USING THE PROCESS OF WRITING TO REVEAL CHANGES IN MIDDLE SCHOOL STUDENTS’ ALGEBRAIC REASONING IN RESPONSE TO OPEN-ENDED WRITING PROMPTS

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Dedication:

To Mumzy – for always giving me that extra nudge.

To Emi – may you have the courage to pursue lifelong learning.
Acknowledgments:

Thank you to those that have helped me along this journey:

To my committee chair: for your wisdom and guidance

Dr. Hannah Slovin

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Dr. Helen Slaughter, Dr. Kathleen Berg,
Dr. Morris Lai, Dr. Melfried Olsen, and Dr. Michelle Manes

To my cheering squad: for without them I would still be writing today

Mumzy and Popsie

Jareus

Emi
Abstract

This qualitative study aims to reveal and describe those changes in middle school students’ emergent algebraic reasoning by analyzing their written responses to open-ended mathematically themed prompts, constructed using the process of writing. In this study, I served as both the teacher and the researcher in my own classroom. All grade eight students in my Algebra I classes responded to seven open-ended mathematically themed prompts throughout the school year. Of those forty-four students, eleven were purposely chosen to participate in this study based on criteria such as academic achievement and proficiency with the English language. Furthermore, the students were deliberately chosen at the end of the school year to minimize any intentional or unintentional bias throughout the school year. Students wrote three drafts for each of the seven open-ended prompts and received teacher-to-student written feedback upon submitting their second draft. Each of the eleven students participated in a one-on-one interview at the end of the school year for the purpose of gaining an insider’s perspective on their experience, validating the emergent themes from the written data, identifying possible contradictions within the written data and/or the students’ shared experiences and providing internal validity during the analysis phase of the study.

A content analysis focused on gleaning meaning suggested by students’ written use of the symbolic and descriptive language of mathematics was conducted; both individual drafts and each set of collective drafts were analyzed. Individual drafts focused on the features that were not seen in the previous drafts and collective sets focused on students’ conceptual maturation for each mathematics theme. This analysis revealed: a)
students recaptured known information using a show-and-tell approach in their freewrite; b) students experimented in the construction and expression of their own mathematical understandings in their second draft; and c) students reconsidered, after receiving feedback, their preexisting ideas in their third draft. A three stage model, knowledge acquisition, mulling over working knowledge, and conceptual evolution, grounded in the data, describes the progression of students’ conceptual understanding.
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Chapter 1

Context of Problem

Mathematics achievement in the state of Hawaii has lagged behind national counterparts in both private and public schools. In 2010, 49% of public school students demonstrated proficiency in mathematics on the Hawaii State Assessment (HSA) (Vorsino, 2010b; Vorsino, 2010d), an increase of 5% from 2009 (Moreno, 2010; Moreno, 2009a). Despite the increase, the Thomas B. Fordham Institute report gave Hawaii public schools a “C” for its educational standards citing that the mathematics learning goals were “mediocre” and “often vague” (Vorsino, 2010c, p. B1).

In 2009, Hawaii’s grade eight students from both private and public schools fell below the national average on the National Assessment of Education Progress (NAEP) even though their mathematics scores improved from 269 to 274. The national average rose from 280 to 282 (Moreno, 2009b). Furthermore, 65% of grade eight students in Hawaii public and private schools demonstrated basic skills, or showed partial mastery, in mathematics compared to the 71% across the United States on the NAEP. Additionally, only 25% of Hawaii’s children were deemed proficient on the NAEP, or demonstrated advanced mastery, in mathematics compared to the 33% nationwide (Moreno, 2009b).

In 2013, the U.S. Department of Education Institute of Education Sciences National Center for Education Statistics (2014) reported Hawaii’s grade eight public school students’ average score on NAEP was 218, an improvement from previous scores. Furthermore, the percentage of students who performed at or above the NAEP Proficient
level increased to 32 percent (U.S. Department of Education Institute of Education Sciences National Center for Education Statistics, 2014). However, they still lag behind their national counterparts in mathematics with the average score being 284 (U.S. Department of Education Institute of Education Sciences National Center for Education Statistics, 2014).

The lack of achievement in mathematics has far reaching implications for students. For example, Moreno (2009a) reported that of the students entering the University of Hawaii in 2009, only 16% scored high enough on a placement test to be placed into a beginning college level mathematics course. Additionally, about half of the 2008 graduates who attended community college required a remedial mathematics course (Vorsino, 2010a).

These assessment data can be viewed in the context of approaches to mathematics curricula used in the United States, which has been focused on teaching students to find solutions primarily through the use of algorithms (Chazan & Yerushalmy, 2003; Findell, 2001). For example, in Algebra I, students traditionally concentrate on manipulating a variety of single variable equations (e.g. \( \frac{x-2}{5} = 9 \)) or inequalities, solving a system of two equations (e.g. \( \begin{cases} 3x + 4x = 12 \\ 7x - y = 10 \end{cases} \)), or graphing linear equations (e.g. \( y = 3 - \frac{5}{2}x \)) on a coordinate plane. Students often gain a false sense of understanding by performing these practiced algorithms to calculate a numerical or graphical solution.

According to the National Council of Teachers of Mathematics (NCTM), this superficial display of mathematical competence has been a bi-product of learning
mathematics in the United States since 1930 (National Council of Teachers of Mathematics, 2000). Students who memorize facts, properties, and procedures without developing conceptual understanding may struggle when asked to apply their learning to a new set of problems or situations previously not encountered. Moreover, many of these same arithmetic and algebraic procedures can now be performed by hand held scientific or graphing calculators (NCTM, 2000), further simplifying procedural manipulations and thus deemphasizing conceptual understanding.

One strategy aimed at combating low mathematics achievement in high school, suggested by research (NCTM, 2006; Spielhagen, 2006a; Spielhagen, 2006b; NCES, 1999; U.S. Department of Education, 2008), is to incorporate algebra concepts, previously reserved for high school Algebra I, into the grade eight mathematics curriculum. Algebra, with a middle school emphasis, is the analysis of patterns and linear functions, the discovery of relationships that model change, the evaluation of systems of linear equations and inequalities, and the symbolic representation of real-world situations (NCTM, 2000; NCTM, 2006; CCSSI, 2010). Both NCTM (2000) and the Common Core State Standards Initiative (CCSSI) (2010) suggest themes students should encounter in algebra that support the integration of algebra into a grade eight mathematics curriculum.

For example, NCTM (2000) suggests middle school students exposed to algebra concepts should be able to, “Use mathematical models to represent and understand quantitative relationships” and “Analyze change in various contexts” (NCTM, 2000, 1

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1 Capitalization of “algebra” will refer to the actual mathematics course, Algebra I or Algebra II; all other references to algebraic concepts will use lower case “a.”

Both NCTM (2000) and Common Core State Standards Initiative Standards for Mathematical Practice (CCSSM) (CCSSI, 2010) emphasize the importance of developing conceptual understandings in mathematics. Alignment with these standards necessitates a shift in mathematics programs for middle school students who are experiencing an algebra curriculum from memorizing facts and procedures towards constructing conceptual understanding and mathematical reasoning.

**Purpose Statement**

The aim of this study is to describe changes in middle school students’ emerging algebraic reasoning by analyzing their written responses to open-ended mathematically themed prompts, constructed using the process of writing. This study is motivated by the potential of writing to fulfill NCTM’s (2000) vision for school mathematics by providing students with the opportunity to learn mathematical concepts and procedures with understanding, explore conjectures using mathematical reasoning, and communicate mathematical ideas effectively. Writing has the potential to encourage students of mathematics to create and revise meaning, communicate discoveries (Fulwiler, 1987;
REVEALING STUDENTS’ ALGEBRAIC REASONING

Mayher, Lester, & Pradl, 1983), construct new ideas based upon experience (Peterson, 2007), and analyze complex situations in creative ways (Sternberg, 1999). For example, in mathematics, writing has the ability to expose students’ conceptual understanding by revealing the clarity of their thoughts indicating areas of redundancy, miscommunication, and mathematical meaning-making.

The National Writing Project (NWP) (National Writing Project & Nagin, 2006) supports writing in mathematics through its Writing-to-Learn and Writing-to-Learn Across the Curriculum (WAC), movements which encourage educators to use writing as a tool for learning (National Writing Project & Nagin, 2006; Mayher, Lester, & Pradl, 1983) by focusing on the process, not the product, of writing (National Writing Project & Nagin, 2006). However, writing’s potential has been overshadowed generally throughout education as it has been relegated to note-taking, fill-in-the-blank exercises, copying, and short answer responses. In this way, writing has been used to measure learning rather than to reveal learning (Fulwier, 1987, emphasis added).

The use of writing in mathematics to reveal students’ developing algebraic reasoning will be explored through the following research questions:

1. How does writing reveal a student’s algebraic reasoning and knowledge of algebraic concepts?
2. In what ways does writing, through the revision of multiple drafts, expose the development of algebraic reasoning?
3. In what ways does teacher-to-student written feedback, provided during the process of writing, prompt students’ written revision of algebraic reasoning?
These three questions provide the framework for the underlying theories for this qualitative study.

**Theoretical Framework**

When used as a process for learning, writing visibly records students’ conceptual understanding and reveals their comprehension of mathematical concepts. Three areas provide the theoretical underpinnings of this study:

1. **Mathematical reasoning:** Reasoning suggests logical thinking. Mathematical reasoning implies rational thinking within the discipline of mathematics; in this study, the emphasis is in the content area of algebra. Identifying and describing changes in students’ construction of mathematical ideas such as creating conjectures, examining mathematical relationships, and logically discussing equations, graphs, or other relevant mathematical ideas, has the potential to reveal the direction of their thinking and conceptual understanding;

2. **Writing-to-Learn:** Using writing as a process, such as drafting, revising, and reflecting on given feedback, encourages students to actively engage with the ideas and concepts of a particular content, particularly mathematics in this study. Writing-to-Learn emphasizes the use of writing for the purpose of learning. It calls for students to intentionally choose the words that mirror their thinking to describe their perception of related ideas;

3. **Teacher-to-student written feedback:** Teachers who provide content-based feedback to students during the process of writing, in reaction to their expressed
ideas, call for students to reflect upon their expressed ideas and prompt them to construct responses to reveal the full extent of their understanding.

**Algebraic Reasoning**

Algebraic reasoning is a process of reasonable thinking (Loewenberg Ball & Bass, 2003; Yackel & Hanna, 2003; Carpenter 2003; MathCounts Foundation, 1984; National Research Council Institute of Medicine, 2004; Yackel, 1997). It entails analysis or judgment in developing logical relationships between known and unknown quantities to make conjectures or draw conclusions based on graphs, functions, equations or expressions within the context of algebra. Students who are capable of demonstrating algebraic reasoning are able to characterize mathematical relationships, model mathematical situations using the language of mathematics both verbal and descriptive, and analyze the nature of change in various scenarios (NCTM, 2000). Algebraic reasoning is a process for thinking about mathematics other than relying on the memorization of decontextualized facts, properties and procedures (Ketterlin-Geller, Jungjohann, Chard, & Baker, 2007). Students capable of demonstrating algebraic reasoning indicate greater conceptual understanding than students who only rely on procedural knowledge to find a solution to a problem.

**Writing-to-Learn**

Writing-to-Learn is the process in which students deliberately choose the words they will use to construct their intended meaning (Mayher, Lester, & Pradl, 1983; National Writing Project & Nagin, 2006). It encourages students to make connections between prior knowledge and new knowledge to gain greater understanding of an idea.
Students who use writing to describe relationships, pose inquires, create meaning, construct arguments, and acquire content-specific vocabulary may reap the additional benefits of writing which include greater comprehension and permanence for conceptual, factual, and procedural knowledge (Graham & Hebert, 2010; Gammill, 2006; Mayher, Lester, & Pradl, 1983).

The process of writing makes thinking more transparent; it reveals what students have learned and points to their understanding of the concepts (National Writing Project & Nagin, 2006). The writing process comprises drafting, revising, seeking feedback and editing (National Writing Project & Nagin, 2006) to assist students in saying what they mean to say while they are creating meaning. These activities occur in a reiterative cycle throughout the composing process until the author designates the piece as complete.

**Teacher-to-Student Written Feedback**

Teacher-to-student written feedback reveals the teacher’s reaction to students’ written expression and meaning with the intent of prompting students to say everything they are capable of expressing. In this and future chapters, it will be referred to as feedback. When feedback is received during the writing process (National Writing Project & Nagin, 2006), it: a) encourages students to think and revise the construction and meaning of their ideas (Patthey-Chavez, Matsumura, & Valdes, 2004; Orrell, 2006; National Writing Project & Nagin, 2006); b) assists students in articulating their thoughts to improve conceptual development and content-area reasoning (Patthey-Chavez, Matsumura, & Valdes, 2004; Orrell, 2006); and c) prompts the giving and receiving of ideas between more- and less-knowledgeable individuals (Treglia, 2009).
Feedback can include recommendations in the form of constructive criticism, specific comments, and positive observations (National Writing Project & Nagin, 2006). Effective feedback that focuses on content and challenges students’ thinking encourages revision; feedback describing superficial errors, such as spelling and capitalization, or vague comments may result in limited revision on subsequent drafts (Patthey-Chavez, Matsumura, & Valdes, 2004). Feedback can assist students in constructing meaning to say what they mean to say with a sense of confidence and clarity.

**Rationale**

Researchers (Baroudi, 2006; Ketterlin-Geller, Jungjohann, Chard, & Baker, 2007; NCTM, 2000) have suggested that most students view mathematical activities as the pursuit of a single solution involving an emphasis on manipulating symbols, performing operations, and memorizing mathematical “truths.” This limited perspective of mathematics may hinder students’ development of conceptual understanding, lead to misconceptions resulting in future difficulties with the subject (Baroudi, 2006), and stifle their ability to reason mathematically (NCTM, 2009).

In contrast to this narrowed view of learning mathematics, writing contains the potential to broaden the scope of learning by engaging students and encouraging them to become more active in the process of constructing meaning (Fulwiler, 1987; Mayher, Lester, & Pradl, 1983; National Writing Project & Nagin, 2006). Writing is capable of revealing gaps and misconceptions in students’ thinking (National Writing Project & Nagin, 2006) by exposing the construction and direction of their thoughts (Mayher, Lester, & Pradl, 1983) and suggesting where students are in their current understanding.
Mixed results have been found with the use of writing in mathematics. Positive results suggest writing can foster richer mathematical understanding (O’Shea, 2004; Hamdan, 2005; Baxter, Woodward, & Olson, 2005; Pugalee, 2004; Clarke, Waywood, & Stephens, 1993). Other results have pointed to the need for further research in the areas related to the role of writing in encouraging and revealing critical thinking and conceptual understanding (Harbaugh, Pugalee, & Adams, n.d.; Shield, n.d.; Akkus & Hand, 2005; Shield & Galbraith, 1998). This research furthers the study of writing in mathematics by focusing on the following areas:

a) Changes in students’ algebraic reasoning will be traced through their responses in a series of three drafts to the same open-ended prompt related to the major ideas presented in each of seven mathematics units;

b) Students will receive teacher-to-student written feedback on their second draft to encourage them to reflect on their ideas and the expression of those thoughts and prompt them to articulate everything they are fully capable of;

c) All grade eight students in my algebra class will respond to seven open-ended mathematically-themed prompts throughout the school year. The students selected to participate in this descriptive qualitative study will be chosen at the end of the school year to prevent any intentional or unintentional bias;

d) Each collection of three drafts will be individually analyzed based on the inclusion of mathematical content and the use of symbolic and descriptive language of mathematics; and
e) Any emergent theory will be grounded in the data based upon students’ written responses and individual interviews.

**Operational Definitions and Assumptions**

The following is a list of terms and their definitions and/or ideas that may be referred to throughout this study.

*Algebra:* Algebra is the understanding of patterns, functions and relationships, analyzing change, modeling quantitative relationships, and representing situations symbolically (NCTM, 2000). In this study, references to algebra and its curriculum will primarily focus on linear algebra.

*Algebraic reasoning:* Algebraic reasoning is a system of logical thinking. It suggests reasonable relationships between known and unknown quantities demonstrated through the construction of conjectures, production of mathematical arguments for and/or against, or creation of conclusions based on graphs, functions, equations or expressions within the context of algebra (NCTM, 2000; Loewenberg, Ball & Bass, 2003; Yackel & Hanna, 2003; Carpenter, 2003, MathCounts Foundation, 1984; National Research Council Institute of Medicine, 2004; Yackel, 1997).

*Common Core State Standards for Mathematics (CCSSM):* Proposed by the Common Core State Standards Initiative (CCSSI) (CCSSI, 2010), the standards stress conceptual understanding indicating students’ readiness for future conceptual development by grade level. This study assumes the premise, as stated in the CCSSM (CCSSI, 2010), that students should be knowledgeable in particular skills and conceptual understandings during their study of mathematics.
Communication of Mathematics: In view of the mathematical communication standard as established by NCTM (2000), to communicate means to use the language of mathematics, both symbolic and descriptive, to express mathematical ideas and relationships, analyze the mathematical reasoning of others, and share individual mathematical thoughts with others.

Conceptual understanding: Conceptual understanding is the comprehension of ideas, awareness of implied and explicit meanings, procedural knowledge, and relationships to other related ideas (NCTM, 2009). In this study, the use of the term conceptual understanding will refer to the subject area of mathematics.

Essential Question: An open-ended inquiry, containing multiple perspectives that addresses an overarching theme (Brown, 2009).

Feedback: Suggestions for revision; these recommendations may appear in the form of constructive criticism, specific comments, and positive observations (National Writing Project & Nagin, 2006).

National Council of Teachers of Mathematics (NCTM) vision of mathematics: Students should be engaged in a curriculum rich in conceptual and procedural understandings offering opportunities for students to learn the following skills: construct mathematical reasoning, become competent problem solvers, and communicate effectively (NCTM, 2000). NCTM has suggested the Principles and Standards for School Mathematics (PSSM) (NCTM, 2000), an approach valued by the mathematics education community reform thinkers, to lead to the kind of substantive learning and understanding in mathematics this particular study addresses.
**Writing process**: The writing process (National Writing Project & Nagin, 2006) consists of a number of strategies used to compose a piece of writing with the following characteristics: a) prewriting—the generation of ideas; b) drafting—the development of content through organization and possible audience awareness; c) revising—re-visioning or re-seeing the content through more focused lenses as well as focusing on structural changes; and d) editing proofreading and focusing on mechanics so that what is being said is what is meant to be said (National Writing Project & Nagin, 2006).

**Chapter 2: Literature Review**

Knowing mathematics is doing mathematics. We need to create situations where students can be active, creative, and responsive to the physical world. I believe that to learn mathematics, students must construct it for themselves. They can only do that by exploring, justifying, representing, discussing, using, describing, investigating, predicting, in short by being active in the world. Writing is an ideal activity for such processes (Countryman, 1992b, p. 2).

Countryman’s (1992b) words suggest writing fosters students’ discovery of mathematics by actively engaging them in constructing meaning through their exposure to a variety of concrete experiences using mathematics. Students who are encouraged to describe their everyday experiences through the symbolic and descriptive language of mathematics may gain richer understanding of the concepts. The aim of this study is to describe changes in middle school students’ algebraic reasoning, suggesting conceptual maturity, as they use the process of writing to respond to open-ended mathematically themed prompts. This chapter presents relevant literature focusing on the themes, both theoretical and empirical, of interest to this study. Each of the relevant themes begins with a discussion of the theoretical construct, followed by supporting empirical research
that provides historical context and then more recent findings. The following is a list of the major themes that are discussed in this chapter:

1. Writing-to-Learn: Writing, when used as a process, can foster and reveal students’ learning of mathematical concepts. The theoretical framework informing this discussion focuses on Vygotsky’s (1978) theory of cognitive development. This will serve as the foundation providing insight into the role of language in cognitive development and how language can be used to suggest students’ understanding in a discussion of word referents and word meanings. This will be followed by a brief historical perspective of the role of writing in schools and the shift in perspective from learning-to-write to writing-to-learn. Examples from research suggest both promising findings and particular concerns regarding writing-to-learn and its potential for fostering students’ cognitive development. This section concludes with a brief discussion on the process of writing.

2. Teacher-to-student written feedback: Engaging in feedback opportunities makes it possible to foster the construction of meaning-making, especially between teachers and students. The theoretical framework informing this discussion focuses on Vygotsky’s (1978) Zone of Proximal Development (ZPD) as the underpinning that encourages the exchange of ideas during the feedback loop throughout the writing process (National Writing Project & Nagin, 2006). This is in contrast to providing feedback solely after a final draft of writing is submitted. This will be followed by a discussion of research identifying potential
characteristics of feedback that encourage revision. Particular concerns by both students and teachers will also be examined.

3. Algebraic reasoning: Encouraging students to think logically and construct mathematical rationale opens up the possibility to reveal students’ conceptual understanding in mathematics. The theoretical framework informing this discussion focuses on Dewey’s (1938/1998) ideas concerning the role experience plays in students’ learning. His theory is used as the foundation suggesting a shift from traditional learning activities to progressive education whereby learning activities in mathematics encourage active participation in meaning-making. This discussion will also include research suggesting how writing can serve as one activity encouraging the development of mathematical reasoning and reveal conceptual understanding.

**Writing-to-Learn**

Writing-to-learn views writing as a process which can encourage and reveal students’ understanding. The following discussion supports this study’s use of writing in a middle school algebra class with emphasis on revealing students’ conceptual understanding as they construct mathematical reasoning using the process of writing to respond to open-ended prompts. Vygotsky’s (1978) theory of cognitive development lays the foundation for understanding the role of language in students’ potential for learning.

**Vygotsky’s Theory of Cognitive Development**

Vygotsky (1978) offers a theory of cognitive development based on students’ experiences and use of language that presents one perspective on students’ thought
development relevant to this study. Cognitive development is the evolution of an individual’s ability to learn (The American Heritage New Dictionary of Cultural Literacy, Third Edition, n.d.). In this study, the term cognitive development will refer to changes in students’ ideas over time whereas the term thinking will refer to a static snap-shot of students’ perspectives at that particular moment in time. Cognitive development is of particular importance because this study is interested in revealing changes in students’ mathematical reasoning as they express their thinking about mathematical concepts using the writing process (National Writing Project & Nagin, 2006) to respond to a variety of topics chosen throughout the school year.

Vygotsky (1978) states cognitive development occurs in stages beginning with a combination of social experiences and oral language; the more complicated the task is, the greater the role language plays in problem solving. Young children begin the process of learning by using egocentric speech, or speech used for their own purposes, in an attempt to understand their physical environment (Vygotsky, 1978). This type of speech can resemble a jumble of words, a brief commentary, or sound like verbal planning (Gredler, 2009). For example, children’s use of egocentric speech as a tool for problem solving in mathematics, alongside the use of manipulatives, indicates verbal planning is assisting them in finding a solution to the problem (Shaw, 2002). The use of egocentric speech to think aloud suggests students are capable of verbalizing their cognitive processes, encouraging the construction of mathematics in logical and meaningful ways.

As children mature, their problem solving strategies may involve more mental functions as opposed to behavioral ones (McDevitt & Ormrod, 2007). According to
Vygotsky (1978), more mature thinkers shift from speaking aloud to speaking internally. Inner speech, or what is referred to as thought, appears to resemble social speech but without the verbalization (Wertsch & Stone, 1985). As these thoughts materialize they are formed on their own plane of existence (Tharp, Jordan, Speidel, Au, et al, 1984). For Vygotsky (1978), there is a difference between inner speech and egocentric speech. When inner speech is used, thought and language are created before action; when egocentric speech is used, speech dictates action (Vygotsky, 1978).

Powell’s (1997) research suggests inner speech, or thought, can be captured through writing. His investigation of the use of writing to expose and respond to students’ mathematical thinking points to the potential writing has to reveal changes in students’ ideas. While Powell’s (1997) research does not indicate the nature of students’ cognitive development, his findings imply that students’ written responses can reveal their conceptual understanding, their own questions, and their acquisition of mathematical vocabulary. Powell’s (1997) research also suggests writing is capable of capturing individual snapshots of students’ thinking about mathematics as they experience mathematical concepts. This study assumes that if writing is capable of exposing an individual’s thoughts at specific moments, then it may also be capable of revealing changes in students’ ideas when asked to respond to the same writing prompt more than once.

Vygotsky’s (1978) theory states language and social experience foster students’ cognitive development. Students’ use of language has the potential to reveal their constructed cognitive understanding as they are exposed to learning activities that
encourage active problem solving. Language, both oral and written, makes it conceivable to foster and reveal students’ aptitude for conceptual development.

**Using Language to Reveal Cognitive Development.**

Students who engage in a variety of learning experiences construct ideas as they try to make sense of the world around them (Tharp, Jordan, Speidel, Au, et al, 1984). Writing has the potential to record and reveal those thoughts and may be able to suggest sources of concept generation. This study is interested in Vygotsky’s (1978) two classifications of language, word referents and word meanings, to reveal how students’ use of them might indicate their conceptual understanding and expose their cognitive development over time.

Word referents refer to a word’s physical object in reality; they are labels individuals give to objects to identify them from their surroundings (Vygotsky, 1978). Students who rely on objects to stimulate their memory create associative links between words and their respective objects (Gredler, 2009). Repetitious interactions and similar experiences assist learners in further constructing word referents. Learners who recall word referents often perceive those remembrances as the act of thinking as opposed to identifying an object of which they have prior knowledge. For example, Robinson (2009) suggests the use of word referents are characteristic of those who are capable of identifying and routinely using basic mathematical operations but may not understand the meaning and relationships associated with these operations.

Word meanings refer to the essence of the word suggesting its connotations and relationships to other words (Wertsch & Stone, 1985). They can also include an
individual’s personal perception of the word and the relationships between the word referent and its environment (Vygotsky, 1978). Maturation in the use of a word from its individual referent to its meaning, including abstract generalizations (Vygotsky, 1978), suggests cognitive development. In this way, the use of words also serves as an identifier for meaning and the construction of concepts (Wertsch & Stone, 1985).

In mathematics, for example, while word referents are useful for providing entrees for students to participate in mathematics, word meanings encourage students to gain greater proficiency in their comprehension of mathematical concepts and their relationships to other concepts (Capraro & Joffrion, 2006). Several researchers (Baroudi, 2006; Falkner, Levi, & Carpenter, 1999) have discussed the difference between word referents and word meanings with respect to students’ understanding of the equal sign. Students familiar with the symbol “=” can name its word referent as “equal.” Many students assume the word and its symbolic referent indicate to complete a mathematical operation. Their faulty understanding could possibly be attributed to the use of calculators; with a push of a button, the calculator automatically produces a numerical answer (Baroudi, 2006). However, a more mathematically precise understanding of the equal symbol, “=,” describes the relationship between two numerical or algebraic expressions (Baroudi, 2006; Falkner, Levi & Carpenter, 1999).

Vygotsky’s (1978) theory suggests students whose language choices include both word referents, symbolic or descriptive, and word meanings may exhibit greater cognitive development as compared to students who primarily use word referents in their articulation of their mathematical experiences. In this study, the language of mathematics
will refer to students’ use of both word referents and word meanings. This includes both the symbolic language of mathematics, the numbers and various symbols used to model a situation and/or find a solution to a problem and the descriptive language of mathematics, using word referents and word meanings to describe mathematical ideas and relationships.

Vygotsky’s (1978) theory provides the underpinnings for a discussion of writing-to-learn and the use of the writing process (National Writing Project & Nagin, 2006) to assist students in saying what they mean to say, thereby expressing their maturing cognitive understanding and gains in cognitive development. The next section will provide an empirical look at the role of writing in learning, the use of the writing process, and how writing has been included in studies about learning mathematics.

**Writing**

Kagesten and Engelbrecht (2006) suggest mathematics assessments do not typically require or reward written explanations, thus limiting students’ opportunities to use writing to express their thinking. This section will trace research related to the role of writing in schools, the shift from learning-to-write to writing-to-learn, and document the potential of writing in understanding more about students’ thinking through their responses and revisions to open-ended writing prompts provided throughout the school year.

“Writing is language choice on paper” (Mayher, Lester, & Pradl, 1983, p. 1). Mayher, Lester, and Pradl (1983) suggest the act of writing involves the conscious choice of selecting words and other written symbols used to construct meaning. Researchers
(Vygotsky, 1978; Vanderburg, 2006; Gredler, 2009; Paul & Elder, 2005) propose a more formal interpretation of writing to include the transfer of language, a systematic collection of symbols associated with words and sounds of a spoken language, into an arrangement of written symbols on paper as a means of transmitting experiences and knowledge. While writing documents students’ thoughts, the ideas themselves are malleable until the student declares the ideas he or she has recorded are permanent (Paul & Elder, 2005).

Writing can play a critical role in students’ learning. It can lead to greater comprehension, be a means for students to record thoughts that can be critically analyzed and revised, and can serve as a permanent record of their conceptual, factual and procedural knowledge (Graham & Hebert, 2010). Writing encourages students to develop an awareness of what they mean to say (National Writing Project & Nagin, 2006). For example, privately, writing reveals the inner thoughts that occur between the writer’s ideas and their use of language; publically, it reveals the writer’s choice of language so that the writer’s intended meaning can be constructed by the reader (Mayher, Lester, & Pradl, 1983). Writing can be used internally by students to record their own thought processes as well as become a social activity to document meaning which has been co-constructed between individuals (Wells, 2007). Despite the potential writing has, it has not always been used in school settings for the purpose of sharing constructed meanings.

**Traditional School Writing.**

Historically, students have written in order to communicate what they have learned (Fulwiler, 1987). Writing-to-communicate is essentially writing to relay
information to someone else (Palmquist, 2011). The nineteenth-century model for writing and language development emphasized memorization and skill-and-drill exercises. This practice continued into the twentieth-century along with an added emphasis on correcting the errors—primarily grammatical, spelling, capitalization, and punctuation, within students’ written compositions (National Writing Project & Nagin, 2006). The emphasis on writing-to-communicate meant writing was taught independently from other subject areas and presented as a tool for learning vocabulary, grammar, writing mechanics, and letter formation. In this model, students were learning-to-write as opposed to writing-to-learn, with the function of writing reduced to the symbolic representation of vocal utterances artificial and disjointed from context (Vygotsky, 1978).

Prior to 1970, emphasis was placed on the product of writing rather than the process of writing (National Writing Project & Nagin, 2006; Fulwiler, 1987). The most common uses of writing in core subject areas were transactional, such as conveying information, and other miscellaneous functions, such as note-taking and copying (Britton, 1975 in Fulwiler, 1987). These uses of writing tended to reward students for rote learning based on how well they regurgitated information (Mayher, Lester, & Pradl, 1983). The exceptions, perhaps, were English classes where students may have engaged in poetic or creative writing.

Learning to write is often associated with learning grammar which involves the structure of a language (Williams, 1998) and has little to do with the process of writing. Moreover, once students learn the mechanics of writing, they are frequently asked to use those skills for assessment purposes and are rewarded for using language to demonstrate
their memorization of specific facts (Wolcott & Legg, 1998; Parker & Goodkin, 1987). Learning to write, when used as a tool for reciting information, decontextualizes learning and deemphasizes the process of writing. Such traditional uses of writing in schools limit students’ opportunities to become involved in deliberate language choice (Dix, 2006) which separates the process of writing from the process of learning.

After 1970, a shift in perspectives regarding the role of writing focused on writing-to-learn as opposed to writing-to-communicate and learning-to-write. Advocates of writing (National Writing Project & Nagin, 2006; Britton, 1975; Fulwiler, 1987; Parker & Goodkin, 1987; Mayher, Lester, & Pradl, 1983) suggest the emphasis should be on writing-to-learn by encouraging writing activities that reveal students’ thinking and their potential for learning. This shift from learning-to-write to writing-to-learn will be further discussed through the use of the writing process (National Writing Project & Nagin, 2006). It encourages students to revise their thinking as they respond to a series of open-ended prompts throughout the school year.

**Writing-to-Learn**

Writing-to-learn is a process in which writers actively choose language to construct meaning, discover connections between prior and new knowledge, raise questions, and problem solve (Mayher, Lester, & Pradl, 1983). This process encourages students to move beyond reproducing and regurgitating facts, procedures, dates and formulas to participating in creating meaning by thinking critically about ideas presented to them (National Writing Project & Nagin, 2006). Writing-to-learn emphasizes writing
for the purpose of representing and understanding students’ perception of reality and focuses on the process, rather than the product, of learning.

Writing-to-learn, as its name suggests, is writing for the purpose of learning. When the focus is on writing as a process rather than writing as a product, writing performs three potential functions: a) writing reveals students’ understanding, including inconsistencies, gaps, and misconceptions (National Writing Project & Nagin, 2006); b) writing encourages the connection between prior knowledge and new knowledge (Mayher, Lester, & Pradl, 1983); and c) writing becomes a process of inquiry and discovery (National Writing Project & Nagin, 2006). Writing used to support and promote understanding encourages students to reflect upon what they have learned, ask questions, and interact with their experiences to construct meaning.

**Writing Reveals Students’ Knowledge.**

The role of writing to reveal students’ knowledge, including relevant ideas, conceptual understanding, gaps, inconsistencies, and misconceptions in their thinking has been documented in the literature. For example, researchers Evans and Houssart (2004) found that writing was able to reveal students’ abilities to identify if a conjecture was true or false even if they were not able to offer a written explanation in support of their thinking, thus revealing the appearance of students’ understanding but also indicating gaps in their thinking. Eleven-year-old students in England were given a written mathematics test asking them to determine whether a given mathematical statement were true and to provide a written explanation supporting their thinking. Evans and Houssart (2004) noticed three emergent themes in students’ explanations as revealed through their
writing: a) students restated given information from the original problem suggesting they understood the mathematical task but failed to construct a mathematical argument in support of their thinking; b) students created numerical examples or continued the original pattern as their explanation suggesting how they may have initially investigated the truth of the original conjecture although they failed to accompany it with a mathematical rationale; and c) students who did offer explanations failed to include enough detail to articulate their intended meaning suggesting they understood the situation but may have had difficulty choosing language to express themselves. While students appeared to understand the mathematics, their inability to construct a mathematical argument suggests their conceptual understanding might not have been as developed as originally anticipated. Evans and Houssart’s (2004) findings are of interest to this study because they suggest writing reveals students’ potential idea generation, such as using language from the original prompt to craft their written expressions and creating examples to investigate the truth of a conjecture. It can also imply areas of misconception and/or gaps in their mathematical understanding, such as failing to construct a mathematical argument.

**Writing Encourages Connections Between Prior and New Knowledge.**

Writing uses language to mediate between prior knowledge and emergent understanding (Gammill, 2006) by encouraging malleable internal dialogue to occur between the student and their experiences (Vygotsky, 1978; Hamdan, 2005; Akkus & Hand, 2005; Mayher, Lester, & Pradl, 1983). Baxter, Woodward and Olson (2005)
examined the claim that writing that encouraged students to justify and support their thinking might assist them in developing mathematical proficiency. Seventh grade students in this study participated in journal writing activities related to mathematical topics being studied in class aimed at promoting an awareness of their thought processes and ownership of knowledge. Baxter, Woodward and Olson (2005) found two emergent themes based on an analysis of students’ writing: a) the way an idea was represented revealed students’ understanding of the concept; and b) students’ use of familiar contexts encouraged them to think about how mathematics was related to their own lives. Baxter, Woodward and Olson’s (2005) findings are significant to this study indicating writing provides a window into students’ ideas and can provide insight into their level of comprehension and the clarity of their mathematical understanding. This suggests that continued opportunities to use the descriptive language of mathematics may further students’ conceptual development and mathematical thinking.

Writing is a Process for Inquiry.

Palmquist (2011) suggests writing-to-learn makes possible the opportunity to foster critical thinking; students who are encouraged to write and write often demonstrate higher levels comprehension and more in-depth understanding than their peers who write less frequently (Graham & Hebert, 2010). One reason for this may be that writing serves as a tool for understanding students’ thinking processes as they engage in inquiry and problem solving. For example, in a comparison between students’ verbal and written problem solving processes, Pugalee’s (2004) findings suggest students who expressed
their thinking through writing were significantly more successful in problem solving tasks than those who rely on symbolic manipulation.

High school algebra students were asked to participate in a pre-study, a two-week enrichment period during which they participated in journal writing to describe their mathematical thinking. Students wrote explanations of their problem solving processes and revised their journals based on feedback from their teacher. Students then participated in the actual study in which they alternated between providing an oral description and a written description of their problem solving processes. An analysis of the data (Pugalee, 2004) showed that successful problem solvers were aware of their own thinking and possessed conceptual understanding of mathematics. The findings were supported by the following students’ actions exhibited during problem solving: a) students who attempted more problem solving strategies had a higher rate of success; b) much of students’ problem solving involved executing tasks such as performing calculations; c) students who created a plan describing their problem solving strategy were more likely to be successful; and d) students who described their problem solving processes in writing generally had a higher rate of success than students who verbalized their thinking on the same problem.

Pugalee’s (2004) research suggests writing can support students developing an awareness of their own thinking but also indicates more research needs to be done on the characteristics of writing activities that will assist students to think about their thinking. Pugalee’s (2004) findings are of interest to this study because they support the use of
writing in mathematics to reveal problem solving processes and suggest writing can help students become aware of their ideas regarding a particular mathematical topic.

**Writing Activities that Support Writing-to-Learn.**

Vygotsky (1978) suggests writing that is relevant, incorporated into meaningful tasks and nurtured, is likely to reveal students’ knowledge and conceptual understanding. The type of writing activity students engage in offers students the opportunity to reveal their knowledge and conceptual understanding. For example, writing that encourages knowledge construction should include tasks that do not ask students to separate ideas from their original context, history, values and/or beliefs (Gammill, 2006). This suggests that open-ended writing prompts may provide students with the opportunity to respond in their own way, utilizing their own experiences, and acknowledges there could be more than one possible solution. Unlike a standardized question in which there is only one correct answer, writing prompts that provide students the opportunity to create their own response are likely to capture students’ thoughts and ideas revealing them as they are without passing judgment on their accuracy or intended meaning.

Research (Theoret & Luna, 2009) suggests students’ content knowledge can be identified from the characteristics that emerge from their writing when given prompts that encourage students to reflect upon their understanding and voice questions they may have. Undergraduate statistics students responded to ten teacher-generated writing prompts in one of two ways: individual journal assignments or computer-based discussion board. Students also completed a survey based upon their experiences at the end of the semester. Theoret and Luna (2009) noticed that four emergent themes
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characterized students’ writing: a) students used worldly examples to illustrate their understanding of statistics concepts; b) students made personal connections to the concepts they were learning; c) discussion boards were more likely to contain students’ questions than their own journal responses; and d) discussion boards indicated students debated and offered suggestions to others.

Theoret and Luna’s (2009) finding that students made personal connections to concepts they were learning is of interest to this study. It suggests that student-generated examples serve as one potential indicator in identifying students’ knowledge and mathematical understanding. These findings (Theoret & Luna, 2009) further suggest journaling encourages students to demonstrate their understanding by giving them the opportunity to make personal connections to their learning and identify relevant examples in their own lives to present unique discussion points. Student-generated examples potentially reveal students’ mathematical understanding when they are given the opportunity to respond freely to the themes addressed in an open-ended prompt and provide one perspective for this study on how to identify students’ conceptual development in mathematics.

Theoret and Luna’s (2009) next two findings suggest an exchange of ideas between individuals may assist students in their construction of meaning through the validation and clarification of their own understanding. The opportunity to receive feedback from someone the student feels is more knowledgeable in the subject than they are supports the construction of meaning and encourages students’ to reveal everything they are capable of with respect to their conceptual understanding. This study will also
utilize the opportunity for feedback during the writing process (National Writing Project & Nagin, 2006) to provide students with a reader’s perspective based on the presentation of their ideas.

**Writing-to-Learn Concerns: Student Anxieties.**

Although writing-to-learn has been shown to lead to students’ richer content understanding and gains in conceptual development, its effects have not always been positive as expressed by both teachers and students. Regarding students’ concerns, there is relevant literature suggesting students’ prior experiences and exposure to formulaic writing models fail to inspire them to write to learn. At the same time, there is research that will respond to teacher concerns and addresses those hesitancies with suggestions encouraging writing-to-learn activities.

Students may become frustrated with writing-to-learn activities if they are unable to develop and organize their ideas (Fulwiler, 1987) or are continually finding fault with their word choices and spelling (Elbow, 1998). If the composing process is hindered by the constant need to make corrections, it retards the meaning-making process (Mayher, Lester, & Pradl, 1983). These unproductive approaches to writing may be attributed to students’ previous writing experiences, both with their own assignments and the published writing they encounter.

For example, Parker and Goodkin (1987) suggest textbooks are rarely good models for student writing, presenting writing in non-creative, monotone, and uninventive ways. Students who are exposed primarily to textbook writing may mimic the language and tone of these books by using mundane language choices (Fulwiler,
1987; Mayher, Lester, & Pradl, 1983; Parker and Goodkin, 1987). Furthermore, Ntenza (2006) suggests rote writing activities, such as copying text that uses objective academic language, might reduce the number of opportunities students have to engage in original thinking and thus, potentially slow the production of meaning-making.

Ntenza (2006) explored the claim that writing-to-learn activities can have a positive effect on learning mathematics. He selected six mathematics teachers and their classes from various schools to study the kinds of mathematical writing students did. After an analysis of students’ textbooks and their written work and teachers’ lesson plans and teacher interviews, Ntenza (2006) found that students with limited exposure to writing activities that prompted mathematical thinking wrote short pieces and were likely to not use their own words to express mathematical ideas. Ntenza’s (2006) findings call attention to the connection between writing activities that primarily involve transcribing and copying information and the quality of student learning. Ntenza, (2006) further suggests writing activities that extend thinking should be introduced regularly in mathematics classrooms; they should include tasks that go beyond mere vocabulary use, expressing procedures in words, and solely focusing on the use and manipulation of mathematical symbols.

**Writing-to-Learn Concerns: Teacher Apprehensions.**

Despite research (Theoret & Luna, 2009; Pugalee, 2004; Baxter, Woodward, & Olson, 2005; Evans & Houssart, 2004; Fulwiler, 1987; Mayher, Lester, & Pradl, 1983; National Writing Project & Nagin, 2006) suggesting writing-to-learn supports thinking, meaning-making, and richer conceptual understanding, teachers have expressed concerns
about implementing writing-to-learn strategies in their classrooms. Troia and Maddox (2004) identified two significant teacher concerns that can negatively impact their use of writing-to-learn activities, namely, a large student load with a wide range of cognitive abilities and teaching demands that include covering subjects with large content areas. Additional areas of concern also include teachers’ claims that they are unqualified to teach writing (National Writing Project & Nagin, 2006) and writing produces too much paperwork (Fulwiler, 1987). These concerns are also concerns of this study: students are responsible for learning a certain amount of algebraic content and the volume of text produced by each student responding three times to each open-ended prompt throughout the school year must be managed by the teacher. Despite these practical concerns, advocates of writing-to-learn (Mayher, Lester, & Pradl, 1983; Gammil, 2006; National Writing Project & Nagin, 2006; Fulwiler, 1987) address these hesitations while continuing to encourage the inclusion of writing-to-learn activities.

Subject area teachers raise a valid concern with the need to balance writing-to-learn activities with content coverage (Troia & Maddox, 2004; Mayher, Lester, & Pradl, 1983) claiming there are limited opportunities to include writing within the academic demands of the curriculum. However, Mayher, Lester, and Pradl, (1983) point out that teachers who place emphasis on test scores and who cover material at a race-pace to satisfy content coverage rather than content mastery may not be providing students with opportunities to integrate new information with previously learned ideas. These teachers, according to the researchers, (Mayher, Lester, & Pradl, 1983; Nicolini, 2007) are not
gaining the insights into their students’ thinking that writing affords and may re-teach material unnecessarily or spend too much time on well-understood topics.

Clarke, Waywood and Stephens (1993) investigated journal writing and how it communicates mathematical knowledge among students in seventh through eleventh grades in a Canadian secondary school. Students were expected to write after each mathematics class and address topics such as, what they did during the lesson, what they learned, and to include any questions and/or examples from their classroom experiences. Students also participated in a survey regarding their journal writing. After analyzing students’ journals, Clarke, Waywood and Stephens (1993) characterized students’ mathematical journal writing in three ways: a) “recount mode” (p. 247) in which students reported on the mathematical tasks they completed; b) “summary mode” (p. 247) indicated students described the mathematics by expressing procedural knowledge, how a solution was found, and by recording their overall progress; and c) “dialogue mode” (p. 248) implied students focused on conceptual ideas and began to connect new knowledge to their prior knowledge. Clarke, Waywood and Stephens (1993) indicated teachers can encourage students to become more independent in their learning of mathematics by encouraging them to use writing to identify what they know, question what they do not understand, and become aware of their own thinking.

Teachers also expressed that they are hesitant to face the large amount of written text students will produce using write-to-learn strategies, some of which, may be poorly written and lack meaningful content (Fulwiler, 1987). Fulwiler (1987) suggests it may not be necessary to evaluate informal writing assignments, such as freewriting or journal
writing, especially if they will serve as potential springboards for more formal assignments subsequently.

Jasmine and Weiner (2007) suggest writing processes that include drafting can support students to become more independent writers, implying students may become less reliant on teachers to inform them on the merit of their ideas. First grade students were exposed to mini-lessons focusing on individual writing strategies and then applied them during writing workshop periods designated by four phases: rehearsal, drafting, revising and editing. After an analysis of students’ survey responses, classroom observations, and samples of their written work, Jasmine and Weiner (2007) indicated students exposed to writing processes that included drafting became more confident in their writing, were able to write more independently, and relied on helping each other revise their ideas during the process.

Teachers also believe they are unqualified to teach writing, especially content area teachers not involved in teaching Language Arts or English claiming they are not English majors or argue they are unqualified to teach writing (National Writing Project & Nagin, 2006). Mayher, Lester, and Pradl (1983) suggest, however, that content area teachers should be familiar with the concepts in their content area and should be able to identify whether students’ ideas are complete or incomplete. Furthermore, Gammil (2006) and National Writing Project and Nagin (2006) indicate that content area writing may expose students to a variety of written genres, both formal and informal, offering students more exposure on how writing expresses ideas.
Writing Across the Curriculum (WAC)

Advocates of writing-to-learn across the curriculum (National Writing Project & Nagin, 2006; Fulwiler, 1987; Mayher, Lester, & Pradl, 1983) support the integration of writing in all content areas encouraging students to relate and connect to those subject-matter concepts. This section includes a brief discussion on Writing Across the Curriculum (WAC) and the Common Core State Standards English Language Arts (CCSSELA) (CCSSI, 2010) which supports writing-to-learn in all content areas, including mathematics.

Writing Across the Curriculum (WAC) emphasizes the use of writing as a process in a variety of content areas for the purpose of learning, developing habits of inquiry, and producing knowledge which demonstrates students’ critical thinking and higher level comprehension (National Writing Project & Nagin, 2006; Hennessy & Evans, 2005). Writing becomes a tool for learning when it encourages critical thinking within a particular discipline (National Writing Project & Nagin, 2006), introducing students to the language, conventions, genres, and literacies of that particular content area. Students exposed to activities fostering connections between prior and new knowledge (Staats & Batteen, 2009) interact with content area concepts. For example, Boyer (2006) encourages the use of writing-to-learn activities in social studies implying these tasks will nurture students’ understanding of social studies concepts and encourage students to think about those ideas beyond the classroom context.

Grimberg and Hand (2009) suggest writing-to-learn activities introduced in science benefit both lower- and higher-achieving students’ reasoning skills. Middle
school students participated in laboratory activities emphasizing particular biological concepts followed by a sequence of writing prompts to investigate and reveal their understanding. Grimberg and Hand (2009) suggest higher-achieving students’ writing displayed more complex critical thinking sooner than lower-achieving students; however, given more time, lower-achieving students were able to construct and express their scientific reasoning as well. Grimberg and Hand’s (2009) findings are of interest to this study demonstrating how writing has the potential to increase both high- and low-achievers content area reasoning skills. It further suggests that all students, regardless of their academic achievement, are capable of participating in writing-to-learn activities in mathematics.

**Common Core State Standards English Language Arts Standards**

This study assumes that the *Common Core State Standards English Language Arts Standards* (CCSSI, 2010) supports current trends and practices in writing across all curriculum areas. The standards provide one perspective for describing the most essential characteristics of students’ writing with respect to their grade level. This study further assumes these standards represent significant learning experiences for students encouraging their communication of content knowledge, conceptual understanding, and construction of relevant arguments to further their knowledge similar to the interests of this study. The relevant standards are listed in appendix C.

**The Writing Process**

Whereas writing is the act of making an individual’s thoughts visible, the writing process assists writers in saying what they mean to say (Fulwiler, 1987; National Writing
The writing process consists of strategies such as prewriting, drafting, revising, and editing (National Writing Project & Nagin, 2006) that encourage writers to transform their ideas by fostering more sophisticated thought processes (Peterson, 2007) and to gain insights into their own thinking (Baxter, Woodward, & Olson, 2005; Gammill, 2006). These strategies form a cyclical process (National Writing Project & Nagin, 2006; Pattey-Chavez, Matsumura, & Valdes, 2004) in which writers can move at their own pace from one strategy to another as they compose their pieces, or they can return to an earlier stage in the process if needed.

The planning stage consists of any prewriting activities that help the writer to generate ideas including brainstorming, diagraming, outlining or concept webbing (National Writing Project & Nagin, 2006). During this stage, writers can make decisions about the topic, purpose and audience (Alber-Morgan, Hessler, & Konrad, 2007). One particular prewriting strategy of interest to this study is freewriting that encourages students to express any idea they feel is relevant in response to the given open-ended prompt without worrying about accuracy or being judged by others. According to Elbow (1998), freewriting is continuous, unrestrained writing that encourages writers to say something on paper without worrying about what is being said and who is going to read it.

Drafting consists of content development; it is the process of shaping content into an organized structure (National Writing Project & Nagin, 2006) to create a more formal expression of their ideas and thoughts. Drafting can reveal students’ understanding (National Writing Project & Nagin, 2006) through language choices indicating
transformations in their thinking (Parker & Goodkin, 1987). Early stages of drafting can focus on inquiry, organization, and exploration of ideas, saving writing mechanics and editing for later stages (National Writing Project & Nagin, 2006).

