "naive" set theory, it was relatively easy to "prove" the existence of an infinite number of objects in the universe. Individuals, classes of individuals, classes of classes of individuals, etc., could be added together, that is, these entities could be accumulated, so that, however few
the individuals were to begin with, one had only to proceed up the logical hierarchy as far as desired in order to reach a stage where there were more than any given number of objects.

As it turned out, the proof in naive set theory is fallacious, since the logical theory of types forbids the lumping together of objects of different levels of abstraction (a "confusion of types" being that which led to the paradoxes). This meant that the authors of *Principia Mathematica* had no recourse but to guarantee the existence of an infinite number of objects by means of an axiom—the axiom of infinity.

The axiom of infinity is an "existential" axiom, not a logical axiom. The characteristic of logical propositions, which some have called *tautology*, does not belong to the assertion that there are a number (any number) of individuals in the world.

Russell reasoned that logic and mathematics deal only with possible, not with actual existence, and he took it for granted that the existence of an infinite number of objects is at least possible, that is, that it is a non-contradictory assumption.

It cannot be said to be certain that there are in fact any infinite collections in the world. . . . At the same time, there is certainly no logical reason against infinite collections, and we are therefore justified, in logic, in investigating the hypothesis that there are such collections.

While Russell accepted the non-contradictoriness of an infinity of objects as self-evident, Hilbert objected. His objection arose out of concern for the logical consistency of infinitistic reasoning, particularly after the discovery of the paradoxes. He demanded
proof that infinitistic reasoning is self-consistent, proof, moreover, by purely "finitistic" methods.

Hilbert admired set theory and emphasized structure and formal reasoning just as much as the logicists did, but he chose to regard the actual infinite as a mere concept of reason, albeit one that is expedient, perhaps necessary, for our thinking. Hilbert became the leading exponent of the "axiomatic" school, according to which classical mathematics is simply a combinatorial game, played with the primitive symbols. Any such axiomatic system demands a proof of self-consistency, but Hilbert believed that through the use of his proof theory the needed consistency proof for classical mathematics could be found. However, it was demonstrated by Kurt Gödel that arithmetic necessarily transcends any given formalism, and so does axiomatic set theory. For within a rigidly logical system, propositions can always be formulated that are undecidable within the axioms of the system. Within the system, there necessarily exist certain clear statements that can be neither proved nor disproved. Hence one cannot be certain that the axioms of arithmetic will not lead to contradictions. 17

Gödel's discovery appears to foredoom hope of mathematical certitude. The mathematical sciences, however, continue to develop according to the ideas of Cantor, Dedekind, et al. Mathematicians, it seems, can proceed quite well without much concern about the "crisis in the foundations." But from the philosophical point of view, it is otherwise. Many philosophers
have based their theory of knowledge to a large extent upon mathematics, and mathematical logic (including set theory) has come to be regarded by many, not only as dealing with a body of knowledge that invalidates much traditional philosophy, but even as a science prior to all other sciences, as philosophy itself was once regarded. The antinomies of set theory therefore created a serious problem, not for mathematics, but for logic and for epistemological platonism.

There seems to have been from the start an intimate connection between epistemological platonism and the methodology in which, by a regression from the immediate object of thought or contemplation, one moves upward or backward, as it were, to abstract reality, to "the True," or to whatever may be conceived as its equivalent. This tendency is marked in the philosophy of Plato and also in the methodology of modern mathematics.

In mathematical philosophy, one takes the ordinary truths of arithmetic as the initial data, and reasons analytically from this data to more remote but far more general and essential premises. This is analogous to the method by which scientists, starting with the data of experience, arrive at general laws of nature. Gödel quotes Russell as saying that "logic is concerned with the real world just as truly as zoology, though with its more abstract and general features," but he notes that Russell's realism was "stronger in theory than in practice."
We have no direct perception of the law of gravity, or other "laws of nature." Russell pointed out that it is only the initial data which can be expected to have the quality of obviousness, not the laws or general premises which are arrived at analytically. Moreover, even the obviousness of the initial data is only relative, it is a matter of degree. The validation of the ultimate premises, however, consists in this: that the original, relatively evident data can be inferred from these abstract premises.19

It is notable that in this methodology, the deduction of the original data from the more remote premises usually (if not always) results in a radical alteration of the original data themselves. For example, in analysis and geometry the continuum, intuitively understood, was at first presumed as a basis for reasoning, but in set theory, the continuum appeared only as a special case of a more general conception. The continuum of analysis and geometry played the role which sense perception plays in science, while set theory represented the abstract formulation necessary for the development of an explanatory theory. The characteristics of space, time, and motion could then be derived from set theory; still, it was never the intuitive characteristics of space, time, and motion which were deduced but only their abstract mathematical properties. As a result the intuitive characteristics were set aside as illusory, or at least as falling outside the province of scientific thought. The same occurred with respect to numbers. That which had been
intuitively understood since the time of Pythagoras as "number" came to be recognized as a mere façon de parler, while certain logical definitions and propositions which belonged to the explanatory theory supplanted it.

The concept of substance, upon which Aristotle built his subject-predicate logic, underwent a fate similar to that of other intuitive concepts. Aristotle's subject-predicate logic may apply in a limited sense where the conditions of human life prevail, but mathematical logic is useful (or at least "true") under all conditions, and can presumably be used to arrive at the general abstract structures underlying those conditions.

Concluding Remarks

In epistemological platonism, the mental processes of individuals are replaced by intersubjective processes of a group of philosophers, scientists, and technicians. These are embodied (possibly they are created, but at least they are properly ordered) in a technical language, the terms of which can be precisely defined, the syntax of which can be precisely prescribed, and which can, at least theoretically, be understood in exactly the same way by all. In such a language, words are often replaced by symbols, as in mathematical logic.

Ordinary language, the human eye, and the concept of substance are all alike intimately connected with the complex mental life of the individual subject. The mental life of the individual is well
served (for example) by the eye, as it could not be served by the microscope. But the subject's mental life is characterized as subjective and idiosyncratic, because not invariant and not precisely communicable; hence it is irrelevant to science, and not only to science, but to philosophy as well. In an argument against Strawson's objections to the use of technical language in philosophy, Russell wrote:

Everybody admits that physics and chemistry and medicine each require a language which is not that of everyday life. I fail to see why philosophy, alone, should be forbidden to make a similar approach to precision and accuracy. . . . For technical purposes, technical languages differing from those of daily life are indispensable.

The roots of epistemological platonism are found in mathematical principles and practices; yet radical objections to it did arise within the field of mathematics itself. These objections, and the philosophy upon which they were founded, will be discussed in the following chapter.
NOTES

4 Ibid., pp. 266-67.
8 Ibid., pp. 75-76.


CHAPTER VIII

INTUITIONISM AS A PHILOSOPHICAL ALTERNATIVE

Intuitionists' Critique of Logic in Relation to Mathematics

In an opening lecture at Cambridge University, Brouwer gave an account of the historic interaction between logicism, formalism, and intuitionism. He recalled that the old formalist school rejected altogether any mathematical elements extraneous to language, in order to establish a rigorous treatment of mathematics and logic. Subsequently, Hilbert established the new formalist school. The new formalism, unlike the old formalism, acknowledged as primordial the intuition of the natural numbers and of the process of complete induction; still, only a very small part of mathematics was recognized as independent of language. But the fundamental thesis of intuitionism is that mathematics is a method of dealing with human experience, a method in which construction constitutes the only proof. By this view, mathematics is prior to both language and logic.¹

Because man is inclined to take a mathematical view of everything, said Brouwer, he has applied this bias to language also, and particularly to the language which expresses mathematical activity. He projects upon words a new mathematical structure,
and this new structure is explained by what is called "the theory
of the syllogism." Classical logic studies the linguistic
counterpart of mathematical reasoning, not mathematical reasoning
itself. The logic of relations studies the linguistic counterpart
of mathematics in general. Any logical system therefore needs
the basic mathematical intuition just as much as mathematics itself
needs it. The idea that mathematical structures can be arrived
at by purely linguistic means, in the absence of construction by
direct intuition, is a mistaken one from the intuitionists'
point of view.

For the intuitionist, the problem of logical consistency
does not arise, since (as with the Greeks) any statement about
a potential infinite can be interpreted as a statement about
a finite extendable structure. In this conception, contradictions
arise only in the linguistic structures which accompany mathematical
activity, and they arise at the point where a decision cannot be
made because the construction cannot be carried further. With
respect to Russell's paradox, Brouwer said that it is evident
to common sense at which point the reasoning "ceases to be alive
and consequently is no longer reliable." The conclusion is that
logical reasonings can be secure only when they are connected with
mathematical systems which have been previously constructed. The
contradictions in logistics must be explained by the lack of such
a construction.
May one neglect the mathematical system under construction in favor of an accompanying linguistic structure, with the confidence that when one returns to the corresponding mathematical construction the application of the Aristotelian laws of thought will prove justified? With respect to one Aristotelian principle, the intuitionists' answer is "no."

By the principle of the excluded middle, it must be true that $p$ or not-$p$, or every proposition is either true or false. In some procedures of analysis and set theory, mathematical objects such as numbers, functions, sets, etc., are shown to exist by means of argumentation based upon the principle of the excluded middle. Intuitionists deny such procedures any binding power; they do not accept the existence of objects which have not been secured constructively. They do accept the principle of the excluded middle in relation to statements about a finite number of objects, each of which can be exhibited individually. In such a case, one may substitute for the general statement the conjunction of a finite number of particular statements $(p_1$ and $p_2$ and ... $p_n)$. In negating such a statement, at least one counter-instance can be constructed. But if the variable $x$ of the general statement ranges over an infinite aggregate, then the statement is no longer logically equivalent to a finite conjunction of particular statements.\(^4\)

Abraham Fraenkel uses the following to illustrate the intuitionistic approach: Take the question $2^{16} + 1$ being
a prime number, does there exist a greater prime number of the form $2^m + 1$, with $m$ greater than 16? A first possible answer would be the discovery of a definite number $m$ greater than 16, for which $2^m + 1$ is prime. If true, this could be demonstrated in a finite number of steps. A second possibility would be a general proof that $2^m + 1$ is a composite number, therefore not a prime, for every $m$ greater than 16. But it may happen that neither has a suitable $m$ been discovered nor has a proof of the non-existence of such an $m$ been obtained.\(^5\)

By the principle of the excluded middle, there must be an objective answer to the above question, whether or not we know what the answer is. But intuitionists allow for the possibility that there is as yet no answer to the question and there may never be one. On the other hand, tomorrow we may discover a suitable $m$ or succeed in finding a general proof that there can be none.

Intuitionists regard as a regular feature of mathematical questions the trifurcation into three possibilities: (1) The solution of a problem by a general proof, (2) its solution by a constructive counter-instance, or (3) lack of any solution at all. The third case has a different character than the other two, and there is no resemblance to a three-valued logic.\(^6\)

For a platonist, a mathematical statement is rendered true or false by a reality which lies outside time. For an intuitionist, a mathematical statement is rendered true or false
by a construction. Constructions take place in time, but intuitionists do not regard mathematical statements as having tense. Intuitionistically, mathematical statements may lack any truth value at all. While they can acquire truth value through an operation occurring in time, the truth value, once it is acquired, is tenseless. Thus, the principle of the excluded middle is not replaced by the introduction of a third case. The "third case" is not coordinate with the other two because it depends upon subjective and provisional circumstances, whereas the other two are objective and permanent. That intuitionists do not force the third possibility out of the mathematical picture has roots which lie deep in the heart of their philosophy.

In 1930, Brouwer's pupil, Arend Heyting, undertook to present intuitionistic logic in symbolic language. Because mathematical activity is regarded by intuitionists as always incomplete, they did not accept Heyting's system as orthodox. But the formalization nevertheless enabled mathematicians and logicians to make a comparison between traditional logic and classical mathematics and their intuitionistic counterparts.

The principle of the excluded middle, as the foundation for indirect proof, has always been regarded as of great importance for mathematics. Mathematicians generally incline to the view that there is an "objective" state of affairs, which implies that any statement must be true or false. But for intuitionists the basis for the principle of the excluded middle...
is not a priori, as for a logical axiom, but is merely empirical. They do not question its use for finite domains, but they deny the legitimacy of applying it where infinite aggregates are concerned.

Intuitionists' Critique of Classical Mathematics, Analysis, and Set Theory

The groundwork of all intuitionistic mathematics is laid in the generation of the natural numbers through a process known as "mathematical induction." It will be remembered that Plato perceived a primordial intuition at work in the repeated bisection of the unit. The intuitionistic idea is that we have a primordial intuition of positive number, and of the possibility of construction through a repeated mental operation, the operation of adding one. Such a process can be continued forever.

What intuitionists accept is the law of constructing the integers; they do not accept the presupposition of all integers existing as an aggregate. In other words, in intuitionistic mathematics all infinity is potential, there is no complete infinite.

From the formal-axiomatic point of view, "constructive" definitions are merely ways to pick out objects which already exist independently of, and prior to, the construction. Against this view, intuitionists argue that no single object which satisfies the definition of "inductive number" can be exhibited without the use of mathematical induction to construct it. For them,
Mathematical induction is the prototype of all mathematical construction.

Intuitionists extend their criticisms to many of the principles and procedures commonly used in analysis, geometry, and set theory. They claim that many dangers have arisen from the platonistic assumption that all infinite structures may be treated as if complete and surveyable in their totality.

In analysis, an assumption is made that an infinite totality of real numbers exists, and the law of the excluded middle is applied almost continually in relation to this totality. Impredicative definitions also assume the existence of such a totality, in terms of which a member of the totality may be defined. For example, an impredicative definition of a real number may appeal to the hypothesis that "all real numbers have a certain property." Such a hypothesis depends upon the assumption that real numbers already exist in a totality before they have been constructed. Impredicative definitions are common in analysis and are used in fundamental proofs.

Constructivist mathematicians generally have refused to admit the mathematical existence of infinite sequences which are not determined in advance by an effective rule for computing their terms. They admit that the continuum cannot be exhaustively reflected by means of constructions, but they believe that one must be satisfied with extracting "atomic drops" from the continuous "pulp." Brouwer, however, introduced a novel idea into
constructivist mathematics. In addition to the infinite sequences determined in advance by an effective rule for computing their terms, Brouwer admitted sequences in whose generation free selection plays a part. Such sequences are not absolutely unrestricted, but a partial restriction is imposed at the outset and again at any stage in the process of generating the sequence. An infinite sequence in the development of which selection of this kind is permitted is known as a free-choice sequence.¹⁰

The generation of a free-choice sequence obviously requires active participation on the part of a reflecting subject. One crucial difference between free-choice sequences and arbitrary infinite sequences is that the free-choice sequence is always taken as in the process of being constructed; it is never regarded as wholly determined in advance. Further, the free-choice sequence does not endeavor to yield a single point but only to produce a partial aggregate which is itself continuous.

The conception of the continuum as an aggregate of existing points (members), which is at the bottom of nineteenth-century analysis and of Cantor's set theory, is replaced by an aggregate of parts which are partially overlapping and each of which is continuous itself. Instead of the member-ship relation the part-whole relation becomes fundamental.

The conception of the continuum as overlapping parts which are themselves continuous is congruent with the ideas of Aristotle and of the ancient Greek mathematicians.

In Cantor's set theory, platonistic assumptions extend even beyond those of the theory of real numbers which make up the
continuum. Cantor developed his theory through the iterative use of a "quasi-combinatorial" concept, based on an analogy of the infinite to the finite. But an infinite cardinal number is a number which does not possess all the properties of an inductively constructed number. For example, such a number is unchanged by adding or subtracting, or by doubling or halving, or by any of a number of other operations which we ordinarily think of as making a number larger or smaller. The ordinary laws of arithmetic, however self-evident they may seem, cannot be taken for granted with respect to infinite numbers. Intuitionists believe that the same holds true with respect to general logical principles. They believe that real number theory and set theory, as developed during the nineteenth century, represent an elaboration on the basis of an erroneous tendency, namely, the tendency to use methods of reasoning in dealing with the infinite which have been developed within the domain of the finite. A notorious example, from their point of view, is to be found in an important axiom of set theory which has been the subject of much debate, the axiom of choice.

The need for the axiom of choice, also called the multiplicative axiom, became apparent in connection with attempts to multiply an infinite number of factors. One of the points at issue is that of making arbitrary selections as against making selections of elements on the basis of a rule or law. In attempting to match the elements of one set with that of
another without a rule, one is faced with infinitely many arbitrary choices and a matching operation which will never end. It is important to remember that in a multiplication problem involving an infinite number of factors, there are no visual or spatial relations to assist in the selections.

Up to the end of the nineteenth century, it was not permitted to make infinitely many arbitrary choices undetermined by a definite law, because it was feared that such procedures would end in meaningless statements. It turns out, however, that some important problems in analysis and set theory cannot be solved without the use of an infinity of arbitrary choices. Therefore the assumption was made explicit that such choices are permissible. This assumption had been made implicitly, without question and unconsciously, by Cantor and most of his successors. At the beginning of the twentieth century, it was realized that a new principle of logic and mathematics, i.e., a new axiom, must be recognized. The axiom of choice asserted that one may choose infinitely many elements and may choose them simultaneously, with each choice being independent of all the others. There are various ways of stating the axiom. Here is one existential way:

Given any class of mutually exclusive classes, of which none is null, there is at least one class which has exactly one term in common with each of the given classes.

The situation is thought of as existential, that is, all sets are thought of as existing simultaneously, including the set of
representative sets which constitutes what is known as the "multiplicative class."

The axiom of choice is equivalent to the assumption that infinite cardinal numbers are multipliable. It is also equivalent to the assumption that every class can be well-ordered, and that of two cardinals which are not equal, one must be the greater; in other words, the axiom of choice is based upon the assumption that infinite cardinal numbers are comparable entities.

Numerous important propositions of set theory depend upon the axiom of choice, for example, proof of the connection of addition and multiplication for infinite cardinals, proof of important parts of Cantor's theory of ordinal numbers, proof that a given class has Aleph Null terms, etc. One important consequence of the axiom of choice is that transfinite induction can be applied to any set. Upon the axiom of choice depends much of the applicability of set theory in mathematics. But as to the truth or falsity of this axiom, nothing seems to be known.

Since its explicit formulation, the axiom of choice has been attacked by mathematicians other than the intuitionists, because it has such far-reaching consequences, and because it leads to proofs which have no constructive elements, that is to say, no definable sequence of operations exists by means of which one could, even in principle, tell whether a statement based on the axiom is true or false by means of exhaustively checking all cases. From what has already been said of intuitionists, it is obvious that they
reject this axiom, as well as other axioms and principles of set theory, on the ground that what is represented is an unwarranted extension of reasoning methods which have been developed within the domain of the finite.

In Brouwer's version of set theory, set elements are given as sequences and are not considered to be given in their entirety but only by stages. The process of construction may be continued indefinitely, either through new choices freely made, or under restriction by some rule. With respect to the infinite regarded as a totality and with respect to the continuum regarded as a system of individualized points, Brouwer asserts that they have no legitimate mathematical existence.

Fraenkel summarizes\textsuperscript{16} the consequences of Brouwer's position for classical mathematics. Those parts of algebra which involve infinite extensions resting upon the axiom of choice become meaningless. In analysis, the comparability of real numbers is restricted in a manner fatal to many classical proofs. The theory of sets is limited to an infinitesimal part of the classical theory and becomes much more complicated within the restricted area.

**Philosophical Ground of Intuitionism**

From a philosophical point of view, intuitionists place the emphasis upon mathematics as an individual's personal appropriation of exactness as an integral aspect of his own mathematical
activity, rather than upon mathematics as an impersonal embodiment of universality and truth. For example, Brouwer wrote that Euclid cannot be charged with the incompleteness of his axioms, if his reasonings were meant to accompany "the exploration of a structure built by himself." He regarded the building of the structure by Euclid, and not the completed external structure, nor the verbal and symbolic reasonings which accompany it, as the essential "pure" aspect of mathematics. 17

Primary, or "first-order" mathematics, according to this conception, is an activity of the human intellect. It is stressed that mathematical activity is neither instinctual nor compulsive, but arises as the result of free choice; therefore, it has nothing in common with physiological psychology. As with Kant, mathematical activity is independent of a posteriori experience, and therefore it is exact and is the origin of the concept of exactness. Its only subject matter is found in the a priori intuition of time.

The name "intuitionism" comes from this thesis, that our concept of natural number develops out of an a priori apprehension of the passage of time. A priori time is not at all the same as scientific time, according to Brouwer. 18 Scientific time is not the basis for, it is rather the product of, mathematical activity. Scientific time is measurable, and passes by for all the points of a three-dimensional space together. In scientific time, there is a kind of absolute equality for adjacent equal parts of an arbitrary
scale. *A priori* time, by contrast, is a measureless, one-dimensional continuum, conceived by a single subject, in which two different time intervals are absolutely different and cannot be measured by each other. The moments of the subject's life fall into qualitatively different parts, separated and reunited by nothing but time. Intuitionists consider this falling apart of the moments of life into qualitatively different parts separated and reunited by time as the fundamental phenomenon of the human intellect. By abstracting from this phenomenon all its emotional content, the intellect acquires the basal intuition of mathematics. Out of it, the intellect creates not only the numbers one, two, three, etc. but also all finite ordinal numbers. In this basal intuition, according to Brouwer, the connected and the separate, the continuous and the discrete are united, giving rise immediately to the intuition of the linear continuum, i.e., of the "between which is not exhaustible by the interposition of new units, and which therefore can never be thought of as a mere collection of units." Therefore, according to Brouwer, continuity and discreteness occur as inseparable complements, both having equal rights and being equally clear. It is impossible to void either of them as a primitive entity, or to construe one of them from the other.

... the continuum as a whole was given to us by intuition;
... An action which would create for the mathematical intuition "all" its points ... is inconceivable and impossible.

Not only the building up of mathematical structures and systems out of the basal intuition, but the projection of such mathematical
structures and systems upon nature is regarded as a spontaneous activity of the human intellect; it is stressed that this projection also is non-compulsive and non-instinctual, a matter of choice. Brouwer insisted that we obtain our initial experiences apart from all mathematics, and hence apart from any conception of space. The simplest causal sequences have only time as a one-dimensional continuum for their substratum, he said. He claimed that the original or a priori form of externality is one-dimensional, and that one can conceive a one-dimensional continuum without being able to compare magnitudes on it, and neither absolute nor relative quantities appear until after "a one-parameter continuum and an arbitrary uniform group" have been constructed. It is only with the construction of a more complex mathematical substratum that objects appear:

... mathematical classification of groups of experiences, hence also the creation of a space conception, are free actions of the intellect, and we can arbitrarily refer our experiences to this catalogization, or we can undergo them unmathematically.

