EXTINCTION AND STABILITY OF BURNER-STABILIZED DIFFUSION FLAMES

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ABSTRACT

A steady state solution was derived and a stability analysis was carried out for a diffusion flame produced by a reactant issuing from a spherical porous burner into a second quiescent reactant. A one-step irreversible reaction using Arrhenius kinetics, an optically thin model for radiation, high reaction and radiation activation energies and nonunity Lewis numbers were used. A small perturbation due to wrinkling produced the trivial solution. The steady state solution showed flame existence and these results suggest that the flame is absolutely stable. Numerical results were produced for steady state burning of ethylene in air. Four different flames with the same stoichiometry and adiabatic flame temperature, varying in flame structure and convection direction were analyzed. At low flow rates, kinetic extinction due to reactant leakage was observed. Increasing radiative heat losses promoted kinetic extinction. Increased flow rates resulted in increased residence times as well as increased radiative heat losses. At high flow rates, radiative heat loss dominated residence time and radiative extinction was observed. A parametric study on the Lewis number (*Le*) was performed. Increases in *Le* of the burner issuing reactant and decreases in *Le* of the quiescent reactant promoted flame extinction. Flame extinction was more sensitive to quiescent reactant *Le* variations than burner issuing reactant *Le* variations.

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NOMENCLATURE

$a_{i,j}$	jth order expansion of the integration constants of variable i in the outer regions
B_K	pre-exponential factor of the combustion reaction
B_R	effective pre-exponential factor of the radiative heat loss
c_i	integration constants
$c_{p,g}$	specific heat of the gas at constant pressure $(J/(g\cdot K))$
c_s	specific heat of the solid material used to build the porous burner $(J/(g\cdot K))$
D_i	mass diffusion coefficient of species i (cm^2/s)
Da_K	Damköhler number of the chemical reaction $(no\ units)$
Da_R	equivalent Damköhler number for the radiative heat loss $(no\ units)$
E_K	activation temperature of the chemical reaction (K)
E_R	effective activation temperature of the radiative heat loss (K)
g_0^-	parameter defined as $(\tilde{T}_{f,s}-\tilde{T}_0)(\tilde{m}/\tilde{r}_{f,s}^2)$
g_0^+	parameter defined as $(\tilde{T}_{f,S}-\tilde{T}_\infty)(\tilde{m}/\tilde{r}_{f,S}^2)/[\exp(-\tilde{m}/\tilde{r}_{f,S})-1]$
Le_i	species i Lewis number defined as $(\lambda_g/c_{p,g})/(\rho_g D_i)$ $(no\ units)$ ratio of thermal diffusivity to mass diffusivity
m	mass flow rate of the species issuing from the burner (g/s)
m_F	fuel consumption rate (g/s)
O	oxidizer
P	combustion products
q_1	heat of combustion per unit mass of the reactant supplied from the burner (J/g)
q_F	heat of combustion per unit mass of the fuel (J/g)
q_{RXN}	energy introduced into the flame by combustion (J/g)
q_{RAD}	energy leaving the flame by radiation (J/g)
r	spatial coordinate in the radial direction (cm)
r_i	inner radius of the porous burner (cm)
r_b	outer radius of the porous burner (cm)
r_f	flame radius (cm)

$r_{f,S}$	steady state flame radius (cm)
t	time (s)
T	temperature (K)
T_0	temperature of the supplied gas at the center of the burner (K)
T_b	temperature at the burner exit (K)
T_f	flame temperature (K)
T_i	temperature at the inner surface of the burner (K)
T_∞	ambient temperature (K)
u_r	radial flow velocity; u (cm/s)
u_θ	flow velocity in the θ direction (cm/s)
u_ψ	flow velocity in the ψ direction (cm/s)
u	radial flow velocity (cm/s)
u_f	radial flow velocity at the flame (cm/s)
V_f	flame volume
ν_i	species *i* stoichiometry coefficient; number of moles of species *i* consumed per mole of reactions
W_i	species *i* molecular weight; unit mass of 1 mole of species *i* (g/mol)
Y_i	species *i* mass fraction; ratio of molecular mass of species *i* consumed to total molecular mass of species *i* $(no\ units)$
$Y_{1,0}$	supplied value of Y_1 at the center of the burner $(no\ units)$
$Y_{2,\infty}$	value of Y_2 at $r=\infty$ $(no\ units)$
$Y_{1,L}$	first order of species 1 reactant leakage past the flame region $(no\ units)$
$\tilde{z}$	coordinate transformation from θ to z for simplified mathematics; $z=\tilde{z}=\cos\theta$ $(no\ units)$

Greek Symbols

α	parameter defined as $\varphi+\{[(\rho_s c_s)/(\rho_g c_{p,g})](1-\varphi)\}$;
δ	order of the radiation region; small expansion parameter defined as $\tilde{T}_{f,S}^2/\tilde{E}_R$ $(no\ units)$
$\bar{\delta}$	order of the disturbance amplitude $(no\ units)$
ε	order of the flame region; small expansion parameter defined as $\tilde{T}_{f,S}^2/\tilde{E}_K$ $(no\ units)$
$\phi_{i,j}$	jth order expansions of the nondim. mass fraction of reactant *i* in the rxn region $(no\ units)$

θ_j expansions of the nondimensionalized temperature function in the reaction region $(no\ units)$

Θ_j expansions of the nondimensionalized temperature function in the radiation regions $(no\ units)$

ψ spatial coordinate along the angular direction

φ porosity (void space/total space) of the porous burner $(no\ units)$

κ Planck mean absorption coefficient $\kappa = x_{CO_2}\kappa_{CO_2} + x_{H_2O}\kappa_{H_2O}$ (cm^{-1}); $\tilde{\kappa} = \kappa / r_b$ $(no\ units)$

x_{CO_2} molar fraction of CO_2

x_{H_2O} molar fraction of H_2O

σ Stefan-Boltzmann constant; $\sigma = 1.36x10^{-8} cal/(m^2 \cdot \sec \cdot K^4)$

Λ_K reduced Damköhler number for the reaction $(no\ units)$

Λ_R equivalent reduced Damköhler number for the radiative heat loss $(no\ units)$

λ_g thermal conductivity of the gas $(W/(cm \cdot K))$

λ_s thermal conductivity of the solid material used to build the porous burner $(W/(cm \cdot K))$

λ combined thermal conductivity defined as $\lambda_g\varphi + \lambda_s(1-\varphi)$; $(W/(cm \cdot K))$

$\tilde{\lambda}$ nondimensionalized combined thermal conductivity defined as $\varphi + (\lambda_s/\lambda_g)(1-\varphi)$ $(no\ units)$

ν_i stoichiometric coefficient of species i $(no\ units)$

ρ_g gas density (g/cm^3)

ρ_s density of the solid material used to build the porous burner

ω frequency of the small disturbance

ξ stretched spatial variable in the reaction region defined as $(\tilde{r} - \tilde{r}_{f,s})/\varepsilon$ $(no\ units)$

ζ stretched spatial variable in the radiation regions defined as $(\tilde{r} - \tilde{r}_{f,s})/\delta$ $(no\ units)$

Subscripts

F fuel

g gas properties

O oxidizer

P combustion products

0 specified value at the center of the burner

1 reactant supplied from the burner

2 quiescent reactant

b	value of variables at the burner exit
f	value of variables at the flame sheet
S	steady state value
i	quantities at the inner radius of the burner
∞	location of the ambient

Superscripts

$\sim$	nondimensional quantities
-	solutions in the region between the burner exit and the flame sheet
+	solutions in the region outside of the flame sheet
–	rescaled variable

CHAPTER I. INTRODUCTION

Experiments and studies of flame behavior have been ongoing for millennia and continue to evolve in relevant and important fields of modern science. One of the major objectives in the study of flames is to obtain improved control of combustion processes. Some of the desired results include improvements in fuel economy, the mitigation of undesired flame extinction such as the power loss in gas turbine engines and industrial furnaces, improved fire safety in both normal gravity and microgravity (outer space) environments, pollution reduction leading to a cleaner Earth and improved health, and more controlled processes for the production of carbon black.

Conventionally, flames are classified as premixed flames or diffusion flames. In premixed flames, the oxidizer and fuel are already mixed together before coming into contact with the flame front. In diffusion flames, the oxidizer combines with the fuel by diffusion at or near the flame front. Combustion can only take place where the fuel meets the oxygen, and burning rate is limited by the rate of mass diffusion. Premixed flames are therefore more explosion prone than diffusion flames. In this study, the focus is on diffusion flames.

The traditional paraffin candle is a classic example of a diffusion flame. The basic mechanism of a candle's flame is as follows. An external heat source is applied to the wax near the candle's wick, causing it to melt. Liquid wax is drawn up the wick by capillary action, further heated and vaporized. The mixing of the heated paraffin vapor with surrounding oxygen results in combustion, emitting heat needed to sustain the process. In normal gravity environments, the heated vapors naturally convect upward due to the buoyant force caused by the effects of gravity, yielding an elongated, vertical flame. Incandescence (emission of electromagnetic radiation, e.g., photons) of soot precursors in the flame is the cause of the yellow glow. In microgravity conditions, natural convection no longer occurs and the flame becomes spherical and more efficient. As a result of the improved fuel efficiency, fewer or no soot precursors are formed and the flame tends to be blue in color from the ionizing of gas molecules in the flame.

Diffusion flame configurations commonly studied include the jet diffusion flame, the counterflow diffusion flame, liquid fuel droplets and most recently, the burner-generated spherical diffusion flame, which is the topic of this study. Convection direction, flame stoichiometry, flammability limits (flame ignition/extinction), and parameters such as the kinetic Damköhler number, the radiative Damköhler number, and the reactants' Lewis numbers are all major components of flame behavior that influence incomplete combustion, unsteady burning and soot production in diffusion flames.

The aforementioned Damköhler number is named after Gerhard Damköhler, who, in 1947, published results from experiments with both laminar and turbulent Bunsen flames [1]. This publication laid the foundation for work on combustion in open-flow systems [5]2]. Following his work, Fendell performed the first theoretical work on ignition and extinction of diffusion flames using perturbation methods applied to axisymmetric stagnation-point flow [3]. He adopted the Damköhler number (Da), as a critical parameter in flame extinction that relates the rate of heat transfer out of the reaction region by diffusion or mass transport into the reaction region by chemical reaction in the form of a ratio.

Kinetic Extinction

In 1974, Liñán first showed analytically that flame extinction occurs at a minimum Damköhler number [4]. Counterflow diffusion flame geometry was used without the inclusion of radiative effects and asymptotic theory was employed via singular perturbation expansions. Da was defined as a measure of the ratio of the characteristic chemical reaction time to the characteristic heat diffusion time, where reaction time is a measure of the rate of heat generated by chemical reaction between the two reactants and diffusion time is a measure of the rate of heat transfer out of the flame region by conduction. Liñán's definition of Da is adopted in this study and specified as the kinetic Damköhler number Da_K. The radiative Damköhler number, Da_R, is discussed below. Liñán showed that flame extinction occurs when there is a drop in flame temperature or excessive leakage of reactants past the reaction region before combustion occurs. Reactant leakage is directly proportional to the residence time in the flame region. Therefore, flame extinction results from a low reaction rate due to low residence time in the flame region, and occurs at a minimum Da_K.

