A DUAL STATE VARIABLE FORMULATION FOR ORDINARY DIFFERENTIAL EQUATIONS

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ABSTRACT

This dissertation defines a new state variable formulation for ordinary differential equations. The formulation allows the systematic identification of eigenvalues for any ordinary differential equation, and leads to parallels with other concepts from linear algebra as well. Furthermore, the eigenvalues described here are generally defined by ordinary differential equations, and as such, the proposed state variable formulation can be reapplied to them. This results in the identification of nested, subsidiary eigenvalues.

As a simple example of its utility, the formulation is applied to the oscillatory motion of the nonlinear pendulum. By modeling the behavior of the eigenvalues for this equation, an approximate solution can be obtained for the period of the pendulum and for its motion. The results are excellent when compared to those of other non-numerical approximation methods.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

CHAPTER 1
INTRODUCTION

Ordinary differential equations, hereafter referred to as ODE's, are often used to describe the motion of discrete dynamic systems. These equations must be solved in some manner to obtain explicit information about the motion of the system. Extracting useful information from differential equations is rarely straightforward, especially if the equations are nonlinear. This dissertation introduces a new and profitable approach to such problems.

The point of view taken in this analysis should be stated before any further discussion. For purposes of exposition, and without loss of generality, the differential equations studied here are assumed to represent the motion of dynamic mechanical systems. Furthermore, most of the dissertation focuses on the analysis of second order ODE's, but the approach is valid for ODE's of any order. Generalization to equations of arbitrary order is the subject of Chapter 9.

Consider a general second order ODE. Commonly the first step in solving such an equation is to reduce it to a set of two first order differential equations, in two variables. These are called the state equations and the state variables. In engineering dynamics the state

equations are almost always formulated in terms of position and velocity, because of their clear physical meaning. However, this choice of state variables is quite arbitrary, and it is proposed here that a different choice may lead to a better understanding of system behavior.

In the state variable form explored here, both variables represent displacement (that is, position). One state variable will represent the solution of interest, and the other can represent some other solution to the same second order differential equation. A different solution, of course, results from a different choice of initial conditions. An alternative point of view is to think of the two state variables as representing the displacement of two identical but separate mechanical systems. For instance, if a mass-spring system is being modeled, one state variable will represent the displacement of that system. An identical but separate mass-spring system is then constructed, and its displacement is used as the second state variable. It is important to note that the motion of the two systems will not generally be the same, since they will likely have different initial conditions. The use of two solutions to the same equation, or of two identical systems, is one reason this is called the dual state variable formulation.

In Chapter 4, it will be shown that requiring the second state variable to be another solution to the same ODE is an unnecessary restriction. The displacement of any dynamic system can be used instead. At present, however, this generalization will only confuse the issue. Until Chapter 4, the reader should assume that the second state variable represents another solution to the ODE under study.

The most striking result of the dual state variable formulation may be that it allows the systematic identification of eigenvalues for any ODE, linear or nonlinear. For some equations, the eigenvalues can be described exactly, in closed form, but they are usually available only in differential form. Even then, their behavior can sometimes be approximated. Approximate models of eigenvalue behavior can be used to find surprisingly accurate approximate solutions for certain nonlinear differential equations, as will be demonstrated for the nonlinear pendulum. Eigenvalues are extremely useful in the analysis of linear equations, and their extension to nonlinear equations is expected to lead to new approaches to the investigation and modeling of many physical systems.

The dissertation begins with a derivation of the basic state variable formulation. Because of the unusual nature of this formulation, it will first be introduced in connection

with linear second order ODE's. The formulation for general second order ODE's can then be presented in a concise and simple manner. The general second order formulation leads to results strikingly similar to those of linear algebra, and these parallels will be discussed in Chapters 5 and 6. As an example, the formulation will then be applied to predict the motion of the nonlinear pendulum. The dissertation will conclude by demonstrating how this formulation can be applied to ODE's of any order.

CHAPTER 2
THE LINEAR FORMULATION

Because this formulation is somewhat unusual, it is instructive to introduce it in connection with second order linear homogeneous ODE's. Once this simple case is understood, its application to nonlinear or nonhomogeneous problems will be easy to follow. The real advantages of the formulation will become apparent in the nonlinear case.

Begin with the differential equation (2.1).

$$x'' = -fx' - gx \tag{2.1}$$

Assume this represents the motion of an elementary damped oscillator, normalized so the mass is unity. The independent variable is assumed to be time, and coefficients f and g can be functions of time. This equation is typically reduced to the following set of first order equations, or state variable form, where the state variables x and v represent the displacement and velocity of the system mass.

$$\begin{aligned} x' &= v \\ v' &= -fv - gx \end{aligned} \tag{2.2}$$

It is clear that equations (2.2) recombine to yield (2.1), and from a practical standpoint that is the only requirement (2.2) must meet to be a state variable representation of (2.1). This particular state variable formulation is almost always used by engineers because of the obvious physical significance of the state variables x and v.

There are, however, an infinite number of other state variable formulations that will also recombine to yield (2.1). A more general linear state variable formulation for (2.1) can be written as follows:

$$x' = \alpha x + \beta y \tag{2.3}$$

$$y' = \gamma x + \delta y . \tag{2.4}$$

Even with the restriction that equations (2.3) and (2.4) must recombine to form (2.1), an infinite number of possible choices exist for the variable coefficients α, β, γ, and δ. The system displacement is still represented by x, but depending on how α, β, γ, and δ are selected, y may not have any physical significance at all. None the less, as will be shown later, this state formulation can still serve to completely identify the state x,v of the system.

The objective of the following analysis is to find, within the infinite number of possibilities permitted by (2.3) and (2.4), a particularly simple and useful state

variable representation of (2.1). This will be done by applying certain conditions that are intuitively selected to impose forms of symmetry on the state space.

Condition 1

Require that x and y both represent the displacement of the same physical system. That is, y must satisfy the equation

$$y'' = -fy' - gy. \tag{2.5}$$

This means x(t) and y(t) are two different solutions to equation (1.1). An alternative view is that, if x represents the displacement of a mass-spring-damper system, then y represents the displacement of another mass-spring-damper system with an identical mass, spring and damper. The two systems are constructed exactly alike, but they will be thought of as separate entities because the motion of the y system will not generally be the same as the motion of x. That is, the two systems will generally have different initial conditions. The system in y is referred to as the dual system.

This condition was selected because it imposes a kind of symmetry on the state space. It means that both axes of

the state space, the x and y axes, measure the same quantity, displacement. Measurements in the state space are made using the same units, regardless of the direction.

To apply this condition, first differentiate the state equation (2.3).

$$x'' = \alpha' x + \alpha x' + \beta' y + \beta y' \tag{2.6}$$

Then use (2.4) to substitute for y', and use (2.3) again to substitute for y. An equation in x results.

$$x'' = \left(\frac{\beta'}{\beta} + \alpha + \delta\right)x' + \left(\alpha' - \alpha\frac{\beta'}{\beta} - \alpha\delta + \gamma\beta\right)x. \tag{2.7}$$

Similarly, find an expression for y''.

$$y'' = \left(\frac{\gamma'}{\gamma} + \alpha + \delta\right)y' + \left(\delta' - \delta\frac{\gamma'}{\gamma} - \alpha\delta + \gamma\beta\right)y. \tag{2.8}$$

Equate (2.1) to (2.7), and (2.5) to (2.8), to find the following four constraint equations for the coefficient functions α, β, γ, and δ.

$$\frac{\beta'}{\beta} + \alpha + \delta = -f \tag{2.9}$$

$$\alpha' - \alpha\frac{\beta'}{\beta} - \alpha\delta + \gamma\beta = -g \tag{2.10}$$

$$\frac{\gamma'}{\gamma} + \alpha + \delta = -f \tag{2.11}$$

$$\delta' - \delta\frac{\gamma'}{\gamma} - \alpha\delta + \gamma\beta = -g \tag{2.12}$$

Solve these for the derivative terms:

$$\alpha' = -\gamma\beta - \alpha^2 - f\alpha - g \tag{2.13}$$

$$\beta' = -\beta\left(f + \alpha + \delta\right) \tag{2.14}$$

$$\gamma' = -\gamma\left(f + \alpha + \delta\right) \tag{2.15}$$

$$\delta' = -\gamma\beta - \delta^2 - f\delta - g \tag{2.16}$$

Note that the equations for γ' and β' have the same form, and have similar exponential solutions, so γ can be expressed in terms of β.

$$\gamma = c_1\beta \tag{2.17}$$

The constant c_1 represents the ratio of the initial conditions, $c_1 = \gamma_0/\beta_0$.

Similarly, consider the following expression:

$$\alpha' - \delta' = -(\alpha - \delta)(f + \alpha + \delta). \tag{2.18}$$

The quantity $(\alpha - \delta)$ also has an exponential solution similar to beta's, and it can be written

$$\alpha - \delta = c_2\beta, \tag{2.19}$$

or

$$\delta = \alpha - c_2\beta. \tag{2.20}$$

The first condition is now satisfied. The constraints it imposes on α, β, γ, and δ are summarized below.

$$\alpha' = -\gamma\beta - \alpha^2 - f\alpha - g \tag{2.13}$$

$$\beta' = -\beta(f + \alpha + \delta) \tag{2.14}$$

$$\gamma = c_1\beta \tag{2.17}$$

$$\delta = \alpha - c_2\beta \tag{2.20}$$

These constraints ensure that the state equations (2.3) and (2.4) are consistent with both (2.1) and (2.5). Variables α,β,γ, and δ are now defined, albeit not uniquely. The next condition serves to identify the two constants c_1 and c_2, but before it can be applied, the state space must be defined.

The State Space Reference Frame

It is important to carefully define an appropriate state space reference frame. For the linear state variable representation (2.3) and (2.4), a plane, orthogonal state space proves to work well, and this simple state space will be used throughout the dissertation. It should be noted, however, that a plane, orthogonal state space is not essential to the formulation.

The state point will be represented by the vector **S**, defined in polar coordinates by length ρ and angle ϕ. The rate of change of **S** is given by the rate vector **R**, whose components (x',y') are defined by the state equations (2.3) and (2.4).

Let $\mathbf{e}_1$ and $\mathbf{e}_2$ be unit vectors in the x and y directions, respectively. Any arbitrary vector **u** represented in this frame will be written $\mathbf{u} = u_1\mathbf{e}_1 + u_2\mathbf{e}_2$. A scalar product must

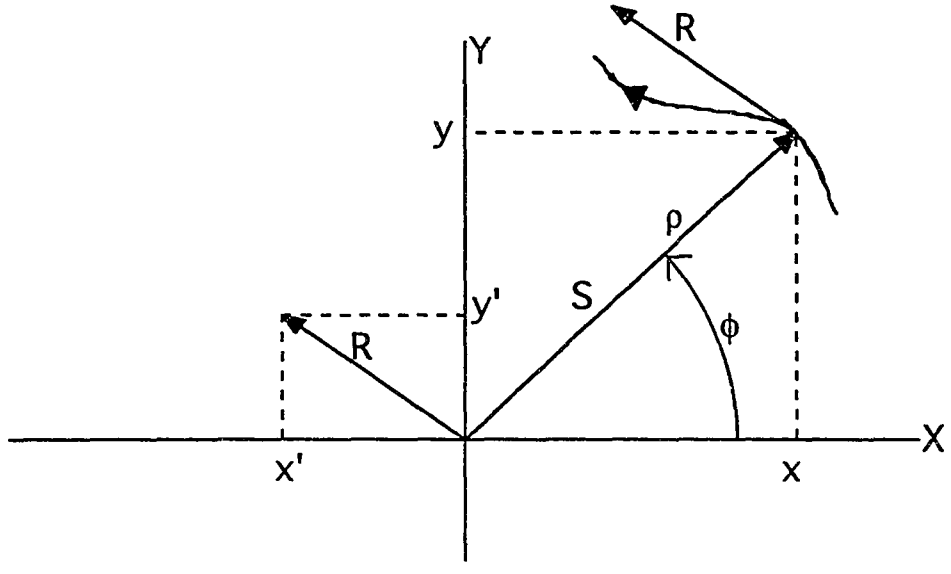


Figure 2-1: The state space. A portion of a state space solution curve is shown, and the rate vector **R** is shown both on the curve and translated to the origin.

also be defined. The usual Euclidean scalar product will be chosen:

$$\mathbf{u}\bullet\mathbf{v} = u_1v_1 + u_2v_2, \tag{2.21}$$

or

$$\mathbf{u}\bullet\mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos\Theta, \tag{2.22}$$

where Θ is the included angle. The values of c_1 and c_2 can now be defined through application of the next condition.

Condition 2

Require the scalar product **S•R** to be invariant. Invariance is defined here as invariance with respect to rotations of the state space reference frame. That is, **S** and **R** must both retain the same lengths and relative orientation no matter how the frame is rotated with respect to **S**. This condition is also chosen because of the symmetry it imposes, as will be seen later.

Another way to present the second condition is to show what happens when a rotation transformation $\mathbf{R}_\theta$ is applied to the state equations. ($\mathbf{R}_\theta$, a 2x2 matrix in this case, is not to be confused with the rate vector **R**.) Let equations (2.3) and (2.4) be written in the matrix form **R=AS**, where

$$\mathbf{R} = \begin{bmatrix} x' \\ y' \end{bmatrix}, \qquad \mathbf{A} = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}, \qquad \text{and} \qquad \mathbf{S} = \begin{bmatrix} x \\ y \end{bmatrix}.$$

If the rotation transformation $\mathbf{R}_\theta$ is applied to both vectors **R** and **S**, then the matrix form of the state equations becomes

$$\mathbf{R}_\theta\mathbf{R} = \mathbf{A}\mathbf{R}_\theta\mathbf{S}, \tag{2.23}$$

or

$$\mathbf{R} = \mathbf{R}_\theta^{-1}\mathbf{A}\mathbf{R}_\theta\mathbf{S}. \tag{2.24}$$

Condition 2 can be restated as the requirement that

$$\mathbf{R}_\theta^{-1}\mathbf{A}\mathbf{R}_\theta = \mathbf{A}. \tag{2.25}$$

To apply condition 2, write $\mathbf{S}\bullet\mathbf{R}$ and expand it using the state equations (2.3) and (2.4).

$$\mathbf{S}\bullet\mathbf{R} = xx' + yy' = \alpha x^2 + \delta y^2 + (\beta + \gamma)xy \tag{2.26}$$

The square of the length of the state vector,

$$|\mathbf{S}|^2 = \rho^2 = x^2 + y^2, \tag{2.27}$$

is invariant with respect to rotations of the frame, so factor that out of (2.26) in a simple manner to find the following two constraint equations,

$$\delta = \alpha \tag{2.28}$$

and

$$\beta + \gamma = 0\ . \tag{2.29}$$

When these constraints are applied to equations (2.17) and (2.20), it is clear constant c_1= -1, and c_2=0. These constraints also yield the following expression for $\mathbf{S}\bullet\mathbf{R}$:

$$\mathbf{S} \bullet \mathbf{R} = \alpha\rho^2. \tag{2.30}$$

This completes the formulation.

Summary of the Linear Formulation

The two conditions imposed above uniquely define the dual state variable formulation for the second order linear homogeneous case. Only the initial conditions for x, y, α, and β remain unspecified. Using the results of these conditions, the formulation can be summarized as follows: The original differential equation

$$x'' = -fx' - gx \tag{2.1}$$

is represented in state form by the equations

$$x' = \alpha x + \beta y \tag{2.31}$$

$$y' = -\beta x + \alpha y, \tag{2.32}$$

where

$$\alpha' = \beta^2 - \alpha^2 - \alpha f - g \tag{2.33}$$

and

$$\beta' = -\beta(f + 2\alpha). \tag{2.34}$$

Alpha and beta can be thought of as variables that relate the solutions of the x and y systems.

A note about initial conditions is in order. Initial conditions are assumed to occur at time t=0. There are only four first order differential equations in the dual state variable equation set listed above, but there are six initial conditions of practical interest: x_0, x_0', y_0, y_0', α_0, and β_0. Only x_0 and x_0' must be specified to identify a solution for equation (2.1), so there is some flexibility in the choice of the other initial conditions. For example, suppose x_0 and x_0' are given. One possibility is to choose α_0 and β_0 at will, and then find y_0 and y_0' from equations (2.31) and (2.32). Once any four of the initial conditions have been chosen, equations (2.31) and (2.32) specify the remaining two. Note also that if the initial conditions for x and y are both set at zero, then the system will remain at the origin, which is an equilibrium point, and the formulation will not be very useful.

In a way, equations (2.31) - (2.34) can be thought of as a four dimensional state representation of equation (2.1). In order to fully identify the state of the x system at some instant, i.e., in order to identify x and x', all four of the quantities x, y, α, and β must be known. Also, alpha and beta can be thought of as functions that relate the behavior

of the x and y systems; that is, they serve to relate two different solutions to equation (2.1).

The state equations (2.31) and (2.32) may seem familiar to some readers. In fact, they can be found in several texts; see, for example, Hirsch and Smale, page 57. The formulation presented here, however, recognizes that the coefficients α and β can be defined in differential form. By so doing, it becomes possible to apply the state equations to a wider range of problems. Equations (2.33) and (2.34), and the way in which the state variable y is defined, appear to set the dual state variable formulation apart from others described in the literature.

At first glance it may seem that this formulation only adds complexity, but in fact it will be shown to contain some very interesting and useful information about the behavior of the system, especially when the nonlinear formulation is studied later. Before proceeding to nonlinear equations, however, there is more to be gained from investigating second order homogeneous linear systems. The remainder of this chapter will present two consequences of the formulation above, and some instructive examples will be presented in Chapter 3.

The Polar Form of the State Equations

Many conditions other than those promoted above can be devised to define the variables α, β, γ, and δ. One of the reasons this particular formulation seemed promising was the particularly simple form of the state equations (2.31) and (2.32) when they are cast in polar form.

The polar coordinates of the state point (see Figure 2-1) are expressed as:

$$\rho = \sqrt{x^2 + y^2} \tag{2.35}$$

$$\phi = \arctan\left(\frac{y}{x}\right). \tag{2.36}$$

By differentiating these and making appropriate substitutions, the polar form of the state equations can be found:

$$\rho' = \alpha\rho \tag{2.37}$$

$$\phi' = -\beta. \tag{2.38}$$

These remarkably simple equations can be anticipated given the form of the state equations (2.31) and (2.32).

They are decoupled, and they show that the oscillatory behavior of the system depends only on beta, and changes in the length of the state vector depend only on alpha. Alpha, therefore, may eventually prove useful in the analysis of system stability.

It will be seen later that the state equations, in both polar and cartesian form, remain unchanged even when the formulation is extended to ODE's in general.

Congruence of Solution Curves

One other property of the formulation (2.31)-(2.34) deserves mention. It is possible to set this formulation up so that all the solution curves in the x,y state space are identical in shape. The solution curves will differ from one another only by a rotation and magnification, as is shown in Figure 2-2. This is accomplished by selecting any set of initial conditions for the α' and β' equations, and then using these same initial conditions in all cases. All possible initial conditions for the x system can still be reproduced.

In this manner, given a single solution curve, all other curves can be generated from it via a rotation and magnification. It is useful in this context to define a master solution curve. Denote points on this curve as (x_m, y_m), and choose initial conditions $x_m(0)=1$ and $y_m(0)=0$, or, in polar

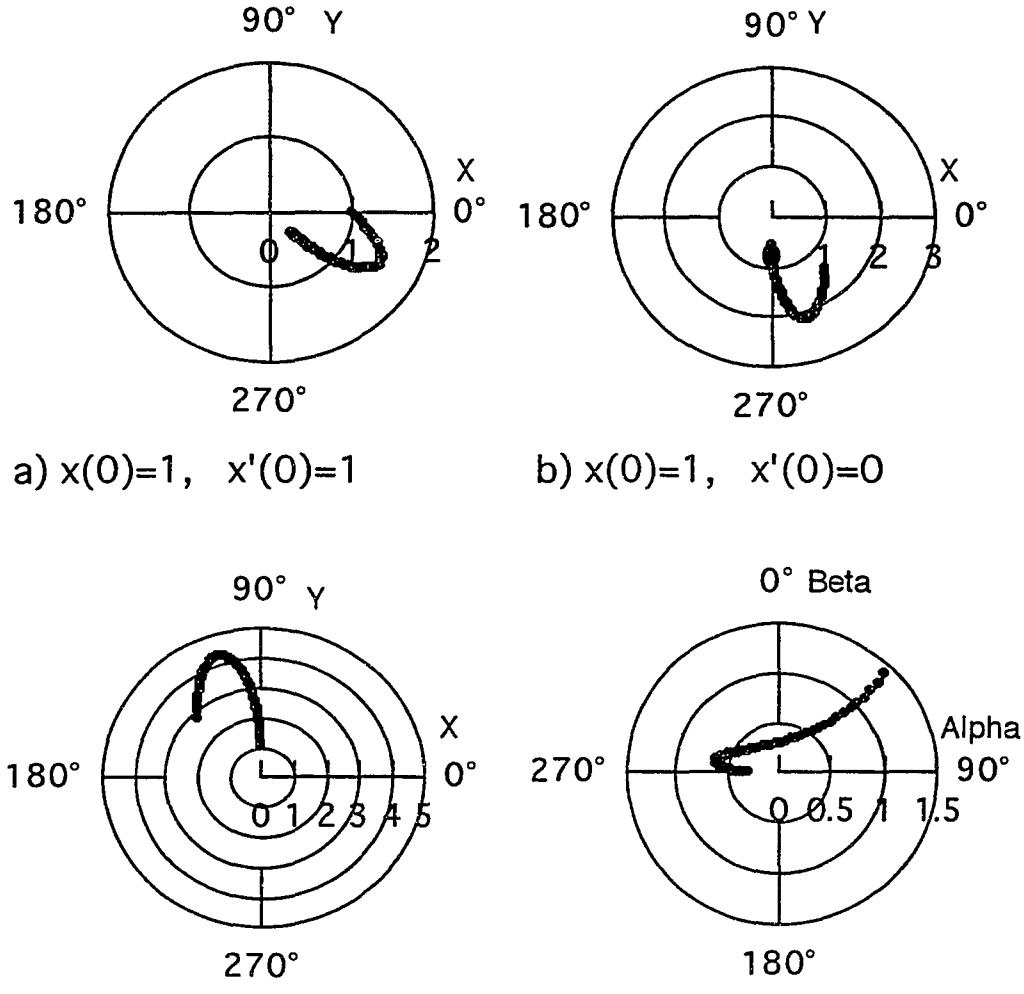


Figure 2-2. Congruent state space solution curves for the equation $x''+tx'+x=0$. The initial conditions for α and β are the same in all cases, so there is only one α,β solution curve, shown in (d). Note that all solution curves in the x,y space are identical except for magnifications and rotations.

form, $\rho_m(0)=1$ and $\phi_m(0)=0$. Once this master curve is computed from equations (2.31)-(2.34), the solution for arbitrary initial conditions (ρ_o, ϕ_o) can be found according to the following formula.

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \rho_o \begin{bmatrix} \cos\phi_o & -\sin\phi_o \\ \sin\phi_o & \cos\phi_o \end{bmatrix} \begin{bmatrix} x_m(t) \\ y_m(t) \end{bmatrix} \tag{2.39}$$

This is a characteristic of the state equations (2.31) and (2.32), and is again recognized in the text by Hirsch and Smale, on pages 55-56. The congruence of solution curves is interesting, but it does not occur in the nonlinear case. It is presented only as a curiosity, and will not be discussed further.

CHAPTER 3
APPLICATION TO
CONSTANT COEFFICIENT DIFFERENTIAL EQUATIONS

The solution of equation (2.1) is almost trivial when coefficients f and g are constant. None the less, study of this simple case will serve to illuminate some intricacies of the dual state variable formulation.

This chapter will focus on the behavior of α and β for second order constant coefficient linear homogeneous equations. Alpha and beta are defined by the coupled differential equations (2.33) and (2.34), and are completely independent of x and y. There is a family of different solutions to these equations, each solution stemming from some choice of initial conditions α_0 and β_0. These solution curves can be plotted in an α, β state space. Figure 3-1 shows typical solution curves for a constant coefficient underdamped system in which the spring rate g is positive. Figures 3-2 and 3-3 show similar solution curves for overdamped and critically damped systems, respectively.

Note that symmetric solution curves exist above and below the alpha axis. This turns out to be the case for any homogeneous linear equation, whether the coefficients are constant or not.

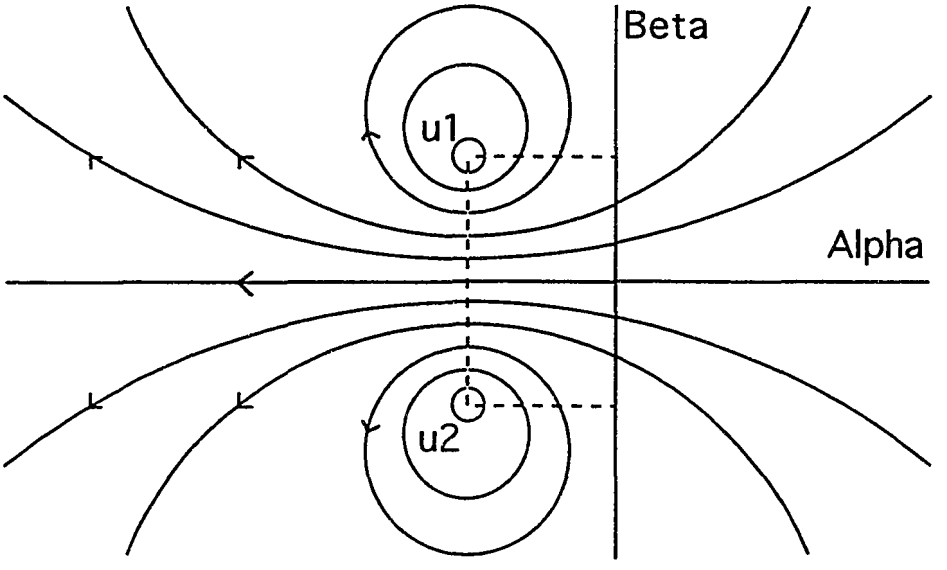


Figure 3-1: Solution curves for a typical underdamped problem, where $f < 2\sqrt{g}$ and $g > 0$.

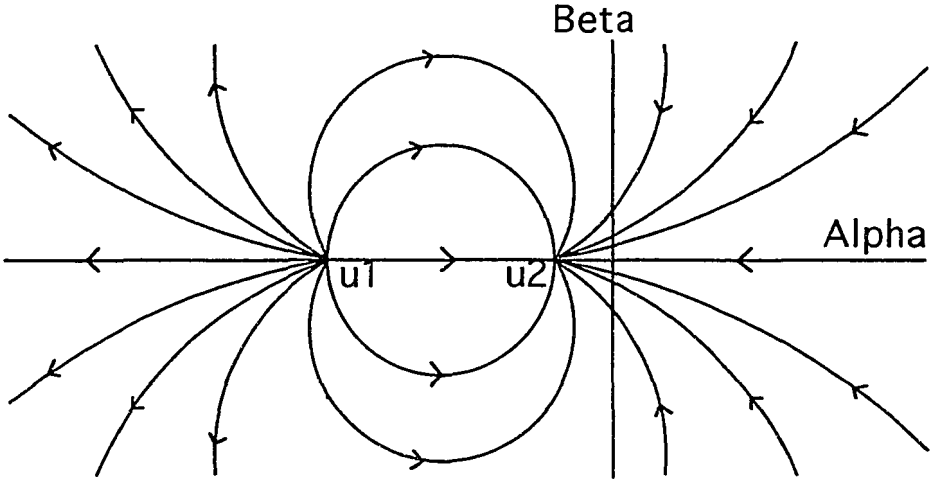


Figure 3-2: Solution curves for a typical overdamped problem, where $f > 2\sqrt{g}$ and $g > 0$.

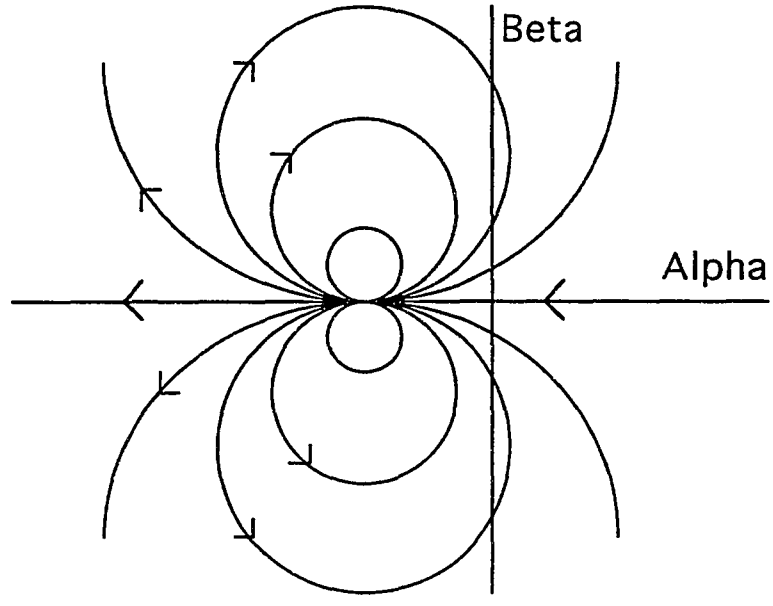


Figure 3-3: Solution curves for a critically damped problem. All paths terminate at a single point on the alpha axis.