Not only is mathematics independent of a posteriori experience, but such experience is also independent of mathematics, unless man chooses otherwise. But no assertions about the external world can be intelligently made besides those that presuppose a mathematical system that has been projected on the external world, and the faculty of taking a mathematical view of his life is a proper activity of man.

Because man is familiar with the setting up of rigid constructions and can control rigid bodies in their behavior, and because
all measurement is referable to rigid bodies, man forms the idea that nature builds only rigid mechanisms. Otherwise nature could build things which man could not imitate. Reliance on the invariability of laws reflecting this idea removes from nature its mystery, and gives man the illusion that "nature can be controlled."

... man makes far more regularity in nature than originally occurred ... he desires this regularity, because it strengthens him in the struggle for life."

Some of the Dutch intuitionists, including Brouwer, were connected with a movement of thought called "Significs," involving a critical examination of language. Language was regarded by them as determining the choice of principles or axioms underlying mathematical language and logic. They regarded language as an activity by which men try to influence others and impose the power of the will upon the world. This motive appears prominent in the choosing of mathematical models of the external world and is reflected in Brouwer's idea of the mathematical image of nature as having been deliberately chosen. Because the intuitionists were concerned for the autonomy and integrity of mathematical thought as such, they distinguished mathematics proper (mathematical activity of the intellect) from second-order mathematics (the study of mathematical structures already constructed, and linguistic activity accompanying mathematics).

Strictly speaking the construction of intuitive mathematics ... is an action and not a science; it only becomes a science, i.e., a totality of causal sequences, repeatable in time, in a
mathematics of the second order, which consists of the mathematical consideration of mathematics, or of the language of mathematics. 22

Within the domain of mathematics proper, contradictions do not arise, said Brouwer. It is when logical structures are created, not as linguistic counterparts of mathematical structures but independently of them, that such structures may be contradictory. Within the limits of the general notion, one cannot predict what special ways of construction may be needed for reaching a particular goal. The most that can be suggested is simply "a mathematical attitude of mind." 23

Many outstanding mathematicians have compared mathematics to music. Intuitionists have carried the analogy further.

... as a composer may suggest to an adept how to compose a symphony—... by trying to describe how he has done it—so a mathematician would initiate a student in the constructive mystery of mathematical production. 24

The attitude of intuitionists toward the scientific enterprise generally, as well as toward the formalization of mathematics, is cautious. Brouwer said that a mathematical system of entities cannot remain reliable as a guide when it is indefinitely extended beyond the perceptions which it made understandable, and he held that in science, while logic often leads to the right result, it cannot be trusted to do so if its application is indefinitely repeated. 25

During and after the late twenties of this century, Brouwer gave expository lectures in intuitionism, mainly in Germany, but
also in Switzerland, Africa, Canada, and the United States. In the years after the Second World War, from 1946 to 1951, he gave various series of lectures on intuitionism at the University of Cambridge. D. Van Dalen, who edited *Brouwer's Cambridge Lectures on Intuitionism*, remarks in his editorial Preface upon Brouwer's consistent refusal to use symbolic notation. He surmises that this aspect of Brouwer's style was a consequence of his personal technique and experience in transferring knowledge and insight, and of his conviction that "intuitionism cannot be taught as if it were, say, linear algebra." The intuitionistic thesis about mathematics and language was unacceptable to the majority of mathematicians, because it makes mathematics a private affair. But philosophers who incline toward the phenomenologico-existentialist trend consider intuitionistic mathematics to be a discipline which can be put into direct relation with human existence and human experience. A number of scholars and mathematicians call themselves intuitionists. This group has been small but has included some outstanding mathematicians from various countries.

**Summary**

Within the field of mathematics itself, intuitionistic mathematicians advocate a revolutionary reform of classical mathematics, and they advocate such a reform on purely philosophical, non-utilitarian grounds.
The first act of intuitionists was the complete separation of mathematical activity from language, on the ground that mathematics proper is a constructive method of dealing with human experience, while the language which accompanies mathematical activity is a secondary and dependent phenomenon.

In classical analysis and set theory, mathematical objects and relations are treated as mathematical existents on the basis of logical argumentation alone, but intuitionists will not accept the existence of objects and relations not secured in a constructive way. In particular, they deny the validity of the principle of the excluded middle as applied to statements about an infinite number of objects.

Mathematics proper is held to be independent of both language and logic, while logic is said to depend upon the basic mathematical intuition. It is when logico-linguistic systems are developed independently of constructive mathematics that contradictions arise, intuitionists contend, and even when such systems are consistent, they cannot establish the existence of mathematical objects and relations.

Intuitionists question the validity of many principles and practices employed in the treatment of infinity, particularly those which are based upon the platonistic assumption that infinite structures can be treated as if complete and surveyable in their totality. The platonistic assumption leads to the application of the law of the excluded middle to the totality of integers and to
the totality of real numbers, and it leads to impredicative definitions, all of which procedures are fundamental to classical analysis and set theory, but which are regarded as questionable by intuitionists. Intuitionists will not admit the validity of arithmetical operations used to construct the classical continuum on the basis of real numbers, i.e., infinite sequences of rational numbers or infinite decimal expansions generated, not according to a constructive rule or law, but by an infinite number of arbitrary choices. Their concept of the continuum is radically different from that of the classical continuum, which is conceived as an aggregate of existing points. Brouwer's view of the continuum is conceived rather as an aggregate of parts which are themselves continuous and partially overlapping, a concept which has something in common with that of the early inventors of the calculus as well as with that of Aristotle. In Brouwer's development of analysis, many of the usual theorems must be abandoned.

Intuitionists especially deplore the use in set theory of methods based on what they regard as an unwarranted assumption of an analogy of the infinite to the finite. As infinite numbers do not obey the laws of ordinary arithmetic, so they may not be assumed to obey the laws of logic, the latter having been developed strictly within the realm of the finite.

Intuitionists insist that all sets must be constructively characterizable. They say that set theory is based upon the iteration of "quasi-combinatorial" concepts. As a case in point
they cite the axiom of choice, an important axiom of set theory, on the basis of which it is assumed, among other things, that infinite cardinal numbers are multipliable, that every set can be well-ordered, that transfinite numbers are comparable, and that the principle of transfinite induction can be applied to any set.

Intuitionistic philosophy emphasizes the primordial significance of the single thinking subject and places the emphasis upon mathematics as an individual's personal appropriation of exactness. By this philosophy, mathematical activity arises from an a priori intuition of the passage of time, and a priori time is not scientific time but is time as conceived by a single subject, in which two different intervals are absolutely different and cannot be measured against each other. The basal intuition of mathematics arises when the passage of time is emptied of emotional content, and it is claimed to be a simultaneous intuition of the continuous and of the discrete. This basal intuition is said to be the substratum of all intellectual activity.

Both the building of mathematical structures out of the basal intuition and the projection of these structures upon external reality are regarded as spontaneous non-compulsive acts of the human intellect. Out of such acts arise the experience of space, the concepts of rigid bodies, of quantitative measurements, of causal sequences, and the idea that nature can be controlled.
Mathematics proper consists in the building of the mathematical structures; the study of such structures and the language which accompanies them both belong to second-order mathematics. While mathematics proper is exact and true, mistakes and inexactness can arise in second-order mathematics, and contradictions can arise when second-order mathematics is treated as if it were independent of mathematics proper.

Intuitionists therefore mistrust formalization in mathematics, and also the over-extension of science through the use of formalized mathematics and logic. They claim that mathematics proper cannot be fully expressed in language, but must be taught as musical composition, art, and other humanities are taught—by suggestion and example, evocation, encouragement, and nurturance of the creative process in the learner.

Mathematical statements which have already acquired a truth value are regarded by intuitionists as objective truths, but the constructive process is conceived as open-ended, and in the mathematical activity by which a truth value is acquired, the emphasis is upon inwardness and upon the thinking subject. Philosophers interested in anthropology, phenomenology, existentialism, and humanistic studies generally regard intuitionistic mathematics as a discipline which can be directly related to human experience.

A small number of outstanding mathematicians from various countries and times have arrived, more or less independently, at intuitionistic ideas, and they seem quite convinced of the ultimate
victory of their ideas and attitudes. Mathematicians generally do not accept intuitionistic mathematics, but they have shown a great deal of interest in working out how far mathematics can actually be carried on the ground of intuitionistically acceptable principles.

Concluding Remarks

There appears to be a close connection between discussions in the philosophy of mathematics and discussions in general philosophy concerning the issue of realism.

Realists argue that it is perfectly intelligible to say that there are infinitely many objects in the universe, whether or not this contention is correct. There is no absurdity, they maintain, in thinking of an infinite totality as already formed, despite the fact that we could never complete the process of generating such a totality, nor could we count the objects forming it. The impossibility of completing the process of generation or the process of counting is not a logical impossibility, according to them, but merely a contingent one, relative to the capacities of human beings. As Russell once put it, it is "a mere medical impossibility."27

From the realist's point of view, the starting point of all thought lies outside the individual subject and is independent of epistemology. Anti-realism, on the other hand, takes its start from epistemology. What Brouwer brings out is the extent to which acceptance of the principle of excluded middle as a logical axiom of universal application rests upon a philosophical presupposition
in favor of realism, or of the absolute objective truth of mathematics independently of the thinking subject. It could be said that, in denying the principle of the excluded middle as a logical axiom, Brouwer does not necessarily deny the realist thesis, but only refuses to presuppose it. But intuitionists do make use of abstract reflections as well as of immediate evidence; for example, in the case of certain very large numbers, such as \(10^{364}\), intuitionists as well as ordinary mathematicians claim that such numbers can be represented by arabic numerals, so that intuitionists are themselves applying a method the use of which they criticize, namely, the general method of analogy whereby one extends to inaccessible objects of thought relations which can only be verified in the realm of objects accessible to intuition. Moreover, there are strong arguments in favor of classical mathematics, based on its elegance and fruitfulness in comparison with intuitionistic mathematics, and it seems on the whole reasonable to hope that doubts about classical procedures will eventually be resolved.

From the point of view of this thesis, the point of greater interest lies in the intuitionists' claim that continuity and discreteness occur inseparably, with consequences for their epistemology. It is as if intuitionists had a vision of mathematics as a strenuous internal activity whereby pure rational structure is created and brought into intimate and strict relation with pure existence, i.e., with subjective a priori time. For such a process, music is indeed an apt analogy.
Intuitionistic mathematics is claimed to be independent of sensuous-emotional experience, and the intuitionistic concept of mathematical statements as objective truth coincides with the classical view. Yet, in the inwardness and the ultimate incommunica- bility of the essential mathematical activity, intuitionistic mathematics has something in common with art. It has also in common with art a distinction between the creative activity which belongs to, and has great significance for, the private individual consciousness, and its completed product, which has significance for society. Indeed, Arend Heyting claims for intuitionistic mathematics the same kind of value as that which may be claimed for the humanities in general, a value which he describes as clearly felt in the activity, although it cannot be defined beforehand. He suggests that this study may prove more valuable for philosophy, history, and the social sciences than for physics, since it is "a phenomenon of life, a natural activity of man."
NOTES


3 Ibid., pp. 89-90.


6 Ibid., p. 219, 227.

7 Ibid., pp. 228-29, 318.


10 Brouwer, Collected Works, 1:477.


14 Ibid., chap. 12.

145
15 Ibid., pp. 123-25, 130; Fraenkel, Set Theory and Logic, p. 87.

16 Fraenkel and Bar-Hillel, Foundations of Set Theory, pp. 253-64.

17 Brouwer, Collected Works, 1:76.


19 Ibid., 1:45.

20 Ibid., 1:68.

21 Ibid., 1:53.

22 Ibid., 1:61.

23 Fraenkel and Bar-Hillel, Foundations of Set Theory, p. 212.

24 Ibid., p. 214.

25 Brouwer, Collected Works, 1:107, 111.

26 D. Van Dalen, Preface to Brouwer's Cambridge Lectures in Intuitionism, p. xi.

27 Dummett, Elements of Intuitionism, pp. 59-60.


PART THREE

LOGIC AND AESTHETICS IN EPISTEMOLOGY
CHAPTER IX

NON-RATIONAL ELEMENTS IN THE LOGIC
OF ARISTOTLE

It is obvious that Aristotle was profoundly influenced by the orientation and practice of Plato's Academy, even where he strenuously disagreed with it. Conspicuously, he accepted the Platonic assumptions that universals are related with each other through a hierarchical arrangement and that scientific knowledge is knowledge of necessary connections. He applied these principles within each genus, and he reasoned that if unambiguous definitions are to be possible, there cannot be an infinity of intermediates between two terms; rather, there must exist immediate or "atomic" connections and disconnections, between which no middle term can be found.

As is well known, Aristotle distinguished the attributes which may be predicated of individuals into "essential" and "accidental" attributes. He argued that essential attributes must necessarily be a finite number, for "this is not like the case of the line, to whose divisibility there is no stop, but which we cannot think if we do not make a stop (for which reason one who is tracing the infinitely divisible line cannot be counting the possibilities of section),
but the whole line also must be apprehended by something in us that does not move from part to part."

On the other hand, accidental attributes are infinite in number, and they cannot be combined together in a unity without relation to a substance.

If one asks whether it is or is not true to say that "This is a man," our opponent must give an answer which means one thing, and not add that "It is also white and large." For besides other reasons, it is impossible to enumerate its accidental attributes, which are infinite in number.

And in general those who say this do away with substance and essence. For they must say that all attributes are accidents and that there is no such thing as "being essentially a man," or "an animal." . . . The predication, then, must go on ad infinitum. But this is impossible; for not even more than two terms can be combined in an accidental predication . . . . For no unity can be got out of such a sum.

While accidental attributes vary from one time to another, essential attributes, being independent of external relations, come into being and pass away along with individual substance. Such substance has an essence consisting of attributes which not only are finite in number, but which belong to a complex unity, not a combination of elements after the fashion of atomism. When Aristotle discussed the complex unity of attributes which constitutes primary substance, he pointed out that such unity cannot be derived from mere conjunction, and that it is not demonstrable. Why, he asks, should man be animal-biped and not merely animal and biped? But a definition demands that the attributes constitute a genuine unity and not merely belong to a single subject as do musical and grammatical when predicated of the same individual.
In Aristotle's view, the concept of unity is abstracted from experience and knowledge of individual substance, "for there is no difference of meaning between 'numerically one' and 'individual,' for this is just what we mean by the individual—the numerically one, and by the universal we mean that which is predicated of individuals." Agreeing that there are different kinds of being—for "Being falls naturally into genera"—Aristotle argued against Plato that there can be neither a "Being-itself" nor a "One-itself," since neither unity nor being is comprised within any one category, and "the one in every class is a definite thing." Neither did Aristotle conclude that the genera are organized in a single hierarchy. He argued against the possibility of a universal science which would be prior to all the individual sciences, that is, against the idea of one science from which the first premises of the other sciences could be deduced. There are indeed certain attributes which belong to being qua being, said Aristotle, especially with regard to pairs of opposites such as oneness and plurality, sameness and difference, and the like; but such attributes, he said, do not include all the attributes of existing things, and the attributes of the changeable are not at all the same as the attributes of the unchangeable and cannot be derived from the same principles. "It is plain that Nature in the primary and strict sense is the essence of things which have in themselves as such a source of movement," said Aristotle. When he speaks of motion or movement in this context, he is most frequently referring to change in the sense of growth.
and development, the actualizing of potential form. Things which have in themselves a source of such movement are "things which come to be for an end."9

In discussing the concept of necessity with relation to "things which come to be for an end," Aristotle said that the "current view," as he called it, placed all necessity in the process of production. But this, said Aristotle, is as if one were to suppose that the wall of a house necessarily comes to be because what is heavy is naturally carried downward and what is light to the top; whereas, while it is true that the wall does not come to be without its stones and foundation being at the bottom and its earth and wood at the top, yet it is not due to these, except as material cause. While it does not come to be without things which have a necessary nature, yet it really comes to be for the sake of sheltering and guarding certain things. Analogously, Aristotle located the necessary in nature in "what is called matter" and compared this with necessity in mathematics. Since a straight line is what it is, he said, therefore it is necessary that the angles of a triangle should equal two right angles; if the line were not straight, the angles would not be equal to two right angles either. This is the necessity which belongs to material cause. But in things which come to be for an end, if the end did not exist, that which precedes it would not exist either, rather than vice versa. In relation to such things, the end, or final cause, is the starting point of the reasoning process, though it is not the start of the action;10 and Aristotle argued against
Plato that the Good cannot be the ground of existing things, in the sense of their material cause, but can only be their final cause, or that for the sake of which they come into existence, and toward which they strive. Therefore God moves the world, not by mechanical force or by the necessity which belongs to material cause, but "as the beloved object moves the lover."¹¹

In Aristotle's system of deduction, causation added non-formal dimensions to the reasoning process, as a means of relating it to reality. Demonstration therefore required both formal validity and correct causal relations, and the system demanded that the conclusion of the demonstrative syllogism be factually as well as formally correct. The middle term of a demonstrative syllogism had to reflect the causal as well as the structural connection between the two premises; it was by this requirement that Aristotle distinguished the "reasoned fact" from the mere "fact." And in Aristotle's eyes, the requirement also distinguished the scientist from both the sophist and the dialectician.¹²

For sophistic and dialectic turn on the same class of things as philosophy, but this differs from dialectic in the nature of the faculty required, and from sophistic in respect of the purpose of the philosophic life. Dialectic is merely critical, where philosophy claims to know; and sophistic is what appears to be philosophy, but is not.

In Aristotle's theory of causation, the word "cause" had a fourfold meaning, and he tried to relate causation to his theory of actuality and potentiality, so that the word "cause" had much richer and broader meaning than is suggested by contemporary usage of the
word. He said that it is the business of the physicist to know all the four causes—the matter, the form, the source of movement, and the that-for-the-sake-of-which, or final cause. But the last three often coincide, he said, since in an individual being, particularly in a living being, the source of movement is in its nature, and what it strives toward is precisely form or essence. All the four causes must therefore be stated by the physicist, but especially the end, for that is the cause of the matter, and not vice versa.\textsuperscript{14}

Aristotle's physics, the study of things which come to be for an end, is a theoretical science completely separate from, and on a par with, both mathematics and theology.\textsuperscript{15} As a consequence, one finds in Aristotle's philosophy two antithetical paradigms, the one in which the lower or less perfect must be derived from the higher or more perfect in a continuous linear series, and the other in which higher states of being and knowledge develop out of lower. These two ideas, according to Lovejoy, operated concurrently, not only in medieval philosophy and in the philosophy of Leibniz, but also in the philosophical development of evolutionary thought, as expressed in romanticism and idealism.\textsuperscript{16} Aristotle himself distinguished the two paradigms as pertaining, respectively, to "things which are prior without qualification" and "things which are prior for us."\textsuperscript{17}

Wayne Thompson's study\textsuperscript{18} of the different uses of the word "induction" in Aristotle's writings, and of the many examples of induction which Aristotle gives, leads him to the conclusion that Aristotle thought of induction as a largely unstructured process
which produces insight, and which is indispensably to the foundations of deductive science. In the Prior analytics, Aristotle defined induction as "the process of attaining certainty through complete enumeration," but there are a number of other places where he illustrated induction with examples in which conclusions are based on two, three, or six instances, and the stress is not upon the number of instances but upon that individual instance, whether coming earlier or later in the process, which induces perception of the universal. The methods of analogy, example, and a fortiori are also included under "induction."

... an analysis of his examples shows that he judged the adequacy of induction by the totality of the argument ... in many instances the looser procedures with their multiscriteria and their "leaps" from data to conclusion are the more useful. ... The verbalization of a general truth shows that the reasoner has perceived a rational relation in his data. 19

Although Aristotle clearly understood the value of abstraction, as witnessed by his eight arguments for the superiority of a deduction based upon the highest level of abstraction attainable in his system, 20 he was concerned with the fact that increase in accuracy and intelligibility gained through ascending to higher levels of abstraction entails a loss or failure of another kind of cognition, that which belongs to closer contact with concrete particulars, and he did not always value exactness and intelligibility at the expense of this other kind of cognition. On the contrary, he said that "the minute accuracy of mathematics is not
to be demanded in all cases, . . . its method is not that of natural science.\textsuperscript{21}

One of Aristotle's favorite examples is the contrast between "concavity" and "snubness." Concavity is the same as its universal formula, he said, but snubness cannot be equated with a universal formula, since it means "concavity of a nose."\textsuperscript{22}

. . . there is the nose, and concavity, and snubness, which is compounded of the two by the presence of the one in the other, and it is not by accident that the nose has the attribute either of concavity or of snubness, but in virtue of its nature.\textsuperscript{23}

Aristotle also explained that while "animal," for example, is a universal, and belongs to the form of man, yet an animal cannot be defined without reference to movement, or without its parts being in a certain state, since, he said, it is not a hand in any and every state that is a part of man, but only "a hand which can fulfill its work." The mathematician treats of attributes of bodies which are, in thought, separable from movement, and in this case no falsity arises from the separation. "Odd" and "even," "straight," "curved," "number," "line," and "figure" do not involve motion, but the same is not true of "flesh" and "bone" and "man"—these are defined like "snub nose," not like "curved." Some attributes, then, are not separable from motion, and we must investigate them as we would the essence of snubness. Yet the holders of the Theory of Forms also separate the objects of nature, which are not separable in the same way as those of mathematics, said Aristotle.\textsuperscript{24}
Abstraction is always possible, said Aristotle, and there is a formula of the essence of a thing. Concrete things can be recognized and referred to by means of such a formula, but they are not identical with it and cannot be known by its means alone; they can be known only through sense perception and intuition combined with the universal formula. It follows that universal knowledge is incomplete knowledge, and the attempt to reduce all things to forms is "useless labor," since some things are particular forms in particular matter, or particular things in a particular state. Aristotel identified the problem of the Meno as a failure to distinguish between "universal knowledge" and "particular knowledge," and he criticized Plato's Theory of Recollection on the ground that "one never has any foreknowledge of particulars."26

On the other hand, sensation cannot give us knowledge of form, nor help us to distinguish what is essential and what is accidental, and "it is in respect of form that we know each thing." If a man apprehends attributes as inhering in their subjects, not in virtue of the subjects' nature, he will possess opinion and not genuine knowledge, for knowledge is the apprehension, e.g., of the attribute "animal," when predicated of man, as incapable of being otherwise.27

Aristotle recognized that there is a first science which studies the laws of thought upon which all men base their proofs, for example, the law of contradiction, but he argued that such laws as these cannot be used as premises for any demonstration.28 He founded the law of contradiction upon the concept of primary
substance, and he regarded this law as joining the science of deduction with the study of being. His deductive theories bear constant reference to the concept of substance.