The year following Liñán's publication, Law performed a similar asymptotic analysis studying flame ignition and extinction of a liquid fuel droplet [5]. In this work, Law showed that appropriate transformations of the structure equation yield the same form used in Liñán's study of counterflow diffusion flames. In 1976, Krishnamurthy et al. performed an asymptotic analysis of diffusion-flame ignition and extinction in the stagnation point boundary layer of vaporizing fuel rods [6]. Experiments were conducted and successfully demonstrated the predicted extinction limits qualitatively. In 1983, Chung collaborated with Law, continuing work in asymptotic theory using counterflow diffusion flames. Their work distinguished between heat transfer by molecular diffusion and heat transfer by conduction with the inclusion of the Lewis number and showed that Lewis number variations have a direct impact on variations in the flame extinction limits [7]. The Lewis number represents the ratio of thermal diffusivity to mass diffusivity. None of these studies [4-7] included the effects of radiation.

Radiative Effects on Kinetic Extinction

Bonne (1971) was the first to experiment with the effects of radiation on ignition and extinction of diffusion flames, predating Liñán's seminal analytical study neglecting radiative effects [8]. Bonne used laboratory flat diffusion flames to model small diffusion flames in zero-gravity environments. Empirically, radiation loss was shown to reduce flame temperature and promote kinetic extinction at higher minimum Da_K than was found when neglecting radiative effects. More than a decade later in 1982, Sohrab et al. included radiative effects in an analytical study on counterflow diffusion flames using asymptotic theory in which an effective radiative Damköhler number Da_R was proposed that relates the heat transfer out of the reaction region by radiation to the heat transfer into the reaction region by chemical reaction in the form of a ratio [9]. However, the existence of radiative extinction was not explicitly observed. In 1986, T'ien included radiative effects in a numerical study on diffusion flame extinction at the stagnation point of a condensed fuel [10]. His numerical results showed that there are lower and upper limits of Da_K for a fixed radiation intensity beyond which extinction occurs. This study was the first to present the possibility of another extinction limit in addition to the extinction limit at minimum Da_K, but additional analytical study was needed to validate the numerical results.

Radiative Extinction

In 1990, Chao et al. conclusively showed the existence of both lower and upper extinction limits in an analytical study on fuel droplets [11]. The terms "kinetic extinction limit" and "radiative extinction limit" were introduced to distinguish between the two types of flame extinctions occurring at minimum Da_K and maximum Da_R, respectively. They are described as follows. When reactants are present, flame extinction is always the result of low flame temperature. However, there are differing mechanisms that cause the flame temperature to drop at the lower and upper extinction limits and these mechanisms differentiate the two types of flame extinction. In kinetic extinction, a reduction in the reaction rates is the primary mechanism that causes the drop in flame temperature. In radiative extinction, the high radiation levels lower the flame temperature, causing the reaction rates to drop, which additionally lowers the flame temperature. Also, as Bonne initially showed experimentally [8], Chao's work analytically verified that small radiative losses (low Da_R) promote kinetic extinction at low mass flow rates. In other words, kinetic extinction corresponds to a higher minimum Da_K when radiative effects are included at low mass flow rates. However, radiation is a volumetric phenomenon directly proportional to the flame radius cubed. Therefore, at high mass flow rates, radiation levels undergo a significant increase and become the primary mechanism for heat loss from the flame.

At the turn of the 21st century, a series of analytical and experimental studies followed that further confirmed the existence of a radiative extinction limit in addition to the kinetic extinction limit. Analytical studies using activation energy asymptotics were performed by Chao and Law on a hot solid fuel surface [12], by Liu et al. on counterflow diffusion flames [13], and by Zhang et al. on spherical diffusion flames around liquid fuel droplets [14]. Microgravity experiments using liquid fuel droplets were performed by Zhang et al. in both Japan's MGLAB 4.5-s drop tower [14] and NASA's Glenn Research Center 2.2-s drop tower, by Nayagam et al. aboard the Space Shuttle Columbia [15], and by Dietrich et al. in the International Space Station [16]. Additionally, Maruta et al. performed numerical and experimental studies on extinction caused by radiative heat transfer for counterflow methane/air flames in microgravity [17]. All studies [12-17] supported the existence of radiative extinction.

Motivated by the inherent transient behavior of the liquid fuel droplet burning as well as the fact that radiative extinction can only be demonstrated for a single flame by increasing flame size, the microgravity spherical porous burner was introduced as a new model for study. In this model, one of the reactants is fed through a porous burner into the second, quiescent reactant surrounding the burner. By adopting this model, not only is steady state achievable, but the flame size can be enlarged by raising the mass flow rate of the reactant issuing from the burner. Additionally, the flow direction and stoichiometric mixture fractions of the reactants can be controlled. In 1994, Patnaik performed numerical simulations using spherical burner geometry that clarified the importance of gravity and heat losses to the burner [18] and Santa et al. performed a numerical study and tested their results by performing experiments at NASA's 2.2-s drop tower, observing radiative extinction during transient flame conditions [19]. Shortly after, Mills and Matalon applied asymptotic theory to the microgravity spherical burner model, including nonunity Lewis numbers in the analysis defined as the ratio of thermal diffusivity to mass diffusivity, for simultaneous diffusion of heat and mass [20]. Multiple additional experimental studies using the spherical porous burner model have since been performed in NASA's 2.2-s drop tower that further addressed the influence of hydrodynamics, flame structure and soot behavior on flame extinction [21-24]. For example, using four different stoichiometric combinations of ethylene and air, Sunderland et al. showed experimentally that the convection direction of soot precursors (hydrodynamic effect) was found to have a smaller impact on soot inception than flame structure (changes in stoichiometry) [22]. In 2011, Wang and Chao performed an analytic study on burner-stabilized diffusion flames including radiative effects and the radiative Damköhler number Da_R , as well as varying hydrodynamics and flame structures, showing the existence of kinetic and radiative extinction limits in steady state conditions for various flame structures and flow directions [25].

CHAPTER II. FORMULATION

The purpose of this study is to verify and extend the work of Wang and Chao [25] by performing a stability analysis, whereby a small periodic disturbance is introduced into the steady state problem in order to determine its effect on flame stability. Additionally, nonunity reactant Lewis numbers are included in the analysis to allow an examination of the distinct roles of heat transfer by thermal diffusivity and by mass diffusivity in flame extinction.

A spherical diffusion flame is modeled by reactant flow issuing from a spherical porous burner into a second, quiescent reactant in a microgravity environment. The gas issuing from the burner is species 1 and the gas in the ambient is species 2. The flow is modeled as uniform in all directions. The darkened region in Figure 1 represents the porous burner. Gaseous reactant is injected into the void core region and passed through the burner into what is classified as the external (or gas) region. An infinitely thin flame sheet is shown as a circle around the burner at the flame standoff location, r_f. The reaction (or flame) region is a very thin region of order ε, surrounding the flame sheet where the chemical reaction occurs. On either side of the reaction region is the radiation region, which is also very small, of order δ, but much larger than the reaction region. Although the radiation region is distinct from the reaction region, radiation is accounted for in both regions. In the external region beyond the radiation region, both the combustion reaction and radiation are neglected because they are insignificant as a result of low temperature. This is specified as the outer region, which is order 1, with $\varepsilon << \delta << 1$.

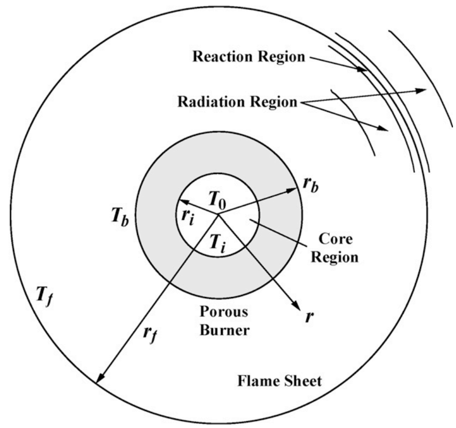


Figure 1: 2-D visualization of 3-D flow model

Energy conservation and mass conservation are the two fundamental laws used in the development of an analytical model for this problem.

Conservation of Energy

Changes in the temperature profile with respect to time and space result from heat transfer into or out of the system by convection, conduction (heat diffusion), radiation, and chemical reaction. This is modeled using the energy equation with spherical coordinates (r,θ,ψ) as shown in Figure 2. Without loss of generality, a two-dimensional perturbation is considered with disturbance imposed along (r,θ) for simplicity in the analysis. Therefore, even in the perturbed state, $\partial / \partial\psi = u_\psi = 0$. Additionally, the disturbance is too small to affect bulk flow velocity such that $u_\theta = 0$. Pressure is assumed to be constant with respect to both time and position and there is no frictional heating because of the low flow velocity as compared to the speed of sound. Under these conditions, the energy equation simplifies to

$$\rho c_p \frac{\partial T}{\partial t} + \rho c_p u_r \frac{\partial T}{\partial r} - \left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(kr^2 \frac{\partial T}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(k\sin\theta \frac{\partial T}{\partial\theta}\right)\right] = \rho\left(\frac{Dq_{RXN}}{Dt} + \frac{Dq_{RAD}}{Dt}\right); \quad (Watts/m^3). \qquad (1)$$

Since the flow velocity is only in the radial direction, subscript r is dropped from u_r. The material derivatives representing the chemical reaction (RXN) and radiation (RAD) terms are replaced with models developed as follows.

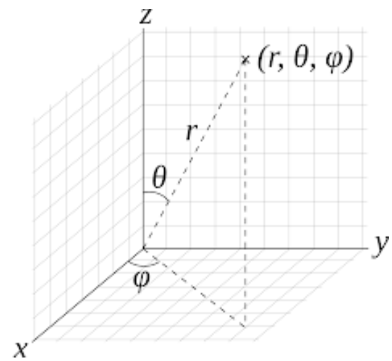


Figure 2: Spherical coordinates

The rate of energy introduced into the system by chemical reaction per unit volume, $\rho(Dq_{RXN}/Dt)$, is modeled using the Law of Mass Action for a one-step, irreversible, pseudo first order reaction following Arrhenius kinetics. That is, $d[A]/dt = -R[A][B]; \quad R = B_o \exp(-E_K/T)$, where $[A]$ and $[B]$ represent the molar concentrations of the two reactant species, R represents the temperature-dependent proportionality relationship between $[A]$ and $[B]$, B_o is a measure of the collision frequency $(m^3/(mol \cdot s))$ and E_K is the chemical activation temperature for the reaction, representing the temperature barrier for the combustion reaction to be significant. It is preferable to convert molar concentrations $[A]$ and $[B]$ into their respective species' mass fractions for the subsequent mathematical analysis as mass is conserved. The mass fraction of species i (1 or 2) is a nondimensional quantity defined as the ratio of the molecular mass of species i to the total molecular mass of the gas. After converting the molar concentrations $[A]$ and $[B]$ to mass fractions Y_1 and Y_2, substituting the temperature-dependent expression for R into the rate equation and absorbing constants into B_o which is therefore renamed as $B_o^{'}$ $(mol/(m^3 \cdot s))$, the reaction rate may be written as $d[A]/dt = -B_o^{'} Y_1 Y_2 \exp(-E_K/T)$. Multiplication by the stoichiometry coefficient v_1 and molecular weight W_1 of species 1, and the energy released q_1 from combustion of 1 kg of reactant 1 yields a final reaction rate in $Watts/m^3$ of $-B_K Y_1 Y_2 \exp(-E_K/T) v_1 W_1 q_1 \rho_g^2; \; B_K = B_o^{'}/\rho_g^2$, which is used in place of $\rho(Dq_{RXN}/Dt)$ in equation 1.