Revision and editing often accompany drafting but are used for different purposes. Revision focuses on the molding and shaping of the actual content; each time a revision is made both contextual and structural changes are made to the text (National Writing Project & Nagin, 2006). Editing focuses on writing mechanics and the use of precise language and symbolic features to ensure the meaning a writer intends to convey is actually expressed (Fulwiler, 1987; National Writing Project & Nagin, 2006). During this process, feedback, especially focused on content and the construction of meaning, can be provided. A more in-depth discussion of feedback, how it can support the process of meaning-making, and its potential relevance to this study, appears in a later section.

The writing process (National Writing Project & Nagin, 2006) has the potential to provide students with the opportunity to shape and reshape their ideas. Staats and Batteen (2009) further suggest prewriting activities may assist students in connecting and integrating prior and new knowledge from different subject areas offering a sense of cohesiveness between the subjects. Staats and Batteen (2009) investigated the claim that writing could assist students to make connections between mathematics and real-world scenarios, those existing beyond the classroom. University students in an introductory developmental algebra class were asked to respond to an article about funding to support the control of malaria in Africa and create a mathematical argument including, mathematical calculations, to support their case. After an analysis of students’ writing,
Staats and Batteen (2009) reported that students who included more contextual detail in their response also included better mathematical arguments to support their conjectures. Staats and Batteen (2009) suggest the following recommendations to support students in creating mathematical models related to a variety of contexts from other subject areas: a) encourage multiple drafts rather than a single response; b) have multiple data sets available for students to use; and c) create prewriting opportunities to assist students in integrating knowledge from different subject areas. Staats and Batteen’s (2009) suggestions for designing and implementing writing in mathematics, especially those focused on the use of the writing process, provide guidance for the type of prompts that students received in this study and the integration of the writing process used in constructing student responses.

While the writing process may provide students with multiple opportunities to become involved with writing-to-learn activities, feedback provided during drafting, can offer students the opportunity to further construct meaning and revise their ideas. Feedback will be discussed in the next section.

**Teacher-to-Student Written Feedback**

Feedback strategies focus on revision; feedback can take the form of constructive criticism, specific comments, and positive observations (National Writing Project & Nagin, 2006). While this study is interested how students can use feedback generally to construct meaning, the specific type of feedback this study is interested in is teacher-to-student written feedback. In this type of feedback, students receive individual written comments from the teacher in the form of suggestions and strategies that assist them in
the revision process to further clarify their understanding (Patthey-Chavez, Matsumura, & Valdes, 2004). In this study, teacher-to-student written feedback will be referred to as feedback.

This discussion on feedback begins with Vygotksy’s (1978) theory of the Zone of Proximal Development (ZPD) with a focus on how interaction between a more knowledgeable individual and a less knowledgeable individual can support students’ acquisition of knowledge and conceptual development.

**Vygotksy’s Theory on the Zone of Proximal Development (ZPD)**

Research (Vygotsky, 1978; Tharp, Jordan, Speidel, Au, Klein, Calkins, et al, 1984) suggests that when interactions occur at a level more sophisticated than an individual’s current ability, the additional assistance a learner receives may enhance the learner’s potential development. Vygotsky (1978) proposes that a learner’s actual development is an accumulation of all of his or her previous learning experiences, social interactions, and prior knowledge and that these tasks and skills can be executed independently and without assistance. A learner’s potential development comprise the tasks and skills that can be accomplished with the assistance of someone more knowledgeable who is able to support the learner. The gap between what a learner can do independently and what he or she can do with assistance is referred to as the Zone of Proximal Development (ZPD) (Vygotksy, 1978). Vygotsky (1978) defines the ZPD in this way:

*It is the distance between the actual developmental level as determined by independent problem solving and the level of potential development as*
determined through problem solving under adult guidance or in collaboration with more capable peers (p. 86).

Vygotksy’s (1978) theory suggests that it is possible for students to have substantive learning experiences when engaged in tasks that are potentially above what they are capable of doing independently.

While Vygotsky (1978) does not name specific activities that foster a learner’s conceptual development and knowledge growth, other literature presents views that extend his theory and suggest potential learning activities. Dewey (1998) proposes that it is the quality of the experience that influences students’ development; the greater the influence of the experience, the more potential knowledge that can be generated. This suggests that activities that engage students in the thoughtful construction of mathematical meaning and reflection upon those ideas encourages the growth of rich conceptual understanding.

Other researchers (Tharp, Jordan, Speidel, Au, Klein, Calkins, et al, 1984; Patthey-Chavez, Matsumoto, Valdes, 2004; Zack & Graves, 2001) suggest students can become involved in joint productive activities. Joint Productive Activities (JPA) focus on collaboration and can range from students working on individual products to teachers and students collaborating together in which the teacher participates as a full collaborator and can model language, skills, and problem solving strategies (Hillberg, Doherty, Epaloose & Tharp, in press). Tharp, Estrada, Dalton, and Yamauchi (2000) suggest activities that: a) encourage the learner to expand within his or her ZPD; b) offer opportunities to use and apply new information to fresh and creative contexts; and c) balance what the student
can do independently with the assistance the student receives, can aid students attain more complex understanding. This implies that writings activities, such as providing feedback, that focus on prompting students to articulate their conceptual understanding in every possible way, reveal the extent of students’ mathematical thinking and understanding.

Researchers (Mercer, 2008; Wood, Bruner, & Ross, 1976) have conducted studies based on Vygotsky’s (1978) ZPD theory suggesting support and scaffolding have fostered students’ conceptual development. For example, Mercer (2008) reported that students who are active participants in dialogue show greater gains in their conceptual thinking from their ability to reason and present logical arguments. Wood, Bruner, and Ross (1976) found that individuals who receive tutoring are more successful in task completion when the tutor exhibits characteristics that provide guidance and modeling to the learner. Crion and Marin (2010) discovered that students who actively participate in giving and receiving feedback showed improvement in the quality of the content expressed in their written drafts, including their initial draft, over the course of the school year. Research (Mercer, 2008; Wood, Bruner, & Ross, 1976; Crion & Marin, 2010) has captured that collaboration between individuals can assist the less knowledgeable individual in negotiating more complex tasks. The more knowledgeable assistant in these joint productive activities provides guidance and may even contribute to performing particular tasks that are beyond the learner’s ability, thus encouraging the learner to concentrate on the activates they are capable of performing. Together, both individuals construct knowledge and foster their cognitive development.
Feedback

One such activity that this study is interested in, influenced by Vygotsky’s (1978) theoretical underpinnings, is the giving and receiving of feedback, encouraged through the writing process (National Writing Project & Nagin, 2006). Obtaining feedback can be one of the most influential steps in revising or re-visioning students’ ideas. The role of feedback during the writing process is to provide an outsider’s perspective on the content and written mechanics to assist writers in creating clarity and sophistication in his or her discussion. The role of language, used during this process, can act as a medium between the writer and the individual providing the feedback for the purpose of constructing meaning (Wells, 2007). For example, a teacher who revoices a student’s written contribution (Lampert & Cobb in Kilpatrick, Martin, & Schifter, 2003) during the feedback process can potentially redirect his or her discussion to include better reasoning and stronger connections between ideas.

For example, Whitington, Glover and Harley (2004) investigated whether university students enrolled in an early childhood pre-service program read feedback on their essays and if they found feedback suggestions to be useful. Analysis of data collected showed that students wanted to receive feedback that contributed to their learning experience, and pointed to four conditions of effective feedback: a) conveys information that is easily understood by the student; b) identifies what improvements need to be made; c) suggests strategies that can be used for improvement; and d) encourages further learning and motivation to improve (Whittington, Glover, & Harley,
These findings suggest potential guidelines for crafting written feedback to students in this study.

Teacher feedback regarding students’ content is generally associated with higher quality compositions (Patthey-Chavez, Matsumura, & Valdes, 2004). Feedback offered during, as opposed to at the conclusion of, the writing process (National Writing Project & Nagin, 2006), may begin a dialogue between teacher and student (Wertsch & Stone, 1985) encouraging students to broaden their learning (McNulty, 2010) and expand conceptual understanding beyond students’ current development. Written feedback enables teachers to reach many students simultaneously and effectively while still communicating their ideas and reactions to each individual student’s writing. Researchers (Silver & Lee, 2007; Vardi, 2009; Bain, Mills, Ballantyne, & Packer, 2002) suggest the type of feedback, the way in which it is delivered, and when it is received might influence how students incorporate those suggestions in revising their written text.

Vardi (2009) investigated the relationship between feedback and changes in students’ writing. Undergraduate students were given a take-home essay and asked to respond to the same prompt three times over the course of the semester, each time trying to improve their previous draft. Written feedback was provided for each draft. Vardi (2009) suggests feedback encourages students to improve their writing if it: a) is content oriented, written with clarity so the student can understand the intent; b) encourages the student to construct meaningful ideas and concept relationships; c) does not focus on writing mechanics such as spelling, grammar, and other writing conventions; and d) is provided in a way that motives students to use it. Furthermore, Vardi (2009) found that
feedback does not need to be extensive if it is meaningful. Vardi’s (2009) findings indicate that feedback can assist students in the revision process when provided in a way that encourages students to continue in the process of illustrating their understanding.

Feedback makes it possible to assist students in re-visioning their thinking, creating greater connections between prior and new knowledge, and encouraging the construction of meaning. However, as with the theme of writing-to-learn, concerns have been raised with the construction of feedback. These concerns offer warnings for the use of feedback in this study.

**Feedback Concerns.**

Despite the plausibility of feedback in encouraging students to revise their thinking, there are some concerns about the type of feedback students receive and inquiry as to why students fail to use the feedback they receive. Writing effective feedback is challenging for many teachers. Weaver (2006) suggests teachers feel the tension between providing useful and meaningful feedback in a timely manner and keeping up with the demands of their student workload. For example, some teachers claimed they did not feel adequately prepared by their teacher-educational programs to provide students with feedback during the writing process (Kellogg, Whiteford, & Quinlan, 2010). These teachers believed that summative comments were appropriate to justify the letter or numerical grade students received. However, researchers (Orrell, 2006; Whitington, Glover, & Harley, 2004) indicated that the least beneficial type of feedback to students is a number or letter grade with only a few statements justifying the letter or number at the conclusion of the writing activity.
Patthey-Chavez, Matsumoto and Valdes (2004) examined students’ drafts before and after receiving feedback to investigate the claim that teacher feedback provides students with the opportunity to revise their thinking, expand their ideas, and reshape their intended meaning. Middle school students from five schools participated in three writing activities given in their language arts class consisting of a multi-draft assignment, reading comprehension assignment, and a project. At the end of this intervention, teachers submitted two projects each that they felt fell into the high, medium, and low categories of achievement. Patthey-Chavez, Matsumoto and Valdes (2004) analyzed the data and found the following: a) students who received content level feedback were more willing to further their ideas than those who received surface level feedback; b) the quantity of feedback students received affected the quality of their writing mechanics; and c) comments on students’ final drafts were not an effective tool for revision. While the results of this study suggest revising feedback during the writing process assisted students in expanding their thinking, Patthey-Chavez, Matsumoto and Valdes (2004) recommend further research on the types of written feedback that may encourage students to make substantive revisions to improve the quality of their written content.

Researchers (Orrell, 2006; Silver & Lee, 2007; Weaver, 2006) have also identified students’ concerns on receiving feedback from their teachers. For example, Orrell (2006) indicated that although teachers may be well intentioned, students may have difficulty interpreting and understanding feedback written in the academic vernacular of a particular subject area. Students may also get discouraged by feedback when it is focused on writing mechanics and errors, particularly if the feedback appears
to imply that students’ authority over their own text is being replaced by the teacher’s (Orrell, 2006; Silver & Lee, 2007; Weaver, 2006). Students suggested comments that are judgmental, superficial or nonspecific (e.g. ‘poor word choice,’ ‘good job,’ ‘did not address the prompt’) offered little guidance for how the content could be revised and, in fact, they may ignore such comments (Weaver, 2006; Silver & Lee, 2007). Additionally, while teachers may be well intentioned, students who view their teacher as an evaluator of their work as opposed to an interested reader who is genuinely trying to assist them in furthering their conceptual understanding (Orrell, 2006; Silver & Lee, 2007) may disregard the feedback or make minimal changes.

Weaver (2006) investigated students’ perceptions of feedback and the claim that feedback provides the opportunity for student-centered learning. Business and Art and Design students were surveyed about their responses to feedback they received on open-ended writing prompts. Students responded negatively to feedback that contained: a) generalized or vague comments lacking details about potential weaknesses in writing; b) comments lacking guidance on how they could revise their thinking; c) comments focused on negative aspects of students’ writing, potentially causing some students to lose motivation; and d) comments unrelated to assessment criteria even if they identified flaws in the writing (Weaver, 2006). Weaver’s (2006) study cautions that students may appear to ignore feedback if they feel it does not offer enough guidance or perceive that the comments are ambiguous.

Despite valid concerns regarding feedback from both teachers and students, feedback continues to open up the opportunity for students to revisit and revise their
thinking and understanding. Feedback has the potential to assist students in taking ownership of their learning, including problems, concepts, procedures, or experiences, and gives them the opportunity to express their intended meaning in their own words (Fulwiler, 1987). Writing, in conjunction with feedback, can be an effective tool for students to construct content understanding, reflect on their prior understanding, support their metacognitive skills, and foster connections between ideas (Dunlap, 2006).

The affordances offered by writing and feedback, as described in the results of relevant studies, were taken into consideration in the design and implementation of the study throughout the school year. In the next section, the need for mathematical reasoning will be addressed and how it might be fostered through the use of writing and the writing process.

**Mathematical Reasoning**

Mathematical reasoning is a system of systematic thinking, analysis or judgment, rooted in the discipline of mathematics. It is used to evaluate new knowledge, problem solve, and make connections to preexisting concepts (Loewenberg Ball & Bass, 2003; Yackel & Hanna, 2003). This study seeks to describe evidence of students’ mathematical reasoning through an analysis of their writing. The ability to utilize mathematical reasoning is one indication that a student is able to apply mathematical concepts to a variety of situations (NCTM, 2000; NCTM, 2009), often in new and creative ways (Sternberg, 1999) and construct logical mathematical arguments to express a rationale (NCTM, 2000). This study assumes that through an analysis of writing, we can trace the
development of students’ mathematical thinking as they respond to open-ended mathematically themed prompts, using the writing process.

In general, students’ mathematics learning involves many more experiences in mastering procedures and developing skills than opportunities to engage in mathematical reasoning. Research (U.S. Department of Education, 2008; NCTM, 2009; Stein, Remillard, & Smith, 2007; Malloy, 1999; Sternberg, 1999; NCTM, 2000) suggests students’ experiences in secondary mathematics classes have been mostly passive. Lecture-style delivery of information tends to encourage the memorization of facts, procedures (Stein, Remillard, & Smith, 2007) and abstract ideas (Malloy, 1999) without understanding, thus, rewarding students for their skilled demonstration of factual and procedural information (NCTM, 2000; Sternberg, 1999). In 1991 and again in 2000, NCTM (NCTM, 1991; NCTM, 2000) called for changing the way mathematics is taught and learned to provide students with opportunities to develop mathematical reasoning. These documents advocated that students explore conjectures, construct logical reasoning, generate creative solutions to non-routine problems, and connect mathematical ideas to each other and to other contexts.

This study looks at writing, not as a treatment to move away from traditional classroom activities, but as a representation of students’ mathematical thinking and reasoning. The study seeks to describe how the reflective nature of writing provides insight into what students have learned and traces changes in students’ mathematical reasoning, thereby describing their conceptual development.
Dewey’s Theory of Experience and Education

This study draws on Dewey’s (1938/1998) belief that learning is an active mental process acquired through the experience of problem solving and other transformative experiences, and applying them in new and creative ways (Oakes & Lipton, 1999). Dewey (1938/1998) believed that not all learning experiences were equal; students must have meaningful experiences in order to understand their current situation and carry them into future learning. Writing reveals the potential origins of students’ recorded thoughts; changes students make to their writing indicates revisions in their thought processes.

Dewey (1938/1998) believed that experiences that encourage intellectual growth are educative; those that retard or hinder growth are ineffective in the learning process. Traditional education may provide students with learning experiences but fails to encourage students to take an active role in their learning (Dewey, 1938/1998). For example, the use of writing in traditional classrooms includes activities such as note taking, copying, and transcribing (Fulwiler, 1987; Mayher, Lester, & Pradl, 1983), not the type of writing that creates the opportunity to connect prior knowledge to new knowledge. By contrast, writing activities, used in conjunction with the writing process, encourage students’ active participation. Such writing includes reflective journaling (Baxter, Woodward, & Olson, 2005), connecting prior knowledge to new knowledge (Clarke, Waywood, & Stephens, 1993), and constructing logical reasoning (Grimberg & Hand, 2009). This type of writing can reveal students’ developing ideas and content understanding. Students’ active participation in writing activities that encourage
significant interaction with content understanding generate the kind of significant experiences Dewey (1938/1998) suggests are crucial for learning.

Dewey (2008) measures the value of an experience by how it continues to appear in future experiences. Students who discover connections between experiences (Dewey, 2008) are more likely to be active learners and make gains in their conceptual understanding. Experiences that engage students in recalling previous experiences provide them with resources in the form of facts, events, relationships, and actions, thus increasing their potential for problem solving. Deciding which strands of an experience are used to inform new experiences is similar to the reflective process writers undergo after receiving feedback. Content-based feedback encourages writers to reflect on their constructed ideas by offering students suggestions for clarifying their intended meaning and revealing gaps and misconceptions in their thinking (Patthey-Chavez, Matsumura, & Valdes, 2004; Silver & Lee, 2007; Vardi, 2009). The decision to maintain or eliminate particular information encourages students to become active members in their writing experiences.

Dewey’s (1938/1998) discussion of experience is relevant to the writing process; both require active student participation to make his or her learning experience meaningful. This study assumes that significant learning experiences, such as the use of writing to record and reflect upon ideas, can predict greater connections between prior knowledge and new knowledge, thus, fostering conceptual development. These meaningful writing experiences are applicable to a variety of content areas, including grade eight algebra.
Algebra

The mathematics content focus of this study is algebra, specifically, algebra as taught in a grade eight course. Algebra focuses on an understanding of patterns, relationships, and functions. Students describe situations using algebraic symbols and mathematical models to represent and understand quantitative relationships and analyze change in various contexts (NCTM, 2000). Algebra has long been thought of as a gatekeeper to advanced mathematics and science courses in high school and college (Nathan & Koellner, 2007; Spielhagen, 2006a, 2006b; Department of Education, 1997). This section will present literature describing algebra for grade eight including presenting some perspectives on the sequential nature of mathematics.

In the United States educational system, mathematics is considered a sequential subject (National Center for Education Statistics, 1999). The majority of secondary students follow a specific course of mathematics classes beginning with algebra; geometry and a second year of algebra often follow. Students who take advanced mathematics often conclude with trigonometry, pre-calculus or calculus. This specific progression of mathematics supports the claim that lower level classes must be completed before mathematical advancement can occur.

Beginning in the 1990’s many educators embraced an “algebra for all” (Viadero, 2010) policy that encouraged students to complete Algebra I by the end of their freshman year in high school. Research (Spielhagen, 2006; Burris, Heubert, & Levin, 2006; NCES, 1999; Morgatto, 2008) suggests grade eight students who complete Algebra I receive more opportunities to pursue advanced mathematics in high school. Additionally, with
the introduction of algebraic thinking encouraged in the elementary grades by the National Council of Teachers of Mathematics (NCTM, 2000), there is more opportunity for schools to incorporate an algebra curriculum into their middle school mathematics program (NCTM, 2000; Spielhagen, 2006b; National Center for Education Statistics (ED), 1999). Even though algebra may be challenging for struggling students, with a proper support system, it is preferable to a low-level mathematics or even arithmetic class (Viadero, 2010).

The introduction of algebra in grade eight mathematics suggests the need for a shift in the focus of learning mathematics from memorizing algorithms to an emphasis on mathematical competence (NCTM, 2000; NCTM, 2009). This shift encourages mathematical competency in three areas: a) conceptual understanding, or theoretical, operational, and relational knowledge; b) mathematical reasoning as revealed through logical thought, justification, explanation and reflection; and c) problem solving within real-world contexts (Findell, 1996; NCTM, 2009). This is in contrast to traditional algebra programs in which students experience the memorization of facts and procedures (NCTM, 2000) with limited opportunities to develop mathematical reasoning and conceptual understanding. Moreover, mathematical assessments in more traditional classrooms focus on computation and solving for unknowns and do little to encourage students to problem solve and apply concepts to new situations (U.S. Department of Education, 2008). The focus on mathematical computation may contribute to the frequent need for teachers to reteach mathematical topics (NCTM, 2009).
This study acknowledges NCTM’s (2000; 2009) suggestion to include richer mathematical experiences for grade eight students enrolled in an algebra curriculum. Further discussion in a later chapter will elaborate on the course of study and classroom context of the grade eight students taking algebra who participated in this study. The inclusion of writing and the writing process, focused on revealing changes in students’ algebraic reasoning, can offer one insight into students’ developing conceptual understanding in such a course.

**Mathematical Reasoning**

Algebraic reasoning focuses on constructing connections between the overarching ideas in algebra (Greens & Findell, 1999) demonstrated by students’ expressions of how mathematical ideas are related to one another (Artzt & Yaloz-Femia, 1999), how they apply their understanding in practical and creative ways (Sternberg, 1999), and how they reflect upon and articulate their mathematical interactions (Draper, 2002). Reasoning, as used in this study, includes thinking in a logical manner involving all the aforementioned characteristics students demonstrate when they engage in algebraic thinking.

NCTM (2000) provides a set of reasoning standards that can be used as a gauge to describe features of students’ logical thinking. Four characteristics indicate the nature of students’ mathematical reasoning: a) creating coherent and systematic mathematical explanations, b) exploring examples, statements that illustrate the qualities of a particular concept, and non-examples, statements that do not demonstrate the qualities of a concept, to justify conjectures, c) verifying or refuting mathematical statements by presenting sound rationale for and against, and d) justifying mathematical arguments.
through agreement or contradiction (NCTM, 2000). Additionally, deductive reasoning, drawing conclusions based on mathematical clues presented, and inductive reasoning, generalizing relationships by examining particular cases (Greenes & Findell, 1999) can be useful to students as they construct mathematical arguments that demonstrate their algebraic understanding.

Research (Malloy, 1999; Bogomolny, 2005; Sternberg, 1999) suggests teachers can support mathematical reasoning by incorporating a variety of tasks into the curriculum that stretch students’ thinking. This is in contrast to tasks and assessments which reward students for their recall of factual and procedural knowledge, thus, leading students to have a false sense of security in their ability to reason mathematically (Sternberg, 1999). Mathematical reasoning can be fostered through tasks that include activities such as asking students to discuss “how” or “why” or find creative solutions to problems (NCTM, 2009). Furthermore, applying mathematics to contexts beyond the classroom (Sternberg, 1999) encourages mathematical reasoning.

Bogomolny (2005) found that students’ construction of relevant mathematical examples is one indicator of conceptual understanding. Students in Bogomolny’s (2005) study were asked to respond in writing to questions asking them to generate an example with respect to the prompt they were given. An analysis of students’ writing indicated that the examples students gave represented their conceptual knowledge, or lack thereof, from which Bogomolny (2005) concluded students need more than procedural knowledge to create relevant mathematical examples.
Writing-to-Learn in Mathematics

Traditional uses of writing in mathematics, including demonstrations of symbolic manipulations, simplification of expressions, solving of systems of equations and inequalities, and the factoring of polynomials (Kieran, 2007) can result in students developing a false sense of mathematical competence. This use of writing emphasizes the communication of mathematical information (Fulwiler, 1987) as teachers often require students to record their calculations and symbolic manipulations during problem solving. Assessments rewarding students for correct use of numerical and symbolic language (Parker & Goodkin, 1987) illustrate their memorization of specific algorithms in decontextualized contexts (Wolcott & Legg, 1998). These conventional uses of writing, recalling procedures and facts, leave students with limited understanding of mathematics; NCTM (2009) indicates mathematics with little or no robust understanding is meaningless and is often forgotten by students.

Writing-to-learn in mathematics differs from traditional uses placing emphasis on revealing students’ conceptual understanding through their ability to demonstrate reasoning and construct mathematical proofs. For example, providing students with opportunities to connect prior knowledge to new knowledge (NCTM, 2000; NCTM, 2009), communicate problem solving processes, and apply mathematics concepts in creative ways (Sternberg, 1999) lessens the emphasis on paper-pencil skill and drills (NTCM, 2000; Stein, Remillard, & Smith, 2007). Learning to write meaningfully in mathematics, both symbolically and descriptively, makes it possible to assist students in developing the vocabulary and tools (Williams, 1998), necessary to convey mathematical
reasoning and proof (NCTM, 2000). The findings from three empirical studies describe how writing to learn encourages students’ mathematical reasoning: a) writing reveals students’ mathematical understanding, including gaps and misconceptions (Akkus & Hand, 2005); b) writing encourages internal dialogue between the student and what is being learned (Hamdan, 2005); and c) writing exposes students’ construction of mathematical arguments (Kagesten & Engelbrecht, 2006).

In a pilot study, Akkus and Hand (2005) investigated students’ mathematical content understanding and reasoning skills using a mathematics reasoning heuristic—a tool encouraging students to express their ideas with respect to problem solving through writing. High school algebra students were assigned to either the control group or a treatment group; pre- and post-tests were assigned. Students in the treatment group were asked to write a letter explaining the area of a ranch-style house whereas the control group received no additional writing tasks. Akkus and Hand’s (2005) analysis of students’ writing revealed the following: a) students solved the problem through intuition, generally restating the information from the problem as part of their explanation; and b) when students attempted to elaborate on their thinking, misconceptions regarding particular mathematical concepts were revealed indicating gaps in their understanding. Akkus and Hand (2005) found students’ writing exposed their mathematical knowledge and conceptual misconceptions suggesting their current understanding of particular concepts.

Hamdan (2005) investigated how writing fostered dialogue between students’ ideas and the mathematics they were learning. Computer science majors were asked to
write in their journals anything they thought was relevant to matrices including examples, definitions, theorems, methods for solving and relevant characteristics. Journal writing was chosen as the type of writing used to record students’ ideas over time. Students were also asked to submit two formal written pieces based upon their journals for the teacher to review, culminating in a final exam question. An analysis of students’ written drafts prior to the final pointed to the following findings: a) students revealed misconceptions in their interpretations when classifying and labeling their ideas; b) students’ failed to support their statements with adequate logical reasoning; and c) students included ideas that were tangential to the goal of the assignment (Hamdan, 2005). Hamdan (2005) found, based on an analysis of the final exam question, that written feedback during the journaling process encouraged teacher-student dialogue, and thus, students’ final submission revealed more complex construction of mathematical meaning.

Kagesten’s and Engelbrecht’s (2006) investigation demonstrated that students’ learning can be enhanced when students are encouraged to supplement their solutions with written explanations. Engineering students at a Swedish university were asked to solve a problem and explain in writing their thinking and rationale. Students were allowed to revise their solutions and resubmit them for a final grading after receiving initial feedback and a score. Students were also interviewed to understand the process they went through and what they may have learned from their experience. Kagesten and Engelbrecht (2006) found the following: a) students’ inexperience with writing may have contributed to their weaknesses in their initial explanations; b) students’ revised responses indicated richer mathematical understanding; and c) the opportunity to receive
feedback further encouraged students to clarify their understanding. Kagesten’s and Engelbrecht’s (2006) findings point to the use of writing to reveal changes in students’ thinking when given the opportunity to receive feedback and revise their intended meaning to better reflect their mathematical reasoning.

Although research (Gammill, 2006; Fulwiler, 1987; NCTM, 2009; Akkus & Hand, 2005; Hamdan, 2005; Kagesten & Engelbrecht, 2006) implies writing has a potential place within mathematics such as encouraging critical thinking and reflection (Gammill, 2006), connecting prior to new knowledge (NCTM, 2009) and revealing students’ self-awareness of their ideas (Fulwiler, 1987), others have expressed concerns regarding the use of writing in mathematics. These concerns are of interest to this study as writing will used by students as the primary means of communication to express their mathematical understanding in response to open-ended prompts.

**Writing-to-Learn Concerns in Mathematics.**

Research (Shield & Galbraith, 1998; Shield, n.d.; Ntenza, 2006; Harbaugh, Pugalee, & Adams, n.d.) reveals instances suggesting some writing activities, such as expository writing and formal report writing, fail to foster true conceptual understanding. Instead, in these instances, students’ writing resembles the organization and show-and-tell approach seen in textbooks. Researchers (Shield & Galbraith, 1998; Ntenza, 2006) implied students who are exposed to textbook-style writing may mimic the structure and tone assuming it is representative of writing in mathematics.

For example, Shield and Galbraith (1998) investigated the claim that writing enhances learning in mathematics. Grade eight students participated in two expository
writing activities, writing a letter to a friend and responding to a students difficulty, and were asked to elaborate on particular mathematical ideas. Shield’s and Galbraith’s (1998) analysis of student writing suggested the following characteristics: a) students’ procedural approach to problem solving was the primary focus of their discussion; b) students failed to provide rationale for their mathematical manipulations; and c) students’ writing appeared to be organized in a tell-show-do approach. Shield and Galbraith (1998) indicated that students’ writing appeared to mimic the writing commonly found in mathematics textbooks and implied textbook writing may be the dominate form of writing in mathematics students were exposed to. Shield and Galbraith (1998) also implied textbook-style writing fails to convey mathematical meaningfulness in students’ writing and higher levels of critical thinking regarding mathematical concepts. While Shield and Galbraith (1998) imply that there is no evidence that suggests teachers purposefully teach students to write in a style mimicking their textbook, they do suggest students are skilled at describing procedural manipulations in writing. Like Shield and Galbraith (1998), other researchers (Shield, n.d.; Ntenza, 2006; Harbaugh, Pugalee, & Adams, n.d.) have called for further research in the area of writing in mathematics to explore the role of writing in encouraging greater mathematical reasoning and higher levels of thinking.

Assumptions About Goals in Mathematics Education

This study hypothesizes that writing may be able to reveal students’ developing algebraic reasoning through the utilization of the writing process and written teacher-student feedback. In addition to research (National Writing Project & Nagin, 2006; Evans
& Houssart, 2004; Baxter, Woodward, & Olson, 2005; Pugalee, 2004; Theoret & Luna, 2009) suggesting writing-to-learn can encourage the development of mathematical understanding, this study is also interested in indicators that convey an accurate representation of the characteristics of students’ mathematical reasoning. The National Council of Teachers of Mathematics Principles and Standards for School Mathematics (NCTM PSSM) (2000) and the Common Core State Standards for Mathematics (CCSSM) (CCSSI, 2010) propose standards that this study assumes are currently believed to represent best practice for students to achieve conceptual understanding and to be college and career ready (NCTM, 2000; CCSSI, 2010).

This study assumes that both the Principles and Standards for School Mathematics (PSSM) (NCTM, 2000) and the Common Core State Standards for Mathematics (CCSSI, 2010) provide insights into students’ attainable achievement in Algebra I and suggest characteristics of demonstrated learning that indicate students’ reasoning in mathematics. Both documents support this study by placing the emphasis on conceptual understanding and suggest these concepts are necessary to construct logical mathematical arguments. This study further assumes these standards represent quality learning experiences in mathematics education and inform this study of potential ideas students should include in their discussion of particular mathematical topics and themes. A discussion of both sets of standards and the ideas this study is interested in follows below.

NCTM’s Principles and Standards for School Mathematics.
NCTM’s *Principles and Standards for School Mathematics* (NCTM, 2000) is guided by the vision that students can be active learners of mathematics, understanding the role of mathematics in the community and expressing their ideas orally and in writing. NCTM’s vision is appropriate for this study as it aims to use writing as an activity to encourage an active construction of mathematical meaning and express those ideas to reveal changes in students’ conceptual understanding.

Of particular interest to this study are the Algebra Standards for Grades 6–8 (NCTM, 2000) that suggest indicators of algebraic content students might reveal in their written discussions in response to open-ended mathematically themed prompts. This study assumes these algebra standards support the cognitive abilities of grade eight students identifying specific and relevant algebra concepts attainable within the classroom context and suggest reasonable performance tasks students might engage in as they construct mathematical reasoning. Appendix B presents the Algebra Standards and the appropriate performance tasks for Grades 6-8 that this study is interested in.

Running in conjunction with NCTM’s (2000) algebra standards are the *Common Core State Standards for Mathematics* (CCSSI, 2010) which offer another perspective for identifying characteristics of students’ conceptual understanding.

*Common Core State Standards in Mathematics* (CCSSM).

The goal of the Common Core State Standards Initiative (CCSSI, 2010) is to express what students are expected to learn in mathematics with emphasis on depth, not breadth, of understanding. The *Common Core State Standards in Mathematics* (CCSSM) (2010) suggest teachers and students should balance procedural knowledge and
conceptual understanding through an organized and continuous progression of sequential mathematics before engaging in an algebraic curriculum.

In addition to the content standards, this study focuses on the CCSSM Standards for Mathematical Practice (CCSSI, 2010). These standards serve two purposes: they describe the characteristic behaviors of students who demonstrate mathematical proficiency and suggest the types of activities that give students opportunities to develop these behaviors as they learn mathematics. The Standards for Mathematical Practice (CCSSI, 2010) are as follows:

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning (CCSSI, 2010).

The standards place emphasis on mathematical process and proficiency suggesting students should be able to demonstrate mathematical reasoning, conceptual understanding, procedural fluency, in addition to expressing their ideas in a variety of ways and making connections between related concepts. Of the eight aforementioned standards, Standards 2, 3, 4, and 6 are more relevant to this study providing guidance for
the analysis of student writing with further discussion to follow in the methods chapter, Chapter 3.

**NCTM’s PSSM and CCSSM: Reasoning and Proof Standards.**

This study is also interested in indicators that suggest characteristics of students demonstrating mathematical reasoning. Reasoning, according to NCTM (2000) is a way for students to demonstrate mathematics has meaning and can be encouraged through the exploration, analysis, and use of mathematical conjectures. Both NCTM’s PSSM (2000) and CCSSM (2010) emphasize the importance of educating students in mathematics by fostering critical thinking and reasoning skills. The NCTM (2000) and CCSSM (2010) standards which discuss mathematical reasoning are as follows:

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**Figure 1. Comparing NCTM and CCSSM Reasoning and Proof Standards**

<table>
<thead>
<tr>
<th></th>
<th></th>
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</thead>
<tbody>
<tr>
<td>Instructional programs from prekindergarten through grade 12 should enable all students to:</td>
<td></td>
</tr>
<tr>
<td>• Recognize reasoning and proof as fundamental aspects of mathematics</td>
<td></td>
</tr>
<tr>
<td>• Make and investigate mathematical conjectures</td>
<td></td>
</tr>
<tr>
<td>Of the eight Mathematics Standards for mathematical practice, three specifically address Reasoning and Proof:</td>
<td></td>
</tr>
<tr>
<td>• Reason abstractly and quantitatively</td>
<td></td>
</tr>
<tr>
<td>• Construct viable arguments and critique the reasoning of others</td>
<td></td>
</tr>
<tr>
<td>• Look for and express regularity in repeated reasoning</td>
<td></td>
</tr>
</tbody>
</table>

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REVEALING STUDENTS’ ALGEBRAIC REASONING

- Develop and evaluate mathematical arguments and proofs
- Select and use various types of reasoning and methods of proof


Shaughnessy (2010) suggests, “The strongest connection between CCSSM and NCTM’s longstanding work in establishing standards for mathematics curriculum and instruction can be found between NCTM’s Process Standards and the Standards for Mathematical Practice articulated in CCSSM” (http://www.nctm.org/about/content.aspx?id=27076). He further suggests NCTM’s PSSM (2000) and CCSSM (2010) standards for mathematical reasoning are complementary and supportive of one another (Shaughnessy, 2010) in encouraging learning activities, such as writing-to-learn activities, that have the potential to foster students’ mathematical reasoning and conceptual development. This study is interested in NCTM’s (2000) and CCSSM (2010) standards for mathematical reasoning as suggested indicators of students’ inclusion of mathematical reasoning in their written responses to open-ended mathematically themed prompts.

The literature suggests that while there are concerns with including writing-to-learn activities within mathematics, the inclusion of these activities has the potential to shift the focus from passive to active learning, encourage mathematical reasoning and
reveal conceptual understanding. Writing that focuses on the process, rather than the product of writing, has the potential to broaden students' limited perspective of learning mathematics and engage them in thoughtful reflection and revision of their own writing revealing gaps and misconceptions in their own thinking (National Writing Project & Nagin, 2006), exposing their idea construction (Mayher, Lester, & Pradl, 1983) and indicating their mathematical understanding.
Chapter 3: Methods

Advocates of writing-to-learn in mathematics (National Writing Project and Nagin, 2006; Mayher, Lester, & Pradl, 1983; Akkus & Hand, 2005; Baxter, Woodward, & Olson, 2005; Pugalee, 2004; Hamdan, 2005; Theoret & Luna, 2009) suggest writing has the potential to assist students in the construction of mathematical meaning by encouraging dialogue between the student and the mathematics learned, revealing gaps and misconceptions in students’ conceptual understanding, expressing mathematical arguments, and connecting prior knowledge to new knowledge. Writing-to-learn activities include explanatory writing (Evans & Houssart, 2004), journal writing (Baxter, Woodward, & Olson, 2005), reflective writing (Theoret & Luna, 2009), and the opportunity to revise after receiving feedback (Pugalee, 2004).

Other research (Shield & Galbraith, 1998; Shield, n.d.; Ntenza, 2006; Harbaugh, Pugalee, & Adams, n.d.), however, indicates inconsistencies and concerns with writing-to-learn activities in mathematics particularly, expository (Shield & Galbraith, 1998) and formal report (Harbaugh, Pugalee, & Adams, n.d) writing as students’ writing resembled the show-and-tell approach of their textbook. These conflicting findings indicate more research is needed to reveal what writing can reveal about students’ construction of mathematical reasoning and their conceptual maturation.

This study aims to reveal and describe changes in middle school students’ emerging algebraic reasoning by analyzing their written responses to open-ended mathematically themed prompts, constructed using the writing process. This chapter
describes and provides a rationale for the data gathering and data analysis methods used in this study. The following is an overview of the major sections of the chapter:

1. Research design: This study was conducted using qualitative research design, appropriate for educational settings and supportive of teacher-researchers. A pilot study was conducted prior to the larger study for the purpose of informing and validating the methods used in the larger study.

2. Participants/sample: A selected group of grade eight students at Mission School\(^2\) were purposefully chosen to take part in this study. The selection process occurred after all students participated throughout the school year in writing-to-learn activities as part of their mathematics curriculum. This section further describes who these participants were, how they were purposefully chosen, and includes some demographics of the students.

3. Instruments: Students responded to open-ended writing prompts throughout the course of the school year as new topics were introduced in mathematics for a total of seven written responses. A second instrument consisting of interview questions was also created. This section further describes how these writing prompts were developed with a complete set appearing in Appendix A and includes a discussion of the construction of the interview questions. The complete list of interview questions appears in Appendix E.

\(^2\) Mission School is a fictitious name to keep the identity of the school anonymous.
4. Procedures: Students responded to each open-ended writing prompt three times throughout their course of study of a particular mathematics theme; they received feedback on their second response. These responses comprised the data for this study in addition to participants’ one-on-one interviews after the written data collection period was complete. This section further describes the data collection process, what types of data were collected, when, how, and how frequently throughout the course of the school year.

5. Data analysis methods: Because of its emergent design flexibility, grounded theory was chosen for the analysis of data in order to produce a theory about the progression of students’ conceptual development in mathematics as revealed through their writing. Each individual draft students produced in response to given open-ended prompts underwent a content analysis for the purpose of revealing trends in their algebraic reasoning. Individual drafts were also analyzed for trends within the drafts as they were produced throughout the school year. Written feedback comments also underwent a content analysis to determine the type of feedback students were receiving. This section further describes how students’ written drafts were analyzed to reveal emergent trends in their mathematical reasoning.

Research Design

This study’s research design draws from the tradition of teacher-as-researcher (Burnaford, Fischer, & Hobson, 1996; MacLean & Mohr, 1999) and qualitative research methodology (Patton, 2002; Hatch, 2002; Merriam, 1998; Charmaz, 2006). The first
section focuses on the appropriateness of qualitative research and how that methodology supports the teacher-as-researcher. The second section focuses on the pilot study that was conducted to inform the data collection instruments and procedures of the study.

**Qualitative Research in Education**

The study utilizes qualitative research methods to produce rich, thick data from which an emergent theory can be generated (Patton, 2002; Hatch, 2002; Merriam, 1998; Charmaz, 2006) to describe the progression of grade eight students’ algebraic reasoning in response to open-ended mathematically themed prompts. It focuses on incorporating the writing process (National Writing Project & Nagin, 2006) and follows the assumption that students’ drafts will capture the development of their thinking, revealing their conceptual understanding, as they are encouraged to revise their constructed meaning as they produce each successive draft. The study seeks to use writing in mathematics to reveal students’ developing algebraic reasoning by considering the following: a) the mathematics classroom context, activities and student interactions; b) students’ engagement in writing-to-learn activities; and c) students’ construction of mathematical understanding as they are revealed through the writing process. An analysis focused on constructing meaning (Patton, 2002; Patton, 1985 in Merriam, 1998) of students’ written content and their in-depth interviews exposing their experiences will be used to reveal their conceptual understanding and maturation in mathematics.

**The Role of Teacher-as-Researcher.**

Descriptive qualitative studies in education allow the teacher to assume the role of a researcher within his or her own classroom. The teacher-as-researcher is granted an
insider’s perspective (Patton, 2002; Merriam, 1998) provided he or she maintains empathetic neutrality (Patton, 2002; Hoepfl, 1997) in interactions with the environment and participants of the study (Hatch, 2002). Empathetic neutrality is maintained by becoming non-judgmental and remaining open, sensitive, aware and responsive to the participants and their environment (Patton, 2002). This is in contrast with the role of a teacher when evaluating and assessing students’ performance using both formative and summative means. The role of the teacher-as-researcher is to ask questions, collect data, and reflect on students’ learning through observations and artifacts, without disturbing the natural flow of the classroom or the school setting (MacLean & Mohr, 1999; Hatch, 2002). Furthermore, the teacher-as-researcher’s educational experiences, both as a teacher and as a student, place him or her at the intersection between teaching and teacher-research, having the ability to look both inward and outward, moving easily between the two perspectives (Hobson, 1996).

In this study I assumed the role of teacher-as-researcher with the intent of viewing students’ work as data to be analyzed and interpreted, with the aim of developing theory (MacLean & Mohr, 1999). Being in the classroom on a daily basis provided me with the opportunity to become fully immersed in the school environment (Kamberelis & Dimitriadis, 2005; Merriam, 1998) thus enabling me to observe the dynamics and complexities of classroom culture including expected and unexpected occurrences.

**Pilot Study**

A pilot study was conducted early in the 2009-2010 school year to inform the data collection instruments and procedures. The pilot study served to improve the larger
study’s effectiveness and validate the use of particular data collection instruments and procedures (National Centre for the Replacement, Refinement and Reduction of Animals in Research, 2006).

An analysis of the pilot study data, collected during the fall semester of 2009, revealed that changes were necessary in the types of writing prompts that would encourage students to construct algebraic reasoning. Thus, the pilot study comprised two parts: Part I, conducted during the fall semester of 2009, focused on textbook based writing prompts. Part II, conducted during the spring semester of 2010, focused on writing prompts based on teacher-constructed essential questions; these open-ended prompts captured the theme of particular textbook chapters. Grade eight students, of various academic achievement levels in mathematics, assigned by school administration to my algebra class, participated in the pilot study.

**Writing in Mathematics Pilot Study: Part I**

I was interested in using writing to reveal students’ algebraic reasoning, content knowledge, and conceptual understanding. At the conclusion of a mathematical theme, usually coinciding with the conclusion of the given textbook chapter, students were asked to choose one problem from a list of problems I generated and discuss how they arrived at their solution. These teacher-chosen word-problems containing contexts and situations existing outside of the mathematics classroom were taken from the students’ textbook *Discovering Algebra: An Investigative Approach* (Murdock, Kamischke, & Kamischke, 2002). Purely algebraic expressions or equations that focused solely on procedural manipulation rather than creative or practical conceptual applications (Sternberg, 1999)
were not included. Additionally these problems were all odd numbered and the students and I both knew that the textbook publishers purposefully included the solution to these odd numbered problems at the conclusion of the textbook whereby students could check the accuracy of their solution. I believed numerical solutions that were readily accessible to students would allow them to focus on demonstrating procedural and conceptual knowledge (Sternberg, 1999) and not worry if he or she arrived at an incorrect solution due to a careless error. Moreover, I made the assumption that writing increases the cognitive demands on students (Urquhart, 2009) and, thus, would encourage them to go beyond a demonstration of manipulating mathematical symbols. Students were given a few days to consider and submit their written responses.

An analysis of students’ written responses revealed the following: a) students’ written responses suggested they believed describing procedural manipulations in writing and/or manipulating mathematical symbols was an acceptable way of demonstrating mathematical reasoning; b) students’ presentation of their rationale mimicked the organization from examples found in their textbook; c) students failed to discuss why they chose to apply a particular concept in finding a solution, and d) students indicated their use of words describing their problem solving processes provided the rationale for their solution. Students patterned their writing after models they saw in their textbook, thus suggesting these examples of writing are dominant in mathematics classrooms (Parker & Goodkin, 1987; Ntenza, 2006). Moreover, I realized that textbooks may not provide optimal writing prompts for students as they fail to encourage writing activities
that ask students to create their own problems or demonstrate their independent conceptual understanding (Ntenza, 2006).

Based on these findings, I proposed that the type of writing prompt given to students could unintentionally affect how students responded to it. These results were corroborated by researchers (Shield & Galbraith, 1998; Shield, n.d.; Harbaugh, Pugalee, & Adams, n.d.) who indicated writing prompts asking students to determine if a mathematical statement is true or false (Evans & Houssart, 2004), expository writing prompts (Shield & Galbraith, 1998), and formal report writing (Harbaugh, Pugalee, & Adams, n.d.) fostered textbook-style writing from students. These findings lead to the revision of the writing prompts aimed to encourage students to reveal greater critical thinking and mathematical understanding (Sheild & Galbraith, 1998).

Writing in Mathematics Pilot Study: Part II

I continued to investigate the use of writing to reveal students’ algebraic reasoning and conceptual understanding in Part II of the pilot study by altering the writing prompts. Furthermore, two additional components were added to the pilot study design: application of the writing process (National Writing Project & Nagin, 3006) and teacher feedback. These additional tools can assist writers in revealing their intended meaning and foster more sophisticated thought processes (National Writing Project and Nagin, 2006; Fulwiler, 1987; Pattey-Chavez, Matsumura & Valdes, 2004).

The new writing prompts were influenced by a series of teacher-generated essential questions from Mission School’s curriculum map designed in 2007-2008. Essential questions are open-ended inquires containing multiple answers (Brown, 2009).
As one of the teachers who participated in the discussions to create these questions, I knew of their existence and witnessed our attempt to craft them in such a way as to suggest there was more than one possible solution. My conjecture was that these questions, focused on particular mathematical themes, would encourage students to consider mathematics in three possible contexts: a) analytical situations that teach students formulas and facts, b) creative situations that encourage students to create their own problems, and c) practical situations where everyday contexts require the application of mathematical concepts (Sternberg, 1999).

In addition to the new prompts, I chose to incorporate the writing process as part of the procedure as students constructed their responses. I assumed the writing process would provide students with multiple opportunities to express and revise their thinking (National Writing Project & Nagin, 2006) revealing their content knowledge and conceptual understanding (Hamdan, 2005; Baxter, Woodward, & Olson, 2005). The design was for students to respond three times during the course of unit study—at the introduction, midway, and conclusion—to the same open-ended prompt whose theme coincided with the topic students were learning in class. I decided to have students complete two revisions follows Graziano-King’s (2007) procedure in which community college students composed a written response at the beginning of the semester and revised it three additional times throughout the semester. Drafting provides insights into students’ construction of knowledge and their developing content understanding (Graziano-King, 2007).
During the revision process, I provided students written feedback on their submitted second drafts. Feedback consisted of hand-written comments to students scattered throughout their written discussion. Feedback encourages students to revise their thinking, construct meaningful content, and improve the overall quality of their writing (Whittington, Glover, & Harley, 2004; Vardi, 2009).

An analysis of students’ written responses in Part II of the pilot study suggested the following findings: a) freewrites, either in bulleted or paragraph form, consisted primarily of definitions from textbooks or in-class notes; some students also included one-to-two sentence examples they believed were relevant; b) second drafts focused on relevant definitions and offered relevant examples but many failed to draw connections between the relevant mathematical themes with respect to those examples; c) final drafts indicated more elaboration of mathematics definitions and integration of mathematics into students’ examples but primarily focused on procedural knowledge. I also noticed most students increased both the quantity and quality of content in their discussion over the course of the revision process.

The findings from the pilot study and those from similar empirical research were adapted for the data collection instruments and procedures of the larger study. In keeping with researchers Shield and Galbraith (1998) whose findings suggest expository prompts encouraged textbook-style writing, the writing prompts for this study were crafted as open-ended inquires to encourage students to respond creatively in their discussions. Additionally, in accordance with Vardi (2009), feedback was content oriented and provided during the process of writing (National Writing Project & Nagin, 2006); this
typed feedback pointed to areas in students’ writing where I suspected gaps and misconceptions existed in students’ thinking (Evans & Houssart, 2004). Furthermore, students participated in the process of writing throughout the school year and, in keeping with Staats and Batteen (2009), were encouraged to revise their ideas after receiving feedback to construct more significant connections between their mathematic model and their student-generated context.

The following research questions were posed to guide the larger study:

1. How does writing reveal a student’s algebraic reasoning and knowledge of algebraic concepts?

2. In what ways does writing, through the revision of multiple drafts, expose the development of algebraic reasoning?

3. In what ways does teacher-to-student written feedback, provided during the writing process, prompt the revision of algebraic reasoning as represented in students writing?

School and Classroom Context and Population

Eleven grade eight students at Mission school were purposefully chosen to participate in the study. This section discusses my classroom context to provide an insight into the types of activities students participate in. This is followed by a description of the participants, their general demographics and how they were chosen to participate in this study.
Background of Mathematics Education at Mission School

Mission School is a kindergarten through twelfth grade co-educational parochial school whose philosophy emphasizes the development of the whole child (School Student/Parent Handbook, 2009). In 1998, the grades six through eight faculty began reforming the then current Junior High into a Middle School. With the guidance of Turning Points 2000 (Jackson & Davis, 2000) and This We Believe (NMSA, 1992), the Middle School was firmly established by 2002.