Therefore, as in syllogisms, substance is the starting-point of everything. It is from "what a thing is" that syllogisms start.

Aristotle argued also that, while it is true that there are axioms common to many of the sciences, e.g., "take equals from equals and equals remain," yet these must be interpreted in terms of the particular genus to be investigated, and they hold good in the various sciences only in the sense of analogy. No demonstration can be founded upon axioms alone, he said, but there must always be at least one premise which is a primary truth concerning the subject matter of the genus to be investigated.

In the final chapter of Posterior analytics, Aristotle summarized his account of scientific knowledge through demonstration and reiterated that it is impossible unless one knows the primary immediate premises. He addressed the question of how the primary premises are apprehended. He argued that knowledge of the primary premises cannot be innate "in a determinate form" and cannot be developed out of higher states of knowledge. His account of the process is as follows:

When one of a number of logically indiscriminable particulars has made a stand, the earliest universal is present in the soul; for though the act of sense-perception is particular, its content is universal . . . . A fresh stand is made among these rudimentary universals, and the process does not cease until the indivisible concepts, the true universals are established; e.g., such and such a species of animal is a step toward the genus animal.
Aristotle explained how one gains insight, which he frequently described as "something better than knowledge," through perspicuous thought about experience. He stated unequivocally that insight is possible because we have a faculty of intuition, and it is intuition which enables us to recognize the universal within our perceptions. It is by means of intuition that we form concepts and establish terms.

Now of the thinking states by which we grasp truth, some are unfailingly true, others admit of error, for instance, opinion and calculation, whereas scientific knowing and intuition are always true; further, no other kind of thought except intuition is more accurate than scientific knowledge. If, therefore, it is the only other kind of true thinking except scientific knowing, intuition will be the originative source of scientific knowledge.

In the Nicomachean Ethics, Aristotle affirmed that "wisdom must be intuitive reason combined with scientific knowledge," and that the first terms and the last are objects of intuitive reason and not of argument, and that "the intuitive reason which is presupposed by demonstration grasps the unchangeable and first terms." Aristotle may with reason be regarded as an empiricist, but this is an over-simplification; it is more accurate to say that his system of deductive science rests on a dual base, for Aristotle held that both universal and particular knowledge are essential to deductive science, since, in his view, if a person has either one without the other, that person falls into error.

Leibniz, who was also concerned with the relations between metaphysics and logic, tried to eliminate the dualism in Aristotle's
system by showing that the difference between factual truth and conceptual, necessary truth is only a matter of degree. In his monadistic philosophy, the reason why any truth is true is to be found in the principle that, in a true proposition, the predicate is always contained in the subject and, contrary to Aristotle, Leibniz considered that a subject may contain an infinite number of predicates, all of which belong to its essence, although such an essence can be clearly known only by God. There are indeed truths which must remain factual and contingent for the human intellect, but these are analytic for the divine intellect, so that, from an ultimate point of view, all truths are analytic. As Couturat and Russell have shown, Leibniz made subject-predicate logic the key to both ontology and epistemology.37

In the early twentieth century, when the American philosopher, G. H. Howison, was developing his speculative philosophy of "personal idealism," in opposition to idealistic monism, which, he said, "is irreconcilable with personality," he wrote of his own system that the high rating which it gave to individuality and final cause was directly derived from Aristotle. Howison remarked upon "the profound ambiguity that marks Aristotle's thought," and he considered his own views as a continuation and development of the individualistic and personalistic tendencies in the philosophy of Aristotle, as well as of the pluralistic tendencies in the philosophy of Leibniz.38
Concluding Remarks

Parmenides and Socrates established, in different ways, that there can be no adequate theory of knowledge without a concept of the Unchangeable. Because Plato was interested primarily in the structural foundations of reason, the Unchangeable in his philosophy became a system of fixed forms and fixed relations, the prototype of lawfulness. On the other hand, Aristotle was interested in the motivational aspects of meaning as well as in the structural aspects of reason, and he described the Unchangeable as a compelling object of love.

Aristotle recognized that there is a difference between the logic of process (his conception of physics or natural science) and the logic of structure (his conception of mathematics), and he apparently believed that the two should be studied separately, although they must ultimately be in relation with each other.

Aristotle repudiated the Platonic doctrine of innate knowledge of Forms, but he proposed that there are innate ideas in indeterminate form and that knowledge is produced when these indeterminate ideas are brought into confrontation with an external reality, presumably already structured. There is a suggestion in Aristotle's writings, albeit an ambiguous one, that he associated structure and lawfulness with the digital and process and motivational meaning with the continuous.
Subsequent philosophies have sought to eliminate duality from epistemology, either by repudiating reason or by repudiating intuition. Similarly, one observes in modern mathematics an attempt to eliminate duality by transforming the continuous without residue into discreteness. Intuitionists deny the legitimacy, as well as the possibility, of such a transformation, and they regard the motivation which makes it necessary as not disinterested and therefore as questionable.
NOTES

1 Posterior analytics 1. 22. 82\textsuperscript{b} 35-83\textsuperscript{a}, 83\textsuperscript{a}20-30; Metaphysics 2. 2. 994\textsuperscript{b} 16-27, 5. 30.

2 Metaphysics 4. 4. 1007\textsuperscript{a} 20-b 10.

3 Categories 5; On Interpretation 11; Posterior analytics 2. 6. 92\textsuperscript{a} 28, 2. 7. 92\textsuperscript{b} 35-93\textsuperscript{a}; Metaphysics 7. 8, 7. 9, 7. 10, 7. 11, 7. 12.

4 Metaphysics 3. 4. 999\textsuperscript{b} 33, 12. 5. 1071\textsuperscript{a} 17-23, 13. 10. 1087\textsuperscript{a}.

5 Metaphysics 4. 2. 1004\textsuperscript{a} 4-5.

6 Metaphysics 7. 16, 10. 2, 11. 2.

7 Posterior analytics 1. 9. 76\textsuperscript{a} 16-30, 1. 28, 1. 32; Metaphysics 4. 2. 1003\textsuperscript{b} 21, 1004\textsuperscript{a} 16-20, 3. 2. 996\textsuperscript{a} 21-32.

8 Metaphysics 5. 4. 1015\textsuperscript{a} 13-19.

9 Physics 2. 8.

10 Physics 2. 9.


12 Posterior analytics 1. 6, 1. 13, 2. 2, 2. 8; Metaphysics 1. 2. 982\textsuperscript{b} 2-10. See also Thompson, Aristotle's Deduction and Induction, pp. 40-41.

13 Metaphysics 4. 2. 1004\textsuperscript{b} 22-27.

14 Physics 2. 3, 2. 7. 198\textsuperscript{a} 22-27, 2. 8. 199\textsuperscript{a} 9-11, 2. 9. 200\textsuperscript{a} 30-35; Posterior analytics 2. 11, 2. 12.

15 Metaphysics 6. 1.

17 Physics 1. 1; Posterior analytics 1. 2. 71b38-72a6; Prior analytics 2. 22. 68b35-36.

18 Thompson, Aristotle's Deduction and Induction, chap. 4.

19 Ibid., p. 84

20 Posterior analytics 1. 24.

21 Metaphysics 2. 3. 99a15-18.

22 Metaphysics 7. 11. 1037b30-b8.

23 Metaphysics 7. 5. 1030b17-19.

24 Metaphysics 7. 11. 1036b25-32; Physics 2. 2. 193b32-194a6.

25 Metaphysics 7. 11. 1037a22-b7, 7. 10. 1036a1-8; Prior analytics 2. 2. 67b1-7; Posterior analytics 1. 1. 7a27-28; Metaphysics 7. 11. 1036b22-24.

26 Posterior analytics 1. 1. 71a29-30; Prior analytics 2. 21. 67a16-26.

27 Posterior analytics 1. 31; Metaphysics 4. 5. 1010a24, 3. 4. 99a28; Posterior analytics 1. 33.

28 Metaphysics 4. 3; Posterior analytics 1. 11. 77a10; 2. 6. 92a12-20.

29 Metaphysics 4. 3, 4. 4. See also Thompson, Aristotle's Deduction and Induction, pp. 103-9.

30 Metaphysics 7. 9. 1034a30-33.

31 Posterior analytics 1. 10. 76a37-b3, 1. 9.

32 Posterior analytics 2. 19. 100a9-11.

33 Posterior analytics 2. 19. 100a15-b4.

34 Posterior analytics 2. 19. 100b4-18, 1. 2. 72a30-b4, 1. 3. 72b18-24, 1. 22. 85b34-38, 84a3-6.
35 Posterior analytics 2. 19. 100b4-18.

36 Nicomachean Ethics 6. 7. 1141a16-20, 6. 6. 1141a2-8, 1141b3-4, 6. 11. 1143a35-b. See also Posterior analytics 1. 23. 84b32-85a2, 2. 19. 102b35.


CHAPTER X

MODERN INTERPRETATIONS OF ARISTOTLE'S
THEORY OF DEDUCTION

The first and most influential of the twentieth-century interpreters of Aristotelian logic was Jan Lukasiewicz, an eminent mathematical logician of the great Polish school, who died in 1956. The title of Lukasiewicz' monograph on Aristotelian logic is Aristotle's Syllogistic from the Standpoint of Modern Formal Logic. As this title indicates, Lukasiewicz interprets Aristotelian logic against the background of, and from the standpoint of, modern mathematical logic.

Lukasiewicz asserts that the syllogistic consists of a set of true propositions, some of which are axioms and the others of which are derived from the axioms by rules of inference which belong to a propositional logic like that of the Stoics. The true propositions of the system are not the simple categorical propositions which appear as premises and conclusions of the syllogisms; rather, they are the syllogisms themselves, stated as compound statements in the form of implications, as in the following:
If A belongs to all B
and B belongs to all C,
then A belongs to all C.

Lukasiewicz characterizes this syllogistic form as a compound proposition which is true independently of the terms which may be substituted for its variables A, B, and C. It is an important point that when a syllogism is stated as a compound proposition, it has a truth value independent of its terms. Lukasiewicz illustrates this point with the following example:

If all crows are birds
and all animals are crows,
then all animals are birds.

This statement, when regarded from the point of view of syllogistic inference, contains one false premise and has a false conclusion, but when regarded as an implicational statement it is true. On the other hand, the traditional syllogism of the form

\[
\text{All B is A;} \\
\text{All C is B;} \\
\text{Therefore,} \\
\text{All C is A}
\]

is a rule of inference rather than a proposition.¹

The syllogisms of the first figure Aristotle himself regarded as self-evidently valid, both indemonstrable and requiring no demonstration. His method of illustrating the validity of the syllogisms of the other figures was to show that all of these syllogisms can be reduced to syllogisms of the first figure. Aristotle did not refer to any of the syllogisms as "axioms," although he did use the term "axiom" in other contexts. He did not
speak of "demonstrating" or "proving" the imperfect syllogisms, but of reducing them to the perfect.

Lukasiewicz regards Aristotle's terminology as improper, and he surmises that Aristotle used it because he was subconsciously aware that he was using a theory of proof which has a fundamental flaw, namely, which supposes all problems can be expressed by categorical syllogisms. Luckasiewicz criticizes Aristotle's procedures generally on the ground that Aristotle tried to prove one syllogism by another. He asserts that Aristotle did not understand the nature of hypothetical reasoning, yet he intuitively used the laws of propositional logic.²

Lukasiewicz draws a comparison between the syllogism "Barbara," stated as

\[
\text{If } A \text{ belongs to all } B \\
\text{and } B \text{ belongs to all } C, \\
\text{then } A \text{ belongs to all } C
\]

and the arithmetical law which reads

\[
\text{If } A \text{ is greater than } B \\
\text{and } B \text{ is greater than } C, \\
\text{then } A \text{ is greater than } C.
\]

He acknowledges that the range of variables is not the same and that the relations differ in the case of the two propositions, but he emphasizes that they are both particular cases of the formula which expresses transitivity of relationship:

\[
\text{If } A \text{ has the relation } R \text{ to } B \\
\text{and } B \text{ has the relation } R \text{ to } C, \\
\text{then } A \text{ has the relation } R \text{ to } C.
\]
On this ground Łukasiewicz concludes that the logic of Aristotle was conceived as "a special theory of relations, like a mathematical theory."\(^3\)

The Kneales affirm Łukasiewicz' basic assumption that the logic of propositions is more fundamental than the logic of terms which Aristotle originated, on the ground that the former is presupposed by the latter.

Aristotle's syllogistic takes its place as a fragment of general logic in which theorems of primary logic are assumed without explicit formulation, while the dialectic of Chrysippus appears as the first version of primary logic.\(^4\)

A minority of logicians in recent years has disputed this assumption, and has maintained that Aristotle's syllogistic is a fundamental logic in its own right.

In Łukasiewicz' interpretation of Aristotelian logic, the syllogistic is perceived as a science, that is, as a body of knowledge based on true statements about a domain of objects, the domain of objects being taken to be names of secondary substances, or universal terms, and the underlying logic of this science being taken to be an axiomatic deductive system based on propositional logic. From John Corcoran's point of view, however, Aristotle's logic embodies only rules of inference and was developed for use as a tool in a number of axiomatic sciences.

Aristotle distinguished between the uses of deduction and the uses of induction in the development of scientific theories. As Corcoran points out, his deductive theory was intended specifically
for use in the inference of scientific propositions from items of knowledge already acquired, either by experience or by other non-deductive means. Such items of knowledge are embodied in primary premises to be used as axioms. Aristotle had no idea of the use of purely logical axioms.\textsuperscript{5}

Aristotle's logic is described in \textit{Prior analytics}; its use as a scientific tool is described in \textit{Posterior analytics}. The system embodies, according to Corcoran, a theory of deduction which is fundamental and which presupposes no other logic. What it does presuppose is a theory of propositional form ultimately based upon the exposition in \textit{Categories}. Here Aristotle referred all predication to the existence of primary substance; he said that if there existed no individual man, "animal" could not be predicated of "man," and if there were no individual body, "color" could not be predicated of anything at all. Łukasiewicz interprets Aristotle's logic as a theory of relations, to be applied to names of secondary substances (species and genus); but Corcoran points out that Aristotle did not regard such connections as relations at all—he considered only relations occurring among primary substances.\textsuperscript{6}

George Engelbretsen claims that Aristotle saw any theoretical science (as distinguished from practical and productive sciences) as an axiomatic system whose axioms depend upon the definitions of the objects of that science. Aristotle developed his syllogistic as a logic of terms, and definitely not as a logic of unanalyzed propositions, such as Łukasiewicz conceives the syllogistic to be,
and the syllogistic was intended as a fundamental logic to be used by the philosopher and the scientist in the teaching of theoretical sciences. 7

Jonathan Lear analyzes Aristotle's approach to logic in the context of a view that Aristotle was not working with two distinct conceptions of consequence, one semantic and one syntactic. Aristotle simply took "follows from necessity" as primitive, and for Aristotle a proof is a syllogism the grasping of which gives knowledge; therefore, its premises must be true, better known than, prior to, and explanatory of the conclusion. Aristotelian methodology recognizes as axioms particular principles which are specific to each science and must be used in formulating syllogisms within that science. While Aristotle regarded the laws of thought as common to all the sciences, they do not appear as premises in any of his proofs. 8

Commenting upon Lukasiewicz' interpretation of Aristotle's syllogistic, Lear says:

Frege's formalization of logic as an axiomatic system with a minimum number of rules of inference and a relatively large number of axioms, taken to be logical truths, has deeply colored the vision of logic held by philosophers and logicians in this century. Twentieth-century interpreters of Aristotelian logic are not out of Frege's shadow—an extreme example is Lukasiewicz' formalization of the syllogistic as an axiomatic system.

Finally, Ian Mueller concludes from his analysis of the syllogistic that mathematics could not have played in its development anything like the role it played in the development of modern
Aristotle included mathematical proofs among syllogisms, and even claimed (without justification) that the conclusion of every proof, including mathematical proofs, is a categorical proposition. But Mueller believes that Aristotle never actually tried to reduce specific proofs from mathematics to his own system. Had he done so, he would have seen its inadequacy for this purpose.10

These interpreters have in common an idea that the historical and cognitive context in which Aristotle's logic was developed, and Aristotle's own intentions and claims for his logic, are important for the correct interpretation of that logic. They insist that Aristotle's epistemology is not separable from his logic. Lear accuses Lukasiewicz of having read his own historical context into Aristotle's work. We have seen, however, that modern logic, in keeping with the Platonic spirit and the spirit of mathematics, seeks to free logical truth from all context, and to divorce abstract thought from connections with any and all thinkers. It was in this spirit that Lukasiewicz, in making his interpretation, determined to ignore all of Aristotle's writings save a few chapters selected from the Prior analytics.

This spirit was as yet foreign to George Boole, even though he was responsible for its direct introduction into logic. In one of his later papers, Boole made a distinction between his calculus of logic and "a higher, more comprehensive logic that cannot be reduced to a calculus."11 The Kneales remark of him that he was
on a wrong track in that he began to look for the basis of logic in the constitution of the human intellect.

It is indeed one of his chief titles to fame that he freed logic from the dominion of epistemology. . . . It was Boole's work which showed clearly by example that logic could be studied profitably, without any reference to the processes of our minds. He believed no doubt that he was dealing with laws of thought in some psychological sense of that ambiguous expression, but he was in fact dealing with some of the most general laws of thinkables.12

It is possible that Boole derived his ideas of the relation between symbolic logic and epistemology from De Morgan's writings, for De Morgan conceived of logic as an instrument of thought which could and should provide a rational foundation not only for scientific generalization but for all modes of thought. De Morgan recognized that not all modes of thought can be made analogous with the mathematical mode, but he believed that they had form which could be symbolized. He thought of logic as both a science and an art and believed that as an art, logic ought to be a preparation for sure and rapid material application. He compared it with the laws of perspective in painting, and the presence of its forms in human thought with the incidental lines which perspective requires, and which are rubbed out before the higher art of the process begins.13 De Morgan tried to implement something of this idea in his notation. Of the attempt his editor, Peter Heath, has this to say:

His treatment of the matter is somewhat perplexed . . . by his attempt to run extension and intension, of both classes and attributes, in double harness throughout, as the "mathematical and metaphysical sides of logic." . . . The fact
is that the logic of intension, whether applied to individuals, classes, or attributes, turns out to be too amorphous to yield very much to any purely mechanical line of attack. . . . The later development of logic has steered clear of it, by concentration almost entirely on the extensional viewpoint.

De Morgan's notation turned out to be a dead end, and this undoubtedly is related to the "perplexed treatment" mentioned by Heath. Peirce, who had a predilection for class expressions, and thereby for the extensional viewpoint, was able to develop a far more satisfactory notation.

Concluding Remarks

In modern logic a distinction is made between deductive systems as instruments of deductive science and logistic (symbolic) systems of uninterpreted symbols with their laws of combination. This perspective upon the foundations of deductive science is the culmination of a long historical evolution, in which the development of modern mathematics and the development of modern logic have gradually merged into one process.

From ancient times there appeared a division within logic. Aristotelian logic was developed as a tool for determining relations among terms, considered as names, in their application to individuals, concepts, or classes. Stoic logic was developed as a tool for determining relations among propositions or states of affairs. The first logic is categorical in form, the second hypothetical. The propounders of each of these logics claimed their
own logic as fundamental and showed that the form of the other logic could be reduced to the form of their own.

That the two forms of logic deal with different kinds of logical relations was explained by Augustus De Morgan around the middle of the nineteenth century. He also perceived that, with respect to modes of inference, both obey the ultimate law of compound ratio, which he called the combination of relations.

Careful examination of the historical foundations of deductive science reveals that the law of compound ratio or combination of relations is at work in the Pythagorean conception of logos, in the dialectical procedures of the Eleatics, in the debating techniques of the sophists, in ancient mathematical proofs, in Socratic method, in the dialectic of Plato's mature years, in Aristotle's syllogistic logic, in Stoic propositional logic, and in the modern logic of relatives. As a general principle of inference, this law became articulate for the first time in the work of De Morgan, who also pointed out that the law is clearly visible in the operations of algebra.

During the nineteenth century, increasing interest in the developing subject of algebra as a symbolic system for expressing and determining relations as pure abstractions, with laws for operations upon them independently of the meaning of the related entities, led first, to George Boole's formulation of Aristotelian logic as an algebra of classes; second, to the investigation of the formal laws governing relations by De Morgan; third, to the
expansion of Boole's calculus, or "algebra of absolute terms,"
to include an algebra of relative terms. The latter expansion was
carried out in the first instance and most notably by Charles S.
Peirce, who defined algebraic relations and operations so as to
include both arithmetical and logical interpretations, devised
symbolic notation including quantifiers and rules of transformation
(a logical syntax) for the new logic, and developed a sign theory
of cognition. A comprehensive theory of the new logic was presented
by Gottlob Frege, and it was put upon a broader and more definite
foundation by Peano, Whitehead, and Russell.

A distinction was now made between what is called "primary
logic" and the logic of relatives. Primary logic, formulated
symbolically, became known as "the algebra of logic." In presenting
this algebra, Louis Couturat emphasized that the symbolic calculus
is a formal system based upon arbitrarily-chosen principles, and
that its logical value as a system of deduction is independent of
interpretation. He showed that the system could be interpreted as
a logic of classes, as a logic of concepts, or as a logic of
propositions, these three interpretations being, up to a point,
isomorphic in form. The logic of propositions, however, emerged as
the most homogeneous and the most useful of interpretations.15

The algebra of logic led to many further developments,
including the development of the modern electronic computer. The
algebra of logic is based upon and derived from mathematical
principles, and it can be called a "mathematical logic," although
it cannot be called the logic of mathematics, for it does not include the logic of relatives developed by Peirce, Frege, Peano, Whitehead, and Russell.

In the early twentieth century, Aristotelian logic as presented by Aristotle fell into such disrepute that it was excoriated by Bertrand Russell as not only useless but injurious.\textsuperscript{16} A little later in the century, Jan Lukasiewicz produced an interpretation of Aristotle's syllogistic logic which showed it as presupposing propositional or Stoic logic and as being itself a special theory of relations similar to the mathematical relation "greater than."

Not everyone agreed with Lukasiewicz' reading of Aristotle's logic, but most agree with the Kneales' assessment of Stoic logic as primary logic, Aristotelian logic as less fundamental in the sense of presupposing propositional logic. However, a minority among twentieth-century logicians argues that Aristotelian logic is a fundamental logic, even though narrower in scope than propositional logic. Their views are based upon an older understanding of the relation between epistemology and logic, a view shared also by Boole and De Morgan. Despite the fact that these latter thinkers made contributions which point directly into the future, they were reformers of the old logic rather than inventors of a new logic.