The rate of energy leaving the system by radiation near the flame per unit volume, $\rho(Dq_{RAD}/Dt)$, is modeled as optically thin such that none of the energy emitted by radiation is reabsorbed into the system. The heat loss rate due to radiation is $-4\sigma\kappa(T^4 - T_\infty^4)$. Only CO_2 and H_2O are considered to emit radiative energy as accounted for in the Planck mean absorption coefficient, κ. Radiation levels are highest in the flame region and decline exponentially with distance from the flame. Therefore, $(T^4 - T_\infty^4)$ is approximated by $B_R \exp(-E_R/T)$, where B_R is the effective pre-exponential factor of the radiative heat loss (units K^4) and E_R is the radiation activation temperature, i.e. the temperature barrier for radiation to be significant. This substitution yields $E_R \approx$ 800K for temperatures between 1600K and 2500K. The resulting rate of radiative heat loss in $Watts/m^3$ is $-4\sigma\kappa B_R \exp(-E_R/T)$, which is used in place of $\rho(Dq_{RAD}/Dt)$ in equation 1.

The chemical activation temperature E_K is much higher than flame temperatures for typical hydrocarbon or carbohydrate fuels so that chemical reaction only occurs in the flame region, where the temperature is highest. Radiation levels are largest in the flame region and decay exponentially with distance from the flame as temperature decreases. Therefore, the equivalent activation temperature for radiation E_R is modeled as high, but much smaller than the chemical activation temperature, so that radiation is accounted for in the flame region as well as the surrounding region, but is negligible further out in the outer region, as was shown previously in Figure 1. The relationships $\varepsilon = \tilde{T}_{f,S}^2/\tilde{E}_K$ and $\delta = \tilde{T}_{f,S}^2/\tilde{E}_R$ are derived later in the analysis with $\varepsilon << \delta << 1$, where the tildes indicate nondimensionalized terms and $\tilde{T}_{f,S}$ represents the nondimensionalized steady state flame temperature.

In the core $(0 < r < r_i)$, energy transport is by convection and conduction, so the energy equation is:

Core Region: $$\rho_g c_{p,g}\frac{\partial T}{\partial t} + \rho_g c_{p,g} u\frac{\partial T}{\partial r} - \left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(kr^2\frac{\partial T}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(k\sin\theta\frac{\partial T}{\partial\theta}\right)\right] = 0. \qquad (2)$$

In the porous burner $(r_i < r < r_b)$, radiation from the burner is assumed to be negligible and there is no burner cooling mechanism. The solid is in thermal equilibrium with the gas such that they share the same temperature profiles. The total energy transport is therefore the sum of the transport by conduction through the solid burner and by conduction and convection by the gas. The porosity factor (ψ), defined as the ratio of void space to total space, accounts for the different cross sections through which the energy is transported.

$$\varphi\left\{\rho_g c_{p,g}\frac{\partial T}{\partial t} + \rho_g u c_{p,g}\frac{\partial T}{\partial r} - \left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(\lambda_g r^2\frac{\partial T}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\lambda_g\sin\theta\frac{\partial T}{\partial\theta}\right)\right]\right\} = 0 \qquad \text{(gas)}$$

$$(1-\varphi)\left\{\rho_s c_s \frac{\partial T}{\partial t} - \left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(\lambda_s r^2 \frac{\partial T}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\lambda_s \sin\theta \frac{\partial T}{\partial \theta}\right)\right]\right\} = 0 \qquad \text{(solid)}$$

After adding the above two equations, the following energy equation is obtained for the burner region.

Burner Region: $$\left[\rho_g c_{p,g}\varphi + \rho_s c_s (1-\varphi)\right]\frac{\partial T}{\partial t} + \rho_g u c_{p,g}\varphi \frac{\partial T}{\partial r} - \left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(\lambda r^2 \frac{\partial T}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\lambda \sin\theta \frac{\partial T}{\partial \theta}\right)\right] = 0 \qquad (3)$$

In the external region, the energy transport equations for the reaction, radiation and outer regions differ only by inclusion or exclusion of chemical reaction and/or radiation.

Outer Region: $$\rho_g c_{p,g}\left(\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial r}\right) - \left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(\lambda_g r^2 \frac{\partial T}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\lambda_g \sin\theta \frac{\partial T}{\partial \theta}\right)\right] = 0 \qquad (4)$$

Rad. Region: $$\rho_g c_{p,g}\left(\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial r}\right) - \left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(\lambda_g r^2 \frac{\partial T}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\lambda_g \sin\theta \frac{\partial T}{\partial \theta}\right)\right] = -4\sigma\kappa B_R \exp(-E_R / T) \qquad (5)$$

Reaction Region:

$$\rho_g c_{p,g}\left(\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial r}\right) - \left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(\lambda_g r^2 \frac{\partial T}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\lambda_g \sin\theta \frac{\partial T}{\partial \theta}\right)\right] = \nu_1 W_1 q_1 B_K \rho_g^2 Y_1 Y_2 \exp(-E_K / T) - 4\sigma\kappa B_R \exp(-E_R / T) \qquad (6)$$

Conservation of Species

The conservation of mass derivation is similar to that of the conservation of energy. Mass is unaffected by radiation and the chemical reaction term is zero everywhere except in the reaction region. For each of the species in the external region, the mass conservation equations for species $i=1$ and $i=2$ are expressed in mass fractions with units $kg/(m^3 \cdot s)$ as follows:

$$\rho_g \frac{\partial Y_1}{\partial t} + \rho_g u \frac{\partial Y_1}{\partial r} - \left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(\rho_g D_1 r^2 \frac{\partial Y_1}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\rho_g D_1 \sin\theta \frac{\partial Y_1}{\partial \theta}\right)\right] = -\nu_1 W_1 B_K \rho_g^2 Y_1 Y_2 \exp(-E_K / T)$$

$$\rho_g \frac{\partial Y_2}{\partial t} + \rho_g u \frac{\partial Y_2}{\partial r} - \left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(\rho_g D_2 r^2 \frac{\partial Y_2}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\rho_g D_2 \sin\theta \frac{\partial Y_2}{\partial \theta}\right)\right] = -\nu_2 W_2 B_K \rho_g^2 Y_1 Y_2 \exp(-E_K / T)$$

The independent variable transformation from θ to z was applied to the energy and mass conservation equations in all regions for simplified mathematics using $z = \tilde{z} = \cos\theta$. Note that $-1 < z < 1$.

Boundary Conditions

In the center of the core, species 1 is introduced at a given, fixed temperature and mass fraction. Penetration of species 2 into the burner is considered negligible. Far from the burner, temperature and mass fraction of species 2 are considered constant, and the mass fraction of species 1 is negligible.

$$r = 0: \qquad T = T_0, \qquad Y_1 = Y_{1,0}, \qquad Y_2 = 0 \qquad \text{(7a,b,c)}$$

$$r \to \infty: \qquad T \to T_\infty, \qquad Y_2 \to Y_{2,\infty}, \qquad Y_1 \to 0 \qquad \text{(8a,b,c)}$$

At the inner surface of the burner, r_i, a control surface energy balance is performed. The only heat transfer mechanism is conduction since convection is a volumetric phenomenon. By Fourier's Law, $k_{core}(\partial T/\partial r)_{r_i^-} = k_{burner}(\partial T/\partial r)_{r_i^+}$, where $k_{core} = \lambda_g$ and $k_{burner} = \lambda$, the combined thermal conductivity of the solid and gas. A similar control surface energy balance is performed at the outer surface of the burner, r_b. The following boundary conditions result:

$$r = r_i: \qquad \lambda_g(\partial T/\partial r)_{r_i^-} = \lambda(\partial T/\partial r)_{r_i^+}, \qquad Y_1 = Y_{1,0}, \qquad Y_2 = 0 \tag{9a,b,c}$$

$$r = r_b: \qquad \lambda(\partial T/\partial r)_{r_b^-} = \lambda_g(\partial T/\partial r)_{r_b^+} \tag{10}$$

The burner boundary temperatures T_i and T_b remain to be determined.

Additionally, control surface mass balances are performed at the outer surface of the burner for both species 1 and 2. There is no diffusive component on the porous burner side for either species since $Y_{1,0}$ and Y_2 are both constant and diffusion is a consequence of the mass fraction gradient. On the external region side, $\partial Y_1/\partial r$ and $\partial Y_2/\partial r$ are non-zero, so non-zero diffusion components exist for species 1 and 2. They are $4\pi r^2 \rho_g D_1(\partial Y_{1b}/\partial r)$ and $4\pi r^2 \rho_g D_2(\partial Y_{2b}/\partial r)$, respectively. The convection of the species mass fraction may be described using mass flow rates. On the burner side, the mass flow rates of species 1 and 2 are $4\pi r^2 Y_{10}\rho_g u$ and 0, respectively. On the external side, the mass flow rates of species 1 and 2 are $4\pi r^2 Y_{1b}\rho_g u$ and $4\pi r^2 Y_{2b}\rho_g u$, respectively. The following control surface mass balances result:

$$r = r_b: \; Y_{1,0}\rho_g u_{r_b} = Y_{1,b}\rho_g u_{r_b} - \rho_g D_1(\partial Y_{1,b}/\partial r)_{r_b} \tag{11}$$

$$r = r_b: \; 0 = Y_{2,b}\rho_g u_{r_b} - \rho_g D_2(\partial Y_{2,b}/\partial r)_{r_b} \tag{12}$$

Stability Analysis Modeling

The steady state solutions to equations (1) through (12) are represented by $T_S(r)$, $Y_{1,S}(r)$ and $Y_{2,S}(r)$. It is observed experimentally that a flame may undergo periodic pulsations or wrinkling due to a disturbance. The flame pulsation or wrinkling may increase in amplitude over time until the flame extinguishes, decrease in amplitude until all periodic motion ceases, or maintain constant amplitude in the condition of neutral stability, which serves as the boundary between stability and instability. If the disturbance is considered to be along (r,z), the flame standoff location r_f may be described using the steady state flame radius $r_{f,S}$, plus a small perturbation that is periodic with respect to both space and time. Because of the pattern of the flame location, the temperature profile is also perturbed by a function that is periodic with respect to space and time. If the initial disturbance at t = 0 is too big such that $T(r,z,t=0)$ is below $T_{\min}$ required for a chemical reaction, then there is automatic flame extinction and a stability analysis cannot be performed. Therefore, the perturbation is established as order $\bar{\delta} << 1$, with a bar superscript used to differentiate from order δ of the radiation region. The temperature solution to the perturbed problem is written in terms of the unknown function F as follows:

$$T(r,z,t) = T_S(r) + \bar{\delta} F(r,z,t) + O(\bar{\delta}^2).$$

Flame oscillations of order ε are already absorbed into the steady state solution shown in Appendix B, so the order of the disturbance must greater than $O(\varepsilon)$ to be relevant. Therefore, $\varepsilon << \bar{\delta} << 1 \sim$ steady state solution.