Note also that the initial conditions α_0 and β_0 can be expressed in terms of the initial conditions for the x and y systems through the use of equations (2-31) and (2-32).

The Constant Alpha, Beta Solution

When f and g are constants (and only then), there is one obvious and simple set of solutions for α and β: the singular points at which α and β remain constant. In Figures 3-1 and 3-2, these are points u1 and u2. These solution points can easily be found by setting α' and β' to zero, and then solving (2.33) and (2.34) simultaneously. For

underdamped systems represented by Figure 3-1, these points are

$$\alpha = -\frac{f}{2} \quad \text{and} \quad \beta = \pm\sqrt{g - \frac{f^2}{4}}. \tag{3.1}$$

For overdamped systems represented by Figure 3-2, these points are

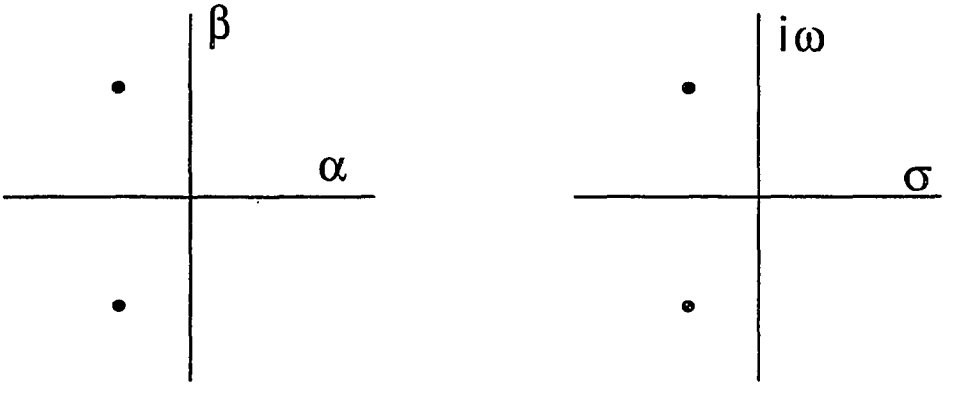
$$\alpha = -\frac{f}{2} \pm \sqrt{\frac{f^2}{4} - g} \quad \text{and} \quad \beta = 0. \tag{3.2}$$

For critically damped systems represented by Figure 3-3, there is only one singular point, at

$$\alpha = -\frac{f}{2} \quad \text{and} \quad \beta = 0. \tag{3.3}$$

The constant solutions for alpha and beta presented above have a unique meaning. Consider the eigenvalues for equation (2.1) when it is written in the traditional x,v (i.e., x, x') state variable form. The eigenvalues are the roots of the characteristic equation, or poles, associated with (2.1), and they are plotted in what engineers call the s-plane. When the constant solutions for α and β are employed, the eigenvalues can be written as $\alpha \pm i\beta$. In fact, there is a

direct relationship between the s-plane and the α,β plane. The α axis can be thought of as coincident with the real axis in the s-plane, and the β axis is coincident with its imaginary axis (Figure 3-4). In both planes, the horizontal location of a point determines how the motion of the system is growing or decaying. The vertical location determines oscillatory behavior. For the α,β plane this is obvious from the polar form of the state equations, (2.37) and (2.38).



a) The α,β plane showing constant α,β solutions.

b) The s-plane, showing eigenvalues derived from the traditional x,v state formulation.

Figure 3-4: The relationship between the s-plane and the α,β plane for a constant coefficient differential equation representing an underdamped system.

Although the two planes are clearly related, it will become obvious that they do not in general contain the same information. For one thing, Figures 3-1,2, and 3 show that the α,β plane contains far more information than the s-plane, but other differences exist as well. Eigenvalues will be discussed more fully in later chapters, but the correspondence is worth introducing here.

Another Special Solution: Beta=0

The α' and β' equations (2.33) and (2.34) always admit the solution $\beta \equiv 0$. Equation (2.33) then becomes:

$$\alpha' = -\alpha^2 - \alpha f - g. \tag{3.4}$$

This will be recognized as the classical Riccati equation, which is a first order nonlinear representation of the original differential equation (3.1). Equation (3.4) can be converted to equation (2.1) with the substitution

$$\alpha = \frac{x'}{x}. \tag{3.5}$$

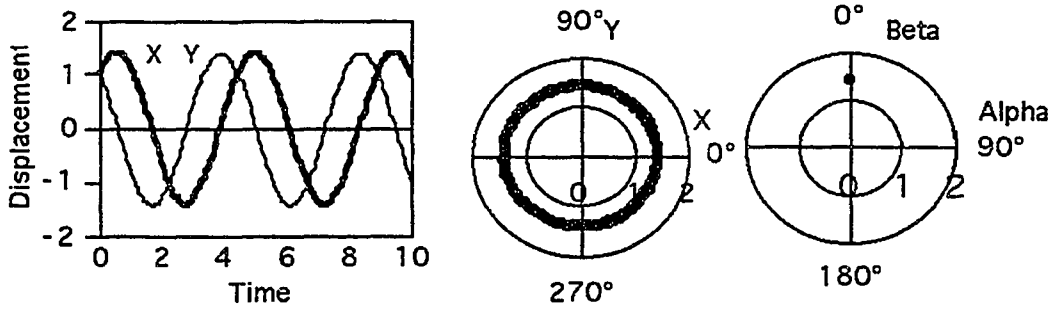
Apparently, the Riccati equation is a degenerate form of equations (2.33) and (2.34). This correspondence with the Riccati equation is the subject of a later chapter.

Behavior in the X,Y State Space

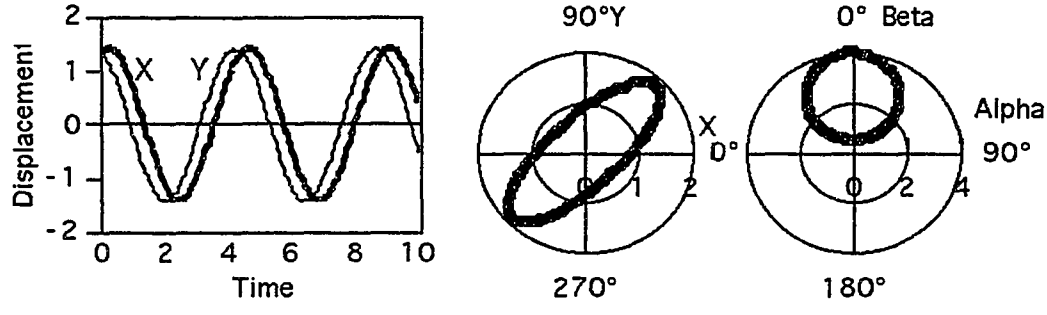
This section is only intended to give the reader a qualitative impression of the behavior of solution curves in the x,y and α,β state spaces. The cases considered are very simple, but certain characteristics of this behavior will be seen later, even in the analysis of nonlinear problems.

Between the constant α,β solution and the $\beta \equiv 0$ solution lie an infinite number of solution curves for α and β (see Figures 3-1, 2, and 3). For each of these solution curves, one can still choose any initial conditions x_0 and y_0. Behavior in the x,y state space can therefore vary considerably, especially for damped systems or systems with time variable coefficients. For the moment, consider a simple undamped oscillator, with initial conditions chosen so that the x and y systems have the same amplitude. Recall that the solutions for α and β are independent of x and y. The behavior for three different α,β solution curves is shown in Figure 3-5.

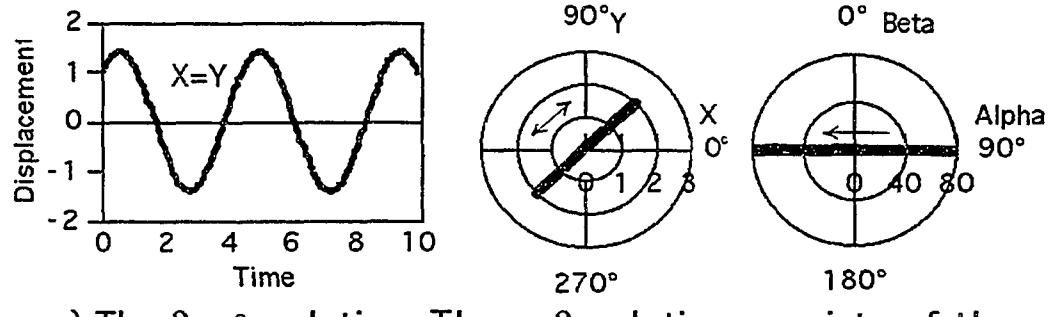
For an underdamped oscillator with constant coefficients, the α,β solution curves are in fact circular orbits (Figure 3-1). The constant α,β solution and the $\beta \equiv 0$ solution are the extremes, the solution for $\beta \equiv 0$ being a circle of infinite radius. When the constant α,β solution is



a) The constant α,β solution; x and y are 90 deg out of phase.



b) This intermediate α,β solution curve is circular, and x and y have a smaller phase difference than in Fig. (a).



c) The $\beta \equiv 0$ solution. The α,β solution consists of the entire α axis, from $-\infty$ to $+\infty$, traversed several times. The x and y systems are exactly in phase.

Figure 3-5: Behavior in the x,y state space for the equation $x'' + 2x = 0$. The data was obtained by numerically integrating equations (2.31)-(2.34).

employed, as in Figure 3-5a, x and y will always be 90 degrees out of phase and of equal amplitude. As the α,β orbits become larger and larger circles (Figure 3-5b), the phase relationship between x and y approaches either 0 or 180 degrees. At the extreme, when $\beta \equiv 0$, the state equations (2.31) and (2.32) lead to the following relationship:

$$\alpha = \frac{y'}{y} = \frac{x'}{x}. \tag{3.6}$$

Equation (3.6) requires y=cx, where c is a constant, and therefore y and x must be exactly in phase, or 180 degrees out of phase (Figure 3-5c). This extreme is difficult to work with since alpha periodically becomes infinite.

In the example above, the initial conditions were selected so that the x and y systems had the same amplitude. This occurs naturally when α and β are constants, but in general it is not necessary. Figure 3-6 depicts a solution for the same example, in which the amplitudes are different. The α, β solution curve will still be circular, since equations (2.33) and (2.34) for α and β are completely independent of x and y.

So far, stable orbits have been shown in both the x,y state space and the α,β plane, because there has been no damping. Now consider a system with constant subcritical

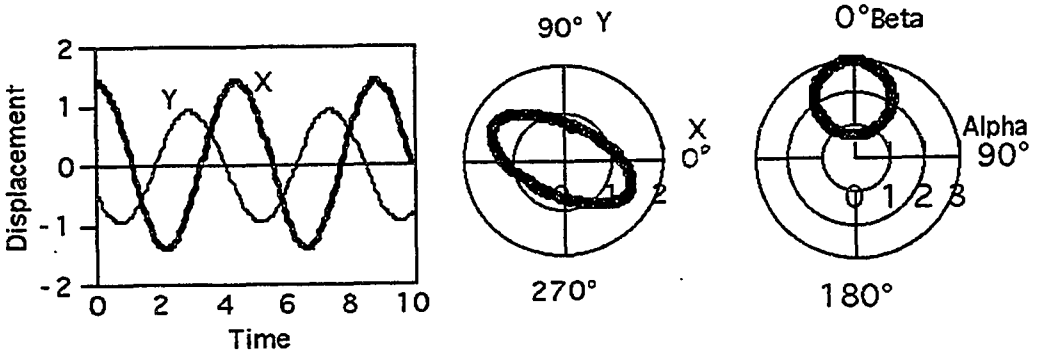


Figure 3-6: Behavior of the equation $x'' + 2x = 0$, with initial conditions chosen so that x and y have different amplitudes.

damping. When the constant solution for α and β is employed (Figure 3-7), the x,y orbit decays as a perfect logarithmic spiral. This can be seen by studying the polar form of the state equation (2.37), $\rho' = \alpha\rho$. From equation (3.1) it is clear α is a negative constant, and the solution to (2.37) will then be a simple decaying exponential.

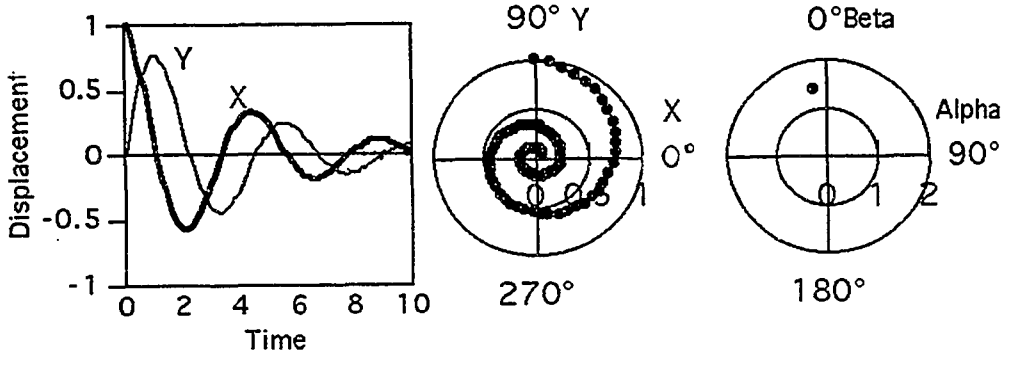


Figure 3-7: Behavior of the damped system $x'' + 0.5x' + 2x = 0$.

When a nonconstant α,β solution is chosen for a damped problem, the x,y solution curve is still a spiral (Figure 3-8). The distance from the state space origin to the state point (ρ) grows or decays depending on the sign of α. Again, the α,β solution curve remains a periodic, closed orbit. That is due to the nature of equations (2.33) and (2.34), when the damping coefficient and spring rate are constant, and because (2.33) and (2.34) are completely independent of x and y.

Overdamped equations have α,β solution curves as shown in Figure 3-2. There is a source and a sink on the alpha axis, defined by the constant α,β solutions of equation (3.2). All α,β solution curves will approach the sink as time passes, unless $\beta=0$; in that case, the solution curve may approach negative infinity. Whenever a solution is chosen in which $\beta=0$, the solution for y will be proportional to the solution for x, as shown in Figures 3-9a and b. Again, this follows from equation (3.6). In such a case, both the α,β path and the x,y state space trajectory will be straight lines.

An example for nonzero beta is shown in Figure 3-9c. The α,β trajectory is approaching the sink, but it will take an infinite amount of time to get there.

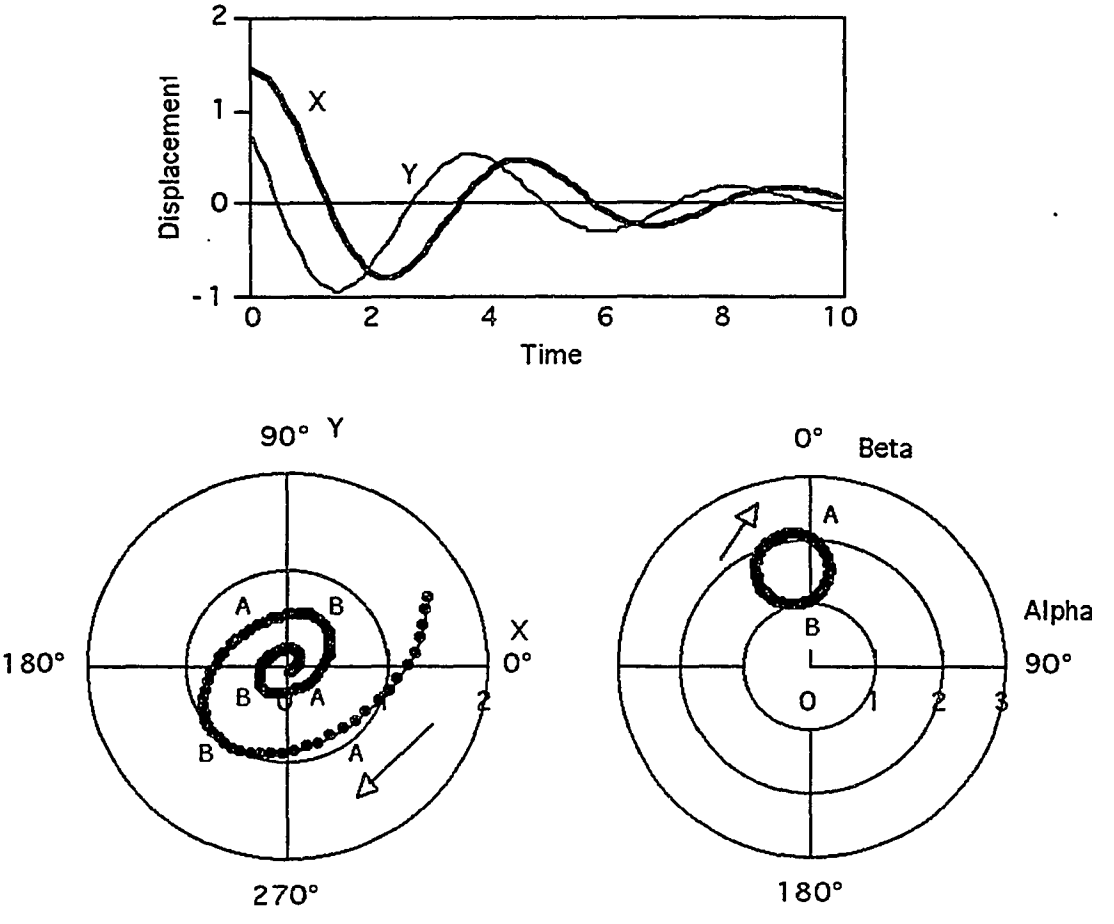
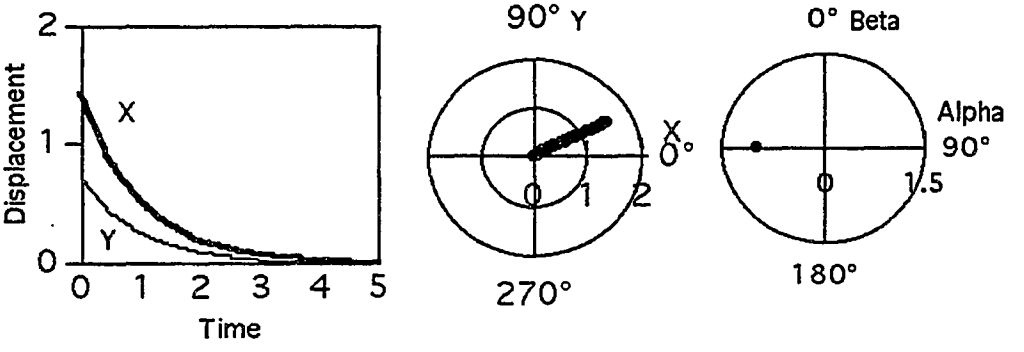
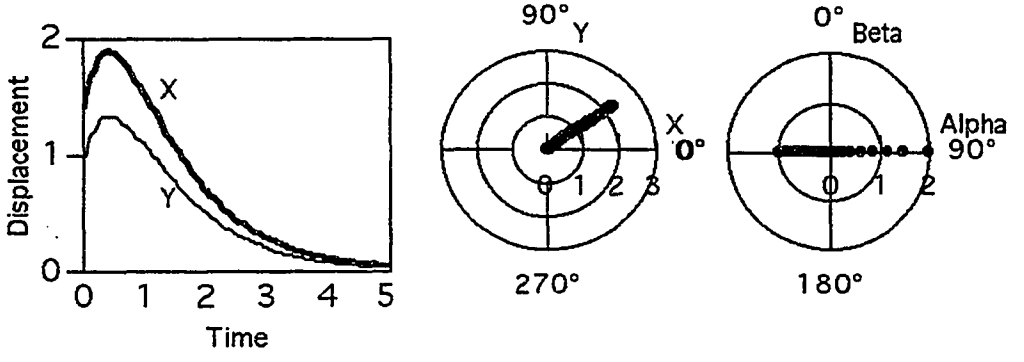


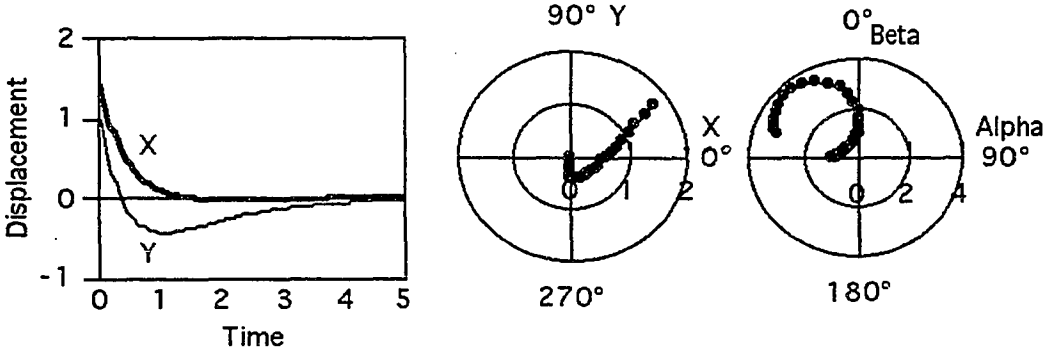
Figure 3-8: Behavior of the damped system $x'' + 0.5x' + 2x = 0$, with a nonconstant α, β solution. Points A and B on the polar curves are matched, and show that the sign of α is directly linked to the growth or decay of the state plane spiral.



a) The constant α,β case. X and y are linearly dependent.



b) $\beta = 0$, but $\alpha_0 = +2$. The α,β point approaches the sink at $\alpha = -1$. Again, x and y are linearly dependent.



c) A more general solution curve. Again, the α,β point moves toward $\alpha = -1$.

Figure 3-9: Behavior of the overdamped system $x'' + 3x' + 2x = 0$.

Closed Form Solutions for Alpha and Beta

The constant α, β solutions given earlier are easy to find, but the remaining solution curves can be identified as well. That is, equations (2.33) and (2.34) can be solved generally, in closed form, when f and g are constant. The derivation will only be outlined below, and the actual solutions are not central to this dissertation.

The first step is to define the geometry of the α,β solution curves. For the solution curves shown in Figure 3-1 (the underdamped case), this is done via empirical study of the

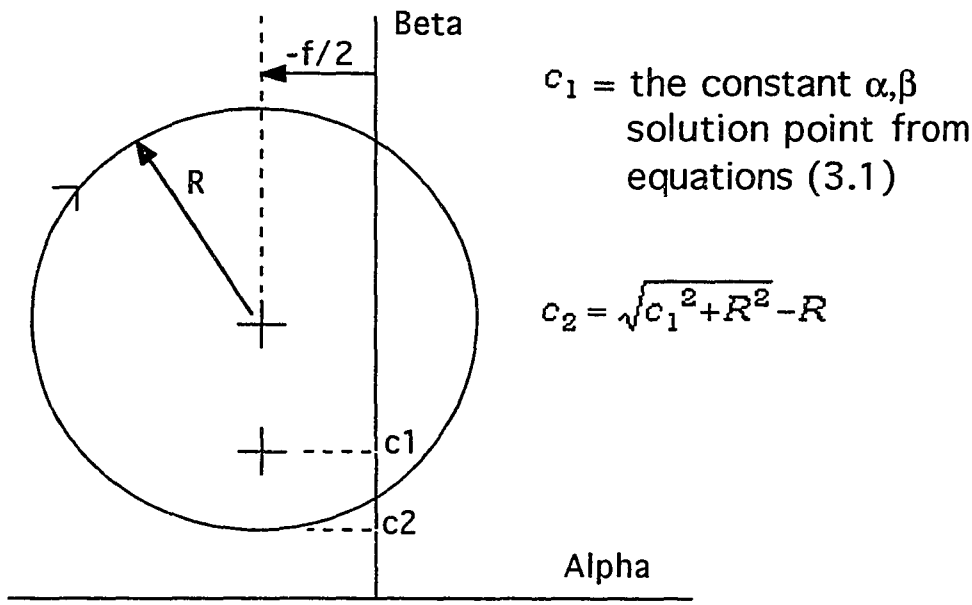


Figure 3-10: The geometry of an α,β solution curve from Figure 3-1. The radius R is chosen arbitrarily.

numerical solutions for some simple problems. The curves are found to be circles, and the geometry is defined in Figure 3-10. For the curves shown in Figure 3-2 (the overdamped case), the task is simpler. Each curve is assumed to be a circular arc. Two points on the arc are defined by the constant α,β solution given by (3.2), and a third point is defined by the initial conditions selected. Three points define the circular arc.

The next step is to put equations (2.33) and (2.34) into a more convenient form with the transformation

$$\alpha = a - \frac{f}{2}. \tag{3.7}$$

With respect to Figures 3-1, 3-2, 3-3 and 3-10, this transformation has the effect of shifting all the circles or arcs left or right, so their centers lie on the beta axis. Equations (2.33) and (2.34) then become

$$a' = \beta^2 - a^2 + \frac{f^2}{4} - g \tag{3.8}$$

and

$$\beta' = -2a\beta. \tag{3.9}$$

Knowledge of the geometry of the solution curves allows an equation to be written relating a and β. If need

be, that equation can be differentiated to generate an equation relating a' and β' as well. These extra relations allow equations (3.8) and (3.9) to be decoupled, and they can then be integrated directly. The process should be clear, so the mathematical steps will not be presented. Instead, only the solutions are given, for three different regimes. The initial conditions are a_0 and β_0. Constants c in this section are not to be confused with those employed elsewhere. The solutions begin on the next page.

The question may arise as to whether or not the results presented below are exactly correct. After all, the assumption that the solution curves are circular arcs is based on numerical results, and might only be an approximation. That would mean the solutions below are mere approximations. Until the analysis was complete, there was no way to know, but now that solutions for a and β are available, they can be checked by direct substitution into equations (3.8) and (3.9). By this method, it can be shown that the solutions for a and β given below are exact, and assumptions about the geometry of the solution curves must have been correct.

The expressions for a and β presented here are not particularly important, but the method by which they were obtained will be used again.

Case 1: $g > 0$, $|f| < 2\sqrt{g}$, and $\beta \neq 0$.

$$\beta = \frac{c_1^2}{c_2 + R + R\cos(c_3 + 2c_1 t)} \tag{3.10}$$

$$a = m\sqrt{2R(\beta - c_2) - (\beta - c_2)^2} \tag{3.11}$$

Where

$$c_1 = \sqrt{g - \frac{1}{4}f^2} \tag{3.12}$$

$$c_2 = \frac{1}{2\beta_0}\left(a_0^2 + \beta_0^2 + c_1^2 - \sqrt{\left(a_0^2 + \beta_0^2 + c_1^2\right)^2 - 4c_1^2\beta_0^2}\right) \tag{3.13}$$

$$R = \frac{c_1^2 - c_2^2}{2c_2} \tag{3.14}$$

$$c_3 = -\frac{a_0}{|a_0|}\arccos\left(\frac{c_1^2 - c_2\beta_0 - R\beta_0}{R\beta_0}\right) \tag{3.15}$$

$$m = -\frac{\sin(c_3 + 2c_1 t)}{|\sin(c_3 + 2c_1 t)|} \tag{3.16}$$

Case 2: $g > 0$ and $|f| > 2\sqrt{g}$. Also, $g \leq 0$ and any f.

$$\beta = \frac{2c_0^2 c_5 \beta_0 e^{2c_0 t}}{c_5^2 e^{4c_0 t} - c_4 c_5 e^{2c_0 t} + c_0^2 \beta_0^2 + \frac{1}{4} c_4^2} \tag{3.17}$$

$$a = \frac{c_0 \left(c_5^2 e^{4c_0 t} - c_0^2 \beta_0^2 - \frac{1}{4} c_4^2 \right)}{c_5^2 e^{4c_0 t} - c_4 c_5 e^{2c_0 t} + c_0^2 \beta_0^2 + \frac{1}{4} c_4^2} \tag{3.18}$$

Where

$$c_0 = \sqrt{\frac{1}{4} f^2 - g} \tag{3.19}$$

$$c_4 = a_0^2 + \beta_0^2 - c_0^2 \tag{3.20}$$

$$c_5 = c_0^2 + c_0 a_0 + \frac{1}{2} c_4 \tag{3.21}$$

Case 3: $g > 0$ and $|f| = 2\sqrt{g}$.

(This case encompasses critical damping.)

$$\beta = \frac{\beta_0}{\left(a_0^2 + \beta_0^2\right)t^2 + 2a_0 t + 1} \tag{3.22}$$

$$a = \frac{\left(a_0^2 + \beta_0^2\right)t + a_0}{\left(a_0^2 + \beta_0^2\right)t^2 + 2a_0 t + 1} \tag{3.23}$$

This concludes Chapter 3. Nonlinear equations are the subject of the remainder of the dissertation.