During the movement from reform of the old logic to the creation of a new logic, a movement which took place, one may say,
between Boole's and De Morgan's generation and that of Peirce, certain decisions were made which reflect a shift in outlook and a change in judgment of the relations between logic and epistemology. In the older view, which is essentially Aristotelian, logic is an instrument which guides human thinking, both in scientific generalization and in every other mode. In the later view, which is essentially Platonic, logic is conceived as having a form and structure analogous to that of arithmetical operations and as having an existence and rationale independent of human processes of thinking. The test of the old logic was its ability to reflect all the forms of human thought, that of the new logic is its ability to transcend the boundaries of human thought and comprehension.
NOTES


2. Aristotle Prior Analytics 1.41. 24b22-26, 1.4. 26b26-34, 1.5. 28a1-9, 1.7. 29a30, 1.23. 40b18-23; Lukasiewicz, Aristotle's Syllogistic, pp. 44-45, 49-59.


9. Ibid., p. 102.


12. Ibid., p. 407.


CHAPTER XI

THE MOVEMENT FROM SUBSTANCE TO FUNCTION:
DE MORGAN, PEIRCE, AND FREGE

In the first half of the nineteenth century, we find Augustus De Morgan, who still believed in the value of the "old logic," distinguishing the "logic of wholes," which applies to the manner in which a given whole may be divided into parts, from the "logic of parts," which applies to the manner in which given parts may be assembled into wholes. He associated the logic of wholes with metaphysics and the logic of parts with mathematics. While being himself a first-rate mathematician, De Morgan deplored the fact that mathematical logic was rapidly crowding metaphysical logic out of awareness. For this reason, he attempted to develop, and even to synchronize, these two different aspects of logic. He did not deny a mathematical substratum to metaphysical logic, for he saw mathematical and metaphysical logic as two different aspects of one discipline; but he pointed out—and thought it relevant—that, in their applications, the two aspects of logic become widely divergent. He wrote:

The proposition of extent remains mathematical to the end: ... The individuals are plain counters in the formal enunciation, and painted counters in the material: but never anything
except counters. But when, in the proposition of intent, the
whole is recognized by the separating attribute, that attribute
coalesces with others in each individual by a process of which
we hide our ignorance when we call it ontological or
metaphysical.

De Morgan asserted that humans are "born metaphysicians,"
whether their metaphysics be true or false. This is more apparent
in children than in adults and even more true of the uneducated
than of the educated. In his view, logic is not a tool of scientific
thought only but is a tool of which the great majority of people
make unconscious, though constant, use. For example, he regarded
even prayer as enunciative and considered that often it is the
tone of voice which predicates. He argued that when someone calls
"John!" the following syllogism is implicitly expressed: "John is
the person I want to speak to; you are John; therefore, you are the
person I want to speak to."²

De Morgan devoted a great deal of space in his five essays
entitled "On the Syllogism" to the explication of his conviction
that the Aristotelian proposition speaks first-intentionally. He
explained that names representing objects and qualities were once
called names of first intention, or first notions; names representing
classes and attributes were names of second intention, or second
notions.

Thus, "every crow is black," considered merely as a collation
of cases, is of first intention; but "the class crow has the
attribute black" is of second intention. Nevertheless, the
first sentence, spoken or written, may be thought under the
second form.
In De Morgan's view, "All men are animals" sums up instances, while "Every man is an animal" tells off one instance after another. In the former, quantity and summation are prominent, but the latter assumes no quantity other than that implied by "There exists that which . . . ," or the notion of unit which exists prior to enunciation. De Morgan asserted that the form "Every man is an animal" was invariably used by Aristotle and by the schoolmen. He claimed that the propositions of Euclid, as well as the propositions of Aristotle, are first-intentional propositions. He pointed out that Euclid makes a selection of some one case, and that the actual demonstration always consists in the reader's ability to perceive that there is nothing to prevent the "someone" from being "any one." This form, which counts off individuals, is enumerative, while the form "All men are animals," in which the concept of class has taken the place of the concept of an individual, is aggregative. De Morgan claimed that these two forms belong to different "logical wholes."4

By De Morgan's analysis, three different logical wholes must be recognized—the enumerative or arithmetical whole, involving individuals, the mathematical whole, involving classes, and the metaphysical whole, involving attributes. Responding to objections that material considerations should have no place in logic, De Morgan said that every inference must after all be due to the presence of something material. "A is B" and "C is D" allow no inference; there must be "A is B" and "B is C," but this is a material difference. He argued that the form of thought is never absolutely separable
from its matter, and that the line of division between formal and material is not fixed but is a moving and relative boundary. But because any statement may be read or thought in a logical whole other than that in which it speaks, De Morgan feared that many important distinctions were about to be lost. 5

Charles Peirce, regarding the concept of unity from a purely relational point of view, and saying that the "logical atom" is analogous to a point in a line, in that it can never be fully determined, applied this analogy to the case of an individual being, "the second Philip of Macedon," and concluded that logical unity can be attributed to such an individual being only by convention, since an infinite number of different predicates (such as "Philip drunk" and "Philip sober") can be attributed to this subject at different times. In a footnote, Peirce noted that the absolute individual cannot, strictly speaking, exist at all, since relations are continuously changing from moment to moment, and there is no such thing as a given moment in time. In the same essay, Peirce asserted that the phrase "any individual man" denotes "an individual by second intention," and that a universal proposition may at any moment be substituted for a proposition about such an individual. 6

As has been previously noted, a movement of thought took place around the middle of the nineteenth century which resulted in a shift in outlook concerning the relations between logic, ontology, and epistemology. As it appeared embryonically in the writings of Peirce, this movement had at least two moments. In the first moment,
the concept of unity, which Aristotle had derived from the existence of primary substance, became a relative concept; in the second moment, the first-intentional proposition, involving individual being, was eliminated from logic in favor of second-intentional propositions involving classes and concepts. In the twentieth century, we find Jan Łukasiewicz giving an ingenious interpretation of Aristotle's syllogistic as a purely formal, second-intentional logic. Such an interpretation could only have become possible through a deliberate severing of those linkages between the formal and the informal aspects of the reasoning process, or between logic and metaphysics, which Aristotle himself had taken great pains to maintain. The severance was, of course, necessary for the development of mathematical and logical rigor in the higher reaches of abstract thought.

It was Frege who succeeded in establishing the principle of complete formalization in mathematics and in extending it to what he called "the wider domain of thought." Frege was first motivated to devise a special language, made entirely of precisely defined symbols and specified rules of procedure, by the difficulties he encountered in attempting to make explicit every step in his proofs of the basic laws of arithmetic. It was the first article in Frege's creed that no symbol should be used until it had been sharply defined, that no hypothesis should remain implicit, that no rule of inference should be tacitly employed. The entire process of reasoning, together with all its foundations, must lie clearly exposed, if the conclusion of
the process is to be regarded as trustworthy, that is, if it is to be asserted as true. To accomplish this end, Frege found it necessary to separate the conceptual content of language from the obscure "psychological" roots of thought altogether. It was in these roots that Frege located the sources of ambiguity and error.7

Frege wrote that when the reference of a sign is an object perceivable by the senses, any idea of this object is an "internal image." Such an idea is often saturated with feeling, and the clarity of its parts varies and oscillates. The same sense is not always connected, even in the same man, with the same idea. The idea is what is to be called subjective; one man's idea is not that of another. This constitutes an essential distinction between the idea on the one hand and the sense of a sign (its meaning) on the other. The sense of a sign may be the common property of many, and therefore it is not part of the mode of the individual mind. Both the "objective reference" and the sense of a sign are to be dissociated from the subjective "idea."

The reference of a proper name is the object itself which we designate by its means; the idea which we have in that case is wholly subjective; in between lies the sense, which is indeed no longer subjective like the idea, but is yet not the object itself.

Thus, language as well as mathematics must be purged of subjectivity. But Frege also made the following rather cryptic statement:

Without some affinity in human ideas, art would certainly be impossible, but it can never be determined how far the intentions of the poet are realized.
Frege seems to say that there are some forms of communication, based upon the workings of individual minds and upon affinities between them, which are irrelevant to science because their meanings are inexact and not invariant, and yet it is in this area that art appears.

Frege wished to symbolize the conceptual content of language so that the same content would always be represented in exactly the same way, and no scope would be left for interpretation or for conjecture. "I follow absolutely the example of the formalized language of mathematics," he said. He did not devise a notation meant to express the truths of arithmetic only, but felt himself bound to identify those structures of thought which he believed to be common to both logic and arithmetic, and to base his symbolic language upon them. His language was meant to be generally applicable in the wider domain of pure thought.

Frege began with the fact that there are certain mathematical formulas, the form of which is fixed while the terms are not determinate. For example, one side of the equation "2 x 2 = 8" may be written "x x^2." In this example, "( ) x ( )^2" is that part of the formula which remains invariant, while the terms used to fill the gaps in this structure may be varied. The invariant structure is known in mathematics as a "function," while any terms fitted into gaps are known as "arguments."

Next, Frege pointed out that the statement "'-1^2 = 1' is true" has the same content as the statement "-1 falls under the concept
'square root of 1,' while the statement '2^2 = 1' is false' has the same content as the statement '2 does not fall under the concept 'square root of 1.' The formula 'c^2 = 1' symbolizes the concept "square root of 1." This shows that what is called a concept in logic is directly connected with what is called a function in mathematics. The concept is identified by Frege as the invariant part of the structure of any proposition, and it is with this invariant part that logic as well as mathematics is concerned.\textsuperscript{12} Frege determined that ordinary language and subject-predicate logic only imperfectly represent the logical content of a thought. He showed that linguistic statements, like mathematical statements, can be split into two parts, which he identified as "concept" and "object." Wherever there is a thought an object is subsumed under a concept, and the bringing of an object under a concept simply means the recognition of a relation; one, moreover, which was already there before it was recognized. In a 1901 essay, Frege made explicit his perception that all functions are simply laws of correlation, and that concepts, like functions, designate abstract relations.\textsuperscript{13} Therefore, the logical term "concept" may be subsumed under the mathematical term "function," and what has been known as an "object" becomes an "argument."

In an article criticizing Schroeder's work on the algebra of logic, Frege wrote that the question whether a proper name stands for something, and the question whether a concept comprehends something under itself, must be kept completely separate. This is because
concepts whose extension is zero must be admitted in logic, as well as in mathematics. It follows that the extension of a concept does not consist of "objects falling under that concept," as Schroeder had assumed. Further, the concept as such takes logical precedence over its extension since, in grasping the notion of an extension, one must appeal first to the notion of a concept. The issues involved in Frege's distinction between concept, extension, and objects falling under a concept are among the most difficult in Frege's work, says Montgomery Furth, and he remarks that Frege's theory is more subtle and more complex than any other theory of the kind.

Frege gave second-intentional terms logical priority over first-intentional terms. Within second-intentional terms, he gave the intensional term "concept" logical priority over extension, and he gave extension logical priority over "class," which designates "the collection of objects falling under a concept." Yet it would be misleading, it seems, to construe Frege's logic as intensional rather than extensional, for Frege wrote:

Someone may get the impression from my procedure that in the battle between extensionalist and intensionalist logicians, I take the side of the latter. I do, in fact, maintain that the concept is logically prior to its extension; and I regard as futile the attempt to take the extension of a concept as a class, and make it rest, not on the concept, but on single things. All the same, in many respects my position may be closer to the author [i.e., Schroeder] than to those who could in contrast to him be termed intensionalist logicians.

According to Hans Sluga's analysis, Frege's main philosophical motivation was the final defeat of both empiricism and subjectivistic
idealism. He wanted to show that thoughts are independent of individual subjects; they are intersubjective and self-subsistent. Yet it was essential for his thesis that thoughts cannot be known empirically. Thoughts cannot be actual; yet they must have the status of being objective. Frege was not explicit about the ontology of what he called "the Objective." What is certain is that he wished to develop a logic which would be completely independent of material and informal considerations.

In order to give content to his pure logic, Frege introduced "the Number One" as a unique logical object, that is, as a kind of second-intentional object which can be used as argument in second-level functions. He made the notion of "the Number One" entirely independent of the notion of "unit."

It is advisable to observe a strict distinction between unit and one. When we speak of "the Number One," we indicate by means of the definite article a definite and unique object of study. There are not divers numbers one but only one. In "One," we have a proper name which as such does not admit of a plural. This makes logic and mathematics completely independent of empirical considerations from the beginning and effectively severs all connections between rational science and concrete individuals.

Frege's analysis of the structure of propositions, derived from the mathematical concept of the function-argument relation, provided the basis for Russell's concept of the propositional function, which, in modern logic, has taken priority over Aristotle's concept of the subject-predicate proposition.
Frege's logic effectively brought to an end the dominance of Aristotelian logic. Post-Aristotelian logic begins only with Frege.

Russell lists as major advantages of the modern purification of logic: (1) It gives a greater chance of isolating a possible element of falsehood, (2) it organizes our knowledge, and (3) it leads to the discovery of things which could not otherwise have been known.

Concluding Remarks

The attempt to develop an adequate and effective "metaphysical logic" appears to have been a failure.

"Digital" is a term used to refer to processes in which values do not change continuously but by discontinuous leaps. Logic is based upon digital thinking, and when Frege says that ideas and intuitions are subjective and incommunicable he is also saying that they belong to continuous non-digital thought processes which cannot be exhaustively analyzed, are not amenable to logical treatment, and cannot be subjected to mechanical manipulation. Many biological and evolutionary processes are of this nature.

It must be admitted that the concept of ultimate reality as digital in nature is fraught with negative implications for life. Without wishing to detract from the grandeur of achievements based upon structural (digital) epistemology, one may deplore the absence of an epistemology better suited to continuous processes.
NOTES

1 On the Syllogism, p. 332.
2 Ibid., pp. 4, 78-79, 84-86, 95, 97-98, 174, 179, 211.
3 Ibid., p. 117.
5 Ibid., pp. 75-81, 89-95, 98-100, 102, 110, 113-14, 120-23, 250-51.
6 'Description of a Notation for the Logic of Relatives,' Writings of Charles S. Peirce, 2:359-432, esp. 2:390-92.
9 Ibid., p. 61.
10 'Begriffschrift,' Translations from Frege, pp. 1-20, esp. p. 3.
12 Ibid., p. 30.

15Introduction to The Basic Laws of Arithmetic by Gottlob Frege, pp. xxxvii-xlvi.


17Foundations of Arithmetic, p. 36; Sluga, Gottlob Frege, pp. 104-5.

18Foundations of Arithmetic, pp. 48-49; see also pp. 67-72, 87-90.

19Sluga, Gottlob Frege, p. 65.

CHAPTER XII

A NEO-KANTIAN INTERPRETATION OF SUBSTANCE AND FUNCTION

In the fifteenth century, Cardinal Nicholas of Cusa (1401-1461) identified ultimate truth, or God, with the actual or completed infinite, which he considered as the terminus ad quem of all knowledge. He taught that finite intelligence could approach nearer and nearer to this ultimate truth, although it could never actually reach it. Cusa's view typifies the fifteenth-century renewal of interest in Plato's mathematical ontology and epistemology. For him, mathematics was not restricted, as for Aristotle, to the science of quantity, but was the necessary form for an interpretation of the universe. 1

Cusa regarded the validity of mathematical propositions as established by the intellect alone, and as being independent of the results of empirical investigations. For this reason, the infinite could be given free rein in mathematics. Carl Boyer sees Cusa's work as an early striving toward the modern concept of limit.

The influence of Cusa's thought was strongly felt by Johann Kepler (1571-1630), who simply accepted analogical inference, based on agreement in conceptual form, as an essential part of the method

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of physics. Galileo organized Kepler's semi-mystical ideas concerning infinite series and summations and gave them greater mathematical precision, and from this time onward, as we have seen, the trend was away from mathematics based upon concrete particulars, and toward mathematics based upon the structural principles of order and series.

The method of Descartes was directed towards establishing a definite order and connection among mathematical intuitions. Descartes rejected syllogistic reasoning, but he perceived that the first rule of thought is that cognitions must be arranged in a single self-contained series, within which there are no unmediated transitions, and into which no member can be introduced as an entirely new element. The neo-Kantian philosopher, Ernst Cassirer (1874-1945), recognizes the same principle of serial order in Aristotle's theory of deductive science, inasmuch as in this theory one does not presuppose the contents as disconnected particulars and afterwards connect them, but tacitly thinks them in the form of an ordered manifold from the first. But in the Aristotelian schema, relations are not independent of "things," they merely add modifications to them.

For modifications and movements and relations and dispositions and ratios do not seem to indicate the substance of anything, for all are predicated of a subject, and none is a 'this.'

In ancient geometry, the geometrical object was an isolated form whose properties were grasped in immediate intuition. Here, ontology provided the model for the construction of logic, and
mathematics could not serve as model for reality. In Aristotle's philosophy, therefore, the mathematical motif receded into the background, and as long as synthetic geometry retained its position as the highest of the deductive sciences, Aristotelian logic remained impregnable.³

Cassirer correlates the shift from Aristotle's subject-predicate logic, based on the concept of substance, to mathematical logic, based on the concept of function, with the shift from spatial form to numerical series in geometry. When the conversion was made from the geometry of spatial form to the geometry of position, the various geometrical forms could be generated out of each other according to a fundamental principle, just as numbers are. And when numbers began to be understood as the most general expression of order and series as such, the goal was to perceive all objects as systematically and strictly connected in series, in the same way as the system of numbers. This could be achieved only if all objects belonged to a manifold that was built up from a self-created beginning according to immanent laws. It was necessary, then, that "objects of the first order" (substances) be replaced by "objects of the second order" (functions), since only the latter have a logical character and can be determined solely by the form of connection and the rule of progression. Thus, the world of sensible things and presentations in consciousness was supplanted by another order, one not derived by abstraction from things.
Here a methodological distinction of great significance appears. The two chief forms of logic . . . are distinguished by the different value which is placed upon thing-concepts and relation-concepts.

By the new principle, abstraction no longer proceeds from things and their properties, but from serial order among concepts.

Awareness of serial relations among concepts per se as the principle of knowledge emerged late in the history of consciousness, yet Cassirer seeks in the Kantian manner to ground this principle in the transcendental nature of consciousness itself. In the presentational life of the individual, he says, the successive images always have a certain inner form of connection with each other, no matter how variegated and diverse; without this connection they could not be grasped as contents of the same consciousness. They stand, at the very least, in an ordered temporal connection, with a definite relation of earlier and later, and even in the loosest succession, the preceding member is not absolutely destroyed by the succeeding members. Temporal succession, like mathematical series, fundamentally involves a relation which includes within itself the members which precede it and becomes itself a unity from which a new construction can proceed, says Cassirer. 5

Cassirer does not subscribe to Brouwer's view that mathematical order is derived in the first place from an a priori intuition of temporality; contrary to Kant, he sees both temporal succession and mathematical series as species of logical form. For such series are based upon a logical principle, the principle that the definition
of any member presupposes the definition of the preceding member. Moreover, according to Cassirer, the creation of the transfinite numbers also issues with absolute necessity from the same serial principle, for the transfinite numbers simply represent a point of view according to which infinite systems themselves can be arranged according to a logical progression.  

Cassirer claims that the logical concept of serial form must lead inevitably to the concept of limit, and that it is not only in mathematics that such a concept is applied. For example, the very concept of "object" demands the perfect filling in of the time series, on the basis of disconnected temporally-separated masses of perception, and thus, strictly speaking, requires the completion of an infinite totality of elements.

It is thus a logical differentiation of the contents of experience and their arrangement in an ordered system of dependencies that constitutes the concept of reality.  

In the same way, the modern construction of scientific concepts is an ordered system of intellectual functions. It is a methodology which constitutes knowledge, that is, it makes knowledge knowledge, and number is its sole perfect fulfillment.  

Cassirer concurs with the logicists in holding that mathematics is a prolongation of logic, and is purely deductive in character. From both points of view, the relation of mathematics to logic is equally close. But according to the logicists, thought discovers an ordering which is independent of the nature of consciousness, while, according to Cassirer, thought generates this ordering out
of its own internal resources. Against the point of view which would eliminate the principle of consciousness from the philosophy of logic and mathematics, Cassirer argues that pure thought cannot be put on a level with passive reception of impressions, because the necessity which attaches to lawful connections cannot be simply found, discovered, or described. "As soon as we ask about the connection and law of the real," he says, "we have already transcended the positivistic demand."9

That serial order presupposes an independent intellectual activity seems manifest to Cassirer above all in the creation of limiting structures. All of our intellectual operations, he says, are necessarily directed upon an ideal limit, the ideal of a fixed and permanent realm of objectively necessary relations, of structures which constitute "the intellectual conclusion of a certain series of presentations, although they can never themselves be presented." This realm "is," although for us it is unattainable; its existence is equivalent to the necessity for idealization. This activity of thought is not arbitrary; it is a strictly regulated and constrained activity, in which the relations between the ideal limit and the members of the series is fixed and cannot be arbitrarily changed; they have the necessity which belongs to the universal. "The will that is here to be satisfied is nothing but the will to logic," says Cassirer, and he quotes William James as saying that our knowledge is subject to a double compulsion; just as we are bound to the properties of our sensuous impressions in our knowledge of facts,
so our thought is determined by an ideal compulsion in the field of pure logic and mathematics.¹⁰

Cassirer postulates that scientific concepts are the ideal limits toward which the sensuous manifold is made to approach. He argues that the very concept of a "thing" is simply an ideal limit. The thing is not an absolute substance, but it is constantly being shaped with the progress of experience and knowledge. As an ideal limit, the thing is never actually realized in our experience, but every attempt we make carries us closer to it, while at the same time demanding a further construction. This explanation of scientific knowledge is admittedly genetic, yet the direction of the progression is not determined by individual needs but by a universal postulate; it does not go arbitrarily from one stage to another, but evolves according to a definite law. The relativity of the object of knowledge at any given moment does not depend either upon the material world or upon particular thinking subjects, but upon a universal principle.¹¹

By an extension of this reasoning, Cassirer seeks to discover the function of ideas of infinite series and ideal limits in cultural activities other than science, such as language formation, myth-making, religion, art, and history. On the basis of his theory that man's peculiar ability to think symbolically and to project his thought toward an ideal limit is not confined to the realm of abstract thought but makes its appearance in every human activity,
Cassirer develops what he calls the logic of the humanities, or the philosophy of symbolic forms.