Prior to the 2006–2007 school year, grade eight students were grouped heterogeneously except for mathematics. A standardized algebra readiness test, given at the end of seventh grade, and a recommendation from the seventh grade mathematics teachers determined whether each child was placed in an Algebra or Pre-Algebra course. Students who needed a longer transition between arithmetic and Algebra were placed in Pre-Algebra. Approximately half of the grade eight, fifty to fifty-six students, enrolled in Algebra with the remaining placed in Pre-Algebra. Despite research (Oakes, 1986; Horn, 2006; Spielhagen, 2006a; Spielhagen, 2006b) suggesting that tracking places students at a severe disadvantage, compromising their access to more sophisticated content knowledge and the opportunity to develop various cognitive skills such as reasoning, school personnel believed this sole exception would continue to promote intellectual safety and camaraderie between classmates of different homerooms.

During the 2006–2007 school year, grade eight students were grouped into homerooms based on their previous year’s mathematics achievement, either algebra or
pre-algebra, due to internal changes within the Middle School personnel. Heterogeneous groups were no longer maintained.

At the end of the 2006–2007 year, the situation was re-evaluated and administrators decided that all students should have an equal opportunity to advance in mathematics; algebra coursework was made available to all students. This decision aligned better with the Middle School Philosophy (Jackson & Davis, 2000) creating a more positive climate of intellectual development and support of high achievement.

Since the 2007–2008 school year, all grade eight students at Mission School receive instruction in algebra. Currently, advancement into high school Geometry for freshman year mathematics is based on three criteria: a) overall performance throughout the students’ achievement in Algebra I; b) a student’s score on an Algebra I placement exam; and c) the current mathematics teacher’s approval for a student’s freshman year mathematics class. This policy affords all grade eight students the opportunity to be exposed to algebra concepts, construct algebraic understanding, and complete Algebra I by the end of their freshman year of high school; those who show academic achievement in Algebra I as an grade eight student are encouraged to progress to more advanced mathematics courses in High School.

**Classroom Context**

A large portion of Mission School’s grade eight Algebra I curriculum focuses on linear algebra. Themes such as rate of change, linear equations, line of fit, and systems of equations and inequalities are just some of the topics students encounter. Students’ textbook, *Discovering Algebra: An Investigative Approach* (Murdock, Kamischke, &
Kamischke, 2002) provides students with examples and practice homework problems. Classroom instruction for each chapter varies but usually includes activities such as homework checks, investigations, group discussion, and use of graphing calculators in addition to note taking and additional practice problems. Students keep all their homework, notes, and investigation results in a spiral notebook.

- **Note Taking:** Students take notes on newly introduced vocabulary words and particular algorithms and equations introduced in a chapter.

- **Practice Problems:** Students receive practice problems in class in addition to those presented in the textbook as a means of applying conceptual and procedural knowledge; the majority of these practice problems contain real-world contexts and are written in the form of words. Practice problems may be done individually, in pairs, or in small groups assisted by the teacher.

- **Homework Check:** In pairs, students compare and discuss answers to the previous night’s homework. Usually the even numbered problems are assigned because the textbook supplies the answers to the odds at the conclusion of the book. Students submit their answers to specific problems chosen by the teacher before the answers are read aloud to allow students to correct their own work. Any questions raised by the students are discussed at this time.

- **Investigations:** Students explore mathematical concepts and principles by doing investigations provided by the textbook, *Discovering Algebra: An Investigative Approach* (Murdock, Kamischke, & Kamischke, 2002). These activities can include real-world situations, purely mathematical contexts and the use of
manipulatives to collect and analyze data for the purpose of creating a mathematical model.

- Group discussions: Groups of students discuss, solve, explain and sometimes present a solution to a particular problem. Some presentations are purely oral; others involve posters containing calculations, mathematical sentences, and relevant figures or graphs. Students are encouraged to ask questions about other groups’ responses after they present their ideas.

- Graphing calculators: Students learn to use appropriate graphing calculator tools after performing various mathematical procedures and operations manually. For example, students find the mean by performing addition and division operations to a given data set; then they learn the calculator command that performs all the operations for them.

Students participate in these activities to gain the factual, procedural and conceptual knowledge necessary to understand the mathematics involved in Algebra I. The writing assignments that are the focus of this study are an additional component of the mathematics class instructional activities.

**Participants**

During the 2010–2011 school year, a total of forty-four students took Algebra I from me at Mission School. Section A had twenty-three students, twelve males and eleven females; section B had twenty-one students, thirteen males and eight females. Both sections of students were of mixed ethnicities including Asian, Caucasian, and Pacific-Islander. Of all the students, only two students, one male and one female were
new to the school; the remainder had received mathematics instruction at Mission School using the textbook *Transition Mathematics* (Viktora, S. S., Thompson, D. R., & University of Chicago School of Mathematics Project, 2008). *Transition Mathematics* functions as a pre-algebra class focusing on topics that include applied arithmetic, algebra and geometry. Student achievement in mathematics ranged from low to high. Each student completed the same types of assignments and participated in similar activities regardless of their academic achievement.

The following procedures were established to maintain empathetic neutrality throughout the school year: a) all students would complete at least seven topics, with emphasis on linear algebra, from their textbook, *Discovering Algebra: An Investigative Approach* (Murdock, Kamischke, & Kamischke, 2002); b) all students would be asked to respond to at least seven open-ended prompts, one for each topic addressed; c) all students would receive content-based feedback on the second draft of their responses; d) students’ quarterly grades would not be affected by agreeing or refusing to participate in the study since students would not be chosen as part of the study population until midway through the fourth quarter; and e) no student would have advanced knowledge of their participation in the study so they would not intentionally or unintentionally alter their written responses or other behavior in the class during the school year.

A purposeful sample of eleven students, one-fourth of the total students that I taught, were selected as the study population. Purposeful samples offer rich sources of data (Patton, 2002; Hatch, 2002) and share similar qualities to the general population (Patton, 2002; Hoepfl, 1997). These eleven students were selected at the end of April,
towards the conclusion of the school year to avoid any intentional or unintentional bias on my part as a teacher and as a researcher (Patton, 2002) or purposeful or unintended changes in students’ behavior towards the teacher because they knew the teacher was also the researcher (Hatch, 2002). In this way, the chosen students could not claim they were given extra work throughout the school year because all students, regardless of their mathematics achievement, participated in the same writing activities.

Potential students were identified using criterion sampling methods to assure a quality sample would be chosen (Patton, 2002). The following criteria were established: a) students must show a history of regularly completing their algebraic homework assignments; b) students must complete at least five full sets of written drafts, one set for each essential question per chapter; c) students must have a firm grasp of the English language and use it to communicate to their teachers and peers throughout the school day. Students who were unable to meet these criteria were not initially considered to be participants in this study. Two additional criteria were applied to narrow the participants further: a) students who received modifications and/or accommodations as stated by their Individualized Education Plan (IEP) per the academic counselor; and b) students who were listed, at any time during the current school year on academic watch or academic probation per the academic counselor, were eliminated from the selection pool. Of the students who remained, a random sampling of approximately the same number of males and females from both sections was chosen. This chosen population represented approximately one-fourth of the total population of students that I taught.
The eleven grade eight students, six male and five female represented the two algebra sections that I taught. The participants ranged in age between thirteen and fourteen years of age. Of the eleven students, two were of Asian-Pacific Islander-Caucasian mix, one was of Asian-Caucasian mixed ethnicity, and the remainder represented various Asian ethnicities, either singly or of mixed Asian ethnicity. The participants had Grade Point Averages (GPA) ranging between 2.5 and 4.0 for each quarter throughout the school year. Their average report card grades in mathematics ranged from C through A with most of the participants consistently earning quarterly grades in the B-range. Of the eleven participants, nine entered Mission School during their elementary years; all eleven participants attended the Mission School’s Middle School during grades 6, 7 and 8. At the end of their grade eight year, only one participant did not continue on to Mission High School for financial reasons.

Data Collection Instruments

Two primary data collection instruments are used for the study: the set of open-ended mathematically themed prompts, one for each of the seven mathematical themes students studied and the set of one-on-one interview questions; observations and field notes are secondary sources of data. This section discusses each data collection instrument, its purpose, and how each was constructed.

Open-Ended Prompts

The open-ended writing prompts used in this study contain the following characteristics: a) writing prompts are crafted as open-ended prompts; b) writing prompts ask students to generate their own examples for discussion; and c) writing prompts are
organized by topic in accordance to themes presented by students’ textbook *Discovering Algebra: An Investigative Approach* (Murdock, Kamischke, & Kamischke, 2002).

Open-ended writing prompts are modeled after essential questions (Brown, 2009) crafted using the words “how,” “why,” or “in what ways,” within the question to encourage students to elaborate on their thinking and provide mathematical rationale to reveal their conceptual understanding. These characteristics are believed to dissuade students from searching for the correct answer or relying on transcribing information from their textbook or notebook and encourages the investigation and discovery of conceptual relationships (Parker & Goodkin, 1987). I hoped that students’ responses to these prompts would demonstrate their level of mathematical understanding suggested through appropriate mathematical applications and models (NCTM, 2000).

These open-ended prompts also asked students to create examples consisting of a context outside of the classroom from which to launch their mathematical discussion to reveal how students applied their mathematical understanding in creative and practical ways (Sternberg, 1999; Bogomolny, 2005). This was in contrast to prompts asking students to respond to the truthfulness of a particular mathematical statement (Evans & Houssart, 2004) or explanatory prompts (Shield & Galbraith, 1998) which tended to encourage procedural explanations rather than mathematical reasoning (Shield & Galbraith, 1998; Evans & Houssart, 2004).

Students responded to the following open-ended writing prompts organized by chapter and topic:
• Chapter 0: Fractions and Fractals — Explain, using the concepts found in Chapter 0, why there is or why there is not a pattern to chaos? Provide an example and/or non-example to justify your reasoning.

• Chapter 1: Data Exploration — In what ways can analyzing and visually displaying data influence your view of the world around you? Provide an example and/or non-example to justify your reasoning.

• Chapter 2: Proportional Reasoning and Probability — Compare and contrast fractions and ratios. Justify how they can be used to describe situations that you’ve experienced outside of mathematics class.

• Chapter 3: Variation and Graphs — How do quantities vary directly and inversely? Discuss and explain an example and/or non-example of a situation outside of mathematics class that illustrates direct and inverse variation. Justify why the situation is direct, inverse or neither.

• Chapter 4: Linear Equations — In what ways can a situation be described as a recursive sequence? Create an example and/or non-example from your life and justify why the situations can be or cannot be described as recursive sequences.

• Chapter 5: Fitting a Line to Data — When is it appropriate (example) and when is it not appropriate (non-example) to create a line of fit to describe human ability and why?

• Chapter 6: Systems of Equations and Inequalities — In what ways do systems of equations and inequalities exist outside of your mathematics class? Provide an example and/or non-example from your life and explain what they mean.
Student Interview Questions

The meaning students construct is only as clear as the language used to convey their intended thoughts (Mayher, Lester, and Pradl, 1983). This limitation comes not only from students’ ability to use words to express their thinking but it is also restricted by the researcher’s analytical interpretation of those words. The interview, in this study, serves two purposes: a) it provides me with another source of data—one that allows for immediate clarification with a clarifying or follow up question. For example, interviewing the participants revealed how they constructed their ideas (Patton, 2002) and the sources they based their ideas on; and b) it allows me to verify any interpretations and analyses of students’ written expressions—any hypotheses formulated regarding students’ understanding could be explored, verified or disputed at this time. For example, descriptions from students on their perspective (Patton, 2002) of their conceptual development and/or thinking could reinforce emergent trends (Hatch, 2002) based on an analysis of their written expressions.

I constructed a semi-structured interview guide based on the following themes: a) writing and constructing algebraic reasoning; b) the writing process (National Writing Project & Nagin, 2006), revision and mathematical reasoning; and c) receiving teacher feedback and revising mathematical reasoning. For a complete list of the questions, see appendix E. These questions were organized in a broad-narrow-broad manner; general questions about past experiences encouraged students to describe their prior experiences before more specific questions focused on their current experiences were posed.
Concluding questions asked students to provide advice based on their experiences and to offer any final thoughts they felt were important to describe their experiences.

Interview questions were crafted as open-ended questions to encourage students to share and elaborate on their experiences (Hatch, 2002) and emphasized there were no right or wrong answers. Three types of questions were asked: a) descriptive questions, such as, “Can you provide any examples (of how writing helps you to learn) from your past or present classes,” focus on eliciting specific knowledge or information; b) structural questions, such as, “What types of feedback benefit you most (and why),” ask the participant to organize information into specific categories; and c) contrast questions, such as, “Compare how writing was used in your previous classes with your current Algebra class,” ask students to create meaning from their experiences. Additional questions were constructed to capture students’ feelings and experiences while others focused on students’ opinions and values. All questions were aimed at understanding how students constructed the mathematical meaning expressed in their writing.

Interviews occurred in late May after students consented to participate in the study. An analysis using constant comparative methods (Patton, 2002) determined the emergent themes from the responses. The protocol for this analysis will be discussed in a subsequent section.

**Observations and Field Notes**

In this study, observations and field notes are secondary sources of data that provide further insight into students’ experiences with writing in mathematics. Observations included students’ behaviors, actions, and interactions that occurred in my
mathematics classroom; these detailed descriptions were recorded in field notes including the context in which the observations were made (Patton, 2003). Observations and field notes were made particularly when students were completing their freewrite as each new open-ended prompt and topic were introduced. Field notes were also taken when students requested additional assistance after receiving feedback on their second draft. Written descriptions based on my experience interacting with each participant were recorded after the sample had been chosen. An example of my observations and field notes appears in Appendix G.

The observations and field notes were used to further confirm the trends that emerged from students’ written responses and their one-on-one interviews. Triangulating the data in this way produces richer descriptions of students’ construction of mathematical meaning. These instruments are key to providing significant data to reveal and describe the changes in students’ algebraic reasoning as they respond using the writing process to open-ended prompts.

**Procedures**

Students responded to seven open-ended writing prompts throughout the school year for a total of twenty-one individual responses. Those chosen to represent the sample population also participated in one-on-one interviews towards the conclusion of the school year. This section describes the data collection process. It describes when, how, and how frequently the writing responses were collected, the protocol for giving teacher-to-student written feedback, and the process for interviewing the participants in this study.
Writing Prompts

Students were given a handout to introduce them to each open-ended prompt after an introduction to the mathematical theme and its corresponding textbook chapter. This handout was teacher generated and given to the students for the purpose of providing them with information they might find useful during the process of writing. The content of the handout, see Appendix A, and any letter and/or numerical grades implied and/or assigned to the final document, will not be considered during the analysis of students’ writing. How students used the handout was entirely their choice as I did not refer to the handout during the process of writing. The handout contained the following information:

a) Essential question — these open-ended prompts were written in bold print so students could clearly identify the topic of discussion and the mathematical theme. Students received a loose context from which to frame their responses to the first two essential questions, as these were students’ first experiences writing in my algebra class. Subsequent open-ended prompts did not provide students with a reference point, allowing them more creativity and emphasizing the appropriateness of multiple responses.

b) Outlaw words — this list of words was generated from a Mission School grade seven Language Arts teacher who identified words she felt students had overused in their writing. Approximately half of the students knew of these outlaw words and were familiar with them. The purpose of ‘outlawing’ these words is to encourage students to use more precise vocabulary to express their ideas; the number of ‘outlaw’ words would increase as students responded to succeeding prompts.
c) Rubric — This teacher-generated rubric was used to remind students of the characteristics they should include in their response and the degree of depth in their discussion they should strive for. It was not meant to suggest there was a correct or incorrect way to complete the writing assignment but rather serve as another reminder to the students to convey everything they were capable of so their writing represented the most complete picture of their conceptual understanding. There were four categories within the rubric: mathematical reasoning, mathematical proof, communication, and conventions. The categories mathematical reasoning and mathematical proof echoed NCTM’s *Standards for Reasoning and Proof* (2000); communication focused on students’ organization of their expressed ideas and their choice of words used to express these ideas, including the language of mathematics, and conventions focused on writing mechanics such as spelling, punctuation and appropriate grammatical structures.

d) Examples — examples, either teacher or student generated, were given to students for the purpose of illustrating how students’ discussion could be crafted. For example, I provided students with a written response to the first open-ended prompt as a way of introducing them to the type of response they were expected to generate. Other examples were student generated used to illustrate how students’ responded to individual prompts by discussing a particular concept or created a mathematical model. These student examples were shown during the second semester after I felt students had a better understanding of expressing their ideas. Students could refer to the examples if they wanted to see how writing could be used to express mathematical ideas and conceptual understanding.
Drafting

Students in my mathematics class responded to the same open-ended prompt three times throughout the course of a particular topic or theme. This use of the writing process (National Writing Project & Nagin, 2006) to create multiple drafts for the purpose of fostering critical thinking (National Writing Project & Nagin, 2006; Lance & Lance, 2006) acknowledges previous studies whose findings indicate writing multiple drafts in response to the same open-ended prompt encourages students to record their ideas and view the process of incorporating new ideas (Pugalee, 2004) by connecting prior knowledge to new knowledge (Baxter, Woodward, & Olson, 2005).

Students wrote drafts after the introductory sections, approximately midway, and at the conclusion of each topic, coinciding with a particular chapter from students’ textbook. The opportunity to create multiple responses encourages students to revisit their previous thoughts and reveal changes in their thinking as captured through their writing as meaning is constructed through students’ learning experiences (Patthey-Chavez, Matsumura, & Valdes, 2004; Singleton & Newman, 2009). Students followed this procedure throughout the school year for each of the seven open-ended prompts to produce a total of twenty-one responses.

Draft 1: Freewrite.

Students composed a ten-minute in-class freewrite following the introductory sections of each chapter, approximately one-third of the way through the chapter based on the total number of sections presented. This pre-writing session is similar to Staats & Batteen’s (2009) recommendation that pre-writing activities may help students connect
mathematics to their experiences and potentially create interdisciplinary connections. The freewrite serves two purposes: first, it introduces the writing prompt to the students. Introducing the writing prompt at this time provides students with a preview of the mathematics topic and the essential question that will guide their learning experiences. Secondly, starting with a freewrite allows students to find whatever words and thoughts are in their minds and record them on paper (Elbow, 1998) without having to worry about formal organization, writing conventions, or specific content. Students are encouraged to write quietly although they will not be restricted if they refer to their textbook or notes. At the conclusion of the ten-minute writing period, students’ freewrite are collected by the teacher to ensure students will have it as a starting place when it comes time to write their second draft.

**Draft Two: Second Draft.**

Students wrote the second draft approximately midway through their exploration of each mathematical topic or about half-way to two-thirds through the coinciding textbook chapter, whichever would be a more appropriate place to pause and reflect upon what was previous learned. At this time, students received their original freewrite and were encouraged to revise it. To revise is to re-vision or reshape the meaning of a piece with a clearer view of the audience and purpose in mind (Elbow, 1998; Alber-Morgan, Hessler, & Konrad, 2007). For example, a student who transcribed mathematical definitions from their textbook in their freewrite may create a context that he or she believes is relevant to discussing a particular mathematical concept. Drafting and revising indicates students are accessing prior knowledge, reflecting and thinking critically about
the content and how to respond to the open-ended prompt (Gammill, 2006). Students were encouraged to create this draft at home allowing themselves think-time and write-time; no other mathematically related homework was assigned that day. Completed drafts were attached and e-mailed to me as a Microsoft Word document.

**Teacher-to-Student Written Feedback.**

Content feedback was provided to each student after they submitted their second draft. I purposefully made the decision not to provide feedback to students after their freewrite since the name ‘freewrite’ suggests it is an opportunity for students to write freely without worrying about specific content, form or other writing conventions (Elbow, 1998). The significance of a freewrite is to encourage students to write continuously recording anything and everything that may possibly come to mind; while it does not have to include topic related material, it is highly encouraged (Elbow, 1998). Thus, it did not seem appropriate to give feedback to students after this draft was produced and the decision was made to provide feedback after the second draft, once the students had more mathematical experience with a particular topic and could engage with the concepts on a more intimate and formal level.

I made the assumption that students’ second drafts would have more content and structure to them after they had greater exposure to the mathematical concepts and that their writing would make their thinking visible by capturing their current thoughts and ideas. I also assumed students would revise their draft once more and made the decision to focus on what they expressed and how they expressed their ideas as opposed to
identifying grammatical and mechanical errors. I anticipated that if their content changed, so might the way they articulated their thoughts.

Feedback that is content-based encourages students to revise their thinking (Patthey-Chavez, Matsumura, & Valdes, 2004). During the drafting process, feedback is generally more valued by students and indicates that the teacher is assisting them in their construction and articulation of their ideas rather than acting as an evaluator. Comments and grammatical markings on students’ final drafts are rarely acknowledged by students, limiting their meaningfulness (Lance & Lance, 2006; Dossin, 1997). Content related feedback in the form of criticism, advice, and guidance encourages writers to co-construct meaning and clarify their thinking and rationale (Vardi, 2009; Silver & Lee, 2007).

Teacher-to-student written feedback mainly focused on students’ content; these comments were crafted as questions aimed to encourage students to reconsider their expression and alter their word choice to say what they meant to say with clarity. I provided feedback comments focused on prompting the students to improve or enhance the clarity of their writing, encouraging them to put forth everything they were capable of expressing in the following ways:

- Prompts that called for clarification (i.e. define the mathematical term or phrase used).
- Prompts that called for further explanation or elaboration (i.e. explain what your rate of change means within the broader context of the situation or example provided).
• Prompts that called for a reexamination of reasoning (i.e. justify why you wrote that a line of fit is an average of the data provided).

• Prompts that called for the creation of a real-world application of the concept that could be found beyond the textbook or mathematics class (i.e. create a real-world example or non-example that illustrates the use or inappropriate application of a particular concept).

Students received a printed copy of their second draft with feedback approximately one week after submission or by the conclusion of the mathematical theme or textbook chapter, whichever provided me with the most time to give quality feedback. They could also request an electronic copy of their feedback if they wanted an additional copy or preferred it digitally. If a student was absent or turned in their document late, I made the effort to ensure all students received feedback before their final draft was due. It was my intent to suggest that feedback is an integral part of the writing process, provide insights into areas of concern for students’ own ideas, and prompt them to reveal the extent of their conceptual understanding.

I typed my comments and inserted them digitally into students’ second draft so they would not misinterpret my feedback due to messy or illegible handwriting. Furthermore, I let it be known to the students that they were able to ask questions to inquire about or clarify any of the comments I had written. Students who sought additional assistance received explanations aimed at assisting them to address the nature of the task (Treglia, 2009) suggested by the comments. These included identifying specific areas of misconception in students’ conceptual understanding and noting areas of
misjudgment in students’ rationale. I also encouraged students to discuss their ideas in response to my feedback so they could voice their ideas aloud. My intention was to encourage students to address the area of concern rather than ignore it and encourage them to reveal everything they were capable of articulating. Students who obtained this additional assistance initiated the conversation and any assistance they received was based upon the original comment and provided orally.

The following safeguards were implemented to ensure all students received the same quality of feedback: a) all students completed the same assignments regardless of academic achievement; b) I maintained a professional and positive working relationship with the students (Hatch, 2002) throughout the school year so any student could potentially be chosen to become a participant in the study; c) students’ written responses to open-ended prompts were not assigned for the sole intent of research (Merriam, 1998) and served foremost as an opportunity for students to discover and construct their own of mathematical knowledge and understanding of algebra concepts; and d) participants in the study were chosen towards the conclusion of the school year so the students and I were both unaware of who would be chosen to participate in the study and could experience any intentional or unintentional bias.

**Draft 3: Final Draft.**

Students’ third draft was their final draft, composed at the conclusion of the mathematical topic and coinciding textbook chapter. Students, having received written teacher feedback on their second draft, were encouraged to reflect on the comments and use them to revise their thinking and mathematical reasoning. The final draft was a
product of two previous revisions in response to an open-ended prompt revealing students’ mathematical understanding at the conclusion of the topic. This final draft received a letter grade from the teacher for the purpose of providing summative assessment to the students. These letter grades, however, had no relationship to the analysis portion of the study as all grades were removed prior to the content analysis of their respective individual drafts.

**Student Interviews**

Students participated in one-on-one interviews with me after they consented to participate in the study. A two-week period was dedicated to interviewing students either before school, at 6:30 a.m. or after school at 2:45 p.m. These times were chosen to accommodate student needs. Students who arrived early to school or had after school commitments could take advantage of the early morning interview times; students who arrived after 7:00 in the morning or left school after 4:00 pm could complete their interview in the afternoon. Additionally, these times were chosen so I could conduct the interviews in my classroom, a familiar location to the students, to put the students at ease and increase their comfort level.

Students were asked to participate in an hour-long interview; an hour was allotted to give me time to ask follow up questions and provide each student time to elaborate on their responses without feeling rushed. I digitally recorded the interviews so I had the freedom to make eye contact with the student, put them at ease, and be attuned to their verbal and non-verbal cues. Recorded interviews were transcribed at a later time. An example of a transcribed interview appears in Appendix H.
During the interview, I remained neutral to students’ responses to encourage students to reveal their thoughts and ideas without worrying about future repercussions, either favorable or unfavorable (Patton, 2002). I formally informed the student at the start of the interview that what they said would remain confidential and would not affect their grades in my algebra class. This was to reassure students they would not be judged for what they discussed or did not discuss during the interview (Patton, 2002). Additionally, the same interview guide was used for each interview so each student would have the opportunity to respond to the questions regardless of their experiences.

Data Analysis

The aim of this study is to use writing to reveal changes in students’ algebraic reasoning to describe their conceptual understanding. The use of the writing process (National Writing Project & Nagin, 2006) and teacher-to-student written feedback provided during the drafting process was intended to expose students’ thinking through the course of studying a particular mathematical theme. A content analysis of students’ writing was conducted using grounded theory methods of coding for emergent trends for the purpose of creating a theory to describe students’ construction of mathematical reasoning and conceptual understanding. This section describes my use of grounded theory methods of coding, content analysis, and triangulation of data.

Content Analysis

Student writing can be viewed in two ways: either identifying what content is missing or analyzing the content that is present. Of the two perspectives, this study takes a similar perspective to Evens and Houssart (2004) and looks at the content that is present
rather than try to identify the content that is missing. A content analysis of students’ writing to discover the emergent trends describes students’ construction of mathematical reasoning and conceptual understanding.

Content analysis was conducted in two ways: a) individual drafts were analyzed for their content; and b) each set of three drafts, freewrite, draft 2 and final draft, was analyzed for content. This served two purposes: a) an analysis of individual drafts revealed students’ generation of their ideas and focused on the features that were not seen in the previous draft; and b) drafts analyzed as a set of three focused on students’ conceptual maturation for that particular mathematics theme. The study employed this analysis method to reduce the large volume of students’ written texts to their core themes and patterns (Patton, 2002).

Each student produced a total of twenty-one drafts, a freewrite, second draft, and final draft for each of the seven open-ended prompts posed throughout the school year. Due to the participant selection at the end of the year, there were three potential concerns that could impact the emergence of themes from a content analysis (Stemler, 2001b; GAO, 1996): a) missing documents (e.g. student absence); b) documents that did not meet the data collection requirements (e.g. student turns in the same written text for the freewrite, second draft, and/or final draft); and c) documents containing missing or ambiguous content (e.g. students’ digression from the topic or theme).

Of greatest concern to the study was the possibility of missing documents. I chose content analysis specifically because it is best utilized on a large volume of text, distilling it to reveal meaning from the words (Krippendorff, 2004). In this study, content analysis
is appropriate to glean emergent themes based on the words used by students to express their mathematical ideas. As a teacher, it has been my experience that the most common reasons for students failing to turn in an assignment is absence from school and/or lack of completion. Students who were absent from school on the day of a writing assignment may or may not complete the assignment, especially if it was done during class, placing their priority on those assignments that are summative as opposed to formative. Additionally, those who are absent prior to a writing activity could potentially be at a disadvantage, failing to participate in a particular classroom experience involving the current mathematic topic or theme; any classroom experience and any discussion that occur are difficult, if not impossible, to reproduce. Furthermore, my previous experiences suggest students who failed to complete assignments for reasons other than absence (e.g. they forgot to do it the night prior to the due date) would quickly draft their ideas, lacking careful thought, just to complete the assignment. These hastily completed pieces of writing do not always reflect a student’s understanding nor everything they are capable of expressing. Regardless of the circumstances, I encouraged all students to complete each writing activity in a timely manner and accepted tardy responses. Accepting drafts without grading them for mastery of content or writing mechanics and offering feedback during the process of writing can further encourage students’ construction and revelation of their knowledge and understanding (Fulwiler, 1987).

I addressed missing documents in the following ways: a) students having less than five complete sets of written documents, a freewrite, second draft, and final draft for each prompt, were eliminated from the pool of potential participants; b) all chosen
participants’ documents were considered for individual draft analysis regardless if it was part of a collective set of three; c) all chosen participants’ second drafts were considered for content analysis of teacher-to-student written feedback; d) any collective set with a missing document was eliminated from content analysis of compiled drafts.

**Content Analysis: Individual Drafts.**

Students were assigned three drafts in response to each open-ended prompt for a total of twenty-one individual drafts. For the purpose of this discussion, individual drafts refer to each of the three drafts separately: freewrite, second draft, and final or third draft. Students were not limited by page length or word count and were encouraged to write, using both descriptive and mathematically symbolic language, as needed. This volume of individual texts was reduced to its core themes for the purpose of addressing the first research question and describing how writing revealed students’ algebraic reasoning and knowledge of algebraic concepts. Content analysis captured students’ use of writing in mathematics by examining their responses to open-ended prompts (Hallagan, 2006) and recorded emergent themes with respect to their mathematical ideas, content knowledge, and conceptual understanding (National Writing Project & Nagin, 2006; Mayher, Lester, & Pradl, 1983; Fulwiler, 1987; Countryman, 1992b).

Individual drafts were analyzed for its content using a systematic approach (Stemler, 2001a; Krippendorff, 2004) to identify, classify, and evaluate students’ use of language (Samaras & Freese, 2006; Patton, 2002). Students could demonstrate, through their use of mathematical language, coherent mathematical explanations, justifications for conjectures, and they could explore examples and create sound mathematical arguments.
(NCTM, 2000). Each draft was treated as a distinct and unique document. Individual
drafts, freewrite, second draft, and final draft were organized in that sequence by chapter
to maintain the order in which they were produced to identify unique features within the
drafts as well as analyze the progression of the drafts throughout the course of the year.

I used content analysis to determine the dominant ideas in each piece of writing
based on students’ use of particular words and phrases and their significance in a text
(Krippendorff, 2004; Patton, 2002). For example, a student’s use of a particular
mathematical term in his or her freewrite revealed possible idea generation if the same
term was used in the open-ended prompt or could be found in the textbook glossary.
Particular interest was given to students’ use of descriptive and symbolic language of
mathematics which suggested students’ mathematical knowledge and conceptual
understanding. As individual drafts underwent content analysis, only newly emergent
themes from successive drafts became objects of focus; carryover themes from the
previous drafts were not included to avoid repetition of ideas.

The use of content analysis to identify students’ conceptual and procedural
knowledge and construction of mathematical understanding is similar to Huang and
Normandia’s (2007) analysis of students’ written descriptions of their problem solving
processes. High school students were asked to explain in writing the standard form of a
quadratic formula using the completing the square technique. Huang and Normandia
(2007) used the following criteria to identify the presence of students’ mathematical
knowledge: a) description — focused on the characteristics of particular contexts, b)
sequence — identified mathematical process, procedures and/or routines, c) choice —
explored alternative methods to approaching the problem, d) classification — identified what concepts are applicable and what relationships they have to one another; e) principles — highlighted use of appropriate mathematical rules, properties, or strategies as a means to achieve an end; and f) evaluation — focused on indicators of why one problem solving strategy was chosen over others. Huang and Normandia (2007) were interested in students’ use of the language of mathematics to identify how students constructed mathematical knowledge. They found there is a connection between a student’s use of words and grammatical devices and his or her construction of mathematical knowledge: a) the more mathematical relationships that were expressed by a student’s writing, the more proficient the student was at constructing knowledge; and b) students’ writing indicated they were apprehensive towards justifying and explaining particular problem solving methods and elaborating on relevant concepts.

This study’s interest in describing students’ algebraic reasoning calls for an identification of words and phrases as indicators of mathematical reasoning. It requires an awareness of particular mathematics concepts that provides me with a sense of reference and direction during the analysis process that give meaning to emergent themes (Patton, 2002) in students’ written responses.

The concepts under consideration are concerned with describing algebraic reasoning and are similar to NCTM’s Reasoning and Proof Standards (NCTM, 2000). According to NCTM (2000), evidence of mathematical reasoning includes the following: a) recognizing reasoning and proof as fundamental aspects of mathematics; b) making and investigating mathematical conjectures; c) developing and evaluating mathematical
arguments and proofs; and d) selecting and using various types of reasoning and methods of proof. While literature (Charmaz, 2006) cautions that outside factors influence how meaning is created, these standards provide one perspective on mathematical reasoning hinting at potential words or phrases but do not force emergent themes into preconceived relationships. Meaning is further extracted by identifying other significant words embedded within students’ texts and by examining them for emergent trends and patterns (Stemler, 2001b).

**Content Analysis: Compilation of Drafts.**

Students responded to a total of seven open-ended prompts throughout the school year on various mathematical topics. For the purpose of this discussion, a compilation of drafts refers to each mathematical topic’s set of three drafts consisting of a freewrite, a second draft, and a third or final draft. This compilation of text utilizes content analysis to describe how the creation of multiple drafts reveals changes in a student’s construction of mathematical reasoning. This corresponds to the investigation of the second research question: in what ways does writing, through the revision of multiple drafts, expose the development of algebraic reasoning? The use of content analysis focuses on a student’s active choice of words and revisions in the language he or she uses to express ideas revealing content knowledge and conceptual understanding.

Written words continue to be malleable until the writer acknowledges his or her definitiveness (National Writing Project & Nagin, 2006; Mayher, Lester, & Pradl, 1983; Fulwiler, 1987); it is implied that a student’s first two drafts continued to prompt exploration and content development with the final draft being his or her completed
response to the open-ended prompt. Viewing a compilation of drafts for a particular open-ended prompt reveals how a student’s use of language communicates his or her content and conceptual knowledge (Mayher, Lester, & Pradl, 1983; Gammill, 2006) as he or she continues to revisit the construction of mathematical ideas. Drafting encourages writing as a process during which inconsistencies, gaps, and misconceptions in a student’s knowledge are revealed (National Writing Project & Nagin, 2006) and exposes how the student addresses those discrepancies. Using content analysis to view changes in a student’s writing indicates the consideration of new ideas (Singleton & Newman, 2009; Fulwiler, 1987) and describes the shaping of mathematical meaning (National Writing Project & Nagin, 2006).

One perspective offering indicators of a student’s communication in mathematics for the purpose of viewing how the expression of mathematical thinking changes over multiple drafts is NCTM’s (2000) Communication standards. According to NCTM (2000), communication in mathematics includes: a) organizing and consolidating mathematical thinking; b) conveying mathematical thinking in a logical manner; c) analyzing and evaluating mathematical thinking; and d) using the language of mathematics to express mathematical ideas. This study takes a similar approach to viewing a student’s communication of mathematical ideas focusing on the student’s revisions from one draft to another indicating use of the language of mathematics, descriptive and symbolic, and revealing changes in his or her thinking.

For the purpose of identifying changes in a student’s drafts, the text is fractured into paragraphs determined by the student’s organization of the response. This serves two
purposes: identifying a student’s additions or deletions of significant words and phrases (Krippendorff, 2004; Patton, 2002; Strauss, 1987) and providing context for the student’s word choice. The conditions a student constructs, whether purposeful or accidental, suggest particular relationships in a student’s word choice and reflects his or her meaning. Identifying significant words and phrases (Krippendorff, 2004; Patton, 2002; Strauss, 1987), for example, leads to inferences suggesting a student’s idea generation. For example, a paragraph composed of mathematical definitions with a show-and-tell approach indicates a student’s reliance on outside sources of information to respond to the writing prompt.

Using significant words and phrases and context to view a student’s idea generation and construction of mathematical reasoning is similar to Evans and Houssart’s (2004) study which analyzed students’ mathematical writing using a framework identifying components such as conclusion, data, warrants, and backing. These four components are described as follows: data are mathematical facts; warrants are explanations describing how a conclusion is reached; backing are statements supporting the warrants, and conclusions are claims requiring statements of rationale and justification. Evans and Houssart’s (2004) categorization of students’ written statements pointed to their failure to describe how they reached their concluding statements despite their attempts at restating the prompt or creating similar or extended examples. While students’ restatement of the original prompt was not considered a mathematical argument, it revealed how students constructed what they believed to be a valid mathematical argument.
Content Analysis: Teacher-Written Feedback.

In addition to students’ written responses, I provided written feedback on students’ second drafts. Content analysis is used to identify characteristics of the comments given to students aimed to prompt students’ revisions in their algebraic reasoning. This analysis corresponds to the research question: In what ways does written teacher feedback, provided during the writing process, prompt the revision of algebraic reasoning as represented in students writing? Written feedback is one way for students and teachers to dialogue about students’ construction of ideas and suggest how students’ meaning is interpreted. Feedback provided by an individual a student believes is more knowledgeable can assist him or her in revising thinking (Vygotsky, 1978) and constructing meaning (Mercer, 2008; Wood, Bruner, & Ross, 1976).

Content analysis of teacher-written feedback identifies words and phrases, thus indicating the type of feedback, such as constructive criticism, specific comments, positive observations (National Writing Project & Nagin, 2006) and suggestions for revision. This study is interested in findings similar to Whittington, Glover, and Harley (2004) and Vardi (2009). They found that content-based feedback identifies areas of improvement and motivates students to revise their ideas. Furthermore, feedback consisting of comments and questions that challenge students’ current perspectives, is more likely to stretch their thinking and prompt the development of logical reasoning (Bain, Mills, Ballantyne, & Packer, 2002).

Content analysis of students’ revisions in response to teacher feedback describes how feedback can assist students to clarify and revise their intended meaning (Fulwiler,
This is similar to Weaver’s (2006) and Treglia’s (2009) examination of how students responded to teacher feedback. Weaver’s (2006) findings, however, point to students’ difficulties in addressing feedback because they did not understand it. Treglia’s (2009) noted obstacles students had in addressing tasks, explaining the relationships between concepts, refining an argument, or demonstrating correlations between concepts and examples.

Content analysis identifies significant words, phrases, and contexts that contain rich meaning. The use of grounded theory coding processes further reduced this collection of ideas identifying relationships that ultimately produced a theory describing changes in students’ algebraic reasoning.

**Grounded Theory**

Grounded theory coding methods used to distill content analysis themes to its significant ideas (Patton, 2002; Merriam, 1998; Charmaz, 2006) offers additional rigor and depth (Denzin & Lincoln, 1994) condensing large volumes of student-generated text into meaningful concepts. This study used these two methods to reveal changes in students’ algebraic reasoning in their written responses to open-ended prompts.

The use of these two methods in concert is similar to Van Sluys, Lewison, and Flint (2006) who used grounded theory coding methods and content analysis to examine classroom conversation involving two girls discussing issues of hairstyle, race, and cultural identity. Content analysis was used to identify important words, phrases, and the meanings those words and phrases had in the context of the conversation to understand how social identities were constructed. A three tiered coding process was further
employed consisting of: a) initial coding labeled important words or phrases in the transcript; b) focused coding further developed and linked categories through possible relationships; and c) axial coding confirmed categorical relationships, supported by the previous coding schemes, to generate an emergent theory based in the data (Van Sluys, Lewison, & Flint, 2006; Charmaz, 2006). Van Sluys, Lewison, and Flint (2006) found cultural identity is revealed through hair style indicating how girls identify themselves within the larger social context. The combination of these analysis methods provided a systematic approach to discovering emergent themes from students’ discussions.

The use of grounded theory analysis methods code and distill the data into a theory (Patton, 2002; Merriam, 1998; Charmaz, 2006) identifying the relationships between content-rich concepts (Strauss & Corbin, 1994). Emergent themes are compared to produce categories based upon particular relationships; codes capture the inferred meaning of particular words, lines, or sections of data (Charmaz, 2006). Each set of individual drafts underwent a tiered coding process to produce thematic relationships that described students’ construction of their mathematical reasoning.

Coding organizes data into specific categories (Charmaz, 2006) and provides context for the students’ intended meaning (Charmaz, 2004). This study used the coding process in the following ways: a) individual drafts, freewrite, draft two, and final, or third, draft were coded independently from one another looking at trends within each draft; b) each set of three drafts with respect to each open-ended prompt were coded to see idea progression from the beginning to the end of the school year; c) teacher written feedback provided after the submission of the second draft were coded looking for trends.
in the type of feedback given; d) individual participant interviews were coded and used for triangulation purposes.

The coding process utilized several tiers of coding, initial, systematic, axial, and theoretical (Charmaz, 2006), to refine and validate any emergent findings. This systematic process maintains the condition of emphatic neutrality (Patton, 2002) by limiting any intended or unintended bias I may place upon the data (Charmaz, 2006). During the coding process, memos, or written notes, were used to identify and define emergent ideas (Charmaz, 2006) and prompted me to reflect on the ideas and relationships suggested by the data (Hobson, 1996).

Initial coding focused on content analysis to identify particular words or phrases that seemed striking to me that might indicate students’ mathematical thinking. Repetitious words and phrases were also considered (Patton, 2002). During the initial stage of coding, I used everyday language to describe the emergent relationships between particular words or phrases students’ use in their mathematical discussions. Although colloquial language can sometimes skip over unusual references or hamper the naming of complex concepts (Krippendorff, 2004), initial coding allowed students’ texts to be viewed from a researcher’s lens as opposed to a teacher’s perspective. The coding process continued until a point of saturation was reached (Charmaz, 2006); no further categories were developed and the meaning gleaned from the data reached a plateau (Kamerelis & Dimitriadis, 2005).

As the coding process progressed, more complex language was used to describe the relationships that existed between codes. The second tier of coding described
emergent relationships from the initial coding (Krippendorff, 2004) reflecting my interpretation of students’ construction of meaning (Charmaz, 2006). Defining standards during the coding process reduced the uncertainty of any inferences gleaned from the data (Krippendorff, 2004). Standards strengthened the emergent theory by validating plausible relationships (Krippendorff, 2004), suggested variations or alternative interpretations (Charmaz, 2006), and indicated potential relationships implying how they mirrored what should be expected (Krippendorff, 2004; GAO, 1996; Truex, n.d).

The standards for this study are similar to NCTM’s (2000) Reasoning and Proof and Communication standards. These tentative criteria for consideration are as follows: a) idea generation — what words, phrases, or chunks of text illustrate a student’s idea generation and influenced their thinking, b) mathematical reasoning — how does a student’s use of the language of mathematics, symbolic and descriptive, express their construction of mathematical rationale; c) conceptual maturity — how does a student’s ideas describe their understanding of mathematics and its potential applications. Newly emergent relationships, regardless of its similarity to NCTM’s (2000) Reasoning and Proof and Communication standards, represents a reduction of the data into mutually exclusive categories (Krippendorff, 2004) further describing how writing reveals changes in students’ algebraic reasoning.

I further categorized these themes and codes into broader relationships discarding the initial categories with more appropriate ones. This third tier of coding interprets the trends emerging from the data suggesting more refined categorical relationships (Charmaz, 2006; Hatch, 2002); these categories described changes in students’ algebraic reasoning.
reasoning and revealed students’ developing conceptual understanding. A final refinement, theoretical coding, of themes containing noteworthy meaning and the formulation of significant relationships generated a theory (Charmaz, 2006) describing how writing revealed changes in students’ algebraic reasoning and conceptual understanding of algebra concepts.

**Interviews: Constant Comparison**

I began the process of constant comparison, used to reveal emergent themes from transcripts of students’ one-on-one interviews, by transcribing the interviews. This process allowed me to record the students’ discussion and listen to how they expressed their ideas all the while providing me with the opportunity to identify significant quotes and possible emergent themes. I used this method of comparing and contrasting to create thematic categories emerging from the data to ensure that any potential themes stay true to the data throughout the analysis process (Hatch, 2002; Tesch, 1990 in Boeije, 2002). Furthermore, constant comparison methods aid in the refinement of categories suggesting relationships that may be incorporated during the development of the emergent theory (Charmaz, 2006; Kamberelis & Dimitriadis, 2005; Stemler, 2001b).

Constant comparison between students’ text and transcriptions of their interviews can confirm or deny patterns discovered in the data (Hatch, 2002). My use of constant comparison methods describing emergent themes from students’ interviews is similar to Boeije (2002) who created a five-tier coding strategy to distill interview data from multiple sclerosis patients and their spousal care providers. The five tiers are as follows: a) comparison within a single interview; b) comparison between interviews within the
same group; c) comparison of interviews from different groups; d) comparison of pairs at the level of the couple; e) comparing couples.

I created this tiered coding process to analyze students’ transcriptions of their responses in one-on-one interviews.

a) Comparisons within a single interview: The aim of this comparison is to identify initial codes to understand students’ experiences writing in mathematics and their development of mathematical reasoning. By comparing different parts of the interview, for example, students reveal how they constructed their ideas since similar questions were asked about students’ idea generation for each draft.

b) Comparisons between interviews of students within the same homeroom: Students’ in the same homeroom are exposed to similar experiences. Comparisons of those experiences identifies the elements that have impacted their use of writing to reveal changes in their thinking, content knowledge, and mathematical understanding.

c) Comparisons between interviews of students from different homerooms: Although I tried to keep instructional activities and discussion questions similar between homerooms, students’ participation may influence their experiences and impact their response to the open-ended prompt. Comparisons between homerooms for congruence and dissimilarity of themes emphasizes potential relationships between emergent codes.
Emergent themes from the data are used to triangulate findings from an analysis of students’ written responses to further validate the emergent theory.

**Triangulation**

I used triangulation to confirm and cross-validate emergent findings to validate the emergent theory (Creswell, 2003) in addition to identifying potential weaknesses within the findings. While the word triangulation implies three methods, its use in this study refers to the combination of methods (Patton, 2002). Two primary sources of data comprise the data in this study: students’ written responses and transcripts of students’ one-on-one interviews. Secondary sources of data include my observations and field notes.

Of the two methods primary methods used to produce data in the study, students’ written responses and one-on-one interviews, students’ written responses are analyzed in two separate ways, both as individual documents and as a compilation of a single set of three successive drafts. These two content analyses in addition to the constant comparison of the interview responses are used in the triangulation process to test for consistency of the findings (Patton, 2002) by validating frequent themes (Kamberelis & Dimitriadis, 2005) and concepts emerging from the data sources (Patton, 2002). The deepening of these understandings serves to create richer meaning (Denzin & Lincon, 1994) describing changes in students’ algebraic reasoning and conceptual understanding. Correlations between ideas reflecting the meaning of students’ text and their experiences expressed in their interview leads to emergent themes and relationships between concepts (Krippendorff, 2004).
Although the goal of triangulation suggests the consistency of the emergent findings from the data from the perspective of validating the emergent themes, it also identifies inconsistencies within the data. These contradictions or inconsistencies suggest a possible strength of a particular method and/or the opportunity to deepen the meaning implied by the data (Patton, 2002). In the event the generalizability of the result becomes questionable (Golafshani, 2003) and/or there is not equality between the findings in the data sources (Creswell, 2003), priority is given to the emergent findings from students’ written responses unless significant contradictions arise from interview findings. My observations and field notes are especially considered in any one-sided interpretations as a way to either strengthen the claims or potentially explain the lack of convergence (Creswell, 2003).
Chapter 4: Analysis

Writing, when used as a process, encourages writers to connect with his or her own experiences (Mayher, Lester, & Pradl, 1983; Giroux, 1978), creates the opportunity to develop rich meaning (Fulwiler, 1987), supports higher order thinking and increased comprehension (Grimberg & Hand, 2009; Mayher, Lester, & Pradl, 1983; National Writing Project & Nagin, 2006; Fulwiler, 1987), and supports the connection between prior and new knowledge (Mayher, Lester, & Pradl, 1983). Students who actively participate in choosing language to express both context and content imply through their words their understanding and reveal inconsistencies, gaps, and misconceptions in their knowledge (National Writing Project & Nagin, 2006). The aim of this study is to describe what writing reveals about students’ mathematical understanding by observing changes in the development of their algebraic reasoning founded upon their written responses to open-ended prompts. I hypothesized that writing is capable of revealing the manner in which students construct their mathematical arguments, thereby making public their emergent understanding of mathematics. The following research questions guide this study:

1. How does writing reveal a student’s algebraic reasoning and knowledge of algebraic concepts?

2. In what ways does writing, through the revision of multiple drafts, expose the development of algebraic reasoning?
3. In what ways does teacher-to-student written feedback, provided during the writing process, prompt the revision of algebraic reasoning as represented in a student’s writing?

Writing reveals the characteristics of students’ construction of algebraic reasoning when used as a process to create meaning and reveal their conceptual understanding.

This analysis, based on the sample of students chosen to represent the entire class, describes in what ways students’ mathematical understanding is revealed through their writing. The first section describes how students’ word choice reveals their mathematical thinking and understanding as suggested by the words and construction of phrases students choose to include in their discussion. The next section reintroduces the sample used in the study and offers further insight into their initial perspective on the role of writing in mathematics. This is followed by a discussion describing the following dominant themes based on an analysis of students’ written responses and transcripts of their one-on-one interviews. The intent is to describe the emergent characteristics from each draft as they were looked at as separate entities. Only newly emergent characteristics are highlighted. What follow below are the themes that capture the features of each draft:

1. Recapture: Recapture describes the ideas students’ presented in their freewrites. It is used to suggest students were prompted by a word or phrase from the open-ended prompt to think about their past experiences as a way to generate ideas and use them to construct their response. An analysis of students’ individual freewrites revealed a show-and-tell approach of recording
mathematical ideas and everyday experiences they believed were appropriate and relevant.

2. Experimentation: Experimentation describes the constructions of students’ second drafts pointing to their attempt to use mathematical language to construct mathematical arguments they believed to be important, develop, from the students’ perspective, relevant conjectures, and create examples and non-examples that offer context to illustrate abstract mathematical ideas. Students’ writing also suggested they continued to be reliant upon other sources of information as references to inform their discussion, evident in their continued use of textbook-like information. This section discusses the emergent characteristics particular to students’ second draft.