CHAPTER 4
THE GENERAL FORMULATION FOR SECOND ORDER ORDINARY DIFFERENTIAL EQUATIONS

The derivation presented in Chapter 2 for homogeneous linear problems will now be extended to second-order ordinary differential equations in general. The derivation presented here is conceptually identical to the one given in Chapter 2, and makes use of the same state space. Begin with the general ordinary differential equation (4.1), where $F_1 = F(x, x', t)$, and retain equations (2.3) and (2.4), the same state variable form used earlier.

$$x'' = F_1 \tag{4.1}$$

$$x' = \alpha x + \beta y \tag{2.3}$$

$$y' = \gamma x + \delta y \tag{2.4}$$

Again, equations (2.3) and (2.4) must recombine to form (4.1). There are an infinite number of choices for the variables α, β, γ, and δ that could satisfy this requirement. The same two conditions applied in Chapter 2 will be used again here to define these variables.

Condition 1

Condition 1 will be more general here than it was in the derivation of Chapter 2. In Chapter 2, x and y were required to represent the motion of the same physical system. Here, x and y must each represent the motion (displacement) of some physical system, but they need not represent solutions to the same system or ODE. That is, x will represent the motion of the system defined by equation (4.1), and y must represent the motion of any independently chosen second order ODE.

$$y'' = F_2, \tag{4.2}$$

where

$$F_2 = F(y, y', t). \tag{4.3}$$

Of course, the y system can still be chosen to be identical to the x system (i.e., y can be defined as some other solution to the ODE in x), but that is no longer a requirement. The variable y can represent the motion of any system the analyst wishes, provided it can be expressed in the form of equation (4.2).

Differentiate equations (2.3) and (2.4) to find equations for x'' and y'', respectively. The object will be to write the x'' equation in terms of x and x', without any y

terms. To do this, use equation (2.4) to substitute for y'. Then solve equation (2.3) for y, and use that to substitute for y. The resulting equation is

$$x'' = \left(\frac{\beta'}{\beta} + \alpha + \delta\right)x' + \left(\alpha' - \alpha\frac{\beta'}{\beta} - \alpha\delta + \gamma\beta\right)x. \tag{2.7}$$

Similarly, find

$$y'' = \left(\frac{\gamma'}{\gamma} + \alpha + \delta\right)y' + \left(\delta' - \delta\frac{\gamma'}{\gamma} - \alpha\delta + \gamma\beta\right)y. \tag{2.8}$$

Now equate (2.7) with (4.1), and (2.8) with (4.2).

$$\left(\frac{\beta'}{\beta} + \alpha + \delta\right)x' + \left(\alpha' - \alpha\frac{\beta'}{\beta} - \alpha\delta + \gamma\beta\right)x = F_1 \tag{4.4}$$

$$\left(\frac{\gamma'}{\gamma} + \alpha + \delta\right)y' + \left(\delta' - \delta\frac{\gamma'}{\gamma} - \alpha\delta + \gamma\beta\right)y = F_2 \tag{4.5}$$

Equations (4.4) and (4.5) are the only constraints that can be drawn from Condition 1. In contrast to the analysis of Chapter 2, the bracketed expressions in (4.4) and (4.5) cannot generally be equated with anything at this stage.

Condition 2

Condition 2, invariance of the scalar product **S•R**, relies only on the geometry of the state space and the state variable formulation (2.3) and (2.4). It is independent of all else. The condition therefore results in the same constraints in the general case as it did in the linear formulation. From Chapter 2, these constraints are:

$$\delta = \alpha \tag{2.28}$$

and

$$\beta + \gamma = 0. \tag{2.29}$$

Apply these to the state equations (2.3) and (2.4) to again find

$$x' = \alpha x + \beta y \tag{2.31}$$

and

$$y' = -\beta x + \alpha y. \tag{2.32}$$

Also, substitute (2.28) and (2.29) into (4.4) and (4.5) to find two constraint equations in α' and β',

$$\left(\frac{\beta'}{\beta} + 2\alpha\right)x' + \left(\alpha' - \alpha\frac{\beta'}{\beta} - \alpha^2 - \beta^2\right)x = F_1 \tag{4.6}$$

and

$$\left(\frac{\beta'}{\beta}+2\alpha\right)y'+\left(\alpha'-\alpha\frac{\beta'}{\beta}-\alpha^2-\beta^2\right)y=F_2. \tag{4.7}$$

Use equations (2.31) and (2.32) to eliminate the x' and y' terms in (4.6) and (4.7), and solve the resulting equations for α' and β'.

$$\alpha'=\frac{F_1x+F_2y}{x^2+y^2}+\beta^2-\alpha^2 \tag{4.8}$$

$$\beta'=\frac{-F_2x+F_1y}{x^2+y^2}-2\alpha\beta \tag{4.9}$$

These equations are the equivalent of (2.33) and (2.34) in the linear case presented earlier, but (4.8) and (4.9) are valid for <u>any</u> second order ODE. If the y system is chosen to be identical to the x system, as it was in Chapter 2, then they are fully consistent with the homogeneous linear formulation of Chapter 2. That is to say, when (4.1) represents a linear homogeneous equation, i.e., when

$$F_1=-fx'-gx, \tag{4.10}$$

and if y represents another solution to the same system,

$$F_2 = -fy' - gy, \tag{4.11}$$

then equations (4.8) and (4.9) reduce to (2.33) and (2.34), respectively.

Summary of the General Formulation

With the exception of equations (4.8) and (4.9), results for the general formulation are identical to those for the linear homogeneous formulation. The differential equation

$$x'' = F_1 \tag{4.1}$$

is represented in state form by the equations

$$x' = \alpha x + \beta y \tag{2.31}$$

$$y' = -\beta x + \alpha y. \tag{2.32}$$

The variables α and β are defined, except for initial conditions, by the equations

$$\alpha' = \frac{F_1 x + F_2 y}{x^2 + y^2} + \beta^2 - \alpha^2 \tag{4.8}$$

$$\beta' = \frac{-F_2 x + F_1 y}{x^2 + y^2} - 2\alpha\beta. \qquad (4.9)$$

The polar form of the state equations remains unchanged,

$$\rho' = \alpha\rho \qquad (2.37)$$

$$\phi' = -\beta, \qquad (2.38)$$

and

$$\mathbf{S \bullet R} = \alpha\rho^2. \qquad (2.30)$$

Linear vs. Nonlinear and Nonhomogeneous Equations

Differential equations are broadly classed as either linear or nonlinear, and linear equations are usually divided into homogeneous and nonhomogeneous categories. Homogeneous linear equations were the subject of Chapter 2. Nonhomogeneous linear equations are of the form

$$x'' = -fx' - gx + h, \qquad (4.12)$$

where f,g, and h are functions of time. When the general dual state variable formulation is applied to a nonhomogeneous linear differential equation, the resulting equations are:

$$x' = \alpha x + \beta y \tag{2.31}$$

$$y' = -\beta x + \alpha y. \tag{2.32}$$

$$\alpha' = \beta^2 - \alpha^2 - \alpha f - g + \frac{x+y}{x^2+y^2} h \tag{4.13}$$

$$\beta' = -\beta(f + 2\alpha) + \frac{y-x}{x^2+y^2} h. \tag{4.14}$$

Suppose solutions for α and β exist that are functions of time only. In this case, the formulation must represent a homogeneous linear equation. This can be seen by studying equation (4.6), and recalling that $F_1 = x''$. If equation (4.6) represents a homogeneous linear equation, then the bracketed expressions in it can only be equated to functions of time. When the bracketed expressions are equated to functions of time, they serve to define α and β, as in Chapter 2. For homogeneous linear equations, then, α and β will be independent of the state space coordinates.

Nonlinear and nonhomogeneous linear equations share one outstanding characteristic that homogeneous linear equations do not exhibit. When the dual state variable formulation is applied to the former, α and β are functions of the state space coordinates x and y. Equations (2.31),

(2.32), (4.8), and (4.9) - or (2.31), (2.32), (4.13), and (4.14) - are all coupled. For nonlinear and nonhomogeneous linear differential equations, it is clear that the x,y state space and the α,β state space cannot in general be considered separately, as they are in the homogeneous linear case. Equations (2.31), (2.32), (4.8) and (4.9) for nonlinear equations, and (2.31), (2.32), (4.13), and (4.14) for nonhomogeneous linear equations, represent behavior in a true four dimensional state space.

This characteristic is very important when working with the dual state variable formulation, and it provides a useful basis for classifying differential equations. Henceforth, nonlinearity will taken to mean that the functions α and β depend on the state space coordinates, and linearity will be taken to mean they do not. Specifically, the term 'nonlinear differential equation' will also encompass nonhomogeneous linear equations, and both types will be discussed in terms of the equation set (2.31), (2.32), (4.8) and (4.9). The term 'linear differential equation' will be applied to homogeneous linear equations only, as discussed in Chapters 2 and 3. Such a classification is unusual, but it makes sense here, and it will simplify the remainder of the text.

This concludes the derivation of the nonlinear formulation. The next two chapters will present some

significant mathematical consequences of the formulation. After that, the formulation will be applied to a practical nonlinear problem.

CHAPTER 5
LINEAR ALGEBRA AND THE DUAL STATE VARIABLE FORMULATION

Linear algebra provides a set of mathematical tools that make it relatively easy to work with linear, ordinary differential equations. It provides techniques for assessing the stability of linear systems, and for solving them in some cases. In particular, for constant coefficient equations, it provides a systematic method for finding exact solutions, based on the use of eigenvalues and eigenvectors. The tools of linear algebra are so useful that scientists and engineers prefer, if at all possible, to work only with linear differential equations.

It is important to remember, however, that linear systems do not exist in nature. In almost all cases, the linear equations engineers work with are approximate and often crude linear models of nonlinear systems. More accurate nonlinear models can always be constructed, but the linearized versions are used solely so that the tools of linear algebra can be applied. In the end, for all its exactness and rigor, linear algebra only sheds light on inexact approximations of better, nonlinear models.

In this chapter, the relationship between the dual state variable formulation and linear algebra will be

investigated. First, the methods of linear algebra will be applied to solve a linear constant coefficient differential equation, expressed in terms of dual state variables. Then it will be shown that a solution of exactly the same form can be written for any second order ordinary differential equation, using only this dual state variable formulation. Finally, working from this solution form, α and β will be shown to comprise the real and imaginary components of an eigenvalue. This interesting result is valid for any second order ordinary differential equation, even if the equation is nonlinear.

In later chapters, it will be demonstrated that solutions for the motion of nonlinear systems can sometimes be obtained by working directly with the nonlinear mathematical model and its associated eigenvalues. Even when these solutions are only approximate, they can still be considerably better than solutions obtained by linearizing the model and using the techniques of linear algebra.

Solution of a Constant Coefficient Equation Via Linear Algebra

A linear transformation can be represented by a matrix of values. The matrix systematically transforms

vectors in one vector space into new vectors in another vector space. Consider the simple case of a second order, linear, ordinary differential equation like (2.1), where f and g are constants.

$$x'' = -fx' - gx \tag{2.1}$$

Put this in (matrix) state variable form similar to that of equations (2.31) and (2.32).

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, \tag{5.1}$$

or

$$\mathbf{x}' = \mathbf{A}\mathbf{x}. \tag{5.2}$$

The matrix **A** is the linear transformation, and it acts on the state vector **x**.

The elements of **A** are functions of time only, by virtue of equations (2.33) and (2.34), so **A** does in fact represent a linear transformation. However, in order to solve this problem using eigenvalues and eigenvectors, the **A** matrix must be constant. When f and g are constants, constant solutions for α and β exist, as given by equations (3.1) - (3.3). Constant values for α and β are assumed here.

An eigenvalue λ is a number, or scalar value, that can replace the entire **A** matrix, but only when the **A** matrix is applied to a very special nonzero vector - an eigenvector (see, for instance, Hirsch and Smale, page 42). When λ_1 is the eigenvalue associated with the eigenvector $\mathbf{x}_1$, this means:

$$\mathbf{A}\mathbf{x}_1 = \lambda_1 \mathbf{x}_1. \tag{5.3}$$

To find the eigenvalues, rewrite (5.3) in slightly more general form,

$$(\mathbf{A} - \lambda \mathbf{I})\mathbf{x} = 0, \tag{5.4}$$

where **I** is the identity matrix. The vector **x** is understood to be an eigenvector in this equation. Since **x** is required to be nonzero, (5.4) is only satisfied when the determinant of $(\mathbf{A} - \lambda \mathbf{I})$ is zero. This requirement is shown in equation (5.5), and it allows the eigenvalues to be identified:

$$\begin{vmatrix} \alpha - \lambda & \beta \\ -\beta & \alpha - \lambda \end{vmatrix} = 0 \tag{5.5}$$

$$\lambda^2 - 2\alpha\lambda + \alpha^2 + \beta^2 = 0 \tag{5.6}$$

$$\lambda_{1,2} = \alpha \pm i\beta. \tag{5.7}$$

Equation (5.6) is also known as the characteristic equation. (The symbol i, of course, represents the square root of minus one.)

Next, find the eigenvectors associated with each eigenvalue. Write out equation (5.4), as follows:

$$\begin{bmatrix} \alpha-\lambda_1 & \beta \\ -\beta & \alpha-\lambda_1 \end{bmatrix}\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{5.8}$$

Solve the two algebraic equations contained in (5.8) simultaneously, for $\lambda_1=\alpha+i\beta$, to find an eigenvector

$$\mathbf{x}_1=\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}=\begin{bmatrix} 1 \\ i \end{bmatrix}. \tag{5.9}$$

Note that any multiple of this eigenvector is also an eigenvector, but the magnitude is unimportant. The one chosen in equation (5.9) will do. To confirm that the eigenvalue and eigenvector function as advertised, simply substitute λ_1 from (5.7) and x_1 from (5.9) into (5.3).

Similarly, for λ_2, find

$$\mathbf{x}_2=\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}=\begin{bmatrix} 1 \\ -i \end{bmatrix}. \tag{5.10}$$

Eigenvalues and eigenvectors can be computed even when the **A** matrix is a function of time, but in that case linear algebra does not provide a way to use them to solve the differential equation. When the **A** matrix is constant, however, a convenient thing happens. For one special solution **x**(t) to the differential equation (5.2), equation (5.3) holds true for all time, not just when the state vector is the (fixed) eigenvector. That is,

$$\mathbf{x}' = \mathbf{A}\mathbf{x}(t) = \lambda_1 \mathbf{x}(t). \tag{5.11}$$

This assertion requires a proof.

Proof:

Consider the function

$$\mathbf{U}(t) = \mathbf{x}'(t) - \lambda_1 \mathbf{x}(t). \tag{5.12}$$

It is a solution of (5.2), as can be seen by substitution:

$$\frac{d}{dt}(\mathbf{x}' - \lambda_1 \mathbf{x}) = \mathbf{A}(\mathbf{x}' - \lambda_1 \mathbf{x}) \tag{5.13}$$

$$\frac{d}{dt}(\mathbf{A}\mathbf{x}) - \frac{d}{dt}(\lambda_1 \mathbf{x}) = \mathbf{A}\frac{d}{dt}\mathbf{x} - \mathbf{A}\lambda_1 \mathbf{x} \tag{5.14}$$

$$\mathbf{A}\frac{d}{dt}\mathbf{x}-\lambda_1\frac{d}{dt}\mathbf{x}=\mathbf{A}\frac{d}{dt}\mathbf{x}-\lambda_1\mathbf{A}\mathbf{x} \qquad (5.15)$$

Of course, $\frac{d}{dt}\mathbf{x}=\mathbf{x}'$, and using equation (5.2), the equation above can be written

$$\mathbf{A}^2\mathbf{x}-\lambda_1\mathbf{A}\mathbf{x}=\mathbf{A}^2\mathbf{x}-\lambda_1\mathbf{A}\mathbf{x}. \qquad (5.16)$$

Now that **U** is proven to be a solution to the differential equation (5.2), rewrite it.

$$\mathbf{U}(t)=\mathbf{A}\mathbf{x}(t)-\lambda_1\mathbf{x}(t) \qquad (5.17)$$

$$\mathbf{U}(t)=(\mathbf{A}-\lambda_1\mathbf{I})\mathbf{x}(t) \qquad (5.18)$$

Again, the identity matrix **I** is inserted to maintain consistency in the matrix form.

The function **U** was written in terms of **x**(t), but there are many solutions **x**(t). A unique function **U** can be found by choosing the eigenvector $\mathbf{x}_1$ as the initial condition for a specific solution **x**(t). That is,

$$\mathbf{U}(0)=(\mathbf{A}-\lambda_1\mathbf{I})\mathbf{x}_1. \qquad (5.19)$$

It was shown previously that $(\mathbf{A} - \lambda_1 \mathbf{I})\mathbf{x}_1 = 0$; that is a defining property of the eigenvalue and eigenvector. Hence

$$\mathbf{U}(0) = \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{5.20}$$

But look at the state equations (5.1) or (2.31) and (2.32). If x and y are zero, then the system must remain at zero always; i.e., $\mathbf{U} \equiv 0$. Furthermore, $\mathbf{x}_1$ is known to be nonzero. Provided $\mathbf{x}_1$ is the initial condition for the solution $\mathbf{x}(t)$, any vector $\mathbf{x}$ taken from the function $\mathbf{x}(t)$ can also be used in equation (5.19). Equation (5.19) can then be expanded and written as follows:

$$\mathbf{A}\mathbf{x}(t) - \lambda_1 \mathbf{x}(t) = 0, \tag{5.21}$$

or

$$\mathbf{x}' = \mathbf{A}\mathbf{x}(t) = \lambda_1 \mathbf{x}(t). \tag{5.11}$$

Hence the assertion of equation (5.11) is verified. Do not forget that (5.11) only works for the solution $\mathbf{x}(t)$ associated with the initial condition $\mathbf{x}_1$. This concludes the proof. ∎

In the proof presented above, the reader can see where the method fails for problems with a variable **A**

matrix. Equation (5.15) will not follow from (5.14) in such a case. The proof above follows one given by Hurewicz, page 60.

Given equation (5.11), one special solution to the differential equation $\mathbf{x}' = \mathbf{A}\mathbf{x}$ is readily obtainable. For the special case in which the initial condition is the eigenvector $\mathbf{x}_1$, and only for that case,

$$\frac{\mathbf{x}'}{\mathbf{x}} = \lambda_1, \tag{5.22}$$

or integrate to find

$$x = x_1 e^{\int \lambda_1 dt}. \tag{5.23}$$

Since λ_1 is a constant, the exponent is usually integrated (from time t_0=0 to time t) and written $\lambda_1 t$. However, in order to make a point later in this analysis, the integral form will be retained here. Written out, the vector equation (5.23) becomes:

$$\mathbf{x} = \mathbf{X}_1 = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1e^{\int \lambda_1 dt} \\ ie^{\int \lambda_1 dt} \end{bmatrix}. \tag{5.24}$$

The designation $\mathbf{X}_1$ is given to this unique solution in honor of its special significance.

A similar solution can be found for the second eigenvalue and eigenvector pair,

$$\mathbf{x} = \mathbf{X}_2 = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1e^{\int \lambda_2 dt} \\ -ie^{\int \lambda_2 dt} \end{bmatrix} \tag{5.25}$$

Because equation (2.1) is linear and second order, theorems from linear algebra show that the two solutions $\mathbf{X}_1$ and $\mathbf{X}_2$ are linearly independent. In other words, $\mathbf{X}_2$ cannot be expressed as a constant multiple of $\mathbf{X}_1$. Furthermore, they form a basis for constructing any other solution to (2.1). They are a complete set of fundamental, or characteristic, solutions for the equation (2.1). That means any solution to (2.1), stemming from any choice of initial conditions, can be written as a linear combination of $\mathbf{X}_1$ and $\mathbf{X}_2$. That is, any solution to (2.1) can be written as

$$\mathbf{x} = c_1\mathbf{X}_1 + c_2\mathbf{X}_2, \tag{5.26}$$

or

$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c_1 e^{\int (\alpha + i\beta) dt} + c_2 e^{\int (\alpha - i\beta) dt} \\ ic_1 e^{\int (\alpha + i\beta) dt} - ic_2 e^{\int (\alpha - i\beta) dt} \end{bmatrix}. \tag{5.27}$$

See Hurewicz, Chapter 3, for further elucidation.

Applying arbitrary initial conditions x_0 and y_0 at $t_0=0$, equation (5.27) yields:

$$c_1 = \frac{x_0 - iy_0}{2} \tag{5.28}$$

$$c_2 = \frac{x_0 + iy_0}{2}. \tag{5.29}$$

Substitute these into (5.27). Then expand the result using the identities

$$e^{iu} = \cos(u) + i\sin(u) \tag{5.30}$$

and

$$e^{-iu} = \cos(u) - i\sin(u).$$

The resulting general solutions are

$$x = e^{\int \alpha dt}(x_0 \cos \int \beta dt + y_0 \sin \int \beta dt) \tag{5.31}$$

and

$$y = e^{\int \alpha dt}(y_0 \cos \int \beta dt - x_0 \sin \int \beta dt). \tag{5.32}$$

Equations (5.31) and (5.32) are the end result of this analysis. Together they are the general solution to equation (2.1) when it is represented in the dual state

variable form. This solution was derived using standard methods from linear algebra, and as such it is inherently limited to linear differential equations with a constant **A** matrix, or, in other words, to linear equations with constant coefficients. For linear differential equations with constant coefficients, the **A** matrix is automatically constant when the state equations are written in terms of the conventional x,v state variable form. However, when the dual state variable formulation is used, the **A** matrix is constant - and the solution (5.31) and (5.32) is valid - only for the very limited case in which α and β are chosen to be constants. Recall that such a choice is possible only for constant coefficient differential equations, as shown in Chapter 3.

An Alternative Derivation

Equations (5.31) and (5.32) were derived for a very limited set of differential equations, as was just noted. However, it turns out that when the dual state variable formulation is used, these equations are in fact valid solutions for any second order ordinary differential equation. Not even linearity is required. In this section, equations (5.31) and (5.32) will be derived using the general dual state variable formulation presented in

Chapter 4. No reference will be made to linearity, nor will any further restriction be placed on the form of the differential equation.

Begin with the polar form of the state equations (2.37) and (2.38), which follow only from the state equations (2.31) and (2.32) and the definition of the state space, and which remain generally valid.

$$\rho' = \alpha\rho \tag{2.37}$$

$$\phi' = -\beta \tag{2.38}$$

From Chapter 4, it is known that these equations represent the general second order ordinary differential equation (4.1).

$$x'' = F(x,x',t) \tag{4.1}$$

Without any knowledge at all about the functions α and β, equations (2.37) and (2.38) can be put in integral form. Use arbitrarily selected initial conditions ρ_0 and ϕ_0.

$$\rho = \rho_0 e^{\int \alpha dt} \tag{5.33}$$

$$\phi = -\int \beta dt + \phi_0 \tag{5.34}$$

The integrals are evaluated from initial time t_0 to time t. Now write the polar-to-cartesian conversion relations, which follow from Figure 2-1.

$$x = \rho\cos(\phi) \tag{5.35}$$

$$y = \rho\sin(\phi) \tag{5.36}$$

Substitute (5.33) and (5.34) into these equations to find the following expressions.

$$x = \rho_0 e^{\int \alpha dt}\cos(-\int \beta dt + \phi_0) \tag{5.37}$$

$$y = \rho_0 e^{\int \alpha dt}\sin(-\int \beta dt + \phi_0) \tag{5.38}$$

Expand these using trigonometric identities.

$$x = e^{\int \alpha dt}\left(\rho_0\cos(\phi_0)\cos(\int \beta dt) + \rho_0\sin(\phi_0)\sin(\int \beta dt)\right) \tag{5.39}$$

$$y = e^{\int \alpha dt}\left(\rho_0\sin(\phi_0)\cos(\int \beta dt) - \rho_0\cos(\phi_0)\sin(\int \beta dt)\right) \tag{5.40}$$

The terms involving initial conditions are recognized from (5.35) and (5.36) as being the cartesian expressions for the

initial conditions, so the equations above can be rewritten as follows.

$$x = e^{\int \alpha dt}\left(x_0 \cos(\int \beta dt) + y_0 \sin(\int \beta dt)\right) \qquad (5.41)$$

$$y = e^{\int \alpha dt}\left(y_0 \cos(\int \beta dt) - x_0 \sin(\int \beta dt)\right) \qquad (5.42)$$

These are exactly the same as equations (5.31) and (5.32), only they have been derived here for any second order ordinary differential equation. Equations (5.41) and (5.42) are even valid for nonlinear differential equations. Numerical investigation using the equation set (2.31), (2.32), (4.8), and (4.9), the general dual state variable formulation, will support this result.

Of course, there is no guarantee that the integrals in this solution can be evaluated, or that α and β are even available as explicit functions of time. Equations (4.8) and (4.9) show that α and β are generally functions of the state space coordinates as well as time. None the less, the formulation remains quite useful.

Linear Algebra for Nonlinear Equations

The same solution form obtained via linear algebra for linear, constant coefficient differential equations has now

been shown to be valid for general, nonlinear equations. It is reasonable to ask whether or not some of the intermediate results from the linear derivation hold true for second order ordinary differential equations in general, when the dual state variable formulation is used. This section will investigate the possibility by working backward from equations (5.41) and (5.42).

Complex Fundamental Solutions

Begin by rewriting the arbitrary initial conditions x_0 and y_0 used in (5.41) and (5.42).

$$x_0 = \frac{x_0 + iy_0}{2} + \frac{x_0 - iy_0}{2} \tag{5.43}$$

$$y_0 = i\frac{x_0 - iy_0}{2} - i\frac{x_0 + iy_0}{2} \tag{5.44}$$

Substitute these into (5.41) and rearrange terms.

$$x = \frac{1}{2}e^{\int \alpha dt}\Big[(x_0 + iy_0)(\cos(\int \beta dt) - i\sin(\int \beta dt)$$

$$+(x_0 - iy_0)(\cos(\int \beta dt) + i\sin(\int \beta dt)\Big] \tag{5.45}$$

$$x = \frac{1}{2} e^{\int \alpha dt} \left[(x_0 + iy_0) e^{-i\int \beta dt} + (x_0 - iy_0) e^{i\int \beta dt} \right] \tag{5.46}$$

$$x = \frac{1}{2}(x_0 - iy_0) e^{\int (\alpha + i\beta) dt} + \frac{1}{2}(x_0 + iy_0) e^{\int (\alpha - i\beta) dt} \tag{5.47}$$

Define two complex constants as follows.

$$c_1 = \frac{1}{2}(x_0 - iy_0) \tag{5.48}$$

$$c_2 = \frac{1}{2}(x_0 + iy_0) \tag{5.49}$$

Define two scalar functions λ_1 and λ_2 as follows.

$$\lambda_1 = \alpha + i\beta \tag{5.50}$$

$$\lambda_2 = \alpha - i\beta \tag{5.51}$$

Now rewrite the solution x, equation (5.47), in terms of these new quantities.

$$x = c_1 e^{\int \lambda_1 dt} + c_2 e^{\int \lambda_2 dt} \tag{5.52}$$

A similar expression for y follows from equation (4.42):

$$y = ic_1 e^{\int \lambda_1 dt} - ic_2 e^{\int \lambda_2 dt}. \tag{5.53}$$

Now write these solutions in vector form.