Since the perturbation function F is periodic, it may be expanded into a Fourier series with the spatial variation written in the form of unknown $T_n'(r,z)$.

$$F(r,z,t) = \sum_{n=-\infty}^{\infty} c_n T_n'(r,z) \quad ; \quad c_n \text{ is the disturbance amplitude}$$

However, since the disturbance may either grow or decay with time, it is reasonable to include the time dependence in the amplitude c_n as a function that reflects the characteristic of exponential growth or decay; $c_n = \exp(\omega_n t)$, where ω_n is the complex frequency parameter.

$$F(r,z,t) = \sum_{n=-\infty}^{\infty} e^{\omega_n t} T_n'(r,z)$$

In order for the total function F to decay with time, every term in the series must also decay with time. Therefore, an arbitrary term in the series is selected as representative of all of the terms, the subscript n is dropped, and function F is simply expressed as $F(r,z,t) = e^{\omega t} T'(r,z).$ Note that for complex $\omega = \omega_0 + i\omega_I$, by Euler's equation we have $\exp(\omega t) = \exp(\omega_0 t)\exp(\omega_I t) = \exp(\omega_0 t)(\cos\omega_I t + i\sin\omega_I t).$ Since both cos and sin vary only between -1 and 1, the exponential growth required for instability depends only on the real component ω_0. In this work, only the cellular instability is examined. As such, $\omega_I = 0$ is considered, and the subscript 0 is dropped from the real component such that $\omega = \omega_0$. Therefore, the temperature and mass fraction profiles are modeled as

$$T(r,z,t) = T_S(r) + \bar{\delta} e^{\omega t} T'(r,z) + O(\bar{\delta}^2) \tag{13}$$

$$Y_1(r,z,t) = Y_{1,S}(r) + \bar{\delta} e^{\omega t} Y_1'(r,z) + O(\bar{\delta}^2) \tag{14}$$

$$Y_2(r,z,t) = Y_{2,S}(r) + \bar{\delta} e^{\omega t} Y_2'(r,z) + O(\bar{\delta}^2) \tag{15}$$

If the growth factor $\omega > 0$, then $\exp(\omega t)$ will increase in magnitude with time and result in cellular instability, whereas if $\omega < 0$, all periodic motion ceases. Therefore, $\omega = 0$ exactly represents the condition of neutral cellular stability in the condition of periodic wrinkling.

Nondimensionalization

The characteristic length used in parameter nondimensionalization is the burner outer radius, r_b. The mass fraction Y_1 is rescaled using $Y_{1,0}$. The characteristic temperature is $Y_{1,0}q_1 / c_{p,g}$, which is the change in temperature of 1 kg of species 1 that results from the energy released from the consumption of 1kg of reactant 1. The stoichiometric mass fraction of Y_2 that corresponds to $Y_{1,0}$ is $Y_{1,0}(v_2W_2 / v_1W_1)$, where v_2W_2 / v_1W_1 is the ratio of the reactant masses consumed in a stoichiometric reaction. The characteristic mass flow rate is different in the burner than elsewhere in the gas phase due to the burner's porosity. Nondimensionalizations and rescalings are as follows:

$$\tilde{Y}_1 = \frac{Y_1}{Y_{1,0}}, \quad \tilde{Y}_2 = \frac{Y_2}{Y_{1,0}}\frac{v_1W_1}{v_2W_2}, \quad \tilde{T} = \frac{c_{p,g}T}{q_1Y_{1,0}}, \quad \tilde{E}_K = \frac{c_{p,g}E_K}{q_1Y_{1,0}}, \quad \tilde{E}_R = \frac{c_{p,g}E_R}{q_1Y_{1,0}}, \quad \tilde{\lambda} = \varphi + (1-\varphi)\frac{\lambda_s}{\lambda_g},$$

$$\tilde{r} = \frac{r}{r_b}, \quad \tilde{t} = \frac{\lambda_g t}{\rho_g c_{p,g} r_b^2}, \quad \tilde{m} = \frac{c_{p,g}m}{4\pi r_b \lambda_g}; \quad m_{burner} = 4\pi r^2 \rho_g u \varphi, \quad m_{gas\,phase} = 4\pi r^2 \rho_g u$$

The nondimensional kinetic and radiative Damköhler numbers and the species' Lewis numbers emerge in the nondimensionalization of the problem and are defined as

$$Da_K = \frac{v_2W_2B_KY_{1,0}\rho_g^2c_{p,g}r_b^2}{\lambda_g}, \quad Da_R = \frac{4\sigma\kappa B_R r_b^2 c_{p,g}}{\lambda_g q_1 Y_{1,0}}, \quad Le_i = \frac{\lambda_g c_{p,g}}{\rho_g D_i}.$$

The nondimensionalized problem is given on the following page.

Nondimensional Problem

Core Region $(0 < \tilde{r} < \tilde{r}_i)$

$$\tilde{\omega}\tilde{T} + \frac{\tilde{m}}{\tilde{r}^2}\frac{\partial \tilde{T}}{\partial \tilde{r}} - \frac{1}{\tilde{r}^2}\left\{\frac{\partial}{\partial \tilde{r}}\left(\tilde{r}^2\frac{\partial \tilde{T}}{\partial \tilde{r}}\right) + \frac{\partial}{\partial \tilde{z}}\left[(1-\tilde{z}^2)\frac{\partial \tilde{T}}{\partial \tilde{z}}\right]\right\} = 0\,, \qquad \tilde{Y}_1 = 1\,, \qquad \tilde{Y}_2 = 0 \tag{16a,b,c}$$

Burner Region $(\tilde{r}_i < \tilde{r} < 1)$

$$\tilde{\omega}\alpha\tilde{T} + \frac{\tilde{m}}{\tilde{r}^2}\frac{\partial \tilde{T}}{\partial \tilde{r}} - \frac{\tilde{\lambda}}{\tilde{r}^2}\left\{\frac{\partial}{\partial \tilde{r}}\left(\tilde{r}^2\frac{\partial \tilde{T}}{\partial \tilde{r}}\right) + \frac{\partial}{\partial \tilde{z}}\left[(1-\tilde{z}^2)\frac{\partial \tilde{T}}{\partial \tilde{z}}\right]\right\} = 0\,, \qquad \tilde{Y}_1 = 1\,, \qquad \tilde{Y}_2 = 0 \tag{17a,b,c}$$

Outer Region

$$\tilde{\omega}\tilde{T} + \frac{\tilde{m}}{\tilde{r}^2}\frac{\partial \tilde{T}}{\partial \tilde{r}} - \frac{1}{\tilde{r}^2}\left\{\frac{\partial}{\partial \tilde{r}}\left(\tilde{r}^2\frac{\partial \tilde{T}}{\partial \tilde{r}}\right) + \frac{\partial}{\partial \tilde{z}}\left[(1-\tilde{z}^2)\frac{\partial \tilde{T}}{\partial \tilde{z}}\right]\right\} = 0 \tag{18}$$

$$\tilde{\omega}\tilde{Y}_i + \frac{\tilde{m}}{\tilde{r}^2}\frac{\partial \tilde{Y}_i}{\partial \tilde{r}} - \frac{1}{Le_i\tilde{r}^2}\left\{\frac{\partial}{\partial \tilde{r}}\left(\tilde{r}^2\frac{\partial \tilde{Y}_i}{\partial \tilde{r}}\right) + \frac{\partial}{\partial \tilde{z}}\left[(1-\tilde{z}^2)\frac{\partial \tilde{Y}_i}{\partial \tilde{z}}\right]\right\} = 0 \tag{19}$$

Radiation Region

$$\tilde{\omega}\tilde{T} + \frac{\tilde{m}}{\tilde{r}^2}\frac{\partial \tilde{T}}{\partial \tilde{r}} - \frac{1}{\tilde{r}^2}\left\{\frac{\partial}{\partial \tilde{r}}\left(\tilde{r}^2\frac{\partial \tilde{T}}{\partial \tilde{r}}\right) + \frac{\partial}{\partial \tilde{z}}\left[(1-\tilde{z}^2)\frac{\partial \tilde{T}}{\partial \tilde{z}}\right]\right\} = -Da_R\exp(-\tilde{E}_R/\tilde{T}) \tag{20}$$

$$\tilde{\omega}\tilde{Y}_i + \frac{\tilde{m}}{\tilde{r}^2}\frac{\partial \tilde{Y}_i}{\partial \tilde{r}} - \frac{1}{Le_i\tilde{r}^2}\left\{\frac{\partial}{\partial \tilde{r}}\left(\tilde{r}^2\frac{\partial \tilde{Y}_i}{\partial \tilde{r}}\right) + \frac{\partial}{\partial \tilde{z}}\left[(1-\tilde{z}^2)\frac{\partial \tilde{Y}_i}{\partial \tilde{z}}\right]\right\} = 0 \tag{21}$$

Reaction Region

$$\tilde{\omega}\tilde{T} + \frac{\tilde{m}}{\tilde{r}^2}\frac{\partial \tilde{T}}{\partial \tilde{r}} - \frac{1}{\tilde{r}^2}\left\{\frac{\partial}{\partial \tilde{r}}\left(\tilde{r}^2\frac{\partial \tilde{T}}{\partial \tilde{r}}\right) + \frac{\partial}{\partial \tilde{z}}\left[(1-\tilde{z}^2)\frac{\partial \tilde{T}}{\partial \tilde{z}}\right]\right\} = Da_K\tilde{Y}_1\tilde{Y}_2\exp(\tilde{E}_K/\tilde{T}) - Da_R\exp(-\tilde{E}_R/\tilde{T}) \tag{22}$$

$$\tilde{\omega}\tilde{Y}_i + \frac{\tilde{m}}{\tilde{r}^2}\frac{\partial \tilde{Y}_i}{\partial \tilde{r}} - \frac{1}{Le_i\tilde{r}^2}\left\{\frac{\partial}{\partial \tilde{r}}\left(\tilde{r}^2\frac{\partial \tilde{Y}_i}{\partial \tilde{r}}\right) + \frac{\partial}{\partial \tilde{z}}\left[(1-\tilde{z}^2)\frac{\partial \tilde{Y}_i}{\partial \tilde{z}}\right]\right\} = -Da_K\tilde{Y}_1\tilde{Y}_2\exp(\tilde{E}_K/\tilde{T}) \tag{23}$$

Boundary Conditions

$$\tilde{r} = 0: \qquad \tilde{T} = \tilde{T}_0, \qquad \tilde{Y}_1 = 1, \qquad \tilde{Y}_2 = 0 \tag{24a,b,c}$$

$$\tilde{r} = \tilde{r}_i: \qquad \tilde{T} = \tilde{T}_i, \qquad (\partial\tilde{T}/\partial\tilde{r})_{\tilde{r}_i^-} = \tilde{\lambda}(\partial\tilde{T}/\partial\tilde{r})_{\tilde{r}_i^+}, \qquad \tilde{Y}_1 = 1, \qquad \tilde{Y}_2 = 0 \tag{25a,b,c}$$

$$\tilde{r} = 1: \qquad \tilde{T} = \tilde{T}_b, \qquad \tilde{\lambda}(\partial\tilde{T}/\partial\tilde{r})_{1^-} = (\partial\tilde{T}/\partial\tilde{r})_{1^+}, \qquad \tilde{m}\tilde{Y}_1 - \frac{1}{Le_1}\frac{\partial\tilde{Y}_1}{\partial\tilde{r}} = \tilde{m}, \qquad \tilde{m}\tilde{Y}_2 - \frac{1}{Le_2}\frac{\partial\tilde{Y}_2}{\partial\tilde{r}} = 0 \tag{26a,b,c,d}$$

$$\tilde{r} \to \infty: \qquad \tilde{T} \to 0, \qquad \tilde{Y}_1 \to 0, \qquad \tilde{Y}_2 \to 0 \tag{27a,b,c}$$

Perturbed Solutions

$$\tilde{T}(\tilde{r},\tilde{z},\tilde{t}) = \tilde{T}_S(\tilde{r}) + \bar{\delta}e^{\tilde{\omega}\tilde{t}}\tilde{T}'(\tilde{r},\tilde{z}) + O(\bar{\delta}^2) \tag{28}$$

$$\tilde{Y}_1(\tilde{r},\tilde{z},\tilde{t}) = \tilde{Y}_{1,S}(\tilde{r}) + \bar{\delta}e^{\tilde{\omega}\tilde{t}}\tilde{Y}_1'(\tilde{r},\tilde{z}) + O(\bar{\delta}^2) \tag{29}$$

$$\tilde{Y}_2(\tilde{r},\tilde{z},\tilde{t}) = \tilde{Y}_{2,S}(\tilde{r}) + \bar{\delta}e^{\tilde{\omega}\tilde{t}}\tilde{Y}_2'(\tilde{r},\tilde{z}) + O(\bar{\delta}^2) \tag{30}$$

CHAPTER III. SOLUTION

Steady State Solution

Initially, the steady state problem was solved with the inclusion of non-unity Lewis numbers. The derivation uses asymptotic theory and numerical methods similar to that which is used in the stability analysis and may be found in Appendix B. In the stability analysis that follows, the steady state solutions are considered to be known functions.