Concluding Remarks

During the twentieth century, there have developed two apparently opposed philosophical approaches. To one group belong the neo-Kantians along with the pragmatists and the positivists. This group emphasizes science and scientific method. To another group belong the intuitionists along with the later phenomenologists and the existentialists. This group mistrusts rational science as the final arbiter of reality.

In a similar manner, the development of the Kantian doctrine has followed two different directions, the one emphasizing the transcendental understanding, the other the transcendental intuition. Paul Natorp (1854-1924) was a prominent member of the neo-Kantian school to which Cassirer belonged. His main interests lay in the area of pure psychology, and he stressed the necessity for maintaining a tension between the domain of scientific thinking (lawfulness) and the domain of conscious experience (meaningfulness).

Edmund Husserl's (1859-1938) phenomenological method shows many points of agreement with Natorp's thought. A commentator has remarked that Cassirer's ideal schema and its role in cognition is quite similar to the eidos in Hussel's phenomenology, and that both are bound up with the possibility of methodical discovery.12
There is this difference, that when Husserl was converted to transcendental idealism, he conceived that the pure Ego must be studied introspectively, after the manner of Descartes. According to Cassirer's main thesis, on the other hand, cognitive processes are known, not through the introspective study of consciousness, but through the study of their realization in cultural forms. For Cassirer, individual experiences cannot be studied as cognitive, since all cognitive aspects of mind are social. While Cassirer was fascinated with the intuitionists' idea that mathematics grows out of an essential tendency of the human intellect, he repudiated the limitation placed upon the development of mathematical concepts by their insistence upon mathematics as a constructive activity that is independent of logic.

Both Husserl's phenomenology and Cassirer's philosophy of culture have their roots in Kant's second and third Critiques and in the whole intellectual tradition in German thought which attempts to give scientific rigor to the humanities. They represent, in part, an effort to bring about a rapprochement of the valid aspects of German idealism with contemporary mathematics and science.
NOTES

1 Boyer, Concepts of the Calculus, p. 90.
2 Aristotle Metaphysics 3. 5. 1001b30.
4 Ibid., pp. 8-9.
7 Substance and Function, p. 280.
9 Ibid., pp. 44, 313-14, 316-17. See also Itzkoff, Ernst Cassirer, pp. 71-75, and Smart, "Cassirer's Theory of Mathematical Concepts," Philosophy of Ernst Cassirer, pp. 261-64.

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In the Kantian view, the form of thought originates with the transcendental Subject, and Kant placed the transcendently subjective form of thinking at the center of his critical philosophy. Yet in Kant's transcendental idealism, experience demands that objectivity be encountered by the empirical ego, even though this objectivity no longer derives from an absolute object. Therefore, the transcendental idealism of the first Critique translates into realism at the level of the empirical ego.

The first Critique was motivated by Kant's concern for the objectivity of scientific thought. In the second Critique, he was concerned with the objectivity of the moral law, and with the authenticity of moral motives, and he recognized that the rational will (pure practical reason) requires analysis from a different perspective than that of the first Critique.

Kant said that only persons can act in accordance with the idea of law, so that the rational idea of lawfulness as such can motivate actions. A person's will must always be determined by a
principle; everyone acts according to a principle of volition. This means that a person does not act merely in accordance with psychological and physiological determinations but acts purposefully, in accordance with and in response to ends (hence the will is called the faculty of desire). It is this which distinguishes human beings from the animals, and constitutes moral being.¹

The moral law is that principle or form of volition (whatever its content may be) which ought to be acted upon by any fully rational being. No person is fully rational, and the principle by which a person's will is actually determined is what Kant called the person's maxim. The maxim must be distinguished from the practical rule, which is merely a variable ad hoc determination of the individual will, more or less loosely connected with the maxim. The maxim is a lasting policy or a settled disposition, of the same order of generality as the moral law. But where the moral law is a priori, both universal and necessary, the maxim is a generalization a posteriori, dependent upon differences among individuals and situations. A study of Kant's writings upon the subject suggests that a person's maxim may undergo infinite change and development in the direction of closer and closer approximation to the moral law.² Hence, the moral law functions as an ideal limit.

In the first Critique, the problem of how to apply the general rule (the a priori or "empty" concept of the understanding) to a particular case (the sensible intuition) was solved by the
schematism of the understanding. What Kant proposed in the second
Critique as an analogue of the schematism was what he called "the
typic of practical judgment." The typic provides a schema of the
moral law, modelled upon the concept of natural law. Through the
typic, I ask myself: Were I to create the world myself, with an
active interest in its welfare and survival, could I include my maxim
as a natural law in that world? The "natural law" of the typic
includes both an idea of the uniformity of nature as a mechanical
system governed by mathematical principles (hence purely rational),
and the idea of a teleological organization of nature as an organic
unity (hence involving purposes or ends). Since the moral law,
as an a priori principle of pure practical reason, is regulative
and not constitutive (unlike the a priori concepts of the
understanding), both these ideas are incorporated into the
typic as regulative principles.3

Kant saw that, on the one hand, there is the phenomenal world
of Newtonian physics, governed by necessary causal laws, and on the
other hand, there is the world of purposes or ends, implying freedom.
There appears to be no relationship between the world of natural
necessity and the world of freedom.

Albeit, then, between the realm of the natural concept, as the
sensible, and the realm of the concept of freedom, as the super-
sensible, there is a great gulf fixed, so that it is not possible
to pass from the former to the latter (by means of the theoret-
ical employment of reason), just as if they were so many separate
worlds, the first of which is powerless to exercise influence
on the second.4
Yet there must be some connecting link, since the latter is meant to influence the former.

There must, therefore, be a ground of the unity of the supersensible that lies at the basis of nature, with what the concept of freedom contains in a practical way.

Kant developed this connecting link in the third Critique. The three cognitive powers which Kant recognized, namely, understanding (Verstand), judgment (Urteilskraft), and (practical) reason (Vernunft), correspond respectively to the power of knowing, the power of feeling pleasure or displeasure, and the power of desiring or willing. Judgment occupies a middle position between understanding and (practical) reason, and feeling occupies a corresponding position between knowing and willing.

There is yet a further ground, upon which judgment may be brought into line with another arrangement of our powers of representation. . . . For all faculties of the soul, or capacities, are reducible to three, which do not admit of any further derivation from a common ground: the faculty of knowledge, the feeling of pleasure or displeasure, and the faculty of desire.

There are, according to Kant, a priori principles of the understanding (e.g., the law of causality) which are given in the very structure of this faculty; judgment has only to subsume particulars under these universals. But the empirical laws of physics, for example, are not given in this a priori way. Neither are they given a posteriori, in the way that particulars are given. The discovery of such laws is the work of what Kant called reflective judgment.
Judgment in general is the faculty of thinking the particular as contained under the universal. If the universal (the rule, principle, or law) is given, then the judgment which subsumes the particular under it is determinant. . . . If, however, only the particular is given, and the universal has to be found for it, then the judgment is simply reflective.

Now, reflective judgment must be guided by the concept of nature as an intelligible unity, where "intelligible" includes the notion "adapted to our cognitive powers." This implies the concept of nature as purposive.

This problem has its seat a priori in our understanding. . . . Albeit, then, it can determine nothing a priori in respect of these (objects) it must, in pursuit of such empirical so-called laws, lay at the basis of all reflection upon them an a priori principle, to the effect, namely, that a cognizable order of nature is possible according to them.

Thus, the concept of purposiveness in nature is a necessary condition for the employment of reflective judgment in investigating these objects of nature. It is not a metaphysical but a heuristic principle. The validity of this principle is subjective rather than objective. In the purposiveness of nature, Kant found the special a priori principle which has its ultimate source in the faculty of reflective judgment.

Thus judgment, also, is equipped with an a priori principle for the possibility of nature, but only in a subjective aspect. By means of this it prescribes a law, not to nature (as autonomy), but to itself . . . , to guide its reflection upon nature.

When purposiveness is judged as accordance of the form of a particular thing with our cognitive faculties, it is a subjective formal teleological judgment, and this is the aesthetic judgment.
But that subjective side of a representation which is incapable of becoming an element of cognition is the pleasure or displeasure connected with it.

In such a case the pleasure can express nothing but the conformity of the object to the cognitive faculties brought into play in the reflective judgment, and so far as they are in play, and hence merely a subjective formal finality of the object. A judgment of this kind is an aesthetic judgment upon the finality of the object.

In the aesthetic judgment, the form of purposiveness is perceived in an object without the representation of any purpose. There is a feeling that the form of a beautiful object is just right, or that it embodies a fulfillment, although we do not conceive any purpose which the beautiful object fulfills. Insofar as the judgment were to involve an objective purposiveness, such as a concept of perfection or a standard of beauty, it would be a cognitive judgment and not an aesthetic judgment, for the aesthetic judgment is made on the basis of feeling, without reference to concepts or conceptual knowledge.

The judgment of taste, therefore, is not a cognitive judgment, and so not logical, but is aesthetic—which means that it is one whose determining ground cannot be other than subjective. Here the representation is referred wholly to the subject, and what is more to its feeling of life. and this forms the basis of a quite separate faculty of discriminating and estimating.

Nevertheless, there is a kind of necessity which attaches to the aesthetic judgment, which Kant called an exemplar necessity. An exemplar necessity attaches to a judgment which exemplifies a universal rule which one cannot state.

What we have in mind in the case of the beautiful is a necessary reference on its part to delight. However, this necessity is
of a special kind. It is not a theoretical objective necessity—such as would let us cognize a priori that everyone will feel this delight. . . . Nor is it a practical necessity, in which case, . . . this delight is the necessary consequence of an objective law. . . . Rather, being such a necessity as is thought in an aesthetic judgment, it can only be termed exemplary. In other words, it is a necessity of the assent of all to a judgment regarded as exemplifying a universal incapable of formulation.  

When someone calls an object beautiful, he does not necessarily believe that everyone will judge it to be beautiful, but he does claim that it is based upon feelings which he attributes to others as well as to himself.

For, since the delight is not based on any inclination of the subject (or on any other deliberate interest), but the subject feels himself completely free in respect of the liking which he accords to the object, he can find as reason for his delight no personal conditions to which his own subjective self might alone be party. Hence he must regard it as resting on what he may also presuppose in every other person; and therefore he must believe that he has reason for demanding a similar delight from everyone.

Kant asserted that the claim to universal assent is an a priori essential feature of the aesthetic judgment as such.

The judgment of taste, with its attendant consciousness of detachment from all interest, must involve a claim to validity for all men, and must do so apart from universality attached to objects, i.e., there must be coupled with it a claim to subjective universality.

The claim to universal assent does not rest on logical proofs, nor upon empirical facts concerning actual common consent or lack of it. The assertion is not that every one will fall in with the aesthetic judgment, but rather that every one ought to agree with it. For the judgment of taste depends upon our presupposing the existence
of a common sense, i.e., an internal sense of the effect arising from the free play of our powers of cognition.\textsuperscript{16}

Here I put forward my judgment of taste as an example of the judgment of common sense, and attribute to it on that account an exemplary validity. Hence common sense is a mere ideal norm. ... For the principle, while it is only subjective, being yet assumed as subjectively universal (a necessary idea for every one), could ... demand universal assent like an objective principle, provided we were assured of our subsumption under it being correct.

This indeterminate norm of a common sense is, as a matter of fact, presupposed by us.\textsuperscript{17}

The beautiful, then, is that which, apart from a concept, is cognized as an object of necessary delight. The claim can be justified if (1) the aesthetic judgment rests purely on the pleasure or displeasure arising from the interplay of cognitive powers, and (2) we have a right to presuppose in all men a similar structure of the cognitive powers and of the relations between them.

For the general ground of this pleasure is found in the universal, though subjective, condition of reflective judgments, namely, the final harmony of an object (be it a product of nature or of art) with the mutual relation of the faculties of cognition (imagination and understanding) which are requisite for every empirical cognition.\textsuperscript{18}

The judgment of taste applies to objects of sense, but it does not determine a concept of them. It is a singular representation of intuition referable to the feeling of pleasure and is, as such, only a private judgment. But the judgment of taste nevertheless contains an enlarged reference on the part of the subject.

The aesthetic judgment ... refers the representation, by which an object is given, solely to the subject, and brings to our notice no quality of the object, but only the final form in the determination of the powers of representation engaged upon it. The judgment is called aesthetic for the very reason that
its determining ground cannot be a concept, but is rather the 
feeling (of the internal sense) of the concert in the play of 
the mental powers as a thing only capable of being felt.

The judgment of taste estimates an object through the free 
conformity to law on the part of imagination. In the apprehension 
of a given object of sense, imagination is tied down to a definite 
form of the object; but the object may supply ready made to the 
imagination just such a form as the imagination would freely 
project, in conformity with the law of the understanding. This 
is conformity to law without there being a law, a subjective 
harmonizing, which Kant called "finality apart from an end."20

The principle of taste exhibits an antinomy, in that, on the 
one hand, the judgment of taste is not based upon concepts, but upon 
feeling. If it were based upon concepts, it would be open to 
decision by means of proofs, which it is not. On the other hand, the 
judgment of taste is based upon a concept, otherwise there could be 
no claim to the necessary agreement of others with this judgment. 
The solution to the antinomy is to be found in the proposal that the 
word "concept" has a different meaning in the thesis than it has in 
the antithesis.21

All contradiction disappears . . . if I say: The judgment of 
taste does depend upon a concept, . . . but one from which 
nothing can be cognized in respect of the object, and nothing 
proved. . . . Its determining ground lies . . . in the concept 
of what may be regarded as the supersensible substrate of 
humanity."22
The thesis, then, should read: The judgment of taste is not based on determinate concepts; the antithesis: It does rest upon an indeterminate concept. 23

The subjective principle, that is to say, the indeterminate idea of the supersensible within us, can only be indicated as the unique key to the riddle of this faculty, itself concealed from us in its sources; and there is no means of making it any more intelligible. 24

An idea, in the comprehensive sense in which Kant used the word, is a representation referred to an object according to a certain principle, which may be either subjective or objective, insofar as such a representation can never become a cognition. An idea may be referred to a concept according to an objective principle, even though being incapable of furnishing a cognition, and in this case, it is called a rational idea. Or, an idea may be referred to an intuition, in accordance with a subjective principle of the harmony of the cognitive faculties (imagination and understanding). In this case, it is to be called an aesthetic idea. 25

An aesthetic idea cannot become a cognition, because it is an intuition (of the imagination) for which an adequate concept can never be found. A rational idea can never become a cognition, because it involves a concept (of the supersensible), for which a commensurate intuition can never be given. 26

The contemplation of an aesthetical idea contributes to the development of our cognitive powers, even though the idea cannot be made fully intelligible. The consciousness of mere formal finality in the play of the cognitive faculties of the subject is pleasure, because it involves a quickening of the cognitive powers in respect of cognition generally, but without being limited to any
definite cognition. We dwell on the contemplation of the beautiful because this contemplation strengthens and reproduces itself through "an inherent causality."27

In the second Critique, it turns out that what effects change for the better in the moral maxim is a "predisposition to good," an inclination to respect the moral law and make it an incentive for the will, which offsets the "predisposition to evil," in the form of self-conceit, in human nature. The predisposition to good is also described as a predisposition to personality, in the form of self-respect, based on an innate and indeterminate apprehension of the noumenal Self as the author of the moral law.28

Kant did not wish to reduce moral judgment to aesthetic judgment, or vice versa, but he did wish to make the point that aesthetic experience promotes the development of the moral sensibilities as well as of the cognitive powers. Moreover, the principle of nature's finality for our cognitive faculties is one which must be of great importance for natural science, since without this principle the understanding could not feel itself at home in nature. The subjective principle, the principle a priori of the faculty of judgment, provides us with a mediating concept between the concept of nature and the concept of freedom.

The spontaneity in the play of the cognitive faculties, whose harmonious accord contains the ground of this pleasure, makes the concept in question, in its consequences, a suitable mediating link connecting the realm of the concept of nature with that of the concept of freedom, as this accord at the same time promotes the sensibility of the mind for moral feeling.29
The form of universality which belongs to an aesthetic judgment is a significant feature for the transcendental philosopher, though not, perhaps, for the logician. "It calls for no small effort on his part to discover its origin, but in return it brings to light a property of our cognitive faculty which, without this analysis, would have remained unknown."30

Kant's analytic of moral and aesthetic experience was seized upon by the romantics, who elevated the heuristic concept of purposiveness in nature to a metaphysical principle. They used this principle to counteract the concentration on critical and analytic thought which was typical of the Enlightenment.

The romantics asserted that form and meaning as such are in the process of being determined on their outward boundaries, and that human beings are very much part of this ongoing process of determination. They regarded history and cultural development as reflecting an unfolding of the infinite potentialities of the human spirit, and they emphasized the importance of the full development of human personality, creative power, and the rich and various possibilities of human experience.31

The romantics looked upon the individual human being as participating in the organic unity of nature through aesthetic feeling, and they advocated the development of both sensible and cognitive powers through aesthetic experience and education. When aesthetic feeling came into conflict with collective norms, the individual, especially the artist, was advised (e.g., by the
pre-romantic poet Friedrich von Schiller (1759-1805) to "overcome the corruptions of his time by ignoring necessity and fortune, and by lifting his eyes to his own dignity and to law."32

Kant's critical philosophy formed the background for post-Kantian idealism also, in which Kant's heuristic concept of teleology was still further inflated, yielding a doctrine that all reality is the manifestation of one Absolute Subject in the process of development toward full self-consciousness and complete spiritual freedom.

Lovejoy remarks that with the ending of the eighteenth century the temporalization of the Absolute emerged for the first time from its background position in Western thought, so that the traditional emanationism and creationism, which had been derived from the Platonic and neo-Platonic world view, were replaced by philosophical evolutionism, according to which the higher develops out of the lower, and there is more in the effect than was contained in the cause, except as an unrealized potentiality. Friedrich von Schelling (1775-1854) in particular called attention to the implications of philosophical evolutionism. According to Lovejoy, the historical significance of Schelling consists chiefly in his introduction of radical evolutionism into metaphysics and theology, and in his attempt to revise even the principles of logic to make them harmonize with an evolutional concept of reality.33
Schelling dwelt upon the thesis that God is Life and not merely Being, and that God has a fate and is subject to suffering and becoming. He justified the evolutionary theology on the ground that it accords with the actual character of experience, and he declared that ordinary theism had given us a God who is alien to Nature and a Nature that is devoid of God. He said further that this had come about as a consequence of a logical doctrine which based everything upon the principle of identity and regarded all judgments as analytical. 34

Despite affinities between the two movements, the philosophy of post-Kantian German idealism presented a marked contrast with the romantic movement, in that the idealists' conception of reality led to an insistence upon systematic rational thought. Where the romantics had emphasized aesthetic feeling and had tended to assimilate philosophy to poetry, the idealists declared that philosophy yields knowledge. In fact, said Johann Fichte (1762-1814), philosophy yields knowledge of knowledge; it is the basic science and not an attempt to say what cannot be said. 35

Since human consciousness was regarded as the vehicle of absolute thought reflecting upon itself, the entire process must be intelligible to the human mind. But cosmic productiveness was not attributed to the individual self, and the German idealists were not subjectivists. Rather, the ultimate principle was regarded as transcending both subject and object. In working out the problematic relationship between the infinite and the finite, the idealists
conceived that the world follows necessarily from the first principle. They tended to assimilate the causal relation to the relation of logical implication, so that the priority of the first principle could be conceived as logical and not temporal priority, and they expanded the idea of logic to include a dialectical process involving a conflict between opposites and leading to their union in transcendence.

Georg Wilhelm Friedrich Hegel (1770-1831) insisted that philosophy is a matter of strictly logical thought, which thinks its subject matter conceptually and makes it plain to view, and he expressed scorn for the approach of the romantics. Both Fichte and Hegel mistrusted individualism and made the striving toward self-realization and spiritual freedom dependent upon society and culture. Hegel especially emphasized the social aspects of ethics and the significance of man as a member of the State.

In Hegel's philosophy of Infinite Reason, romanticism and idealism became thoroughly intellectualized. But Schelling came to doubt the adequacy of reason, and he abandoned the attempt to deduce the existence of empirical reality in an a priori manner. He developed a philosophy of aesthetic intuition which expressed so well the spirit of romanticism that Josiah Royce called Schelling "the Prince of Romanticists." However, it was not Schelling's aestheticism but Hegel's rationalistic monism which had the greater historical impact.
Already before the death of Hegel in 1831, the Hegelian system had come under attack on several fronts simultaneously. It had become aligned with the authoritarian regime of Prussia, while Germany had begun to change to a pre-industrial society, based on technological and scientific progress. The deductive conclusions of both Hegel and Schelling conflicted with the results of new observational and experimental techniques in science, and the idealists, after the first generation, were unable to produce anything new. Idealism, as well as romanticism, began to appear philosophically weak and vague. 37

When Charles Darwin (1809-1882) proposed his theory of non-purposive mechanism as the propelling force in evolutionary development, the effect upon the romantic-idealistic line of thought was devastating. In the latter half of the nineteenth century, romanticism and idealism were superseded by scientific naturalism and materialism, and philosophy was all but absorbed into science, as in the evolutionary theories of Herbert Spencer (1820-1903), according to whom evolutionary progress means not that human beings are becoming better or happier, but that they are becoming increasingly adjusted to environment, and that this occurs in a lawful and predictable manner, independently of their efforts and even without their cooperation. But men like Hermann von Helmholtz (1821-1894) and Friedrich Lange (1828-1875) criticized the covert metaphysical presuppositions of such views. They preferred to regard the material world as an interpretation of
phenomenal experience and recommended psychological, physiological, and historical analyses of human processes of perceiving and knowing. William James (1842-1910) was profoundly critical of Spencer's philosophy of mind. Spencer had done a service, he said, in showing that mind has a biological basis, yet his notion of mind as adjustment to the environment is absurd. James regarded knowledge of reality as a function of man's active, selective interests, and he followed Peirce in making the criterion for the truth of a thought the actual occurrence of that very train of consequences which it projects or predicts.\textsuperscript{38}

Post-Kantian idealism took new root in England around the middle of the nineteenth century, when its influence in Germany was already on the wane. The two British names most often heard in discussions of idealism are those of Francis Herbert Bradley (1848-1924) and Bernard Bosanquet (1848-1923). It was against Bradley's idealistic philosophy in particular that Bertrand Russell and G. E. Moore were in rebellion around the end of the nineteenth century.\textsuperscript{39}

Frege's work provided philosophical grounding, corroboration, and support for Russell's work in the systematic development of modern symbolic logic. Naturalism and materialism had had a tendency to dissolve into phenomenalism, pragmatism, and various forms of subjectivism, and it was this movement which Frege deplored, rather than Hegelianism. It was because of this movement that Frege was led to a defensive emphasis upon the sharp boundary
line between subjective psychological conditions of thinking and the objective content of thought, a distinction which had already been made by Kant. It was out of a motivation quite similar to Frege's that the neo-Kantians raised the cry "Back to Kant!"