3. Transition: Transition describes the shift in students’ thinking after receiving teacher-to-student feedback prompting them to reveal everything they are capable of thinking of with respect to their discussion in a subsequent draft. Teacher-to-student feedback was analyzed from two perspectives: a) it describes the type of feedback, prompted by students’ word choice, which was given to students; and b) it describes students’ response to the feedback as viewed through the construction of their third and final draft.

4. Reconsider: Reconsider suggests students reviewed their previous draft after receiving teacher-to-student feedback and reassessed what they said and how they said it when constructing their third and final draft. Students’ reflections and reexaminations of their previous ideas were evident in their discussion as
they described the relevancy of particular mathematical models with respect to the examples they generated. This section discusses significant characteristics as seen in students’ third and final draft.

**Word Choice**

Writing and the writing process (National Writing Project & Nagin, 2006) assist students in saying what they mean to say by encouraging them to participate in choosing the words used in their discussion. Students’ conscious choice of selecting words used to convey meaning is significant because these words are indicators of their conceptualization of the mathematics. Words expose students’ mathematical understanding and point to their content knowledge. This analysis identifies specific words and phrases used by the students, implying their mathematical understanding in the following ways: a) Students’ voice is revealed in their use of transcribed versus paraphrased words; and b) students’ idea generation indicates if they have copied from some source or if their ideas are based on personal experiences that have informed their thinking.

**Students’ Voice: Imitated Versus Paraphrased**

Students’ voice refers to their use of language to express their thinking in such a way that emphasizes the person behind the writing. If the readers close their eyes, it is as if they can hear the student speaking directly to them. Students craft their voice through the words they choose to express their ideas, thus, it is evident when students transcribe
information from other sources because the “voice” becomes sterile or mechanical in nature.

Students’ imitation of words points to their failure to express an original thought or opinion. While copying assumes a negative connotation, purposeful borrowing from a particular source suggests: a) students’ reliance upon some other resource for information including their textbook and class notes; b) students’ hesitancy about their own mathematical understanding such that, rather than err, they side with caution and transcribe what has been previously given to them; c) copied words serve as an entryway for discussing mathematics; and d) consistent access to the same definition of a mathematical term. Students copy so that someone else can articulate what they are unable to express suggesting they recognize the tentativeness and uncertainty that lies within their own understanding.

Students’ reliance upon other sources for information is suggested by Maccy who reveals her sources for the words and phrases she chooses to include in her discussion. She states: “I mostly look from my notes from my notebook that I take from the power point and sometimes I put the math book definitions,” (Maccy, Interview, May 20, 2011). Maccy identifies consistent sources of information she can rely upon in her use of the words, “notes,” “power point,” and “math book definitions.” She suggests that these sources offer her words and phrases she believes are pertinent to her discussion or

3 For the purpose of confidentiality, all names have been changed to pseudonyms. This includes any names used by the participants in reference to themselves and to other known individuals in their written responses or oral interviews.
can serve as an entryway if she is unsure of how to begin her discussion. Maccy’s admission to using transcribed words implies an awareness of the difference in voice between copied phrases and when she actively participates in the process of choosing her own words to express an idea.

Maccy implies that borrowing words from sources she trusts provides her with a springboard for reflecting upon her mathematical experiences. She states, “… I at least try to think of half or one example so it can at least help me to remember what we’ve learned in the past because sometimes notes aren’t enough or the textbook...” (Maccy, Interview, May 20, 2011). Maccy indicates that resources such as her textbook guide her thinking as she ponders how a concept describes a particular context. This is further indicated by her use of the phrase, “help me to remember what we’ve learned in the past,” implying given information fails to be meaningful until it is set within meaningful context. Maccy’s reliance upon text-based sources for information suggests she is not confident in her own understanding and references them to keep her discussion relevant.

Students’ use of paraphrasing contrasts with their use of transcription of given information. Paraphrasing suggests students choose their own words to express an idea based upon previously given information. Students’ use of paraphrasing implies they possess some working knowledge of the concept enabling them to participate in choosing words to express their ideas. Their word choice reveals both understanding and misconceptions or gaps in their knowledge as they attempt to express their thinking. For example, students who express a mathematical concept in their own words believe they are capable of capturing the intended meaning of the concept. This is illustrated by Jolly
who expresses his understanding of a fraction by stating, “A Fraction is simply a piece of a whole” (Jolly, Written Response Draft 2, November 14, 2010). In this statement, Jolly expresses his understanding of a fraction but also exposes his assumption that all fractions represent values between zero and one. Students, such as Jolly, who use paraphrasing to voice their ideas, reveal the degree to which they understand a particular concept more openly than those who rely on transcription to inform their discussion.

**Students’ Ideas: Copied versus Informed**

Students’ choice of words describes the manner in which their ideas were generated. For example, words that point to a show-and-tell approach indicate they were copied from a text-based or other reliable source. In contrast, words that are chosen by the student that describe, for example, relevant contexts that can be modeled by a particular mathematical concept, suggest more original thinking.

Students who rely on other sources of information, including peers, teachers, and text-based materials, do not feel that they are copying but rather view it as a way of becoming aware of mathematical ideas. They suggest these sources inform their thoughts in the following ways: a) identifying particular definitions for mathematical terms; b) finding consistent examples of how the language of mathematics can be used to describe a mathematical idea; and c) using social experiences to influence or extend their thinking. Copying for the purpose of increasing mathematical awareness suggests one way students approach abstract mathematical ideas and attempt to make them more concrete.

Tutty echoes these ideas as he identifies the sources he uses to inform his ideas by stating, “I look at my first freewrite and then I just build on top of that, I look in the
textbook for definitions or any terms I forget or like any equations...uh, sometimes I’ll call my friends and see what they wrote, I’ll get some ideas from that” (Tutty, Interview, May 24, 2011). Tutty reveals the go-to resources he feels are trustworthy in his use of the words, “my first freewrite,” “textbook,” and, “call my friends.” Naming his freewrite as a potential source of information suggests this draft contains relevant ideas such as transcribed mathematical definitions or other notions that have potential for his discussion. He also points to his textbook as a means to supply him with relevant, “equations,” that can be used to model particular contexts. He loosely implies that his textbook also influences the type of example or non-example he constructs or that he may pattern his example after ones he has previously seen.

Tutty, however, also indicates that communicating with others serves as a springboard for creating his own ideas in his use of the phrase, “I’ll call my friends and see what they wrote.” While it is difficult for students to claim an idea solely as their own, they do use their conversations with others to create their own ideas as suggested by Tutty’s phrase, “I’ll get some ideas from that.” He suggests utilizing social contact with others is a way of becoming more informed, thus supporting him in developing similar ideas to be expressed in his own words. Tutty’s participation in deliberately choosing words to express ideas suggests he is taking ownership of the idea and claiming it as his own. Tutty’s description of his writing process illustrates the notion that although no idea may be originally his own, through participating in choosing words to express his ideas, he has reflected, expanded upon, and refashioned his draft to reflect his thinking and thus he takes ownership of that particular statement.
Initial Perceptions of the Role of Writing in Mathematics

At the onset of the 2010–2011 school year, I asked her algebra students to reflect on the role of writing in mathematics by responding to the prompt: “Based on your experiences in mathematics, what do you feel is the role of writing?” Students described the type of writing they experienced in their previous mathematics classes and suggested writing was used to communicate information. Their past experiences led them to believe that any use of letters and words, including mathematical operational symbols and numbers, constituted writing in mathematics. Students indicated putting a writing instrument to paper was writing regardless of the nature of the written expression or the purpose of the writing task. Students pointed to three uses of writing in their previous mathematics classes:

1. To communicate mathematical procedural knowledge from student to teacher with and without the use of numbers and operational symbols;
2. To communicate a mathematical context or situation; and
3. To communicate mathematical information, especially from teacher to student.

Communicating Procedural Knowledge

Students indicated writing was used to communicate their procedural knowledge in a step-by-step approach. This display of mathematical symbols and numbers was often prompted by the directions, “show your work” which suggested to students that a symbolic demonstration of their problem solving approach was appropriate to reveal how they found a solution to the problem. These directions encouraged students to find
numerical solutions to problems regardless if they understood the meaning of their manipulations or the application of a particular concept as suggested by Maccy’s response to the prompt, “Based on your experiences in mathematics, what do you feel is the role of writing?”

1. Writing is very important in mathematics because you have to be able to explain how you solved the problem…It is important to show the steps to solving a problem so everyone can understand your thought process. Writing out the steps will help us to see different ways of solving a problem. …Using equations, operations, and formulas you can combine these elements together and you can solve any mathematical problem (Maccy, Written Response Initial Perception of Writing, August 26, 2010).

Maccy states mathematical explanations communicate, “the steps to solving a problem,” in line 2. These “steps,” she refers to involve, “using equations, operations, and formulas,” in line 4 indicating numerical and symbolic manipulations illustrating how she arrived at the solution. In line 3, Maccy’s phrase, “so everyone can understand your thought processes,” emphasizes the role of writing in conveying an approach to problem solving. However, she fails to discuss how this type of writing reveals students’ conceptual understanding associated with the manipulations and formulas used in their problem solving processes. Maccy indicates this show-and-tell approach to mathematics is the predominate function of writing in her previous mathematics classes.

The use of writing in a step-by-step approach to record procedural knowledge is further emphasized by Cissy. He suggests that writing documents each procedural
Manipulation done in a multi-step or complex problem when students must use paper and pencil to perform them.

1 …There are mathematical problems that are just too difficult to answer using mental math or calculating an answer in your head. This is where writing comes in. Instead of doing the problem in your head you can write the equation or expression you are trying to solve on a piece of paper and figure it out that way. (Cissy, Written Response Initial Perception of Writing, August 26, 2010).

Cissy suggests mental math is a students’ ability to perform mathematical calculations without the use of tools such as paper-and-pencil. He indicates his use of mental math is determined by the complexity of the calculations required to find a solution to a problem. Cissy thus suggests writing serves two purposes: it records, “equations or expressions,” in line 3 so these mathematical models can be referred to during the problem solving process and it lists in an orderly fashion the procedures used to find a solution, as implied by the phrase, “figure it out that way” in line 4. This show-and-tell use of writing, communicating procedural processes, further emphasizes the role of writing to express completing a numerical solution.

Communicating Mathematical Contexts and/or Situations

Students believed that any use of written words or letters constituted writing in mathematics. This perspective suggests that in their previous experience, writing was used as a means to convey contextual information that numbers and mathematical symbols alone could not express. For example, Rally interprets the directions, “please label your answer with the appropriate units,” to mean he should use words to identify the
context a particular numerical value was associated with. He states, “Writing has a big
portion in algebra or math, because we have to use the writing to label some of our
answers like centimeter or millimeter” (Rally, Written Response Initial Perception of
Writing, August 26, 2010). Rally’s use of the word, “label,” implies he views writing as
independent from the mathematics of the problem, used to communicate the context of a
numerical solution.

The perspective that writing provides context is further emphasized in word
problems. Word problems use words to describe a situation that can be modeled by
numbers and other mathematical symbols. Kitty indicates that these words communicate
to students the context of the problem and indicate how to construct the appropriate
mathematical expression or equation whose solution can be calculated.

1. In math there are word problems such as: If Sharon had five apples, then gave
2. some apples to her friend, and ended up with three apples. How many apples did
3. she give to her friend? …You can take any mathematical problem and turn it into
4. a word problem, so as you can see writing plays an important role in math” (Kitty,
5. Written Response Initial Perception of Writing, August 26, 2010).

Kitty indicates words supply the context for mathematical models. Her use of
words such as, “Sharon had five apples,” in line 1, and “ended up with three apples,” in
line 2, suggests the original number of apples and how many remained after giving some
away. The phrases, “how many,” in line 2 provides the context for the solution while the
phrase, “gave some apples to her friend,” suggests the mathematical operation subtraction
should be used to model the situation. Kitty implies, in her use of the phrase, “you can
take any mathematical problem and turn it into a word problem,” in line 4 that words
describe the context associated with a mathematical model and thus give the numbers
meaning. Writing in this instance is used to express mathematical situations to provide
the context for mathematical models and numerical values.

Communicating Mathematical Information

Students also perceived writing as a communicator of mathematical information. Mobby indicated students use writing to record information given to them by their
teacher. “When you take notes in a class you have to write down what you are learning”
(Mobby, Written Response Initial Perception of Writing, August 26, 2010). Mobby
assumes note taking supports learning but it often means recording information from the
board or what is said by the teacher. Students further suggested this writing, both by the
teacher to convey information and the students to record given information, are common
uses of writing in mathematics class. Writing for the purpose of note taking and recording
given information often fails to lead students to reflect upon their classroom experiences
or to construct knowledge. Students’ reflections on the purpose of writing in mathematics
suggested writing serves to communicate step-by-step mathematical processes, describe
the context for a mathematical model, and communicate mathematical information.

Freewrite: Recapture

Writing is a communicator of both product and process, capable of recording
changes in students’ thoughts and ideas during their construction of knowledge. Students’
initial responses to the open-ended prompts recaptured information they thought was
relevant to the prompt. The content of the writing revealed either transcribed information
from the students’ textbook or snippets of students’ personal experiences they thought might provide some context for the mathematics topic they were discussing. This section focuses on how students recaptured information, revealed through their word choice, as expressed in their freewrites.

Students’ first draft, a freewrite, emphasizes the freedom they have as they write. In this first draft, students are encouraged to start from where they are in their knowledge and understanding (Mayher, Lester, & Pradl, 1983). As ideas come into focus, students may recall their own experiences, ask questions, or reflect on new information. Students used a show-and-tell approach to identify relevant mathematical information and explore personal experiences they felt might illustrate the mathematical theme they were studying.

Idea Generation

Characteristic of students’ freewrites is the transference of knowledge from one source to another suggesting narrow understanding and limited mathematical creativity in their response to the open-ended prompt. It implies students are giving information that they previously received from another source such as their textbook, or their current or past teacher. Students suggested they were capable of naming mathematical terms and concepts but were unable to elaborate on their meaning, thus relying on other sources to provide them with the appropriate words. Tutty indicates he generates ideas by seeing what others have to say based on their introduction to the topic in class.

Well, I talk to my tablemates to see what they’re writing and then I’ll try to make something similar but not totally the same ‘cause I just write down my first ideas
Tutty reveals two sources for the ideas he records in his freewrite: his peers and the introductory lesson. “I talk to my tablemates” in line 1, suggests he relies on other students’ ideas to inform his discussion. He indicates, through his reference to his classmates, that he may have difficulty generating his own ideas and struggles to address the prompt. He sees his inquiry as an entryway to greater insights that might spark an idea or help him recall some experience he might have had that seems appropriate to the discussion. His second source of information is “ideas we learned from the lesson” in line 2, suggesting his introduction to the mathematical theme provides him with some reference from which to list “equations and some examples” in line 3. He implies that the “examples” he creates are based upon personal experiences that he re-crafts to take on the characteristics and organization of examples from the textbook. This is further suggested by his reference to “equations” found in his textbook.

Tutty’s idea generation is echoed in his freewrite response to the open-ended prompt, “How can a mathematical situation be described as a recursive sequence?” He writes:

1 Recursive sequence – starting number, and apply the rule over and over and you
2 will get a sequence.
Most sequences are linear when you put them down on a scatter plot. On the graph, the $x$ and $y$ variables will increase by a definite amount.

Counter example: How a human grows (Tutty, Written Response Freewrite, January 25, 2011).

Tutty’s definition of “recursive sequence” in line 1 frames the ideas for his discussion. His definition of recursive, “starting number, and apply the rule over and over,” in line 1 suggests he used phrases either from his textbook or in-class notes in describing the term. He relies on his prior knowledge of graphing when he writes, “Most sequences are linear,” in line 3 implying that his previous experiences with graphing consist of ordered pairs and straight lines. Tutty also suggests a non-example, “how a human grows,” as a discussion point pointing to a situation that may not be recursive based on what he knows about the people’s growth patterns. He refers to it as a “counter example;” believing he is disproving mathematical statements relevant to recursion by naming a situation he believes does not contain the elements relevant to the concept. However, he should have referred to it as a non-example because he suggests that the situation does not illustrate specific characteristics relevant to the concept of recursion and recursive sequences.

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4 Any use of letters representing a variable, a quantity whose value can vary or change, will be italicized.

5 Students often referred to non-examples in their writing as counter examples. Whereas a non-example does not demonstrate the characteristics of a concept, a counter example disproves a statement. Know that when students use counter example they are really speaking about a non-example.
Nevertheless, Tutty’s word choice used to express his ideas indicates he referenced prior knowledge and relied upon concrete sources of information. His recapture of previously known information demonstrates his reliance upon textbooks, in-class notes, and personal experiences to respond to the prompt in the freewrite.

Students used the freewrite to explore and identify content knowledge. Their recapture of previously introduced ideas and experiences serves as an entryway for them to investigate an abstract idea and interact with new concepts in a non-threatening way. Two significant themes emerged from an analysis of students’ freewrites suggesting students’ recaptured their prior knowledge:

1. Students identified personal experiences beyond the classroom that either shared some similarity with examples they encountered in class or were prompted by a word or phrase from the open-ended writing prompt. Students expressed these ideas in the form of examples and non-examples that may or may not have included any mathematical references.

2. Students identified specific mathematical words or phrases they felt were significant or related to the mathematical theme as suggested by the open-ended writing prompt. Their identification of these particular words lead them to transcribe known information such as mathematical definitions or discuss word problems previously seen in class as indicated by their choice of words and expression of ideas.
Examples and Non-examples

Students recaptured a variety of situations from their own experiences outside of mathematics class to investigate abstract concepts through a concrete event. This was evident in the students’ use of the pronoun “I” in reference to themselves and their experiences. Students usually patterned their examples after similarly presented problems from class suggesting they were attempting to see if it was possible for their situation to be organized in the same way as a previously encountered word problem. For example, Mobby considers her participation at a car wash to determine if she can organize her experience in such a way that it can be modeled by a system of two equations as she responds to the prompt, “Where are systems of equations in the real world and what do they mean?”

1 I charge $10 for a car and $15 for an SUV/truck that I wash. I made $600 that day. I also washed a total of 49 vehicles. How many cars and how many SUV/trucks did I wash?” (Mobby, Written Response Freewrite, April 19, 2011).

Mobby chooses to use the word “I” in her example, which could indicate that she participated in a similar type of event. While there could be several reasons why students choose to use the word, “I,” the assumption is that students who choose “I” are attempting to bring an abstract idea closer to a personal perspective. Furthermore, Mobby’s organization of her example appears to be patterned after similar problems provided by me in class when students were introduced to the topic. Mobby’s example demonstrates the ease with which she molds her example into a word problem similar to
the one from class, suggesting another way students attempted to understand abstract ideas.

Mobby’s use of, “How many cars and how many SUV/trucks did I wash?” in line 2, reveals her knowledge of the characteristics of a system of equations, implying that they contain more than one unknown quantity. Her specific use of the phrase, “how many,” is significant because it indicates Mobby has identified which quantities are unknown and that the potential mathematical model must provide a solution for the number of cars and the number of SUV/trucks washed on the day in question. Mobby’s personalization of the context is an example of how students’ use of personal experience, both in and out of the classroom, aided them in the exploration of abstract mathematical ideas.

Students who felt confident in describing a personal experience containing particular mathematical characteristics accompanied the context with a symbolic model. Cissy illustrates this with his use of a mathematical expression in his response to the prompt, “Compare and contrast fractions and ratios and how they can be used to describe real world situations.”

1  I was at a party for my sister and there were twelve girls and three boys so a ratio

2  for this data would be four girls per one boy. \[
\frac{12g}{3b} \div \frac{3}{3} = \frac{4g}{1b} \]  (Cissy, Written

3  Response Freewrite, November 2, 2010).

Cissy’s response fails to address the comparison of fractions and ratios; rather, he chooses to construct an example illustrating his use of ratios. His use of the word “ratio”
in line 1 suggests he believes he can compare the two quantities, girls and boys, to each other. Cissy creates a mathematical model of the situation in his use of the phrase, “a ratio for this data would be four girls per one boy,” in line 2. He attempts to justify his example with a procedural demonstration. He constructs an initial model of the context with his use of the expression, \( \frac{12g}{3b} \) in line 2 to illustrate his comparison of girls to boys. While Cissy’s use of the numbers twelve and three can be reflected by his context, the use of the letters “g” and “b” accompanying each number is unclear. My assumption is that Cissy’s intent was to use the letters as labels, as opposed to mathematical symbols, to represent the quantities, “the number of girls,” and “the number of boys.” However, “g” and “b” could also be interpreted as variables representing an unknown quantity of girls and an unknown quantity of boys or some other unnamed context.

Despite Cissy’s failure to clarify his use of the letters “g” and “b,” he continues to discuss his expression suggesting it leads him to an equivalent ratio, “four girls per one boy,” in line 2. He supports this by offering a procedural explanation in his use of the expression, \( \div \frac{3}{3} \) indicating his belief that three is a factor of both twelve and three and dividing the numerator and denominator by the same factor will produce an equivalent ratio, \( \frac{12g}{3b} \div \frac{3}{3} = \frac{4g}{1b} \). Cissy’s procedural manipulation of his original expression indicates he believes a procedural approach offers the appropriate mathematical support to justify his thinking. While his expression is initially flawed, Cissy illustrates how students use procedural manipulations to support and expresses their mathematical thinking.
In addition to numerical and symbolic models, students also suggested drawings as models to reveal their interpretations of abstract mathematical concepts. Tobby demonstrates this in his response to the prompt, “How can a mathematical situation be described as a recursive sequence?”

1 Your heart rate is an example of a recurring sequence. It is also linear because it is showed [sic] in the medical field with a line that looks like

2

3

4 (Tobby, Written Response Freewrite, January 25, 2011).

Tobby recaptures the mental image he has of seeing a heartbeat on a monitor to investigate recursive sequences. He interprets the word, “recursive” from the original prompt to mean, “recurring” in line 1. Tobby, as suggested by his drawing, interprets the word “recurring” to mean “returning over a given time interval” in the same way the monitor captures the heart pumping at particular intervals. Tobby’s interpretation of “recurring” suggests his understanding of the word recursive, which involves a repetitive process, may not be as clear as he believes.

Tobby repeats this figure, in line 3, seven times to complete his drawing suggesting the repetitive nature of his heart beating is similar to his interpretation of recursion. He implies that sequences are produced when the same pattern is duplicated. This indicates his assumption that the ending of the previous repetition becomes the starting point for the next repetition.

Tobby also claims his drawing is “linear” in line 1 and assumes that “linear” in means drawn with straight line segments as indicated by his figure, . Tobby’s
interpretation of the word linear from the perspective of his drawing suggests his literal assumption that it refers to lines and line segments. Tobby illustrates one way in which students interpret abstract mathematical terms based upon their known experiences.

Students’ also created non-examples reflecting their prior experiences to aid them in exploring abstract mathematical concepts. Non-examples include contexts that appear to contain the characteristics of a particular mathematical concept but are unable to produce an appropriate mathematical model when that particular concept is applied. Rally illustrates this in his response to the prompt, “How can a mathematical situation be described as a recursive sequence?”

A mathematical sequence can be a recursive sequence because it is repeating again and again, for example height is a recursive sequence but it’s a counterexample, people can grow, but it alters from year to year. For example you grow most during adolescents [sic] to adult, however it is different when you are an adult, sometimes you shrink… (Rally, Written Response Freewrite, January 25, 2011).

Rally defines “recursive sequences” in line 1 as, “repeating again and again” implying the duplication of some process. He proposes that although human growth appears to continue each year, he refers to it as a, “counterexample” indicting he believes recursive procedures involve the repetition of the same procedure. He supports his rationale with, “it [growth] alters from year to year,” in line 3 suggesting that humans do not grow by the same amount each year. This idea is further reinforced in his discussion with, “you growth most during adolescents,” in line 3 indicating greater growth in height during this
time as compared to the previous year. He also states, “when you are an adult, sometimes you shrink,” in line 4 to suggest negative growth. Rally’s non-example illustrates how a student could explore a mathematical concept but reflecting on why it would not be appropriate to model a particular situation.

The contexts students choose to discuss reveal their interpretation of mathematical terms from the perspective of their recaptured personal experiences. These experiences serve as entryways for students to investigate abstract concepts in concrete ways. Students’ inclusion of models and drawings with their experiences further serve to reveal their understanding and interpretations of the concept.

**References to Information Presented in Mathematics Class**

Students used their discussion and activities from class to inform their responses to the open-ended prompts identifying key words they thought would be relevant in their discussion. One approach to integrating this information was to transcribe mathematical definitions from a given source, either students’ textbooks or their notebooks. This was revealed by Haily as she addressed the prompt, “How can a mathematical situation be described as a recursive sequence?”

1. Recursive – describes a procedures that’s applied over and over again
2. • Starting w/ a # / geometric fig. [to produce a sequence of #’s and figures.]
3. • Each stage of a recursive procedure builds on the stage before → (prev. stg.)
4. • Resulting sequence is “generated recursively” & so the procedure is known as recursion.
Haily identifies the theme of the open-ended prompt with her use of the word “recursive” in line 1 as the focal point of her discussion. She goes on to list ideas she has previously encountered in her mathematics textbook as content she believes to be relevant to the topic. For example, her use of the phrase, “Starting w/ a # / geometric fig. [to produce a sequence of #’s and figures.]” in line 2 is similar to the definition listed in the glossary of her textbook. Haily’s reliance on her textbook is further emphasized by her word choice with phrases such as, “Each stage of a recursive procedure,” in line 3 and, “Resulting sequence is ‘generated recursively’ & so the procedure is known as recursion” in line 4 that mimics the language in her textbook.

Haily reveals her own idea with her use of the phrase, “I think” in line 6 implying it is based upon the given information she previously recorded. Her capitalizing the word, “PATTERN” in line 6 indicates her belief that this word is somehow significant to the concept of recursion. She supports her belief by referencing the number sequences she encountered in class but does not suggest what is meaningful about the patterns she found. Haily demonstrates how recapturing potential information from a given source provides her with the content for her freewrite.

In addition to transcribing definitions, students also reflected on problems presented in class as another way of exploring an abstract concept. These problems usually contain real-world contexts as opposed to a ‘naked’ equation or expression. For
example, Keggy turns to her textbook to generate ideas given the prompt, “How do real-world quantities vary directly and inversely.”

Bernard Lavery, a resident of the United Kingdom, has held several world records for growing giant vegetables. The graph shows the relationship between weight in kilograms and weight in pounds.

Quantities have direct variation when the ratios have a constant. Examples [sic] of direct variation is lb [pound] to kg [kilogram]. The quantities when divided all come close to 2.2. 2.2 is the constant and the amount of lbs. in a kg (Keggy, Written Response Freewrite, December 6, 2010).

She cites the textbook definition of direct variation stating, “when the ratios have a constant,” in line 1 and recaptures her classroom experience, “Examples of direct variation is lb to kg,” in lines 1 and 2 suggesting her reliance upon a primary source to generate ideas she is hesitant about. Keggy’s use of the phrase, “The quantities when divided all come close to 2.2,” in line 2 implies she recalls the procedure used to find the
ratio of pounds to kilograms. While Keggy’s calculations have thus lead her to conclude, “2.2 is the constant and the amount of lbs. in a kg,” in line 3, she fails to elaborate on her use of the word “constant” in reference to the ratio pounds per kilogram. This suggests she is capable of identifying the physical appearance of a constant but reveals her limited understanding of the concept and what a constant means within the context of the situation. Keggy’s freewrite illustrates the reliance students have upon their classroom experiences to serve as entryways for thinking about abstract mathematical concepts.

Students recaptured their experiences in and out of the classroom to inform their mathematical discussions in their freewrites. Their use of mathematical expressions, equations, drawings and textbook references suggest the manner in which they reflected upon the concept and attempted to investigate it in a more concrete manner. The ideas in students’ freewrite indicated their current perception of the concept. Additionally, they became a starting point as students responded to the same open-ended prompt midway through the topic. Features unique to students’ second draft are described in the next section.

**Draft Two: Experimentation**

Changes in students’ discussions are a natural consequence after more experience and exposure to any mathematical concept. However, this study is interested in describing changes in students’ mathematical discussions as they continue to experiment in constructing meaning to understand a particular mathematical concept. Students attempted to integrate mathematical content into real-world contexts as a way of expressing their mathematical understanding based upon their working knowledge of
abstract mathematical concepts. Creating a second draft encouraged students to become aware of changes in their conceptual understanding in comparison to their initial freewrite as they experimented with expressing their content knowledge.

**Writing Characteristics: Independent Elements**

Students’ second drafts were characterized by the inclusion of content that they believed would support particular mathematical ideas, create mathematical conjectures, and use the language of mathematics, both symbolic and descriptive. Although the addition of these elements better informed their mathematical discussions, students still did not elaborate on the connections and relationships between their examples and the mathematics concepts, treating each as if they were independent from one another. For example, students followed their transcribed definitions with real-world examples implying that their context illustrated the intent of the mathematical concept, thus assuming what was abstract was now concrete. Furthermore, students’ failure to pause to discuss the meaning associated with their symbolic referents related to a particular equation or expression points to their assumption that their procedural manipulations would stand alone as an independent mathematical argument related to their discussion. Characteristic of the type of discussion in the second drafts, students who included these types of stand-alone features implied they were relevant based on the overall theme of the discussion.

The perspective that mathematics is separate from real-world contexts suggests students are thinking about the meaning of mathematics but have yet to solidify those relationships. This is illustrated in Cissy’s response to the prompt, “How can a
mathematical situation be described as a recursive sequence,” in which he reveals changes in his thinking through his second draft as compared to his freewrite. Cissy’s freewrite is as follows:

1 In chapter 4 for Algebra we learned about the order of operations, number tricks, working backwards, recursive routines and linear equations. But in this paper I am going to talk all about recursive routines and linear equations.

2 A recursive routine is like your foundation and instructions to build a recursive sequence (Cissy, Written Response Freewrite, January 25, 2011).

The most meaningful statement Cissy suggests in his freewrite is his perspective that there is a connection between recursive routines and recursive sequences. His use of the words, “foundation,” and “instructions,” in line 4 point to his belief that a relationship exists based on his logic that recursive sequences are produced using a recursive routine. However, Cissy’s failure to elaborate on this idea implies that his thinking is limited by his lack of experience with the concept.

Cissy’ freewrite also indicates his knowledge is currently limited to a wordlist of mathematical terms he encountered through his initial exposure to the given chapter. Using words without elaborating on their meaning suggests information giving and a reliance on other sources of information. It also points to his assumption that using word referents indicates understanding by revealing his failure to elaborate on the meaning the concept conveys when applied to a particular context. Cissy’s writing reveals his current perspective on the concepts, as he understands them.
Cissy’s second draft represents changes in his thinking in responding to the same open-ended prompt approximately midway through the chapter. While he chooses to keep the same introductory paragraph as his freewrite, his second paragraph illustrates his revised understanding of the concept as suggested by his elaboration of individual word referents. Cissy writes:

A recursive routine is like your foundation and instructions to build a recursive sequence. The foundation is what you call the starting number (the first number in the recursive sequence) and the instructions are the rule. The rule is like the operation you perform on the previous number to get the next number in your recursive sequence, which is a sequence of numbers produced by a recursive routine (Cissy, Written Response Draft 2, January 28, 2011).

Cissy repeats the first sentence in this paragraph from his freewrite indicating he continues to believe there is a relationship between recursive routines and recursive sequences. Unlike his freewrite, he defines what he refers to as the “foundation,” by calling it, “the starting number,” in line 2. His use of metaphor for this word referent suggests he believes that this number is significant in producing a recursive sequence. He further defines what the starting number is in his use of the phrase, “the first number in a recursive sequence,” in line 2. Defining mathematical terms through this use of metaphor suggests he has more experience working with the concept as well as greater procedural knowledge.

Cissy continues his use of metaphor as he elaborates on what he means by, “the rule,” in his use of the phrase, “the operation you perform on the previous number to get
the next number in your recursive sequence,” in lines 3 and 4. Cissy describes the procedure he believes should be followed to produce a recursive sequence by elaborating on what he means by, “instructions,” in line 1. This indicates he is reflecting on his procedural knowledge and its relevance to producing a recursive sequence. Cissy implies that the content in this paragraph serves to illustrate his working knowledge of the concepts as he proposes an example in his third paragraph to illustrate what he believes is a concrete application of an abstract idea.

Cissy’s proposed example suggests he is attempting to make the abstract more concrete based on his understanding of the concept. He treats his example as its own entity suggested by his use of mathematical word referents he assumes the reader will understand within the context of his discussion based upon his previous description of them. Cissy focuses on the production of a recursive sequence through his emphasis on the procedural manipulation of the numbers he includes.

For example let’s say Bob is working at a pet store. He has saved up 400 dollars. After one month he now has 550 dollars. After two months he has 700 dollars, after three months he has 850 dollars, and after four months he has 1,000 dollars. If you have already noticed that these numbers are a recursive sequence you are doing great. 400, 550, 700, 850, and 1,000 is your recursive sequence. But what is your recursive routine? The starting number is 400 but the rule is unknown. In order to find that out, subtract the starting number, 400, from the second number, 550. You should get 150. If you add 150 to 400, you get 550 and if you add 150 to
Cissy suggests there is a pattern to the amount of money “saved” with his use of the phrases, “After one month he now has 550 dollars. After two months he has 700 dollars, after three months he has 850 dollars, and after four months he has 1,000 dollars” in lines 8 and 9. His use of the phrase, “If you have already noticed that these numbers are a recursive sequence,” in line 10 suggests he purposefully created a scenario with a recursive sequence in mind. Cissy continues to imply, through this question, “But what is your recursive routine,” in line 11, that he is working under the assumption that his reader will automatically grasp his train of thought based on his descriptions of each month’s new total. He includes the phrase, “subtract the starting number, 400, from the second number, 550,” in line 13 to highlight his procedural knowledge thus leading him to determine, “the rule for this number sequence is to add 150 to the previous number,” in line 14 and 15. Cissy’s implied conclusion points to his assumption that the “rule” refers to a repetitious mathematical operation, but he does articulate the critical importance that the change from one quantity to the next be static. The example Cissy creates illustrates students’ assumptions that using mathematical word referents and procedural manipulations is part of a meaningful mathematical discussion.

Cissy indicates that he has acquired greater procedural knowledge in his second draft. His inclusion of an example implies his ability to determine, from a procedural perspective, if a series of values is a recursive sequence. The inclusion of specific uses of the language of mathematics to describe particular mathematical procedures suggests his
efforts to validate his thinking by describing procedural manipulations. Cissy’s second
draft represents changes in students’ approach to responding to the open-ended prompt
and can be further characterized by the emergent themes describing students’
construction of mathematical arguments and meaning-making strategies. They are:

1. Expansion of his wordlist of mathematical terms to include statements that
   elaborate on particular mathematical terms suggesting more familiarity with
   these terms.

2. Inclusion of an example that he believes illustrates a situation that has the
   potential to model particular mathematical concepts implying the possibility
   for interdisciplinary connections.

3. Use of the language of mathematics for descriptive purposes, such as
   discussing mathematical procedures in words, indicating procedural
   understanding.

These three emergent themes describe the shift in students’ mode of
communication from information-giving to one of experimentation by constructing
mathematical arguments that include any of the following components: conjectures,
relevant scenarios, and the language of mathematics, both descriptive and symbolic.

Unique to these features is their seemingly independent nature, related only by the overall
theme of the response, suggested by students’ failure to identify meaningful connections
between the mathematics in their discussion and their application of those mathematical
concepts to other contexts and situations. The independent nature of these particular
elements suggests students are experimenting in their construction of mathematical arguments and their use of mathematical reasoning to support their ideas.

**Emergent Meaning-Making Strategies**

Students’ second drafts contain student-generated contexts, mathematical factual, and some procedural knowledge. This suggests the emergence of mathematical reasoning as they experiment on how to express their mathematical thinking. Students included in their discussions proposals for conjectures, suggestions for equations modeling particular scenarios, and offered examples that they believed were relevant to a particular mathematical concept or theme. Despite these developments, they often failed to draw explicit connections between the mathematics and their real-world contexts. Students’ emergent meaning-making strategies suggested they had developed more content and procedural knowledge but were still somewhat immature in their conceptual understanding based on their word choice and construction of mathematical arguments.

This is further indicated in the following ways:

1. **Inclusion of mathematical reasoning:** Students experimented with creating mathematical reasoning by suggesting statements containing factual and procedural knowledge they believed would lend credibility to their discussion.

2. **Proposed mathematical conjectures:** Students indicated the direction of their thinking through the creation of conjectures revealing how they perceived particular mathematical concepts.

3. **Descriptive and symbolic use of the language of mathematics:** Students attempted to use the descriptive and symbolic language of mathematics to describe and/or
model a particular situation suggesting their attempt to make sense of quantities and their relationships.

**Mathematical Reasoning.**

Students’ second drafts demonstrate the experimentation in their thinking from a show-and-tell approach to creating what they believed was mathematical reasoning. Students suggested mathematical reasoning was any statement, both descriptive and/or symbolic, they felt would reinforce their ideas. For example, Keggy illustrates how students perceived that mathematical reasoning consists of written statements describing a procedure that would justify the solution to a given context in her response to the prompt, “How can a mathematical situation be described as a recursive sequence?” Keggy’s use of words to describe mathematical operations implies she continues to rely on the idea that “show your work” refers to procedural manipulation, and that this presents a valid mathematical argument to justify a numerical solution. In her response to the prompt, she explains when a mathematical situation can be described as a recursive sequence based on an object falling.

A mathematical term is described as recursive when a routine is applied over and over again to a number or a geometric figure. To apply a recursive routine to a number you must start with a starting number. The next step is to find your routine rule. It could be add 5 or subtract 2…

One example for a recursive routine is the force of gravity on a falling object. If you ignore air resistance, any object will fall at the constant rate of 9.8 m/s/s. The first second the object falls it drops 9.8 m/s. The second will fall...
Keggy makes the assumption that 9.8 meters per second per second is a constant rate, thus believing she can model the height of a falling object through a recursive sequence. Despite her misinterpretation of the formula, assuming that a constant acceleration is synonymous to a constant rate, she illustrates how she uses words to describe procedural manipulations as a way of revealing her ideas with respect to the concept.

Keggy begins her discussion by defining the concept “recursive” in line 1. Her definition is transcribed from to her textbook suggesting she is reliant upon this resource to provide her with concrete information about an abstract mathematical term. Her organization of the phrases, “To apply a recursive routine to a number you must start with a starting number,” in line 2 and, “The next step is to find your routine rule,” in line 3 indicate her understanding of the procedure for creating a mathematical sequence. Keggy’s example, “any object will fall at the constant rate of 9.8m/s/s” in line 6 further illustrates her procedural knowledge as she describes how to calculate the force of gravity on a falling object. Keggy means to suggest by this sequence of values, “the first second the object falls it drops 9.8 m/s. The second will fall another 19.6m/s and the third would drop another 29.4m/s,” in lines 7 and 8 the resultant value after applying a recursive rule, “to add 9.8” in line 9. However, her words say, “the second will fall another 19.6m/s,” which means an additional 19.6 not a total of 19.6 suggesting faulty writing as opposed to
a failure in her procedural knowledge. Statements such as these illustrate Keggy’s procedural knowledge and her failure to describe her conceptual understanding by integrating the context with her mathematical explanation.

Keggy’s second draft reveals how students use their procedural knowledge as mathematical reasoning. Her inclusion of mathematical calculations interspersed with relevant mathematical terms suggests that she believes that procedural knowledge demonstrates the meaning of a mathematical concept. This use of procedural knowledge suggests students’ perception that mathematical reasoning is a procedural demonstration producing a solution based on a series of calculations.

Students also attempted to construct mathematical reasoning by restating information that was previously given to them from the prompt or a given definition most likely transcribed from their textbook. Students believed that transferring information from one source to another, as suggested by Rally, “I usually just get it from the textbook,” (Interview, May 25, 2011) was a valid form of justifying their ideas under the assumption that information from a published source added creditability to their discussion. Students assumed the repetition of similar phrases throughout their discussion provided the necessary support because they were similar to the original transcribed phrase. Kitty illustrates this characteristic in her second draft as she uses similar phrases to construct mathematical arguments in her response to the open-ended prompt, “Compare and contrast fractions and ratios and how they can be used to describe real world situations.”
…A ratio is an expression that compares quantities relative to each other… A ratio has a comparative relationship between two separate quantities… In another instance, when you look at the nutrition facts on a can of soda one of the sections tells you the number of calories per serving. That right there, the calories per serving is a ratio. It is comparing two quantities that are relative to each other. (Kitty, November 11, 2010).

Kitty begins by defining “ratio,” as, “an expression that compares quantities relative to each other,” in line 1. Her definition is similarly rephrased as, “a ratio has a comparative relationships between two separate quantities,” in line 2. She repeats the word, “compare,” such as, “comparing two quantities,” in line 5, implying her belief that this word is significant when justifying why something is a ratio. The repetitious use of similar phrases and words suggests her attempt to construct mathematical reasoning, but her reliance upon words and phrases from a given definition to justify her thinking implies her knowledge of ratios is limited to identifying applicable situations. This is further suggested by her use of the phrase, “calories per serving in a ratio,” in line 4 where she names the ratio but fails to elaborate on what features of the situation justify her use of the word ratio in this situation. Kitty’s repetitious use of mathematical words and phrases, although transcribed, illustrates the manner in which students’ approached mathematical reasoning.

Students’ reliance upon preexisting statements to justify their thinking is further emphasized by Haily who describes her reliance upon her preexisting interactions with known mathematical content to inform her idea generation. She states:
REVEALING STUDENTS’ ALGEBRAIC REASONING

Mostly my notebook and worksheets we get like all the side notes and comments and what we say [in class discussion]. In the introduction I usually start off with the definitions I have in my freewrite and in my conclusion I usually have the answers, most of the answers, or at least some of them that I had questions to in my freewrite…just the core example and counter example\(^6\) to address the main open ended question (Haily, Interview, May 23, 2011).

Haily suggests she relies on her “notebook and worksheets” in line 1 indicating those documents record mathematical content and interaction with particular concepts and procedures. She continues to rely on her textbook implied by the phrase, “I usually start off with the definitions I have in my freewrite,” but indicates her working knowledge of the topic with the use of the phrase, “in my conclusion I usually have the answers…or at least some of them that I had questions to in my freewrite” in line 3. Haily’s belief that she is capable of finding numerical solutions suggests growth in her procedural knowledge indicating one change in her thinking as she progresses from her freewrite to her second draft. A more significant change in her thinking is expressed by her inclusion of, “the core example and counter example\(^7\) to address the main open ended question,” in line 5 pointing to conceptual understanding despite the implication that these contexts are related more by name to the overall theme than by contextual mathematical relationships. Haily illustrates that students’ approach to mathematical

\(^6\) Haily really means to use the term non-example to describe his situation as opposed to a counter example.

\(^7\) Haily’s discussion suggests a non-example rather than a counter example.
reasoning is often reliant upon expressing procedural knowledge in words and repetitious phrases from given definitions.

**Conjectures.**

A conjecture, in this study, is a prediction regarding some aspect of a mathematical concept that indicates the direction of students’ thinking. The conjectures students propose in their second drafts are supported by students’ use of factual and procedural knowledge reflective of their current mathematical experiences and understanding. Maccy’s draft two response illustrates how a conjecture represents her thinking in her response to the prompt, “How do real-world quantities vary directly and inversely?” Her initial proposal in her freewrite stated, “Direct variation is a relationship in which the ratio of two variables is constant. I think inverse variation is a relationship in which the ratios of two variables are not constant. (Maccy, Written Response Freewrite, December 6, 2010),” but she continues to explore her ideas in her second draft with the following discussion:

1. If the quantities are lower the price is lower, but if the quantity is higher the price is higher. For direct variation the points and the lines are supposed to be a slant and go through the origin. In the stock market on TV, if the price goes up that is how much it increases. If the arrows go down then it shows how much money is lost (Maccy, Written Response Draft 2, January 4, 2011).

Maccy’s original conjecture suggests that direct and inverse variation have opposing characteristics, particularly with a constant ratio. The continuation of this train of thought from her freewrite suggests that her current understanding contains misconceptions.
pertaining to inverse variation. This misconception in her understanding is further revealed through the vagueness of her word choice in phrases such as, “if the quantity is higher the price is higher,” in line 1, which makes it difficult to determine what she means by the phrase. She attempts to support her thinking with the phrase, “For direct variation the points and the lines are supposed to be a slant and go through the origin,” in line 2, in which she tries to describe the graph of a situation to match the phrase, “If the quantities are lower the price is lower, but if the quantity is higher the price is higher,” in line 1. However, she fails to draw connections between her description of the graph and the context of pricing which leads me to assume that she does not understand the concept of a constant ratio even though she uses the term in her conjecture. Finally, Maccy’s assumption that inverse and direct variation are opposites, as implied by her conjecture, is further indicated by her inclusion of the phrase, “If the arrows go down then it shows how much money is lost,” in line 4 as she suggests a situation that appears to contrast the phrase, “if the price goes up that is how much it increases,” in line 3. It is difficult to determine her intended meaning from the use of these phrases because she fails to describe whether these situations are related to inverse variation or direct variation and in what manner are they related. Maccy reveals how her conjecture reveals the direction of her thinking as she attempts to construct mathematical arguments and propose relevant contexts to support her ideas.

**Integration of Prior Knowledge.**

Students also relied on prior knowledge to inform their discussions and support their mathematical arguments and models. Prior knowledge, in this study, refers to any
past learning experience that students can draw upon to inform their current understanding and make meaning from the concepts they are in the process of learning. Students’ use of prior knowledge reveals their perception of previously learned concepts and content knowledge of mathematics suggesting the lens from which they are approaching new ideas. Cissy suggests how prior knowledge aids students in their exploration of new mathematical understanding in his response to the prompt, “Compare and contrast fractions and ratios and how they can be used to describe real world situations.”

1 Fractions and ratios may seem the same but they’re actually two different mathematical concepts. A fraction is a number represented by $a/b$. …But remember that the numerator which is the number on the top must be less than the denominator which is the bottom number for the fraction to be a fraction and not a mixed or whole number. Fractions are if you remember part of a whole number...

2 …Ratios compare two different numbers. They can be written as fractions like $a/b$ or they can be like $a:b$ (Cissy, Written Response Draft 2, November 2, 2010).

3 Cissy indicates the direction of his thinking in his use of the phrase, “Fractions and ratios may seem the same but they’re actually two different mathematical concepts.” He believes that there are no similarities between fractions and ratios other than their written appearance as expressed by, “$a/b$” in lines 2 and 6. He describes the characteristics of a fraction based on his prior understanding by stating, “the numerator which is the number on the top must be less than the denominator which is the bottom number” in line 3. Although Cissy correctly identifies the, “numerator,” and
“denominator” of a fraction, his understanding of their relationship based on prior knowledge is expressed by his use of the phrase, “the top must be less than…the bottom number.” His belief that all fractions express values between zero and one suggests that his experience with fractions may have been limited and that perhaps more emphasis was placed on manipulating fractions than on understanding the meaning they conveyed.

Cissy illustrates how prior knowledge reveals his understanding of previously encountered mathematical terms and is relied upon as an entryway to understanding new concepts.

Tohly’s comment during an interview further demonstrates the significance of having a mathematical foundation from which to build new knowledge.

1 I guess sometimes background knowledge, like if you know, like, what it’s like from before, you could, like, think back it or see what the other teachers taught you and put that into your writing (Tohly, Interview, May 12, 2011).

Tohly refers to prior knowledge as, “background knowledge,” in line 1. He suggests these understandings come from previous learning experiences originating from, “what the other teachers taught you,” in line 2. Tohly suggests these prior understandings support current ideas and conjectures expressed in his discussion through the use of his phrase, “put that into your writing” in line 3. Tohly implies that prior knowledge can be integrated into the construction of new mathematical understanding by offering a foundation from which to support developing ideas.

**Descriptive and Symbolic Uses of the Language of Mathematics.**
Students used descriptive and symbolic mathematical language to experiment with creating mathematical arguments, describe mathematical ideas and model a variety of situations. The inclusion of the language of mathematics indicates the attempt to identify interdisciplinary connections between mathematics and other contexts. Students used the language of mathematics by borrowing transcribed definitions as mathematical reasoning to explain their mathematical thinking based on their prior knowledge. They also described what they believed would be a mathematical representation of a situation. This is illustrated by Mobby who included descriptive language in her response to the prompt, “How can a mathematical situation be described as a recursive sequence?”

1. The recursive rule is to start at 0 miles and keep adding 57.6 miles until you reach 807 miles. If this were to be put on a graph, the $x$-axis would be the each day that I traveled and the $y$-axis will be how many miles I traveled. It will stay in the 1$^{st}$ quadrant and will be a straight, diagonal line that is increasing at a constant rate.

2. (Mobby, Written Response Draft 2, January 24, 2011).

Mobby uses the term, “recursive rule,” in line 1 to describe the procedure of, “adding 57.6 miles” suggesting she understands the procedure for producing a recursive sequence. Her description of the graph of the situation further demonstrates her inclusion of the language of mathematics as she appropriately identifies the units represented by the axes with the phrases, “the $x$-axis would be the each day that I traveled and the $y$-axis will be how many miles I traveled in lines 2 through 3. This suggests she believes there is a relationship between the quantities of days traveled and the total mileage despite her failure to articulate what that relationship is. She goes on to suggest the most appropriate
quadrant to model her scenario in her use of the phrase, “it will stay in the 1st quadrant,” in line 3 suggesting that the numerical values produced by her procedure are positive, another indication of her procedural knowledge. Mobby further describes her graph as, “a straight, diagonal line that is increasing at a constant rate,” in line 4 indicating her belief that straight-line graphs have constant ratios. Her use of appropriate mathematical language to describe the graphical model of the situation suggests she possesses procedural knowledge to generate and graph a recursive sequence. Mobby illustrates how students use the language of mathematics to express procedural knowledge and describe appropriate graphs of situations believing its inclusion suggests appropriate mathematical reasoning to support their discussion of particular ideas.