Separation of Variables

Homogeneous equations 16a, 17a, 18, 19 and 21 were all solved similarly by separation of variables. First, the nondimensional perturbed solutions given in equations 28, 29, and 30 were substituted into the homogeneous equations. Equations 16b,c and 17b,c were also be solved at this time by substituting in the nondimensional perturbed solutions. After rearranging and separating orders, the first order problem emerged as the steady state problem, and the $O(\bar{\delta})$ problem was as follows:

Core Region: $$\tilde{\omega}\tilde{T}' + \frac{\tilde{m}}{\tilde{r}^2}\frac{\partial \tilde{T}'}{\partial \tilde{r}} - \frac{1}{\tilde{r}^2}\left\{\frac{\partial}{\partial \tilde{r}}\left(\tilde{r}^2\frac{\partial \tilde{T}'}{\partial \tilde{r}}\right) + \frac{\partial}{\partial \tilde{z}}\left[(1-\tilde{z}^2)\frac{\partial \tilde{T}'}{\partial \tilde{z}}\right]\right\} = 0, \qquad \tilde{Y}_1' = 0, \qquad \tilde{Y}_2' = 0 \tag{31a,b,c}$$

Burner Region: $$\tilde{\omega}\alpha\tilde{T}' + \frac{\tilde{m}}{\tilde{r}^2}\frac{\partial \tilde{T}'}{\partial \tilde{r}} - \frac{\tilde{\lambda}}{\tilde{r}^2}\left\{\frac{\partial}{\partial \tilde{r}}\left(\tilde{r}^2\frac{\partial \tilde{T}'}{\partial \tilde{r}}\right) + \frac{\partial}{\partial \tilde{z}}\left[(1-\tilde{z}^2)\frac{\partial \tilde{T}'}{\partial \tilde{z}}\right]\right\} = 0, \qquad \tilde{Y}_1' = 0, \qquad \tilde{Y}_2' = 0 \tag{32a,b,c}$$

Outer Region: $$\tilde{\omega}\tilde{T}' + \frac{\tilde{m}}{\tilde{r}^2}\frac{\partial \tilde{T}'}{\partial \tilde{r}} - \frac{1}{\tilde{r}^2}\left\{\frac{\partial}{\partial \tilde{r}}\left(\tilde{r}^2\frac{\partial \tilde{T}'}{\partial \tilde{r}}\right) + \frac{\partial}{\partial \tilde{z}}\left[(1-\tilde{z}^2)\frac{\partial \tilde{T}'}{\partial \tilde{z}}\right]\right\} = 0 \tag{33}$$

Outer & Radiation Regions: $$\tilde{\omega}\tilde{Y}_i' + \frac{\tilde{m}}{\tilde{r}^2}\frac{\partial \tilde{Y}_i'}{\partial \tilde{r}} - \frac{1}{Le_i\tilde{r}^2}\left\{\frac{\partial}{\partial \tilde{r}}\left(\tilde{r}^2\frac{\partial \tilde{Y}_i'}{\partial \tilde{r}}\right) + \frac{\partial}{\partial \tilde{z}}\left[(1-\tilde{z}^2)\frac{\partial \tilde{Y}_i'}{\partial \tilde{z}}\right]\right\} = 0 \tag{34}$$

As a case example for all of the above homogeneous relations, homogeneous equation

$$\tilde{\omega}\tilde{Y}' + \frac{\tilde{m}}{\tilde{r}^2}\frac{\partial \tilde{Y}'}{\partial \tilde{r}} - \frac{1}{p\tilde{r}^2}\left\{\frac{\partial}{\partial \tilde{r}}\left(\tilde{r}^2\frac{\partial \tilde{Y}'}{\partial \tilde{r}}\right) + \frac{\partial}{\partial \tilde{z}}\left[(1-\tilde{z}^2)\frac{\partial \tilde{Y}'}{\partial \tilde{z}}\right]\right\} = 0 \tag{35}$$

was used, where constant p is representative of either Le_i or $1/\tilde{\lambda}$. To solve by separation of variables, a solution was sought in the form $\tilde{Y}'(\tilde{r},\tilde{z}) = \hat{Y}(\tilde{r})A(\tilde{z})$. Note that the first term was eliminated for the condition of neutral cellular stability because $\omega = 0$. The separated equation with arbitrary constant $-c$ is

$$-\frac{1}{\hat{Y}}\frac{d}{d\tilde{r}}\left(\tilde{r}^2\frac{d\hat{Y}}{d\tilde{r}}\right) + \tilde{m}p\frac{1}{\hat{Y}}\frac{d\hat{Y}}{d\tilde{r}} = \frac{1}{A}\frac{d}{d\tilde{z}}\left((1-\tilde{z}^2)\frac{dA}{d\tilde{z}}\right) = -c$$

By selecting the arbitrary constant $-c = -n(n+1)$ for any integer $n \geq 0,$ the separated equation for A became Legendre's differential equation, and solutions for A were found by the power series method to be finite nth Legendre polynomials $P_n(\tilde{z})$. Since all Legendre polynomials $P_n(\tilde{z})$ are linearly independent for $n \geq 0$, the general solution for $\tilde{Y}_1'$ was found to be $\tilde{Y}'(\tilde{r},\tilde{z}) = \sum_{n=0}^{\infty} \hat{Y}_n(\tilde{r}) P_n(\tilde{z})$. Only one arbitrary term in the series was needed as a representative particular solution for $\tilde{Y}'$ such that

$$\tilde{Y}'(\tilde{r},\tilde{z}) = \hat{Y}(\tilde{r}) P_n(\tilde{z}) \,, \tag{36}$$

with subscript n dropped on $\hat{Y}$. After rearranging, the following problem remained to be solved for $\hat{Y}$.

$$\frac{d}{d\tilde{r}}\left(\tilde{r}^2 \frac{d\hat{Y}}{d\tilde{r}}\right) - p\tilde{m}\frac{d\hat{Y}}{d\tilde{r}} - n(n+1)\hat{Y} = 0 \tag{37}$$

The detailed derivation of the solution of (37) is presented in Appendix C. Since (37) is second order, two linearly independent solutions Ψ_1 and Ψ_2 form the general solution $\hat{Y} = \hat{c}_1\Psi_1 + \hat{c}_2\Psi_2$, with unknown constants $\hat{c}_1$ and $\hat{c}_2$. The first linearly independent solution Ψ_1 was sought by power series method, substituting $\hat{Y} = \Psi_1 = 1 + \sum_{k=1}^{\infty} c_k \tilde{r}^k$ into (37). Orders of $\tilde{r}$ were equated, and Ψ_1 was found by recursion.

$$\Psi_1(\tilde{r},p) = 1 + \sum_{k=1}^{n}\left\{\left(\prod_{i=1}^{k}\left[i(i-1) - n(n+1)\right]\right)\frac{\tilde{r}^k}{k!\,p^k\tilde{m}^k}\right\} \tag{38}$$

Reduction of order was used to find the second linear independent solution Ψ_2. That is, solution $\hat{Y} = \Psi_2 = B\Psi_1$ was sought in (37) where unknown $B = B(\tilde{r})$. Solving for B gave $B = \int_{\infty}^{\tilde{r}} \frac{\exp(-p\tilde{m}/\tilde{r})}{\tilde{r}^2\Psi_1^2} d\tilde{r}$.

Therefore, $\Psi_2 = \Psi_1 \int_{\infty}^{\tilde{r}} \frac{\exp(-p\tilde{m}/\tilde{r})}{\tilde{r}^2\Psi_1^2} d\tilde{r}$. By the use of recursion (letting $n = 1,2,3...$ in Ψ_1), Ψ_2 simplified to

$$\Psi_2(\tilde{r},p) = \exp(-p\tilde{m}/\tilde{r})\left\{1 + \sum_{k=1}^{n}\left(\prod_{i=1}^{k}\left[n(n+1) - i(i-1)\right]\right)\frac{\tilde{r}^k}{k!(p\tilde{m})^k}\right\} - (-1)^n\Psi_1 \,. \tag{39}$$

The above results gave solutions to $\hat{T}(\tilde{r})$ in the core, burner and outer regions as well as solutions to $\hat{Y}_1(\tilde{r})$ and $\hat{Y}_2(\tilde{r})$ in the outer and radiation regions. Constants $\hat{c}_1$ and $\hat{c}_2$ differed from region to region and remained unknown.

$$\hat{T}_{core}(\tilde{r}) = c_{1c}\Psi_1(\tilde{r},1) + c_{2c}\Psi_2(\tilde{r},1) \,, \qquad \hat{T}_{burner}(\tilde{r}) = c_{1b}\Psi_{1b}(\tilde{r},\tilde{\lambda}^{-1}) + c_{2b}\Psi_{2b}(\tilde{r},\tilde{\lambda}^{-1}) \tag{40a,b}$$

$$\hat{T}_{out}^{-}(\tilde{r}) = \hat{c}_{T,1}^{-}\Psi_1(\tilde{r},1) + \hat{c}_{T,2}^{-}\Psi_2(\tilde{r},1) \,, \qquad \hat{T}_{out}^{+}(\tilde{r}) = \hat{c}_{T,1}^{+}\Psi_1(\tilde{r},1) + \hat{c}_{T,2}^{+}\Psi_2(\tilde{r},1) \tag{40c,d}$$

$$\hat{Y}_{1,out}^{-}(\tilde{r}) = \hat{Y}_{1,rad}^{-}(\tilde{r}) = \hat{c}_{1,1}^{-}\Psi_1(\tilde{r},Le_1) + \hat{c}_{1,2}^{-}\Psi_2(\tilde{r},Le_1) \,, \qquad \hat{Y}_{1,out}^{+}(\tilde{r}) = \hat{Y}_{1,rad}^{+}(\tilde{r}) = \hat{c}_{1,1}^{+}\Psi_1(\tilde{r},Le_1) + \hat{c}_{1,2}^{+}\Psi_2(\tilde{r},Le_1) \tag{40e,f}$$

$$\hat{Y}_{2,out}^{-}(\tilde{r}) = \hat{Y}_{2,rad}^{-}(\tilde{r}) = \hat{c}_{2,1}^{-}\Psi_1(\tilde{r},Le_2) + \hat{c}_{2,2}^{-}\Psi_2(\tilde{r},Le_2) \,, \qquad \hat{Y}_{2,out}^{+}(\tilde{r}) = \hat{Y}_{2,rad}^{+}(\tilde{r}) = \hat{c}_{2,1}^{+}\Psi_1(\tilde{r},Le_2) + \hat{c}_{2,2}^{+}\Psi_2(\tilde{r},Le_2) \tag{40g,h}$$