Concluding Remarks

The third Critique establishes a distinction between meaningfulness and lawfulness, for the beautiful is that which has meaning independently of law. Such meaningfulness depends upon the concept of teleology and bears a direct reference to process and to the development of cognitive powers on the one hand and of moral sensibilities on the other.

Where meaningfulness in its relation to process and to development is emphasized, private judgments become significant. Where lawfulness in its relation to structure is emphasized, thoughts which have become collectively sanctioned subsume and replace private judgments and private thought processes.

In Kant's philosophy, tension is maintained between meaning and law, as between process and structure. In romanticism, an attempt is made to dissolve law into meaning, structure into process. In post-Kantian idealism, and particularly in Hegelianism, an attempt is made to conflate the two, with lawfulness gaining the upper hand.
NOTES


2 Ibid., p. 88.


5 Ibid., par. 176 (p. 14).

6 Ibid., par. 177 (pp. 15-16).

7 Ibid., par. 179 (p. 18).

8 Ibid., pars. 184-85 (p. 74).

9 Ibid., pars. 185-86 (p. 25).

10 Ibid., par. 189 (p. 29).

11 Ibid. pars. 189-90 (p. 30).

12 Ibid., pars. 203-4 (pp. 41-42).

13 Ibid., pars. 236-37 (p. 81).

14 Ibid., par. 211 (pp. 50-51).

15 Ibid., par. 212 (p. 51).

16 Ibid., par. 238 (p. 83).

17 Ibid., par. 239 (pp. 84-85).
18 Ibid., par. 191 (p. 32).
19 Ibid., par. 228 (p. 71).
20 Ibid., par. 220 (p. 62); pars 240-41 (pp. 85-86).
21 Ibid., pars. 338-40 (pp. 206-8).
22 Ibid., par. 340 (pp. 207-8).
23 Ibid., pars. 340-41 (p. 208).
24 Ibid., par. 341 (pp. 208-9).
25 Ibid., par. 342 (pp. 209-10).
26 Ibid.
27 Ibid., par. 222 (p. 64).
29 Aesthetic Judgment, par. 197 (p. 39).
30 Ibid., par. 213 (p. 53).
34 Friedrich Wilhelm Joseph Schelling, "Ideas on a Philosophy of Nature as an Introduction to the Study of This Science," trans. Priscilla Hayden-Roy, Philosophy of German Idealism,


37 Sluga, Gottlob Frege, pp. 13-14.

38 Wilshire, Introductory Note to "Lamarck and Darwin," Romanticism and Evolution, pp. 214-17.

39 Sluga, Gottlob Frege, p. 15.

40 Ibid., pp. 36, 43-44, 51-55, 61-63, 104-5.
Cassirer originally belonged to the Marburg School of neo-Kantianism and was under the influence of its leader, Hermann Cohen (1842-1918). Cohen saw himself as carrying on the work of Kant and expanding it. He sought to develop a purely logistic interpretation of the Kantian philosophy by abandoning the Kantian doctrine of intuition and making the principles of the calculus fundamental to experience as well as to thought. Under the influence of Hegel, Cohen shifted attention away from the study of individual minds and toward the study of society and culture as historical objectified expressions of mind.¹

Cassirer's bent, like Cohen's, is toward logical and structural explanations of cultural phenomena. His ambition is to bring about a rapprochement between German idealism and contemporary mathematics and science and so to bring scientific rigor to the humanities. He makes his starting point the Kantian doctrine of knowledge as an a priori structure which thought contributes to experience, but he imports into this doctrine the Hegelian idea of an objectively determined historical evolution.
from the primitive modes of consciousness to the rational modes of consciousness typical of mathematical logic.

Kant recognized three dimensions of human existence, the logical, the ethical, and the aesthetic. He drew a distinction between the objective validity which belongs to logic and mathematics and the kind of validity which derives from universal subjectivity. In keeping with his own program, Cassirer seeks a common root for both forms of validity and hits upon the idea of a prelogical concept of structure as this common root. He then seeks evidence of such a prelogical concept of structure in data drawn from historical, anthropological, and ethnographical sources. Despite his idealism he constantly appeals to specific factual and scientific evidence to support his theoretical points. This attitude links his work with that of the logical positivists and with that of the pragmatists.

Cassirer adopts the Kantian doctrine that man is distinguished from the animals by his power to originate purpose and to be guided by the idea of lawfulness as such. He traces the origins of both these powers to the nature of symbolic thought. He determines that symbolic thought stands in relation to rational thought as genus in relation to species, in the sense that symbolic thought is the more comprehensive form. He determines that the power of symbolic thought has belonged to consciousness from its earliest beginnings, while rational thought is in the nature of a culminating and relatively recent development.
Assuming that an organism is a system of mutually conditioning circuits which interact with the environment according to a determinate pattern, Cassirer asserts that in the human species the functional circle of the organic system is interrupted by a symbolic system. The power of creating symbols is that which distinguishes man from the rest of nature, and this power in its most primitive manifestation must belong universally to man. Since a long road must be travelled between the forms of apprehension found in primitives and the earliest and simplest forms of rational construction, and since all stages in the evolution of consciousness have in common the power of creating symbols, man must be defined as the symbolic animal rather than as the rational animal.\(^2\)

Even the organization of perceptual materials displays the work of guiding principles beyond those inherent in perceived materials as such. This is because man has an intrinsic drive to shape his experience and uses the power of symbolizing to this end. His symbols do not merely reflect the actual as passively received but incorporate a vision of the possible as well and express the drive to transform envisioned possibility into actuality. Symbolic thinking is by its nature teleological, concerned with meaning as well as with form. It must be the case that man envisages orders and relations without reference to external criteria.\(^3\)

Cassirer considers that the human intellect has a differentiating power that enables it to produce different kinds of symbolic
forms in response to different kinds of purposes and problems. In envisaging orders and relations and in seeking to transform the possible into the actual, man makes use of many different stratagems, including those which depend upon psychological factors such as feeling and imagination. Every form of mental energy participates in the effort, but man's ordering relations acquire an objective status only in the recognition of a common cosmos, a shared world which takes precedence over private experience. The end goal of the entire process is the self-objectification of the human intellect. 4

Thus knowledge is a product of the human intellect epitomized in history. The true nature of knowledge is not given all at once but is unfolded progressively and is revealed by the form of the process itself. Therefore, the study of creative achievements in myth, language formation, religion, art, etc., gives an understanding of the nature of knowledge and of how mind functions in its collective and cooperative aspects.

Not only in different stages of development but also in separate spheres of cultural activity, both verbal and non-verbal—e.g., in language formation, in tool-making, in art, in music and dance—there may be exhibited distinct possibilities of symbolic representation, none of which is an imitation of an already-existing reality, but each of which constitutes a separate mode of development toward a distant ideal. In Cassirer's usage, symbols do not merely indicate realities but bring about or condition what is meant by
a reality; they represent a formative power and have a factual meaning only within a determinate symbolic context.

By the Kantian thesis, the creation of alternative modes of perception and conception implies the possibility of alternative phenomenal worlds, i.e., differently constituted "real" worlds. The problems of reality and objectivity take on a different shape, says Cassirer, if, instead of treating the object of knowledge as firmly fixed from the beginning, we view it as from an infinitely distant point toward which all knowledge and understanding aim.⁵

In sum: Cassirer introduces into the Kantian idea of immanent structure both the idea of a variety of symbolic forms or modes of thought, governed by an immanent law, conditioned by but not determined by environment and culture, and producing different phenomenal worlds, and the idea of an inner dialectic leading to a continual lawfully determined development towards an end which functions as an ideal limit.

Cassirer identifies space, time, and number as common structural elements which underlie various manifestations of symbolic expression. With relation to each of these elements, he tries to identify its role in the prelogical structuring of experience and to document a continual evolutionary progress toward abstract and universal thought, as increasingly logical (relational) patterns emerge with greater and greater clarity from emotional expressions and from perceptions. The earliest acts of symbolic thinking, by Cassirer's account, are expressions of aspects of human character
as well as of aspects of the environment; they are suffused with motivations arising from within the human psyche. But certain common structural elements underlie such expressions and an abstract structure is eventually disentangled from the symbolic thinking which originally included sensuous and emotional experience. This implies that all cultural activities are evolving toward abstract thought forms like those of mathematical logic and that the latter represent the paradigm for cognitive activity in pure form. The process may be studied, for example, in Cassirer's account of man's changing relations with space.

Primitiv e space is neither objective nor theoretical; rather, the idea of space for primitive man is bound up with the subject and is an affective and concrete idea. One anthropologist-philosopher, Gregory Bateson, has given an account of the Balinese attitude toward space which strikingly confirms Cassirer's point.

The Balinese are markedly dependent upon spatial orientation. In order to be able to behave they must know their cardinal points, and if a Balinese is taken by motor car over twisting roads so that he loses his sense of direction, he may become severely disorientated and unable to act (e.g., a dancer may become unable to dance) until he has got back his orientation by seeing some important landmark, such as the central mountain of the island around which the cardinal points are structured.

This account illustrates what Cassirer calls organic space, the space of action and self-expression. By contrast, the space of geometry is representative rather than expressive, and the common cosmos which it presents has become a theoretical system. There is a vast abyss between organic space and geometrical space.
For example, primitives are sensitive to every change of location of objects and can almost always find their way from one point to another, but they are quite unable to read, much less to draw, a map. The map depends upon a general concept of space and of objects and relations as determined by a general system. Such a concept the primitive does not have; yet such a concept does eventually develop. Cassirer postulates that the development would not be possible without a concurrent development of language from the expressive to the representative mode.9

Cassirer adopts a notion derived from Hermann Lotze (1817-1881) that there is a primary universal contained in perception itself. This primary universal is different from logical concepts but it must be presupposed by them. While language is born of the need for expression and is rooted in particulars, every particular is accompanied by an intuition of the fact that there are other particulars which could equally well perform the same expressive function.

As Suzanne Langer explains it, whatever evokes emotion may receive a name, even if it is not a "thing." The name gives unity, permanence, and stability; it is not simply tacked on but is an expressive act and at the same time a way of coming to know, for it transcends the multiplicity and diversity of momentary responses and constitutes a determinate reality.10 In a similar manner, predication expresses the conceptualization of relations which have been intuited beforehand. As illustrated by the evolution of man's
understanding of space, the ability to isolate and symbolize relations per se develops only very gradually. Language, then, projects and anticipates objectivity even when it does not reflect an already self-subsistent reality.

Cassirer regards language and myth as root symbolic forms, originating simultaneously and developing along different lines. It is in his concept of myth as an independent thought form that Cassirer breaks away to some extent from orthodox neo-Kantianism. As he describes it, myth belongs to a phase of consciousness in which the real and the imagined are indistinguishable. It is a form of aesthetic fantasy of which the products are like dream elements, all manner of shifting, fantastic images charged with feeling. In his discussions of the mythical consciousness, Cassirer refers to phenomena culled from psychiatry, and Langer points out that the dream world of Freud's unconscious mental mechanism has much in common with the mythic mode that Cassirer describes as the primitive mode of ideation. Abstract and rational developments in religion and ethics are supposed to have emerged from the mythical Urstratum, but Freud has shown that the mythical mode persists as a non-discursive thought form underlying both practical and theoretical activity and maintaining an integrity of its own.

In Freudian terms the operations of the unconscious are structured in terms of primary process, while critical thinking is structured in terms of secondary process. Primary process is involuntary and cannot be manipulated by the will or used with
intention to deceive. It refers metaphorically to unidentified things, and its subject matter is relationship between self and other people or self and environment. Feelings and emotions are outward signs of these relationship patterns.

In pure primary process the metaphors are not distinguished from objects, thought is not distinguished from reality, subject is not distinguished from object. Everything is simply an element of the system, all elements having equal status. In pure secondary process, on the other hand, the elements not only are sharply distinguished but are organized in a logical hierarchy representing different levels of abstraction.

In the mythic apprehension everything belongs to one unbroken whole; the boundaries between various phases and spheres of thought are non-existent, or at best they fluctuate, so that nothing has an unvarying shape but everything may turn into everything else or be connected with everything else. But mythic symbolism has its own inner logic which depends upon a unity of feeling rather than upon an ordering among objects. For example, the religion of totemism is a system in which animals, plants, and people are equal parts of one cosmos, without sharp boundary lines dividing one kind from another.

Cassirer conceives of each symbolic form as having its own peculiar meaning and significance. Art is a symbolic form that is categorically different from the other symbolic forms. In the other forms, word and image must ultimately become separated as the
symbol frees itself from the concrete. In art, however, the image is accepted for itself. Art is neither an abbreviation nor a generalization of reality but is an intensification and at the same time a transformation of concreteness itself.\textsuperscript{13}

The artist, unlike the scientist, does not set out to translate a particular intuition into a more general symbolism but rather to create a symbolism suited to express the particularity of a given intuition. In expressing his intuition, the artist does not merely reproduce what he expresses but transforms it.

The image of a passion is not the passion itself. At a Shakespearean play we are not infected with the ambition of Macbeth or with the jealousy of Othello. The great dramatists show us the form of our inner life.\textsuperscript{14}

Participation in the aesthetic experience frees us from the domination of emotions, Cassirer claims, just as the concept of geometrical space frees our understanding from the domination of things. But aesthetic freedom does not consist in absence of passion but in intensification of it and at the same time in change of form.\textsuperscript{15}

Yet ultimately, Cassirer's views are such as to suggest that all thought should free itself from the subjective elements which cling to ordinary language and should translate itself into abstract symbolisms.\textsuperscript{16} Cassirer acknowledges that this is never completely possible. "The issue, as I see it," comments Wilbur Urban, "is not whether this is possible, but whether it is desirable."\textsuperscript{17}
Concluding Remarks

In a study entitled Primitive Man as Philosopher, the anthropologist Paul Radin cites the case of a man who, after describing an incident in an opposite way to the way it was usually described, declared: "This is my way of telling it. Others have different ways." Radin is struck by what he calls "freedom of thought" among savages. It appears that thinking gives to the primitive a sense of the reality of his own subjective life, while having little or nothing to do with what we call objective reality. It is an expression of personality and not an instrument for acquiring or communicating information.  

Psychoanalysis is based upon the idea that there are two forms of thinking, one of which is historically earlier than the other, both of which are carried on concurrently, with or without consciousness of the fact. The earlier form of thinking is non-discursive and certain of its characteristics have been identified. Primary process thinking has no negatives, no tense, no linguistic moods. It is iconic or metaphoric and is not digitally coded. It moves in circular rather than in linear thought patterns. Those hierarchical levels of generality which Russell recognized as belonging to different logical types are not distinguished from one another. Thought is not distinguished from reality nor subject from object. Symbols and things may be equated with each other or
simultaneously equated and discriminated, as when an insult to a country's flag is treated as an insult to the country itself. 19

Gregory Bateson has proposed that primary process presents us with an ongoing and dynamic iconic representation of that self-regulating system constituted by what we call the "self" together with what we call "environment." To say that the representation is iconic is to say that its pattern and structure correspond with the pattern and structure of the system which it represents metaphorically. 20

Can such a thought system have knowledge value? May it provide us with a source of truth about human realities the validity of which is exemplary in nature and cannot be made fully articulate?

It is possible to read this idea into Cassirer's exposition of the meaning and significance of mythical and artistic symbolisms, since at times he leaves an impression that there are different forms of knowledge associated with different forms of symbolic expression. But at other times he maintains that the ideal of all knowledge must be found in logico-mathematical thought forms. At one point in his writings Cassirer appears to distinguish between meaningfulness and lawfulness; at another point he seems to conflate the two and to give priority to lawfulness in the form of structure. No doubt the conflict arises from his neo-Kantian inheritance on the one hand and from his intellectual honesty in relation to his data on the other. This creates an inconsistency which remains unresolved in the philosophy of symbolic forms.

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NOTES

1 Itzkoff, Ernst Cassirer, pp. 28, 33-38; Felix Kaufmann, "Cassirer's Theory of Scientific Knowledge," Philosophy of Ernst Cassirer, p. 185.


3 Logic of the Humanities, p. 75; Essay on Man, pp. 24-25, 56-57; Itzkoff, Ernst Cassirer, pp. 25, 122, 157; Clarence Smith Howe, Foreword to Logic of the Humanities, p. xvii.


15. Ibid., pp. 148-49, 168-69; *Logic of the Humanities*, p. 84.


20. Ibid.
In his exposition of *De Anima*, Franz Brentano (1838-1917) brings out the fact that in Aristotle's model of cognitive activity, imagination plays the role of mediator between the unconscious creative intellect and the conscious intellect, and between the conscious intellectual will and the body's changes and movements. Imagination provides, as it were, a two-way mediation between body and mind. It is the connecting link between the formal and the informal aspects of cognition which Aristotle believed to be of first importance not only for epistemology but for logic. It is therefore not surprising that Aristotle counted imagination as thinking; he often called it *nous* or a kind of knowledge (*noesis*).2

The passive-receptive intellect (the *nous pathetikos*) was compared by Aristotle with a sense organ; in this aspect, the intellect is related to images as sense is related to external sensible things. Sense receives its pictures by turning toward external objects; intellect receives its ideas by gazing, as it were, upon images. And just as seeing and hearing are no longer possible when the seen or heard object is removed from the visual field...
or the range of hearing, so thought is no longer possible when the appropriate images are no longer present.\(^3\)

Plato thought that we recognize, for example, actual flesh and the Idea of flesh by apprehending two different things which are separate from each other. Aristotle raised the question of how the object through whose reception we recognize individual flesh is related to the object through whose reception we recognize the Idea of flesh. One of them is obviously something sensory and corporeal; the other must either be something supersensible and incorporeal, as Plato held, or it must be the same object as the first, but found in the intellect in a different state, that is, the same object must be found in the intellect as abstract which is found in sense perception as individual, material, and concrete.\(^4\)

If the intellect were able to grasp directly a supersensible being, said Aristotle, knowledge would not come to it in images. Such is not the case with any concepts, not even the mathematical, with the apparent exception of self-knowledge. But the intellect cannot even know itself until after it has become something which thinks actually rather than merely potentially; hence Aristotle concluded that this knowledge, too, must come to it by way of images.\(^5\)

In Aristotle's schema, potentiality becomes actuality only through an active principle separate from itself, and an actuality must precede and be the cause of every potentiality. If the intellect were in itself an active principle, it would be always
thinking, which it is not. Therefore there must be an active principle by which the potentiality of thought is brought to actuality.\textsuperscript{6} Aristotle reasoned that the active principle cannot be found in the senses; at the same time, it cannot be a principle which directly affects the potential intellect, for in this case no images would be necessary. Aristotle concluded that there is an active intellectual power, separate from the \textit{nous pathetikos}, the activity of which is directed toward sensory representations. He designated this principle the \textit{nous poiētikos}, the creative intellect,\textsuperscript{7} and he compared its role in thinking with the role of light in visual perception. As color is made visible only through light, so the intelligible forms that are in the sensory representations are received into the \textit{nous pathetikos} through the agency of the \textit{nous poiētikos}, whereby the potentially intelligible is made actually intelligible. Thus the \textit{nous poiētikos} "brings forth thoughts" in the \textit{nous pathetikos} by means of images.\textsuperscript{8} Brentano wrote:

The active intellect without images would be like a bow without an arrow; the images without the active intellect, like an arrow without the propelling force of the bow; it would be impossible for either of them alone to reach the target, for they would be incapable of generating thought.\textsuperscript{9}

Since the creative intellect illumines images and makes noticeable to the mind's eye the intellectual within the sensory, and in this way it brings forth thought, it must act before all thought, hence without consciousness. But since we are capable of free and methodical progressions of thought which are not tied to
the senses, Aristotle had to recognize also an active intellectual power which is conscious. He identified this power with the higher intellectual faculty of desire quite apart from sensory appetite. To this higher power (which Kant will call the practical reason) he assigned freedom, and he noted that it is able to intervene in the sphere of the sensitive and must be able to exert an influence upon the body and to modify the body's activities. He pointed out that that which moves in an immediate way is not the general judgment but the single or particular judgment, and that it is the task of the senses, not of the intellect, to recognize particulars. Thus the intellectual faculty of desire must influence the body by acting upon images. As images change, so change the body's movements. Hence reason can exert an influence that determines sensory desires and bodily movements only through the transformation of images, and every sensory desire and passion and every movement which can be influenced by reason bears a direct reference to imagination, which stands under the influence of the intellect. 10

In British empiricism, and particularly in the philosophy of David Hume (1711-1776), there developed a tendency to see imagination as a productive power of the intellect, an activity of the mind which plays a central role in stabilizing perceptions, making them intelligible as parts of an organized world. The word "imagination" was recurrently used by Hume in many contexts, and the interpretations given to his usage of this word are bound up with interpretations of Hume's philosophy as a whole.
Hume wished to make evident that ideas alone cannot afford a sufficient basis for belief. After he had established the point that value judgments are not based upon rational insights but upon feeling, Hume carried over the same point of view to epistemology, and he made belief rest upon feeling rather than upon cognition. Hume equated imagination with the "vivacity of conception" which constitutes belief, and he used the general word "passion" for beliefs as well as for instincts, propensities, feelings, emotions, and moral sentiments.\(^\text{11}\)

Insofar as epistemology, morals, and aesthetics all involve imagination and passion, they were all brought under the same general principles, and Hume recognized relations between imagination and reason, between imagination and feeling, and between imagination and will. He looked upon the principles which govern imagination as involuntary and spontaneous in their operation, and he recognized that the action of the imagination can take place both consciously and unconsciously.\(^\text{12}\)

Hume's theory of mental action reached its fullest development in his treatment of our belief in material objects. In Hume's treatment the imagination unites certain qualities to form the idea of substance, and our belief in the continued and independent existence of such bodies arises from a concurrence of some of these qualities with the qualities of the imagination.\(^\text{13}\)

H. H. Price\(^\text{14}\) suggests a reconstruction of Hume's theory which can be stated as follows: The material world is just an
imaginative construction incorporating actual sense impressions, a gigantic piece of imaginative extrapolation. There is no sense in asking whether the imaginative construction corresponds to the facts, for there is no conceivable way of getting to a realm of facts outside it. In developing his interpretation of Hume, Price proposes that the function of material-object statements is simply to give expression to some of our mental processes, and that this is a fundamental tendency of human nature.