Students also used the symbolic language of mathematics, numbers, symbols and operation signs, to suggest models and procedures associated with various situations in their discussions. This symbolic language of mathematics points to students’ ability to translate words to mathematical symbols to produce a relevant equation and/or expression thereby decontextualizing a given situation (CCSSI, 2010b). Often times these mathematical models are followed by procedural manipulation which are either described using a series of symbolic manipulations or descriptive phrases of mathematical operations. This is seen in Rally’s response to the open-ended prompt, “Where are systems of equations in the real world and what do they mean?”

1 One example in like is both my mom and Dad are driving but they are at different
2 locations. My dad is driving to Keehi from my house, but he is only 90 miles
3 away from Keehi and is driving at a speed of 20 miles/hr. My mom is driving
home from Keehi at 10 miles/hr. The two equations look like this $y = 10x$ for my mom, and $y = 90 - 20x$ for my dad. $Y$ is the distance in miles from Keehi; $x$ is the hours they are driving. My mom is $y = 10x$ because she is driving at a speed of 10 miles/hour. My dad is $y = 90 - 20x$ because he is 90 miles away from Keehi and is driving at a speed of 20 miles/hour (Rally, Written Response Draft 2, May 6, 2011).

Rally constructs two equations based upon the situation he created. He uses the symbolic language of mathematics to model his parents’ travel by describing their place of origin, “My dad is driving to Keehi from my house,” in line 2 and “My mom is driving home from Keehi,” in line 3. He then translates that information into each parent’s respective equation, “$y = 10x$ for my mom, and $y = 90 - 20x$ for my dad” in line 4. Rally goes on to define the meaning he associates with his variables $x$ and $y$ in his use of the phrase, “$Y$ is the distance in miles from Keehi; $x$ is the hours they are driving,” in line 5. This indicates that he understands that a variable represents an unknown quantity whose units he has designated based on his understanding of the context of his situation.

Rally further demonstrates his use of the symbolic language of mathematics by manipulating it to find a numerical solution appropriate for his context. He includes a verbal description of various symbolic manipulations articulating his procedural understanding revealing insights into his thought processes.

9 $y = 10x$

10 $y = 90 - 20x$

11 $10x = 90 - 2x$ \hspace{1cm} 1. Substitute $y$ for $10x$
12  $10x + 2x = 90 - 2x + 2x$  2. Add $2x$ to both side to cancel out the negative $2x$
13  $12x = 90$  3. Add $2x$ to $10x$ to equal $12x$
14  $12x$ divided by $12 = 90$ divided by $12$  4. Divided by both side by $12$ to cancel out the $12x$
15  $x = 7.5$  5. It takes $7.5$ hours for them to meet and at $75$ miles the ordered pair is $(7.5, 75)$ in which $7.5$ is $x$ and $75$ is $Y$

Rally makes a copying error when he substitutes the expression, “$90 - 20x$” in line 10 into a new equation “$10x = 90 - 2x$” in line 11. His failure to accurately copy the equation leads him to an inaccurate solution based on the context of his original problem. This error, however, suggests this mistake is an oversight since he follows through and manipulates the equation and should, thus, be recognized as a careless calculation error having little affect his mathematical reasoning.

Rally accompanies each symbolic manipulation with its respective phrase such as, “$10x + 2x = 90 - 2x + 2x$. Add $2x$ to both side to cancel out the negative $2x$,” in line 12, indicating what was procedurally done to the equation to generate the next one in the sequence until reaching a solution. In this procedural exposition, Rally uses both symbolic and descriptive language indicating his ability to go between both to better clarify his procedural knowledge. Each step of the manipulation is accompanied by its respective description with one exception; Rally fails to symbolically justify this solution for $Y$, "$75$ is $Y$. However, Rally suggests the values he calculated have meaning, as implied by the phrase, “It takes $7.5$ hours for them to meet and at $75$ mile,” in line 16.
This suggests he used the symbolic language of mathematics to justify the accuracy of the equations used to model the context of his situation.

Students commented that the language of mathematics aided them in their construction of mathematical arguments and conjectures by describing the procedural manipulation of equations and expressions to find a reasonable solution, model scenarios, and create situations they felt would best illustrate a particular mathematical concept. Students’ use of the language of mathematics in these instances demonstrated their procedural knowledge of particular concepts and indicated their ability to translate words into mathematical symbols.

Students revealed how their thinking had progressed in their second drafts by indicating the direction of their thinking, including examples they believed could be modeled by particular mathematical ideas, thus revealing insights into their procedural knowledge. However, their failure to describe significant connections between these features and any relevant concepts suggests they still view these elements independently. Moreover, the continued use of outside, reliable sources including textbook information and experiences from class to inform their discussion implies limited independent thinking. To assist students in furthering their own thinking and encouraging internal dialogue between themselves and their written content, I commented on their second drafts with teacher-to-student written feedback, as discussed in the next section.

**Transition: Teacher-to-Student Written Feedback**

The intent of feedback is to prompt students to probe their own thinking and accurately describe their conceptual understanding such that their responses contain
everything they are capable of producing. In this study, I provided feedback not to prompt improvement per se, but to see what students’ reflection on their expressed content could generate. Feedback figures prominently during the writing process by offering an outsider’s perspective on what was said and how it was said. While feedback can take the form of constructive criticism, specific comments, and positive observations (National Writing Project & Nagin, 2006), I generally crafted feedback as questions to raise students’ awareness of how their expression was interpreted and to prompt reflection of their ideas. Vygotsky (1978) suggests that input, such as questions or comments, from a more knowledgeable individual supports substantive learning experiences when students are engaged in activities that are potentially above what they are capable of doing independently. In terms of feedback, asking students to reflect and consider how their expressed meaning is being interpreted by someone else helps them to consider if that intent is really all they are capable of producing.

After seeing the changes in students’ ideas after receiving feedback following their second drafts, I was prompted to reflect upon the type of feedback she provided. For example, Tohly’s second draft, in response to the open-ended prompt, “How do real-world quantities vary directly and inversely?” states, “You have to divide for direct and multiply for inverse,” (Tohly, Written Response Draft 2, January 4, 2011). His use of the words, “divide,” and “multiply,” refer to specific mathematical operations associated with direct and inverse variation, but it is difficult to determine in what way he is using these operations. This suggests he has some working knowledge of the procedural aspects
of direct and inverse variation but fails to elaborate on their significance and usage within each equation.

After receiving feedback, Tohly’s response changed, describing characteristics he felt were associated with the concepts of direct and inverse variation.

Direct variation is when two variables have a constant, which is an unchanged ratio. As one quantity increases, the other also increases. Also, if you graph direct variation, you would have a straight line that goes through the origin of the graph.

Inverse variation is a relationship between two variables in which when you multiply them together the product is the constant. And as one quantity increases, the other decreases so that the product is unchanged. If you graph inverse variation, you would have a curved line that doesn’t go through the origin or the $x$- and $y$-axis. (Tohly, Written Response Draft 3, January 7, 2011).

In his third draft, Tohly states, “a constant, which is an unchanged ratio,” in line 1 describes the relationships between the variables $x$ and $y$. His use of the word, “ratio,” is more descriptive than his use of the word “divide” in his second draft suggesting he was prompted to describe the mathematical relationship as opposed to naming a mathematical operation. This is again echoed in his discussion of inverse variation where he goes beyond naming operations, “multiply,” to identifying the significance of the operation, “when you multiply them together the product is the constant,” in line 4. Additional changes include the description of the graphs of both direct variation and inverse variation such as, “If you graph inverse variation, you would have a curved line that doesn’t go through the origin or the $x$- and $y$-axis,” in lines 6 and 7. These changes are
significant and while they could be attributed to further conceptual interaction, I hypothesized it was receiving feedback in tandem with these additional experiences that prompted students’ to reveal additional mathematical understanding.

Students suggest feedback prompted them to reflect upon their original thinking and expression of their ideas after realizing how someone else interpreted their meaning. For example, Tutty comments that receiving meaningful feedback encourages him to reexamine his ideas and reconsider how he presents his ideas. “Well, it [feedback] makes me think in a different way… I don’t like those [feedback that asks you to explain what was said]… ‘cause I don’t understand the chapter myself” (Tutty, Interview, May 24, 2011). Tutty implies feedback, “makes me think in a different way,” prompting him to reflect on his own understanding and reexamine how he expressed his intended meaning. While he expresses his displeasure for feedback that questions his rationale, “I don’t like those,” he admits these comments raise his awareness of his lack of understanding suggested by the phrase, “I don’t understand the chapter myself.”

While an analysis of my feedback was not originally planned, I observed significant changes between drafts two and three triggering an investigation to describe the characteristics of the feedback given to students that may have prompted revisions in their thinking expressed through their writing. Thus, this section is organized in the following manner: a) it identifies particular words and phrases students used in their second drafts that prompted teacher-to-student written feedback; b) it identifies my response to students’ use of those particular words and phrases; and c) it describes changes students made to their writing after receiving feedback from me.
Content-based feedback prompts students to have an internal dialogue between themselves and their content (Patthey-Chavez, Matsumura, & Valdes, 2004; Silver & Lee, 2007; Vardi, 2009) to reflect on how their chosen words convey their mathematical understanding. An analysis of teacher-to-student written feedback found that most of the comments were written in the form of questions. Questioning suggests neither correct nor incorrect ideas but indicates areas where students’ choice of words fails to convey clarity in their intended meaning. It prompts students to reflect on whether their failure to convey meaning is attributed to their lack of understanding or a poorly written statement or both. Three themes emerged from the feedback students received:

1. Feedback asking for clarification: These comments asked students to refine and/or explain their preexisting ideas to clarify their intent including: a) students’ understanding of a mathematical term or concept; b) students’ rationale supporting conclusion-like statements; and c) students’ rationale decontextualizing contexts and modeling them mathematically with the appropriate numerical symbols and operations.

2. Feedback to further idea development: These comments prompted students to think more deeply, going beyond the most obvious or superficial mathematical argument, focusing on the following: a) the need to present a logical rationale supporting a particular idea; and b) the need to describe meaningful mathematical relationships beyond textbook definitions or words from the original prompt. These comments also suggested topics to include that would further the discussion and assist students in elaborating on their ideas in a meaningful way.
3. Feedback asking students to consider alternate perspectives: These comments asked students to consider alternate situations or possible exceptions to their preexisting ideas such as: a) drawing a conclusion based on a single case or example; and b) assuming a general statement will suffice to describe specific characteristics or attributes of a particular concept.

These types of feedback are aligned with research (National Writing Project & Nagin, 2006; Whittington, Glover, & Harley, 2004; Silver & Lee, 2007; Vardi, 2009; Bain, Mills, Ballantyne, & Packer, 2002; Patthey-Chavez, Matsumura, & Valdes, 2004) suggesting effective feedback is content oriented, encourages students to revise their thinking, and takes the form of constructive criticism, specific comments, and positive observations (National Writing Project & Nagin, 2006). Maccy also expresses her thoughts on the type of feedback she finds meaningful:

I think that’s [feedback’s] one of the best parts of writing because you’re able to expand more after feedback is given. When teachers give feedback most of the time we know what we need to change…someone who’s older than you and has more knowledge will be able to guide you with their feedback in the situation or topics. I like feedback like when they’re truly honest about what they think about your writing… they’ll tell you what parts were really horrible or maybe you even had a small section that was good and they told you they liked how you explained it or what example you gave and if your writing was terrible they could tell you how you could improve that section of writing or think of what else you learned (Maccy, Interview, May 20, 2011).
Maccy suggests that feedback is valuable when it is given by, “someone who’s older than you and has more knowledge,” in line 4. This implies she trusts that the feedback she receives will prompt her to reflect upon her writing and reexamine her understanding for the purpose of capturing everything she is capable of expressing. Maccy points to two ways feedback prompts her to reconsider her thinking: a) the development of her ideas—“you’re able to expand more after you’re feedback is given,” in line 2; and b) identify areas of misconception—“we know what we need to change,” in line 3. Maccy expresses her appreciation of receiving feedback that is, “truly honest,” in line 5 because it highlights specific areas of strength and weakness, “they’ll tell you what parts were really horrible or maybe you even had a small section that was good,” in line 6. She also suggests feedback prompts her to reflect and, “think of what else you learned,” in line 9 and stretch her thinking to its greatest capacity. She feels that any changes, “improve that section of writing” in line 8 believing these revisions make her response more meaningful.

Feedback Asking for Clarification

Asking for clarification suggests that students either failed to distinctly communicate their understanding or assumed their reader will use their prior knowledge to decode their meaning. Approaching students’ drafts as if one possesses little to no knowledge of the topic helps the reader to avoid making assumptions about what the student is trying to say and brings to light these clouded areas. For instance, mathematical arguments that do not support suggested conjectures or other conclusion-like statements point to students’ omission in describing significant ideas related to a particular concept.
This is evident in Maccy’s second response to the open-ended prompt, “Compare and contrast fractions and ratios and how they can be used to describe real world situations.”

Maccy initiates her discussion with the statement, “A fraction is a number that is expressed with a numerator and a denominator,” (Maccy, Written Response Draft 2, November 13, 2010) suggesting that fractions are defined solely by their appearance. This definition limits the meaning conveyed by any fraction she uses to model a situation to just another number and prompts me to respond (see Figure 2).

I responded to Maccy in the following way: “What do the numerator and denominator express in a fraction? Be sure your definition of a fraction would include fractions such as ½ but also 3/2 as well” (Teacher-to-Student Written Feedback, November 21, 2010).
November 21, 2010). My use of the word, “express” suggested that Maccy failed to articulate the meaning represented by the values in the numerator and the denominator. In the event Maccy had difficulty understanding what is meant by the word, “express,” I also used the word, “definition,” to prompt Maccy to describe what a fraction was beyond its physical appearance. Moreover, I asked Maccy to reflect on, “fractions such as ½ but also 3/2,” and consider both proper and improper fractions if she chose to revise her original statement.

Maccy’s third draft illustrates her response to my feedback suggesting feedback prompted her to express a more thoughtful understanding of fractions.

1 A fraction is when you divide a whole into equal parts. Fractions are expressed as numerators and denominators. The denominator is how many equal pieces the whole has been divided into. The numerator is how many pieces you have out of the whole. (Maccy, Written Response Draft 3, November 13, 2010).

Maccy keeps her original description of the physical appearance of a fraction but goes on to express the meaning each number conveys. She suggests the denominator conveys, “how many equal pieces the whole has been divided into,” in lines 3 and 4 illustrating how she reveals her understanding of fractions. This serves as her starting point for a discussion on the numerator, “how many pieces you have out of the whole,” in lines 4 and 5 and further implies a relationship exists between the numerator and the denominator. Although Maccy does not specifically address the feedback asking her to consider, “fractions such as ½ but also 3/2,” she suggests that her revisions consist of everything she is capable of sharing during the process of crafting this draft.
Students who made conclusion-like statements without further rationale or decontextualized symbolic representations (CCSSI, 2010) that failed to reflect the original situation received feedback asking them to illustrate their thinking. Students who propose these statements imply they have working procedural knowledge but only tentative conceptual understanding. Maccy demonstrates this as she continues her discussion in her second draft of fractions and ratios in response to the prompt, “Compare and contrast fractions and ratios and how they can be used to describe real world situations.”

1 Fractions can be used in situations like cooking and baking, sewing, and buying candy. It is necessary for baking because you need to know how to measure three-fourth cup of sugar or two and a half cups of flour (Maccy, Written Response Draft 2, November 13, 2010).

2 Maccy offers a laundry list of contexts, “cooking and baking, sewing, and buying candy,” in line 1, she believes are appropriate for the use of fractions. This list fails to demonstrate her conceptual understanding of fractions or their application to modeling particular contexts. She, however, believes it is informative by suggesting, “three-fourth cup,” in line 2 referencing an existing quantity based upon the action of measuring. Her use of the phrase, “you need to know how to measure,” in line 2, further describes the manner in which various amounts are physically determined.

I respond to Maccy with the following questions: “What does ¾ mean as a fraction? What does this [two and a half cups] mean as a fraction?” My use of the word,
“mean,” suggests Maccy’s use of \( \frac{3}{4} \) and 2 \( \frac{1}{2} \) was unclear and wants to prompt her to reconsider their use within the context of her example.

Maccy responds to the feedback by including new discussion points in her next draft:

5 Fractions can be used in situations like cooking and baking, sewing, and buying candy. It is necessary for baking because you need to know how to measure three-fourth cup of sugar or two and one-half cups of flour. Three-fourths means that the whole was broken into four pieces and you have three pieces. As a fraction, two and one-half means that there are two whole pieces that were not divided.

6 You also have one half of a whole, which means that the whole was divided into two equal pieces and you have one of those two pieces (Maccy, Written Response Draft 2, November 13, 2010).

Maccy chooses to keep the original context for her discussion and interprets my use of the word, “mean,” to imply she should explain her use of fractions based on her newly revised definition in the previous paragraph. She even uses the word, “mean” in her explanation suggesting the manner in which she interpreted its use in the feedback. She states, “Three-fourths means that the whole was broken into four pieces and you have three pieces,” in line 7 and 8; her use of the words, “whole,” and, “pieces,” point to her interpretation of a fraction’s part-to-whole relationship. While Maccy’s description has improved, her description does not clearly convey if or how she understands fractions in interpreting the context of dividing the capacity of a one-cup measure and only filling it to \( \frac{3}{4} \) of its capacity.
Maccy’s efforts suggest her attempt to address the feedback she received, however, these revisions also indicate that in reflecting on her explanation in the second draft, she may have relied on a more elementary interpretation of fractions to inform her discussion. At the same time, Maccy’s revision implies that I may not have been specific enough in asking for clarification regarding how she thought she was decontextualizing the proposed context. Nevertheless, Maccy’s response indicates feedback prompted an awareness of her discussion and encouraged her to attempt to clarify her expressed ideas.

Haily also suggests feedback asking for clarification raises an awareness of how she expresses meaning.

1 …Feedback is important because…when we get it back we know… Like what
2 they want you to explain more, what they want you to go in depth with more. I
3 think feedback is important in that kind of sense because you …have what they’re
4 looking for, you just have to, have to go more in depth with it (Hailey, Interview,
5 May 23, 2011).

Haily suggests feedback asks her to consider if what she expressed was all she is capable of producing in phrases such as, “explain more,” and “in depth,” in line 2. She interprets these phrases to suggest areas she failed to express her mathematical reasoning clearly by stating, “when we get it back we know…what they’re looking for,” in lines 1 and 3. This statement further suggests Haily believes feedback indicates areas she needs to reflect upon to clarify her articulation of the ideas and conceptual understanding she wants to express.
Feedback to Further Idea Development

Students’ deliberate choice of words reveals their thinking; in this way, I am able to glean their understanding regarding particular mathematical concepts. Broad approaches to mathematics, such as making general statements that could be applicable to a variety of concepts prompted feedback to narrow students’ ideas and suggest a manner in which to express their understanding. The difference between furthering idea development and asking for clarification is the addition of concrete talking points for students to include in their discussions. These talking points suggested specific ways students could probe their own thinking and expand their discussion in a meaningful manner. Although these suggestions were given for various prompts, they were more frequently offered when the prompt was vague and students’ responses indicated the need for guidance to focus their response. For example, students could have discussed a variety of single-variable data displays, any one of which would have been appropriate, in response to the open-ended prompt, “In what ways can analyzing and visually displaying data influence your view of the world around you.” Jolly’s response illustrates his uncertainty of how to approach the prompt and demonstrate his mathematical understanding of a specific concept.

1 When you see something visually it helps you see information and see the obvious differences. Your visualization of the information may be good or bad.
2 When you hear a description it is different than seeing it because when you see it you can study it and see its color and shape. When you see something visually you can how [sic] the data may be similar or may vary. The data may be clearer
when you see it than when it is described to you. When you see data visually you can see the range of the data and where the majority of the data falls. Also when you can see the data it may be easier to work with because in some cases you can rearrange the data or move it (Jolly, Written Response Draft 2, October 4, 2010).

Jolly’s response to this open-ended prompt was to focus on the word “visually” indicated by his use of the word, “see” and its related derivatives throughout his discussion. “See,” suggests pictures communicate meaning, an idea which appears to generalize his discussion through statements such as, “it helps you see information,” in line 1, “see its color and shape,” in line 4, and “[see] how the data may be similar or vary,” in lines 4 and 5. These broad statements fail to illustrate Jolly’s conceptual understanding of single-variable data and their graphical displays or use them to appropriately model individual contexts.

My response to Jolly suggests talking points to narrow his generalizations and broad sweeping statements to include a specific example illustrating his conceptual understanding (see Figure 3).
Figure 3. Jolly’s Second Draft after Receiving Teacher-to-Student Feedback

When you see something visually, it helps you see information and see the obvious differences. Your visualization of the information may be good or bad. When you hear a description, it is different than seeing it because when you see it, you can study it and see its color and shape. When you see something visually, you can bow the data may be similar or may vary. The data may be clearer when you see it than when it is described to you. When you see data visually, you can see the range of the data and where the majority of the data falls. Also when you can see the data it may be easier to work with because in some cases you can rearrange the data or move it.

Comments

Comment [Teacher]: What obvious difference do you see referring to the conclusion that can be drawn from the data?

Comment [Teacher]: I’m not sure what point you’re trying to make—this sentence is confusing to me. Please clarify your thoughts.

Comment [Teacher]: What type of data analysis specifically are you referring to?

Comment [Teacher]: When you mention “averaging data amount of,” what do you mean by this? Please explain your thoughts.

Comment [Teacher]: Create a real-world example that can help justify your position on how analyzing and visually displaying data can influence your viewpoint.

When can your example, or even a counter example, include:

- What type of data could be collected?
- How would the data be analyzed? Would different analysis techniques produce different conclusions about the data?
- What conclusions might drawn from various graphs?
- How would the conclusions affect your viewpoint?

Figure 3. This is the second draft written by Jolly (Written Response Draft 2, October 4, 2010) in response to the question, “In what ways can analyzing and visually displaying data influence your view of the world around you?” The feedback was teacher generated and given in the form of teacher-to-student written feedback. The focus of this feedback concerns prompts to further idea development. Adapted from Teacher-to-Student Written Feedback, October 6, 2010.

I offered these talking points to Jolly prompted by his overall response; these comments are not attached to a particular word or phrase in his actual writing but follow at the end suggesting one approach to reflecting on his content knowledge and understanding.
Create a real-world example that you can use to help justify your position on how analyzing and visually displaying data can influence your viewpoint. When discussing your example, or even a non-example, include:

1. What type of data could be collected?
2. How would the data be analyzed? Would different analysis techniques produce different conclusions about the data?
3. What conclusion(s) could be drawn from various graphs?
4. How would the conclusion(s) you created affect your viewpoint? (Teacher-to-Student Written Feedback, October 6, 2010).

I state, “create a real-world example,” in line 1, implying that Jolly’s generic discussion could benefit from a specific example demonstrating his content knowledge. To assist Jolly in constructing his “example,” in line 1 or, “non-example,” in line 3, I ask him to reflect on specific questions, such as, “What type of data could be collected,” in line 4 to prompt, at the very least, his decontextualization of the situation using the appropriate single-variable graph. Other questions, such as, “Would different analysis techniques produce different conclusions about the data,” in line 5 are suggested as additional ways to prompt him to reflect on the appropriateness of his constructed single-variable graph and rationalize why other graphs would be less appropriate. My questions offer one approach for Jolly to narrow his focus and reflect on whether his discussion illustrates all of his conceptual understanding.

Jolly crafts an example in his third draft suggesting he considered the feedback he was given.
If you see something visually then you can be influenced on your judgment of the situation. You can use a box plot to visualize the statistics of a NBA team. It will help you to visually see where the data lies on a number line compared to the other players. If you compare the stats of two players then you can see whose points are overall the higher of the two. If the range of the players is bigger there may be an outlier. The outlier may tell you that the player may have had an off game one-day. This type of graph can help you determine which basketball player is the better of the two (Jolly, Written Response Draft 3, October 10, 2010).

In keeping with his previous discussion, Jolly uses the phrase, “if you see something visually,” in line 1 to emphasize graphs are visual models of a situation. However, he reveals his reflection on the feedback he was given in his phrase, “You can use a box plot to visualize the statistics of a NBA team,” in line 2 suggesting feedback prompted him to include an example to support his discussion. His naming of mathematical terms such as, “number line,” in line 3, and “range,” and “outlier,” in line 5 suggests some working knowledge of these terms but also points to his failure to discuss their significance. He alludes to his rationale with statements such as, “If the range of the players is bigger there may be an outlier,” but without further detail it is difficult to determine if an outlier truly exists. Additionally, his use of the phrase, “compare the stats of two players,” in line 4 raises more questions about Jolly’s understanding than it clarifies because he does not specifically name those statistics and demonstrate how he is using them to decontextualize the situation. Despite the flaws in Jolly’s discussion, his
revisions suggest feedback prompted him to rethink his original response as he attempted to include an example he felt would illustrate his thinking.

Feedback offering advice to students is geared toward prompting them to construct responses that accurately represent the full extent of their conceptual understanding. In her interview, Mobby suggests how feedback aids her in generating and expressing her ideas. She states:

1 Well, the role of feedback for me really helps me to better my writing. …I’m not good with grammar and the mathematical concepts so when I explain something wrong, there’s always a bubble [feedback] next to it that really helps me to understand. …Well, when you ask an open ended question and then you have the feedback and they’re also open ended questions. That really helps me to generate my ideas (Mobby, Interview, May 20, 2011).

Mobby’s use of the phrase, “role of feedback,” in line 1 implies feedback is given for a purpose. She expresses that purpose as, “helps me to better my writing,” in line 1. It is important to note that she uses the word, “better,” as opposed to the word, “improve,” to suggest that feedback prompts her to reflect upon areas of concern, “when I explain something wrong,” in line 2, regarding what she wrote and how she expressed her ideas. The revisions she makes after generating ideas based on feedback crafted as, “open ended questions,” in line 5, she views as changes made to her previous draft.

Feedback Considering Alternative Perspectives

Students who expressed tentativeness in their conceptual understanding or narrow reasoning through their chosen words were asked to reflect on alternative perspectives.
For example, students who stated conclusion-like statements based on little or no situational context were asked to consider other viewpoints to prompt their demonstration of additional reasoning. For example, students who used one premise from their example on which to base their argument failed to consider if the entire context they proposed is capable of being modeled by that particular concept. Feedback given in these instances questioned the students’ rationale by proposing “why,” or “what if,” type inquiries to prompt them to consider other relevant ideas. My response to Kitty’s second draft (see Figure 4) illustrates this type of feedback.

*Figure 4. Kitty’s Second Draft*

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How you ever noticed a pattern in the things you do? Let’s say that every day
you brush your teeth for five minutes. If you wanted to figure out the total amount of

| time you brushed your teeth per day you would continuously add five to the starting
| value, five. This was an example of a recursive sequence where the starting value is
| five and the rule is to add five to the result. Any situation that has a rule applied over
| and over again and has an ordered list of numbers is a recursive sequence.

Figure 4. This is the second draft written by Kitty (Written Response Draft 2, Written Response Draft 2, January 30, 2011) in response to the question, “How can a mathematical situation be described as a recursive sequence” The feedback was teacher generated and given in the form of teacher-to-student written feedback. The focus of this feedback concerns prompts to consider alternative perspectives. Adapted from Teacher-to-Student Written Feedback, February 1, 2011.
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Kitty’s response to the open-ended prompt, “How can a mathematical situation be described as a recursive sequence,” is to decontextualize an example situation and express it as a mathematical model.
REVEALING STUDENTS’ ALGEBRAIC REASONING

Let’s say that every day you brush your teeth for five minutes. If you wanted to
figure out the total amount of time you brushed your teeth per day you would
continuously add five to the starting value, five. This was an example of a
recursive sequence where the starting value is five and the rule is to add five to
the result (Kitty, Written Response Draft 2, January 30, 2011).

Kitty states a “recursive sequence,” in line 3 is capable of describing, “the total
amount of time you brushed your teeth,” in line 2. She reasons, “you would continuously
add five to the starting value,” in lines 2 and 3 illustrating her belief that recursive
sequences are constructed using a repetitious procedure; in this case that method is, “add
five.” However, her use of the phrase, “per day,” in line 2 is unclear. “Per day,” appears
to suggest the cumulative amount of time she brushes her teeth throughout a single day;
thus, a recursive sequence consisting of multiples of five could indicate the number of
times she brushed her teeth during the day. But this interpretation does not appear to be
consistent with Kitty’s initial statement, “let’s say that every day you brush your teeth for
five minutes,” suggesting the previous interpretation of her use of the phrase, “per day,”
is not her intention.

I assume the recursive sequence Kitty intends to express reflects the cumulative
time spent brushing over a period of several days. She responds, therefore, to Kitty’s
claim, “This was an example of a recursive sequence where the starting value is five,” in
line 3 and questions her use of the number five by asking, “Why is 5 the starting value?”
(Teacher-to-Student Written Feedback, February 1, 2011). This implies that I am not
convinced that this value accurately reflects the context of the situation other than Kitty’s
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use of five to represent, “every day you brush your teeth for five minutes,” in line 1. I call into question its relevance in by asking Kitty to consider an alternative perspective stating, “One could argue that if you don’t brush your teeth, the time is 0,” (Teacher-to-Student Written Feedback, February 1, 2011) to imply a potential flaw exists in the way she decontextualized the situation.

While length does not necessarily correlate to better mathematics content, Kitty’s response is to expand her rationale suggesting feedback prompted her to reflect on her expressed ideas.

1 Have you ever noticed a pattern to the things you do? Let’s say that every day you
2 brush your teeth for five minutes. A recursive routine is composed of a starting
3 value and a rule. Zero is the starting value because if you’ve brushed for zero days
4 then you would have brushed for zero minutes. The starting value is the number
5 you start with in the beginning of your sequence. A recursive rule is the
6 instructions for producing each stage of a recursive sequence from the previous
7 stage. In this situation to get to the next stage you would add five to the previous
8 minutes. For ever[y] time the number of minutes increases by five the number of
9 days increases by one. Because it’s five minutes per day five is the rate of change.
10 The linear equation for this problem is $Y = 5X$. $Y$ represents the total number of
11 minutes you’ve brushed your teeth, $X$ represents the total number of days you’ve
12 brushed your teeth, and five is the rule in your recursive routine (Kitty, Written
Kitty’s use of the word, “pattern,” in line 1 suggests her context “every day you brush your teeth for five minutes,” leads her to believe it can be modeled by a recursive sequence. She rationalizes, “if you’ve brushed for zero days then you would have brushed for zero minutes,” in line 3 to illustrate how she considered the feedback she received in the context of her example. In addition to supplying a contextual reference for her decision, “Zero is the starting value,” in line 3, she also includes a mathematical perspective in her use of the phrase, “The starting value is the number you start with in the beginning of your sequence,” in line 4. This line of reasoning indicates her consideration of the feedback she received and reveals her thinking through her revisions.

Kitty attempts to reflect on the expression, “Zero is the starting value,” in line 3 in her model, “\( Y = 5X \),” in line 9. However, she appears to center her discussion solely on the, “rate of change,” in line 8 with statements such as, “For ever[y] time the number of minutes increases by five the number of days increases by one,” in line 7. While this statement clarifies the confusion of her use of “per day,” from her previous draft it fails to acknowledge how her model reflects this statement and her discussion regarding zero as the starting number.

In her response to the interview question about feedback, Cissy comments that reflection raises his awareness regarding how he thinks about a concept.

1. It helps… it shows you what you did wrong and it really helps you to watch out for stuff like that the next time you’re writing, so yeah, it makes you more aware of and what to be careful about. …Uh, feedback where it’s like, why is this; why is this like so and so; it doesn’t make sense; is this really; are you sure about this;
cause it gets me thinking, and then like, oh! ... Um it shows you like where you
have to put more effort into it so yeah, it really helps you out. ... It helps you
better cause it refines what you thought and that makes it better. (Cissy, Interview,
May 17, 2011).

Cissy suggests feedback, “makes you more aware,” in line 2. His interpretation of
this “awareness,” is suggested through, “it shows you what you did wrong,” in line 1
implying that he believes feedback indicates areas of concern. However, he also indicates
it prompts him to consider his own line of reasoning,” it gets me thinking,” in line 4, in
the following manner: “why is this; why is this like so and so,” in line 3 and, “is this
really; are you sure about this” in line 4. This reflection appears to lead him to
realizations about his own understanding pointed to in his use of the phrase, “and then
like, oh!” in line 5.

Students’ revisions made after receiving feedback suggest they considered the
feedback they received. For example, changes in students’ choice of words to clarify their
meaning or the inclusion of specific mathematical rationale contextualizing mathematical
models indicate students were prompted by the feedback they received to express
everything they were capable of offering at the time of the revision. What follows is the
construction of a third and final draft.

**Final Draft: Reconsider**

Students had one final opportunity to respond to the open-ended prompts. For
many students, this last draft indicated closure to the topic and the textbook chapter as
well as one last attempt to demonstrate their mathematical understanding. Whereas the
freewrite recaptured students’ experiences to make what was abstract concrete, and the
second draft suggested students were experimenting with the inclusion of some form of
mathematical reasoning, the final draft is characterized by students’ reconsideration of
their previous ideas as suggested by their attempts to express and make connections
among mathematical ideas. Students’ expression of their ideas in draft three represents
their reflection on the feedback. Two unique features indicated students’ reconsidered
their previous ideas thus implying they engaged in reflective internal dialogue. These
features were: a) students suggested greater understanding of particular mathematical
concepts by integrating them into their discussion; and b) students’ writing contained
more substantive mathematical arguments in which they expressed their mathematical
thinking through symbolic and descriptive mathematical language.

Integration of Mathematical Concepts

Students expressed their understanding of mathematical concepts throughout their
discussion by integrating them in a variety of ways. For example, students described their
interpretation of a particular mathematical term as opposed to solely relying on a
transcribed definition. Students also made references to characteristics of particular
terms, suggesting how they interpreted the meaning of a specific context through that
lens. Furthermore, constructing mathematical arguments alluding to logical connections
between mathematical ideas implied conceptual understanding. Using mathematical
ideas, arguments and language in their third draft responses suggests students shifted
from a show-and-tell approach to one that indicates mathematical reasoning.
This development is demonstrated by Cissy as he responds to the open-ended prompt, “How can a mathematical situation be described as a recursive sequence?” Cissy begins his discussion by describing the relationship between recursive routines and recursive sequences with phrases such as, “A recursive routine is your starting values and instructions to build a recursive sequence,” (Cissy, Written Response Draft 3, February 16, 2011). He also introduces other pertinent mathematical terms he feels are significant to his discussion implying he will later refer to these ideas throughout his discussion thus informing his reader that he understands them. Cissy illustrates this in the following discussion:

Did you know that this recursive routine is also linear? It is linear because when converting data in graphs you use an intercept equation to find points you use to plot on a graph. An intercept equation looks like \( y = a + bx \). \( X \) and \( y \) are the values for a recursive routine. \( A \) is the \( y \)-intercept which is the value of \( y \) when \( x \) equals zero, and \( b \) is the rate of change. So in intercept form the recursive routine would look like \( y = 400 + 150x \) if you used this equation you would see that the outcomes would match the numbers on your recursive sequence. Do you notice that the starting number for \( y \) is the same as the \( y \)-intercept? Also that the rate is the same value you add to \( y \)? These similarities is [sic] the reason why this recursive sequence is linear (Cissy, Written Response Draft 3, February 16, 2011).

Cissy purposefully asks his reader questions during his discussion, as if prompting them to consider mathematical relationships he already believes exist between concepts.
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He opens his discussion by asking his audience if they know there is a connection between recursion and linear relationships in his use of the phrase, “Did you know that this recursive routine is also linear,” in line 1 implying that he believes one exists. To make his point, Cissy demonstrates he knows the generic form of a linear equation in intercept form in his use of the phrase, “An intercept equation looks like $y = a + bx$,” in line 3. His attempt to explain the relationships that exist between the two concepts are noted in his use of the phrase, “in intercept form the recursive routine would look like $y = 400 + 150x$,” in line 5 pointing to his knowledge of the physical equation. He attempts to describe the connections between the starting number in a recursive sequence and the y-intercept in intercept form of a linear equation in his use of the phrase, “the starting number for $y$ is the same as the y-intercept,” in line 7 suggesting linear equations offer a way to symbolically model the values produced by a recursive routine. Finally, his use of the word, “similarities,” in line 8 suggests he believes the concepts share common features which is what allows him to use them both to model the same situation and produce similar solutions.

Cissy’s attempt to integrate mathematics into his discussion for the purpose of illustrating the relationship between two concepts suggests his understanding stretches beyond a textbook definition. Tobby also demonstrates this development as he revealed how his final draft discussion is more complex than his previous two drafts by saying, “I can understand and comprehend and explain the terms,” (Tobby, Interview, May 24, 2011). Tobby’s use of the words, “understand,” and “comprehend,” implies he has developed a degree of confidence about his conceptual understanding of the mathematics
he is learning. This is further implied in his use of the word, “explain,” suggesting he is able to discuss how particular concepts can be applied to model contexts existing outside of mathematics class.

**Substantive Mathematical Arguments**

Students’ writing demonstrated a more fluid and purposeful use of the language of mathematics, both descriptive and symbolic, to construct their mathematical arguments. This deliberate choice of words reflecting students’ ideas was revealed in the following ways:

1. Students described the connections between their mathematical model and the context of their proposed situation suggesting contextualization and the meaning they believed was associated with the mathematical symbols used.
2. Students revealed their thought process in writing by offering a series of statements based on prior knowledge or their current understanding to justify particular mathematical ideas.

**Contextualization.**

Contextualization refers to the inclusion of the description of any symbolic referents used in a mathematical model of a particular context (CCSSI, 2010b). Students who make connections between their mathematical models and the context of the situation, and give meaning to relevant values suggest a more mature understanding of the concepts (Loewenberg Ball & Bass, 2003).

Maccy illustrates students’ inclusion of mathematical statements implying contextualization as she responds to the essential question, “How can quantities vary
directly and inversely?” She begins her discussion by briefly introducing direct variation to provide some mathematical context for her example and mathematical model.

For example…you plan to invite seven friends to an outing at Ice Palace. The admission cost is $8.25 per person, and the cost of admission for your friends will be $57.75. After realizing that you didn’t include yourself in the cost of admission, you have to recalculate if you have earned enough for yourself to skate too. This would be an example of direct variation because as the number of people who will be skating increases, so does the total amount you pay in admission fees. The directly proportional formula you would use in this situation would be \( Y = kx \). The constant, \( k \), would be 8.25 because the admission fee per person will always be $8.25. I found the constant by using the method \( k = \frac{y}{x} \). \( Y \) would be the total amount you would pay for the admission of your seven friends and yourself and \( x \) would be the number of people in your party that will be skating, which is eight. So, if \( k = 8.25 \) and \( x = 8 \), then \( Y = 66 \). You have earned enough money to pay for yourself and seven friends to skate at Ice Palace (Maccy, Written Response Draft 3, January 8, 2011).

Maccy suggests a scenario she believes is appropriate to be modeled by direct variation to calculate, “the total amount you pay in admission fees,” in line 7. She claims this situation is appropriate for the use of a mathematical model involving direct variation in her use of the phrase, “This would be an example of direct variation,” in line 6 and justifies her belief with the phrase, “as the number of people who will be skating increases, so does the total amount you pay in admission fees,” in lines 6 and 7. Although
it is difficult to justify the use of direct variation based solely on the assumption that two
quantities in an increasing relationship must be direct variation, Maccy suggests this
observation is an indicator that a direct relationship exists and continues to discuss other
connections between her context and her mathematical model to support her rationale.

Maccy proceeds to discuss her proposed equation, “\(Y = kx,\)” in line 8. Her word
choice suggests her understanding of the concept is still developing but she attempts to
describe the connections between the context and her model implying contextualization.
Her use of the phrase, “directly proportional,” in line 7 indicates she believes a
relationship exists between the variables \(x\) and \(y\), but she does not elaborate on why her
choice of words is relevant. She also refers to the equation as a, “formula,” in line 8
suggesting her understanding of this equation is more procedural than conceptual.
However, she goes on to suggest the meaning of each variable based on the context of the
situation in her use of the phrases, “\(Y\) would be the total amount you would pay for the
admission,” in line 10 and “\(x\) would be the number of people in your party,” in line
11. This implies that she believes “\(y\) depends on \(x\)” and further supports her belief with the
use of the phrase, “I found the constant by using the method \(k = \frac{y}{x},\)” in line 8. Her use of
the word, “constant,” in line 8 to describe the, “admission fee per person,” in line 9
implies that she believes a constant ratio exists in the situation, thus implying another
reason for her use of direct variation to model this situation.

Maccy identifies the value of each variable based on her context in her use of the
phrase, “\(k = 8.25\) and \(x = 8,\)” in line 12 and suggests it is possible to use her equation, “\(Y
= kx\)” to calculate the total amount of money needed to pay for admission in her use of the
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phrase, “$Y = 66,”” in line 12. Although Maccy does not include the actual calculations leading to her calculated value of “$Y$” she implies that by substituting the values for “$k$” and “$x$” into the equation, it is possible to complete the multiplication and find a reasonable solution. Maccy illustrates students’ attempt to contextualize by suggesting connections between the context and the mathematical equation to give meaning to the values she discusses.

Students’ word choice indicates constructed algebraic reasoning by revealing the ideas that lead them to particular statements in their discussion. Haily describes how her writing reveals the ways in which she has approached and thought about mathematics.

1 It makes me think more about algebra in a more detailed kind of way instead of just formulas and numbers. I do think about what’s behind it all, like explaining why this or that or why that isn’t this and I think it helps me just remember so when I’m having an exam or quiz or just a simple worksheet I can always refer back to what I think (Haily, Interview, May 23, 2011).
Haily’s use of the word “detailed,” in line 1 suggests her writing reveals her ideas regarding particular mathematical ideas that extend beyond naming formulas and equations. She suggests her third draft encourages her to, “think more about algebra,” in line 1 since this is the second time she has revisited her response to the open-ended prompt. She indicates that her thinking about mathematics extends beyond a procedural approach to understanding the “why” and “how” of particular concepts in her use of the phrase, “I do think about what’s behind it all, like explaining why this or that or why that isn’t this,” in lines 2 and 3. The benefit to thinking in this way, Haily reveals, is, “I think it helps me just remember,” in line 3 implying that her understanding of mathematics is enriched by her awareness and recall of the finer details when applying those particular concepts to a variety of problems. Haily suggests detailed thinking with respect to mathematical ideas extends her understanding beyond numbers, equations, and procedures.

Thought Process Revealed.

Systematic and logical thinking is revealed by students who offer a series of statements as their way of constructing mathematical reasoning. Students’ choice of words distinguishes their ideas from those that were transcribed from another source. More skilled writers who are particularly knowledgeable in mathematics generally convey fluid expression, but any use of students’ own words to convey their intended meaning provides insights into their thought processes. A series of statements suggesting how one rationale follows another reveals insights to students’ thinking.

Students’ mathematical reasoning was included in their third draft in the following ways: a) conjectures were supported with statements describing particular
characteristics of a mathematical concept; b) mathematical models based upon the contexts of examples and non-examples were supported by describing their symbolic meaning and connection back to the original context; and c) relationships between related concepts were discussed in a logical manner.

These characteristics are demonstrated by Haily as she responds one final time to the prompt, “When is it appropriate to create a line of fit to describe human ability or other real world data?” Haily suggests her overall conclusions are based on her reflection on the data she modeled describing trends from the Men’s 400-meter relay gold medalist teams in the Summer Olympics. The paragraph demonstrates her rationalization of the symbolic model she created by integrating particular mathematical ideas and contains a justification and the connections she made between the context and mathematical model.

First of all, you’d need to define your ‘x’ and ‘y’ variables. My two variables are winning time and Olympic year. Time depends on the year it has been run. ‘Y’ is always the dependent variable. So therefore, because the winning time depends on the year the Olympics were held in, x is years and y is time (in seconds)…. The line of fit shows the trend of the data. My line of fit goes through at least two points and it shows that the general direction of my data is decreasing and it shows that as the years go by, the times are getting faster…. Also, the number of ordered pairs above and below the line is the same—there are ten points below and ten points above…. Next, I found my slope. Slope is the steepness of a line or the rate of change of a linear relationship…. The slope of my data would be about -0.053. The slope means that when the year increases by 1, the time decreases by -0.053 seconds….
Finally, your slope-intercept form is $y = 143.184 + (-0.053x)$ (Haily, Written Response Draft 3, May 14, 2011).

Haily offers the rationale she believes will justify her mathematical decisions and reveal her thinking based on her understanding of the ideas. She begins by defining the quantities represented by the variables “$x$” and “$y$” in her use of the phrase, “Time depends on the year it has been run,” in line 2. Her use of the phrase, “depends on,” indicates her belief that a specific relationship exists between the winning time and the Olympic year alluding to the implication that the winning time is based on the Olympic year. Haily continues to support her use of the phrase, “depends on” by stating, “‘$Y$’ is always the dependent variable,” in line 3 implying that the output, “$y$” is dependent upon a particular input for “$x$.”

Haily alludes, in her use of the phrase, “the general direction of my data is decreasing...as the years go by, the times are getting faster,” in lines 5 through 7. Her description of the line of fit is based on her previous discussion on the relationship between the variables $x$ and $y$. She rationalizes that her line of fit is accurate in her use of the phrase, “the number of ordered pairs above and below the line is the same—there are ten points below and ten points above,” in lines 7 and 8. Haily suggests that if there are an equal number of ordered pairs both above and below the line of fit, the line must go through the middle of the data. This reasoning points to the inclusion of her thought process despite her failure to discuss the implications of the ordered pairs that either fall above or below the line of fit and what those ordered pairs mean in relationship to the line of fit.
Haily’s justification of her line of fit leads her into a discussion the meaning of her slope from the trend she found from graphing her data. Her use of the phrase, “when the year increases by 1, the time decreases by -0.053 seconds,” in lines 11 and 12 suggests her interpretation of the value \(-0.053\) within her equation, implying that time, in seconds, is negative. Although her interpretation attempts to reveal the meaning she associates with that value, her misinterpretation of the negative is evident; the negative describes the decrease since time itself cannot be negative. However, her use of this statement implies she believes her calculations are accurate and further supports her previously suggested relationship, “as the years go by, the times are getting faster,” in lines 6 and 7. Moreover, her use of these descriptive phrases reveal the manner in which she is uses her previous line of reasoning as a springboard to discuss her beliefs regarding a new idea. This suggests her attempt to link the two ideas to imply they are related and relevant to her discussion. Haily’s discussion points to her thinking which serves as her reasoning to justify the significance of particular symbolic notations and describe mathematical relationships between concepts.

Students revealed, in their third and final drafts, evidence of mathematical reasoning and contextualization through their construction of mathematical arguments reflecting the various situational contexts. Their attempts to present statements of support justifying their ideas reveal their thinking and the way in which they interpret mathematics. More inclusion of descriptive and symbolic language of mathematics indicates greater mathematical comprehension pointing to the depth of their understanding. Students’ third and final drafts also demonstrated the presence of logical ideas implying previous ideas lead to subsequent ones. This fluidity of students’ writing
suggests greater confidence in their ability to construct this draft, as opposed to previous ones, and implies greater understanding of the factual, procedural and conceptual aspects of the mathematical ideas.

There was a noticeable shift in students’ use of writing in mathematics after using the writing process to produce their final draft. Although they admitted the task was difficult and was at times stressful, they indicated they attained a greater understanding of mathematics than they had prior to using writing as a process to construct and revise their ideas. Students also indicated writing shifted their approach towards learning mathematics by revealing to them gaps and misconceptions in their understanding, especially after receiving feedback with the intent of assisting them to clarify or expand their thinking.

Maccy suggests that the use of writing in mathematics shifted her perspective of learning mathematics from procedural skillfulness to conceptual understanding, whereby she is able to better understand relationships between concepts.

1 I thought in the beginning it was very difficult for me because I never had to write
2 before so it was kinda just like add, solve, and divide and all the operations. But
3 we were never really asked to explain why and how it’s applied to real life
4 because before we had to solve and no one asked for writing. …Now I see things
5 from a different perspective…. [and] how it does apply to the new chapters that
6 we’re learning today (Maccy, Interview, May 20, 2012).

Maccy’s use of the word “perspective” in line 5 is significant because it suggests a shift in her understanding of mathematics. This is further emphasized by her use of the words, “explain,” in line 3 and “apply,” in line 5 indicating her ability to use the language
of mathematics to both express her thinking and understand the meaning associated with particular mathematical symbols. Maccy suggests through this revelation that writing encouraged her to understand and find meaning to mathematics as opposed to only manipulating numbers.

Tohly’s perspective is similar to Maccy’s in that he also suggested writing encouraged him to view mathematics in ways other than memorizing formulas or mathematical properties without understanding them. He points to the notion that mathematics is more than just a show-and-tell demonstration of the manipulation of mathematical facts and procedures to find a solution to a problem.

1 It really helps us to learn because it’s another way of thinking rather than just thinking math, math, math, or all equations. Like putting it into words sometimes it like helps take off lots of stress knowing I have to know this equation. (Tohly, Interview, May 12, 2011).

Tohly implies that understanding mathematical relationships, “helps take off lots of the stress knowing I have to know this equation,” in lines 2 and 3. He suggests that he can rely on his conceptual knowledge as opposed to memorizing numbers and variables in a particular order in the hope he can recall it accurately. He indicates that memorization without understanding presents him with a narrow perspective of mathematics in his use of the phrase, “just thinking math, math, math, or all equations,” in lines 1 and 2. Tohly implies he appreciated the opportunity to grow in his use of the language of mathematics to gain richer conceptual understanding of mathematical ideas. Like Tohly and Maccy, students indicated that the opportunity to write in mathematics encouraged them to actively participate in their learning of new concepts, draw
connections between ideas both past and present, and become aware of their mathematical thinking.

As a whole, students suggested that writing in mathematics revealed their thinking and/or understanding of mathematics. Students such as, Cissy, Kitty and Mobby implied writing made them aware of their mathematical understanding. Cissy expresses it best when he states, “This year it really helped me…it really shows how far I’ve come…and gives me a better understanding too” (Cissy, Interview, May 17, 2011). Cissy implies he was able to see the development of his own math ideas in his use of the phrase, “it really shows how far I’ve come.” It also indicates he saw the progression of his thinking based on individual responses to a particular topic. But it also implies a sense of awareness of his own mathematical understanding looking back over the course of the school year and recognizing his own progress. He continues to suggest writing, “gives me a better understanding,” implying the process of writing encouraged him to reflect upon what he thought he knew and how to best articulate those ideas.