Boundary conditions were applied to the results in (40), resolving constants $c_{1,c}$, $c_{2,c}$, $c_{1,b}$, $c_{2,b}$, completing the solution for the temperature profiles in the core and burner regions. Constants $\hat{c}_{T,2}^{-}$, $\hat{c}_{T,1}^{+}$, $\hat{c}_{1,2}^{-}$, $\hat{c}_{1,1}^{+}$, $\hat{c}_{2,2}^{-}$ and $\hat{c}_{2,1}^{+}$ were also resolved, but constants $\hat{c}_{T,1}^{-}$, $\hat{c}_{T,2}^{+}$, $\hat{c}_{1,1}^{-}$, $\hat{c}_{1,2}^{+}$, $\hat{c}_{2,1}^{-}$ and $\hat{c}_{2,2}^{+}$ remained unknown and were expanded, along with the respective temperature and mass fraction profiles, and Lewis numbers Le_i, as shown below. Note that $\hat{T}_b$ was expanded in place of constant $\hat{c}_{T,1}^{-}$, defined as $\hat{c}_{T,1}^{-} = A_{T2}\hat{T}_b$, where A_{T2} was used as shorthand notation for a lengthy expression of a known constant, given in Appendix A.

$$\hat{T}_b = [\hat{T}_{b,0} + \varepsilon\hat{T}_{b,1} + O(\varepsilon^2)] + \delta[\hat{T}_{b,2} + O(\varepsilon)] + O(\delta^2), \qquad \hat{c}_{T,2}^{+} = [\hat{a}_{T,0}^{+} + \varepsilon\hat{a}_{T,1}^{+} + O(\varepsilon^2)] + \delta[\hat{a}_{T,2}^{+} + O(\varepsilon)] + O(\delta^2) \qquad (41a,b)$$

$$\hat{c}_{i,1}^{-} = [\hat{a}_{i,0}^{-} + \varepsilon\hat{a}_{i,1}^{-} + O(\varepsilon^2)] + \delta[\hat{a}_{i,2}^{-} + O(\varepsilon)] + O(\delta^2), \qquad \hat{c}_{i,2}^{+} = [\hat{a}_{i,0}^{+} + \varepsilon\hat{a}_{i,1}^{+} + O(\varepsilon^2)] + \delta[\hat{a}_{i,2}^{+} + O(\varepsilon)] + O(\delta^2) \qquad (41c,d)$$

$$\hat{Y}_i^{-} = [\hat{Y}_{i,0}^{-} + \varepsilon\hat{Y}_{i,1}^{-} + O(\varepsilon^2)] + \delta[\hat{Y}_{i,2}^{-} + O(\varepsilon)] + O(\delta^2), \qquad \hat{Y}_i^{+} = [\hat{Y}_{i,0}^{+} + \varepsilon\hat{Y}_{i,1}^{+} + O(\varepsilon^2)] + \delta[\hat{Y}_{i,2}^{+} + O(\varepsilon)] + O(\delta^2) \qquad (41e,f)$$

$$\hat{T}^{-} = [\hat{T}_0^{-} + \varepsilon\hat{T}_1^{-} + O(\varepsilon^2)] + \delta[\hat{T}_2^{-} + O(\varepsilon)] + O(\delta^2), \qquad \hat{T}^{+} = [\hat{T}_0^{+} + \varepsilon\hat{T}_1^{+} + O(\varepsilon^2)] + \delta[\hat{T}_2^{+} + O(\varepsilon)] + O(\delta^2) \qquad (41g,h)$$

Applying these expansions to (40c-f) and to boundary conditions gave expressions for $\hat{Y}_{i,0}^{\pm}$, $\hat{Y}_{i,1}^{\pm}$, $\hat{Y}_{i,2}^{\pm}$, $\hat{T}_1^{\pm}$, $\hat{T}_2^{\pm}$ and $\hat{T}_3^{\pm}$ in terms of remaining unknown constants. Notably, $\hat{T}_1^{-}(\tilde{r})$ and $\hat{T}_1^{+}(\tilde{r})$ are shown below.

$$\hat{T}_1^{-}(\tilde{r}) = \hat{T}_{b,1}\{A_{T2}\Psi(\tilde{r},1) + [1 - A_{T2}\Psi_1(1,1)]\Psi_2(\tilde{r},1)/\Psi_2(1,1)\}, \qquad \hat{T}_1^{+}(\tilde{r}) = \hat{a}_{T,1}^{+}\Psi_2(\tilde{r},1) \qquad (42a,b)$$

The full solution to the homogeneous equation (35) is found in Appendix B.

Matching at the Outer and Radiation Region Boundary

Since the width of the radiation region is order δ, the stretched variable $\zeta = (\tilde{r} - \tilde{r}_f)/\delta$ was defined for use in this region such that $\tilde{r} = \tilde{r}_f + \delta\zeta$. The flame location was defined as $\tilde{r}_f = \tilde{r}_{f,S} + \bar{\delta}\hat{r}_f P_n(\tilde{z})e^{\tilde{\omega}\tilde{t}}$ where $\hat{r}_f = \hat{r}_{f,0} + \varepsilon\hat{r}_{f,1} + O(\varepsilon^2)$. The exponential dependence $e^{\tilde{\omega}\tilde{t}}$ dropped off since $\omega = 0$. An inner expansion for the temperature profile in the radiation region was defined as

$$\tilde{T}_{rad}^{\pm} = [\tilde{T}_f - \varepsilon\tilde{\Theta}_2^{\pm} + O(\varepsilon^2)] - \delta[\tilde{\Theta}_1^{\pm} + \varepsilon\tilde{\Theta}_3^{\pm} + O(\varepsilon^2)] + O(\delta^2) \quad ; \quad \tilde{\Theta}_j^{\pm} = \Theta_{S,j}^{\pm} + \bar{\delta}P_n e^{\tilde{\omega}\tilde{t}}\,\hat{\Theta}_j^{\pm} + O(\bar{\delta}^2), \quad j = 1,2,3. \qquad (43a,b)$$

Functions $\tilde{\Theta}_j^{\pm}$ were assumed to be separable following the solution to homogeneous equation (35) such that $\Theta_{S,j}^{\pm} = \Theta_{S,j}^{\pm}(\tilde{r})$ are known from the steady state solution, $P_n = P_n(\tilde{z})$ is the Legendre polynomial, and $\hat{\Theta}_j^{\pm} = \hat{\Theta}_j^{\pm}(\zeta)$ are unknown functions. An inner expansion for $\tilde{T}_{out}^{\pm}$ was defined as

$$\tilde{T}_{out}^{\pm} = [\tilde{T}_0 + \varepsilon\tilde{T}_1 + O(\varepsilon^2)] + \delta[\tilde{T}_2 + \varepsilon\tilde{T}_3 + O(\varepsilon^2)] + O(\delta^2) \quad ; \quad \tilde{T}_j^{\pm} = \tilde{T}_{S,j}^{\pm} + \bar{\delta}P_n e^{\tilde{\omega}\tilde{t}}\hat{T}_j^{\pm} + O(\bar{\delta}^2). \qquad (44a,b)$$

$\tilde{T}_{S,j}^{\pm}$ and $\hat{T}_j^{\pm}$ represent steady state solutions and unknown functions respectively. (This notation is similar in all preceding profile expansions.) The next step was to equate (43a) with (44a) at the shared boundaries of the outer and radiation regions in a step referred to as 'matching'. Before matching could take place, a Taylor series expansion of the outer region temperature expansion (44a,b) about the steady state flame radius $\tilde{r}_{f,S}$ was performed, resulting in the following equation

$$
\begin{aligned}
\tilde{T}^{\pm}_{out}\Big|_{\tilde{r}_{f,S}} &= \left\{ \tilde{T}_f\Big|_{\tilde{r}_{f,S}} + \bar{\delta}\, P_n \left[\hat{T}^{\pm}_0\Big|_{\tilde{r}_{f,S}} + \hat{r}_{f,0} \frac{d\tilde{T}^{\pm}_{S,0}}{d\tilde{r}}\Big|_{\tilde{r}_{f,S}} \right] \right\} + \varepsilon \left\{ \tilde{T}^{\pm}_{S,1}\Big|_{\tilde{r}_{f,S}} + \bar{\delta}\, P_n \left[\hat{T}^{\pm}_1\Big|_{\tilde{r}_{f,S}} + \hat{r}_{f,0} \frac{d\tilde{T}^{\pm}_{S,1}}{d\tilde{r}}\Big|_{\tilde{r}_{f,S}} + \hat{r}_{f,1} \frac{d\tilde{T}^{\pm}_{S,0}}{d\tilde{r}}\Big|_{\tilde{r}_{f,S}} \right] \right\} + \delta \left\langle \left[\tilde{T}^{\pm}_{S,2}\Big|_{\tilde{r}_{f,S}} + \frac{d\tilde{T}^{\pm}_{S,0}}{d\tilde{r}}\Big|_{\tilde{r}_{f,S}} \zeta \right] \right. \\
&\quad \left. + \bar{\delta}\, P_n \left\{ \left[\hat{T}^{\pm}_2\Big|_{\tilde{r}_{f,S}} + \hat{r}_{f,0} \frac{d\tilde{T}^{\pm}_{S,2}}{d\tilde{r}}\Big|_{\tilde{r}_{f,S}} \right] + \left[\frac{d\hat{T}^{\pm}_0}{d\tilde{r}}\Big|_{\tilde{r}_{f,S}} + \hat{r}_{f,0} \frac{d^2\tilde{T}^{\pm}_{S,0}}{d\tilde{r}^2} \right]_{\tilde{r}_{f,S}} \zeta \right\} \right\rangle + \varepsilon\delta \left\langle \left[\tilde{T}^{\pm}_{S,3}\Big|_{\tilde{r}_{f,S}} + \frac{d\tilde{T}^{\pm}_{S,1}}{d\tilde{r}}_{\tilde{r}_{f,S}} \zeta \right] + \bar{\delta}\, P_n \left\{ \left[\hat{T}^{\pm}_3\Big|_{\tilde{r}_{f,S}} + \hat{r}_{f,0} \frac{d\tilde{T}^{\pm}_{S,3}}{d\tilde{r}}\Big|_{\tilde{r}_{f,S}} \right. \right. \right. \\
&\quad \left. \left. \left. + \hat{r}_{f,1} \frac{d\tilde{T}^{\pm}_{S,2}}{d\tilde{r}}\Big|_{\tilde{r}_{f,S}} \right] + \left[\frac{d\hat{T}^{\pm}_1}{d\tilde{r}}\Big|_{\tilde{r}_{f,S}} + \hat{r}_{f,0} \frac{d^2\tilde{T}^{\pm}_{S,1}}{d\tilde{r}^2}\Big|_{\tilde{r}_{f,S}} + \hat{r}_{f,1} \frac{d^2\tilde{T}^{\pm}_{S,0}}{d\tilde{r}^2}\Big|_{\tilde{r}_{f,S}} \right] \zeta \right\} \right\rangle + O(\delta^2, \varepsilon^2, \bar{\delta}^2).
\end{aligned}
$$

Matching the above expansion with (43a) at $\zeta \to \pm\infty$ and equating orders resolved constants $\hat{T}_{b,0}$ and $\hat{a}^{+}_{T,0}$, and yielded relations for use later in the analysis (e.g. simplifying relations, substitutions, etc.).