$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c_1 e^{\int \lambda_1 dt} + c_2 e^{\int \lambda_2 dt} \\ ic_1 e^{\int \lambda_1 dt} - ic_2 e^{\int \lambda_2 dt} \end{bmatrix} \tag{5.54}$$

The right side can be rewritten as follows.

$$\mathbf{x} = c_1 \begin{bmatrix} e^{\int \lambda_1 dt} \\ ie^{\int \lambda_1 dt} \end{bmatrix} + c_2 \begin{bmatrix} e^{\int \lambda_2 dt} \\ -ie^{\int \lambda_2 dt} \end{bmatrix} \tag{5.55}$$

Equations (5.54) and (5.55) are merely restatements of equations (5.41) and (5.42), in vector form. This analysis shows that the scalar solutions x and y are always linear combinations of two exponential functions. In vector form, the solution **x** is a linear combination of two vector functions, which will be designated as follows.

$$\mathbf{X}_1 = \begin{bmatrix} e^{\int \lambda_1 dt} \\ ie^{\int \lambda_1 dt} \end{bmatrix} \tag{5.56}$$

$$\mathbf{X}_2 = \begin{bmatrix} e^{\int \lambda_2 dt} \\ -ie^{\int \lambda_2 dt} \end{bmatrix} \tag{5.57}$$

Equation (5.55) then becomes:

$$\mathbf{x} = c_1\mathbf{X}_1 + c_2\mathbf{X}_2. \tag{5.58}$$

Of course, $\mathbf{x}$ is a solution to the matrix form of the state equations, equation (5.2), $\mathbf{x}' = \mathbf{A}\mathbf{x}$. An obvious question is whether or not the component functions $\mathbf{X}_1$ and $\mathbf{X}_2$ are, individually, solutions to equation (5.2) as well. It can be proven that they are by substituting $\mathbf{X}_1$ and $\mathbf{X}_2$, one at a time, into (5.2). Differentiation of the function $\mathbf{X}_1$ yields

$$\mathbf{X}_1' = \begin{bmatrix} \lambda_1 e^{\int \lambda_1 dt} \\ i\lambda_1 e^{\int \lambda_1 dt} \end{bmatrix}. \tag{5.59}$$

Substitution of (5.56) and (5.59) into (5.2) yields

$$\begin{bmatrix} \lambda_1 e^{\int \lambda_1 dt} \\ i\lambda_1 e^{\int \lambda_1 dt} \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix} \begin{bmatrix} e^{\int \lambda_1 dt} \\ ie^{\int \lambda_1 dt} \end{bmatrix}, \tag{5.60}$$

or the identity (5.61).

$$\begin{bmatrix} (\alpha+i\beta)e^{\int(\alpha+i\beta)dt} \\ i(\alpha+i\beta)e^{\int(\alpha+i\beta)dt} \end{bmatrix} = \begin{bmatrix} (\alpha+i\beta)e^{\int(\alpha+i\beta)dt} \\ i(\alpha+i\beta)e^{\int(\alpha+i\beta)dt} \end{bmatrix} \qquad (5.61)$$

This shows that $\mathbf{X}_1$ is indeed a solution to the matrix differential equation (5.2). Similarly, it can be shown $\mathbf{X}_2$ is a solution as well.

The next question is whether or not these two solutions, $\mathbf{X}_1$ and $\mathbf{X}_2$, are linearly independent. This may seem an unusual question to ask about two functions that can be solutions to a nonlinear differential equation, which is what (5.2) generally represents, but the test for linear independence is simple and clear. When C_1 and C_2 are constants, if the equation

$$C_1\mathbf{X}_1 + C_2\mathbf{X}_2 = 0 \qquad (5.62)$$

implies C_1 and C_2 must both be zero, then the two solutions $\mathbf{X}_1$ and $\mathbf{X}_2$ are linearly independent. The vector equation (5.62) yields two simultaneous scalar equations in C_1 and C_2.

$$C_1e^{\int\lambda_1dt} + C_2e^{\int\lambda_2dt} = 0 \qquad (5.63)$$

$$iC_1e^{\int\lambda_1dt} - iC_2e^{\int\lambda_2dt} = 0 \qquad (5.64)$$

With lambda expressed as $\lambda = \alpha \pm i\beta$, these become

$$e^{\int \alpha dt}\left(C_1 e^{i\int \beta dt} + C_2 e^{-i\int \beta dt}\right) = 0 \qquad (5.65)$$

$$ie^{\int \alpha dt}\left(C_1 e^{i\int \beta dt} - C_2 e^{-i\int \beta dt}\right) = 0 \qquad (5.66)$$

or, in trigonometric form,

$$e^{\int \alpha dt}\{(C_1 + C_2)\cos[\int \beta dt] + i(C_1 - C_2)\sin[\int \beta dt]\} = 0 \qquad (5.67)$$

$$ie^{\int \alpha dt}\{(C_1 - C_2)\cos[\int \beta dt] + i(C_1 + C_2)\sin[\int \beta dt]\} = 0. \qquad (5.68)$$

Constants C_1 and C_2 can be nonzero if α has reached minus infinity, but in this case equation (5.33) shows that the state point (x,y) is at the origin of the state space: x=y=0. This is a known singular point, and indicates the achievement of equilibrium, since it means x'=y'=0 as well (see the state equations equations 2.31 and 2.32). Provided the system is not at this singular point, the constants C_1 and C_2 must be zero, and the two solutions $\mathbf{X}_1$ and $\mathbf{X}_2$ are in fact linearly independent. Hence the complex solutions $\mathbf{X}_1$ and $\mathbf{X}_2$ constitute a fundamental set associated with the solution **x**.

Real Fundamental Solutions

When a complex solution to a real, linear differential equation exists, the real and imaginary portions of the solution must each individually be solutions to the differential equation (Hurewicz, Chapter3, Theorem 14). This assertion remains valid here. Consider $\mathbf{X}_1$ and $\mathbf{X}_2$ as defined by equations (5.56) and (5.57). These can be expanded and rewritten as follows.

$$\mathbf{X}_1 = \begin{bmatrix} e^{\int \alpha dt} \cos \int \beta dt \\ -e^{\int \alpha dt} \sin \int \beta dt \end{bmatrix} + i \begin{bmatrix} e^{\int \alpha dt} \sin \int \beta dt \\ e^{\int \alpha dt} \cos \int \beta dt \end{bmatrix} \tag{5.69}$$

$$\mathbf{X}_2 = \begin{bmatrix} e^{\int \alpha dt} \cos \int \beta dt \\ -e^{\int \alpha dt} \sin \int \beta dt \end{bmatrix} - i \begin{bmatrix} e^{\int \alpha dt} \sin \int \beta dt \\ e^{\int \alpha dt} \cos \int \beta dt \end{bmatrix} \tag{5.70}$$

The real and imaginary portions of these are each, individually, solutions to the matrix differential equation (5.2). This can be verified by substituting each into (5.2). However, inspection reveals that only two of the four expressions are independent. This is to be expected for a second order differential equation. Choose the imaginary portion of $\mathbf{X}_1$ and the real portion of $\mathbf{X}_2$ to define two real solutions to equation (5.2), and designate these as $\overline{\mathbf{X}}_1$ and $\overline{\mathbf{X}}_2$.

$$\bar{\mathbf{X}}_1 = \begin{bmatrix} e^{\int \alpha dt} \sin \int \beta dt \\ e^{\int \alpha dt} \cos \int \beta dt \end{bmatrix} \tag{5.71}$$

$$\bar{\mathbf{X}}_2 = \begin{bmatrix} e^{\int \alpha dt} \cos \int \beta dt \\ -e^{\int \alpha dt} \sin \int \beta dt \end{bmatrix} \tag{5.72}$$

These real solutions are linearly independent. Now consider the following linear combination of (5.71) and (5.72), where x_0 and y_0 are the initial conditions of the differential equation (5.2):

$$y_0 \bar{\mathbf{X}}_1 + x_0 \bar{\mathbf{X}}_2 = \mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}. \tag{5.73}$$

This linear combination exactly duplicates solutions (5.41) and (5.42), when they are written in vector form. That is why the linear combination of $\bar{\mathbf{X}}_1$ and $\bar{\mathbf{X}}_2$ above is equated with the vector solution **x**.

Solutions as Vectors in a Vector Space

The defining requirements for a linear vector space follow. It must consist of:

1. A field of scalars, in this case the real numbers.
2. A set of vectors, in this case real solutions to the differential equation (5.2). All solutions can be written as $\mathbf{x}$, but of course $\bar{\mathbf{X}}_1$ and $\bar{\mathbf{X}}_2$ are solutions too.
3. Commutative vector addition, such that the sum of any two vectors in the space is itself a vector in the space.
4. Scalar multiplication of vectors, with certain requirements. Specifically, for arbitrary vectors $\mathbf{x}$ and constants c:
   a) $1\mathbf{x} = \mathbf{x}$
   b) $c_3c_4\mathbf{x} = c_3(c_4\mathbf{x})$
   c) $c(\mathbf{x}_1 + \mathbf{x}_2) = c\mathbf{x}_1 + c\mathbf{x}_2$
   d) $(c_3 + c_4)\mathbf{x} = c_3\mathbf{x} + c_4\mathbf{x}$.

Almost all of these requirements or characteristics are present, as simple computations will verify. The only one that is sometimes lacking is number 3. If solutions to the differential equation are to be vectors in a vector space, then any two of them must add together, and the result must be another vector in the vector space (i.e., it must be another solution to the differential equation). Any two solutions derived from the dual state variable formulation can easily be added, but the result may not

itself be a solution. The ability to add any two solutions and arrive at another solution is the key to defining a vector space of solutions. It is also the essence of the principle of superposition, but superposition is only known to exist for linear equations.

Consider the homogeneous linear case. Note that $\bar{\mathbf{X}}_1$ and $\bar{\mathbf{X}}_2$ are functions of α and β. For homogeneous linear equations, it was shown that α and β are only functions of time, and of their own initial conditions α_0 and β_0. Once a set of initial conditions α_0 and β_0 are selected, the functions α and β are fixed functions of time, and so are the fundamental solutions $\bar{\mathbf{X}}_1$ and $\bar{\mathbf{X}}_2$. No matter what solution **x** to equation (5.2) is considered (i.e., no matter what initial conditions x_0 and y_0 are selected), $\bar{\mathbf{X}}_1$ and $\bar{\mathbf{X}}_2$ will not change. Any solution **x** can be written as a linear combination of the same two fundamental solutions $\bar{\mathbf{X}}_1$ and $\bar{\mathbf{X}}_2$. Furthermore, as equation (5.73) indicates, any linear combination of $\bar{\mathbf{X}}_1$ and $\bar{\mathbf{X}}_2$ is also a solution to the differential equation. This means the solutions **x** to a linear homogeneous differential equation of the form (5.2) are vectors in a vector space, and the fundamental solutions $\bar{\mathbf{X}}_1$ and $\bar{\mathbf{X}}_2$ are basis vectors from which any other solution can be constructed.

Now consider a nonlinear or nonhomogeneous differential equation, with the formulation presented in

Chapter 4. In this case, α and β must be functions of the state space coordinates x and y. For every different set of initial conditions x_0 and y_0, the functions α and β will be different. This means the fundamental solutions $\bar{\mathbf{X}}_1$ and $\bar{\mathbf{X}}_2$ are different as well, since they change with every set of initial conditions. Every solution curve for **x** is associated with a pair of linearly independent fundamental solutions, but these fundamental solutions are unique to that specific solution curve. There is no guarantee that the fundamental solutions associated with two different solutions **x** will add together to yield another solution **x**; in fact, generally they will not. The solutions may still be thought of as vectors, but they are not part of a linear vector space, because the sum of two of them is not also a solution.

To summarize, in the linear case α and β can be defined as fixed functions of time valid for all solutions **x**, provided the same initial conditions for α and β are used for all solutions **x**. Hence the fundamental solutions $\bar{\mathbf{X}}_1$ and $\bar{\mathbf{X}}_2$ are the same for all solutions **x**, and can be considered basis vectors in a vector space of solutions. The principle of superposition holds, and the analysis is entirely consistent with the results of linear algebra.

In the general case, α and β change with every new set of initial conditions for **x**. Equations (4.8) and (4.9) make this clear. That means $\bar{\mathbf{X}}_1$ and $\bar{\mathbf{X}}_2$ change with every new set

of initial conditions too. All solutions **x** can be expressed by equation (5.73), but since $\bar{\mathbf{X}}_1$ and $\bar{\mathbf{X}}_2$ are different for each solution **x**, two solutions cannot generally be added to get another solution. Solutions are not vectors in a linear vector space, and superposition does not hold.

Eigenvalues and Eigenvectors

Continue the analysis with the complex fundamental solutions $\mathbf{X}_1$ and $\mathbf{X}_2$ developed earlier. The initial conditions for these linearly independent solutions $\mathbf{X}_1$ and $\mathbf{X}_2$ are obvious from inspection of (5.56) and (5.57). They are:

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ i \end{bmatrix}, \qquad \mathbf{x}_2 = \begin{bmatrix} 1 \\ -i \end{bmatrix}. \tag{5.74}$$

The initial conditions $\mathbf{x}_1$ and $\mathbf{x}_2$ and the associated scalar functions λ_1 and λ_2 of equations (5.50) and (5.51) are consistent with the eigenvectors and eigenvalues found in the earlier linear algebraic solution, shown in equations (5.2)-(5.10). They can be identified with that technique, and the linearly independent solutions (5.56) and (5.57) can then be written.

The final - and key - question to be asked here is whether the expression (5.11) remains valid generally.

$$\mathbf{x}' = \mathbf{A}\mathbf{x}(t) = \lambda_1 \mathbf{x}(t). \qquad (5.11)$$

When (5.11) was first written, it was noted that it was valid only for one special solution from among the family of all solutions $\mathbf{x}(t)$. It was then shown that the special solution was the fundamental solution $\mathbf{X}_1$ associated with the eigenvalue λ_1. Rewriting (5.11) to reflect this, find

$$\mathbf{A}\mathbf{X}_1(t) = \lambda_1 \mathbf{X}_1(t). \qquad (5.75)$$

The essence of this equation is that the matrix $\mathbf{A}$ can be replaced with the scalar λ_1, for the one special solution $\mathbf{X}_1(t)$. This is what qualifies λ_1 to be called an eigenvalue (Hirsch and Smale, page 42). A similar expression holds for the other fundamental solution,

$$\mathbf{A}\mathbf{X}_2(t) = \lambda_2 \mathbf{X}_2(t). \qquad (5.76)$$

Is equation (5.75) valid for the general dual state variable formulation? Substitute $\mathbf{X}_1$ and λ_1 into (5.75) to see.

$$\begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix} \begin{bmatrix} e^{\int \lambda_1 dt} \\ ie^{\int \lambda_1 dt} \end{bmatrix} = (\alpha + i\beta) \begin{bmatrix} e^{\int \lambda_1 dt} \\ ie^{\int \lambda_1 dt} \end{bmatrix} \tag{5.77}$$

This is easily shown to be an equality, so equation (5.75) must be generally true. The same can be shown for equation (5.76). Since these relations are true for all time, they must be true for the initial conditions $\mathbf{x}_1$ and $\mathbf{x}_2$ as well.

The terms eigenvalue and eigenvector seem almost inextricably linked to linear systems. They are virtually always used in that context. However, there does not seem to be any inherent reason why these terms cannot also be applied in a nonlinear context, provided the functions they represent meet certain criteria. Nonlinearity does not automatically preclude the existence of eigenvalues and eigenvectors.

For the general dual state variable formulation it is clear that the scalar functions λ_1 and λ_2 completely replace the **A** matrix in the differential equation (5.2), for all time, when the associated vectors $\mathbf{x}_1$ and $\mathbf{x}_2$ are used as the initial conditions. The two functions λ_1 and λ_2, and the two associated vectors $\mathbf{x}_1$ and $\mathbf{x}_2$, can be systematically identified for all second order ordinary differential equations. Each function and vector pair can be used to identify a fundamental solution to the original differential equation, and the two fundamental solutions $\mathbf{X}_1$ and $\mathbf{X}_2$ so

identified are linearly independent, provided α is finite. Any solution to the matrix differential equation (5.2) can be written as a linear combination of these two fundamental solutions. The analysis is completely consistent with the results of linear algebra. These findings seem to qualify λ_1 and λ_2 as eigenvalues, and to qualify $\mathbf{x}_1$ and $\mathbf{x}_2$ as eigenvectors. For readers who feel more comfortable reserving these terms for linear situations, they may be designated 'extended' eigenvalues and 'extended' eigenvectors instead.

In the general nonlinear case, the eigenvalues defined here exist in a four dimensional space. They are functions of x, y, α, β, and t. In the (homogeneous) linear case, the dependence on x and y disappears.

It is worth noting that if all four initial conditions for the x and y systems (x_0,x_0',y_0, and y_0') are specified, then there is one unique pair of eigenvalues associated with the resulting x,y solution. However, often only the initial conditions for the x system are specified, and in this case the initial conditions α_0 and β_0 (or, alternatively, y_0 and y_0') can be chosen at will. Each different choice will lead to a different pair of eigenvalues, all of which are associated with the same solution to the x system. (Initial conditions were discussed in the section summarizing the formulation

in Chapter 2.) The eigenvectors $\mathbf{x}_1$ and $\mathbf{x}_2$ defined by equations (5.74) are constants that are valid in all cases.

Chapter Summary

When linear algebra is applied to a linear, constant coefficient differential equation, it leads to the identification of one unique pair of eigenvalues, which are then used to allow the integration of the differential equation. Only one unique pair of fundamental solutions can be found this way. However, because the differential equation is linear, any solution to the differential equation can then be found as a linear combination of the fundamental solutions. By contrast, the dual state variable formulation allows eigenvalues and fundamental solutions to be identified for any second order ordinary differential equation. Once a pair of fundamental solutions has been found, they cannot generally be used to express all other solutions to the differential equation. However, the dual state variable formulation compensates for this shortcoming by allowing the identification of unique eigenvalues and fundamental solutions for each and every solution to the differential equation.

The ability to define eigenvalues and eigenvectors for general second order ordinary differential equations, and

to write a valid solution in terms of them, may be unique to the dual state variable formulation. This property clearly does not exist for the x,v state representation.

The eigenvalues associated with the dual state variable formulation differ from those traditionally used in several respects. For one thing, they are defined in differential form, or, more properly, their components α. and β are defined in differential form. They are also generally functions of the state space coordinates. Finally, they are not associated solely with the second order ODE in x that is being studied. Instead, they are associated with a pair of ODE's, the one in x and the one in y. The ODE in y may be the same as the one in x, but it need not be. Because the analyst can define the y system and its initial conditions at will, an infinite number of different eigenvalues can be associated with the x system. This may at first appear to be a drawback, but the flexibility it permits offers many possibilities for manipulating and attacking a particular problem.

It might be said that the dual state variable formulation casts nonlinear differential equations into a form for which the results of linear algebra hold true. There are some difficulties with this point of view. For one thing, the standard derivation using linear algebra is invalid, and another approach must be taken. That is because the **A**

matrix is generally nonlinear, not to mention time variable. It generally represents a truly nonlinear transformation, and it changes with the initial conditions of the solution. Furthermore, solutions are not generally vectors in a linear vector space, and the principle of superposition does not hold true. For these reasons, it is tempting to think of the preceding analysis as a kind of "nonlinear algebra".

The dual state variable formulation and associated analysis is seen to differ markedly from the traditional linear algebraic approach. It provides a systematic way to approach any second order ordinary differential equation, and it allows the engineer to work directly with the nonlinear mathematical model, instead of working with the usual linear approximation. In a later chapter, it will be shown that considerably better approximate solutions to nonlinear equations can sometimes be had by working directly with the nonlinear equations. Furthermore, the formulation is entirely consistent with linear algebra, when it is applied to linear differential equations.

CHAPTER 6
THE RICCATI EQUATION

In 1723 Count Jacopo Riccati described a nonlinear, first- order differential equation that is related to a linear, second-order differential equation via a simple transformation (Hille, page 103; Bittanti, Chapter 1). Specifically, the Riccati equation

$$\alpha' = -\alpha^2 - \alpha f - g \tag{6.1}$$

is related to the second order linear differential equation

$$x'' + fx' + gx = 0 \tag{2.1}$$

by the transformation

$$\alpha = x' / x. \tag{6.2}$$

The Riccati equation has proven useful in solving a number of engineering problems. The relationship between the dual state variable formulation and the Riccati equation is not central to this dissertation, but it is interesting enough to warrant a cursory discussion.

Linear Equations

When the dual state variable formulation is applied to a linear differential equation, as represented by equations (2.31)- (2.34), it is clear that $\beta \equiv 0$ is always a valid solution to the equation set.

$$x' = \alpha x + \beta y \tag{2.31}$$

$$y' = -\beta x + \alpha y \tag{2.32}$$

$$\alpha' = \beta^2 - \alpha^2 - \alpha f - g \tag{2.33}$$

$$\beta' = -\beta(f + 2\alpha) \tag{2.34}$$

When $\beta \equiv 0$, the state variable representation is degenerate, since there is only one independent state equation,

$$x' = \alpha x, \tag{6.3}$$

and the α' equation becomes

$$\alpha' = -\alpha^2 - \alpha f - g. \tag{6.1}$$

The initial condition for α, for equation (6.1), comes from the initial conditions for x and x', and is defined using equation (6.3). Equation (6.1) is exactly the Riccati equation, and (6.3) is the transformation Riccati used to relate it to a linear first order equation.

The Riccati equation is apparently a degenerate form of the dual state variable formulation, and the Riccati variable, called α here, is a degenerate form of the eigenvalues identified in the previous chapter. This special case of the dual state variable formulation (i.e., the case in which $\beta \equiv 0$) is not often used here. One of its drawbacks is that, for many problems, α periodically becomes infinite; the transformation $\alpha = \tan(\psi)$ can help in this regard.

A complex combination of α and (nonzero) β also yields the Riccati equation. For the linear formulation given by equations (2.31)-(2.34), consider the eigenvalues $\lambda = \alpha \pm i\beta$. The equation

$$\lambda' = -\lambda^2 - \lambda f - g \tag{6.4}$$

is valid for both eigenvalues, as can be shown by substitution. This is, of course, the Riccati equation. Furthermore, define a new variable Z,

$$Z = x \mp iy, \tag{6.5}$$

where the choice of sign must be opposite that selected for the eigenvalue. Then it can be easily demonstrated that

$$Z' = \lambda Z. \tag{6.6}$$

(This representation is known, at least for linear systems; see, for instance, Hirsch and Smale, page 57.) When this equation is written out in terms of x,y,α, and β, for example

$$x' - iy' = (\alpha + i\beta)(x - iy)$$

$$= \alpha x + \beta y - i(-\beta x + \alpha y), \tag{6.7}$$

it is clear that the real and imaginary portions of the equation are consistent with the state equations (2.31) and (2.32).

Nonlinear Equations

Some similarity between the Riccati equation and the dual state variable formulation extends to the nonlinear case, although the Riccati equation per se does not appear. The general equations for the dual state variable formulation are repeated below.

$$x' = \alpha x + \beta y \tag{2.31}$$

$$y' = -\beta x + \alpha y \tag{2.32}$$

$$\alpha' = \frac{F_1 x + F_2 y}{x^2 + y^2} + \beta^2 - \alpha^2 \tag{4.8}$$

$$\beta' = \frac{-F_2 x + F_1 y}{x^2 + y^2} - 2\alpha\beta \tag{4.9}$$

F_1 and F_2 come from the differential equations in x and y, respectively. That is, $x'' = F_1$ and $y'' = F_2$, as defined at the beginning of Chapter 4.

The solution $\beta \equiv 0$ exists in association with the relation y=x for any second order ordinary differential equation, not just for linear equations. (Note that if y(t) is to equal x(t), then the ODE defining y must be identical in form to the ODE in x, which is a restriction of the analysis of Chapter 4.) When y(t)=x(t), then $F_2 = F_1$. When, additionally, $\beta \equiv 0$, equation (4.9) is satisfied and equations (2.31) and (2.32) become redundant. Under these conditions, equation (4.8) becomes

$$\alpha' = \frac{F_1}{x} - \alpha^2. \tag{6.8}$$

This equation can be transformed into the original second order differential equation $x''=F_1$ with the same transformation (6.2), $\alpha = x'/x$.

A complex representation can also be constructed. Again, this complex form does not assume $\beta \equiv 0$. Let

$$\lambda = \alpha \pm i\beta$$

and

$$Z = x \mp iy. \tag{6.5}$$

Then it can be shown that

$$Z' = \lambda Z \tag{6.6}$$

and

$$\lambda' = \frac{F_1 \mp iF_2}{Z} - \lambda^2. \tag{6.9}$$

If the transformation

$$\lambda = \frac{Z'}{Z} \tag{6.10}$$

is applied to equation (6.9), the result is

$$Z'' = F_1 \mp iF_2. \tag{6.11}$$

The real and imaginary portions of equation (6.11) correspond to the original differential equations $x''=F_1$ and $y''=F_2$.

In the previous chapter , for second order differential equations in matrix form, it was shown that the A matrix could be replaced with a scalar eigenvalue λ. Equation (6.6) shows that such a matrix equation can be written entirely in scalar form when the dual state variable formulation is employed.

The occurrence of the Riccati equation as a special case of the dual state variable formulation is interesting, but it will not be discussed further in this dissertation.

CHAPTER 7
NONLINEAR EIGENVALUES

Thus far, the discussion of eigenvalues has been entirely theoretical. This chapter is an introduction to the practical investigation of eigenvalues for nonlinear second order ODE's. It will be assumed here that the ODE defining the y system is chosen to have the same form as the one representing the x system. That is, x and y will be different solutions to the same ODE. The undamped, nonlinear pendulum will be used as an example, and meaningful, qualitative portraits of its eigenvalues will be developed here. These portraits will be put to use quantitatively in the following chapter.

In engineering, eigenvalues are useful for more than just finding solutions to linear equations. They are also known to contain essential information about the behavior of the physical system under study. The real part of the eigenvalue is associated with the growth or decay of system motion over time, and the imaginary part indicates the oscillatory behavior of the system. Eigenvalues from the dual state variable formulation provide similar information, as can be seen from the polar form of the state equations:

$$\rho' = \alpha\rho \tag{2.37}$$

$$\phi' = -\beta. \tag{2.38}$$

As noted previously, these equations are always valid, even for nonlinear differential equations. Alpha is a direct measure of the rate of growth of ρ, the length of the dual state vector; this can be seen in the simple linear example of Figure 3-8. Beta is a direct measure of the angular velocity of the state vector.

When engineers plot the poles of a linear system in the s-plane, they are plotting the eigenvalues from the usual x,v (i.e., x, x') state formulation. In Chapter 5 it was proven that $\alpha \pm i\beta$ are eigenvalues, which supports the claim that the α,β plane is analogous to the s-plane. The information contained in the two planes is not the same, however; there are significant differences that must be recognized and considered when studying eigenvalues in the α,β plane.

To begin with, eigenvalues do not depend solely on the differential equation of motion. They also depend on the state variable form chosen to represent the motion. The α,β plane contains eigenvalues from the dual state variable formulation, in which the two state variables x and y are both displacements. Eigenvalues from this state

formulation generally do not look the same as eigenvalues from the traditional formulation, in which the state variables are displacement x and velocity v.

Another difference is that, in the traditional s-plane, eigenvalues can only be defined for homogeneous linear equations. By comparison, eigenvalues from the dual state variable formulation can be plotted in the α,β plane for any second order ordinary differential equation. When the differential equation is nonlinear or nonhomogeneous, α and β are functions of the state point (i.e., x and y; see equations 4.8 and 4.9). Hence these eigenvalues exist in a 4-dimensional space defined with axes x,y,α, and β. The α,β plot of such an eigenvalue is really a 2-dimensional projection of a 4-dimensional curve. Therefore plots of the eigenvalues can sometimes appear quite confusing and complicated.

Yet another difference is that there is only one unique set of eigenvalues for any linear differential equation represented in the traditional x,v state form. No matter what particular solution is being studied, the eigenvalues remain unchanged. In the dual state variable formulation, this is not the case. Furthermore, the solutions to two different differential equations are involved (one in x and one in y), or, as is assumed in this chapter, two different solutions to the same differential equation are involved

(x and y). Generally, there is a different pair of eigenvalues associated with each particular set of solutions x and y. This is because α and β, the components of the eigenvalues, are defined by differential equations. [See equations (4.8) and (4.9), or, for linear problems, (2.33) and (2.34).] Once the initial conditions x_0,x_0',y_0, and y_0' are chosen, the initial conditions for α and β are defined by the state equations (2.31) and (2.32). Different initial conditions mean different solutions for α and β, and hence, different eigenvalues.

An interesting consequence is that, unlike the s-plane, the α,β plane can be completely filled with information. Generally, only the x solution is of interest. If the y solution doesn't matter, then an infinite number of different y solutions can be associated with any particular x solution, since any initial conditions y_0 and y_0' (or, alternatively, α_0 and β_0) can be selected. Hence, an infinite number of eigenvalues can be associated with any particular solution x. This can be seen in Figures 3-1, 3-2 and 3-3 for constant coefficient linear equations. One of the purposes of this chapter is to demonstrate a way of finding a single, useful eigenvalue in the α,β plane from among the infinite number of possibilities. (The 'single' eigenvalue referred to here will not be a fixed point in the α,β plane, but rather a point moving on a path, a function of

time. Its dependence on the state space coordinates x and y will not be apparent, since it will be viewed in the α,β plane only.)

Selecting an Eigenvalue for the Nonlinear Pendulum

The undamped, nonlinear pendulum, represented by equation (7.1), will be used throughout for illustration.

$$x'' + \sin(x) = 0 \tag{7.1}$$

All of the plots shown in this section can be generated numerically using the general dual state variable formulation of Chapter 4, which consists of the following four equations.

$$x' = \alpha x + \beta y \tag{2.31}$$

$$y' = -\beta x + \alpha y \tag{2.32}$$

$$\alpha' = \frac{F_1 x + F_2 y}{x^2 + y^2} + \beta^2 - \alpha^2 \tag{4.8}$$

$$\beta' = \frac{-F_2 x + F_1 y}{x^2 + y^2} - 2\alpha\beta \tag{4.9}$$

The functions F are explained in Chapter 4, and are defined from equation (7.1): F_1=-sin(x), and F_2=-sin(y). It is not necessary for F_2 to be defined this way, but this choice will be advantageous here. Attention will be restricted to the oscillatory motion of the pendulum only; rotational motion will not be considered.