Other students, Tohly, Keggy, and Tobby, expressed they believed writing prompted them to think about mathematics. Tobby captures this by stating, “Think deeply, don’t, don’t just think equations and everything. Think about word problems and what you can do with the formulas and apply it to whatever you can” (Tobby, Interview, May 24, 2011). Tobby uses the word “deeply,” to suggests writing prompts him to go beyond surface knowledge of mathematics and encourages him to reflect and accurately express to the best of his ability the full extent of his conceptual understanding and content knowledge. He further supports this by saying, “don’t just think equations,” implying he is prompted to go beyond rattling off a given formula or some other piece of
information towards explaining his thinking. His use of the word “apply” also points to more in-depth thinking and how mathematical concepts can be used to model a variety of situations. These students suggested they gained a richer understanding and more thoughtful as they used the process of writing to express and reflect upon their ideas.

Chapter 4 Findings

To say that students do not mature in their thinking due to an increase in their exposure to a particular concept would go against conventional teacher experience that suggests greater interaction and immersion can lead to greater conceptual understanding. However, the dominant themes from an analysis of students’ writing, recapture, experimentation, and reconsider, also suggest that writing was able to capture how students construction of their ideas changed throughout their learning experiences including using the process of writing to record their constructed thoughts and receiving feedback as a means of prompting them to reveal as much of their ideas as they can possibly capture on paper.

Students looked to recapturing their previous experiences, prompted by the open-ended inquiry, to record anything they believed was relevant to the topic in their freewrite including personal experiences in and out of the mathematics classroom and particular definitions associated with mathematical terms. Freewrites suggested students’ idea generation was reliant upon words, phrases and experiences that would allow them some insight into the topic. Even if their ideas lacked accuracy, the simple fact that they were able to produce something written revealed how they created entryways into thinking about mathematics.
Students’ second draft was all about their experimentation with the language of mathematics to form what they believed was a meaningful discussion that included conjecture, mathematical reasoning, and examples and non-examples. The manner in which they constructed these pieces suggests students were attempting to articulate what they thought would be a significant revealing their understanding of the concepts. What was evident from this draft from their treatment and presentation of their ideas was that each portion of their discussion was treated almost independently from the rest of the discussion implying there were still gaps in their understanding form their failure to connect all of the pieces together. However, this draft revealed greater understandings than their initial freewrite suggesting writing captured a more mature awareness of the concepts.

Students’ transition between what they were capable of in the second draft and the production of their third draft was prompted by the teacher-to-student feedback they received encouraging them to reveal everything they were capable of articulating to produce their final draft. This feedback was prompted by the words students used to articulate their ideas in their second draft suggesting their content revealed areas of uncertainty, confusion, contradiction, or a failure to go beyond a superficial discussion of an idea. Transition implies a shift; students did not necessarily change what they thought about a particular mathematical concept but the way they thought about that concept. For instance, rather than create an example that would appear to illustrate a particular concept, students would then discuss more intimately the connections between particular words and their mathematical model they created based upon them.
Students’ final draft revealed the full extent of their understanding by the conclusion of the mathematical theme. Their writing suggests their ideas have matured from the way in which they create mathematical arguments focused on drawing connections between examples and/or non-examples and the mathematical model they created based upon that particular situation. Their mathematical reasoning is substantive such that it focuses on specific words from the context that suggest to them an appropriate translation to mathematical symbols and what that implies with regard to the application of a particular mathematical concept. Any procedural manipulations accompanying the model only further support the appropriateness of the model and serve to demonstrate the accuracy of their applied procedural knowledge.

This analysis suggests that students experience a progression in the development of their conceptual understanding based upon the emergent themes from an analysis of students’ mathematical reasoning of individual drafts. A model describing this progression based on these findings will be discussed in the next chapter.
Chapter 5: Findings

Similar to the initial views of the students in this study, many students believe the purpose of writing in mathematics is to present a solution to a problem using the symbolic language of mathematics. The process of writing, however, has the greater potential to prompt students to express their thinking and reveal their conceptual understanding especially through the creation of multiple drafts. The aim of this study is to describe changes in students’ construction of mathematical ideas as they respond three times to a series of open-ended prompts. The following research questions were proposed to investigate the use of writing as a process in mathematics:

1. How does writing reveal a student’s algebraic reasoning and knowledge of algebraic concepts?
2. In what ways does writing, through the revision of multiple drafts, expose the development of algebraic reasoning?
3. In what ways does teacher-to-student written feedback, provided during the writing process, prompt the revision of algebraic reasoning as represented in a student’s writing?

The discussion in this chapter comprises the emergent theory grounded in the data: a model, and its indicators, describing students’ expressed understanding in response to mathematically themed prompts, the limitations of the study, and suggestions for future research.

1. A model describes students’ progression of conceptual maturation based on a series of three drafts crafted with the use of the writing process (National Writing Project & Nagin, 2006). This model proposes three stages and their
respective indicators suggesting students’ potential stage of conceptual maturation.

2. The limitations of the study address areas that have impacted the findings and generalizability of the study. These areas are: a) the teacher also serves as the primary researcher; b) the sample was purposely chosen rather than randomly derived; c) triangulation was limited to two primary sources of data; and d) students’ hesitancy in addressing and/or interpreting the feedback they received.

3. I am proposing two areas for further research: a) a longitudinal study describing how students’ mathematical reasoning progressed successively within individual and collective sets of drafts written over the course of the school year; and b) a study describing students-to-student (peer) written feedback and how that feedback prompts the receiver to revise his or her writing. In such a study, the nature and content of the written feedback’s provider could be another indicator of the author’s conceptual understanding.

**Findings and Interpretation**

Individual drafts produced using the writing process (National Writing Project & Nagin, 2006) revealed a general shift in students’ mode of communication from information giving to the construction of mathematical arguments evidenced by the inclusion of mathematical reasoning and the use of descriptive and symbolic mathematic language. Written drafts, generated using the writing process (National Writing Project & Nagin, 2006), suggest students’ conceptual understanding progresses in the following manner: a) acquiring knowledge—students explore abstract ideas by identifying known
information and/or recalling prior experiences as a means of grounding their learning within a concrete context; b) mulling over working knowledge—students propose the direction of their thinking by attempting to use the language of mathematics to express tentative mathematical rationale; and c) conceptual evolution—students indicate conceptual understanding and maturation through the construction of mathematically appropriate arguments. The evolution of students’ conceptual understanding are described by the following model:

This model is based on emergent themes following students’ use of the writing process (National Writing Project & Nagin, 2006), which includes drafting and receiving teacher-to-student written feedback. The writing process (National Writing Project & Nagin, 2006) provides consistent structure as students craft their response to each open-
ended prompt. Drafting captures students’ thinking in each of the twenty-one responses they composed consisting of: a) ten-minute freewrites created in class after an introduction to the theme; b) second drafts occurring approximately at the midway point composed outside of class to allow for more think-time; c) receiving teacher-to-student written feedback on the second draft; and d) revisiting the prompt at the conclusion of the topic crafting their third and final drafts.

In general, first draft responses suggested students were creating entrees into thinking about mathematics. These ‘doorways’ appeared as students relied on concrete experiences to gain access to abstract mathematics content. Students indicated they used intermediary drafts (the second draft) to experiment with creating meaning. These drafts suggest a sense of messiness as students attempt to define the direction of their thinking characterized by the tentativeness of their expression. The full extent of students’ conceptual maturation at the conclusion of the topic was revealed in their final drafts after receiving feedback to craft this response.

Feedback is not part of the model, but it is included in the graphic because it is an important component of the writing process (National Writing Project & Nagin, 2006); it was critical to the choices I made in creating the conditions for the writing data that was analyzed to describe students’ conceptual understanding. Feedback prompts inner dialogue between the student and their content (Patthey-Chavez, Matsumura, & Valdes, 2004; Silver & Lee, 2007; Vardi, 2009) and helps them to say what they mean to say (National Writing Project & Nagin, 2006).

All students received teacher-to-student written feedback after submitting their second drafts. The purpose of the feedback was to prompt students to probe their own
thinking, reflect on how their constructed meaning was received, and accurately describe their conceptual understanding to the best of their ability. I crafted feedback in the form of questions suggesting this was not a summative evaluation of students’ responses. The nature of students’ thinking and understanding expressed in their responses are reflected in the model, further illustrated by the indicators that are discussed in a later section.

**Organization of the Model**

The configuration of this model is intended to convey the following ideas: a) conceptual development is not linear or predictable; thus the assumption should not be made that each stage of the model corresponds to a particular draft constructed during the process of writing; and b) indicators describing students’ presentations of their ideas and use of written expressions discussing mathematical ideas are reflected in the model.

**Non-Linear Progression.**

The purposeful overlap of each stage in the model suggests that the previous stage becomes the starting point for the next stage; students can linger for as long or as little time as needed in each stage. While changes in students’ discussions are expected with more experience and exposure to a mathematical concept, how they express that understanding in their writing reveals their progression of conceptual maturation. For example, a student whose later drafts primarily use verbatim phrases and/or information giving suggests the students need to spend more time acquiring knowledge.

The purposeful overlap of the stages suggests it is possible for students to straddle two stages indicated by the content they express in their writing. For example, students who express ideas that fall into adjacent stages suggest their progression is headed out of one stage and into the next stage. This is illustrated by Rally whose discussion straddles
two stages of maturation in response to the prompt, “How can a mathematical situation be described as a recursive sequence,” based on his definition and inclusion of an example and non-example.

A mathematical sequence can be a recursive sequence because it is repeating again and again, for example height is a recursive sequence but it’s a counter example\(^8\), people can grow, but it alters from year to year. for example, you grow most during adolescence [sic] to adult, however it is different when you are an adult, sometimes you shrink, it is a sequence but not linear. However a recursive sequence is the same. Time however is a recursive sequence because it is always ticking at a constant rate (Rally, Freewrite, January 25, 2011).

It is difficult to determine Rally’s intended meaning when he describes a recursive sequence with the phrase, “repeating again and again,” in line 1. He fails to describe what is repetitious and the significance of that repetition suggesting, his attempt to summarize the original definition, “Describes a procedure that is applied over and over again, starting with a number or geometric figure, to produce a sequence of numbers or figures,” (Murdock, Kamischke, & Kamischke, 2002, p. 699). Deliberately choosing some words over others points to a failure to understand all aspects of the concept indicating his knowledge base is reliant on given information.

Rally’s failure to capture the meaning of the concept, “recursive sequence,” does not dissuade him from explaining why he believes human height is not recursive. His use

\(^8\) Rally’s discussion suggests a non-example rather than a counter example.
of the phrase, “alters from year to year,” in line 3 suggests human growth cannot be described by his definition, “repeating again and again.” in line 1. He supports this by stating, “you grow most during adolecens [sic],” in line 3 and, “sometimes you shrink,” in line 4, illustrating variance in human growth throughout an individual’s lifetime. In contrast to his non-example, he states, “Time however is a recursive sequence because it is always ticking at a constant rate,” in lines 5 and 6. Rally’s examples reveal how he envisions how the phrase, “repeating again and again,” in line 1, appears in contexts beyond the classroom. His failure to discuss significant connections between the concept and the context point to the tentativeness of his ideas and the messiness associated with the existence of multiple ideas. Moreover, his discussion suggests his thinking straddles two stages, knowledge acquisition and mulling over working knowledge.

**Describing Conceptual Maturation.**

This model assumes that the typical student will undergo three stages of conceptual maturation: knowledge acquisition, mulling over working knowledge, and conceptual evolution. The assumption should not be made that students are skipping stages even if it appears that their responses suggest they are at a stage beyond what is expected of the typical student. Rather, if some students skipped over a first stage that does not mean that they never displayed characteristics of that first stage. It does suggest, however, that at the time they were asked to express their understanding, they moved beyond the previous stages placing their discussion at a higher stage.

Tobby’s freewrite suggests his thinking has matured beyond the initial stage prior to starting this freewrite as illustrated by his response to the open-ended prompt, “When is it appropriate to create a line of fit to describe human ability or other real world data?”
In his freewrite he describes the meaning associated with the $x$- and $y$-intercepts and discusses whether or not he believes they are appropriate in describing his scenario.

Most of the time, the $y$-intercept & $x$-intercept in these (Tour de France) graphs don’t make sense because at a year 0, the time for the race would be too long to be even possible, and at year 0, the bicycle wasn’t yet invented, nor the Tour de France was created as a race. And for the $x$-intercept, time for the race would never be 0 hours, 0 minutes because it is humanly impossible (Tobby, Freewrite, March 7, 2011).

Tobby hypothesizes the $x$- and $y$-intercepts, “don’t make sense,” in line 2, based on his understanding of them within the context of the situation. He rationalizes, “at year 0, the time for the race would be too long to be even possible,” in line 2, revealing his interpretation of the $x$-coordinate and suggesting a relationship exists between the Olympic year and the winning time. He also reasons, “at year 0, the bicycle wasn’t yet invented, nor the Tour de France was created as a race,” in lines 3 and 4; while these statements are logical, they are not as significant as if he had identified a specific $y$-coordinate correlating with his proposed $x$-coordinate of zero.

Tobby’s discussion of the “$x$-intercept,” echoes his interpretation of the $y$-intercept. Although he fails to name the specific $x$-coordinate corresponding to a winning time of zero, suggested by the $y$-coordinate, this discussion appears to be more advanced than transcribing relevant definitions associated with the prompt, suggesting he is processing his working knowledge of the concepts. While the factors pertaining to his more developed understanding cannot be determined by this study, the model suggests he is in the process of mulling over whatever working knowledge he has accumulated.
A comparison of the initial and final drafts in each collective set of responses indicated students were more conscious of their mathematical understanding. While this model presents an overview of the stages of students’ conceptual maturation, specific indicators provide descriptors that characterize what served as evidence of where students’ writing fell in the model. These indicators are discussed in the section that follows.

**Indicators**

The term indicator is used to describe the characteristics of students’ writing that give evidence of their thinking at each stage in the model. There are eleven indicators, each associated with a particular stage in the model. These descriptions address the construction of mathematical content, students’ idea generation, and/or students’ word choice using descriptive and/or symbolic mathematical language. Each set of indicators is accompanied by examples from students’ writing. Earlier indicators will demonstrate students’ reliance upon given information; latter indicators suggest greater conceptual understanding and expression of more mature idea development. While these indicators are not exhaustive, they point to the maturation of students’ mathematical understanding based on their deliberate choice of words used to represent their thoughts. The indicators, including illustrative examples, are listed with their respective stages.
### Level of maturation

<table>
<thead>
<tr>
<th>Knowledge Acquisition</th>
<th>Indicator(s) and Examples</th>
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</table>
|                       | • Copying verbatim from a published source (i.e. textbook)  
|                       | • Copying information from students’ notebooks  
|                       | Example: Recursive – describes a procedure that is applied over and over again starting with a number or geometric figure, to produce a sequence of numbers or figures (Mobby, Freewrite, quoting from textbook).  
|                       | • Summarizing the context of a story problem from class  
|                       | • Paraphrasing a personal experience after a similar story problem from class  
|                       | • Naming a content term, but it is difficult to determine the writer’s intended meaning  
|                       | Example: A recursive consist of a starting number (Jolly, Freewrite). |
### Level of maturation

<table>
<thead>
<tr>
<th>Indicator(s) and Examples</th>
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<tbody>
<tr>
<td>Creating a tentative conjecture using mathematical language suggested by the initial writing prompt</td>
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</table>
| Prompt: When is it appropriate to create a line of fit to describe human ability?  
Student response: “It is appropriate to create a line of fit because you can see the average performances” (Tutty, Freewrite). |
| Using the symbolic language of mathematics to find a numerical solution to a proposed situation existing outside of the classroom that appears to be relevant to the topic |
| Example:  
...you want to purchase one and a fourth pound of caramel squares… One pound of caramel squares cost $2.49. …So you set up your equation: $1 \frac{1}{4} \times 2.49. 1 \frac{1}{4} \times 2.49 = 3.11$ (Maccy, Draft 2). |
| Using previously stated definitions as mathematical reasoning to support proposed mathematical models or conjectures |
| Example:  
If you wanted to figure out the total amount of time you brushed your teeth per day you would continuously add five to the starting value, five. …Any situation that has a rule applied over and over again and has an ordered list of numbers is a recursive sequence (Kitty, Draft 2). |
| Stating a personal experience or narrative to accompany a previously stated mathematical definition to illustrate a “real-world” example of the definition. |
| Example:  
A ratio is an expression that compares quantities relative to each other. An example of a ratio is twenty-five miles per hour (Kitty, Draft 2). |
| Loosely suggesting a relationship exists between two or more concepts by association in reference to the same example or non-example. |
| Example:  
There are recurring sequences…that are linear, and/or constant or non-constant (Tobby, Draft 2). |
<table>
<thead>
<tr>
<th>Conceptual Evolution</th>
<th>Indicator(s) and Examples</th>
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<tbody>
<tr>
<td>Level of maturation</td>
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</table>
| • Supporting mathematical models by describing the “real-world meaning” associated with particular numbers or symbols used in the model illustrating specific connections between context and the model. | Example:  
The directly proportional formula you would use in this situation would be $Y=kt$. The constant, $k$, would be 8.25 because the admission fee per person will always be $8.25 (Maccy, Final Draft). |
| • Formulating mathematical reasoning and/or arguments using mathematically appropriate language both descriptive and symbolic to discuss the appropriateness of modeling an example existing outside of the classroom | Example:  
So in intercept form the recursive routine would look like $y = 400+150x$ if you used this equation you would see that the outcomes match the numbers on your recursive sequence. Do you notice that the starting number for $y$ is the same as the $y$-intercept? Also that the rate is the same value you add to $y$? These similarities is [sic] the reason why this recursive sequence is linear (Cissy, Final Draft). |
| • Formulating mathematical reasoning and/or arguments using mathematically appropriate language both descriptive and symbolic to discuss non-examples that may appear to lend themselves towards the application of a particular concept but do not based upon particular characteristics of the situation | Example:  
A good counter example would be a receipt you get after you paid for your clothes at a store or paid the bill at a restaurant. You start with zero and add the prices, but there is no pattern or rule to the numbers you have. That is because all of the items have different prices. If there is no rule that is repeating then it is not recursive. On a graph the line my move up but it does not move at a constant rate (Keggy, Final Draft). |

9 Keggy’s discussion suggests a non-example as opposed to a counter example.
Knowledge Acquisition.

Knowledge acquisition refers to the act of obtaining and using given information with little or no alteration from sources students can consistently rely on to inform their thinking. These sources include published resources such as textbooks and secondary sources such as notes and other personal mathematical experiences during class. Acquiring knowledge occurs as students gather a collection of words; words offer entryways into thinking about a particular concept. For example, a student who attempts to string a series of mathematical terms together in a sentence in such a way as to present inadequate or undistinguishable meaning suggests limited interaction with that particular concept.

The transcription of information is an indicator that students’ conceptual maturation has not gone beyond knowledge acquisition. For example, Mobby’s response to the essential question, “How can a mathematical situation be described as a recursive sequence?” is to copy the definition of recursive from her textbook. She writes, “Recursive—describes a procedure that is applied over and over again starting with a number or geometric figure, to produce a sequence of numbers or figures” (Mobby, Freewrite, quoting from textbook, January 25, 2011). Copying suggests students’ reliance on given information to inform their thinking especially when their written expression does not indicate conceptual understanding.

Mimicking is another indicator suggesting students are in the knowledge acquisition stage. Mimicking, as used here, differs from copying; the end result imitates the original whereas copying suggests verbatim transcription. Nevertheless, both are signals that students are relying on someone or something else as a basis for their
discussions. For example, students mimic given story problems by keeping the same organizational structure but changing the context to reflect their own experiences. Tohly demonstrates this in his response to the essential question, “Where are systems of equations in the real world and what do they mean,” by patternning his example after a problem found in the textbook.

1 Original problem:

2 “The school’s photographer took pictures of couples at this year’s prom. She charged $3.25 for wallet-size pictures and $10.50 for portrait-size pictures.

3 …Crystal and Dan bought a total of 10 pictures for $61.50. (Murdock, Kamischke, Kamischke, 2002, p. 329).

4 Tohly’s student-generated example:


Tohly demonstrates how a problem involving portrait and wallet sized pictures becomes a template for constructing an example he feels reflects a situation that can be modeled by a system of equations. Following the structure of the original problem, Tohly informs the reader of the cost of each item, “burgers cost $3 fries cost $1,” in line 6, the total amount of items purchased, “20,” in line 7, and the total amount of money spent, “$30,” in line 6. Mimicking suggests students are using their classroom experiences to assist them in constructing scenarios that can be modeled by particular equations, graphs or other representations. Copying and mimicking are indicators that students are in the process of acquiring knowledge, relying on other sources of information to furnish information for them.
Students who use content terms in name but whose choice of words to describe the content do not make clear their intended meaning are in the knowledge acquisition stage. For example, Jolly states, “A recursive consist[s] of a starting number,” (Jolly, Freewrite, January 25, 2011), in response to the open-ended prompt, “How can a mathematical situation be described as a recursive sequence?” It appears he is attempting to relate the terms “starting number,” and “recursive,” but his choice of words conveys his failure to identify if the word “recursive,” is the actual term he means to use. The vagueness of this expression suggests his content knowledge is limited to naming word referents.

**Mulling Over Working Knowledge.**

The second stage is referred to as “mulling” to emphasize the variety of ways students attempt to create and express their mathematical understanding. Mulling suggests that students’ drafts in this stage convey a sense of messiness as they attempt to express their rationales. This messiness appears in the form of statements where it is difficult to determine students’ intended meaning, gaps and misunderstanding in their responses, or errors in their mathematical expression, both symbolic and descriptive. This stage has the most indicators, illustrating the variety of ways students demonstrated how they mulled over their understanding and expressed them in their responses.

The awkwardness of expression in Tobby’s second draft responding to the essential question, “How can a mathematical situation be described as a recursive sequence,” illustrates how he mulled over creating a non-example in discussing the concept of recursion.
Some examples that are not recurring in life are the way you move. One example is when you are base running in baseball. First you hit a ball that ends up being a ground ball, then you sprint to first base to try to be safe, you slow down after you hit the bag, but then the ball goes sailing over the first baseman’s head, and then you turn then speed up to try to reach second. The graph for this would look like a gradual line going up, then going down gradually, but then slowly sloping up again. This example would be non-recurring because it is not at a constant speed (Tobby, Written Response, January 29, 2011).

Tobby’s writing focuses on describing the game of baseball. He implies there is mathematics in speeding up and slowing down, but he does not make explicit connections between the situation and his discussion of recursive routines. His use of the words, “not,” in lines 1 and 7 and “non” in line 6 suggest his intention to convey the opposite of the mathematical terms he uses to describe the situation. Tobby hints at these ideas by portraying the graph he imagines would model the situation. However, it is difficult to determine his intended meaning with his use of the word, “sloping,” indicating that each section of the graph has its own unique steepness, thus implying a rate of change. Additionally, the phrase, “not at a constant speed,” contradicts his use of the word, sloping,” thus creating further confusion for the reader. While this non-example reveals Tobby’s use of his personal experience to illustrate a mathematical concept, he fails to provide enough of an explanation to clearly present his intended meaning.

Students whose written expressions suggest they are in the mulling period are usually ready for feedback that will assist them as they experiment with constructing meaning. Feedback for these students prompts reflection on their understanding and
expression of their thinking. This response to feedback may differ from students who are in the knowledge acquisition stage. Students who copy verbatim, for example, may regard feedback as too complex for their current knowledge base, and they may not know how to use the feedback to reflect on what they have written. For example, a student who finds the need to copy given definitions may have difficulty creating an example to illustrate an appropriate context for the application of the concept when they suggest their own awareness of the concept is limited. However, feedback in this instance would provide students with a next-step approach to prompting their thinking.

This is echoed by Keggy who suggests her thinking would be limited without receiving feedback. “...when I get another draft I have to go over it cause that’s what I was taught. But then if I don’t get feedback I don’t know what to do,” (Keggy, Interview, May 16, 2011). Her response points to the significance of receiving feedback—providing her with the opportunity to extend her thinking beyond its current state. This implication is illustrated in her the statement, “if I don’t get feedback I don’t know what to do,” suggesting feedback provides her with insights into her current state of thinking and the clarity of the content she expressed.

**Conceptual Evolution.**

The use of the word, “evolution,” describes the maturation of students’ ideas while suggesting the opportunity to continue gaining knowledge and mathematical experiences to further enrich their understanding. A student who reaches conceptual evolution expresses logical thinking, constructs significant mathematical arguments, and uses the language of mathematics, both symbolic and descriptive, in meaningful ways. This does not, however, imply that students’ expressions or ideas are flawless.
example, Haily responds to the essential question, “Where are systems of equations in the real world and what do they mean,” by describing her understanding of how to find a solution to a system of equations.

1. The substitution method involves solving one of the equations for one variable and then substituting the resulting expression into the other equation (Haily, Written Response Draft 3, May 18, 2011).

The succinctness of Haily’s definition suggests she referred to her textbook to borrow words that she felt would assist her in presenting her understanding with clarity. Her direct approach indicates she wanted to distinguish this approach from other possible methods used to solve a system of equations. This is followed by a description of how she understands the concept of substitution.

3. If both equations are in intercept form, which is $y = a + bx$, then you can just take the value of ‘$y$’ from one of the equations and substitute it in for ‘$y$’ in the second equation. However, if the equations are in standard form, $ax + by = c$, then you must first convert the equation into intercept form and then substitute the value of ‘$y$’ and plug it in for ‘$y$’ in the second equation (Haily, Written Response Draft 3, May 18, 2011).

Haily demonstrates her conceptual and procedural understanding with phrases such as, “plug it in,” in line 6, implying one value can replace another and still maintain the relationship between the left and right sides of the equation and, “first convert the equation into intercept form,” in lines 5 and 6, suggesting the symbolic manipulation of one equation into its equivalent form. Haily supports this discussion by describing
significant connections between the context, “Clifford and Emily Elizabeth are co-owners of ‘Very, Very Large Pet Store,’ and her mathematical model.

First, you need to define your variables (x, y). ‘X’ is the number of dog toys and ‘y’ is the number of collars. This is because you want to find out how many of each product was sold and also you have two variables—dog toys or collars. Your system of equations would be: \( x + y = 220 \) and \( 12x + 5y = 2,010 \). After you have solved for ‘x’ and ‘y’, you should’ve gotten \( x = 130 \) and \( y = 90 \). This means that Clifford and Emily Elizabeth sold 130 dog toys and 90 dog collars.

\[
\begin{align*}
\text{Rates:} & \quad \text{dog toy: $12 per} \\
& \quad \text{collar: $5 per} \\
\end{align*}
\]

\[
\begin{align*}
x & +1y = 220 \\
12x + 5y & = 2,010
\end{align*}
\]

\[
\begin{align*}
x + y & = 220 \\
y & = 220 - x
\end{align*}
\]

You can now substitute the value of ‘y’ from this equation into the value of ‘y’ for the other equation.

\[
\begin{align*}
12x + 5(220 - x) & = 2,010 \\
12x + 1,100 - 5x & = 2,010
\end{align*}
\]

\[
\begin{align*}
7x + 1,100 & = 2,010 \\
7x & = 2,010 - 1,100 \\
7x & = 910 \\
7x & = 910 \div 7 \\
x & = 130 \\
130 & \text{ dog toys}
\end{align*}
\]

(Haily, Written Response Draft 3, May 18, 2011).
Haily suggests there is a logical reason supporting her mathematical decisions. For example, her use of the word, “because” in line 9 suggests that what follows, “you want to find out how many of each product was sold,” is her rationale justifying the definition of her variables $x$ and $y$. However, this sentence in its entirety, “This is because you want to find out how many of each product was sold and also you have two variables—dog toys or collars,” in lines 9 and 10 illustrates her conceptual understanding is still in formation. The first part suggests the variables represent the number of dog toys and collars but the second portion refers to the dog collars and toys themselves. While this could signal faulty writing, it also illustrates the way in which she conceptualizes the meaning of the variables suggesting, through her inconsistency to describe them in the same way that her conceptual understanding of them is still evolving.

She continues to share her rationale by establishing specific connections between her mathematical representation and the context of her scenario. She accompanies the mathematical sentence, “$12x + 5y = 2,010$,” in line 11, with the phrase, “Rates: $12$ per dog toy $5$ per collar,” in line 14, to illustrate the significance of her use of numbers 12 and 5 on the left side of the equation and “Total amount of money earned from selling dog toys and collars,” in line 15 to support her use of the number 2,010 on the right. Haily also points to specific procedural manipulations with phrases, “You can now substitute the value of ‘$y$’ from this equation into the value of ‘$y$’ for the other equation,” in line 17 to describe her manipulation the equation, “$x + y = 220 \Rightarrow y = 220 - x$,” in line 16. This systematic inclusion of Haily’s rationale reveals her thinking and implies the evolution of both conceptual and procedural understanding allowing her to justify her model of the situation.
Implications Regarding Student Thinking

The intent of this model was not to evaluate a student’s ability but rather to describe the progression of their conceptual maturation over the course of three written drafts to the same open-ended prompt. Sources that contributed to students’ formation of ideas as well as the direction of their thinking were revealed through their written responses. For example, Jolly’s responses to the open-ended prompt, “How can a mathematical situation be described as a recursive sequence,” illustrates how the model suggests the progression of his conceptual understanding over the course of his writing on this theme.

Jolly’s initial response (draft 1) to the essential question, “How can a mathematical situation be described as a recursive sequence,” is captured this sentence, “A recursive consist of a starting number & how that number changes (add or subtract) [sic]” (Jolly, Written Response Freewrite, January 25, 2011). It is difficult to determine the specific mathematical term he intended to express in use of the word, “recursive,” but his vagueness suggests he is unfamiliar with the concept and is relying on the prompt to for key words. Jolly’s failure to describe the phrase, “starting number,” further points to the tentativeness of his understanding. His use of the word, “changes,” to describe what happens with the “starting number,” is unclear. Change could refer to creating a new recursive sequence by altering the starting number or it could suggest the number following the starting number is based upon manipulating the previous number. Jolly’s discussion suggests his conceptual progression is in the knowledge acquisition stage as the vagueness of his discussion could benefit from seeking a primary or secondary source of information to help him express himself.
A maturing in students’ thinking is a natural consequence of more exposure and interaction with a particular concept. This change is reflected in Jolly’s second draft through his attempt to identify situations that illustrate the concept.

A recursive routine consists of two things. First, a number that starts the pattern (starting number) and second, the rule of the routine (add, subtract…etc.)… For example, time is a recursive routine. If I was to say that each minute is a stage and there are sixty seconds in each minute then you can relate the situation to the description. Your recursive rule would be to add sixty seconds to the previous step. Instead of counting the seconds in 30 minutes you could use this recursive routine to get the answer. A recursive routine can go on forever and you can have an infinite amount of stages. (Jolly, Written Response, January 30, 2011).

Jolly expresses the messiness of his thinking in the way he attempts to justify why he believes time is recursive. He initially states, “each minute is a stage and there are sixty seconds in each minute,” in lines 3 and 4 suggesting he would like to identify the total number of seconds after $x$-number of minutes have passed. It is difficult to understand his intent, however, from the statement, “Instead of counting the seconds in 30 minutes you could use this recursive routine to get the answer,” in lines 5 and 6. There are several points of confusion within this statement: a) what does he mean by, “counting seconds?”—it is unclear if he is referring to the physical action of counting each individual second or some other method of counting seconds; b) why did he choose, “30 minutes,”—was this an arbitrary amount of time or was this in reference to some context he previous failed to mention; and c) how does he intend to use his, “recursive routine,”
to, “get the answer?” Jolly’s failure to express clarity in his thinking suggests he is mulling over the working knowledge he possesses.

Jolly’s third draft suggests he continues to mull over his working knowledge as he fails to describe significant relationships between concepts and decontextualize the context.

1 Recursive sequences are patterns that generate data sets. Some of the data sets can be represented as a rate of change. …A car traveling at the same pace can be described in an intercept equation. An intercept equation is \( y = a + bx \). (Jolly, Written Response Draft 3, February 22, 2011).

Jolly’s naming of several concepts in succession, “rate of change,” in line 2 and “intercept equation,” in line 3, implies his assumption of an existing relationship by making mention of these concepts to describe the same situation. His failure, however, to explain how these terms, “recursive sequences,” line 1, “rate of change,” in line 2, and “intercept equation,” in line 3 are related, suggests his understanding of these concepts has yet to mature beyond working knowledge. This is echoed in his use of the phrase, “a car traveling at the same pace,” in line 2 pointing to his belief that a rate of change is constant and can be expressed by one of the variables in the equation, “\( y = a + bx \),” in line 3.” Once again, his failure to describe how his equation reflects this situation by decontextualizing the situation indicates his need to continue to mull over his working knowledge of the concepts.

Students convey their mathematical understanding through their deliberate choice of words used to express their thinking. The use of the writing process (National Writing Project & Nagin, 2006) reveals the changes students make in their thinking through their
writing. These characteristics of students’ thinking that emerge from each draft are
described through the indicators of the model, thus pointing to the potential stage of their
conceptual maturation.

**Limitations**

The limitations of the study address situational and methodological factors that
have impacted the findings and generalizability of the study. These areas consist of the
following: a) the teacher also serves as the primary researcher; b) the sample was
purposely chosen rather than randomly chosen; c) triangulation was limited to two
primary sources of data; d) students may have been unsure of how to address or interpret
the feedback they received; and e) the meaning suggested by the students’ written and
oral responses as determined by my interpretation of their words.

**Teacher-as-Researcher**

In this study, I was both the teacher and the primary researcher, which was both
an advantage and a limitation. It was advantageous because I was fully immersed in the
study, could act as the means of data collection, and had developed a trusting relationship
with the students, thus enabling her to gain access to their perspective. However, being
both a teacher and a researcher within the same classroom creates limitations. One can
argue that being a teacher-researcher is a conflict of interest whereby the researcher
becomes too involved in the classroom experience and then tries to create the perfect data
for the research. For example, the teacher may intentionally or unintentionally use her or
his influence to shape the classroom experience or influence students in particular ways
that may have the potential to influence the data collected. Keeping the teacher and the
researcher roles distinct yet balanced is challenging because it is easy to get caught up in one role or the other.

To address this limitation, all students participated in the same writing activities throughout the school year. Each student had the opportunity to receive feedback from the teacher on his or her second draft. The participants of the study were purposefully chosen at the end of the school year so I could not impart any purposeful bias towards the students throughout the school year. The late determination of participants also allowed for more authentic data to be collected since the students did not know they would be asked to participate in a qualitative study.

Even with these checks in place, the balance between teacher and researcher may not have always been equal. For example, students who wanted additional feedback or clarification on their feedback came to talk to the teacher of their own volition. The teacher continued in the role of a mentor to assist and guide the student without offering specific answers but letting the student propose and participate in the discussion to further his or her ideas. Maccy’s comments illustrate this, “…definitely come in for help because a lot of the times it’s very difficult if you’re on your own trying to figure out what the bubbles [teacher’s feedback comments] mean of if it’s unclear to you…but if you come in for help at recess or in the morning or after school we could ask you…what does this mean and they could think of ideas” (Maccy, Interview, May 20, 2011). While the researcher would be concerned with the additional assistance, the teacher would welcome the idea of students asking questions and taking the initiative to better their understanding.
Participant Selection

Another limitation of this study concerns participant selection—what themes, if any at all, would emerge from a small population of eleven students? For example, would three drafts be sufficient to find emergent themes or would additional drafts be needed to support all students, regardless of their academic achievement, express their conceptual understanding? This concern arose after a finding by Grimberg and Hand (2009) suggested lower-achieving students are capable of higher-order thinking given more time to interact with the concepts. I made the decision to choose students who evidenced the capability for strong mathematical development and sound use of the written language.

However, due to the purposeful sample, the model may have limitations regarding its applicability to certain groups of students. These groups include: a) students where English is a second language (ESL); b) lower achieving students; and c) students who have mild or moderate learning disabilities that have been mainstreamed. The generalizability of this study may be limited to mainstream students who are higher achievers based on the population chosen to participate in the study.

Data Sources

A third limitation is the number of primary data sources used for triangulation. Triangulation suggests that the emergent theory is grounded in more than one source of data and the data sources support each other’s findings. More sources of data that point to similar conclusions might lend additional credibility to the emergent theory. In this study, there were two primary sources of data: a) students’ written responses to seven open-ended prompts, three per prompt for a total of twenty-one responses, gathered over the course of the school year; and b) one-on-one interviews that I conducted with each
Although these two data sources were compared and the emerging trends were supported by each source, an additional source would provide further validation to the resultant emergent theory.

The following are examples of additional data sources relevant to this study: a) asking students to think-aloud before and/or during drafting their written response to hear their idea generation and construction of meaning—“I talk to my tablemates to see what they’re writing,” (Tutty, Interview, May 24, 2011); b) students’ reflections on how writing assisted them in preparing for an upcoming chapter assessment in the form of a quiz or test—“This year it really helped me because I went from a C average for my tests and projects to an A so it really shows how far I’ve come and my improvements for writing and yeah, it gives me a better understanding too,” (Cissy, Interview, May 17, 2011); and c) analyzing samples of writing produced using the writing process (National Writing Project & Nagin, 2006) in a different content area to determine if the emergent theory on idea maturation is similar—“Well, like in social studies when we write essays about people we’d have to know about them and how their stories fit in with the timeline like the civil war or something,” (Kitty, Interview, May 26, 2011). These data sources would provide additional data to verify the emergent findings based on the collection of student responses to open-ended mathematically-themed prompts.

**Student Interpretation of Teacher-to-Student Written Feedback**

A fourth limitation resulted from students’ uncertainty about how to interpret and address the content-based feedback they received. This concern arose as students took the initiative to ask for teacher assistance to interpret and understand the feedback they received; additionally students also suggested they asked each other for assistance. Based
on these observations, I hypothesized that students’ difficulties stem from the following:

a) insufficient working-knowledge of the concepts of mathematics—“I just had to ask the [previous] teachers and stuff, and depends on what they said…like they just tell us…the formula and they didn’t really tell us why it was like that,” (Haily, Interview, May 23, 2011);
b) misinterpretation or failure to understand what the feedback is asking of them—“I get a little confused on [sic] but when I talk to you about it I get more understanding,” (Kitty, Interview, May 26, 2011);
c) hesitancy of how to address the content the feedback was referring to—“Before there really wasn’t any writing you had to do, it was just take tests and study for quizzes,” (Cissy, Interview, May 17, 2011);
d) difficulties using feedback to construct meaning in mathematics—“Yeah, it’s hard to word it sometimes cause you’re trying to put math into English,” (Tohly, Interview, May 12, 2011).

This limitation could also be in part due to how the feedback was presented to the students. The use of content-based language may have been difficult for students to interpret if they had a tentative grasp on mathematical ideas. Additionally, students may have had difficulty addressing the analytical tasks suggested by the feedback if their own understanding failed to offer the knowledge needed to address the task. Although feedback has the potential to assist students in saying what they mean to say (Fulwiler, 1987; National Writing Project & Nagin, 2006), students who fail to use it are unable to reap any benefits while constructing meaning.

Construction of Meaning

The final limitation stems from the language students use to express their ideas and my interpretation of their constructed meaning. Meaning is constructed by the reader
based on the writer’s use of language (Mayher, Lester, and Pradl, 1983). Students whose grasp of language may be limited due to a variety of factors such as English as a second language or low achievement writing may fail to communicate their intended meaning with limited knowledge of words, the meaning expressed by the word, and the word’s connotations, both literal and figurative. Thus, although students deliberately choose the language they want to use to articulate their ideas, a failure to have a firm grasp on language itself, regardless if it’s mathematical in nature, may result in an express that may not accurately reflect their intended meaning.

Additionally, the meaning students’ construct is also restricted by my analytical interpretation of their words. My interpretation is based on the words students use to convey their ideas limited by my own understanding of how students use language to construct meaning. For example, I may misinterpret a students’ meaning due to their use of colloquial or dialectical language and slang not fully understanding the connotation they are attempting to express. Furthermore, as a teacher I can sometimes make the assumption that a student is expressing a particular idea based on previous conversations or discussions in school as opposed to viewing their meaning strictly from the words they use to express themselves.

To address this limitation I analyzed students’ written responses in two unique ways, both individually and as a collective set and held one-on-one interviews with the students as a way to gain more insight into their experiences and, if necessary, follow up on something they wrote to better verify or refute a particular emergent theme. While there were only two primary sources of data, I could also rely on my field notes and observations to help me to reflect on the possible meaning I was constructing within the
context of what I knew about the school conditions the student was experiencing during
the process of writing or my own interaction with them. The combination of data sources
assisted me in the validation of my interpretation of students’ written and oral expression.

**Suggestions for Future Research**

This study was concerned with how writing reveals changes in students’ algebraic
reasoning when responding to an open-ended prompt three times over the course of a
particular mathematics theme. It was not concerned with whether or not writing caused
changes in students’ meaning making; writing was used so I could create the conditions
used to describe the changes students made in their thinking and mathematical reasoning.
An area of further study would concern if/how the process of writing, along with all of its
elements, such as drafting, revising, and feedback, affected their growth.

Another area of inquiry that was not addressed in this study is the role of peer
feedback. Peer feedback refers to the content-based written feedback which can come
from an individual’s peers. Effective peers who provide feedback to one another to
determine the presence and accuracy of content material should have some working
knowledge of the content being assessed. Peers can also assume the role of a guide,
offering content-based suggestions in the form of questions and statements. Furthermore,
students who provide content feedback to others may benefit more from being a co-
constructor of meaning (Crinon & Marin, 2010) than a recipient because they must
actively participate in recognizing logical mathematical arguments or identifying
misconceptions in another’s thinking. As an additional benefit, feedback providers may
include more diversity in how they approach revisions in their own thinking and focus
less on writing mechanics such as grammar, spelling, or punctuation (Crinon & Marin, 2010).

This study was concerned with describing changes in students’ algebraic reasoning recorded in individual draft responses to open-ended prompts that reveal their conceptual understanding. Another area of research to further understand the role of writing in revealing the development of students’ understanding is to investigate the repetition of the process of writing. For example, an analysis of successive freewrites may reveal trends in students’ approach towards thinking about mathematics throughout the school year by describing changes in the characteristic between the previous freewrite and the succeeding one. A similar analysis would also be performed on students’ second and final drafts. A second approach investigating the repetition of the process of writing would focus on a describing collective drafts as they are sequentially written throughout the school year. Discovering emergent trends over the course of the school year may provide additional insights into how students’ thinking changes when actively engaged in using the process of writing regularly.

Furthermore, while this study was interested in using writing to reveal students’ understanding to their reader, another study could focus on what does writing reveal to students about their own thinking and understanding of concepts. For example, a research question could focus on what writing reveals to students about their own meaning making strategies, idea generation, and/or conceptual understanding as they review their previous draft prior to writing their next draft. During the interview process, students could be asked to discuss their thinking leading to a particular draft and reflect upon what they believe the draft reveals about their thinking and conceptual understanding.
Recommendations

Students’ written responses to a series of mathematically-themed open-ended prompts over the course of the school year point to the idea that writing has the potential to reveal their thinking and indicate the evolution of their conceptual understanding over the course of three written drafts. They suggested seeing their thoughts on paper made them more aware of their own thinking and understanding, especially after receiving feedback prompting them to reveal everything they were capable of with regard to that topic. These positive findings are similar to researchers (Akkus & Hand, 2005; Kagesten & Engelbrecht, 2006) that also found that the process of writing has the potential to encourage students to construct mathematical reasoning and reveal their mathematical understanding. Based on the emergent findings of this study, the following recommendations are suggested:

1. Students should have the opportunity to use the process of writing to construct content based, including mathematical, arguments with the emphasis on constructing meaning to supplement their everyday classroom experiences.

Students who use writing to reflect upon their own ideas and revise them based on their experiences have the opportunity to develop confidence in their understanding, acquire content specific vocabulary, and use the language particular to that content area to construct reasonable arguments. The emphasis of the use of writing should be to reveal the process of thinking or pause for reflection and revision of new knowledge as opposed to communicating prior knowledge. During this process students should be allowed some degree of creativity to construct their own examples and/or non-examples using them to support their content specific reasoning. Moreover, personally connecting the content
area to the world outside of the classroom can assist students in creating meaning through particular content area concepts as it is now integrated into a larger context.

These ideas are echoed by students in mathematics who participated in this study. Tobby states, “Think deeply, don’t, don’t [sic] just think equations and everything. Think about word problem[s] and what you can do with the formulas and…apply it to whatever you can” (Interview, May 24, 2011). Tobby implies through this statement that thinking should go beyond the textbook presentation and classroom experience and extend to practical applications and real life contexts and situations. He suggests making connections, in his use of the phrase “apply it to whatever you can,” makes learning meaningful.

2. The writing process should be utilized fully, including drafting and feedback, to assist students as a means of investigating and refining their own thinking.

The writing process assists students in planning, re-visioning, and developing the clarity of their intended meaning through active language choice and opportunities to revise. The use of the writing process and drafting supports higher order thinking and active construction of knowledge as writing becomes a tool for learning by supporting metacognitive processes. Both content and word choice are considered in the articulation of ideas over the course of time as students become aware of their thinking by being able to visualize it on paper.

This is echoed by Haily who states, “I guess we just write what we think is right and when we get it back we have like those correction things and we write another draft after and turn it in with the final. I think it helps because you understand what you have to do and in the beginning if you didn’t know anything at all then when you draft you
begin to start thinking and the gears are turning,” (Interview, May 23, 2011). Time must also be allotted to produce subsequent drafts allowing students to describe relationships, wrestle with understanding, present inquires, reflect upon content, and acquire content vocabulary. Greater conceptual maturity can occur when students receive feedback as part of the writing process and are asked to produce a draft based upon the comments they received.

3. Teachers who provide feedback in the form of teacher-to-student written feedback with emphasis on idea development can encourage students to revise their thinking in meaningful ways.

Feedback, when it is received from someone who is considered to be more knowledgeable in the content-area, can assist students in constructing meaning. Receiving feedback can be meaningful when it shifts the role of the individual giving the feedback from evaluator seeking a specific answer to a guide indicating a more appropriate direction. Feedback is especially appreciated by students when it is content-specific and relevant to aiding them in clarifying, elaborating, or considering alternative perspectives.

This is echoed by Haily as she discusses how she feels about receiving feedback:

1 Feedback is important because…when we get it back we know you have like an idea and you know…what they want you to explain more, what they want you to go in depth with more. …I think [open ended questions] it’s because that way
2 instead of just specifically defining what it is like, what it is in the book you really
3 have a chance to explain why you think this is true. …I feel like the open ended
Haily suggests through the phrase, “they really do help you to explain why,” in line 6 that feedback that is content based and open-ended prompts her to reveal everything she is capable of expressing regarding a particular topic or idea. This is further implied in lines 4 and 5, “instead of just specifically defining what it is like, what it is in the book you really have a chance to explain why you think this is true,” as she suggests feedback gives her the opportunity to explain her ideas and elaborate on her thinking as opposed to borrowing someone else’s words and phrases from a textbook or other source. Haily suggests feedback of this nature is beneficial for her as a student, giving her the opportunity to clarify her intended meaning and also prompt her to fully reveal the full extent of her content knowledge and conceptual understanding.

4. Open-ended, content-based writing prompts can encourage the development of students’ understanding and content language use beyond a show-and-tell approach to include higher levels of comprehension and critical thinking.

Specific, yet open-ended tasks asking students to use writing in particular content areas can shift the focus of writing from transferring knowledge to using writing to encourage higher-order thinking such as evaluating, analyzing, or synthesizing information in new and relevant ways. Tasks which are open ended allow students to use writing in creative ways to construct content specific arguments in support of their numerical models, graphs, and equations. This is echoed by Cissy who states, “I think that there’s an advantage for like an open-ended question [be]cause it’s like you can use your creativity and how you can apply what you learned” (Interview, May 17, 2011).
When prompted, students can use the words within the prompt to gain access to entryways that allow them deeper access to information for the purpose of constructing new understanding and bridging prior knowledge with new knowledge.

Classroom teachers, regardless of the content area they teach can incorporate these recommendations into their classroom at a level they feel comfortable with. For instance, classroom teachers can shift one writing assignment from a closed prompt to an open ended prompt giving students the opportunity to express their ideas and opinions on the topic. Or teachers could incorporate the writing process but on a smaller scale by having students write two drafts, the first of which could receive content based feedback. In this way, students would have the opportunity to revise their ideas and the teacher would have the opportunity to preview their students’ thinking. These recommendations are applicable to classroom teachers who are willing to incorporate the process of writing, or some aspect of it, into their classroom.

**Summary and Conclusion**

The purpose of this study was to reveal and describe changes in students’ algebraic reasoning as they responded to open-ended prompts utilizing the process of writing. Three research questions drove the study:

1. How does writing reveal a student’s algebraic reasoning and knowledge of algebraic concepts?
2. In what ways does writing, through the revision of multiple drafts, expose the development of algebraic reasoning?
3. In what ways does teacher-to-student written feedback, provided during the process of writing, prompt students’ written revision of algebraic reasoning?
Findings from an analysis of students written documents and transcripts from one-on-one interviews suggest the development of students’ conceptual understanding follows a non-linear progression that follows the following three stages: acquiring knowledge, mulling over working knowledge, and conceptual maturation. Students may spend as much or as little time within the three stages as needed, producing multiple drafts if necessary, before suggesting through their writing movement to the next stage. Although feedback is not actually a part of the model itself, it is an important component in the writing process (National Writing Project & Nagin, 2006) and must be considered as a transition between drafts that can aid students in shifting the way in which they think about an idea and present it to reveal the full extent of their understanding.

Students suggested their mathematical understanding benefited from the opportunity to use the writing process to address topics presented by open-ended prompts. Writing revealed students’ thinking as their ideas matured from draft to draft, shifting from information giving to creating mathematical arguments supported by logical reasoning. Students suggested writing encouraged them to recognize and reveal their mathematical ideas and think about how those ideas could be used to address contexts beyond mathematics class that would illustrate the appropriate application of a particular concept to describe or model a situation.

Furthermore, students reported feedback assisted them by suggesting areas of misconception or gaps in their thinking by questioning the meaning they implied in their written thoughts. Offering feedback in the form of teacher-to-student written feedback afforded students the opportunity to return to those comments at any time to continue the dialogue with the teacher even if the teacher was not physically present suggesting the
students could benefit from think-time before altering their actual meaning as revealed on paper. Overall, students suggested that writing does have a place within mathematics as it encourages students to reveal their thinking by creating logical arguments and reasoning to support mathematical statements and models. This is summed up by Haily who states, “…Math isn’t a spectator sport, it’s something you have to do” (Haily, Interview, May 23, 2011).