Radiation Region Temperature Profile Solution

Following the matching, the radiation region temperature expansion (43a) was substituted into the radiation region energy equation (20), variable transformation from $\tilde{r}$ to ζ was applied, and the equation was arranged according to order of magnitude. The three leading orders were kept, yielding the following system of equations:

$$\partial^2\tilde{\Theta}^{\pm}_1 / \partial\zeta^2 = -\Lambda_R \exp(-\tilde{\Theta}^{\pm}_1) \tag{44}$$

$$\partial^2\tilde{\Theta}^{\pm}_2 / \partial\zeta^2 = -\Lambda_R \tilde{\Theta}^{\pm}_2 \exp(-\tilde{\Theta}^{\pm}_1) \tag{45}$$

$$\frac{\partial^2\tilde{\Theta}^{\pm}_3}{\partial\zeta^2} - \frac{\tilde{m} - 2\tilde{r}_{f,0}}{\tilde{r}^2_{f,0}} \frac{\partial\tilde{\Theta}^{\pm}_2}{\partial\zeta} = \Lambda_R \left(\tilde{\Theta}^{\pm}_3 + \frac{2\tilde{\Theta}^{\pm}_1\tilde{\Theta}^{\pm}_2}{\tilde{T}_f} \right) \exp(-\tilde{\Theta}^{\pm}_1) \tag{46}$$

with reduced radiative Damköhler number $\Lambda_R = \delta Da_R \exp(-\tilde{E}_R / \tilde{T}_f)$ defined for simplification. The solutions for $\tilde{\Theta}^{\pm}_1$, $\tilde{\Theta}^{\pm}_2$ and $\tilde{\Theta}^{\pm}_3$ were found by solving this system of equations. The solutions are lengthy and are therefore included in Appendix A. The derivation can be found in the full problem solution.

Matching at the Radiation and Flame Region Boundary

Since the width of the flame region is order ε, the stretched variable $\xi = (\tilde{r} - \tilde{r}_{f,0}) / \varepsilon$ was defined for use in this region such that $\tilde{r} = \tilde{r}_{f,0} + \varepsilon\xi$ where $\tilde{r}_{f,0} = \tilde{r}_{f,S} + \bar{\delta}\hat{r}_{f,0} P_n(\tilde{z}) e^{\tilde{\omega}\tilde{t}}$. Additionally, the stretched variable ξ was expanded as $\xi = \hat{\xi} + \bar{\delta}\hat{r}_{f,1} P_n(\tilde{z}) e^{\tilde{\omega}\tilde{t}}$. Inner expansions for the temperature and mass fraction profiles in the radiation and flame regions were defined as

$$\tilde{T}_{flame} = [\tilde{T}_f - \varepsilon\tilde{\theta}_1 - \varepsilon^2\tilde{\theta}_2 + O(\varepsilon^3)] + O(\varepsilon / \delta), \qquad \tilde{\theta}_j = \theta_{1,S} + P_n(\tilde{z}) e^{\tilde{\omega}\tilde{t}} \hat{\theta}_1 \tag{47a,b}$$

$$\tilde{Y}_{i,flame} = [\varepsilon\tilde{\phi}_{i,1} + \varepsilon^2\tilde{\phi}_{i,2} + O(\varepsilon^3)], \qquad \tilde{\phi}_{i,j} = \phi_{i,j,S} + P_n(\tilde{z}) e^{\tilde{\omega}\tilde{t}} \hat{\phi}_{i,j} \tag{48a,b}$$

$$\tilde{Y}^{\pm}_{i,rad} = [\tilde{Y}^{\pm}_{i,0} + \varepsilon\tilde{Y}^{\pm}_{i,1} + O(\varepsilon^2)] + \delta[\tilde{Y}^{\pm}_{i,2} + O(\varepsilon)] + O(\delta^2), \qquad \tilde{Y}^{\pm}_{i,j} = \tilde{Y}^{\pm}_{i,j,S} + P_n(\tilde{z}) e^{\tilde{\omega}\tilde{t}} \hat{Y}^{\pm}_{i,j} \tag{49a,b}$$

where $\tilde{T}_f$ represents the flame temperature, $\theta_{1,S}$, $\phi_{i,1,S}$ and $\tilde{Y}^{\pm}_{i,j,S}$ represent the steady state solutions, and

$\hat{\theta}_2$, $\hat{\phi}_{i,2}$ and $\hat{Y}^{\pm}_{i,j}$ are unknown error functions. In order to match flame region (47a) and (48a) with radiation region (43a), and (49a) at the shared boundaries of the flame and radiation regions, Taylor series expansions and coordinate transformations were performed on radiation region (43a) and (40e,f,g,h). As the radiation region approaches the flame region boundary, the stretched variable ζ goes to zero. Therefore, the radiation region temperature equation (43a) was Taylor series expanded about $\zeta = 0$, and mass fraction equations (40e,f,g,h) were Taylor series expanded about $\tilde{r}_{f,S}$. The radiation region temperature equation Taylor series expansion is shown below,

$$\tilde{T}^{\pm}_{rad}\Big|_{\substack{TSE \\ about\,\zeta=0}} = \left\{ \tilde{T}_f - \varepsilon\left[\tilde{\Theta}^{\pm}_2\Big|_{\zeta=0} + \frac{\partial\tilde{\Theta}^{\pm}_2}{\partial\zeta}\Big|_{\zeta=0}\left(\frac{\varepsilon}{\delta}\hat{\xi}\right) + \ldots + O(\varepsilon^2)\right]\right\} - \delta\left\{ \tilde{\Theta}^{\pm}_1\Big|_{\zeta=0} + \frac{\partial\tilde{\Theta}^{\pm}_1}{\partial\zeta}\Big|_{\zeta=0}\left(\frac{\varepsilon}{\delta}\hat{\xi}\right) + \frac{\partial^2\tilde{\Theta}^{\pm}_1}{\partial\zeta^2}\Big|_{\zeta=0}\frac{1}{2}\left(\frac{\varepsilon}{\delta}\hat{\xi}\right)^2 + \ldots \right.$$

$$\left. + \varepsilon\left[\tilde{\Theta}^{\pm}_3\Big|_{\zeta=0} + \frac{\partial\tilde{\Theta}^{\pm}_3}{\partial\zeta}\Big|_{\zeta=0}\left(\frac{\varepsilon}{\delta}\hat{\xi}\right) + \ldots \right] + O(\varepsilon^2)\right\} - \delta^2\left[\tilde{\Theta}^{\pm}_4\Big|_{\zeta=0} + \frac{\partial\tilde{\Theta}^{\pm}_4}{\partial\zeta}\Big|_{\zeta=0}\left(\frac{\varepsilon}{\delta}\hat{\xi}\right) + \frac{\partial^2\tilde{\Theta}^{\pm}_4}{\partial\zeta^2}\Big|_{\zeta=0}\frac{1}{2}\left(\frac{\varepsilon}{\delta}\hat{\xi}\right)^2 + \ldots + O(\varepsilon^3)\right] + O(\delta^2)$$

which was matched to (47a) at $\xi \to \pm\infty$. The radiation region mass fraction equation Taylor series expansions were matched with flame region equations (49a) at $\xi \to \pm\infty$. These expansions are lengthy and can be found in Appendix C. Equating orders of the matched boundary relations resolved constants $\hat{a}^{\pm}_{1,0}$, $\hat{a}^{\pm}_{1,2}$, $\hat{a}^{\pm}_{2,0}$, $\hat{a}^{\pm}_{2,2}$, $\hat{T}_{b,2}$ and $\hat{a}^{+}_{T,2}$. Additionally, applying inner expansions (47b) and (48b) to the results yielded the relations for $\hat{\phi}_{i,1}$ given in the preceding problem summary (69b) and (69c).

Flame Region Solution

Inner expansions (47a) and (48a) were substituted into temperature profile (22), and the variable transformation from $\tilde{r}$ to ξ was performed. A reduced kinetic Damköhler number Λ_K was defined for simplification. The resulting leading order for the temperature profile was

$$\frac{\partial^2\tilde{\theta}_1}{\partial\xi^2} = \Lambda_K\tilde{\phi}_{1,1}\tilde{\phi}_{2,1}\exp(-\tilde{\theta}_1), \qquad \Lambda_K = \varepsilon^3 Da_K \exp(-\tilde{E}_K/\tilde{T}) \tag{50a,b}$$

Species 1 mass fraction (23) was subtracted from species 2 mass fraction (23), inner expansions (47a) and (48a) were substituted into the resulting relation, and the transformation from $\tilde{r}$ to ξ was performed. The resulting two leading-order relations were integrated, yielding the two following relations with unknown integration constants $\tilde{c}_1$, $\tilde{c}_2$ and $\tilde{c}_3$.

$$\frac{\tilde{\phi}_{1,1}}{Le_1} - \frac{\tilde{\phi}_{2,1}}{Le_2} = \tilde{c}_1\xi + \tilde{c}_2 \tag{51}$$

$$\left[\frac{1}{Le_1}\frac{\partial\tilde{\phi}_{1,2}}{\partial\xi} + \frac{2\xi}{Le_1\tilde{r}_{f,0}}\frac{\partial\tilde{\phi}_{1,1}}{\partial\xi} - \frac{\tilde{m}}{\tilde{r}^2_{f,0}}\tilde{\phi}_{1,1}\right] - \left[\frac{1}{Le_2}\frac{\partial\tilde{\phi}_{2,2}}{\partial\xi} + \frac{2\xi}{Le_2\tilde{r}_{f,0}}\frac{\partial\tilde{\phi}_{2,1}}{\partial\xi} - \frac{\tilde{m}}{\tilde{r}^2_{f,0}}\tilde{\phi}_{2,1}\right] = \tilde{c}_3 \tag{52}$$

Species 1 mass fraction (23) was added to temperature profile (22), inner expansions (47a), and (48a) were invoked, and the variable transformation from $\tilde{r}$ to ξ was performed. The resulting two leading-order relations were integrated, yielding the two following relations with unknown integration constants $\tilde{c}_4$, $\tilde{c}_5$ and $\tilde{c}_6$.