As noted previously, many different eigenvalues can be defined when the dual state variable formulation is used, and they can sometimes be quite complicated. To illustrate the point, a randomly chosen α,β curve, or eigenvalue, is plotted in Figure 7-1. The initial conditions given with the figure are stated in terms of x, y, α, and β, in order to be consistent with the general dual state variable equation set presented above. The initial conditions x'_0 and y'_0 can be found from the values given, by use of the state equations (2.31) and (2.32). The initial conditions for the plots could just as easily have been written in terms of x, x', α, and β, or in terms of x, x', y, and y'.

Figure 7-1 is meant to be intimidating. It is difficult to see how to extract any useful information from these plots. However, only the behavior of the x system is of interest here. The initial conditions for the dual y system can be selected quite arbitrarily. This flexibility allows the analyst to pick and choose from an infinite number of x,y

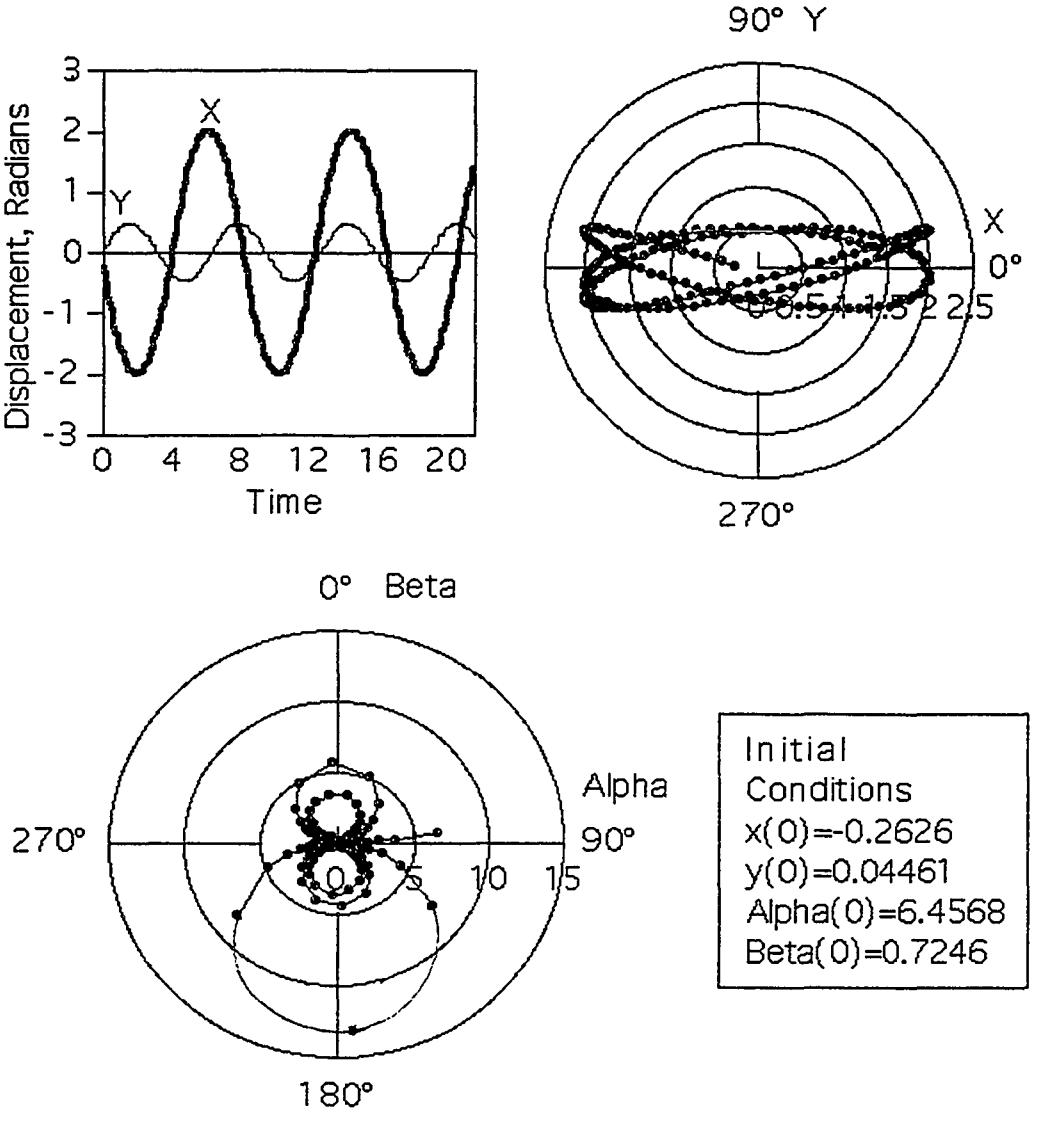


Figure 7-1: A randomly chosen x,y and α,β plot for the nonlinear pendulum, $x''+\sin(x)=0$. There are different plots for every set of initial conditions. Note that the x,y and α,β plots are 2-dimensional projections of behavior in a 4-dimensional x,y,α,β space.

possibilities there are often several solutions that appear to be rather simple. This section will show one way to find particularly simple and useful solutions for periodic oscillatory systems. It is about how to look at nonlinear eigenvalues.

Amplitudes

A clear advantage can be gained by arranging for the motion of the y system to have the same amplitude as the motion of the x system. The period of a pendulum depends on its amplitude. If the x and y solutions have the same period, their phase relationship will be fixed, and simpler behavior can be expected. The y solution can be chosen to have the same amplitude as the x solution by assuring that the two systems have the same energy at initial time t=0.

Selecting the initial conditions for the y motion in this way greatly simplifies the plots, and begins to reveal some interesting behavior. This situation is shown in Figure 7-2, for an amplitude of 2.5 radians. Both the x,y and α,β solutions become periodic - the orbits shown in the polar plots will not change over time. Also, the double loop that forms the α,β orbit is traversed twice for every x,y orbit. The α,β orbit appears to be a limacon, but it is not, exactly. It is stretched in the direction of the beta axis.

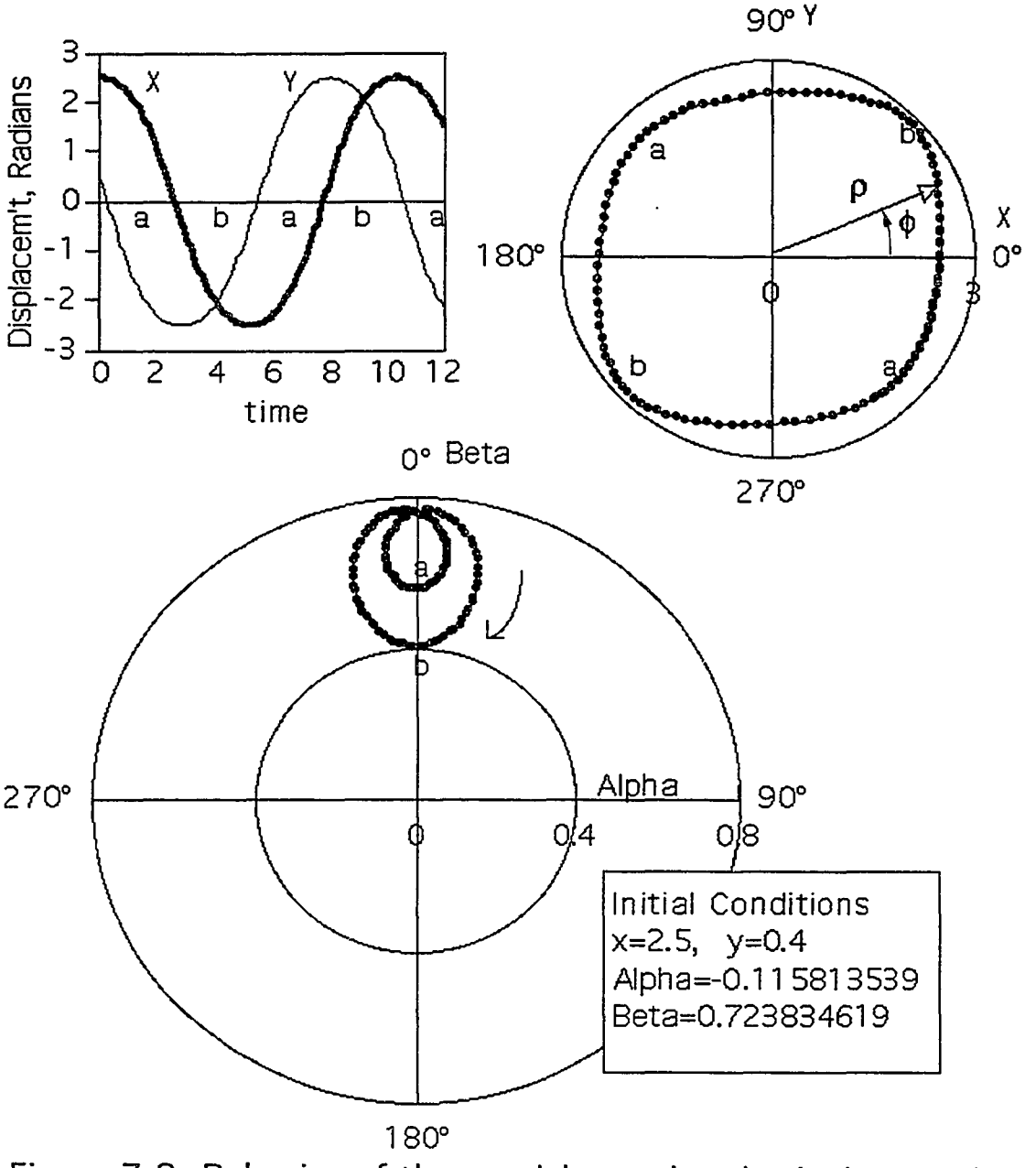


Figure 7-2. Behavior of the pendulum, when both the x and y solutions have an amplitude of 2.5 radians. The y solution leads x by approximately 85 degrees. The two minima of the α,β path are labeled a and b, and corresponding points on the other graphs are so labeled.

The curves in the x,y and α,β plots of Figure 7-2 are really two dimensional projections of a single solution curve in the four dimensional x,y,α,β solution space. That is because each variable depends on the other three. The choice of initial conditions made here leads to a closed four dimensional solution path, or orbit, that can then be simply represented in separate x,y and α,β plots.

Phase Relationships

The next step is to investigate what happens when the phase relationship of x and y is varied. Figure 7-3 shows a series of polar plots, for six different phase relationships. The time traces of x and y are not really necessary to understand the behavior, and will no longer be included.

Figure 7-3a is the degenerate solution to the dual state variable formulation, wherein $\beta=0$ and $y=x$. The α,β solution curve is a straight line along the α axis that periodically passes from $-\infty$ to $+\infty$. In this limiting case, the x,y phase space essentially collapses and becomes one-dimensional. This solution is called degenerate because the x and y state equations (2.31) and (2.32) become redundant. None the less, it still yields the correct solutions for the x and y motion, although that is difficult to show since α becomes infinite periodically.

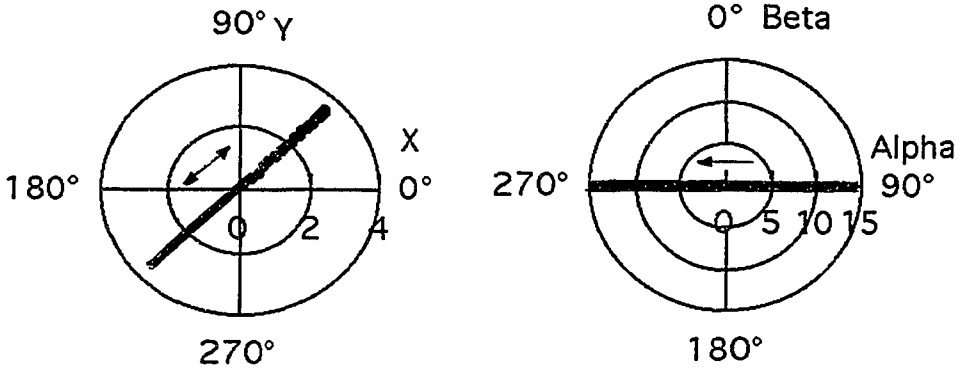


Figure 7-3a. Phase difference 0 degrees (y=x).

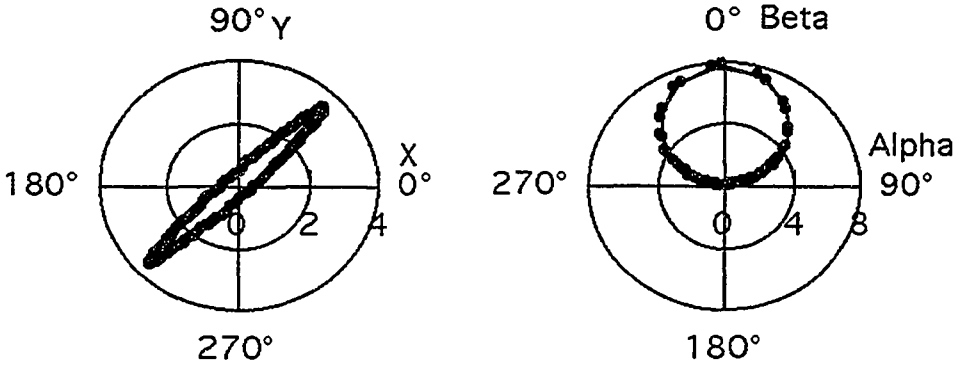


Figure 7-3b. Phase difference approx. 10 degrees.

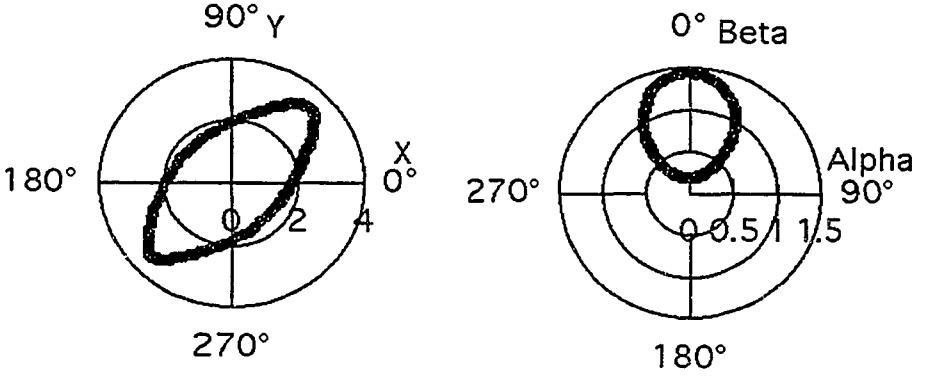


Figure 7-3c. Phase difference approx. 45 degrees.

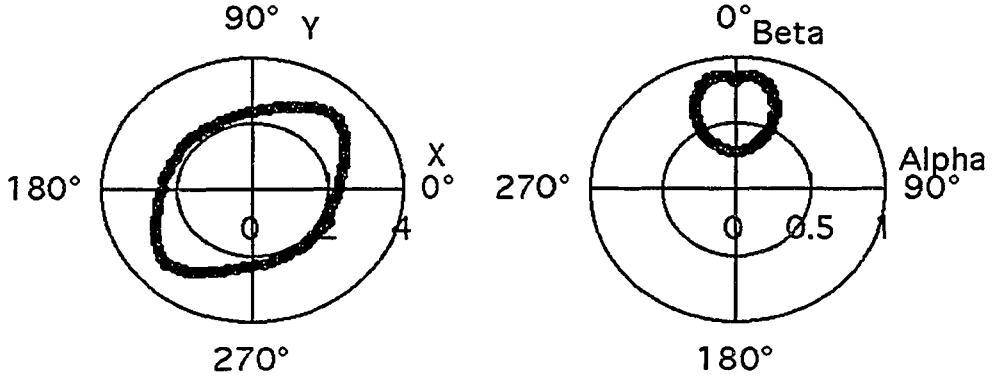


Figure 7-3d. Phase difference approx. 65 degrees.

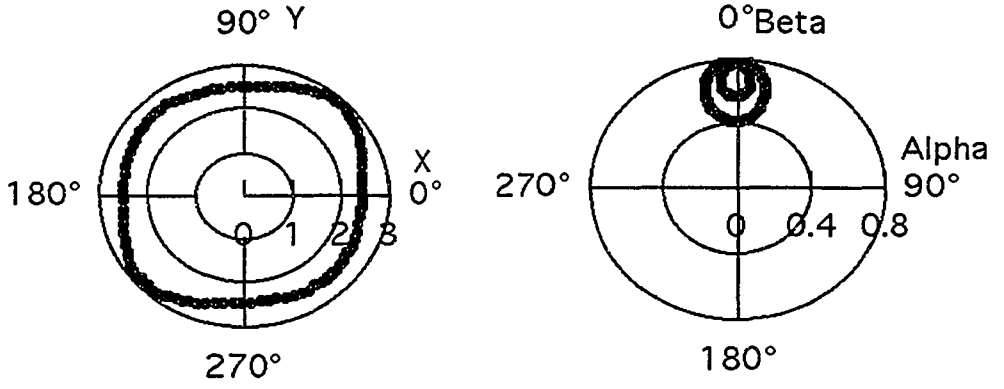


Figure 7-3e. Phase difference approx. 85 degrees.

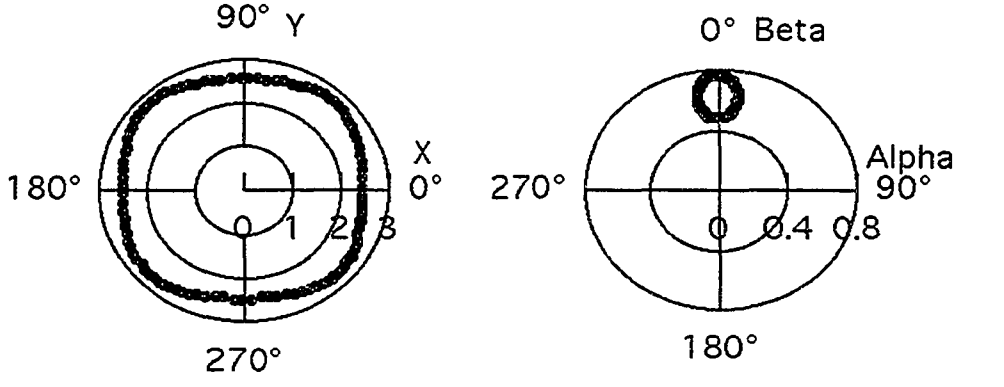


Figure 7-3f. Phase difference 90 deg. (y leads x by 90 deg.)

Figure 7-3. Behavior of the nonlinear pendulum x''+sin(x)=0 for various phase relationships. Amplitude is fixed.

As the phase difference between x and y increases to 10 degrees (Figure 7-3b), the α,β solution curve resembles a circle of large radius. In fact, the α,β orbit approaches a perfect circle of increasing radius as the x,y phase relationship approaches zero degrees, i.e., as y approaches x. The α,β solution shown in Figure 7-3a, which is a straight line, can be thought of as a perfect circle of infinite radius.

As the phase difference between x and y increases (Figure 7-3c), the α,β orbit becomes smaller and begins to look somewhat egg-shaped. The α,β orbits, so far, are traversed twice for every orbit of the x,y solution.

As the phase difference between x and y is increased further, the α,β orbit becomes reminiscent of a cardioid (Figure 7-3d). Eventually, the cardioid shape becomes a double loop similar to a limacon (Figure 7-2 and Figure 7-3e). The double loop is traversed twice for each orbit in the x,y plane.

As the phase relationship reaches 90 degrees (Figure 7-3f), the two loops of the α,β orbit merge into one. The resulting single loop orbit is traversed four times for each orbit in the x,y plane. When x and y are separated by a 90 degree phase difference, the x,y plot assumes its greatest symmetry, and the α,β orbit assumes its minimum area. This is reminiscent of the behavior of the constant

coefficient equations of Chapter 3: the α,β solution curve becomes a fixed point, the same point as the traditional x,v state space eigenvalue, exactly when x and y are 90 degrees out of phase (Figure 3-5a).

The dual state variable formulation, when applied to a nonlinear differential equation, can never lead to a constant α,β solution (a solution curve that is a fixed point in the α,β plane). If it did, the solution would have to represent the motion of a constant coefficient linear equation, since every constant α,β solution can be associated with the eigenvalue of such a linear equation. Apparently, the α,β solution curve shown in Figure 7-3f is as close as the nonlinear pendulum equation gets to having a constant eigenvalue.

To sum up this discussion, general solutions to nonlinear problems, as represented in the x,y,α,β space, can be very confusing. However, some flexibility exists in the choice of initial conditions. Since the motion of the x system is the focus of the analysis, and the motion of the y system is generally of no real concern, the initial conditions for the y motion can be chosen to provide a simpler picture of behavior. Specifically, when the amplitude of the y motion is chosen to be the same as that of the x motion, the four dimensional x,y,α,β solution curve becomes a closed orbit, with relatively simple behavior.

Additional flexibility remains, as the phase relationship of x and y can also be chosen.

Any particular phase relationship could be used as a basis for further work, but the remainder of this chapter will employ the 90 degree relationship of Figure 7-3f. This is a logical choice, in a way, since it most nearly approximates the simple, constant α,β solution found for linear, constant coefficient equations.

Characterizing the Behavior of the Pendulum

In this section, the behavior of the simple pendulum x''+sin(x)=0 will be investigated as a function of its amplitude. A phase difference of 90 degrees and identical amplitudes for x and y will be consistently employed.

Plots of the x,y state space behavior will be considered first. Figure 7-4 shows the x,y orbits for various amplitudes all nested on the same graph. At small amplitudes, the orbits are almost circular. That is consistent with the observation that the pendulum can be successfully linearized for small oscillations.

Rotational behavior of the pendulum is not really considered here, but it may help to compare the entire x,y state space to the more familiar x,v state space. Comparable state spaces are shown in Figure 7-5.

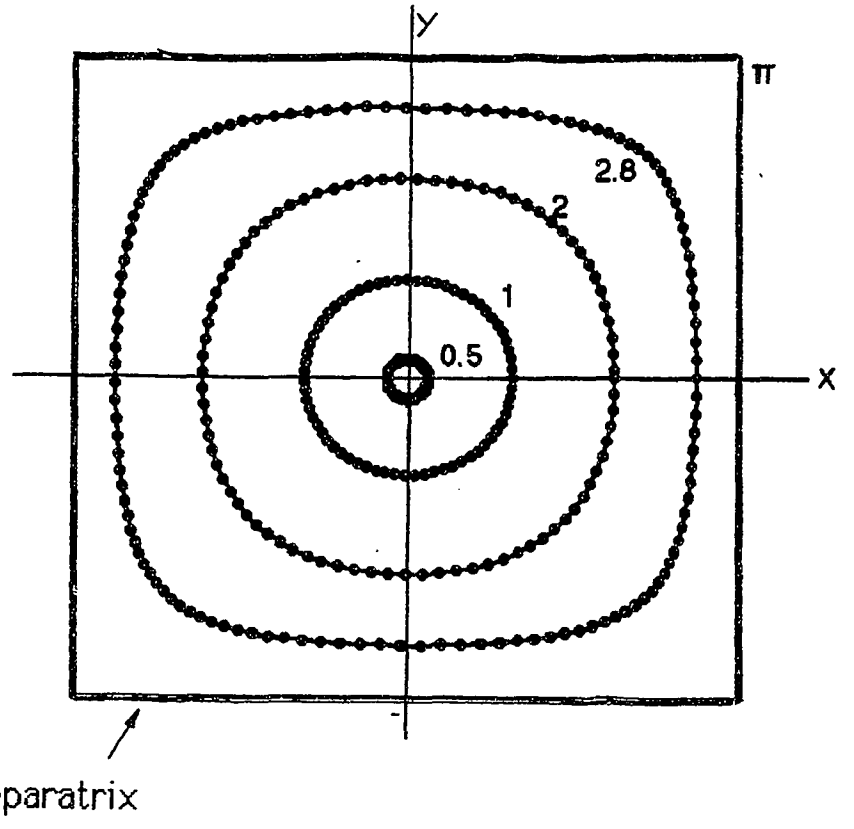


Figure 7-4. Solution curves in the x,y state space for the nonlinear pendulum, $x''+\sin(x)=0$. Five plots are nested together on the same graph, each for a different amplitude. The square outer orbit is a solution as well: it is the separatrix, and its edges are located at $\pm\pi$. Amplitudes are given in radians.

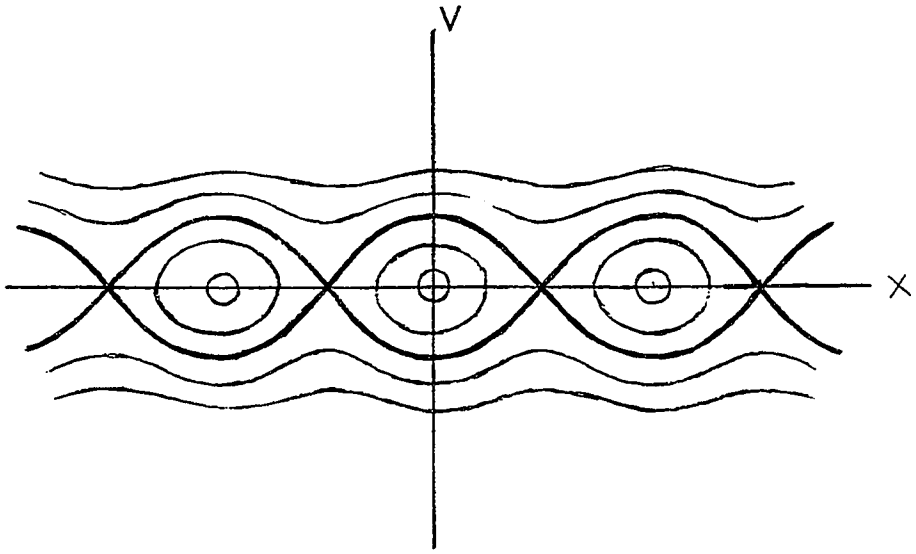


Figure 7-5a. The traditional x,v state space.

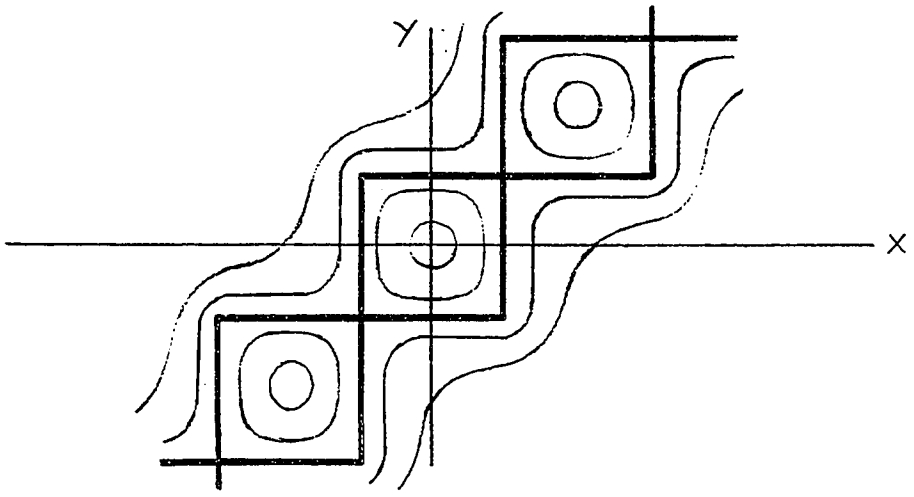


Figure 7-5b. The dual state variable x,y state space.

Figure 7-5. Comparison of two state representations for the undamped, nonlinear pendulum.

The separatrix in Figures 7-4 and 7-5b is a square. This can be understood by considering that the two systems (x and y) are 90 degrees out of phase. For oscillations approaching ±180 degrees, one system will spend most of its time in the nearly upright position, near the unstable equilibrium point. That system will be almost stationary while the other system is moving rapidly. For oscillations of exactly ±180 degrees, this results in a perfectly square orbit, but of course it will then take an infinite time to reach any corner.

Now turn to an investigation of the α,β space; that is, look at the eigenvalues. Alpha-beta orbits are shown in Figure 7-6 for various amplitudes of pendulum motion. It is more instructive, however, to look at this in three dimensions as in Figure 7-7. Figure 7-6 is a vertical view of the graph in Figure 7-7.