As for my own personal practice, this study has given me the opportunity to investigate the use of writing and what it reveals about students’ thinking and construction of ideas in a curriculum other than English and/or Language Arts. I have always suspected that what students submit on a formative assessment may not always reveal the extent of their thinking and understanding. For example, as a teacher of mathematics I have always wondered if the student who aces a mathematics test can truly explain to someone else how and why they did what they did. The findings of this study confirm for me, personally as a teacher, that the depth or nature of students’ understanding is not always apparent from the test responses they give.

As a teacher I have gained additional perspective on how writing can be incorporated as a regular part of the curriculum within a non-English or Language Arts type of class. For example, I can use writing to capture at-that-moment thinking by asking students to engage in quiet and thoughtful reflection with an open-ended prompt. I believe it can also be used as formative assessment to reveal students’ understanding given a particular topic.

Moreover, this study has given me pause to reflect on how writing can be incorporated for the purpose of understanding how students process their learning.
experiences as opposed to solely using writing to communicate the product of their learning in a summative assessment. While it may not always be possible for me to engage students in the process of writing, writing itself is a powerful way to have one-on-one communication from student to teacher. In addition to writing, providing students with content based feedback, even if it is only one or two comments, gives me the opportunity to suggest areas where they might want to reconsider their thoughts or dig deeper to clarify their meaning and reveal everything they are capable of expressing.

On a personal note, while the thought of collecting another set of papers on my desk is simply abhorrent, not to mention one of my least favorite aspects of teaching, I can’t help but value how writing gives me an insider’s perspective by providing a new level of awareness and insight into revealing students ideas. So as an advocate of writing, bring on the paperwork, and let the thinking flow.
Appendix A: Open-Ended Writing Prompts

Chapter 0: Fractions and Fractals

On Wednesdays past teachers who have been on yard duty during big recess have exclaimed, “This is chaos!” Write a business letter to Ms. Tangerine either supporting or disproving the teachers’ claim.

- Explain, using the concepts found in chapter 0, why there is or why there is not a pattern to chaos. Provide an example to justify your reasoning.

Use your draft with comments from Ms. Strawberry to help you to write your letter to Ms. Tangerine. Remember, Ms. Tangerine does not know anything about chapter 0 so it is important that you are clear and specific. The comments and feedback should help you better create your argument.

Requirements

_____ Content: Did you state your position: “Is there or is there not a pattern to chaos?” and explain why. How well did you incorporate the concepts and vocabulary from chapter 0? Did you include an example? A non-example? How convincing was your mathematical explanation? Did you explain your ideas well? Did you use the correct mathematical vocabulary?


_____ Business Letter Format Requirements

___ Heading ___ Salutation ___ Signature

___ Correct date ___ Body ___ Typed signature

___ Inside address ___ Closing

_____ Outside Envelope requirements (letter sized envelope)

___ Letter is tri-folded ___ Destination address

___ Addresses are clearly written/typed ___ Correct postage (DRAW your own stamp)

___ Return address

10 Names used in these examples have all been changed to protect the identity of the individual(s) who wrote the sample.
SAMPLE LETTER FORMAT:

Ms. Strawberry
Mission School
1234 Avocado Street
Pineapple, Papaya 54321
September 13, 2010

Homeroom Apple-Banana
Mission School
1234 Avocado Street
Pineapple, Papaya 54321

To Whom It May Concern:

I would like to provide you with some hints for successfully writing your business letter.

Think creatively yet critically. You will need both creativity and critical thinking to tie together all that you have learned from chapter 0 to create a sound argument and write to Ms. Tangerine.

Address the situation and clearly state whether you believe there is a pattern to chaos or not. Do not be wishy-washy.

Create one example that illustrates your position. Explain your example by using the concepts and vocabulary involved in chapter 0.

Create an illustration that supports your position. Your illustration can be on its own page but must come with an explanation. Be sure your illustration is supported by the concepts and vocabulary from chapter 0.

Create a non-example. Once again, use the concepts and vocabulary from chapter 0 to demonstrate why this represents a non-example.

Be mindful of your spelling, grammar, punctuation, and capitalization. Say what you mean to say and be specific.

Create a reasonable solution for Ms. Tangerine. Aliens from outer space cannot kidnap half of the student population and return them at 2:30, just in time for dismissal.

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11 Permission has been granted by the individuals whose writing samples have been used.
Thank you for reading the hints listed in this letter. If you have any further questions, please see me as soon as possible.

Sincerely,

Ms. Strawberry

p.s. Good luck.
SAMPLE LETTER:

Ms. Strawberry
Homeroom Apple-Banana
Pineapple, Papaya 54321
September 15, 2010

Ms. Tangerine
Mission School
1234 Avocado Street
Pineapple, Papaya 54321

Dear Ms. Tangerine:

It has been noted that during professional days, big recess can be chaotic. Students from first through eighth grade are all playing together at one time. However, be reassured that there is a pattern to chaos and what teachers see is merely a snapshot of the pattern.

Let me explain why there is a pattern to chaos. Although the definition of chaos is lack of organization and complete disorder, we have discovered in Algebra that it is possible to find a recursive pattern to chaos given enough time or repetitions of the pattern.

For example, when playing golf, an amateur like myself has very poor skill in hitting the golf ball. Hence, chaos, or disorder, emerges since the ball does not always fly in a straight line. This is the chaotic element. However, no matter where the ball lands, that is the new starting point in which I will hit the ball again. And if I hit the ball the same way each time, a pattern emerges despite my lack of skill. I also know that the goal is to get the golf ball into the hole. The hole serves as an attractor, or the result of many repetitious procedures that may appear individually to be chaotic. Eventually, after many, many recursive procedures hitting the ball, I will get closer and closer to the hole, or the attractor. This demonstrates that there is a pattern to chaos.

Many people will argue that there is no pattern to chaos. For example, when going to Las Vegas, many people who try to play the slot machines to win a jackpot leave empty handed. They claim that even though they performed the same procedure over and over again – they put in a five dollars and pull the lever 10 times – they still did not find a pattern to their winnings or eventually hit the jackpot. This is something similar that occurred in an algebra experiment that we did in class. Even though we completed the same procedure forty times by rolling a dice, finding the midpoint between our starting value and one of the vertices of a triangle, no pattern emerged and we were left with a collection of dots. However, let me reassure you that if you do enough recursive procedures over a long period of time, a pattern will emerge despite what may appear to be initial chaos.
Therefore, based on what I know about patterns, recursive procedures, and chaos, this is my recommendation to you: during recess you should watch which grade level has the most students playing in the courtyard. The entire grade level should then walk to the park where they have more space to play. This also gives the other grade levels more playing room on the courts at school.

Thank you for reading my letter.

Sincerely,

Ms. Strawberry
Chapter 1: Data Exploration

You are a “merchant of cool” – someone who looks for and markets the newest trends for young people your age. As a “merchant of cool” you must consider this essential question very carefully:

- **In what ways can analyzing and visually displaying data influence your view of the world around you? Provide an example to justify your reasoning.**

Based on what you know from chapter 1, discuss the essential question. Specify TWO ways data analysis and visual displays of data can influence your view of the world around you. Justify your reasons with real-world examples incorporating various data analysis techniques and visual displays.

Outlaw Words: words you cannot use in your explanation

- Easier
- Better
- Harder
- Stuff
- thing
### Rubric

<table>
<thead>
<tr>
<th>Score</th>
<th>Mathematical Reasoning</th>
<th>Mathematical Proof</th>
<th>Communication</th>
<th>Conventions</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>The writing:</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>- Accurately and concisely defines mathematical vocabulary in precise terms using the writer’s own words.</td>
<td>- Discusses a relevant ORIGINAL example justifying the real world application of concepts, equations, tables and / or graphs.</td>
<td>- Contains words and phrases that accurately demonstrate the language of mathematics</td>
<td>- Is almost error-free, generally containing correct spelling, punctuation, capitalization, grammar, and paragraph breaks.</td>
</tr>
<tr>
<td></td>
<td>- Thoroughly discusses and explains relationships between concepts.</td>
<td>- Discusses a relevant ORIGINAL non-example justifying an inappropriate application of concepts, equations, tables and / or graphs.</td>
<td>- Organizes ideas in a logical fashion to enhance the theme</td>
<td>- Requires limited editing.</td>
</tr>
<tr>
<td></td>
<td>- Accurately describes connections between relevant equations, figures, tables, and / or graphs to support concepts.</td>
<td>- Contains an introduction, body and conclusion that adds interest and is appropriate for purpose and audience.</td>
<td>- Requires minimal editing.</td>
<td></td>
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<tr>
<td>3</td>
<td>The writing:</td>
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<tr>
<td></td>
<td>- Satisfactorily defines mathematical vocabulary using the writer’s own words in general terms.</td>
<td>- Discusses a satisfactory example illustrating the real world application of concepts, equations, tables and / or graphs.</td>
<td>- Satisfactorily uses the language of mathematics but in general terms.</td>
<td>- Contains few errors in the realm of spelling, punctuation, capitalization, grammar and paragraph breaks.</td>
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<td></td>
<td>- Satisfactorily discusses relationships between concepts with predictability.</td>
<td>- Discusses a satisfactory non-example illustrating an inappropriate application of concepts, equations, tables and / or graphs.</td>
<td>- Satisfactorily organizes relevant ideas to develop the theme.</td>
<td>- Requires minimal editing.</td>
</tr>
<tr>
<td></td>
<td>- Satisfactorily develops connections between relevant equations, figures, tables, and / or graphs to support concepts.</td>
<td>- Contains an introduction, body and conclusion that are predictable for purpose and audience.</td>
<td>- Contains an introduction, body and conclusion that are predictable for purpose and audience.</td>
<td></td>
</tr>
</tbody>
</table>
| 2 | The writing:  
- Somewhat defines mathematical vocabulary.  
- Somewhat developments concept relationships.  
- Somewhat discusses equations, tables, figures, and / or graphs to support concepts. | The writing:  
- Contains an underdeveloped example with limited discussion of the real world application of concepts, equations, tables, and/or graphs.  
- Contains an underdeveloped non-example with limited discussion of the inappropriate application of concepts, equations, tables and / or graphs.  
- Somewhat uses the language of mathematics but it may contain flaws.  
- Presents several ideas that loosely support the theme.  
- Contains an introduction, body, and conclusion that are underdeveloped for purpose and audience. | The writing:  
- Somewhat uses the language of mathematics but it may contain flaws.  
- Presents several ideas that loosely support the theme.  
- Contains an introduction, body, and conclusion that are underdeveloped for purpose and audience.  
- Contains errors in spelling, punctuation, capitalization, grammar, and paragraph breaks.  
- Requires moderate editing. | The writing:  
- Contains errors in spelling, punctuation, capitalization, grammar, and paragraph breaks.  
- Requires extensive editing. |
|---|---|---|---|---|
| 1 | The writing:  
- Minimally or does not define mathematical vocabulary mentioned.  
- Mentions concepts by name but does not discuss relationships.  
- Lists or does not make mention any equations, tables, figures, and / or graphs. | The writing:  
- Includes a flawed example OR non-example with inaccurate mention of the real world application of concepts, equations, tables, and / or graphs.  
- Does not provide an example OR non-example at all. | The writing:  
- Minimally uses the language of mathematics or does not use it correctly  
- Minimally includes few, if any, ideas that may or may not support the theme.  
- May lack an introduction and / or conclusion.  
- Contains multiple errors of a variety of types including spelling, punctuation, capitalization, grammar, and paragraph breaks.  
- Requires extensive editing. |
Chapter 2: Proportional Reasoning and Probability

Essential Question:

- Compare and contrast fractions and ratios. Justify how they can be used to describe situations that you’ve experienced outside of math class.

Outlaw words:

- Easier
- Better
- Harder
- Stuff
- Thing

- Come, comes, coming, came
- Go, goes, going, went, gone
- Very

- A lot
- A few
### Rubric

<table>
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<th>Mathematical Proof</th>
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Chapter 3: Variation and Graphs

Essential Question:

- How do quantities vary directly and inversely? Discuss and explain an example found outside of mathematics class of a situation that illustrates direct and inverse variation. Justify why the situation are direct or inverse variation or neither (non-example) using the equations and characteristics of direct and inverse variation.

Outlaw words:

- Easier
- Better
- Harder
- Stuff
- Thing
- Come, comes, coming, came
- Go, goes, going, went, gone
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Chapter 4: Linear Equations

Based on what you know from Chapter 4, discuss,

- In what ways, a mathematical situation be described as a recursive sequence? Create an example and a non-example from your life and justify why the situations can be or cannot be described as recursive sequences.

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- Stuff
- Thing, things
- Come, comes, coming, came
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**Student Examples from previous essays:**

**Communication: uses the language of mathematics**

Fractions and ratios can be expressed in many different ways. They are similar in some ways, while different in others. A fraction is when you divide a whole into equal parts. Fractions are expressed as numerators and denominators. The denominator is how many equal pieces the whole has been divided into. The numerator is how many pieces you have out of the whole. A ratio is when you compare two quantities that do not need to be equal. Fractions and ratios are similar; they can both be expressed with a fraction bar. They are different because a ratio is used to compare quantities while a fraction is used to divide the whole into equal parts. Three-fourths means that the whole was broken into four pieces and you have three pieces. As a fraction, two and one-half means that there are two whole pieces that were not divided. You also have one half of a whole, which means that the whole was divided into two equal pieces and you have one of those two pieces.

**Mathematical proof:**

A person’s pay is a key example of showing direct variation. Suppose you work as a babysitter and your pay is $8.00 an hour. The amount of money you earn for one hour, which is $8.00, is the constant because the pay you earn for an hour will not change. The equation for direct variation is $y = kx$, ‘$y$’ and ‘$x$’ are variables and ‘$k$’ is the constant. The ‘$x$’ variable is the number of hours you work and the ‘$y$’ variable is the total amount of money you earn (how many hours you work multiplied by the pay rate or constant equals your total amount of money). If you work for one hour, you have made $260$.
$8.00. And if you worked for 2 hours, then you'd have made $16.00. This is direct variation because the more hours that you baby-sit, your pay increases $8.00 more, and in direct variation, as one variable increases, the other increases. However, it is important to know that if you worked for zero hours or you haven’t worked at all, then you wouldn't make any money.

**Mathematical reasoning:**

Direct variation and inverse variation are opposites. In direct variation, the $x$ and $y$ values increase or decrease at a constant rate. If one of them is getting bigger, so is the other; and if one of them is getting smaller, so is the other. The variable $k$ represents this constant rate. The method of finding the value $k$ is vastly simple. You use the equation $k = \frac{y}{x}$. The Direct Variation equation is $y = kx$ or $y = k \times x$. In Inverse Variation, as the $x$ value increases, the $y$ value decreases and vice versa at a constant rate of $k$. $x$ and $y$ set off in opposite directions. The method of finding the value of the variable $k$ is also incredibly simple. You use the equation $k = \frac{xy}{k}$ or $k = y \times x$. The Inverse Variation equation is $y = \frac{k}{x}$.

Direct and inverse variations are definitely not the same. One difference between the two is the equations of both variations. The Direct Variation equation is $y = kx$, while $y = \frac{k}{x}$ is the Inverse variation equation. The equation for direct variation uses multiplication while the inverse equation uses division.

Another difference is in how they appear on graphs. A direct variation appears to always progress at a positive incline or negative decline while traveling through the
origin. It is a straight line. An inverse variation seems to always curve and sometimes traverse through the origin.
Chapter 5: Fitting a Line to Data

Think about the Olympics and other events that demonstrate human ability.

- **When is it appropriate and when is it not appropriate to create a line of fit to describe human ability and why?**

Find actual data and use it to justify when a line of fit would appropriately describe human ability (example) and when it wouldn’t appropriate describe human ability (non-example).

Discuss the best method to finding the line of fit (There are 2 to choose from in this chapter – you’ll have to decide on the BETTER method based on your data). Include a graph and your data table.
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<td>The writing:</td>
<td>The writing:</td>
<td>The writing:</td>
<td>The writing:</td>
</tr>
<tr>
<td></td>
<td>• Minimally or does not define mathematical vocabulary mentioned.</td>
<td>• Includes a flawed example OR non-example with inaccurate mention of the real world application of concepts, equations, tables, and / or graphs.</td>
<td>• Minimally uses the language of mathematics or does not use it correctly</td>
<td>• Contains errors in spelling, punctuation, capitalization, grammar, and paragraph breaks.</td>
</tr>
<tr>
<td></td>
<td>• Mentions concepts by name but does not discuss relationships.</td>
<td>• Does not provide an example OR non-example at all.</td>
<td>• Minimally includes few, if any, ideas that may or may not support the theme.</td>
<td>• Requires extensive editing.</td>
</tr>
<tr>
<td></td>
<td>• Lists or does not make mention any equations, tables, figures, and / or graphs.</td>
<td></td>
<td>• May lack an introduction and / or conclusion.</td>
<td></td>
</tr>
</tbody>
</table>
Student Examples from previous essays:

Example of using mathematical language & defining vocabulary

A recursive routine is your starting values and instructions to build a recursive sequence. The starting number is the first number in a recursive sequence. The instructions or rule is the operation you perform on the previous or starting number to get the next number in your recursive sequence. A recursive sequence is a sequence of numbers produced by a recursive routine.

Did you know that this recursive routine is also linear? It is linear because when converting data in graphs you use an intercept equation to find points you use to plot on a graph. An intercept equation looks like \( y = a + bx \). \( X \) and \( y \) are the values for a recursive routine. \( a \) is the \( y \)-intercept which is the value of \( y \) when \( x \) equals zero, and \( b \) is the rate of change. So in intercept form the recursive routine would look like \( y = 400 + 150x \) if you used this equation you would see that the outcomes would match the numbers on your recursive sequence. Do you notice that the starting number for \( y \) is the same as the \( y \)-intercept? Also that the rate is the same value you add to \( y \)? These similarities is the reason why this recursive sequence is linear.

Example

An example of a recursive sequence outside of math class would be as you grow up your height increases. From birth there are children who are born taller than others. Say that there is child number one and she is born at the height of 16 inches. Then there is child number two and he is born at the height of 12 inches. If a baby grows at a rate of 2 inches per month for 6 months, you would use a recursive sequence to find out the next
height for each child for every new month. For child number one you starting value would be 16 because that’s how tall she is from zero months old. Then for month one you would add 2 to 16 and get 18 inches and so on. Then for child number two you would start from 12 because that is the height he was born at and to get month one you would add 2 to 12 and get 14 inches. So a table of both babies would be:

<table>
<thead>
<tr>
<th>Age (months)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Child #1</td>
<td>16</td>
<td>18</td>
<td>20</td>
<td>22</td>
<td>24</td>
<td>26</td>
<td>28</td>
</tr>
<tr>
<td>Child #2</td>
<td>12</td>
<td>14</td>
<td>16</td>
<td>18</td>
<td>20</td>
<td>22</td>
<td>24</td>
</tr>
</tbody>
</table>

The table shows the monthly growth of the two babies. Each month, each baby grows by 2 inches and child number one would be taller than child number two because she was born taller than the other baby. But if you were to use a linear equation and intercept form the equation for child number one would be $y = 16 + 2x$. The number 16 represents the $y$-intercept and the number 2 represents the rate of change. The linear equation and intercept form equation for child number two would be $y = 12 + 2x$, and again the 12 equals $y$-intercept and the 2 equals the rate of change. But if you were to use a recursive sequence you would take the height of the babies at zero months, and the rule would be to add 2 to each new month.

**Example of a non-example:**

However, a recursive sequence doesn’t always create a linear equation. When making free throws you won’t always get the ball in the basket and that affects the linear equation. You make two baskets during your first two tries but on your third try you miss the basket and on your fourth try you make it in. Because you did try to shoot...
the ball into the basket you would add one to the number of times the ball was shot. But because you didn’t make it you can’t add two points. This means that there’s no longer any rule, rate of change, or a constant ratio. If you graphed these points on a graph the points wouldn’t line up to form a straight line and when there’s no straight line it proves that the equation isn’t linear. Because there’s no rule there’s no recursive routine.

**Example of a non-example:**

Sometimes the sequences aren’t recursive. For example, if I went back to my driving example and instead of measuring each day, I measured every minute in each day. This would change the relationship from being linear to not linear because each minute I could be doing something different like for example one minute I could be going 20mph and the second minute I could be traveling at a speed of 30mph. So I wouldn’t have a recursive rule or sequence because I wouldn’t be traveling at a constant rate each minute. What makes it recursive in my first example is the fact that at the end of every day I will travel the same amount of miles. But in this example if I were to measure each day by the minute it wouldn’t be at a constant rate because I could be stuck at a traffic light one minute, then the next minute I could be driving. If I were to graph this on a graph, the x-axis would be the minutes and the y-axis will be the miles. The line would not be a straight; it would increase and decrease depending on if I was traveling at a constant rate. So in conclusion, in this example, it would not be a linear relationship because I measuring by the minute instead of by the day. The difference in measuring in days than by measuring in minute is that in days by the end I would have traveled the
same distance. But in minutes I could be going at 40mph one minute then at a speed of 35mph the next. So, in life there are many recursive routines that we may not realize but they are everywhere, we just have to look.
Use the following questions to help you understand your data:

1. Graph YOUR data and draw the most appropriate line of fit (use both methods – which method is better and why?)

2. Find the slope

   a. write an equation in point slope form

   b. write an equation in intercept form

3. Where is the y-intercept?

   a. What is the real world meaning of the y-intercept?

   b. Does the y-intercept make sense in the context of the data? Why or Why not?

4. Where is the x-intercept?

   a. What is the real world meaning of the x-intercept?

   b. Does the x-intercept make sense in the context of the data? Why or why not?

5. What conclusions can you draw about your line of fit based on the situation and the data? Between what years would your line of fit make the most sense? (You may need to investigate other years besides those that you’ve graphed)
Chapter 6: Systems of Equations and Inequalities

In what ways do systems of equations and inequalities exist outside of your mathematics class? Provide an example of a system of equations and a system of inequalities from your life and justify their solution using the concepts and procedures from chapter 6. Also include and justify a reasonable non-example.

Outlaw Words:
- Easier
- Better
- Harder
- Stuff
- Thing, things
- Come, comes, coming, came
- Go, goes, going, went, gone
- Very
- A lot
- A few
- Many
- More
- Less
- Like
- Really
- Always
- Bad
- Good
- something
<table>
<thead>
<tr>
<th>Score</th>
<th>Mathematical Reasoning</th>
<th>Mathematical Proof</th>
<th>Communication</th>
<th>Conventions</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>The writing:</td>
<td>The writing:</td>
<td>The writing:</td>
<td>The writing:</td>
</tr>
<tr>
<td></td>
<td>• Accurately and concisely defines mathematical vocabulary in precise terms using the writer’s own words.</td>
<td>• Discusses a relevant ORIGINAL example justifying the real world application of concepts, equations, tables and / or graphs.</td>
<td>• Contains words and phrases that accurately demonstrate the language of mathematics</td>
<td>• Is almost error-free, generally containing correct spelling, punctuation, capitalization, grammar, and paragraph breaks.</td>
</tr>
<tr>
<td></td>
<td>• Thoroughly discusses and explains relationships between concepts.</td>
<td>• Discusses a relevant ORIGINAL non-example justifying an inappropriate application of concepts, equations, tables and / or graphs.</td>
<td>• Organizes ideas in a logical fashion to enhance the theme</td>
<td>• Requires limited editing.</td>
</tr>
<tr>
<td></td>
<td>• Accurately describes connections between relevant equations, figures, tables, and / or graphs to support concepts.</td>
<td>• Contains words and phrases that accurately demonstrate the language of mathematics</td>
<td>• Contains an introduction, body and conclusion that adds interest and is appropriate for purpose and audience.</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>The writing:</td>
<td>The writing:</td>
<td>The writing:</td>
<td>The writing:</td>
</tr>
<tr>
<td></td>
<td>• Satisfactorily defines mathematical vocabulary using the writer’s own words in general terms.</td>
<td>• Discusses a satisfactory example illustrating the real world application of concepts, equations, tables and / or graphs.</td>
<td>• Satisfactorily uses the language of mathematics but in general terms.</td>
<td>• Contains few errors in the realm of spelling, punctuation, capitalization, grammar and paragraph breaks.</td>
</tr>
<tr>
<td></td>
<td>• Satisfactorily discusses relationships between concepts with predictability.</td>
<td>• Discusses a satisfactory non-example illustrating an inappropriate application of concepts, equations, tables and / or graphs.</td>
<td>• Satisfactorily organizes relevant ideas to develop the theme.</td>
<td>• Requires minimal editing.</td>
</tr>
<tr>
<td></td>
<td>• Satisfactorily develops connections between relevant equations, figures, tables, and / or graphs to support concepts.</td>
<td>• Contains an introduction, body and conclusion that are predictable for purpose and audience.</td>
<td>• Contains an introduction, body and conclusion that are predictable for purpose and audience.</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>The writing:</td>
<td>The writing:</td>
<td>The writing:</td>
<td>The writing:</td>
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<td>---</td>
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</tr>
<tr>
<td></td>
<td>- Somewhat defines mathematical vocabulary.</td>
<td>- Contains an underdeveloped example with limited discussion of the real world application of concepts, equations, tables, and/or graphs.</td>
<td>- Somewhat uses the language of mathematics but it may contain flaws.</td>
<td>- Contains errors in spelling, punctuation, capitalization, grammar, and paragraph breaks.</td>
</tr>
<tr>
<td></td>
<td>- Somewhat developments concept relationships.</td>
<td>- Contains an underdeveloped non-example with limited discussion of the inappropriate application of concepts, equations, tables and / or graphs.</td>
<td>- Presents several ideas that loosely support the theme.</td>
<td>- Requires moderate editing.</td>
</tr>
<tr>
<td></td>
<td>- Somewhat discusses equations, tables, figures, and / or graphs to support concepts.</td>
<td>- Contains an underdeveloped non-example with limited discussion of the inappropriate application of concepts, equations, tables and / or graphs.</td>
<td>- Contains an introduction, body, and conclusion that are underdeveloped for purpose and audience.</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>The writing:</td>
<td>The writing:</td>
<td>The writing:</td>
<td>The writing:</td>
</tr>
<tr>
<td></td>
<td>- Minimally or does not define mathematical vocabulary mentioned.</td>
<td>- Includes a flawed example OR non-example with inaccurate mention of the real world application of concepts, equations, tables, and / or graphs.</td>
<td>- Minimally uses the language of mathematics or does not use it correctly</td>
<td>- Contains multiple errors of a variety of types including spelling, punctuation, capitalization, grammar, and paragraph breaks.</td>
</tr>
<tr>
<td></td>
<td>- Mentions concepts by name but does not discuss relationships.</td>
<td>- Does not provide an example OR non-example at all.</td>
<td>- Minimally includes few, if any, ideas that may or may not support the theme.</td>
<td>- Requires extensive editing.</td>
</tr>
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<td></td>
<td>- Lists or does not make mention any equations, tables, figures, and / or graphs.</td>
<td></td>
<td>- May lack an introduction and / or conclusion.</td>
<td></td>
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</tbody>
</table>
Student Example:

Chapter 5 Essay

A line of fit is a line used on a scatter plot to show the general direction of the data on the graph. There are two ways in which you can create a line of fit one way is called the two-point method and the other is called the Q-points method.

Using the two-point method you must choose two ordered pairs from your data in which your line of fit goes through. For example if the general trend of your data is decrease you must choose two points on your graph that also are decreasing. If the general trend of the data is increasing you must choose two points that are also increasing. Your line of fit should follow the general direction of the data but there should be the same amount of points on both sides of the line.

The advantages of using the two-point method is that it takes less time to do and that it is based on your observations of the data. But you have sure thing that those two points are the best points to base your line of fit on.

When using the Q-point method you must first establish which column of data is going to be represented by x and which column is going to be represented by y. Once you have established that, you must find the five number-summary of the data from both columns. If the slope of the data is negative use the first quartile from column x and the third quartile from column y to create an ordered pair like this (Q1x, Q3y) then use the other quartiles to create a second ordered pair like so (Q3x, Q1y). When the slope of the data is positive you have to use the first quartile from column x and the first quartile from column y to create your first ordered pair like this (Q1x, Q1y) the other ordered pair
would be (Q3x, Q3y). Once you have graphed your points draw a line going through both points.

The advantages of using the Q-point method is that it may give you a better line of fit and that you may check with another person to see if both your answers are correct since there is only one right answer. But doing this method requires more time and work than using the two-point method.

I chose to create my line of fit based on the two-point method. When I compared both lines of fit I noticed that most of the data was below the line of fit that was created using the Q-point method. The line of fit that was created using the two-point method went right through the middle of the area with the most ordered pairs in it.

It is appropriate to create a line of fit describing human ability when it seems to make sense based on your observations and your equation for the data. For example let’s say that I graphed data on the times of the men’s 100 meter dash for the Olympics. X is the years when the competition took place and y is the winning times of the Olympics that year in seconds. The general trend of the data shows that as the years go by the time it takes men to run 100 meters decreases. I chose x to represent the years and y to represent the winning times of the Olympics that year because in algebra y depends on x and for this data the winning times for the Olympics depended on the year that Olympics took place.

My equation for the data using the two point method is $y = 49.76 + (-0.02x)$. The slope of the data is -0.02 this means according to my equation that the time it takes man to run 100 meters decreases by 0.02 seconds each year. The y-intercept is the value of $y$ when $x$ is zero and the x-intercept is the value of $x$ when $y$ is zero. For the data (0, 49.76)
is the y-intercept so according to my equation at year zero it took man 49.76 seconds to run 100 meters. This does not make sense because when measuring human ability men are trying their hardest to run 100 meters and the greatest recorded time for the 100 meter dash in the Olympics is twelve seconds. With a time of 49.76 that is much greater than twelve seconds, which is not a good time considering that most of the times were from nine to twelve seconds. The x-intercept according to my equation is (2496, 0) this does not make sense because you cannot even move an inch much less than 100 meters without time passing by, it is impossible.

It is inappropriate to create a line of fit based on human ability when trying to predict future events passed the year 2230 because when substituting $x$ with 2230 in the equation based on the two point method I got five which is saying that it will take man five seconds to run 100 meters in the year 2230. I think it is inappropriate to create a line of fit passed the year 2230 because that would mean that the runners would have to run 50 meters in 2.5 seconds which I think is unreasonable. Even if they did their pace would slow down as an effect of running that fast.

It is inappropriate to graph data before the year 1738 because according to my equation based on my line of fit the time it would take man to run 100 meters is 15 seconds which I believe would be the greatest time it would take man to run 100 meters since it is only three seconds greater than the greatest time in my data.

To conclude this paper I would just like remind you that no line of fit is accurate before or after a certain point in time. Also that just because my line of fit using the two-point method worked better in representing the general trend of the data, then the line of fit using the Q-point method that may not always be the case. Using the two-point
method worked better for this specific data I think. Either method may work better than one another in different data sets so remember to create a line of fit using both methods first. Then it is up to you to decide which line of fit is better.
Appendix B: Standards for Mathematics

National Council of Teachers of Mathematics: Algebra Standards

Figure B.1. NCTM Algebra Standards for Grades 6-8

<table>
<thead>
<tr>
<th>Instructional programs from prekindergarten through grade 12 should enable all students to:</th>
<th>In Grades 6-8 students should:</th>
</tr>
</thead>
</table>
| Understand patterns, relations, and functions | • Represent, analyze, and generalize a variety of patterns with tables, graphs, words, and symbolic rules  
• Relate and compare different forms of representation for a relationship  
• Identify functions as linear or nonlinear and contrast their properties from tables, graphs or equations |
| Represent and analyze mathematical situations and structures using algebraic symbols | • Develop an initial conceptual understanding of different uses of variables  
• Explore relationships between symbolic expressions and graphs of lines, paying particular attention to the meaning of intercept and slope  
• Use symbolic algebra to represent situations and to solve problems, especially those that involve linear relationship  
• Recognize and generate equivalent forms of simple algebraic expression and solve linear equations |
| Use mathematical models to represent and understand quantitative relationships | • Model and solve contextualized problems using various representations, such as graphs, tables, and equations |
| Analyze change in various contexts | • Use graphs to analyze the nature of changes in quantities in linear relationships |

Common Core State Standards Initiative: Standards for Mathematical Practice

*Figure B.2. CCSSM Mathematics Grade 8*

<table>
<thead>
<tr>
<th>Domain Standards (larger groups of related standards) for Mathematical Practice:</th>
<th>Standards for Mathematical Content: Students should be able to:</th>
</tr>
</thead>
</table>
| Expressions and Equations | • Understand the connections between proportional relationships, lines, and linear equations  
• Analyze and solve linear equations and pairs of simultaneous linear equations |
| Functions | • Define, evaluate, and compare functions  
• Use functions to model relationships between quantities |

**Figure B.3. CCSSM High School: Algebra**

<table>
<thead>
<tr>
<th>Domain Standards (larger groups of related standards) for Mathematical Practice:</th>
<th>Standards for Mathematical Content: Students should be able to:</th>
</tr>
</thead>
</table>
| Seeing Structure in Expressions | • Interpret the structure of expressions  
• Write expressions in equivalent forms to solve problems |
| Creating Equations | • Create equations that describe numbers or relationships |
| Reasoning with Equations and Inequalities | • Understand solving equations as a process of reasoning and explain the reasoning  
• Solve equations and inequalities in one variable  
• Solve systems of equations  
• Represent and solve equations and inequalities graphically |
| Mathematical Practices | 9. Make sense of problems and persevere in solving them  
10. Reason abstractly and quantitatively  
11. Construct viable arguments and critique the reasoning of others  
12. Model with mathematics  
13. Use appropriate tools strategically  
14. Attend to precision  
15. Look for and make use of structure  
16. Look for and express regularity in repeated reasoning |

References


Appendix C: English Language Arts Standards

*Figure C.1. CCSSI English Language Arts Standards Writing Grade 8*

<table>
<thead>
<tr>
<th>Domain Standards:</th>
<th>Standards for Writing: Students should be able to:</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Text Types and Purposes:</strong></td>
<td>• Write arguments to support claims with clear reasons and relevant evidence</td>
</tr>
<tr>
<td></td>
<td>• Write informative/explanatory texts to examine a topic and convey ideas, concepts, and information through the selection, organization, and analysis of relevant content</td>
</tr>
<tr>
<td></td>
<td>• Write narratives to develop real or imagined experiences or events using effective technique, relevant descriptive details, and well-structured event sequences</td>
</tr>
<tr>
<td><strong>Production and Distribution of Writing</strong></td>
<td>• Produce clear and coherent writing in which the development, organization, and style are appropriate to task, purpose, and audience.</td>
</tr>
<tr>
<td></td>
<td>• With some guidance and support from peers and adults, develop and strengthen writing as needed by planning, revising, editing, rewriting, or trying a new approach, focusing on how well purpose and audience have been addressed</td>
</tr>
<tr>
<td><strong>Range of Writing</strong></td>
<td>• Write routinely over extended time frames (time for research, reflection, and revision) and shorter time frames (a single sitting or a day or two) for a range of discipline-specific tasks, purposes, and audiences</td>
</tr>
</tbody>
</table>


**References**

Appendix D: Pilot Study

Purpose

The purpose of this pilot study was to develop the instruments used in data collection and clarify the procedures for implementing the instruments for the larger study. It was crucial that my pilot study aid in the process of producing a methodology that was reliable, rigorous and improved the validity of the study (Hatch, 2002). The benefit of completing a pilot study was to develop, adjust, and test the practicality of the larger study so any flaws could be addressed and corrected prior to performing the larger study (Hopkins, 2000). It also provided me with the opportunity to become involved with the prospective age group of students and the classroom context as a researcher to capture artifacts from the participants (Patton, 2002); my current association with the students and the school setting was only of a teach prior to the pilot study. Furthermore, it established an opportunity for me to observe the interactions of the students within the context, setting, and activities established by the design of my study. Any gaps in the design could be addressed prior to the larger study taking place the following school year.

My purpose for conducting a pilot study for this qualitative study was twofold: first, I wanted to identify the methodology used to collect data, and secondly, I wanted to identify the type of questions that would best demonstrate the development of students’ algebraic reasoning. The pilot study was conducted in two parts; part I was conducted during the fall semester of 2009 and part II was conducted during spring semester of 2010.
Writing in Mathematics Pilot Study: Part I

When I began my pilot study, I only had one research question: how does writing reveal a student’s algebraic reasoning and knowledge of algebraic concepts? I knew this was to be the overarching theme of the study but at the time, I was unsure of the best methodology to collect and analyze the data. Malloy (1999) suggested that teachers should extend student learning by creating activities that allow them to make and evaluate conjectures and apply mathematical reasoning to a variety of situations. Additionally, Nathan & Koellner (2007) proposed that mathematical skills and reasoning is best taught through problem solving contexts in conjunction with prior knowledge; prior knowledge being an assistant in facilitating algebraic understanding. Finally, Staats & Batteen (2009) recommend that students who write in mathematics are able to better apply mathematical ideas to real-world contexts. Based on these researchers’ suggestions, I created a protocol to provide students with the opportunity to further extend their mathematical reasoning using writing as the primary medium of communication.

Data Collection Instrument: Closed-Ended Prompts

Initially, I asked my students to choose a designated problem from the chapter and discuss how the solution was derived. NCTM (2000) suggests that one possible method of encouraging students to write is to allow them to explain how they found the solution to a problem. I thought that this might be one way in which the students could demonstrate their understanding of procedural and conceptual understandings from the chapter. The problems that the students could choose from contained contexts and situations that existed outside of mathematics class; purely algebraic expressions or equations were not included. This is because I felt that Algebraic expressions and
equations measure analytical thinking rather than creative or practical mathematical reasoning in everyday contexts (Sternberg, 1999) and are therefore would be unable to demonstrate the full range of a student’s mathematical understanding.

Additionally, I specifically chose odd numbered problems whose solutions would be accessible to the students. Even though the students completed the even numbered problems for homework and received the answers the following day, the odd numbered problems contained textbook supplied answers found in the conclusion of their textbook *Discovering Algebra: An Investigative Approach* (Murdock, Kamischke, & Kamischke, 2002). My rationale for providing the students the solution to their chosen problem was to shift their focus from finding the answer to explaining how and why the solution was achieved. In this way, students could no longer find fault in their solution due to a careless error or lack of number sense (Sternberg, 1999) and had the opportunity to demonstrate their procedural and conceptual knowledge. Moreover, the problems that were chosen were ones that the students did not encounter in class so they could not rely on their notes or homework and had to go through the process of finding the solution for themselves.

Students were asked to choose their preferred question and respond to it at the conclusion of the chapter. I chose to assign a writing task at the conclusion of the chapter as a way of increasing the cognitive demands (Urquhart, 2009) on the students to stretch them beyond applying the correct procedural steps to finding a solution. Then they were given a few days in which to submit their written response. As a teacher, I did not provide any feedback to the students on their written response between the time the assignment was given and the collection of their written mathematical explanation. I
rationalized that since the students had a number of experiences with various problems throughout the chapter, they would be able to demonstrate their understandings by expressing their thoughts on paper. Anything the students submitted was accepted as a final piece of writing. Each assignment was later returned to the students with a letter grade and a few comments in the margins that may have pointed out gaps or discrepancies in the students’ conceptual or procedural rationale. Although the students were given the option to rewrite their explanation, since it was not required most students looked at the grade and moved on with the class as long as they earned a C- or above.

**Field Notes**

My findings for this portion of the pilot study were based on the field notes I had taken during the semester. These field notes are the written descriptions of what I had observed in students’ writing and thought were noteworthy of noting (Patton, 2002). Inclusive in these field notes were my own experiences reading the students’ work and what I had thought about what I had read. These field notes also served as a brief, beginning analysis (Patton, 2002) looking at how students may have approached writing in mathematics and how the writing was used to convey their thinking. As a new researcher, I was cautious when recording my experiences so as to not impose any preconceived notions on the texts I was observing, jotting down only what I thought was significant.

**Analysis**

May illustrates how she uses writing to explain a solution to this given problem: “Pasta Lover has 4/9 of a piece of spaghetti. It is 1 3/5 of a meter long. How long is the whole piece of spaghetti?” She approaches this teacher-generated problem by explaining
in words and pictures her solution indicating she believes words and pictures describe her approach to the problem are mathematical reasoning.

1. **Step 1:**

   Since $1 \frac{3}{5} = 1 \times 5 + 3 = \frac{8}{5}$, four pieces (numerator) of the nine pieces that make up the spaghetti (denominator) equals $\frac{8}{5}$ of a meter.

   ![Diagram showing spaghetti pieces]

   \[ \frac{1}{9} \quad \frac{1}{9} \quad \frac{1}{9} \quad \frac{1}{9} \quad \frac{1}{9} \quad \frac{1}{9} \quad \frac{1}{9} \quad \frac{1}{9} \quad \frac{1}{9} \]

   \[4/9 = \frac{8}{5} \text{ meters}\]

2. **Step 2:**

   You know that four pieces of the spaghetti plus four pieces equal eight out of the nine pieces, which is not yet a whole, so you can multiply the length of $\frac{4}{9}$ of the spaghetti by 2.

   \[ \cdots \cdots \cdots \frac{8}{9} = \frac{16}{5} \text{ meters} \cdots \cdots \cdots \]

   \[\frac{8}{5} \times 2 = \frac{8}{5} \text{ (length of } \frac{4}{9} \text{ of spaghetti) } + \frac{8}{5} = \frac{16}{5} \text{ meters} \]

   So $\frac{16}{5}$ meters = 8 pieces out of the nine in the spaghetti

3. **Step 3:**

   Now you need to find the length of just one ninth of the spaghetti since you already found the length of 8 out of the nine.

   Going back to step one, if $\frac{8}{8}$ equals $\frac{4}{9}$ then you should be able to figure out one ninth by dividing $\frac{8}{5}$ by four.

   \[\frac{8}{5} \text{ (of a meter) } / 4 \text{ (ninths in the } \frac{8}{5} ) = \frac{8}{5} \times \frac{1}{4} = \frac{8}{20} = \frac{2}{5}\]
Finally, you add $\frac{2}{5}$ (one ninth of the spaghetti) to $\frac{16}{5}$ (eight ninths of the spaghetti) and you get $\frac{18}{5}$ of a meter which is equivalent to $3 \frac{3}{5}$ of a meter. So the length of the total spaghetti is $3 \frac{3}{5}$ meters long (May, Written Response, September 8, 2009).

May’s use of the word, “step” in lines 1, 7, and 15 suggest her explanation demonstrates her procedural knowledge and lists the manipulations she performs in a specific order. This is further emphasized by her inclusion of expressions and equations such as, “$1 \frac{3}{5} = 1 \cdot 5 + 3 = \frac{8}{5}$.” She accompanies these symbolic manipulations with verbal descriptions, such as, “Now you need to find the length of just one ninth of the spaghetti since you already found the length of 8 out of the nine,” in lines 16 and 17 pointing to her procedural approach to finding the solution. May illustrates through her response that her thinking focuses on the procedural aspect of problem solving.

Moe demonstrates how students justify an equation used to model a situation by referring back to the original context of the problem.

Question: The year after a dictionary was published, two librarians notified the publisher of mistakes. One librarian found 43 errors, and the other found 62. The reprint editor noticed that 35 of the errors were mentioned by both librarians. He used the capture-recapture method to estimate the actual numbers of errors. How many errors are there likely to be? (Hint: The errors found by either librarian can be used as the number of tagged errors in the capture phase. The other librarian represents the recapture phase with the errors found by both being tagged errors) (Murdock, Kamischke, & Kamischke, 2002, p. 103).

Answer:
First, write an equivalent proportion for the capture-re-capture\[sic\] phases,

Capture: Tagged/Total = Re-capture: Tagged/Total, which is, \(\frac{43}{x} = \frac{35}{62}\) OR \(\frac{62}{x} = \frac{35}{43}\). The first proportion in the first equation, \(\frac{43}{x}\), represents the capture phase. 43 is the tagged number of errors one librarian found. The \(x\) represents the variable, which is an estimation of the total number of errors there are in the dictionary. The second at proportion in the first equation, \(\frac{35}{62}\) represents the recapture phase. Of the 43 errors that the first librarian found in the capture phase, 35, can be considered the tagged number of errors in the re-capture phase because the 2\textsuperscript{nd} librarian found the same 35 errors out of a total of 62 errors in the re-capture phase. Since the problem says that neither errors found by the librarians can be used as the number of tagged errors in the capture phase and the other librarian represents the recapture phrase with errors found by both librarians being tagged errors, the 2\textsuperscript{nd} equivalent proportion, \(\frac{62}{x} = \frac{35}{43}\), can also be used to find an estimation of the total number of errors there are in the dictionary.

Secondly, Solve for \(x\), using either of the above equations. (Moe, Written Response, October 26, 2009).
Moe uses phrases such as, “Of the 43 errors that the first librarian found in the capture phase, 35, can be considered the tagged number of errors in the re-capture phase because the 2nd librarian found the same 35 errors out of a total of 62 errors in the re-capture phase,” in lines 14 through 17 to restate portions of the original problem suggesting these phrases prompt him to symbolically represent the situation in a particular way. However, these phrases also indicate his failure to express his conceptual understanding of the concepts used to model the situation.

Both Mae and Moe’s response suggest their written responses use words to describe the procedural manipulations used to find a solution to a particular problem. Their use of pictures and phrases restating the original context of the problem indicate their attempt to express their rationale supporting their procedural decisions and models of the original context.

**Findings**

My observations revealed a pattern of students’ explanations resembling textbook-like examples in the way they demonstrated how they found the solution. I noticed that as a whole, students rewrote mathematical procedures they had performed to solve the problem in words. Their use of mathematical terms and operations suggested that students’ believed writing words as opposed to numbers to describe a mathematical procedure was an acceptable way of demonstrating mathematical reasoning and explaining why they felt their solution to the problem was accurate. Parker and Goodkin (1987) suggest that because students often encounter textbooks in their academic career, they pattern their writing after those models. But they note that textbooks are rarely ever
good models for student writing because they present writing in non-creative, monotone, and uninventing ways.

Many students also failed to include their conceptual understandings as part of their written explanation. They failed to discuss, for example, why they choose to apply a particular mathematical idea in specific manner or suggest a rationale for the meaning associated with the solution to the problem within the context of the given situation. This suggests that students view writing as a way to express a final product; in this case, their final product was a procedural justification of their solution. Wolcott and Legg (1998) and Parker and Goodkin (1987) imply that students are frequently asked to use writing for assessment purposes, rewarding students who demonstrate their memorization of specific knowledge. This is further emphasized when writing is taught as an isolated skill; words only convey meaning when they are used in specific contexts (Williams, 1998). Since students failed to reference the context of the problem or explain the meaning of the equation they derived from the situation, it appears that their sole purpose for using writing was to describe a mathematical procedure in words.

It also appeared from the students’ submissions that although using textbook problems provided the students the opportunity to write mathematical formulas and procedures using the language of mathematics they may not have prompted students to demonstrate their algebraic reasoning. Ntenza (2006) suggested that textbooks do not provide optimal writing prompts for students. His research implied that textbooks do not provide the type of extended writing activities that ask students to demonstrate their
mathematical reasoning or ask students to create their own story problems to demonstrate their mathematical understanding of a concept.

My initial rational for using textbook problems was to have students focus on exposing their mathematical reasoning and conceptual understandings in their writing. Instead, it appeared that these types of problems prompted explanatory-like responses similarly to the way in which students find the solution for homework. Researchers Shield and Galbraith (1998) suggest that teachers need to rethink how they encourage students to write in mathematics class if their purpose is to increase higher levels of thinking and comprehension of mathematical concepts. This emphasized to me that well written prompts can engage students in personal reflection and dialogue through writing to discuss how mathematical meaning is derived in creative ways. Thus, based on observations from Part I of the pilot study, I made changes to the prompt design and made the decision to use writing as a process, instituting the writing process, as opposed to use writing as a product as I had previously done in this pilot study.

**Writing in Mathematics Pilot Study: Part II**

Part II of the pilot study was conducted during the spring semester of the 2010-2011 school year. With the introduction of the writing process, I decided to pose two additional research questions during Part II of the pilot study. These questions addressed the inclusion of the writing process and teacher feedback to describe how writing reveals algebraic reasoning:

- In what ways does writing, through the revision of multiple drafts, demonstrate the development of algebraic reasoning?
In what ways does written teacher feedback, provided during the writing process, prompt the revision of algebraic reasoning as represented in students writing?

With the inclusion of these two research questions, I shifted the focus of writing from being product oriented to process oriented. When writing is viewed as a process rather than a product it serves three potential functions: a) writing reveals inconsistencies, gaps, and misconceptions in students’ knowledge (National Writing Project & Nagin, 2006); b) writing encourages the connection of prior and new knowledge (Mayher, Lester, & Pradl, 1983); and c) writing becomes a process of inquiry and discovery (National Writing Project & Nagin, 2006). Additionally, the inclusion of feedback is also supported by the writing process as another means of assisting writers say what they mean to say. Feedback is the suggestions for revision; feedback can take the form of constructive criticism, specific comments, and positive observations (National Writing Project & Nagin, 2006).

Data Collection Instrument

I adjusted the type of writing prompt given to the students during this portion of my pilot study. Sternberg (1999) suggested that students develop their mathematical reasoning if they are exposed to three different types of situations: a) analytical situations teach students formulas and facts and when to apply them to abstract situations; b) creative situations insist on students creating their own problems and imagining how the mathematical concepts may be applied; and c) practical situations where the context is situated in an everyday context and requires the use of mathematical concepts.
To address Sternberg’s (1999) first suggestion, analytical situations teach students formulas and facts and when to apply them to abstract situations, students would continue to experience analytical situations, formulas, and mathematical facts during class. In addition to this exposure, students are also asked to practice applying their procedural knowledge to a variety of story problems provided by their textbook. NCTM (2000) suggests that when students are exposed to factual, procedural and conceptual knowledge, they can become more proficient in finding solutions to problems they have never encountered before. Gaining a deeper understanding of mathematics by building new knowledge upon prior knowledge can assist in their development of algebraic reasoning.