In the Critique of Pure Reason, there is a passage where Kant said that the function of pure reason has need of another function which he called the "productive imagination." Cassirer calls this "one of the deepest and most fruitful sections of the first Critique." It reinforces Cassirer's view that man must first retreat into the world of play and fantasy in order ultimately to conquer the world of reality.

This formative activity always begins by holding off the world, as it were. . . . the tension between the I and its environment . . . is mediated. . . . by way of creative formation. Despite this insight, Cassirer in the end subordinates the principles of imagination to the principles of understanding. It was upon this point that he parted company with Martin Heidegger. In the year 1929, Cassirer and Heidegger met at Davos in Switzerland to debate their divergent views of the philosophy of Kant. Heidegger's part in the debate was founded upon his recently completed work entitled Kant and the Problem of Metaphysics. In this work Heidegger claims that Kant, in the first edition of the Critique

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of Pure Reason, showed that the pure intuition of time must be directly related to the transcendental imagination, and that this remains the inner truth of the first Critique.

The transcendental imagination must be understood as the power which is able "to give to itself that which it intuits." The giving is a giving of form at the transcendental level, while the intuitions occur at the empirical level in the presence of an intensive reality which is independent of the human intellect. An "image," however, is not a likeness of what is given but is that which constitutes the appearance of an object as object. Kant specifically set forth imagination as formative of images in exactly three ways, namely: (1) It forms images which are representations as present, (2) it forms images which are representations as reproductions, and (3) it forms images which are representations as anticipations. These three forms are representations of images as present, past, and future, respectively. Such images are always and necessarily related to time, and the form given by the transcendental imagination is primordial time itself.19

In the first edition Kant described the transcendental imagination as "an indispensable faculty of the soul without which we should have no knowledge whatsoever,"20 and Heidegger maintains that at this time Kant himself conceived of imagination as a unique and primordial power out of which both intuition and thought function as elements. Heidegger presents arguments based on Kant's
account in the first edition to show that intuition at its root is simultaneously receptive and spontaneous and that at its root, understanding, too, is simultaneously spontaneous and receptive, so that the two are not discernible at their common root. Transcendence is a unity and not a duality, and the transcendental imagination as originally conceived by Kant is the root of the two stems, a power which Kant referred to as "the unknown common root of which we are hardly ever conscious."²¹

This was Kant's original and radical insight, but Kant was still influenced by the traditional notion according to which imagination must be regarded as a faculty subordinate to reason. If reason were transformed into a product of the transcendental imagination and were shown to be dependent therefore upon primordial time, Kant would have been "brought before an abyss," Heidegger declares. From the abyss, Kant recoiled; in the second edition of the first Critique, Kant had made a decision in favor of the priority of the principles of the understanding over those of the imagination.²²

Heidegger argues that by giving the transcendental principles of the understanding priority over those of the imagination, Kant over-emphasized the principle of consciousness, and that this is the general thrust of all of modern epistemology. Where idealism sought to overcome the split between reality and consciousness by identifying all reality with the pure absolute Ego, Heidegger attempts to re-introduce the epistemological dualism favored by
Aristotle and to establish a linkage between the two aspects of human knowledge, based upon their common root in the transcendental imagination. Heidegger interprets the Kantian philosophy in such a way that the activity through which our reality is constituted is an imaginative and not a conceptual one. This point of view is reminiscent of the spirit of romanticism.

Rationalism demands the elimination of emotion, but for the romantics there are aspects of the world, let alone of the self, which cannot be known without correct emotion. The romantics believed that the natural man experiences life in a primal way in which a spontaneous overflow of feeling is inextricably bound up with thought. The romantic thesis might be expressed by saying that human beings require for their organic well-being, for their mental and moral health, and even for their correct relations with external reality, the kind of inspiration which accompanies the poetic thought which is far more natural to them than is rational thought.

An important theme of romanticism is that feeling is cognitive, and that feeling and fantasy are morphogenetic with respect to human thought as well as to human behavior. The pre-romantic German poet and Kantian scholar, Friedrich von Schiller, expressed this theme by saying:

"It is . . . not going far enough to say that the light of the understanding only deserves respect when it reacts on the character; to a certain extent it is from the character that this light proceeds; for the road that terminates in the head must pass through the heart."
Many romantics expressed an anxiety that, as a result of the Enlightenment's emphasis upon a purely rationalistic and/or positivistic approach to human experience, man's emotional life was in danger. They called attention to the original meaning of the word "religion," which is "to reconnect" or "to tie back." They encouraged a rekindling of implicit religious feeling in unorthodox forms. For example, Friedrich Schleiermacher (1768-1834) conceived of religion as "the greatest love of the greatest beauty."

Where conceptual thought demands fixed boundaries, romantic thought tended to dissolve all boundaries in the flow of emotional life. But there must be criteria for the validity of the emotions as well as for the operations of the intellect. False or inadequate emotional responses lead to misunderstanding of reality as surely as does a false or inadequate perception or false or inadequate reasoning.

William Wordsworth (1770-1850) gave voice to the romantic perception that the modern consciousness was becoming sick and distorted when he wrote in the Preface to his Lyrical Ballads:

... a multitude of causes unknown to former times are now acting with a combined force to blunt the discriminative powers of the mind, ... The most effective of these causes are the great national events which are daily taking place, and the accumulation of men in cities, where the uniformity of their occupations produces a craving for extraordinary incidents which the rapid accumulation of intelligence hourly gratifies.

Wordsworth here anticipated the Marxian theme of man's alienation from himself as a consequence of the industrialization and the urbanization of society, but he traced the ultimate cause
of this illness not to economics and sociology but to epistemology, and the remedy which he proposed was quite different from that proposed by Marx; it was that the individual must recover "primal sympathy," "emotive perception," and "feeling intellect."

Wordsworth asserted that the human mind is capable of being excited, indeed of becoming exalted, without the aid of "gross stimulants," and that the endeavor to produce or enlarge this capacity is the best service in which anyone can engage, especially, he said, "in the present day."\textsuperscript{26}

There emerged in romantic thought the idea that religion, art, and even science have their source in a spiritual aspect of man's nature which is not fully conscious, but which functions as guide and ally. For example, Friedrich Schlegel (1772-1829) wrote of the artist's relation to his "unknown self":

He would not dare to appropriate to himself what his most passionate exertions have failed to effect, but he cannot attribute to any extraneous power that of which he is so intimately conscious as his own peculiar possession; he has gained a new portion of his unknown self.\textsuperscript{27}

Schlegel goes on to say that all during our apparently vain efforts, subconscious allies are being let loose behind the scenes, but the unknown self functions in this helpful way only so long as the ego maintains the right relationship with the unconscious, i.e., with nature. If our efforts are misdirected, there will be let loose not allies but agents destructive to the personality.\textsuperscript{28}

The romantics emphasized creative imagination as part of their general emphasis upon the whole person and upon the evolution of
individual personality. They insisted that knowledge must include an intimate acquaintance with things known, intuitions of them, involvement with them. We can participate in what we encounter and become part of it; if one's knowledge, however scientific, fails to affect one's nervous system, it is partial and incomplete knowledge.

Obviously romanticism could not be expressed in a systematic philosophy but could only be understood as a general attitude toward life and the universe. Nevertheless the romantic spirit found a philosophical voice in Schelling. Schelling built a philosophy around the romantic idea, derived from Kant, that true and appropriate emotions are elicited by aesthetical ideas, because it is these ideas which bind together the self and the not-self.

Under the influence of Fichte, Schelling started out from the pure absolute Ego as the highest principle, but he was soon led to the conclusion that the rational, conscious, and free dimensions of the intellect require complementation from another and very different sphere.

In Schelling's system, philosophy must be composed of two separate spheres which are opposed to each other in principle but which nevertheless reciprocally supplement each other. These two spheres were visualized by Schelling as the sphere of the natural philosopher (the scientist) who is wholly intent upon the objective, and the sphere of the transcendental philosopher who seeks the entire exclusion of the objective from the purely subjective
principle of knowledge. Schelling posed the question: By what means are the two separate but fundamental spheres related?

How, at the same time, the objective world conforms to representations in us, and representations in us conform to the objective world, cannot be conceived, unless there exists a pre-established harmony between the two worlds of the ideal and the real. But this pre-established harmony is itself not conceivable unless the activity by which the objective world is produced is originally identical with that which displays itself in volition, and vice versa.

This original activity must be productive, as is the activity which manifests itself in voluntary action. But there are then two forms of productive activity, one of which is productive with consciousness, in our free volitions, and the other of which is productive without consciousness. Schelling held that it is products of one and the same imaginative activity which appear to us beyond consciousness as real and on the hither side of consciousness as ideal. He maintained that if this is the case, there must be an activity also within the subjective which is both conscious and unconscious. This activity is aesthetic.

Schelling postulated that a great work of art is brought to completion by two thoroughly different activities. One of these—the conscious—involves deliberation and reflection, can be taught and learned, can be received from others, can be attained by practice. The other activity, the unconscious, cannot be influenced by any voluntary effort but can only be a spontaneous gift. Either without the other has no value.
...the gods have so firmly tied the exercise of that original power to painstaking human effort, to industry and deliberation, that without art, poetry, even where it is innate, produces only products that appear lifeless, in which no human understanding can take delight.32

Aesthetic intuition is that intuition the product of which is continuous on the one side with the product of nature and on the other side with the product of freedom; it unites within itself the characteristics of both. Nature was equated by Schelling with objectivity and necessity, while consciousness was equated with subjectivity and freedom. Aesthetic action is action in which freedom (consciousness) and necessity (unconsciousness) are united, and in the product of which, conscious and unconscious activity appear fused into one. It would be futile, therefore, to ask which of the two constituents is prior to the other, since either without the other has no value.

In the work of art, conscious and unconscious activity are presented as one, not as they are in themselves but as they are for the ego. But if the ego is to be conscious of both, they must appear separately. Here there is a contradiction, and Schelling averred that this contradiction "seizes upon the ultimate" in the whole personality. In the process of creation consciousness must be "carried along" until it reaches the very point where the two activities, the necessary and the free, become one. Just here conscious control of the production must stop, for the ultimate identity cannot appear in consciousness. Yet it can be reflected in the product, the work of art.33

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In its higher function, science has the same problem as art, since there is a contradiction in the fact that the idea of the whole can grow distinct only by an unfolding of individual parts, while the unfolding of individual parts is possible and conceivable only through the idea of the whole. This contradiction can be resolved only through a confluence of conscious and unconscious activity. Science cannot achieve this confluence within the scientific framework, since, in science, any problem whose solution can be discovered by genius must also be soluble mechanically, and there can be no mechanical solution to the contradiction. The problem must remain for science what Schelling called "an infinite one."34

Schelling concluded that the productive faculty by which art "attains to the impossible" is the sole capacity by which we are able to think and to comprehend, and that this faculty is reflected back to us nowhere but in the work of art. Art, then, stands alone as the connecting link between the soul and nature, and it must be recognized as the true organon of philosophy.35

Like romanticism and idealism in general, Schelling's philosophy of the aesthetic quickly faded out of the mainstream of philosophical thought and appeared to end in failure and disappointment. On the one hand, unconscious forces came to be seen either as mechanical and reflexive or else as including an element of the demonic and the destructive, and on the other hand, human freedom came to be seen, even by Schelling himself, as a capability
for perverting and distorting the relationship between nature and spirit, as much as a capability for maintaining and furthering that relationship.

**Concluding Remarks**

In Aristotle's psychology, reason can exert an influence on image formation, and imagination to this extent stands under the influence of the intellect. Motivation is the issue; reason cannot motivate, because it does not deal with particulars, and motivation is tied to particulars.

Is the question of motivation relevant for epistemology? Aristotle apparently thought so, and Hume certainly thought so. It is at least arguable that Hume's epistemology laid the groundwork for Kant's third *Critique* and for the romantic movement.

But how are we to understand the relation between reason and imagination? Is the productive imagination simply responsible for the generation of structures that determine our "actuality" in such a way that that actuality gradually evolves in the direction of a Platonic vision of perfect rationality, leaving behind passions and emotions as well as individual perceptions and experience? This would imply that the direction of evolutionary process is determined in advance. In such a version of idealism, motivation is transferred to the transcendental level, and individuals become mere instruments for the fulfillment of rational goals.
Heidegger claims that Kant evaded the central question which his own philosophy had raised. He argues that insofar as it is imagination which gives rise to reason and to human consciousness, imagination must comprehend more than both together. Schelling, too, had argued that reason is a necessary but insufficient instrument for the understanding of nature and of life. Thus that "profound ambiguity" which was latent in the philosophy of Aristotle became explicit in modern philosophy, and it revolves around a debate about the relations between reason and imagination, about the problems of motivation, i.e., of how and toward what the world is evolving, and about the nature of temporality.

We have observed a philosophical conflict arising within the ranks of mathematicians, for whom also the central issue is that of the relations (if any) between mathematics and temporality.

If questions of motivation and evolutionary process are, indeed, of importance for epistemology, have these questions any bearing upon mathematics?
NOTES


3. De Anima 3. 7. 431a14, 3. 8. 432a1-8; Brentano, Psychology of Aristotle, pp. 96-100.


5. De Anima 3. 4. 429a10-429b9; Brentano, Psychology of Aristotle, pp. 96-98.

6. De Anima 3. 5. 430a10.


10. De Anima 3. 7. 431b, 3. 9. 432b20, 3. 10; Brentano, Psychology of Aristotle, pp. 100-5.


16. Ibid.

17. Ibid., p. 870.


26. Ibid., pp. 242-43.


28. Ibid.

30 Friedrich Wilhelm Joseph von Schelling, "Deduction of a Universal Organon of Philosophy, or Main Propositions of the Philosophy of Art According to Principles of Transcendental Idealism," *Philosophy of German Idealism*, pp. 203-16.

31 Ibid.

32 Ibid., p. 208.

33 Ibid., pp. 203-10.

34 Ibid., pp. 211-12.

CHAPTER XVI

AESTHETICS IN MATHEMATICS AND SCIENCE

Topology is a mathematical discipline concerned with the visualization of problems, including problems in analysis. Experts in topology claim that their methods render complicated structures in analysis spatially intuitive, because topology can provide a "language" which directly "speaks about" the nature of space, considered as a medium in which continuous processes occur.¹

For some time after its initiation as an independent discipline by Poincaré in 1895 and its brilliant elaboration and development by Brouwer in 1911, topology existed in isolation from other mathematical disciplines, but this initial isolation has since been replaced by what one writer calls "an almost embarrassing involvement of topology in the general development of mathematics."²

In a general way, topology is the study of qualitative properties of geometric configurations which remain invariant under certain kinds of transformations, and these properties are closely associated with the concept of continuity. For example, if one imagines a circle as made up of some elastic, deformable material, one can easily picture this elastic circle as contracting and expanding, as undergoing deformation to the shape of an ellipse,
or to the perimeter of a triangle or of a square. Such transformations do not at all affect the neighborhoods of the points which make up the circle. Configurations into which the elastic circle (or any other geometric configuration) may be transformed without affecting the neighborhood relations of the points are called "homeomorphic" or "topographically equivalent" structures. By contrast, making the circle into a line by cutting the perimeter does affect the neighborhood relations of the points, and therefore an open curve such as a line is not topographically equivalent to a closed curve such as a circle. Plane surfaces may be similarly treated, and topology has sometimes been called "rubber sheet geometry."

It has been found that certain properties are regularly associated with topological equivalence. For example, the property of dividing the plane into two regions belongs to the circle and to all of its topological equivalents, but it does not belong to the line. Another important property associated with topological equivalence is that of dimension. A one-dimensional, two-dimensional, three-dimensional, or n-dimensional configuration will retain its dimensionality throughout all "one-to-one bicontinuous transformations." This was first discovered by Brouwer.\(^3\)

The generality of topology, which he called analysis situs, was graphically explicated by Poincaré.

... it is here that geometric intuition truly comes into play. ... it is often said that geometry is the art of careful reasoning from badly drawn figures. ... What is a badly drawn
figure? It is one which our clumsy draughtsman might have done; he changes the proportions more or less grossly; his straight lines have alarming zigzags; his circles have disagreeable bumps; all this makes no difference. . . . But this inexperienced artist must not represent a closed curve by an open one; three lines intersecting in a single point by three lines which have no point in common; a surface with a hole in it by one without. . . . Intuition would not have been impeded by defects in the drawing which are of concern only in metric or projective geometry; it will become impossible in case defects have reference to analysis situs.

Since geometric intuition is concerned with just those features which are studied by topology, Poincaré regarded this branch of mathematics as embodying the essence of geometrical intuition. It is no accident that the pioneer work in topology was done by the leaders of the intuitionistic school of mathematics, Poincaré and Brouwer.

One fundamental concept in topology is that of a topological structure or a "topological space." In order to grasp this concept one must first return to a suggestion made by George Cantor (whom we have met as the founder of set theory) that any geometric configuration may be conceived as a set of points in Euclidean space. A set of points is a completely amorphous and structureless object that may be given any number of structures.

A given set may possess more than one kind of structure. . . . Much of the day-to-day work of contemporary mathematics consists precisely of interrelating various structures in an intimate and interdependent manner. Functional analysis, for example, superimposes a topological and/or measure structure on algebraic systems. . . . On the other hand, one may consider the algebraic composition of certain given topological structures.
Topology now has many strong interactions with modern algebra, algebraic analysis, and the theory of partial differential equations. It is currently impossible, says Carl Boyer, to imagine a theory in analysis which would not be based on a previous topological study, and in spite of the apparently vague results to which it sometimes leads, topology is closely linked with the most precise mathematical questions.

Topological thinking is of another character than that demanded by abstract algebra, although it may appear in some ways similar. The greatest difference arises from the fact that in algebraic structures, one is concerned with combinations according to rule of certain set elements to produce other set elements, and the emphasis is upon logical analysis with little or no regard for spatial intuition. Again, in metric structures one is concerned with notions related to extension or measurement, involving such terms as areas, angles, volumes, mass. In projective structures, one is concerned with proportionality. In topological structures, one is concerned with relations between set elements (which need not be points, but may be anything at all) involving the purely qualitative concept of "neighborhood." The neighborhood concept has been described as the generalization of the pictorial notion of a ball, but completely liberated from the concept of distance.

It is possible to make an association between the topological concept of "neighborhood" and the psychological concept of continuity between events in time which we think of as relevance or context.
Relevance is that species of connectedness which makes a series of otherwise disconnected events into what we call "a story." I quote a fable taken from Gregory Bateson's work in philosophical anthropology:

A man wanted to know about mind. He asked his private large computer: Do you compute that you will ever think like a human being? The machine set to work to analyze its own computational habits, and finally printed out its answer. The man ran to get the answer and found neatly typed the words: THAT REMINDS ME OF A STORY.8 The moral of this tale is that natural human thinking is continuous and story-like.

Efraim Fischbein has recently (1987) undertaken a study of the role of "intuition" in human cognition. The word "intuition," he finds, has been given many different, often conflicting interpretations, and it has special connotations in various fields, as in art, in morals, and in scientific and professional usages. In spite of this, certain features are common to all usages. Intuition is immediate and presents itself as being self-evident. Intuitive representations have a powerfully coercive quality. They always imply a global element, that is, a sense of an indeterminate totality within which the particulars are related and to which they all belong.9

His study leads Fischbein to the conclusion that intuition is rooted in the syncretistic, constructive thinking which is typical of children and of human beings in the early stages of
civilization. In other words, intuition belongs with the story-making propensities of the human mind. Its function is to establish continuity or relevance, to make stories out of the facts at hand.

Hume and Kant recognized that there are certain forms of lawful connectedness among events, e.g., cause-and-effect relations, which cannot be shown to prevail independently of the human intellect. Both neo-Kantians and intuitionists saw mathematical structures as products of the inherent creativeness of the human intellect. Apart from lawfulness, an important function of the story-making propensities of the human intellect has to do with motivation.

Aristotle perceived that motivation is tied to the representation of particulars, i.e., to the presence of images. We now perceive that motivation requires not only particulars but also continuity and context. It is not only decision and action which must be so motivated, but thinking itself as well.

Fischbein's studies have shown that even mathematical thinking is stimulated by imagery as it cannot be stimulated by purely formal presentations. Mathematically stimulating imagery is not static but dynamic; it is the imagery which accompanies a constructive internal activity and which leads to an intuitive rather than to a purely rational understanding of the problem at hand. We are familiar with the intuitionists' demand for constructivity in mathematical proofs, and with their insistence that mathematical activity is inventive, unpredictable, and dynamic. The intuitionists
have recognized the necessity for non-deductive, informal elements in mathematical work, and it appears that this approach is realistic from the psychological point of view, if from no other.

That reason and experience do not alone suffice in scientific work has often been remarked. Claude Bernard has been quoted as saying that the experimental method is based upon a tripod consisting of feeling, reason, and experience, and that feeling takes the initiative and engenders the idea. The problem is not to eliminate psychological processes but to learn how to develop them and to use them creatively, to learn to understand the separate uses of, the pitfalls of, the interactions between those aspects of scientific research for which no complete formulation is possible and those formal structures for which no adequate intuitive interpretation is possible.

Augustus De Morgan long ago made a plea for the development of a logic which would be suited to the needs of human thinking in general. The need for a methodological logic is now being recognized by some thinkers, that is, the need for a logic suitable to the actual language used in scientific research and mathematical invention, a language including such terms as "plausible," "understandable," and the like. It has also been pointed out that many, if not most, students and historians of the humanistic disciplines, e.g., of serious reflective literature, lack the training which would allow them to recognize and critically evaluate the epistemological and logical presuppositions upon which such
Intuitionists may be closer to devising a workable methodological logic than are the formalists or the logicists.