Figure 7-7 provides a visual image of how the pendulum departs from linear behavior as amplitude increases. The first step in the traditional approach to the nonlinear pendulum equation is to linearize it, recognizing that $\sin(x)\approx x$ for small amplitudes. The linearized equation, $x''+x=0$, is then put into the traditional x,v state variable form, and eigenvalues are computed. The eigenvalues found this way are $\lambda=0\pm i$, and they are completely independent of the pendulum's amplitude. Although they are found via

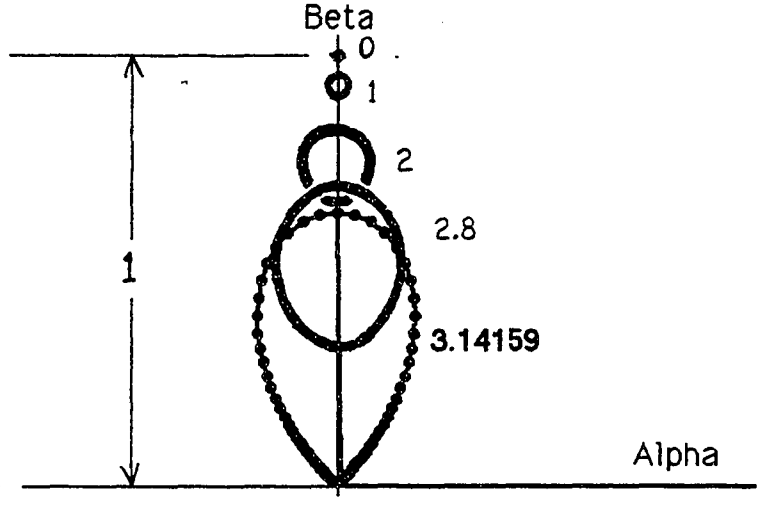


Figure 7-6. Alpha, beta solution curves (eigenvalues) for the undamped, nonlinear pendulum. Eigenvalue orbits are shown for 5 different amplitudes, which are given in radians.

traditional methods and are plotted in the s-plane, these same eigenvalues can be found by applying the dual state variable formulation to the linearized equation. (Recall from Chapter 3 that to find this specific eigenvalue pair with the dual state variable formulation, x and y must have the same amplitude and be 90 degrees out of phase.) Hence it makes sense to plot an eigenvalue of the linearized equation in the α,β plane, along with those of the nonlinear equation. The eigenvalue $\lambda=i$ (i.e., $\beta=1$) is plotted together with the eigenvalues of the nonlinear equation in Figure 7-7

for comparison. It is clear that for small amplitudes, the eigenvalues of the nonlinear pendulum equation and its linearized model are nearly the same. As amplitude increases, the orbits of the eigenvalues of the nonlinear equation diverge rapidly from the fixed eigenvalue of the linear model.

The α,β orbit associated with the separatrix is unique in that it passes through the origin of the α,β space. When it is at this point, the x,y state point is not moving (see equations 2.37 and 2.38). This corresponds to the corners of the square separatrix in the x,y space. It means the x and y systems are both stuck at the unstable, upright equilibrium point, one system having arrived and the other ready to depart. This α,β path can still be thought of as a continuous orbit, it is just that it takes an infinite amount of time to pass through the origin.

The angular velocity of the state vector (Figure 2-1) is given by the polar state equation $\phi' = -\beta$. Note that as the amplitude increases, eigenvalue orbits display a decreasing average β value. This means that the larger the amplitude, the slower the state vector rotates around the origin of the x,y state space; it means the period increases as the amplitude increases. Other observations about the motion can also be made by studying the way alpha changes.

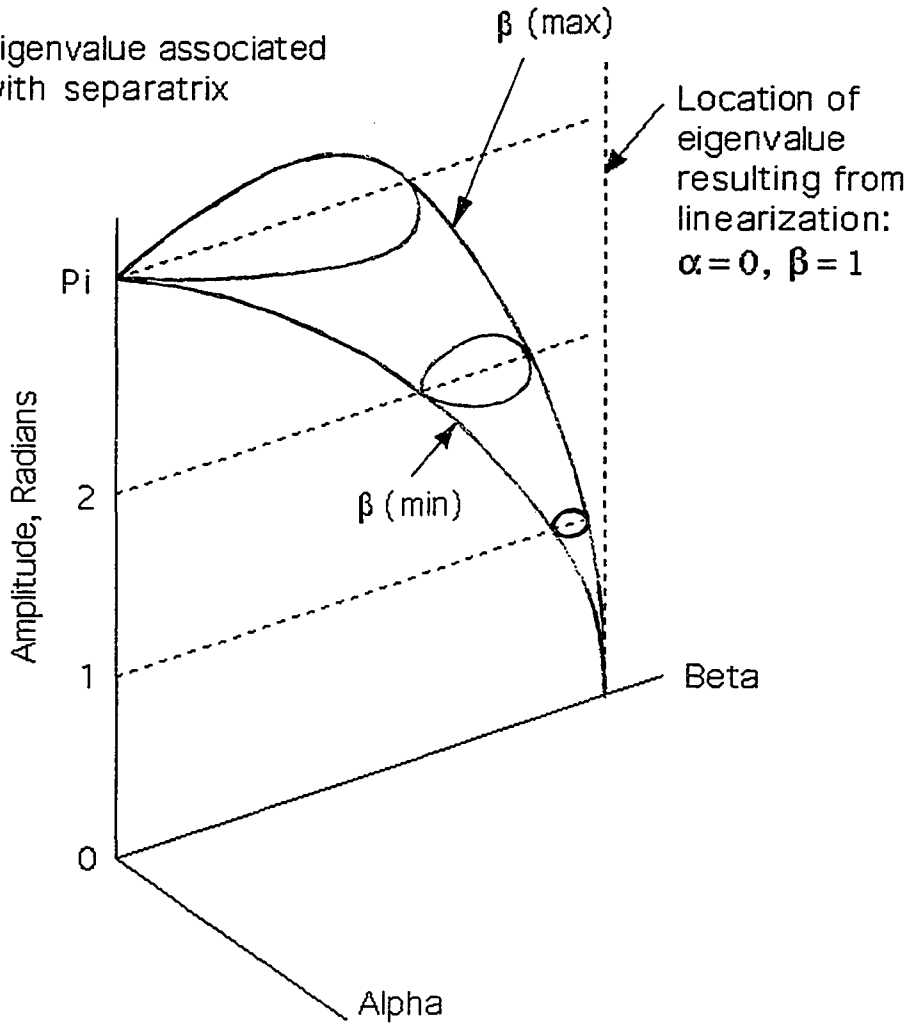


Figure 7-7. A 3-D graph of eigenvalues vs. amplitude. The α,β curves are the same ones shown in Figure 7-6. For small amplitudes, the actual nonlinear eigenvalues are approximated by the constant eigenvalue of the linearized equation.

Figure 7-7 is the main attraction in this chapter. It demonstrates that by judicious use of the flexibility inherent in this formulation, nonlinear eigenvalues can sometimes be found that are simple and instructive. The approach taken here to simplify the x,y and α,β state representations is not the only productive approach. Others will come to mind upon reflection, and creativity is definitely called for when applying the formulation.

In the next chapter, it will be shown that the nonlinear pendulum equation and the associated nonlinear eigenvalues can be used directly to obtain an approximate solution for the period and motion of the pendulum. The use of the actual nonlinear equations leads to a much better solution than one could obtain with a linearized model, and the results are even better than can be expected with perturbation methods or other advanced techniques.

CHAPTER 8
THE NONLINEAR PENDULUM

$$x'' = -\sin(x) \tag{8.1}$$

As simple as it appears, this equation for the motion of an undamped pendulum has never been solved in closed form. All known solutions are approximations based on linearized models, numerical methods, or several other advanced techniques that often involve infinite series. An approximate solution for the motion and period of the nonlinear pendulum will be developed here using an entirely new approach. The solution will be found by working directly with the nonlinear equation (8.1) and its associated nonlinear eigenvalues. Only simple integration, differentiation, algebra, and clues provided by numerical integration will be used. No infinite series or advanced mathematical techniques will be called upon. All observations drawn from numerical investigation will be vindicated by the accuracy of the final result. This example will be sufficient to demonstrate some of the advantages of the dual state variable formulation.

As in Chapter 7, the y system is chosen to have the same form as the x system; that is, since $x'' = F_1 = -\sin(x)$, let $y'' = F_2 = -\sin(y)$. In the dual state variable formulation,

four equations define the motion, except for initial conditions.

$$x' = \alpha x + \beta y \tag{2.31}$$

$$y' = -\beta x + \alpha y \tag{2.32}$$

$$\alpha' = \frac{x\sin(x) + y\sin(y)}{x^2 + y^2} + \beta^2 - \alpha^2 \tag{8.2}$$

$$\beta' = \frac{-x\sin(y) + y\sin(x)}{x^2 + y^2} - 2\alpha\beta \tag{8.3}$$

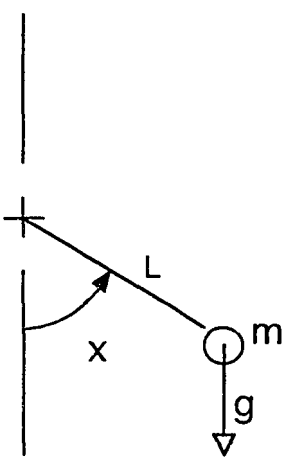


Figure 8-1. The pendulum.

Equations (8.2) and (8.3) are drawn from equations (4.8) and (4.9), respectively. They are written out to correspond to the present problem, a treatment of equation (8.1). That is, appropriate substitutions have been made for F_1 and F_2. Three of these equations will be solved, approximately, in this chapter. The solution for y(t) could be found as well, but it is irrelevant here.

Modeling Eigenvalue Behavior

The first step is to characterize the α,β orbit as well as possible. Since α and β are defined by nonlinear ODE's, investigation of their behavior will be done numerically. A typical α,β orbit and the associated x,y state space orbit are shown in Figure 8-2. Points a and b are matched on both plots. It is assumed that the orbit is centered on the β axis (α=0); this is confirmed by numerical investigation, and it is naturally expected for an undamped system. The locations of the maximum and minimum beta values β_1 and β_2 need to be determined. These beta values are associated with points a and b in the figure, respectively. Once their locations are identified, a model, or approximation, of the eigenvalue path will be constructed.

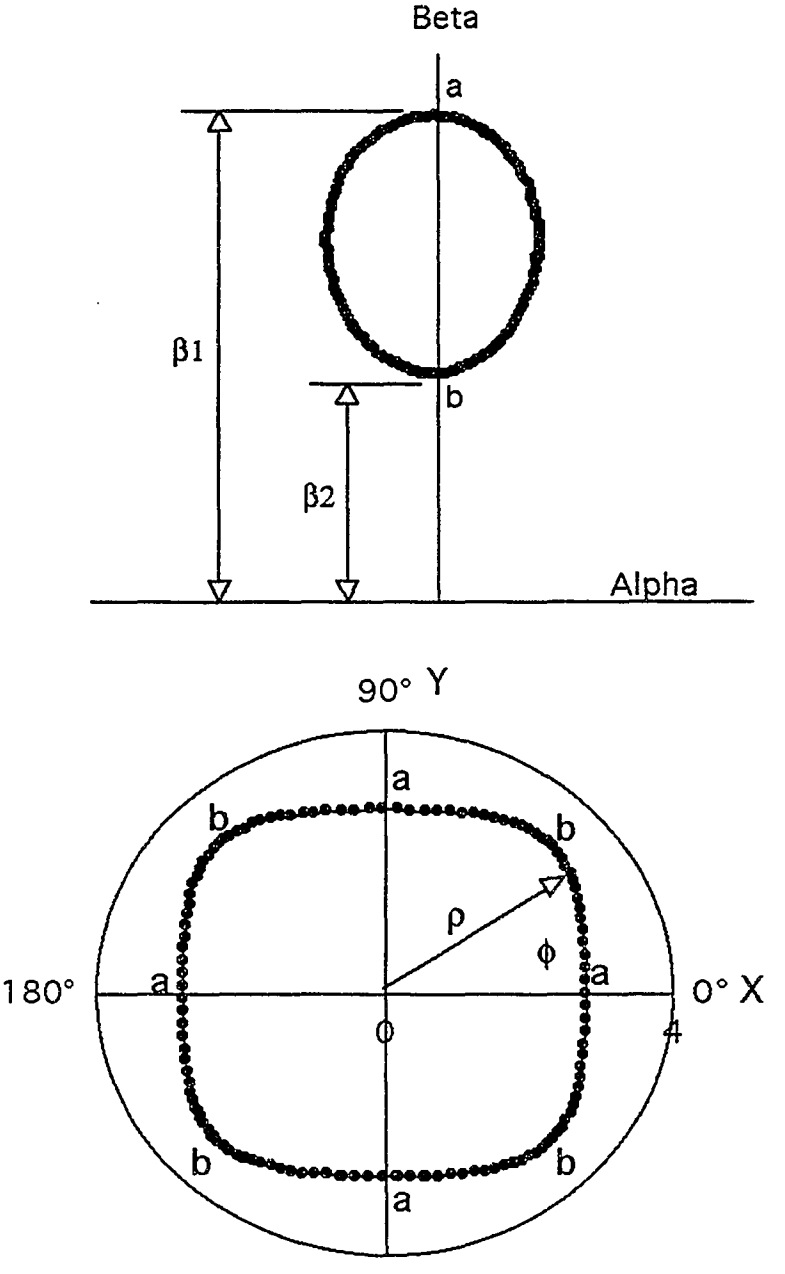


Figure 8-2. The state space solution curve, and the associated α,β orbit, or eigenvalue, for an undamped pendulum with amplitude 2.8 radians. Both the x and y systems have the same amplitude, and they are 90 degrees out of phase, as in Chapter 7.

The Identification of β_1:

The maximum beta value β_1 occurs when $\alpha=0$ (at points a in Figure 8-2). The amplitude of pendulum motion is designated X. By the symmetry evident in the plots, one of the points β_1 is associated with is x=0 and y=X (see points a in Figure 8-2). At this point the state equation (2.31) for x' becomes

$$x' = \beta_1 y = \beta_1 X,$$

or

$$\beta_1 = \frac{x'}{X}, \tag{8.4}$$

where x′ must be the maximum velocity. Energy relations allow x′ to be identified.

The equation is of course normalized so the mass is 1, and the coefficient of the sine term in (8.1) is 1, indicating the acceleration of gravity g equals the length of the pendulum arm L, g=L. The total energy of the x (or y) system, based on the maximum potential energy, is

$$E_x = L^2 [1 - \cos(X)]. \tag{8.5}$$

The linear velocity of the pendulum bob is Lx′, and the maximum kinetic energy is then

$$E_X = \frac{1}{2}L^2x'^2 = L^2[1-\cos(X)]. \qquad (8.6)$$

From this find the maximum angular velocity of the x (or y) system,

$$x' = \sqrt{2[1-\cos(X)]}, \qquad (8.7)$$

and

$$\beta_1 = \frac{\sqrt{2[1-\cos(X)]}}{X}. \qquad (8.8)$$

The Identification of β_2:

The identification of β_2 requires some numerical sleuthing. The empirically derived result is

$$\beta_2 = \frac{\sin(x_2)}{x_2}, \qquad (8.9)$$

where x_2 is the associated state coordinate, at point b in Figure 8-2. Although this is an empirically derived result, it is believed to be exact. Numerically observed errors are exceedingly small. Even if an exact expression is not available, the analysis can easily be continued with an approximation.

It is interesting to note that β_1 and β_2 can both be written in the same form,

$$\beta_1 = \frac{\sin\left(\frac{x_1+y_1}{2}\right)}{\frac{x_1+y_1}{2}} \quad \text{and} \quad \beta_2 = \frac{\sin\left(\frac{x_2+y_2}{2}\right)}{\frac{x_2+y_2}{2}}, \tag{8.10}$$

where x_1 and y_1 are the values at point a in Figure 8-2, and x_2 and y_2 are the values at point b. A trigonometric half angle identity, equation (8.8), and knowledge of x_1 and y_1 from the figure will lead to the first of these relationships. The second relationship, for β_2, comes from equation (8.9), and the observation from Figure 8-2 that $y_2 = x_2$ at point b. Unfortunately, this relationship does not hold for intermediate points on the α,β orbit.

Beta 2 is expressed in terms of x_2, and it is desirable to know what it is in terms of the amplitude X. Work with point b in the first quadrant of the x,y plot. At this point, $y_2 = x_2$, $y' = -x_2'$, and $\alpha = 0$. From the state equation (2.31) find $x_2' = \beta_2 x_2$. Then, using equation (8.9), write

$$\beta_2 = \frac{x_2'}{x_2} = \frac{\sin(x_2)}{x_2},$$

or

$$x_2' = \sin(x_2). \tag{8.11}$$

This equation can be used to relate the potential and kinetic energy of the system, just for the point b. From equation (8.5),

$$KE + PE = L^2 [1 - \cos(X)]. \tag{8.12}$$

Substitute for the kinetic and potential energy terms, and cancel L to find

$$\frac{1}{2} x_2'^{\,2} + [1 - \cos(x_2)] = [1 - \cos(X)]. \tag{8.13}$$

Replace the derivative term with equation (8.11) and solve for x_2:

$$x_2 = \arccos\left\{-1 + \sqrt{2\left[1 + \cos(X)\right]}\right\}. \tag{8.14}$$

Beta 2 can now be written in terms of the amplitude X,

$$\beta_2 = \frac{\sqrt{-2 - 2\cos(X) + 2\sqrt{2(1 + \cos(X))}}}{\arccos\left\{-1 + \sqrt{2\left[1 + \cos(X)\right]}\right\}}. \tag{8.15}$$

Note that β_1 is known exactly, and the expression for β_2 is believed to be exact. That is nice, but it is not necessary. Good approximations would be just as useful.

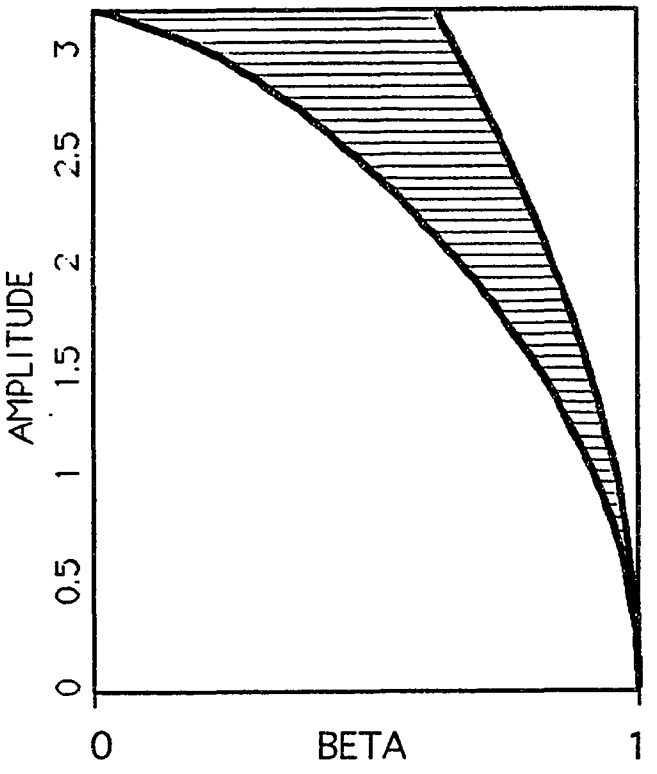
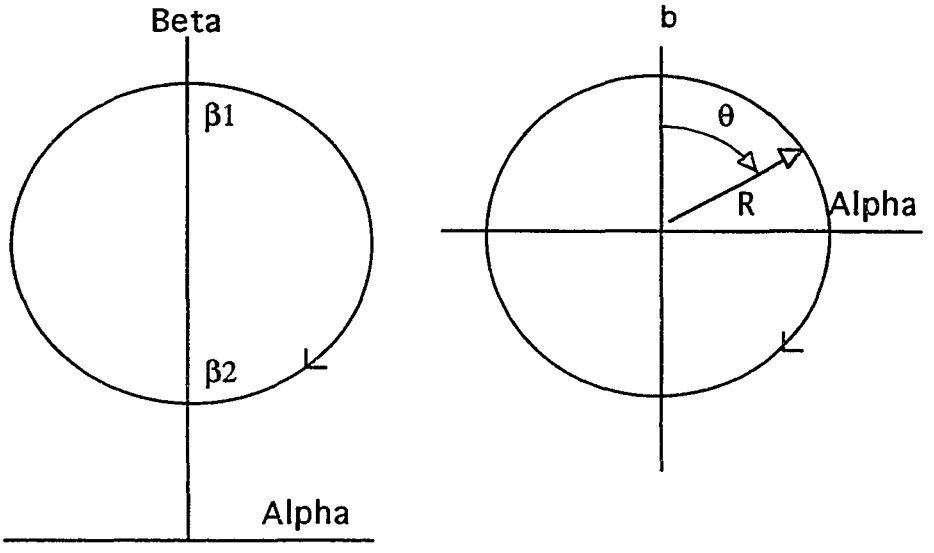


Figure 8-3. Computer generated plots of β_1 and β_2, the maximum and minimum values of beta, respectively. This plot is a slice of the 3-D graph shown in Figure 7-7, made at α=0. This graph again shows how the eigenvalue departs from the linear approximation as the amplitude increases.

A Model for the Path of the Eigenvalue:

Figure 8-2 shows the path of an actual eigenvalue. Although it is egg shaped, it will be modeled as a circle, as shown in Figure 8-4a. To simplify the analysis, the α axis is shifted up to the circle center, and the β axis becomes the b axis. (Do not confuse the coordinate axis b with the points designated a and b in earlier figures.) In all computations to follow, motion is assumed to begin at the top of the eigenvalue orbit: t=0 at α=0 and $\beta = \beta_1$. Also, x=0 and y=X at t=0.



a) The model in the α,β plane

b) The model, with the α axis shifted to the circle center.

Figure 8-4. The model of the eigenvalue path.

The geometry of the model will now be defined analytically. The center of the circular orbit is designated B,

$$B = \beta(\text{center}) = \frac{\beta_1 + \beta_2}{2}, \tag{8.16}$$

and the b coordinate can be written

$$b = \beta - B. \tag{8.17}$$

The radius of the orbit is

$$R = \frac{\beta_1 - \beta_2}{2}, \tag{8.18}$$

and the equation of the circle is

$$\alpha^2 + b^2 = R^2. \tag{8.19}$$

The derivative of the circle equation may also be used.

$$2\alpha\alpha' + 2bb' = 0 \tag{8.20}$$

The polar form of these equations will often be used in the analysis as well.

$$\alpha = R\sin(\theta) \tag{8.21}$$

$$b = R\cos(\theta) \tag{8.22}$$

$$\alpha' = \theta' R\cos(\theta) \tag{8.23}$$

$$b' = -\theta' R\sin(\theta) \tag{8.24}$$

The geometry of the eigenvalue path has now been modeled. A circle may seem to be a crude approximation of the α,β path shown in Figure 8-2a. It <u>is</u> crude, but it is radically better than the fixed, constant eigenvalue of the linearized equation shown in Figure 7-7. It will also be necessary to choose a model for the speed of the eigenvalue on its path, but that will be left for a later section.

The Period of the Pendulum

The only things necessary to approximate the period of the pendulum are the maximum and minimum beta values β_1 and β_2, respectively. Beta is a direct measure of the angular velocity of the state point (recall equation (2.36), $\phi' = -\beta$). If beta was a constant, the integral of equation (2.36) would be $\phi = -\beta t$. One complete orbit in the x,y state space implies $\phi = 2\pi$, and the period could then be written

$$T = \frac{2\pi}{\beta}. \tag{8.25}$$

The minus sign drops out because rotation in the x,y state space, for $\beta>0$, is in the $-\phi$ direction. Of course, β is not constant in the model considered here, but it is possible to find an average value that will yield a good estimate of the period.

Two different averages will be computed. The first is simply the arithmetic average of the maximum and minimum values defined by equations (8.8) and (8.15), respectively. From this average, find

$$T = \frac{4\pi}{\beta_1 + \beta_2}. \tag{8.26}$$

A geometric average can also be computed for beta. This is simply

$$\beta_{avg} = \sqrt{\beta_1 \beta_2}\,. \tag{8.27}$$

This average beta value can be related to earlier analysis. Recall the linear underdamped example of Chapter 3. A circular orbit for the eigenvalue was related to the constant α,β solution point, as shown in Figure 3-10. This

constant point (c1 in the figure) is not at the center of the circular orbit. If the equations of Chapter 2 are applied to the circular orbit modeled here, a corresponding beta value (i.e. c1) can be found, and it is the geometric average given by equation (8.27). The geometric average leads to the approximate period given below.

$$T = \frac{2\pi}{\sqrt{\beta_1 \beta_2}} \tag{8.28}$$

These two estimates of the period are shown in Figure 8-5.

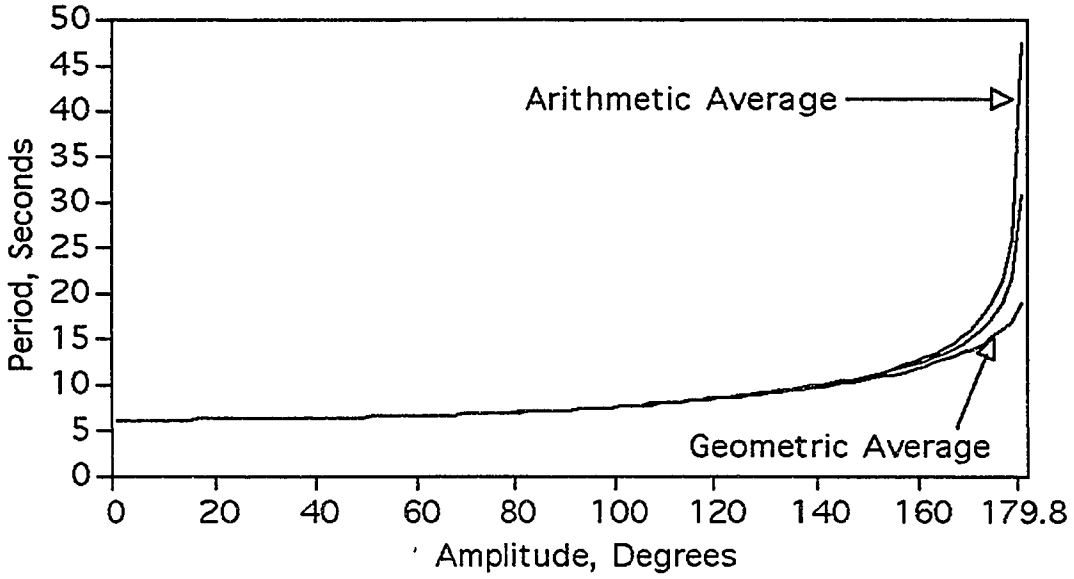


Figure 8-5. Estimates of the period of the pendulum, from equations (8.26) and (8.28). The actual period is the curve shown between the two estimates.