Secondly, to explore the idea of encouraging students to create either creative or practical situations in which to apply their algebraic reasoning, I would ask each student to create an example justifying a situation outside of mathematics class in which the concepts from the chapter could be applied in finding a solution. Sternberg (1999) continued to imply that students who create different contexts to represent a situation mathematically benefit from seeing these variations and can better understand the circumstances of when an algebraic concept can be applied. Furthermore, advocates of Writing Across the Curriculum (WAC) suggest that writing can be integrated into other contexts, including mathematics, from which students can relate to and share their experiences (Fulwiler, 1987; Mayher, Lester, & Pradl, 1983).

Lastly, the student-generated examples had to be based in realistic situations existing outside of the mathematics classroom which could be described through the
mathematics they were currently learning. Although writing prompts from the textbook were convenient, they did not provide the depth needed to engage students in revealing their algebraic reasoning. Gammill (2006) suggests that knowledge should not be separated from its original context, history, values, and beliefs implying that mathematical ideas should be applied to and discovered in everyday contexts. Therefore, I sought pre-existing prompts that asked students to consider contexts outside of the classroom as another way of encouraging mathematical reasoning.

Mission School’s two Algebra I teachers, one of which was me, created a set of questions for the school’s Algebra I curriculum map designed in 2007-2008. These questions were constructed to incorporate the themes and subthemes from each chapter. They were also questions that could not be answered by a simple yes/no response or only the display of procedural knowledge. These open-ended inquiries, containing multiple answers, are termed as essential questions (Brown, 2009).

I noticed that each essential question we constructed contained the following characteristics: a) each question, one per chapter, focused on the overall theme of the chapter; b) each question was open-ended and could be responded to in a variety of ways; c) each question required conceptual and procedural understanding to produce an answer. By introducing a system that invited open response, students could be rewarded for their creativity in their mathematical reasoning as opposed to only receiving recognition for their ability to memorize analytical procedures (Sternberg, 1999). However, until this pilot study was implemented, these essential questions had not been formally
incorporated into the class structure nor used as a formal means for students to demonstrate their algebraic reasoning.

In addition to seeking new prompts, I also implemented the writing process. The writing process consists of the strategies used to compose a piece of writing (National Writing Project & Nagin, 2006). These strategies such as prewriting, drafting, and revising have the potential to expose students’ mathematical reasoning as they use the process to revise previous drafts. Hamdan (2005) and Baxter, Woodward and Olson (2005) suggests writing offers students the opportunity to dialogue between themselves and the abstract mathematical material by providing an entryway for the students into their own thought processes. Moreover, writing can provide concrete evidence of students’ mathematical understandings over a period of time and expose their thoughts (Hamdan, 2005; Baxter, Woodward, & Olson, 2005) for themselves as well as a public audience.

I decided that my students would compose three drafts throughout the course of the chapter: once at the beginning, approximately midway through, and after the close of the chapter. This was similar to Graziano-King’s (2007) research on the self-revised essay in which the methodology asked student to initially compose their response at the beginning of the semester and then revise it two additional times throughout the course. Asking my students to write three drafts would allow me the opportunity to view their thinking over time as I exposed them to the mathematical material with various activities. Furthermore, it provided the students the opportunity to build their conceptual and procedural knowledge as they gained more experiences manipulating the various
concepts. Each time a new draft was assigned, the previous draft was returned so students could review what they wrote. In this way, students had the opportunity to witness their own thinking, reflect upon their previous ideas, and revise them accordingly as they experienced more mathematical content. I projected that by last revision, the students would be able to write a more elaborate response, using the language of mathematics, to the initial prompt given.

As part of the writing process, students would receive feedback from me, from the perspective of a teacher, on their second draft. Whittington, Glover, and Harley (2004) propose that students want to receive feedback because feedback enables students to increase their learning and motivate them to revise their thinking. Thus, I felt that providing feedback on students’ freewrites may not be as productive because the freewrite served as a springboard for the students to jot down possible ideas. The second draft was ideal for feedback because students were exposed to multiple experiences midway through the chapter and we would most likely finish the chapter before the final draft was assigned. Additionally, researchers such as Vardi (2009) suggest that feedback that is content specific and encourages students to rethink the meaning they have constructed results in improved writing quality. Thus I felt that providing feedback on the second draft allowed the students to observe where gaps or redundancies may lie in their thinking before they completed the final draft.

Analysis

I again conducted a content analysis on the final products of the written responses produced by the students to their chosen textbook problem. Content analysis is a method
used to analyze a large volume of text to identify significant themes (Patton, 2002). In this analysis, I viewed students’ final texts to see what words and phrases were used by students to describe their thinking and mathematical reasoning. I chose

Although I read each student’s written responses from the perspective of a teacher, I chose to analyze two students’ written responses more closely. Both students were in my homeroom and my algebra class which meant they were easily accessible as students. Also, both were diligent about completing their assignments on time and neither had Individualized Educational Plans (IEP) drawn up by the academic counselor. I selected these two students for the purpose of taking full advantage what I could learn from these specific students’ texts (Stake, 1995). Although these two students’ will have their text analyzed more closely, student observations of the other students’ texts in the class will be conducted.

After an analysis of the results, I discovered that the essential questions on the Algebra I curriculum map produced two types of responses: explanatory responses and creative responses which may have been prompted by how the initial question was worded. Explanatory responses included more mathematical facts and appeared to be procedural in nature, assuming an information-giving-and-receiving tone. Although some students may have used the language of mathematics appropriately, many times they failed to discuss the conceptual nature of the ideas involved, preferring instead to define
the mathematical terminology. This is demonstrated by Pal who responded to the question, “How can mathematical situations be described as recursive sequences?”

Math (Mathematics) can be a recursive sequence (sequence of numbers), because the word recursive means continuous and repeating. For example, in math there are repeating decimals, and this is an example because repeating decimals are a sequence of numbers, like the fraction 1/3, and the decimal is .33 and that is a repeating decimal. The word sequence means in order, an arrangement, or continuous series, one example of a sequence is when you count; you usually count in order like 1, 2, 3, 4, 5, 6, 7, 8… There are many different examples of a sequence.

Math can also be a recursive sequence, by Pascal’s triangle. In, Pascale’s triangle the numbers are in a certain order, and are continuous. Down the sides of Pascal’s triangle are 1’s, the second row diagonally is going by one’s, for example, 1, 2, 3, 4, 5, 6, 7, 8, 9…

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12 All names have been changed to pseudonyms to keep the confidentiality of the students.
Pal’s use of the phrase, “sequence of numbers,” in line 1 to define what she feels is a recursive sequence suggests that her discussion will be based on defining each individual word, recursive and sequence, as opposed to the concept recursive sequence. This is further emphasized as she states in line 2, “the word recursive means continuous and repeating,” which she feels is the definition of the word recursive. Moreover, Pal’s example, “in math there are repeating decimals, and this is an example because repeating decimals are a sequence of numbers, like the fraction 1/3, and the decimal is .33 and that is a repeating decimal” in lines 3 and 4 emphasize that she continues to view the concept through its individual components. Her focus on the fraction 1/3 and it’s decimal, $0.\overline{3}$ continues to emphasize her belief that a recursive sequence is a sequence of repeating numbers based on how she defined each term individually.

Furthermore, her use of the phrase, “Math can also be a recursive sequence, by Pascal’s triangle,” in line 8 suggests that she is focusing more on numerical examples associated with mathematics than on how recursive sequences can describe situations beyond the mathematics classroom. This is further emphasized by the two diagrams she includes of Pascal’s triangle implying that she believes this constitutes a mathematical situation which contains a recursive sequence according to her definition of the term. Pal fails, however, to discuss the meaning of the concept recursive sequence as a whole and focuses more on the explanatory nature of the question placing emphasis on mathematical situation and the definition of each individual word, recursive and sequence.
By contrast, student responses to questions that were seemingly more open-ended demonstrated a variety of creative responses. Creative responses were inclusive of student-generated examples and were more likely to contain some form of mathematical reasoning after given the opportunity to use the writing process. Some students were even able to accompany their student-generated example with a non-example as demonstrated by May as she responded in her final draft to the question, “When is it appropriate to fit a line to data to describe human ability (or some other related real-world situation).

1 Though there are many ways you could fit a line to data, there are some times
2 where it would not be appropriate to do this, for example no matter how hard you
3 train and how healthy you eat, at one point there will be a time where the human
4 body just can’t run any faster. This is where a line of fit (a straight diagonal line)
5 becomes inappropriate. The same goes for your growth chart. After you hit
6 puberty, you will stop growing. A line of fit in these situations would only be
7 useful when, as mentioned above, predicting the next couple of years. (May,
8 Written Response, March 23, 2010).

May’s use of the phrase, “there are some times where it would not be appropriate to do this,” in line 1 and 2 suggests she is capable of offering a non-example in her discussion. She believes that her non-example will illustrate a situation in which the particular mathematical concept would be inappropriate to describe the context. Her use of the scenario, “no matter how hard you train and how healthy you eat, at one point there will be a time where the human body just can’t run any faster” in lines 2 through 4 emphasizes that she believes that her situation describes when a line of fit can no longer
predict the skill level of a human. Although she fails to discuss specific ordered pairs that would further support her belief, her discussion suggests that she reveals some supportive statements for her reasoning.

May’s use of the phrase, “line of fit (a straight diagonal line) becomes inappropriate,” in line 4 suggests she is able to use the language of mathematics to discuss when her mathematical model of the situation would be unsuitable. To further emphasize her perspective, May includes a second non-example. Her use of the phrase, “A line of fit in these situations would only be useful when, as mentioned above, predicting the next couple of years” in lines 5 through 7 summarizes her belief of the time frame in which her mathematical model would be appropriate for a particular situation. May’s discussion reveals students are able to demonstrate their mathematical reasoning after being encouraged to use the writing process when discussing their response to a particular open-ended question.

Findings

When writing is used to communicate a product such as the solution to an equation, the definition of a mathematical term, or a description of mathematical procedure, it separates the process of writing from the process of learning. These traditional products, while using words to convey information, encourage the idea of learning to write in mathematics as opposed to writing to learn in mathematics (National Writing Project & Nagin, 2006; Parker & Goodkin, 1987; Fulwiler, 1987; Mayher, Lester, & Pradl, 1983). The difference is that writing to learn in mathematics encourages higher-order thinking by engaging students in active construction of their understandings.
and knowledge (Mayher, Lester, & Pradl, 1983; National Writing Project & Nagin, 2006; Fulwiler, 1987; Parker & Goodkin, 1987; Gammill, 2006). The analysis suggests that students are encouraged to use writing for learning when the writing prompt suggests they can generate a creative response as opposed to a procedural response and the writing process is supported.

Based on the observations and content analysis of student written responses in the pilot study from Part I and Part II, the following items influenced the data collection instrument for this qualitative study:

- Prompts will be crafted as open-ended inquires to encourage creative responses
- Students will write three drafts throughout the course of the chapter
- Students will receive teacher feedback on the second draft
Appendix E: Interview Questions

Thank you for participating in this study. Your participation in this interview will help me get a better understanding of the role of writing in mathematics. I want to reassure you that your answers to the interview questions will remain confidential and will have no impact on your grades in class. Let me also assure you there are also no wrong answers as your responses will help me to plan for next year’s algebra class.

These are the research questions I am interested in:

- How does writing reveal a student’s algebraic reasoning and knowledge of algebraic concepts?
- In what ways does writing, through the revision of multiple drafts, expose the development of algebraic reasoning?
- In what ways does teacher-to-student written feedback, provided during the process of writing, prompt students’ written revision of algebraic reasoning?

How does writing describe a student’s algebraic reasoning and knowledge of algebraic concepts? [THEME: WRITING & ALGEBRAIC REASONING]

1. How does writing help you to learn?
   a. Can you provide any examples from your past or present classes?

2. How did your previous mathematics teachers use writing in in their classes?
   a. What types of writing assignments did you have?
   b. How did those writing assignments help you to learn mathematics?
c. Compare how writing was used in your previous classes with your experiences in your current Algebra class.
   i. What is the same?
   ii. What is different?

d. How does writing in this class help you to learn algebra?

e. What can I learn about your understanding of algebra from your writing?

f. What have you learned about your understanding of algebra from your writing?

3. What do you struggle with most when asked to write in mathematics?

4. What do you feel is your strength when asked to write in mathematics?

5. What do you think about having to answer open-ended questions in mathematics?
   a. What do you feel are the benefits of answering these types of questions in regards to learning mathematics?

In what ways does writing, through the revision of multiple drafts, describe the development of algebraic reasoning? [THEME: REVISION & REASONING]

The writing process consists of the strategies used to compose a piece of writing such as prewriting, revising, editing and repeating the process in drafting.

6. In your experiences with writing, regardless of the subject or class, how often do teachers ask you to compose multiple drafts before submitting the final?
   a. How many drafts would you compose?
7. If you weren’t asked to submit more than one draft to your teacher, how often would you use the writing process on your own before turning in your final product?

8. What do you feel are the benefits of writing multiple drafts before submitting the final?

9. Specifically to your own learning of algebra, how does the writing process help you to respond to the chapter’s open-ended question?

   a. What do you normally write in your first draft (and why)?
   
   b. When you get the first draft back, what do you do with it?
   
   c. What do you do to your first draft to produce your second draft (and why)?
      
      i. Where do you get your ideas from?
   
   d. When you get the second draft back, what do you do with it?
      
      i. Where do you get your ideas?
      
      ii. What do you do with the feedback?
   
   e. What do you do to your second draft to produce your third draft (and why)?
   
   f. Do you feel you are satisfied with your third draft when you turn it in (and why)?
      
      i. If you’re not satisfied, how many more drafts would you like to complete?
      
      ii. What would you like your teachers’ role to be in the completion of these additional drafts?

10. Have you noticed any changes in the way in which you write as a result of repeating the writing process for each chapter?
11. Have you noticed any changes in the way in which you think about mathematical ideas as a result of repeating the writing process for each chapter?

12. How do you know when a piece of writing you’ve done is finished?

In what ways does written teacher feedback, provided during the writing process, promote the revision of algebraic reasoning as represented in students writing?

[THEME: FEEDBACK & REASONING]

13. What role do you think feedback has in writing?
   a. What types of feedback benefit you most (and why)?
   b. How does feedback help you to revise your ideas?
   c. How would you prefer to get feedback: written or oral (and why)?
   d. Do you value written feedback you receive from teachers (and why)?

14. When you receive feedback on draft 2, what does it tell you about your ideas about the chapter?
   a. How do you use the feedback you’ve been given on draft 2 for to revise your ideas for draft 3?
   b. What would your third draft look like if you did not receive any feedback on your second?

15. How does getting feedback on your second draft change how you think about your mathematical ideas?

16. What is the biggest contributor towards developing your ideas in your writing?
17. What advice would you give to someone, who was given an open-ended question like the ones you’ve received in Algebra, on how to develop a successful response?

18. Is there anything else you’d like me to know about your experience with the writing process in this class?
Appendix F: Teacher-to-Student Written Feedback Example

Figure 4. Tohly’s Second Draft

<table>
<thead>
<tr>
<th>Chapter 4 Project</th>
</tr>
</thead>
<tbody>
<tr>
<td>What is a recursive sequence? A recursive sequence is a list of numbers defined by a starting value and a rule. A mathematical situation can be described as a recursive sequence by if a bicyclist rides his bike for 6 hours, for every hour he rides he goes 20 miles.</td>
</tr>
</tbody>
</table>

Your starting numbers are 0,0 and the rule is add 20. The situation can be described as a recursive sequence because if you graph this equation, the coordinates would be (0,0), (1,20), (2,40), (3,60) and so on. This is also a recursive sequence because there are both starting numbers and a rule. You can apply the rule to the starting value, then apply it to the resulting value, and repeat this process.

One way a situation can be a recursive sequence but not linear, is if the x coordinate increase as y coordinate increases or vice versa. An example of this is if a swimmer goes diving for an hour, for every 15 minutes, he goes 30 feet below the sea.

Comment: (RS13): Define starting value and rule. What do you mean by those two things and how do they help you to produce a recursive sequence?

Comment: (RS21): How is the second situation a recursive sequence? How does it produce a recursive sequence if a rule and starting value are defined?

Comment: (RS14): Why ask a question about your starting numbers? What are they in the situation?

Comment: (RS15): Why is this the rule? What does the rule actually mean in the context of the problem?

Comment: (RS16): How useful is to graph an equation—what equation?

Comment: (RS17): If you’re going to talk about intercept form, make the connection between recursive, sequence and intercept form. Be sure to discuss the equation y = 2x + 1 and how it is related to recursive sequence.

Comment: (RS18): Avoid using phrases like this.

Figure 4. This is the second draft written by Tohly (Written Response 2, January 30, 2011) in response to the question, “How can a mathematical situation be described as a recursive sequence?” The feedback was teacher generated and given in the form of teacher-to-student written dialogue. The focus of this feedback concerns questions asking for clarification. Adapted from Teacher, Written Feedback, February 2, 2011.
Appendix G: Field Notes and Observations of Student Participants

Purpose

These field notes and observations were taken throughout the school year as needed as a secondary source of data. They were present and considered in the event of any one-sided interpretations as a way to either strengthen the claims or potentially explain the lack of convergence (Creswell, 2003).

Tohly

- Tohly is the student in class who is always the first to answer the teacher’s question – and doesn’t raise his hand. Anytime procedural knowledge is asked for, he’ll volunteer by blurting out an answer. Sometimes, his answers are welcomed by others in the class but other times they are taken offense to as students respond back, “can you wait” or “I didn’t figure it out yet.”

- Tohly can’t sit still in his seat either – he likes to get up and walk around the room, especially if one of his friends needs help. Actually, he seems to be willing to help anyone in class as long as he can get out of his seat and walk to another table.

- Tohly can come across to others as cocky because he knows that his answers are usually correct. But he can demonstrate his patience when helping other students

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13 Names have been changed to keep the confidentiality of the student who participated in the study.
Tohly doesn’t always take notes and if he does they may not be word for word as presented in a definition; he may simply write down an equation or a few words rapidly; he’s usually one of the first to finish his note taking.

Tohly seems to be the kind of student who constantly needs a challenge. He can provide himself with his own entertainment if he’s feeling bored or has finished practicing before his peers are finished.

He’s fairly confident in mathematics and seems to intuitively know what to do; usually he can hear an explanation once before catching on.

Tohly never comes to ask for help on any of his drafts.

Tohly writes as if he assumes his reader will know exactly what he’s thinking – he doesn’t take the time to elaborate his thoughts.

One word to sum up Tohly: impulsive.

Haily

Haily is the last one to finish taking notes and copying – she is very methodical and will write down everything. She will even write down notes and references from the previous day if the class is referring to them.

Haily is very methodical – she has a compulsion to continuously ask for clarification for everything. She wants to know and she wants to be able to repeat back to you the explanation. She’ll even stay after class to finish copying or asking her questions.

Haily always asks for help on the second draft once she’s gotten her feedback. She wants to clarification on the comments she’s received. She also wants to
know if her answers to the feedback are correct or at least point her on the right track. Depending on when the final draft is due in relationship to receiving the second draft back, Haily will go home, rewrite a few paragraphs or make notes to herself and come back the next day asking if her ideas are correct or if she’s explained her ideas with clarity.

- One of the things that Haily needs to work on in her writing is organization. she has a lot to say but sometimes she jumps from topic to topic.
- Haily is always striving for perfection
- Haily is conscious about her abilities – when she needs help, she will ask a teacher or a peer; many times once she is pointed in the right direction, she is able to finish solving the problem or understands it well enough to explain the solution and/or procedure to another classmate asking the same question
- One word to sum up Haily: perfectionist? Conscientious?

Cissy

- Cissy seems to strive for the A all the time; he may be more driven to get the A than to learn for the sake of learning. However, because of his drive, he will ask the questions to deepen his understanding or find the connections he needs to get the best grade possible
- Cissy can get off task when he feels the material is easy for him; but he’ll also help his peers when they ask for help and he can be patient with them
• Cissy always asks for help between the second and third draft. He wants clarification on the feedback, he will rewrite certain paragraphs or answer questions posed by the feedback and check the next day to see if he is correct

• Cissy can be impulsive – sometimes he’ll do or say things without realizing what he’s done or said and sometimes his peers get irritated at the behavior

• Cissy likes to show off a bit when he gets grades that are higher than most of his peers or classmates

• Cissy has no problems sharing in class and depending on who else is on his table, he’ll be the spokesperson

• One word to sum up Cissy: achiever

Jolly

• Jolly goes with the flow in class; nothing seems to faze him from day to day. He’ll take notes when it’s time to copy from the board, he’ll practice problems when he’s given problem sets, he’ll participate and work with his tablemates when doing an activity

• Jolly may sometimes volunteer in class but only it seems when he’s confident in his answer

• Jolly’s typically quieter in class even if he is distracted by his peers around him

• Jolly’s writing is usually very short – perhaps the most he’ll produce on a final is 1 page, double spaced, typed.

• Jolly will always turn his drafts in on time

• One word to sum up Jolly: laid-back
Kitty

- Kitty will always ask the question if she needs help but prefers to do so one-on-one rather than in front of the entire class.
- Kitty will always ask for help between the second and third draft; she always looks for clarification and tries to address each comment / feedback received.
- Kitty is generally quiet and on task in class; she’ll copy notes, work problems, and generally has a good work ethic.
- Kitty will help her peers one-on-one if they ask her a question.
- One word to sum up Kitty: conscientious.

Tutty

- Tutty is lazy – he’ll do things to suit his purposes and he’ll do just enough to get by.
- Tutty asks good questions when he wants to know more about something in class; usually he asks the questions based on concepts rather than procedural knowledge.
- Tutty is somewhat of a goofball, attention seeker, and maintains that image by complaining in class when things are hard or finding ways to distract himself & others if he’s bored.
- Tutty will on occasion, sit and listen to others ask their questions between draft 2 and 3 for clarification. He won’t ask any questions on his own but he’ll listen to others and the discussion that’s occurring at the table. If he asks a question, it’s one-on-one had he initiates the conversation because he wants to know.
• Tutty mentioned one day in class, as a side note that he wants to know how things are connected to each other; that’s what he’s interested in. I’m assuming he was referring to the composition of the equation and why it was the most appropriate to use in that situation or any like it, rather than the procedure of how to use the equation to solve.

• One word to sum up Tutty: two-sides to the same coin? Multi-facet?

**Tobby**

• Tobby appears to be the conscientious student; will take notes in class; doesn’t cause any disruptions; participates as necessary or when called upon but there may be the hidden-complainer as others have reported him complaining about this and that outside of class.

• Tobby works better when the pace of the class is slower but may never admit it to his peers (he may address it with his teacher, however, especially if he doesn’t understand something or he may also be the “voice” of the class).

• Tobby may ask questions as needed in class but mainly for clarification purposes.

• Tobby generally puts in the effort to try.

• Tobby one word to sum up Tobby: Willing-to-try

**Mobby**

• Mobby will ask questions for clarification; she’s not afraid to speak up individually or in class.
• Mobby will come to ask or at least listen to any conversation / discussion between the second and third draft

• Mobby is a conscientious student; she’ll do what is necessary to help her to succeed; she usually works hard to earn her grades

• when Mobby is confident she will help others and does a good job explaining procedural knowledge to them; sometimes she can explain conceptual knowledge as well

• Mobby’s writing, over time, become longer and longer as she tried to include more explanation into her responses to the prompts

• one word to sum up Mobby: hard-working

Maccy

• Maccy knows she is weak in mathematics and must work extra hard to understand not only procedural knowledge but conceptual knowledge

• Maccy will ask questions but prefers to do so one-on-one

• Maccy needs to see her success to be encouraged by it, she can beat herself up over her mistakes; Maccy thrives when she feels confident in her abilities

• Maccy demonstrates her confidence in her understanding by volunteering to help others

• Maccy always comes to ask for help between the second and third draft; especially in the second semester although she started to towards the end of the first semester

• One word to sum up Maccy: perseveres
Rally

- Rally seems to intuitively understand mathematics; show him a procedure once and he’ll easily pick up on it
- Rally will easily finish a set of practice problems without too much difficulty before his classmates are done
- Rally is second language; his first language is Chinese which he does speak regularly at home
- Rally does have some difficulty articulating his ideas which may in part be due to the language barrier
- Rally will sometimes ask redundant questions in class or questions for things that were just explained, mostly dealing with factual or procedural knowledge
- One word to sum up Rally: confident

Keggy

- Keggy is a serious student; she seems to know what must be done in order for her to succeed in a class and do well
- Keggy will ask questions but mostly one-on-one or for clarification purposes in class
- Keggy will often say that she understand a concept but can’t explain it but she will help others if it’s procedural or factual knowledge they’re asking for
- Keggy will on rare occasion ask for help between the second and third draft or she may just listen on others discussion if she’s around
• Keggy appears to be an independent learner; she’ll do what she has to do to learn the material regardless of the distractions around her

• One word to sum up Keggy – diligent?
Appendix H: Sample Interview Transcript

Thank you for participating in this study. Your participation in this interview will help me get a better understanding of the role of writing in mathematics. I want to reassure you that your answers to the interview questions will remain confidential and will have no impact on your grades in class. Let me also assure you there are also no wrong answers as your responses will help me to plan for next year’s algebra class.

These are the research questions I am interested in:

- How does writing prompt a student’s algebraic reasoning and knowledge of algebraic concepts?
- In what ways does writing, through the revision of multiple drafts, prompt the development of algebraic reasoning?
- In what ways does written teacher feedback, provided during the writing process, prompt the revision of algebraic reasoning in students writing?

How does writing describe a student’s algebraic reasoning and knowledge of algebraic concepts? [THEME: WRITING & ALGEBRAIC REASONING]

How does writing help you to learn?

*Um, well, um when you write stuff you it really shows that you know what you’re talking about because you could be saying stuff that you don’t know but if you write it down everything you know, you summarize it then you make it a good solid paragraph then that really shows that you know everything*
Can you prove and example from any of your classes, it doesn’t matter if it’s this year or last year, any former teacher, current teacher, of what you just spoke about in terms of writing and learning?

Well um like your math essays because like before like, like I thought that everything that my math essay was good but when I get it back every single time there’s something that I have to redo so yeah I sort of just got the picture that every time I do it there’s something that’s going to be wrong, so yeah

Now when it’s wrong, does it mean that you, um… what does it mean to you when it’s wrong?

Um, um I guess it’s like oh, it should have been that and it helps me understand why it should have been that answer and why it shouldn’t have been the one I wrote

Does it also tell you where things are missing? Like gaps?

Yeah, cause when I read my stuff I didn’t realize it cause when I type I just write down everything I think and sometimes um miss stuff that I forget to do, yeah

So how did your previous math teachers use writing in their classes?

Um…um Mrs. Veggie what she did was that she made us write down um show our work and write down all the equations we did and when I didn’t do that I would get like a bad grade so after I wrote down everything it helped me like when I got an answer wrong it helped me realize were I made my mistake and so like showing your work really

\[14\] All names have been changed to maintain the confidentiality of any individuals mentioned in the interview, including the interviewee.
helped me out and like so um, you can see what you’re thinking so it makes it a lot easier to find your mistakes and your answers and stuff

So besides showing your work, are there any other types of writing assignments that you’ve had in your former math classes?

*Um*... no I don’t think so just 8th grade is when I started writing for math

And when you did show your work, and I’m assuming that you wrote down every single step to find the solution, how did that help you to learn the mathematics better?

*Um well it helped me because step by step I could see like I knew what to do afterwards and I learned about order of operations and like I said it helped me to see what was wrong with my answer if I got it wrong on a test or something so I would go back to where I wrote everything down and check and I’d just have to think what I did wrong I could actually see and look over my stuff

Could you compare how writing was used in your former math classes with your experiences now in algebra?

*Uh well before there really wasn’t any writing you had to do it was just take tests and study for quizzes but like now that we have to actually write everything down and show how everything is done yeah, it gives you like a better understanding, it’s hard but it gives you a better understanding of everything

So you’re saying that writing helps you to get a better understanding of the overall concepts in algebra and procedures too?

Yeah, procedures too

Definitions?
Definitions? Oh yeah, definitions, definitely

What can I learn about what you know about algebra from reading your writing?

Um well you can learn the different concepts and the equations that um, that I read about, and like the examples it, it’s like easy to um understand, it’s not really like all these different words it’s like um it flows like introduction and examples of certain things like stuff

What do you struggle the most with when you have to write in algebra?

Um...[pause] I think it’s um the just getting examples or something cause it’s you have to think about uh possible like uh situations where you can use like the uh what you’re taught

So the application portion?

Yeah

What do you feel is your strength when you write in algebra?

Um I think it’s the introduction is like the, like explaining what these things what the stuff is about and like what the different things you can do and use it for

So are you referring to defining...?

Yeah

Like you define your terms well, is that what you mean by your introduction?

Yup

What do you think about having to answer an open ended question in algebra?

Uh... sorry...what’s the....
So in other words normally on a test you’re given a problem and you write the equation and you do the procedure and you come up with an exact solution. And everyone in the class should come up with the exact solution.

Right.

When I give you your prompt, your question, there are multiple ways to answer the question because everyone doesn’t turn in to me exactly the same paper. When you turn in a test to me everyone should have exactly the same answer; I should go down every question and number one is three, number two is ten, number four is eight. But when you do writing for me, everyone can come up with something different in their paper

Oh yeah

So I’m asking you, what do you think about having to answer an open ended question?

I think it’s um it’s, it’s pretty easy or so because you can come up with whatever you want to come up with and not just have to stick with one like um question you could just like multiple amounts like

Do you think there’s an advantage to asking an open ended question or an essay as opposed to an essay question that has to have a very specific answer like your test?

Um...[pause]...um I don’t know, I think that there’s an advantage for like an open ended question cause it’s like you can use your creativity and how you can apply what you learned so yeah it sort of helps you out

Are there any other benefits besides your creativity and your ability to demonstrate you can apply what you’ve learned? Are there any other benefits besides those two in regard to learning the mathematics, concepts, procedures, definitions?
Um…. I don’t really know I’m sure there’s a lot more but those two are just the ones I know off the top of my head

In what ways does writing, through the revision of multiple drafts, describe the development of algebraic reasoning? [THEME: REVISION & REASONING]

The writing process consists of the strategies used to compose a piece of writing such as free writing, which we do in class, prewriting, revising, editing and repeating the process over and over again until you get a finished piece. So in your experiences with writing in general, any class that you want, regardless of the subject or the teacher, how often are you asked to do multiple drafts before turning in the final?

Um, I think, I did, um, well not really now, do we do that we just take notes and they tell us we have to summarize it into our like um into our summary or like a paragraph with that information so not a lot it’s just like take notes and write your paper Is that for my class or for other classes?

It’s for other classes

So maybe your English class, your social studies class…

Its soc. and science

Soc. and science? What about English?

English…uh….we haven’t really done English a lot lately, it’s mostly the soc. cause of all the projects

When you did have to compose, how many drafts did you do?
Well we had to make a web with all the information and then we had to make like a first draft of uh what we thought, usually like we had to make four drafts including the web so like the first draft we had to make corrections and the second draft was like a little more corrections and the third was like the final

Okay so you did a web, and then you wrote a first draft, based on the web, and then you got / made corrections on the first draft and then you wrote the second draft, and then on the second draft you got / made corrections again and then you wrote the final draft?

Yup

Okay. And is this for an English or social studies or science or last year’s English, this year’s…

Like last year

Last year, okay. If you weren’t asked to do more than one draft, let’s say last year or even this year, how often would you use the writing process? Or how many drafts would you compose before you turned in your final?

Um…. [Pause]

If no one asked you to complete any drafts….

I think I would, I would still make like probably like two drafts, so...

So you would write one

Yeah, revise it

Revise it, and then turn it in the second one

Yeah

What do you feel are the benefits of doing multiple drafts before you turn in a final?
The benefit is that you have someone that will correct and show you what’s wrong in your thinking and that helps you out in your second draft when you just know what’s wrong and then you know what you’re supposed to correct so, yeah.

So specifically to this class in algebra, how does the writing process help you to respond to the prompt that I’ve given you? You get a prompt every chapter, so how does the writing process help you to respond to the prompt?

Well uh, it helps me because I learned to make um not make like um become repetitive and not to make my essays too long and just like introduction, like a couple of examples, and counter examples and the conclusion.

When you write your free write, which is your first draft, what kinds of things do you write in there?

Um... the free write I don’t really I’m not able to write that much because after we’re done with the introduction I’m starting to write my example and then time’s up so I don’t really write a lot for the first draft; the second draft is where I write down everything at home and stuff.

So the first draft you at least try for an introduction and you try for an example.

Yeah.

When you get your first draft back from me, what do you do with it?

Um I go over like what I did that was wrong and I see like why it was wrong so I correct it; stuff that I don’t understand I go to you for help.

15 Cissy is referring to non-examples as opposed to counter examples.
So when you read your first draft, you might find errors in there, and then you correct those and that’s what gets turned into me for the second draft

_Yup_

Where do you get your ideas from when you write the second draft besides your corrections?

_Why, like you, like you help like with stuff I can do like give examples like you can do this or do that and I just like think of things that would like I would like go with things I was trying to apply so yeah, just thinking about it, yeah, just thinking about everything_

And where might you get some of these ideas that you’re just thinking about?

_I don’t know, I just, I’m not really sure, I just, it’s like I think about everything, I don’t really know where I get them from_

Now, you give me the second draft and I give it back to you with some feedback. What do you do with the feedback?

_Um I try to like um apply it and read what you said and for the things you don’t understand I try to make it more understandable and yeah, just correct whatever that’s wrong_

Is there anything else that you do with the second draft when you get it back from me besides try to fix the little comments that you get in the bubbles?

_Well I look over what you commented and then I start thinking like oh I understand why now that’s wrong yeah, and stuff_

Do you feel that you’re satisfied with the third draft when you turn it into me?
Um, I think so, yeah, yeah, I’m satisfied because after all of that hard work and when you get the feedback when it’s good then I know that there’s no corrections and everything’s pretty good

How do you know when you’re done with your third draft? You’re working on the third one, fixing up the second one, using the comments, you’re asking questions, how do you know it’s done?

Um well after I correct everything that you commented on I just go over it and go over my corrections and see if it makes sense and if they don’t then I just revise it if it does I just e-mail it

Have you noticed any changes in the way you write as the result of doing the writing process for each chapter?

Yeah I did because former math projects I got like C’s and stuff and now I got like B’s and A’s so I’ve noticed lots of different changes like my introduction to you is like the vocabulary words from the textbook and make short and to the point examples and especially counter examples, it’s hard to think of those but, yeah, I provide counter examples.

Do you notice any differences in the way you think about mathematics and the algebra that you’re learning as a result of writing for every chapter over and over and over and over?

Um not really, but, um not really I really didn’t notice anymore that I think any changes in the way I think about mathematics
So you don’t think differently? So when you first started writing for me, your thinking is the same as you’re writing now for me? From chapter 0 to chapter 6? The way you think about mathematics is still the same?

*Oh wait, no, it’s not the same because I have to be specific about all these different examples and explain what Y is or what X is and why this is the way it is and stuff like that*

So are you saying you’re more aware of the mathematics….

*Yup*

Now after writing so many times than what you’ve used to be before?

*Yup*

In what ways does written teacher feedback, provided during the writing process, promote the revision of algebraic reasoning as represented in students writing?

[THEME: FEEDBACK & REASONING]

So I also want to talk about feedback now. So I want to know from you, what role does feedback play or have in writing?

*It helps, like I said, it helps you it shows you what you did wrong and it really helps you to watch out for stuff like that the next time you’re writing, so yeah, it makes you more aware of and to be careful about stuff*

What type of feedback benefits you the most?
Uh, feedback where it’s like why is this... why is this like so and so it doesn’t make sense, is this really... are you sure about this cause it gets me thinking and then like, oh...

So you like the kinds of feedback that make you think about what you’ve said

Yeah

How does feedback help you to revise your ideas?

Um it shows you like where you have to put more effort into it so yeah, it really helps you out

How would you prefer to get feedback: written, like the way I do for you with the little comment bubbles or oral as if I talked to you?

Um I prefer written because it’s a hard copy and you don’t really have to um remember everything you say so you have a hard copy in case you forget what you did wrong

So you can go back to it later on…

Um hum

And you know what it is. Do you value the feedback that you get and receive from your teachers in terms of your writing?

Yeah

Why

Uh it helps me get a better grade; it also helps me to learn more about writing and the structure of it like five paragraph essays like introduction and stuff like that
So when you receive your feedback on your second draft from me, what does it tell you about your understanding of the ideas in the chapter?

*Uh... that I still have a long way to go cause I thought that correcting the first draft was like the hard part but then there’s like more stuff to correct and I’m like awwww...*

And so you like to use the feedback on the second draft to write the third:

*Yup...it’s sort of like, um, it helps you better cause it refines what you thought and that makes it better*

What would your third draft look like if I didn’t give you any feedback on the second one?

*Um.... I don’t know... I think, I think there’d be a little, a little corrections but not a whole lot cause the first one is where you’d have most of the corrections but if you didn’t correct the second one it wouldn’t be that [vital? Mumbled word]....*

Okay, now are you talking about when I give you back the first draft because I don’t give you any corrections back on your free write. so I’m talking about if I didn’t give you any bubbles on the second one, that’s the one I’m talking about, what would your third draft look like if I didn’t give you any bubbles on your second one at all

*Oh....*

I just give a check on it and gave it back to you for credit… what would your third draft look like?

*I’d be pretty bad from past experience cause every time I get it back it’s like awww, a whole lot of bubbles and I’m like wow, so honestly it’d be pretty bad*
Does getting feedback on your second draft, so all the little bubbles, does it help you to change your thinking about the way you perceive the concepts or the ideas in the math…

Yeah it makes me aware about like all these different things and when I think about one thing I think that’ll work and then I remember like wait no, last time that didn’t’ work so well so it just helps me know when it’s correct and when it’s not

What is the biggest contributor towards developing your ideas for your written pieces?

[Pause] … sorry can you…

What helps you the most to develop and fine tune your ideas to get to that final, third, or last draft, that you’re satisfied with?

Corrections yeah cause like what happens is like cause when you write stuff you think, cause you know what you’re writing about so you think it’s correct but if another person were to read your like paper they would get, it’s like a really good paper they’d get like a real world thing like what people would think cause they don’t know what you’re thinking about so yeah, it’ll help you a lot if someone else is correcting it

What advice would you give to someone, who was given an open-ended question like the way you were given them this year, let’s say a seventh grader, coming into algebra next year, what advice would you give them, on how to write a successful response to a chapter’s question?

Um well, I think the first thing you have to do is just um define just to like write out what the question is and you have to define it and then you have to give like, like good examples and a counter example so they know what to watch out for, so the examples, has to be like, not necessarily short but long enough to where they get the point
Any other pieces of information you want to tell the seventh grader?

    Um... oh yeah, never think that you have the perfect paper because there’s always
going to be a correction

And is there anything else you want to tell me about your experiences with the writing
process in your algebra class this year?

    This year it really helped me because I went from a C average for my tests and
projects to an A so it really shows how far I’ve come and my improvements for writing
and yeah it gives me a better understanding too
Appendix I: Administrator, Student and Parent Consent Forms

Administrator Consent Form (A)

Mrs. Spinach\textsuperscript{16}, Teacher
apple.sauce@missionschool\textsuperscript{17}.org

Administrator’s Consent for the School to Participate in Research Project

*Describing Revisions in Algebraic Reasoning in Middle School Students through the Writing Process*

Dear Mrs. Turnip:

I am requesting permission to conduct research with voluntary members of the grade eight class at Mission School. My research focuses on the following topic: *Describing Revisions in Algebraic Reasoning in Middle School Students through the Writing Process*.

Algebra I is often viewed as the gateway to higher mathematics in High School as well as college (Nathan & Koellner, 2007, Spielhagen, 2006a, 2006b, Department of Education, 1997). However, it is no longer sufficient to merely know a particular set of algorithms; rather understanding when they should be applied, how to apply them, and why this particular situation warrants their application are equally, if not more important. Conceptual development and reasoning are becoming forerunners within mathematics curricula. My focus is on developing algebraic reasoning alongside algorithmic procedures.

This study investigates how writing can foster algebraic reasoning. Algebraic reasoning is a system of systematic thinking, analysis or judgment, extending beyond personal ideas of awareness, allowing individual students to think logically about relationships, make conjectures or draw conclusions based on graphs, functions, equations or expressions within the context of algebra (Loewenberg Ball & Bass, 2003; Yackel & Hanna, 2003; Carpenter, 2003; MathCounts Foundation, 1984; National Research Council Institute of Medicine, 2004; Yackel, 1997). Each chapter will pose an essential question, an open-ended inquiry, which captures the theme and concepts within the chapter. All students in 8A and 8B, as part of the curriculum, will be asked to compose a written response and will be given the opportunity to revise their thinking as the chapter progresses.

\textsuperscript{16} For the purpose of confidentiality, all names and e-mail addresses have been changed to pseudonyms.

\textsuperscript{17} Mission School is a fictitious name to keep the identity of the school anonymous.
Data will be collected in the following ways:

- An analysis of written journals in response to each chapter’s essential question throughout the school year.
- Possible interview in the first semester to investigate the process of writing and transformation of thinking between drafts. Interviews may last approximately one hour.
- Possible interview in the second semester to investigate the transformation of algebraic reasoning and writing over the course of the school year. Interviews may last approximately one hour.
- Collection and use of students’ Algebra I journals written in the 2009-2010 school year.

Students will be interviewed on campus in a private location either before or after school at a time arranged for their convenience. I will audio-record all interviews. Confidentiality of all findings is assured to all participants.

The school’s privacy will be respected. During my research project, I will keep all data in a secure location and have sole access to the data. Pseudonyms will be given to all participants to protect confidentiality in any reporting or possible publication. I will not use students’ names or the school name in the final dissertation report but use a pseudonym instead. Summarized results and recommendations based on the study may be shared with the school. In the event that the dissertation is published, all names of participants and the school will be changed in all publications and reports resulting from this study.

If you agree to participate and to allow the study to be conducted at Mission School, please sign the consent form. A copy of your signed consent form will be returned to you. Please keep it as a reference for your information. If you have questions, please contact me at apple.sauce@missionschool.org or by phone at (808) 952-7171 (voicemail). Thank you.

Sincerely,

Mrs. Spinach
Teacher
I give my consent for Mission School to participate in this study: *Describing Revisions in Algebraic Reasoning in Middle School Students Through the Writing Process*

Name (print)_________________________________________ Title: __________________________

Signature:____________________________________________

Date:_________________________________________________

If you have any questions about your rights or that of the school in this project, you can contact the University of Hawaii, Committee on Human Studies (CHS), 1960 East-West Rd. Biomedical Bldg, B-104, Honolulu, HI 96822. Phone: (808) 956-5007. E-mail: www.hawaii.edu/irb
Dear Parents of Grade Eight students:

A teacher is a lifelong learner and often finds herself in the role of student. To that end, I have returned to graduate school to attain a Ph.D. in Education at the University of Hawaii, Manoa. As a requirement of my course of study, I am about to conduct a dissertation study. Having already obtained approval from school Administration, I am now seeking your assistance.

My research focuses on the following topic: *Describing Revisions in Algebraic Reasoning in Middle School Students Through the Writing Process*. Algebra I is often viewed as the gateway to higher mathematics in High School as well as college (Nathan & Koellner, 2007, Spielhagen, 2006a, 2006b, Department of Education, 1997). However, it is no longer sufficient to know a particular set of algorithms; rather understanding when they should be applied, how to apply them, and why this particular situation warrants their application are equally, if not more important. Conceptual development and reasoning are becoming forerunners within mathematics curriculums. My focus is on developing algebraic reasoning alongside algorithmic procedures.

This study investigates how writing can foster algebraic reasoning. Algebraic reasoning is a system of systematic thinking, analysis or judgment, extending beyond personal ideas of awareness, allowing individual students to think logically about relationships, make conjectures or draw conclusions based on graphs, functions, equations or expressions within the context of algebra (Loewenberg Ball & Bass, 2003; Yackel & Hanna, 2003; Carpenter, 2003; MathCounts Foundation, 1984; National Research Council Institute of Medicine, 2004; Yackel, 1997). Each chapter will pose an essential question, an open-ended inquiry, which captures the theme and concepts of the chapter. All students in 8A and 8B, as part of the curriculum, will be asked to compose a written response and will be given the opportunity to revise their thinking as the chapter progresses.

All students will be asked to compose three drafts; the final draft will be submitted for grading. A rubric (a systematic grading tool) has been designed to establish the expectations of each written assignment and ensure that grading is fair for all students. Students will also receive teacher feedback on their second draft and can utilize all feedback comments to revise their written work before final submission. Participation, or
deciding not to participate in this research project, will have no affect on student grades or overall class standing.

I am asking your permission to involve your child in this study that will take place in the 2010-2011 school year. Your consent to allow your child to participate would mean they would take part in the following:

☐ Analysis of written journals in response to each Algebra chapter’s essential question throughout the school year.

☐ One-on-one interview to investigate the transformation of algebraic reasoning and writing over the course of the school year. Interviews may last approximately one hour.

☐ Collection and use of your child’s Algebra I journals written in the 2012-2013 school year.

If students consent to be interviewed, all interviews will take place either before or after school at a time arranged at their convenience in a private location on campus. I will audio-record all interviews.

Confidentiality of all findings is assured. Individual identities will not be disclosed. Research data will be confidential to the extent allowed by law. Agencies with research oversight, such as the UH Committee on Human Studies, have the authority to review research data. All research records will be stored in a locked file in the Primary Investigator’s office for the duration of the research project. Audio tapes will be destroyed immediately following transcription. All other research records will be destroyed upon completion of the project.

Your child’s privacy will be respected. During my research project, I will keep all data in a secure location and have sole access to the data. Pseudonyms will be given to all participants to protect confidentiality. I will not use your child’s name or your name in the final dissertation report but use a pseudonym instead. Summarized results and recommendations based on the study may be shared with the school. In the event that the dissertation is published, all names of participants and the school will be changed in all publications and reports resulting from this study.

Participation in this project is voluntary. If you do graciously consent to have your child participate, but later change your mind, please feel free to withdraw from the research without prejudice. Participation, or deciding not to participate in this research project, will have no affect on student grades or overall class standing.
If you or your child has any questions, please contact me at apple.sauce@missionschool.org or by phone at (808) 952-7173 (voicemail). If you consent to participate, please sign the consent forms attached and return them to me by: ______________. A full copy of the signed consent form will be returned to you. Please keep it as a reference for your information. Thank you for considering this.

Sincerely,

Mrs. Spinach
Teacher
I certify that I have read and that I understand that my child will be participating in the research project: *Describing Revisions in Algebraic Reasoning in Middle School Students through the Writing Process*. I have been given satisfactory answers to my questions concerning project procedures and other matters. I have been advised that I am free to withdraw my consent and discontinue my child’s participation in this project at any time without prejudice. I hereby give my consent for my child

______________________________________________________
(Print Name of Child)

to participate in this study *Describing Revisions in Algebraic Reasoning in Middle School Students Through the Writing Process*.

Name of parent/Guardian (print):

______________________________________________________

Name of parent/Guardian Signature:

______________________________________________________

Date: _______________________________________________________

If you have any questions about your rights or that of the school in this project, you can contact the University of Hawaii, Committee on Human Studies (CHS), 1960 East-West Rd. Biomedical Bldg, B-104, Honolulu, HI 96822. Phone: (808) 956-5007. E-mail: www.hawaii.edu/irb
Staff Assent Form (C)

Mrs. Spinach, Teacher
apple.sauce@missionschool.org

Student’s Agreement to Participate in Research Project

Describing Revisions in Algebraic Reasoning in Middle School Students Though the Writing Process

Dear Grade Eight Students:

It is important that teachers continue to learn; thus, I am enrolled in graduate school at the University of Manoa to continue my education. As part my graduation requirement, I must conduct a research project, and need your help.

My research is on the following topic: Describing Revisions in Algebraic Reasoning in Middle School Students Though the Writing Process. This study investigates how writing can foster algebraic reasoning and improve understanding of mathematical concepts. My focus is to develop your algebraic reasoning alongside algorithmic procedures. My hope is that this study will help other students better understand mathematics and build a firm foundation for their future mathematics courses.

All students in 8A and 8B will be asked to compose three drafts; the final draft will be submitted for grading. A rubric has been designed to establish the expectations of each written assignment and ensure that grading is fair for all students. Students will also receive teacher feedback on their second draft and can utilize all feedback comments to revise their written work before final submission. Participation, or deciding not to participate in this research project, will have no affect on student grades or overall class standing.

If you decide to participate, it would mean taking part in the following:

☐ I will conduct an analysis of your written journals in response to each chapter’s essential question throughout the school year.

☐ One-on-one interview to investigate the transformation of algebraic reasoning and writing over the course of the school year. Interviews may last approximately one hour.

☐ Collection and use of your Algebra I journals written in the 2012-2013 school year.
If you consent to be interviewed, all interviews will take place on campus either before or after school at a time arranged at your convenience in a private location. I will audio-record all interviews.

Your privacy will be respected. During my research project, I will keep all information in a secure location, and only I can use it. Pseudonyms (false names) will be used to insure confidentiality. Summarized results and recommendations based on the study may be shared with the school. In the event that the dissertation is published, neither participant names nor the school name will be used in all publications and reports resulting from this study.

Participation in this project is voluntary. If you do graciously consent to participate, but later change your mind, please feel free to withdraw from the research without penalty. Participation, or deciding not to participate in this research project, will have no affect on student grades or class standing.

If you have any questions, please contact me at apple.sauce@missionschool.org or by phone at (808) 952-7173 (voicemail).

If you agree to participate, please sign the form attached and return it to me by: __________________. A copy of your signed consent form will be returned to you. Please keep it as a reference for your information. Your parents will also be asked for their consent for you to participate.

Thank you.

Sincerely,

Mrs. Spinach
Teacher
I have read and understood that I will be participating in the research project:

*Describing Revisions in Algebraic Reasoning in Middle School Students Though the Writing Process*

I have been given satisfactory answers to my questions about the project and understand what I must do if I agree to participate. I have been told that I am free to stop my participation in the project at any time without penalty. I hereby give my agreement to participate.

Student name: ____________________________________________________________
(Print name)

Student signature: __________________________________________________________

Date: ____________________________________________________________________

If you have any questions about your rights or that of the school in this project, you can contact the University of Hawaii, Committee on Human Studies (CHS), 1960 East-West Rd. Biomedical Bldg, B-104, Honolulu, HI 96822. Phone: (808) 956-5007. E-mail: www.hawaii.edu/irb
References


McNulty, R. J. (2010, July). *Quadrant D leadership*. Power Point presented at Hawaii Model Schools Conference, Honolulu, HI.