The question remains whether an intuitionistic approach to mathematics and logic can be made congruent with the formal-axiomatic point of view. There are intimations that it can. For example, Fraenkel reports that the neo-intuitionistic logical system is incomplete. What this means is that the addition of an underviable formula does not make the system inconsistent. From a formal-axiomatic point of view, this incompleteness permits the formula for the principle of the excluded middle to be added or dropped, so that there are two kinds of logic one of which includes the other, and this may be regarded as analogous to the dropping of the axiom of parallels from Euclidean geometry.\(^\text{14}\) It is true that intuitionists do not accept the formal-axiomatic attitude according to which the choice of axioms can be arbitrary, but the fact that such an interpretation is nevertheless possible indicates that the two systems need not be technically incompatible. Again, Kurt Gödel has given an interpretation of terms and symbols according to which classical arithmetic actually appears to be part of intuitionistic arithmetic. Gödel's interpretation is not acceptable to the intuitionists, but classical mathematicians have concluded from it that intuitionistic arithmetic is not narrower than classical arithmetic.\(^\text{15}\)
Some mathematicians (for example, Paul Bernays) have already adopted an eclectic position and have admitted the legitimacy of both intuitionistic and classical approaches in mathematics, since many propositions may be regarded from complementary points of view, and it is not always necessary to choose among the logical, the epistemological, and the psychological, although it may indeed be wise to distinguish among them. The productivity of such an alliance seems to be demonstrated by the example of topology. And a formal definition of topological space, making use of the concept of neighborhood, has been given by Felix Hausdorff (1888-1942), known as "the high priest of point-set topology." He postulated that by a topological space one understands a set $E$ of elements and certain subsets $S_x$, known as neighborhoods of $x$. The neighborhoods are assumed to satisfy four axioms, known as the "Hausdorff axioms." Using this definition, Hausdorff was able formally to introduce continuity and convergence as qualitative concepts.\textsuperscript{16}

As a writer interested primarily in the philosophy of education, Fischbein complains that curricula and textbook writers have tended either to emphasize the intuitive approach or to confine their strategies to the purely formal, ignoring the fact that every thinker must learn how to abstract formal structures from practical realities and intuitive interpretations, how to describe them explicitly, how to evaluate each step and to accept erroneous guesses; for this is how everybody solves problems.\textsuperscript{17}
The student of epistemology should be brought into direct confrontation with didactical situations, and not merely exposed to verbal explanations and discussions. Such situations should present aspects of research and invention which demand intuitive strategies, and they should also present formal structures for which no intuitive interpretation is possible. It is a matter of becoming intuitively and rationally convinced of the fundamental roles of both intuitive strategies and logical constructs in every serious intellectual endeavor.

Intuitionistic mathematics and logic may prove useful tools in the creation of such didactical situations. Students may well become enlightened as to the complementary roles of intuition and logic through the comparative study of intuitionistic and purely formal systems of reasoning, together with unprejudiced philosophical discussion of the epistemological issues involved. Such preliminary training would surely prove invaluable as a preparation for study and research in many disciplines, including psychology and the social sciences as well as the humanities.
NOTES


7 Roman, Some Modern Mathematics pp. xxv, 203.


10 Ibid., p. x.

11 Ibid. pp. 16-21.


13 Lovejoy, Great Chain Of Being, p. 17.


15 Ibid., pp. 233-34.


According to the thesis of this dissertation, the original and primordial intuition of the continuum is precisely that which distinguishes individual thinking from intersubjective and scientific thinking, and in excluding the intuition of the continuum from science, scientists tacitly accept the presupposition that life, in the form of continuous change, is extrinsic to ultimate truth and ultimate reality. This presupposition legitimizes the exclusion of concrete individuals from consideration in the search for scientific truth. Structure is thus conceived as ontologically and epistemologically prior to continuous process.

In our interpretation the Pythagoreans, who made no distinction between thinking and cosmogony, identified primordial intuition with what they called the Unlimited. In their original cosmogonical myth, an interaction between the Unlimited and the principle of Limit was recognized as necessary for the production of the world, but in their mathematical ontology, they attempted to assimilate the Unlimited entirely to the principle of Limit. The discovery of incommensurable magnitudes defeated the attempt and revealed that it produced self-contradiction in their system. Their ontology and their mathematical methods were all but destroyed by this discovery.
In the metaphysics of Heraclitus, the eternal flux was conceived as governed by a structure. In the rational metaphysics of Parmenides, the flux was adjudged illusory because of its lack of intelligibility, and with the flux, both time and motion were adjudged unreal. Parmenides accepted the continuum only insofar as it could be rationalized, i.e., could be conceived as homogeneous and indivisible extension. Hence, logic in ancient Greece was born as geometry, the science of ratios and relations among ideal figures in ideal space.

Under the influence of Parmenides, Greek mathematicians chose to rationalize the Unlimited in the form of ideal space, retaining certain organic aspects which were known through geometric intuition. Greek philosophers, prominent among them Democritus and Plato, addressed themselves to the metaphysical and epistemological dilemmas which had appeared with the discovery of incommensurable magnitudes.

The atomists re-introduced the digital, and with it the problems of infinity and the dangers of self-contradiction. After Plato had conjoined the mathematical with the conceptual and the linguistic, the idea of the structured as a separate realm apart from, and superior to, the continuous, as unstructured, was born. Plato noted that this paradigm left geometry in limbo, since geometry partakes of both the structured and the continuous. He classified geometry as a science inferior to arithmetic on this
account. He then proceeded to build a great philosophy around efforts to reconcile the structural with the continuous.

Alone among the ancients, Aristotle located the meeting ground of the Unlimited and the principle of Limit in the psyches of existent individuals. He regarded image-making as an indispensable form of two-way mediation between metaphysics and logic, object and subject, body and mind, necessity and freedom. He sought to maintain the link through his theory of causality and through what he called a postulate of reality. This thought paradigm exerted a powerful influence upon the formation of his subject-predicate logic and created what appears to modern philosophers as a profound ambiguity in his thought.

Plato's vision of structure encouraged Aristotle to enunciate the principles of logic. Had Aristotle also accepted the Platonic metaphysics, he would have invented logic along the lines of Eleatic thinking, as the Stoics did later. Logic might then have been from the beginning mathematical, i.e., based upon abstract relations. But Aristotle did not ignore the related entities. On the contrary, he considered that it is the nature of these entities that gives rise to logical relations in the first place. He did not conceive these entities as ideal but as actual and temporal, involved in continuous change. He attempted to resolve the problematic relation between temporality and structure through the concept of teleology. He conceived of continuous change as having purpose and direction, a purpose
and direction inherent in the nature of things. Innate ideas of structure and lawfulness are mere potentialities which require to be actualized within time; form and structure result from development on the part of actual temporal beings.

The difference between Platonic and Aristotelian epistemology is well brought out in the fact that Aristotle valued geometry for reasons quite different from those of Parmenides and Plato. An image of constructive activity guided by innate and indeterminate ideas of lawfulness dwelt in the back of Aristotle's mind, and this accounts for the otherwise inexplicable fact that he constantly used geometric illustrations in expounding his logic, even though the reasoning used in Greek geometry was not syllogistic. Aristotle saw both geometry and logic in terms of structure arising out of constructive activity.

In this connection, it is interesting to compare Aristotle's notion of geometric constructivity with that of Proclus, who prided himself upon being a strict Platonist. Proclus recognized constructivity as an indispensable feature of geometric reasoning and identified this constructive action as taking place in imagination. His idea of an "intelligible matter" as the stuff with which imagination works is derived directly from Aristotle. But Proclus regarded the geometer and his thinking as mere instruments of disembodied Reason (Nous). Aristotle, on the other hand, stated unequivocally that the potential for geometric construction arises from the actuality of the geometer's thinking.
in the same way that the actuality of a living being gives rise to the potentiality for parenthood. The geometer's thinking is by no means confined to purely logical or rational operations according to an already given law; in Aristotle's view, it includes the power of creation.

In Aristotle's logic the structural and the continuous, the rational and the experiential, have only partly been distinguished. Structure exists as a potentiality in the real entities of the temporal world; his logic is tied to temporality. It is founded upon his concept of primary substance, his theory of actuality and potentiality, and his theory of causation, which includes the all-important notion of final cause. His logic is inseparable from his metaphysics.

It has been mentioned that two antithetical thought paradigms operated concurrently in Aristotle's philosophy, in one of which the lower or less perfect is derived from the higher or more perfect, in the other of which higher states of being and knowledge develop out of lower. In the first case the structural is given epistemological priority, in the second case the continuous is awarded that position.

Aristotle assumed that logic is merely a guiding instrument in human thought. In Augustus De Morgan's view, the Aristotelian proposition is first-intentional, which means that it describes a telling-off of individual instances. In other words, Aristotelian logic is based not only upon the idea of constructivity but also
upon the idea of potential infinity, unlike second-intentional logic, which is based upon the concepts of aggregation and of the actual infinite. De Morgan's valuable insight that these two kinds of logic belong to different logical wholes and that they should not be treated as convertible has largely been ignored.

Plato had inherited from the Pythagoreans a dynamic and constructive method of generating the natural numbers. In his efforts to reconcile the digital with the continuous, Plato replaced the merely additive operations visualized by the Pythagoreans with a dynamic process involving multiplication and division. His effort to arithmetize the continuum could not succeed because he lacked an algebra, a generalized concept of number, and the concepts of variability and functionality, but he moved toward a vision of the infinite ideal numbers as already present from all eternity, representing all the possible relations among the Forms.

In the early modern era, the Democritean solution gave birth to mechanism and the Platonic solution gave birth to rationalism; both played a prominent role in the development of modern scientific and mathematical epistemology. The arithmetization of the continuum became possible when the limit was conceived as an ordinal idea and the actual infinite was accepted as a legitimate mathematical concept.

In modern empiricism, as in the philosophy of Cratylus, the continuous is closely tied to sense perception, but Frege himself called attention to the fact that it is the whole of man's
inner life that is in continuous flux, and he therefore identified continuous change with what he called "the psychological." Russell describes the psychological as "that essence of individuality which always eludes words and baffles description, but which for that very reason is irrelevant to science."^{2}

When the contents of experience are differentiated and ordered in a mathematical system, the psychological subject must be left out of the series, yet the ordering of the parts presupposes a totality which is not itself a member of the series but must be grasped in advance of the members of the series. As Aristotle put it, "the whole line must be apprehended by something in us that does not move from part to part."^{3} When the series is taken as a complete actuality, the nature and constitution of the subject is derived from the nature and constitution of the series, rather than vice versa; hence one concentrates attention upon the products of constructive activity rather than upon the activity itself.

The product, excluding the subject, presents an established law. The activity, on the other hand, depends upon a meaningfulness which belongs to subjectivity alone and which motivates the thinking of the rational and the objective. Meaningfulness, as distinguished from lawfulness, arises when, as in the case described by Kant, the representation is referred wholly to the subject, "and what is more to its feeling for life." This "forms the basis of a quite separate faculty of discriminating and understanding."^{4} a faculty which Kant identified as aesthetic judgment.
Kant distinguished between the mathematical judgment and the aesthetic judgment on the ground that the former is an objective formal judgment while the latter is a subjective formal judgment. In mathematical as well as in aesthetic judgments the judgment is attended by a consciousness of detachment from personal interest and involves a claim to universal validity. But aesthetic judgments are not subject to proof.

In mathematical constructions, objectivity derives from the fact that while the construction is carried out by an individual thinker on the basis of free choice, yet every person who chooses to carry out such a construction must arrive at the same result. What is brought about through construction is that an indeterminate idea of lawfulness resolves itself into determinate law, which must be the same for everyone.

Kant analyzed the aesthetic judgment as depending upon a free conformity to law on the part of the imagination. What does this mean? In the case of aesthetic activity, expression does not convert meaningfulness into specific meaning which must be the same for all; rather the aesthetic expression, which is individual and concrete, evokes the universal idea of meaningfulness itself, because it expresses a state of the individual subject in which the idea of meaningfulness has been brought into harmony with the idea of lawfulness. The goal of aesthetics, then, is to bring about a harmonious relation between private thought processes, based on the private idea of meaning, and publicly sanctioned thought processes,
based on the idea of law, or to bring about a harmonious relation between individuals and the natural law. Aesthetics aims at reconciliation between individuals and the larger wholes to which they belong. Hence it is said that the contemplation of aesthetic ideas elicits true and appropriate emotions because these ideas bind together the self and the not-self.

The neo-Kantians, including Cassirer, regarded the aim of evolution as the resolution of the material and the informal into the fully structured. This would entail that the continuous must become an infinite aggregate of points; it means regarding boundaries as fixed and determinate and all relations as serially ordered in a hierarchical structure. Even granting that this outcome is conceived as an unattainable ideal, it is in contrast with Brouwer's description of a priori time as a one-dimensional continuum in which the moments of time simply fall apart because they are qualitatively different from each other.

As constructivists, the intuitionists do not conceive that the continuous can be exhaustively converted into the structured, or even that it should be. They maintain that there will never be an end to the process of conversion, and that those aspects of the continuous which do become structured do not entirely lose their continuous character. The structuring of the continuous takes place through a creative activity on the part of individual subjects and involves acts of free choice. This process has been recognized by some philosophers as an ideal model for the
possibility of creatively structuring time and experience, just as Greek geometry is an ideal model for structuring our intuitions of space.

Intuitionists attempt to confine mathematical operations within the first-intentional logical whole, where material and informal elements must play their part and where there is a linkage between the structured and the continuous. They have attempted to claim equal epistemological status for both. They follow Aristotle in asserting that the continuous is a primordial given, that intuition is an active power, and that individual thought processes, occurring in time, are of central significance, even in mathematical philosophy. Here one finds again that there is an association between one's attitude toward individual thinking and one's attitude toward the continuous. The intuitionists believe that objectivity and permanent truth value can be established within time by means of constructions. This accords with Aristotle's idea that what exists initially as indeterminate potentiality can become determinate actuality.

As mathematical activity is an expression of the indeterminate idea of lawfulness, its product is determinate law. Yet intuitionists see mathematical activity as having more in common with artistic activity than with science. One important parallel is that in intuitionism a distinction is made between the significance of the activity and the significance of its product. Intuitionists refer to the activity as "first-order mathematics,"
and they refer to the study of its product as "second-order mathematics." The former is of greater significance for individual thinkers, the latter of greater significance for society. This intuitionistic model is of importance for epistemology because it provides a workable description of how the two may be related, as well as by what characteristics they must be distinguished.

Throughout the history of epistemology, two kinds of thinking have been recognized, under various names. Examples of one kind of thinking are found in dreams and in sensory representations. Examples of another kind of thinking are found in arithmetic. The former kind of thinking is concerned at a less conscious level with the qualitative concept of connectedness or relevance, and hence with continuity, and at a more conscious level with internal relations of parts within metaphysical wholes, hence with a partially structured reality in which the parts retain a measure of continuity. The prototype of such thinking is found in organic processes, and it lies at the root of Aristotle's concept of primary substance and the logic which he built upon that foundation. The second kind of thinking is concerned with external relations among discrete, determinate concepts with fixed boundaries, and it can be identified with the digital. Its prototype is found in mechanical models, and it lies at the base of analytic recursive procedures used in mathematical logic. This kind of thinking is characteristic of intersubjective or publicly
verifiable thinking, insofar as such thinking tends toward the objective and the scientific, the structured and the lawful.

The former kind of thinking is identified in anthropology with the mythical, in psychoanalysis with primary process, and in other fields with intuition, just as the latter kind of thinking is associated with the fully conscious ideation known as pure reason or secondary process. Primary process is historically the earlier form. When understood as passive-receptive in relation to an already subsistent external reality, it may be identified either with sense perception or with intuition, but there is evidence that primary process occurs in advance of, and separately from, what we usually think of as knowledge of reality, that it is bound up with internally motivated (subjective) factors as much as with the external world, that it is a syncretistic and constructive process of image-making which ordinarily takes place unconsciously, or which is, at any rate, largely beyond conscious control, and that its presuppositions are necessarily built into our knowledge. Organic space, the space of movement and action, for example, belongs to primary process, while the theoretical space of modern physics belongs to secondary process. Similarly, personal time, in which the parts are qualitatively different from each other, belongs to primary process, while scientific time, the parts of which are equal and homogeneous, belongs to secondary process.
Image formation is thought of as concrete and particular rather than as abstract and general, and there is a tendency to associate it more or less exclusively with the visual sense. This may be too narrow a view of primary process. The anthropologist Paul Radin speculates that an over-emphasis on the visual sense appeared, along with an exclusive preoccupation with the external world, as one of several momentous consequences which followed upon the invention of the written word.\(^5\) Gregory Bateson points out that the self-understanding which accompanies a belief such as totemism is profoundly non-visual.\(^6\) The same is true of our personal, pre-scientific apprehensions of space, time, and causality.

Primary process played a conspicuous role in the development of the calculus, and its influence is evident also in Hume's discovery of the important roles played by imagination and feeling in all forms of empirical knowledge. It was to the work of primary process that Kant appealed in his second and third Critiques when he spoke of the role played in moral and aesthetic judgments by an indeterminate concept of noumenal reality, and of aesthetical ideas as representations of the imagination to which no concept can be adequate, but which have an exemplar necessity resulting from the harmonious interplay of our cognitive powers.

One can still recognize the influence of primary process in De Morgan's concern for the preservation of first-intentional propositions and metaphysical wholes in the vocabulary of logic,
but in the nineteenth century, one finds in Frege's work the first decisive and successful severing of the linkage between primary process and secondary process. Since Frege, many philosophers belonging to the mainstream of Western thought have sought to eliminate primary process not only from mathematics and logic but also from language analysis and from philosophy.

It is a troublesome feature of primary process that its judgments cannot be directly communicated, but each individual subject must make use of his or her own powers in order to acquire them. Another troublesome feature of primary process is that it tends to move in circular patterns. For example, because someone dislikes me, I will behave in certain ways, and because I behave in these ways, the other person dislikes me; or the syllogism may involve more premises but come back to the first in the same circular way. Such cycles are characteristic of self-regulating eco-systems.

That circular chains of cause and effect are the rule rather than the exception in natural processes was noted by Aristotle, who cited as an example the cycle by which rain is produced. It was also noted by Aristotle that there are circular patterns in human cognition. He said that circular demonstration is not possible in the unqualified sense of the term, but it is possible if the word "demonstration" is extended to include "that other method of argument which rests upon a distinction between truths prior for us and truths without qualification prior." He added
that "such is the the syllogism which establishes the first and immediate premises."\(^7\)

The romanticists considered that the images and fantasies produced by primary process are in some sense morphogenetic, they are in some sense built into our knowledge of ourselves and things. They also held that a transformation of these images, while hard to achieve, can produce a new ordering which directly affects our emotions, value judgments, and actions, if not our very thoughts and perceptions.

In attempting to combine ethnographic studies with information theory, some anthropologists\(^8\) have concluded that the iconic component in private thinking corresponds with the analogic or continuous nature of certain biological processes such as those involved in growth and maturation, while publicly sanctioned concepts and valuations tend to be digitally coded, as for example by rituals which mark the boundary lines between youth and maturity, or between the seasons. That process and structure mutually affect each other is reflected in the thought of the German idealists, who converted circular movement into an ever-ascending spiral. In the circular thought pattern, contradiction is avoided by the introduction of time, and since circular chains of thought are typical of primary process, Bateson suggests that while secondary process provides guiding principles with respect to structure, primary process provides guiding principles with respect to
growth and development, and that both are needed to provide an adequate model for learning, discovery, invention, and evolution. 9

Private or continuous thought processes may be broadly described as the spontaneous ongoing construction of a partly changing, partly stable iconic representation which embodies a particular point of view, along with a symbolic representation of an indeterminate global reality. It creates a context for choice, decision, and formative action, both mental and physical. It is of the essence of such processes that they involve a principle of choice and some degree of personal autonomy, and that in them universal form and particular actuality are presented simultaneously. It is just this feature which enables the intellect and the sensitive-emotional sphere to exert a reciprocal influence upon each other.

In an adequate epistemological model, the two kinds of thinking can be brought into relation with each other, as shown by the successful and fruitful combination of topology with formalism in contemporary mathematics. History shows, however, that there has been a prevailing tendency in Western thought to convert iconic or continuous thinking into digital thinking wherever this is possible.

There have been in the history of the West no more outspoken advocates of the importance of individuals than the romantics, and the great logician, Alfred North Whitehead, has called attention to the failure of Western scientists to take the romantic movement
with the seriousness which it deserves. It has been remarked that in modern times the individual consciousness encases itself and shuns what was once regarded as natural, out of fear of anthropomorphizing or psychologizing objective reality, and that these patterns of Western thought may prove inadequate for directing the tools of technology. The need for a corrective influence may be both greater and of another nature than has been supposed. Identifying the humanities in general as disciplines in which continuous and iconic thinking exercises a predominant influence may add a new content to our understanding of their significance and their role in human culture, but while compartmentalization prevails, the corrective influence can be but minimally effective. It must be through re-establishing the linkage between private and public thought processes that a better balance can be achieved. This linkage is equally important in the humanities and in science, for, as Schelling saw, poetic thinking alone cannot fulfill the human personality, while scientific research demands the exercise of imagination just as much as do the arts.

Evocation of the idea of meaningfulness brings about a quickening of cognitive powers in the absence of a definite cognition, and it brings about a quickening of moral sensibilities in the absence of any specific demand for moral decision or action. Religious symbols and works of art, while being in themselves highly concrete and individual, express an aesthetic idea which remains forever indeterminate and open to interpretation. The more
indeterminate such an idea is, the more successfully it can fulfill its role, which is to evoke the idea of meaningfulness itself.

Meaningfulness, while contemplated in the form of specific imagery, remains itself always non-specific, and it is for this very reason of great value for evolutionary development, since it militates against what Roy Rappaport has identified as the canonization of specific content. The canonization of specific content, says Rappaport, undermines evolutionary flexibility, the ability to adapt to every kind of change, even change in the specific content of rules and values themselves. Since loss of evolutionary flexibility means diminution of the power to survive, the non-specific idea of meaningfulness has great survival value. It has the power to motivate toward indeterminate long-term goals as against determinate short-term goals, and this, too, makes for survival. If the non-specific idea of meaningfulness evoked by religious symbols and works of art has both great motivating power and great survival value, it deserves to be taken seriously by philosophers in general and by epistemologists in particular.

If the outcome of evolutionary development depends in any measure upon our choices, then the problem is not only epistemological but pre-eminently philosophical. There was a function which philosophy originally fulfilled, one not easily delegated, namely, the function of correcting one-sided situations in the collective ethos. At the time of Socrates, the mythico-religious
thought forms had long predominated, to the point that the
development of rational, scientific thought forms had been
crippled. Given the supposition that it is possible for another
kind of one-sidedness to develop, it must be the task of philosophy
to correct this imbalance as well.
NOTES

1Metaphysics 9. 8, 9. 9.
2Introduction to Mathematical Philosophy, p. 61.
3See chap. 9 above.
4See chap. 13 above.
5Primitive Man as Philosopher, pp. 53-62.
6Mind and Nature, p. 156.
7Prior analytics, 2. 23; Posterior analytics 1. 3. 72b25, 30;
2. 12. 95b37-96a8.
8Bateson, Steps to an Ecology of Mind; Mind and Nature;
9"From Classification to Process," Mind and Nature,
10Science and the Modern World (New York: Macmillan Co.,
120-22.
11Wilshire, Introductory Note to "Hölderlin and Baudelaire:
Human Estrangement and World-Weariness," Romanticism and Evolution,
p. 260.
12Rappaport, Ecology, Meaning, and Religion,
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