Both of these estimates are reasonable, but because one is high and the other is low, a simple average of the two yields an even better result. An arithmetic average of equations (8.26) and (8.28) leads to the following estimate.

$$T = \pi\left[\frac{2\sqrt{\beta_1\beta_2} + \beta_1 + \beta_2}{\sqrt{\beta_1\beta_2}\left(\beta_1 + \beta_2\right)}\right] \tag{8.29}$$

Equation (8.29) is a good estimate for the period, but other estimates can also be constructed. Equations (8.26) and (8.28) can be averaged geometrically as well, and the result, for instance, can be arithmetically averaged with equation (8.29). The result in this case is the estimated period of equation (8.30). For some amplitudes near π radians, it is better than equation (8.29). The results of equations (8.29) and (8.30) are shown in Figure 8-6.

$$T = \frac{\pi\left[2\sqrt{2\beta_1\beta_2}\left(\beta_1 + \beta_2\right) + \left(\beta_1 + \beta_2 + 2\sqrt{\beta_1\beta_2}\right)\sqrt{\left(\beta_1 + \beta_2\right)\sqrt{\beta_1\beta_2}}\right]}{2\sqrt{\beta_1\beta_2}\left(\beta_1 + \beta_2\right)\sqrt{\left(\beta_1 + \beta_2\right)\sqrt{\beta_1\beta_2}}} \tag{8.30}$$

These estimates for the period of the pendulum appear to be very good. They will be compared with a solution published by S. J. Liao in 1992. Liao used a new

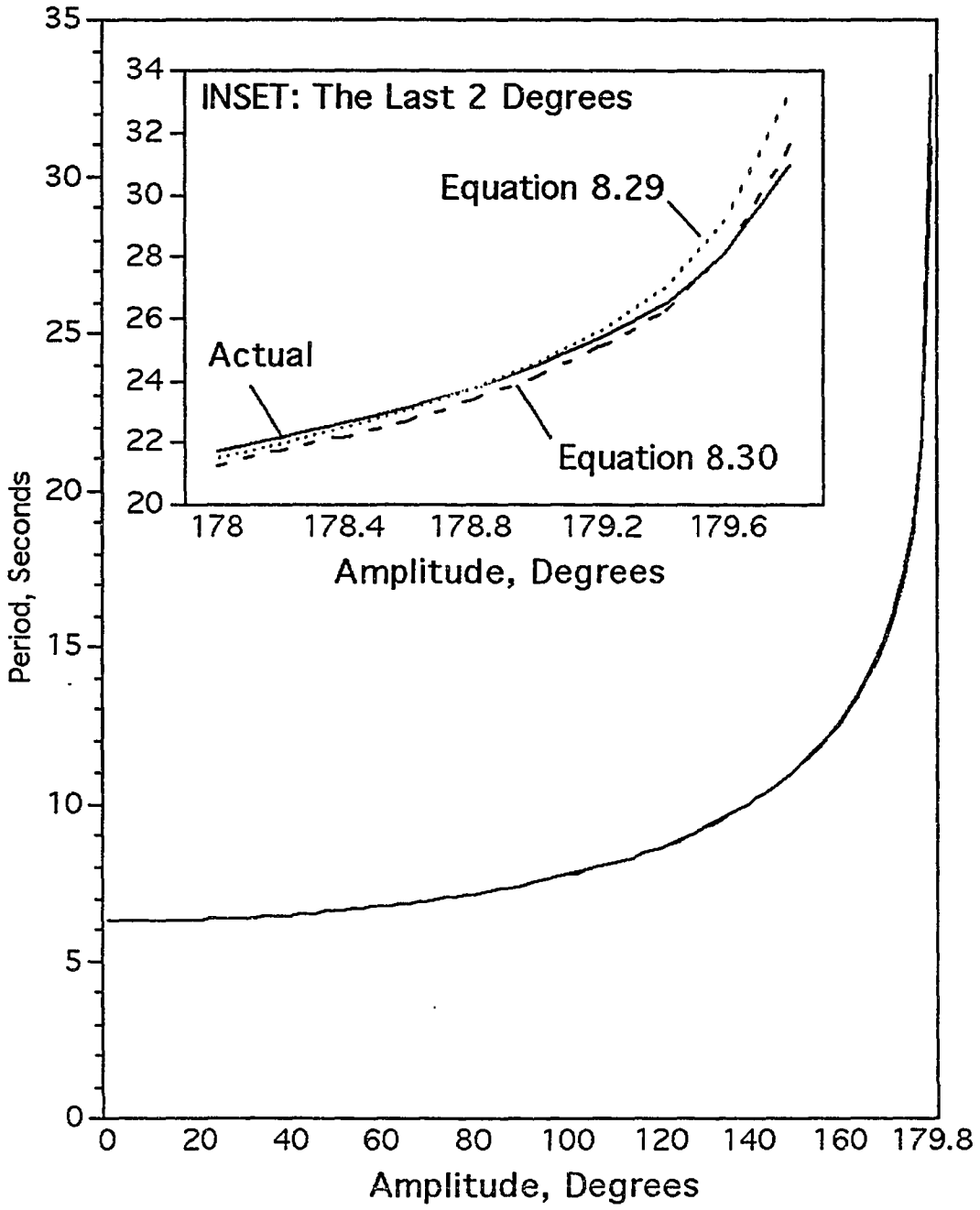


Figure 8-6. Estimates of the period from equations (8.29) and (8.30), shown with the actual period. The three curves overlay each other and appear as one, except in the inset.

technique called the process analysis method (PAM), and compared his results with those from a perturbation method. Table 8-1 compares equations (8.29) and (8.30) with these published results. For the range and accuracy shown, the results of (8.29) and (8.30) are identical.

The PAM method, as applied by Liao, requires the use of infinite series and Bessel functions. The results published by Liao are the best the author could find that originate directly from the equation of motion. Better predictions exist (see, for instance, Abramowitz and Stegun), but they rely on numerical methods or curve fitting techniques

Table 8-1: Comparison of the estimated period with the results of perturbation and PAM methods (see S. J. Liao).

	Normalized Period $T/2\pi$			
Amplitude Degrees	Exact	Perturb'n	PAM	Eqn. (8.29) and (8.30)
20	1.0076	1.008	1.008	1.0077
40	1.0313	1.030	1.031	1.0313
60	1.0732	1.069	1.07	1.0732
80	1.1375	1.122	1.127	1.1375
100	1.2322	1.190	1.201	1.2321
120	1.3729	1.274	1.296	1.3723
130	1.4698	1.322	1.351	1.4687

Equation (8.29) is two orders of magnitude more accurate than the results of the PAM analysis at an amplitude of130 degrees. A review of the derivation of the PAM results will confirm that the solution given here is at least an order of magnitude less complicated as well.

Engineers will often find the period from a table of complete elliptical integrals of the first kind. A typical table (see Tuma, "Engineering Mathematics Handbook") lists values equivalent to 1/4 the period, for angles $\omega = 0$ to 90 degrees. One degree in ω is equivalent to 2 degrees of amplitude for the system represented by equation (8.1). A comparison with exact, tabulated values is given in Table 8-2.

Table 8-2. The pendulum period predicted by equations (8.29) and (8.30), compared to exact, tabulated values.

Amplitude degrees	Exact seconds	Eqn (8.29)	Error %	Eqn. (8.30)	Error %
60	6.743	6.743	0.003	6.743	0.003
90	7.416	7.416	0.007	7.416	0.007
120	8.626	8.623	0.039	8.623	0.039
150	11.072	11.043	0.266	11.042	0.275
160	12.613	12.549	0.510	12.546	0.540
170	15.326	15.171	1.02	15.151	1.15
178	21.739	21.529	0.97	21.299	2.03
179	24.511	24.585	0.30	24.118	1.61
179.8	30.948	33.271	7.50	31.641	2.24

Either equation (8.29) or (8.30) can be easily used in many practical engineering computations. A correction factor could be devised to improve the accuracy of these equations even further, but that is not the object of this text. The uncorrected results are evidence enough of the improved accuracy that can be obtained by working directly with nonlinear equations and their eigenvalues.

An Approximate Solution for Pendulum Motion

To fully characterize the behavior of the eigenvalue, the speed of the eigenvalue on its path must also be determined. That is, before a solution can be found, θ' must be known; θ is shown in Figure 8-4. It is desirable to determine the speed analytically, without further approximation. In other words, it would be nice to use the equations developed previously, and some additional energy relations, to solve for θ' directly. For some differential equations, with some eigenvalue path geometries, this is possible. Unfortunately, for the pendulum, with the eigenvalue path geometry chosen, such an analytic determination is difficult.

It turns out, however, that even exceptionally crude approximations for θ' lead to very reasonable solutions for the motion of the pendulum. An actual plot of θ' is shown in

Figure 8-7. A word about how the plot was generated is in order. First, the dual state variable equation set, consisting of equations (2.31), (2.32), (8.2) and (8.3), was numerically integrated. The solution for β was expressed in terms of the coordinate b using equation (8.17). The value of θ was then computed for points around the orbit (from Figure 8-4b, note $\theta = \arctan(\alpha/b)$), and then differentiated. However, as will be seen, the solution does not depend on this numerical work. The 'actual' θ' curve in Figure 8-7 is only shown for comparison.

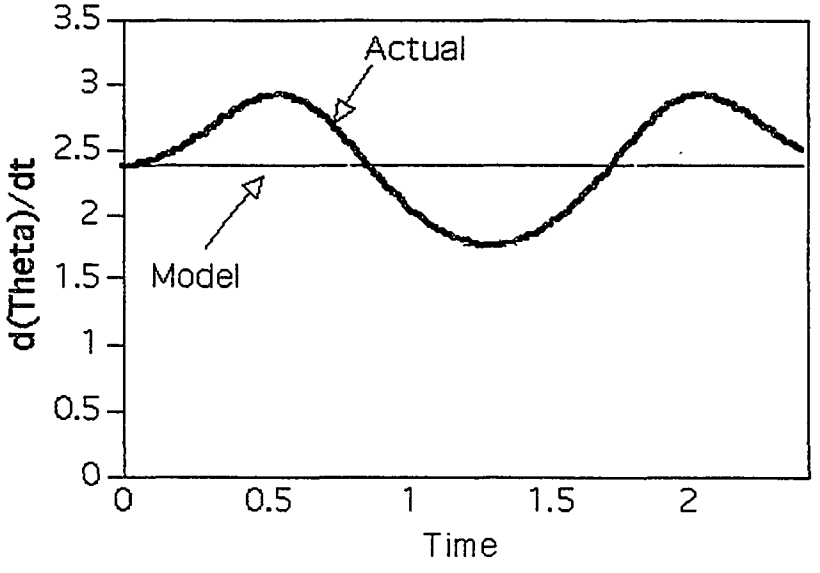


Figure 8-7. An actual plot of θ' for an amplitude of 2.5 radians. The behavior is modeled as a straight line.

The simplest possible model will be used here: θ' will be modeled as a constant, as shown in Figure 8-7. The assumed constant value of θ' is arbitrarily chosen to be its initial value. This value is easily computed. Refer to Figure 8-4b, and recall that the initial conditions selected for the analysis are α=0, $\beta = \beta_1$, x=0, and y=amplitude X. The initial condition for β comes from equation (8.8). From Figure 8-4b, at the initial point, the value of the angular velocity θ'_0 can be written

$$\theta'_0 = \frac{\alpha'_0}{R}. \tag{8.31}$$

The value of α'_0 comes from equation (8.2) with appropriate substitutions for the initial conditions. The resulting constant is

$$\theta'_0 = \frac{2 - X\sin(X) - 2\cos(X)}{RX^2}. \tag{8.32}$$

Integrate this to find

$$\theta = \theta'_0 t. \tag{8.33}$$

Alpha and beta can now be written using the polar form of equations (8.21) and (8.22), and also using (8.17).

$$\alpha = R\sin(\theta_0' t) \tag{8.34}$$

$$\beta = R\cos(\theta_0' t) + B \tag{8.35}$$

The integrals of α and β will also be needed, and can easily be found.

$$\int_0^t \alpha dt = \frac{R(1-\cos[\theta_0' t])}{\theta_0'} \tag{8.36}$$

$$\int_0^t \beta dt = \frac{R\sin[\theta_0' t]}{\theta_0'} + Bt \tag{8.37}$$

Recall the general solution form from Chapter 5, equation (5.41). This applies to all second order ordinary differential equations, and it is repeated below.

$$x = e^{\int \alpha dt}\left(x_0 \cos[\int \beta dt] + y_0 \sin[\int \beta dt]\right) \tag{5.41}$$

The solution x(t) can now be written directly using this equation.

$$x = e^{\left(\frac{R(1-\cos[\theta_0' t])}{\theta_0'}\right)}\left[X\sin\left(\frac{R\sin[\theta_0' t]}{\theta_0'} + Bt\right)\right] \tag{8.38}$$

This concludes the derivation of the solution.

The approximate solutions for α and β are plotted with the numerical solutions in Figure 8-8. The associated solution x(t) for 1/4 period of the pendulum is shown in Figure 8-9. A large amplitude, 2.5 radians (143 degrees), has been used in all these plots in order to make deviations from the true solutions more obvious. The numerically obtained curves in these figures are verified in the appendix.

This solution was based on a simple, crude model for the behavior of an eigenvalue. It was modeled as traveling at constant speed in a circular orbit. This leads to an exceptionally simple formulation of the problem, but the results are still considerably better than those achieved with conventional linearization. Linearization is essentially the process of modeling the eigenvalue as the fixed point $\beta=1$ and $\alpha=0$, shown in Figure 7-7. Whatever its weaknesses, modeling the eigenvalue as a properly located circular orbit is radically better than modeling it as a fixed point.

Better modeling can be expected to lead to better results. The most readily attainable improvement will be to model the orbital speed θ' better. Additional improvements might be expected by modeling the orbit as an ellipse. There are many ways to improve the model, and the creativity of the analyst may be richly rewarded.

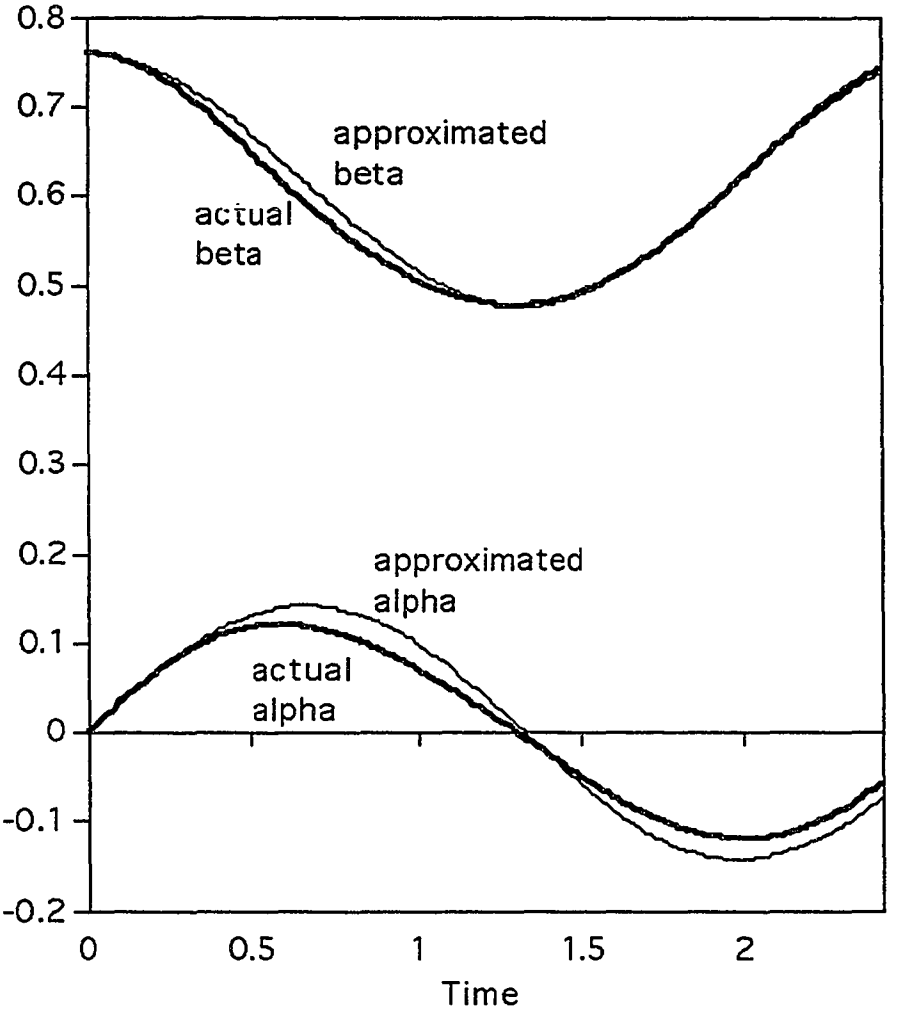


Figure 8-8. The approximate solutions for α and β, compared to numerical results, for an amplitude of 2.5 radians.

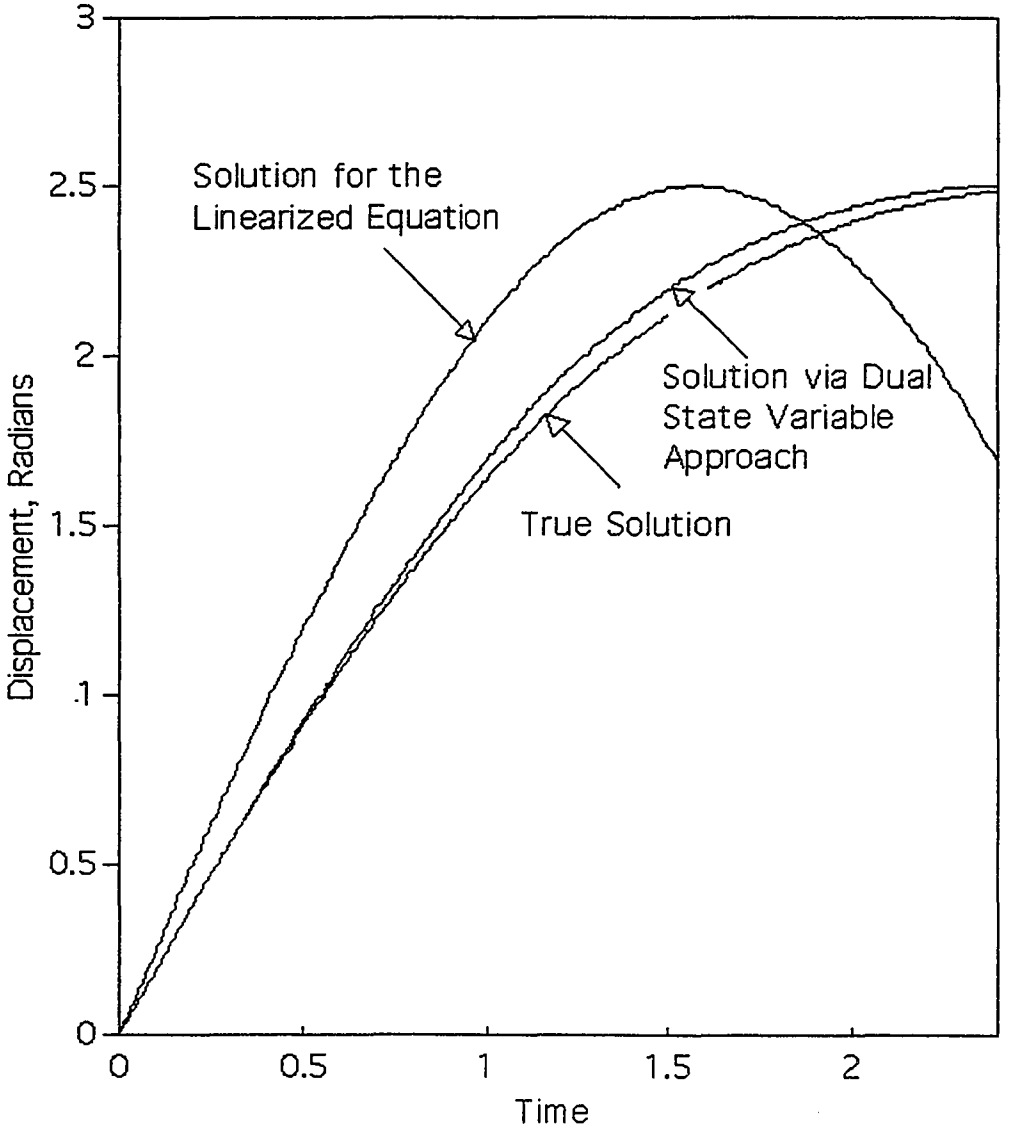


Figure 8-9. The approximate solution for the pendulum, compared to the numeric solution, and the solution obtained by classic linearization. The amplitude for this example is 2.5 radians.

Incomplete elliptical integrals of the first kind are linked to the motion of the pendulum. Solutions for them are available in tabular form only. It may be possible to use this approach to find a very accurate solution for the motion of a pendulum, and then write an effective closed form approximation for these integrals, especially for, say, the first half of the tables.

The point of this chapter was not to find the best possible solution for the pendulum. Better solutions can surely be found by refining the method used here. Instead, this chapter serves to demonstrate how the dual state variable formulation can be used to generate an approximate solution for a nonlinear differential equation, by working directly with the nonlinear ODE and its associated eigenvalues. It shows that even crude models of the nonlinear behavior will yield results far better than can be expected when working with the linearized equation. Finally, it shows that all this can be done with the basic mathematics generally taught in an undergraduate engineering curriculum. Numerical integration is used as a tool for visualizing behavior, but no reference to advanced mathematical concepts or techniques is required.

CHAPTER 9
EXTENSION TO HIGHER ORDER DIFFERENTIAL EQUATIONS

The dual state variable formulation will now be extended to represent ordinary differential equations of any order n. The approach taken here will be different than that taken in Chapters 1 and 3, but the derivation will still be based on a physical interpretation of the phase space. The two conditions applied earlier will be used again, but not in the same order, nor with the same condition numbers. The results will be entirely consistent with the second order formulation presented in Chapter 4.

A derivation will be carried out for third order ODE's first; this will serve to explain the logic involved. Once this is done, it will be possible to present the extension to n-th order ODE's in relatively simple terms. A brief example of one of the consequences of this extension will be presented as well.

Third Order Ordinary Differential Equations

Begin with the general third order equation

$$x''' = F_1(x'', x', x, t). \tag{9.1}$$

Represent this with the following general linear state variable equation set.

$$x_1' = a_1x_1 + a_2x_2 + a_3x_3 \tag{9.2}$$

$$x_2' = a_4x_1 + a_5x_2 + a_6x_3 \tag{9.3}$$

$$x_3' = a_7x_1 + a_8x_2 + a_9x_3 \tag{9.4}$$

In matrix form this is

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \tag{9.5}$$

or

$$\mathbf{R}=\mathbf{AS}, \tag{9.6}$$

where **S** is the state vector and **R** is the rate vector. Assume these vectors are expressed in a three dimensional orthogonal phase space in which all three axes (x_1, x_2, and x_3) represent displacement. The coefficients a_i are completely undefined, and an infinite number of possible definitions could be found for them such that equation (9.5) accurately represents equation (9.1). As with

the second order formulation, certain conditions will be applied that will uniquely identify coefficients a_i.

Condition 1

The first condition applied here is that the scalar product $\mathbf{S} \bullet \mathbf{R}$ must be invariant with respect to rotations of the state space reference frame. This was designated as Condition 2 in Chapter 2, where further discussion can be found. The scalar product is expressed as follows.

$$\mathbf{S} \bullet \mathbf{R} = x_1 x_1' + x_2 x_2' + x_3 x_3' \tag{9.7}$$

Expand this using the state equations (9.2), (9.3), and (9.4).

$$\begin{aligned} \mathbf{S} \bullet \mathbf{R} = {} & a_1 x_1^{\,2} + a_5 x_2^{\,2} + a_9 x_3^{\,2} \\ & + (a_2 + a_4) x_1 x_2 + (a_3 + a_7) x_1 x_3 + (a_6 + a_8) x_2 x_3 \end{aligned} \tag{9.8}$$

In practice, this condition is applied by factoring the square of the length of the state vector (i.e., ρ) out of the scalar product above.

$$\rho^2 = x_1^{\,2} + x_2^{\,2} + x_3^{\,2} \tag{9.9}$$

Factoring this out of (8.8) in as simple a manner as possible, find the following constraints:

$$a_5 = a_1, \quad a_9 = a_1$$
$$a_4 = -a_2, \quad a_7 = -a_3, \quad a_8 = -a_6. \tag{9.10}$$

The A matrix becomes

$$\mathbf{A} = \begin{bmatrix} a_1 & a_2 & a_3 \\ -a_2 & a_1 & a_6 \\ -a_3 & -a_6 & a_1 \end{bmatrix}. \tag{9.11}$$

Having applied this condition, it is now profitable to compute the expression for ρ', and also to compute the eigenvalues associated with the **A** matrix. Differentiate equation (9.9) and solve for ρ'. Then make appropriate substitutions using (9.2), (9.3), (9.4), and (9.10) to find

$$\rho' = a_1 \rho . \tag{9.12}$$

The eigenvalues are

$$\lambda_1 = a_1 \tag{9.13}$$

$$\lambda_2 = a_1 + i\sqrt{a_2^{\,2} + a_3^{\,2} + a_6^{\,2}} \tag{9.14}$$

$$\lambda_3 = a_1 - i\sqrt{a_2^{\,2} + a_3^{\,2} + a_6^{\,2}}\,. \qquad (9.15)$$

Because a_1 is the real part of each eigenvalue, and equation (9.12) is consistent with one of the (polar) state equations from Chapter 4, the symbol α will be substituted for a_1: $a_1 = \alpha$.

Condition 2

Given the results of condition 1, it is reasonable to ask whether or not the imaginary portion of the eigenvalues can be related to some single quantity that represents the rotational motion of the state vector **S**. If so, that single quantity will be analogous to the variable β used in connection with second order equations. This idea leads to the second condition.

The rotational motion of the state vector **S** must first be described. The velocity of the tip of the state vector is given by the rate vector **R**, and both of these vectors exist in a three dimensional orthogonal space, the x_1, x_2, x_3 phase space. Only the magnitude of the angular velocity of the state vector will be computed, since we are looking for a single scalar value we can compare to the imaginary portion of the eigenvalues. The magnitude of the angular velocity is designated ϕ'. The vector product **S**x**R** provides

a way to find it. Recall that ρ is the length of the state vector; then

$$\phi' = \frac{|\mathbf{S} \times \mathbf{R}|}{\rho^2}. \tag{9.16}$$

Let i, j, and k be unit vectors associated with the x_1, x_2, and x_3 state space axes, respectively. Then the cross product is

$$\mathbf{S} \times \mathbf{R} = i(x_2 x_3' - x_3 x_2') + j(x_3 x_1' - x_1 x_3') + k(x_1 x_2' - x_2 x_1'), \tag{9.17}$$

and

$$|\mathbf{S} \times \mathbf{R}| = \sqrt{(x_2 x_3' - x_3 x_2')^2 + (x_3 x_1' - x_1 x_3')^2 + (x_1 x_2' - x_2 x_1')^2}. \tag{9.18}$$

Substitute equations (9.2), (9.3), (9.4), and (9.10) to find

$$|\mathbf{S} \times \mathbf{R}| = \rho \sqrt{(a_3 x_1 + a_6 x_2)^2 + (a_2 x_1 - a_6 x_3)^2 + (a_2 x_2 + a_3 x_3)^2}, \tag{9.19}$$

where

$$\rho = \sqrt{x_1^2 + x_2^2 + x_3^2}. \tag{9.20}$$

Substituting into equation (9.16), find

$$\phi' = \frac{\sqrt{(a_3x_1 + a_6x_2)^2 + (a_2x_1 - a_6x_3)^2 + (a_2x_2 + a_3x_3)^2}}{\sqrt{x_1^2 + x_2^2 + x_3^2}}. \quad (9.21)$$

Condition 2 is simply the requirement that ϕ', the magnitude of the angular velocity of the state vector, be equal to the imaginary portion of the eigenvalues. That is,

$$\phi' = \sqrt{a_2^2 + a_3^2 + a_6^2}. \quad (9.22)$$

Substitute (9.21) into (9.22) and then simplify to find the following constraint equation.

$$a_6x_1 - a_3x_2 + a_2x_3 = 0 \quad (9.23)$$

This constraint can be satisfied in many ways, but the simplest solution appears to result when the following choices are made:

$$a_2 \equiv 0, \quad a_6 \equiv 0, \quad \text{and} \quad x_2 \equiv 0. \quad (9.24)$$

Given this solution for equation (9.23) , the matrix form of the state equations (9.5) can be expressed as follows.

$$\begin{bmatrix} x_1' \\ 0 \\ x_3' \end{bmatrix} = \begin{bmatrix} \alpha & 0 & a_3 \\ 0 & \alpha & 0 \\ -a_3 & 0 & \alpha \end{bmatrix} \begin{bmatrix} x_1 \\ 0 \\ x_3 \end{bmatrix} \tag{9.25}$$

The eigenvalues become

$$\lambda_1 = \alpha \tag{9.26}$$

$$\lambda_2 = \alpha + ia_3 \tag{9.27}$$

$$\lambda_3 = \alpha - ia_3, \tag{9.28}$$

and the expression for ϕ' reduces to

$$\phi' = -a_3. \tag{9.29}$$

The sign of a_3 is optional, and the choice selected above will ensure consistency with the original second order formulation of Chapter 4.

With the equations in this form, it is now advantageous to change the names of some variables. Make the following changes.

$$a_3 = \beta\,, \quad x_1 = x, \quad \text{and} \quad x_3 = y \tag{9.30}$$

With these changes, the formulation becomes nearly identical to the second order formulation. Only two state equations remain, and they are consistent with earlier results in both cartesian and polar form. The state equations become:

$$x' = \alpha x + \beta y \tag{1.31}$$

$$y' = -\beta x + \alpha y, \tag{1.32}$$

or

$$\rho' = \alpha\rho \tag{1.37}$$

$$\phi' = -\beta. \tag{1.38}$$

All that remains is to define expressions for α and β. That will be accomplished through application of the third and final condition.

Condition 3

This third condition corresponds to the first condition used in the analysis of Chapter 4. It was not applied first

here because that would unnecessarily complicate and obscure the analysis.

Recall equation (9.1), and write a dual equation to define y'''.

$$y''' = F_2(y'', y', y, t) \tag{9.31}$$

As suggested in earlier analysis, it is often convenient to construct this equation in y such that it represents the same system as the equation in x. However, as pointed out in Chapter 4 and elsewhere, that is not essential. Any third order equation in y can be chosen, and, in fact, it is even possible to choose an equation of some other order. For the moment, however, let y be defined by a third order equation.

The third condition is to require the twice-differentiated state equations (2.31) and (2.32) to be identical to (9.1) and (9.31), respectively. That is, the state equations are required to duplicate the equations of motion, when they are differentiated appropriately. Successively differentiate(2.31) and (2.32) to find the following relations.

$$x'' = \alpha x' + \beta y' + \alpha' x + \beta' y \tag{9.32}$$

$$y'' = -\beta x' + \alpha y' - \beta' x + \alpha' y \tag{9.33}$$

$$x''' = \alpha x'' + 2\alpha' x' + \beta y'' + 2\beta' y' + \alpha'' x + \beta'' y \tag{9.34}$$

$$y''' = \alpha y'' + 2\alpha' y' - \beta x'' - 2\beta' x' + \alpha'' y - \beta'' x \tag{9.35}$$

The application of the third condition yields the following constraint equations.

$$\alpha x'' + 2\alpha' x' + \beta y'' + 2\beta' y' + \alpha'' x + \beta'' y = F_1 \tag{9.36}$$

$$\alpha y'' + 2\alpha' y' - \beta x'' - 2\beta' x' + \alpha'' y - \beta'' x = F_2 \tag{9.37}$$

These can now be solved for α'' and β''.

$$\alpha'' = \frac{F_1 x + F_2 y}{x^2 + y^2} - \alpha^3 + 3\alpha\beta^2 - 3\alpha\alpha' + 3\beta\beta' \tag{9.38}$$

$$\beta'' = \frac{-F_2 x + F_1 y}{x^2 + y^2} + \beta^3 - 3\alpha^2\beta - 3\alpha'\beta - 3\alpha\beta' \tag{9.39}$$

In order to arrive at these equations, all derivatives of x and y must be eliminated by appropriate substitution of equations (2.31), (2.32), (9.32), and (9.33).

Summary of the Formulation for a Third Order Equation

With this formulation, the general third order equation (9.1) is represented with an equation set that is nearly identical to that of the second order formulation. The only difference is that the differential equations defining α and β are of higher order.
The defining equations for the third order formulation are collected below.

$$x' = \alpha x + \beta y \tag{2.31}$$

$$y' = -\beta x + \alpha y \tag{2.32}$$

$$\alpha'' = \frac{F_1 x + F_2 y}{x^2 + y^2} - \alpha^3 + 3\alpha\beta^2 - 3\alpha\alpha' + 3\beta\beta' \tag{9.38}$$

$$\beta'' = \frac{-F_2 x + F_1 y}{x^2 + y^2} + \beta^3 - 3\alpha^2\beta - 3\alpha'\beta - 3\alpha\beta' \tag{9.39}$$

N-th Order Ordinary Differential Equations

The sequential application of the three conditions in the previous section helps illuminate the physical significance of the formulation, but it need only be done

once. The extension of the dual state variable formulation to n-th order equations can now be carried out in a much more straightforward manner.

Begin with the n-th order ordinary differential equation

$$x^{[n]} = F_1\left(x^{[n-1]}, x^{[n-2]}, \ldots, x, t\right). \tag{9.40}$$

To apply the dual state variable formulation, begin by letting the equation be represented by just two state variables x and y, in an orthogonal, two-dimensional state space. Construct or define a dual system in the variable y. While any differential equation will do, and the equation in y does not even have to be the same order as the x equation, it will be assumed here that the y equation will have the same form as equation (9.40) in x.

$$y^{[n]} = F_2\left(y^{[n-1]}, y^{[n-2]}, \ldots, y, t\right) \tag{9.41}$$

Because of the simple polar form resulting from them, adopt the following state equations.

$$x' = \alpha x + \beta y \tag{2.31}$$

$$y' = -\beta x + \alpha y \tag{2.32}$$

Differentiate these equations [n-1] times.

$$x^{[2]} = \frac{d}{dt}(\alpha x + \beta y) \tag{9.42}$$

$$y^{[2]} = \frac{d}{dt}(-\beta x + \alpha y)$$

•

•

•

$$x^{[n]} = \frac{d^{[n-1]}}{dt^{[n-1]}}(\alpha x + \beta y) \tag{9.43}$$

$$y^{[n]} = \frac{d^{[n-1]}}{dt^{[n-1]}}(-\beta x + \alpha y) \tag{9.44}$$

Equate (9.43) and (9.44) with equations (9.40) and (9.41), respectively:

$$\frac{d^{[n-1]}}{dt^{[n-1]}}(\alpha x + \beta y) = F_1 \tag{9.45}$$

$$\frac{d^{[n-1]}}{dt^{[n-1]}}(-\beta x + \alpha y) = F_2 \quad . \tag{9.46}$$

Then solve these for the highest order derivatives of α and β, which will be of order [n-1], assuming, of course, that the equations defining the x and y systems are of the same order.

$$\alpha^{[n-1]} = H_1(F_1, F_2, x, y, \alpha, \beta, t) \tag{9.47}$$

$$\beta^{[n-1]} = H_2(F_1, F_2, x, y, \alpha, \beta, t) \tag{9.48}$$

Various derivative terms will occur on the right side of these equations. Derivatives of x and y can be removed by judicious substitution of equations (2.31), (2.32), and their successive derivatives given by (9.42).

The dual state variable equation set in this case is made up of equations (2.31), (2.32), (9.47), and (9.48). It is consistent with the following matrix representation of equation (9.40) in state variable form.

$$\begin{bmatrix} x' \\ 0 \\ \bullet \\ \bullet \\ \bullet \\ 0 \\ y' \end{bmatrix} = \begin{bmatrix} \alpha & 0 & \bullet & \bullet & \bullet & 0 & \beta \\ 0 & \alpha & 0 & \bullet & \bullet & 0 & 0 \\ & & & \bullet & & & \\ & & & \bullet & & & \\ & & & \bullet & & & \\ 0 & 0 & \bullet & \bullet & 0 & \alpha & 0 \\ -\beta & 0 & \bullet & \bullet & \bullet & 0 & \alpha \end{bmatrix} \begin{bmatrix} x \\ 0 \\ \bullet \\ \bullet \\ \bullet \\ 0 \\ y \end{bmatrix} \tag{9.49}$$

This n-dimensional representation can therefore be reduced to a two-dimensional representation by application of the dual state variable formulation. That is, equation (9.40) can be represented by the matrix differential equation (9.50) below, which is equivalent to equation (9.49). Again, the accompanying equations for α and β will be of order [n-1] when the equations (9.40) and (9.41) defining x and y are of the same order.

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \qquad (9.50)$$

Linear Algebra and the N-th Order Formulation

The 'linear algebra' described in Chapter 5 depends only on the state equations (2.31) and (2.32), and on the definition of the state space set forth earlier. These things do not change at all when the formulation is applied to higher order differential equations, and the fact that the equations defining α and β are of higher order is of no consequence. Therefore all the analysis and results presented in Chapter 5 remain valid for the general n-th order formulation. That is to say, every solution is composed of two fundamental solutions, and all solutions to equation (9.40) can be written in the form

$$x = e^{\int \alpha dt}\left(x_0 \cos(\int \beta dt) + y_0 \sin(\int \beta dt)\right). \qquad (5.41)$$

The differential equation (9.40) can be associated with exactly two eigenvalues,

$$\lambda_{1,2} = \alpha \pm i\beta. \qquad (9.51)$$

The simple polar form of the state equations also remains valid,

$$\rho' = \alpha\rho \qquad (2.37)$$
$$\phi' = -\beta. \qquad (2.38)$$

Expansion of the Equation Set

Application to First Order ODE's

The extension of the dual state variable formulation to n-th order ODE's developed above allows for the formulation to be applied to first order ODE's as well. By choosing a dual y system for a first order ODE in x, and applying the formulation, eigenvalues can be associated with the behavior of x and y. When x and y are both described by first order ODE's, a review of the derivation

above will show that the components of the eigenvalues, α and β, are described algebraically, not as differential equations.

Expansion of the Equation Set: General Description

Note that equations (9.47) and (9.48), which describe α and β, are themselves higher order ordinary differential equations. The dual state variable formulation can be independently applied to each of them. A potentially more useful approach when applying the formulation to the α and β equations, though, is to choose equation (9.48), for β, to be the dual equation associated with equation (9.47) for α. The dual state variable formulation can then be applied to this pair of equations. In this manner, the dual state variable formulation can be successively applied until the formulation consists of a set of 2n first order differential equations. Of these, [2n-2] will be similar in form to equations (2.31) and (2.32), and 2 will be of the form (4.8) and (4.9).

One more stage of expansion is possible. Consider the last two first order differential equations in the set, which, as just noted, will have the form of equations (4.8) and (4.9). The dual state variable formulation can be applied to these as well, resolving them into two first order

differential equations of form (2.31) and (2.32), and two algebraic equations. The net result is a formulation consisting of 2n equations of form (2.31) and (2.32), and 2 (nonlinear) algebraic equations.

Alpha and beta are known to be the components of the eigenvalues for the paired x and y systems. If the dual state variable formulation is then applied again to the differential equations representing α and β, it will result in the generation of eigenvalue components - call them η and κ - associated with the α and β systems. The essence of this is that eigenvalues can be nested. Eigenvalues can have eigenvalues. In other words, α and β are the real and imaginary portions of the two eigenvalues associated with the differential equation (9.40). Since α and β are generally described by differential equations, though, eigenvalues can be associated with them as well.

A Conceptual Example of Expansion

As an example, the dual state variable equation set for third order equations will now be expanded. Consider equations (9.38) and (9.39), which are both ODE's to which the dual state variable formulation can be applied.

$$\alpha'' = \frac{F_1 x + F_2 y}{x^2 + y^2} - \alpha^3 + 3\alpha\beta^2 - 3\alpha\alpha' + 3\beta\beta' \qquad (9.38)$$

$$\beta'' = \frac{-F_2 x + F_1 y}{x^2 + y^2} + \beta^3 - 3\alpha^2\beta - 3\alpha'\beta - 3\alpha\beta' \qquad (9.39)$$

For simplicity, let the right side of each equation be designated as a function G. That is, let the equations above be written as follows.

$$\alpha'' = G_1 \qquad (9.52)$$

$$\beta'' = G_2 \qquad (9.53)$$

The beta equation (9.53) will be used as the dual equation associated with equation (9.52). Equations (9.52) and (9.53) are second order, so the formulation of Chapter 4 can be applied directly to them. In the state equations (2.31) and (2.32), α and β will replace x and y as the state variables. The coefficients in these new state equations will be designated η and κ.

The state equations for the (subsidiary) state variables α and β follow. They represent the second application of the dual state variable formulation, this time applied to the second order equations for α and β.

$$\alpha' = \eta\ \alpha + \kappa\beta \tag{9.54}$$

$$\beta' = -\kappa\alpha + \eta\beta. \tag{9.55}$$

The expressions for the coefficients η and κ follow from equations (4.8) and (4.9).

$$\eta' = \frac{G_1\alpha + G_2\beta}{\alpha^2 + \beta^2} + \kappa^2 - \eta^2 \tag{9.56}$$

$$\kappa' = \frac{-G_2\alpha + G_1\beta}{\alpha^2 + \beta^2} - 2\eta\ \kappa \tag{9.57}$$

One more stage of expansion will be carried out. The dual state variable formulation will be applied to equations (9.56) and (9.57). The equation for kappa, (9.57), will be chosen as the dual to equation (9.56) for eta. Again for simplicity, designate the right side of equations (9.56) and (9.57) as functions H_1 and H_2, respectively.

$$\eta' = H_1 \tag{9.58}$$

$$\kappa' = H_2 \tag{9.59}$$

The coefficients used in the state equations for η and κ will be designated μ and σ. The form of these equations is taken directly from equations (2.31) and (2.32):

$$\eta' = \mu\ \eta + \sigma\ \kappa \tag{9.60}$$

$$\kappa' = -\sigma\ \eta + \mu\ \kappa \tag{9.61}$$

By use of the ideas presented in the section "Application to First Order ODE's" above, algebraic equations describing the coefficients μ and σ can be written. That is, by equating (9.60) with (9.56), and (9.61) with (9.57), and then solving simultaneously for μ and σ, the following equations are found.

$$\mu = \frac{H_1\eta + H_2\kappa}{\eta^2 + \kappa^2} \tag{9.62}$$

$$\sigma = \frac{-H_2\eta + H_1\kappa}{\eta^2 + \kappa^2} \tag{9.63}$$

The expanded dual state variable equation set for this case is summarized below. The equations (2.31) and (2.32) represent the initial application of the dual state variable formulation to the third order equation (9.1). The second

order equations for α and β, (9.38) and (9.39), are then written in dual state variable form, resulting in (9.54) and (9.55). The coefficients η and κ used in (9.54) and (9.55) are then also expressed in dual state variable form, and their state equations are (9.60) and (9.61). The coefficients used in these equations are μ and σ.

$$x' = \alpha x + \beta y \tag{2.31}$$

$$y' = -\beta x + \alpha y \tag{2.32}$$

$$\alpha' = \eta\ \alpha + \kappa\beta \tag{9.54}$$

$$\beta' = -\kappa\alpha + \eta\beta. \tag{9.55}$$

$$\eta' = \mu\ \eta + \sigma\ \kappa \tag{9.60}$$

$$\kappa' = -\sigma\ \eta + \mu\ \kappa \tag{9.61}$$

$$\mu = \frac{H_1\eta + H_2\kappa}{\eta^2 + \kappa^2} \tag{9.62}$$

$$\sigma = \frac{-H_2\eta + H_1\kappa}{\eta^2 + \kappa^2} \tag{9.63}$$

The eight equations above represent equation (9.1) in a manner entirely consistent with the four equation set (2.31), (2.32), (9.38), and (9.39) developed earlier. The only

difference is that the dual state variable formulation has now been successively applied to the equations for the coefficients of all the state equations. The original four equation representation has been expanded.

The eigenvalues associated with the x and y equations are

$$\lambda_{1,2} = \alpha \pm i\beta. \tag{9.51}$$

These will be regarded as the primary eigenvalues. Similarly, the (different) eigenvalues associated with the α and β equations are

$$\lambda_{1,2} = \eta \pm i\kappa. \tag{9.64}$$

These will be called the secondary eigenvalues. Furthermore, the (different) eigenvalues associated with the η and κ equations, the tertiary eigenvalues, are

$$\lambda_{1,2} = \mu \pm i\sigma. \tag{9.65}$$

When attempting to represent an n-th order ordinary differential equation in state variable form, the ability to expand it from a set of four equations to a set of 2n+2 equations may be less significant than the ability to write a

set of 2n state equations as a set of just four equations. Because of the clear meaning of the four variables involved in the unexpanded dual state variable formulation, it may be easier to work with the four equation set. However, experience with the dual state variable formulation is limited, and the ability to expand the equation set may prove quite useful as well.

A Practical Example of Expansion: The Nonlinear Pendulum

The dual state variable formulation for the nonlinear pendulum of Chapters 7 and 8 will be expanded here. The standard four equation dual state variable equation set for the pendulum is repeated below.

$$x' = \alpha x + \beta y \tag{2.31}$$

$$y' = -\beta x + \alpha y \tag{2.32}$$

$$\alpha' = \frac{x\sin(x) + y\sin(y)}{x^2 + y^2} + \beta^2 - \alpha^2 \tag{8.2}$$

$$\beta' = \frac{-x\sin(y) + y\sin(x)}{x^2 + y^2} - 2\alpha\beta \tag{8.3}$$

This equation set is expanded by replacing equations (8.2) and (8.3) with state equations for α and β. Begin by writing state equations to represent α and β; let the coefficients be η and κ.

$$\alpha' = \eta\ \alpha + \kappa\beta \tag{9.66}$$

$$\beta' = -\kappa\alpha + \eta\beta. \tag{9.67}$$

Equate the right sides of (8.2) and (9.66), and of (8.3) and (9.67). Then solve the resulting equations simultaneously for η and κ.

$$\eta = \frac{(\alpha x + \beta y)\sin(x) + (-\beta x + \alpha y)\sin(y)}{(x^2 + y^2)(\alpha^2 + \beta^2)} - \alpha \tag{9.68}$$

$$\kappa = \frac{-(-\beta x + \alpha y)\sin(x) + (\alpha x + \beta y)\sin(y)}{(x^2 + y^2)(\alpha^2 + \beta^2)} + \beta \tag{9.69}$$

Alpha and beta, the components of the primary eigenvalues, are the real and imaginary components of the eigenvalues associated with the differential equations for x and y. Eta and kappa are the real and imaginary components of the (secondary) eigenvalues associated

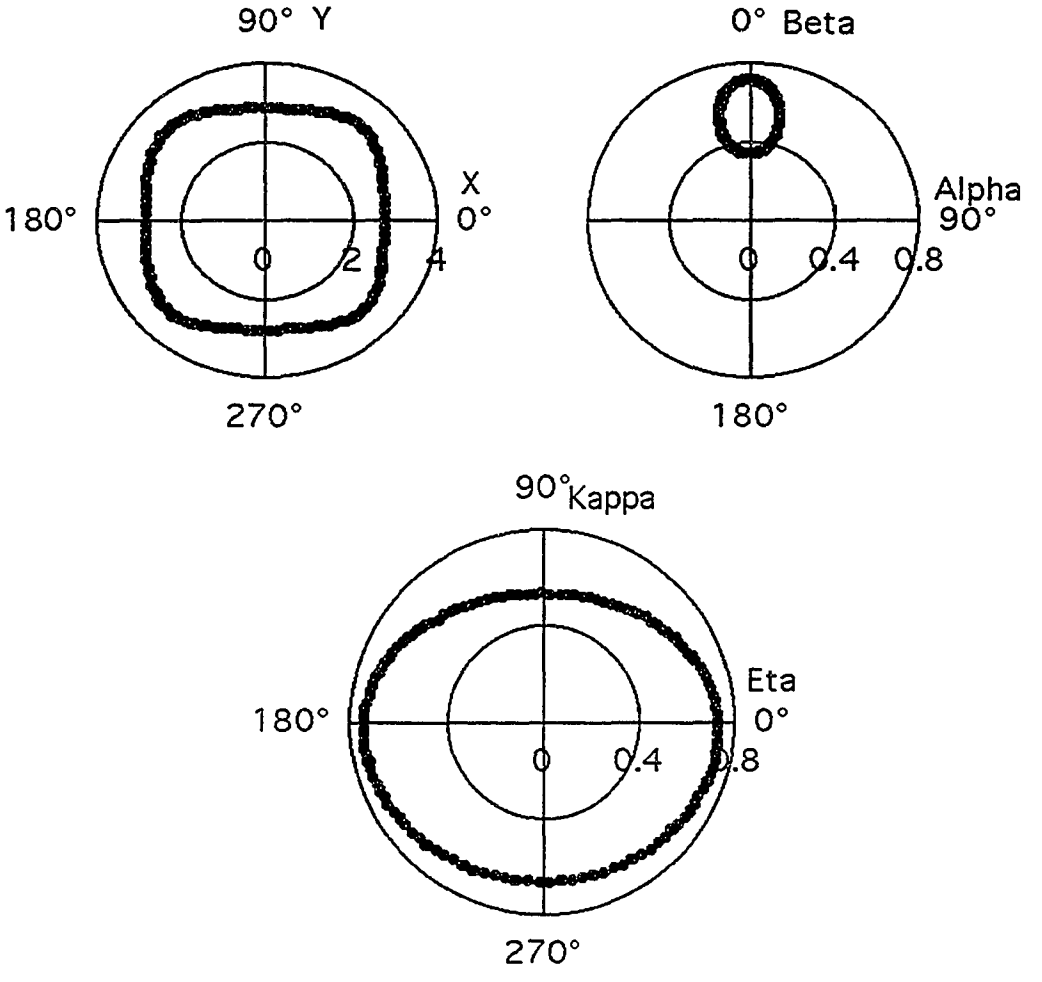


Figure 9-1: An eigenvalue associated with an eigenvalue for the nonlinear pendulum. The graphs at top come directly from Chapters 7 and 8, for a pendulum amplitude of 2.8 radians. The figure at bottom, representing the eigenvalue associated with α and β, is drawn from equations (9.68) and (9.69).

with the differential equations for α and β. Eta and kappa comprise the eigenvalues of the eigenvalues.

In Chapter 8, a particularly useful eigenvalue $\lambda = \alpha + i\beta$ was identified. It is depicted in Figure 8-2 for an amplitude of 2.8 radians. Portions of this figure are reproduced in Figure 9-1 along with a plot of the associated eigenvalue $\eta + i\kappa$. The η,κ plot is intimately related to the α,β plot. At any instant η identifies the rate of change of the length of the α,β state vector, and κ identifies the rate of rotation of the α,β state vector.

Possible uses for the secondary eigenvalues have not been explored, but it is possible that in some situations it may be difficult to model the primary eigenvalues. In such a case, it may be possible to model a secondary eigenvalue, then integrate the real and imaginary portions of it, and then find an approximate solution for the primary eigenvalue by use of appropriately adapted versions of equations (5.41) and (5.42).

Observations

The n-th order formulation allows the selection of exactly enough initial conditions to arbitrarily specify the motion of both the x and y systems. Equations (2.31) and (2.32) require one initial condition each, and equations

(9.47) and (9.48) require [n-1] initial conditions each, for a total of 2n initial conditions. That is exactly the number required to fully and independently specify the initial conditions for equations (9.40) and (9.41). The same number of initial conditions are required in the expanded equation set. As in the second order case, it may be advantageous to specify the initial conditions for the x and y systems, and then work backward to identify the requisite initial conditions for the α and β equations.

This formulation allows the behavior of any n-th order system to be visualized in a four dimensional space (x, y, α, and β). Practically, as was done in the preceding chapters, behavior can be visualized in a two-dimensional x,y phase space, with an accompanying two-dimensional α,β plot. The fact that $\alpha$and β are defined by higher order nonlinear differential equations is not of great concern, since they will often be evaluated numerically.

Other solutions exist that will satisfy the conditions set forth in the n-th order derivation above, and in fact other conditions could be chosen as well. None of these have been explored, even though they may in some cases prove useful. Other solutions, however, are unlikely to lead to the simple physical interpretation for the variables α and β found here, and may not lead to clear parallels with linear algebra as the present formulation does.

CHAPTER 10
CONCLUSION

The dual state variable formulation originally arose out of an attempt to develop an approach to stability analysis without reliance on linear algebra. The symmetry it involved, and the particularly simple polar state variable equations that resulted, sustained interest in the approach early on. It has led to the replication of some key results of linear algebra, but by other means, and these results are applicable even to nonlinear problems. The most important result may be the systematic definition of eigenvalues associated with any second order ordinary differential equation. The formulation appears to be a completely new approach for dealing with ordinary differential equations in state variable form. No reference to such a formulation, or to the results it leads to, has been found in the literature.

One of the fundamental ideas behind the dual state variable formulation is that both state space axes, or state variables, should represent the same type of variable. That is, they should both represent the displacement (for example) of some physical system. It is often convenient to think of the state variables x and y as two different, but complementary, solutions to the same differential equation, or, alternatively, as solutions to two identical

physical systems, whose initial conditions are not generally the same. Some less important conditions are applied to complete the formulation. The results have been rewarding.

This dissertation demonstrates that eigenvalues can be systematically defined for any ordinary differential equation. These eigenvalues exist generally as curves in four dimensional space - that is, they can be represented in the usual two dimensional space with a real and imaginary axis, and in the general nonlinear case, they are functions of the state space coordinates x and y as well. Furthermore, every solution curve in the state space (the x,y space) has associated with it a unique pair of eigenvalues. The motion of only one state variable, say x, is usually all that is important, and the motion of the other variable y can then be chosen at will. This flexibility means that there are many solution curves (in x, y, α, and β) associated with a given set of initial conditions for x and x'. Therefore, there are many different pairs of eigenvalues associated with a given set of initial conditions for x and x'.

Furthermore, since the eigenvalue components are themselves defined by differential equations, the dual state variable formulation can be iteratively applied to them as well. By this method it has been shown that eigenvalues have associated with them eigenvalues of

their own. By successive applications of the formulation, these nested eigenvalues are identified, and the n-th order ODE initially under study is reduced to a set of 2n first order ODE's, and two additional algebraic relations.

It has been shown, for some nonlinear problems at least, that certain relatively simple eigenvalues can be found from among the many possibilities associated with any given set of initial conditions for x and x'. A simple eigenvalue is, for practical purposes, one that can be effectively modeled. This dissertation has shown how to devise such a model, and then use it to generate an approximate solution to a nonlinear differential equation. The example given demonstrates that the results so obtained will be far better than the results to be expected by working with the linearized equation and linear algebra.

Dr. Joel Weiner of the Department of Mathematics at the University of Hawaii has made an interesting observation about the relationship of the dual state variable formulation to Lie groups and Lie algebras. Lie algebras are presently used to categorize differential equations; the analyst working with an ODE looks for a Lie algebra it can be associated with, and thereby gains some understanding of the equation. The A matrix used in this formulation is a Lie algebra. The dual state variable formulation apparently imposes a Lie algebra on a given

ODE. Dr. Weiner suggests that it may be possible to impose other Lie algebras on ODE's for other purposes, and thereby gain some advantage. It may well be useful to view this formulation from the standpoint of Lie algebras, but because that is one of the more esoteric branches of mathematics, as far as engineers are concerned, it has been avoided in this dissertation.

The implications and uses of the dual state variable formulation remain largely unexplored. Since the dual y system and its initial conditions can often be chosen at will, the formulation allows considerable flexibility; it is sometimes possible to select the y system to yield certain path geometries for the eigenvalues, or to impose other restrictions on the formulation. It may be possible to find methods for setting up and modeling the eigenvalues of nonoscillatory systems, and thereby develop approximate solutions for them. The formulation also appears to have potential for use in stability analysis and control system design. Finally, the careful interpretation of this formulation as it relates to the concepts of linear algebra is also worthy of attention. As Committee Member Dr. George Wilkens once said, "When working with nonlinear equations, one desperately wants to hold onto everything one knows about linear equations." The approach developed here may represent a foot in that door.

APPENDIX
COMPUTER USAGE and NUMERICAL METHODS

Computers are obviously essential for this kind of analysis, but relatively simple hardware and software sufficed for the work presented here. All work was done using a Macintosh SE. It was accelerated to 25 Mhz, and equipped with a math coprocessor, 4 Mb of RAM, and a 50 Mb hard drive. Numerical routines were written in Basic, and most of the analysis was done on the computer using software for symbolic mathematics.

All numerical integration was carried out using a fourth order Runge-Kutta routine written by the author. Numerical integration was used extensively to provide insight into behavior. Some of the results are in fact based on models derived from numerical investigation, but in all cases the final results can and have been checked independently. An example is the derivation of the solutions for alpha and beta in the linear case, presented in the last section of Chapter 3. The geometry of the alpha, beta trajectory, obtained numerically, is essential for solving the equations. However, once the solutions were found, they were verified by substitution in the differential equations for alpha and beta.

The numerical results presented in Figures 8-8 and 8-9 could not be confirmed with non-numerical methods, so they were verified by comparison with results from the commercially available software package known as Mathematica. The appropriate equations were integrated using the author's Runge-Kutta program and again using Mathematica; the results are compared in Table A-1, for two arbitrarily selected points in time.

Mathematica uses a computational approach known as the Risch algorithm, which Wolfram Research, the publisher, has apparently modified. Results from Mathematica were obtained with all normal default settings in place. In particular, the default settings allow for six digit accuracy. The results from the Runge-Kutta routine were obtained with a constant step size of 0.0005 seconds. The differences shown in the table are not surprising, and are inconsequential with regard to the data presented in the figures.

In addition to a numerical integration program, a symbolic mathematics software package was essential. Some of the symbolic computation required for this work would be extremely tedious if done by hand, and prone to error. Certainly, the use of such software cut the time required for symbolic analysis radically.

Table A-1: Comparison of numerical results obtained via the author's Runge-Kutta integration routine and from Mathematica. Alpha and beta correspond to points on the curves labeled "actual alpha" and "actual beta" in Figure 8-8, and x corresponds to the curve labeled "True Solution" in Figure 8-9.

t=0.7 sec	x(t)	$\alpha(t)$	$\beta(t)$
Runge-Kutta	1.23086	0.114762	0.581237
Mathematica	1.23086	0.114762	0.581237
Percent Difference	0	0	0
t=1.6 sec			
Runge-Kutta	2.19410	-0.0727739	0.507053
Mathematica	2.19411	-0.0727699	0.507049
Percent Difference	0.00045	0.0055	0.00079

REFERENCES

1. Milton Abramowitz and Irene Stegun, Handbook of Mathematical Functions (New York: Dover Publications, 1965).

2. Sergio Bittanti, Alan Laub, and Jan Willems,*The Riccati Equation* (Berlin: Springer-Verlag, 1991).

3. Einar Hille, *Ordinary Differential Equations in the Complex Domain* (New York: Wiley, 1976).

4. Morris Hirsch and Stephen Smale, *Differential Equations, Dynamical Systems, and Linear Algebra* (New York: Academic Press, 1974).

5. Witold Hurewicz, *Lectures on Ordinary Differential Equations* (New York: Dover, 1990).

6. S. J. Liao, A second-order approximate analytical solution of a simple pendulum by the process analysis method, in the ASME Journal of Applied Mechanics, vol. 59 (December 1992), 970-975.

7. Jan Tuma, *Engineering Mathematics Handbook* , 3d ed. (New York: McGraw-Hill, 1970)