$$\tilde{\theta}_1 - \frac{\tilde{\phi}_{1,1}}{Le_1} = \tilde{c}_4 \xi + \tilde{c}_5 \tag{53}$$

$$\left(\frac{\partial \tilde{\theta}_2}{\partial \xi} + \frac{2\xi}{\tilde{r}_{f,0}} \frac{\partial \tilde{\theta}_1}{\partial \xi} - \frac{\tilde{m}}{\tilde{r}_{f,0}^2} \tilde{\theta}_1 \right) - \left[\frac{1}{Le_1} \frac{\partial \tilde{\phi}_{1,2}}{\partial \xi} + \frac{2\xi}{Le_1 \tilde{r}_{f,0}} \frac{\partial \tilde{\phi}_{1,1}}{\partial \xi} - \frac{\tilde{m}}{\tilde{r}_{f,0}^2} \tilde{\phi}_{1,1} \right] = \tilde{c}_6 \tag{54}$$

Boundary conditions at $\xi \to \pm\infty$ obtained from matching were applied to (51-54), resolving integration constants $\tilde{c}_1$ through $\tilde{c}_6$, providing a system of four relations that were used to resolve constants $\hat{a}_{1,1}^-$, $\hat{a}_{1,1}^+$, $\hat{a}_{2,1}^-$ and $\hat{a}_{2,1}^+$, and yielding the boundary conditions shown below

$$\tilde{\theta}_1\Big|_{\xi \to -\infty} = -[(g_0^-)^2 + 2\Lambda_R]^{1/2} \{ [(\tilde{T}_{b,S,A} + \overline{\delta} P_n \hat{T}_1^- \Big|_{\tilde{r}_{f,S}}) / g_0^-] + \xi \} + O(\overline{\delta}^2) \tag{55a}$$

$$(d\tilde{\theta}_1 / d\xi)\Big|_{\xi \to -\infty} = -[(g_0^-)^2 + 2\Lambda_R]^{1/2} + O(\overline{\delta}^2) \tag{55b}$$

$$\tilde{\theta}_1\Big|_{\xi \to \infty} = -[(g_0^+)^2 + 2\Lambda_R]^{1/2} \{ [(a_{T,A}^+ + \overline{\delta} P_n \hat{T}_1^+ \Big|_{\tilde{r}_{f,S}}) / g_0^+] - \xi \} + O(\overline{\delta}^2) \tag{56a}$$

$$(d\tilde{\theta}_1 / d\xi)\Big|_{\xi \to \infty} = [(g_0^+)^2 + 2\Lambda_R]^{1/2} + O(\overline{\delta}^2). \tag{56b}$$

Constants $a_{1,1}^+$, $\tilde{T}_{b,S,A}$, and parameters g_0^- and g_0^+ were defined in the steady state solution and introduced into the stability analysis during the solution of the energy equation in the radiation region; $a_{1,1}^+$ and $\tilde{T}_{b,S,A}$ remained unknown; g_0^- and g_0^+ are defined in (63a) and (63b) below.

Problem Summary

Equation 50 represents the flame structure equation in the flame region with boundary conditions given in (55) and (56). Inner expansion of variables $\tilde{\theta}_1$, $\tilde{\phi}_{1,1}$, and $\tilde{\phi}_{2,1}$ (47b) and (48b) were substituted into the flame structure (50) and boundary conditions (55) and (56). After equating orders, the problem separated into the steady state problem and the perturbed problem given as follows.

Steady State Problem:

$$\frac{d^2 \theta_{1,S}}{d\xi^2} = \Lambda_K \phi_{1,1,S} \phi_{2,1,S} \exp(-\theta_{1,S}) \tag{57a}$$

$$\theta_{1,S}\Big|_{\xi \to -\infty} = -[(\tilde{T}_{b,S,A} / g_0^-) + \xi][(g_0^-)^2 + 2\Lambda_R]^{1/2}, \qquad (d\theta_{1,S} / d\xi)\Big|_{\xi \to -\infty} = -[(g_0^-)^2 + 2\Lambda_R]^{1/2} \tag{58a,b}$$

$$\theta_{1,S}\Big|_{\xi \to \infty} = -a_{1,1}^+ [1 - \exp(\tilde{m} / \tilde{r}_{f,S})] + [(g_0^+)^2 + 2\Lambda_R]^{1/2} \xi, \qquad (d\theta_{1,S} / d\xi)\Big|_{\xi \to \infty} = [(g_0^+)^2 + 2\Lambda_R]^{1/2} \tag{59a,b}$$

Perturbed Problem

$$\frac{d^2\hat{\theta}_1}{d\xi^2} = \frac{d^2\theta_{1,S}}{d\xi^2}\left[\frac{\hat{\phi}_{1,1}}{\phi_{1,1,S}} + \frac{\hat{\phi}_{2,1}}{\phi_{2,1,S}} - \hat{\theta}_1\right] \tag{60a}$$

$$\hat{\theta}_1\Big|_{\xi\to-\infty} = -\hat{T}_1^-\Big|_{\tilde{r}_{f,S}}[(g_0^-)^2 + 2\Lambda_R]^{1/2}/g_0^-, \qquad (d\hat{\theta}_1/d\xi)\Big|_{\xi\to-\infty} = 0 \tag{61a,b}$$

$$\hat{\theta}_1\Big|_{\xi\to\infty} = -\hat{T}_1^+\Big|_{\tilde{r}_{f,S}}[(g_0^+)^2 + 2\Lambda_R]^{1/2}/g_0^+, \qquad (d\hat{\theta}_1/d\xi)\Big|_{\xi\to\infty} = 0 \tag{62a,b}$$

Relations for g_0^-, g_0^+ and $\tilde{r}_{f,S}$ are known from the steady state analysis and given below.

$$g_0^- = (\tilde{T}_{f,S} - \tilde{T}_0)(\tilde{m}/\tilde{r}_{f,S}), \qquad g_0^+ = (\tilde{T}_{f,S} - \tilde{T}_\infty)(\tilde{m}/\tilde{r}_{f,S})/[\exp(\tilde{m}/\tilde{r}_{f,S}) - 1], \qquad \tilde{r}_{f,S} = Le_2\tilde{m}/\ln(1+\tilde{Y}_{2,\infty}) \tag{63a,b,c}$$

Constants $\tilde{T}_{b,S,A}$ and $a_{1,1}^+$ remained unknown in the steady state problem and constants $\hat{T}_{b,1}$ and $\hat{a}_{T,1}^+$ remained unknown in the relations for $\hat{T}^-$ and $\hat{T}^+$ given in (42a,b). The steady state flame temperature, $\tilde{T}_{f,S}$, was resolved numerically using equations 63a,b,c, as follows.

Steady State Flame Temperature and Adiabatic Flame Temperature

The steady state analysis showed that

$$\tilde{m}/\tilde{r}_{f,S}^2 = [(g_0^+)^2 + 2\Lambda_R]^{1/2} - [(g_0^-)^2 + 2\Lambda_R]^{1/2}, \tag{64}$$

which was used with expressions (63a,b,c) for g_0^-, g_0^+, and $\tilde{r}_{f,S}$ to numerically determine $\tilde{T}_{f,S}$ by the Newton iteration method and to analytically approximate the steady state adiabatic flame temperature. The adiabatic flame temperature was found in the limit of perfectly efficient combustion and an infinitely thin flame. In this approximation, the flame is adiabatic $(\Lambda_R = 0)$ since radiation is a volumetric effect, and notation $\tilde{T}_f^0$ was substituted for $\tilde{T}_{f,S}$ in (63a,b) and (64), yielding $\tilde{T}_f^0 = 1 + \tilde{T}_0 - (1 + \tilde{T}_0 - \tilde{T}_\infty)(1 + \tilde{Y}_{2,\infty})^{-1/Le_2}$. A reference adiabatic flame temperature $\tilde{T}_{ad}$ was fixed using $Le_2 = 1$, so that

$$\tilde{T}_{ad} = 1 + \tilde{T}_0 - (1 + \tilde{T}_0 - \tilde{T}_\infty)(1 + \tilde{Y}_{2,\infty})^{-1}.$$

Liñan's Form

The resulting problem was converted into Liñan's form following the method described by Law [5]. That is, $\bar{\theta}$ and $\bar{\xi}$ were defined as $\bar{\theta} = \alpha^{1/3}(\theta_{1,S} + A\xi + B)$ and $\bar{\xi} = \alpha^{1/3}(D\xi + G)$ with unknown constants α, A, B, D and G; $D > 0$. Shorthand notation $\gamma = -A/D$ was adopted and boundary conditions $(d\bar{\theta}/d\bar{\xi})\Big|_{\bar{\xi}\to-\infty} = -1$ and $(d\bar{\theta}/d\bar{\xi})\Big|_{\bar{\xi}\to\infty} = 1$ were assumed. The variable transformations $\theta_{1,S} \to \bar{\theta}$ and $\xi \to \bar{\xi}$ were applied to the steady state problem (57-59), and the assumed boundary conditions were used to solve for constants α, A, B, D, G and γ, which are given in Appendix A. Variable transformations

$\theta_{1,S} \to \bar{\theta}$ and $\xi \to \bar{\xi}$ were also applied to the perturbed problem (60-62), yielding the summarized problem below.

Steady State Problem in Liñan's Form:

$$\frac{d^2\bar{\theta}}{d\bar{\xi}^2} = (\bar{\theta} - \bar{\xi})(\bar{\theta} + \bar{\xi})\exp[-\alpha^{-1/3}(\bar{\theta} - \gamma\bar{\xi})] \quad (65)$$

$$\bar{\theta}\Big|_{\bar{\xi}\to-\infty} = -\alpha^{1/3}\left\{\frac{\tilde{T}_{b,S,A}[(g_0^-)^2 + 2\Lambda_R]^{1/2}}{g_0^-} + a_{1,1}^+\left(1 - \mathrm{e}^{\frac{-\tilde{m}}{\tilde{r}_{f,S}}}\right) - \frac{a_{1,1}^+\tilde{Y}_{2,\infty}}{Le_2(1+\tilde{Y}_{2,\infty})}\right\} - \bar{\xi}, \quad (d\bar{\theta}/d\bar{\xi})\Big|_{\bar{\xi}\to-\infty} = -1 \quad (66a,b)$$

$$\bar{\theta}\Big|_{\bar{\xi}\to\infty} = \alpha^{1/3}(a_{1,1}^+ / Le_1)[1 - \exp(-Le_1\tilde{m}/\tilde{r}_{f,S})] + \bar{\xi}, \qquad (d\bar{\theta}/d\bar{\xi})\Big|_{\bar{\xi}\to\infty} = 1 \quad (67a,b)$$

Perturbed Problem in Liñan's Form:

$$\frac{d^2\hat{\theta}_1}{d\bar{\xi}^2} = \frac{d^2\bar{\theta}}{d\bar{\xi}^2}\left[\frac{\hat{\phi}_{1,1}}{Le_1(\bar{\theta} - \bar{\xi})} + \frac{\hat{\phi}_{2,1}}{Le_2(\bar{\theta} + \bar{\xi})} - \frac{\hat{\theta}_1}{\alpha^{1/3}}\right] \quad (68a)$$

$$\hat{\phi}_{1,1} = \hat{a}_{1,1}^+\Psi_2(\tilde{r}_{f,S}, Le_1) + Le_1\{[(g_0^+)^2 + 2\Lambda_R]^{1/2}(\hat{a}_{T,1}^+ / g_0^+)\Psi_2(\tilde{r}_{f,S}, 1) + \hat{\theta}_1\} \quad (68b)$$

$$\hat{\phi}_{2,1} = \hat{a}_{2,1}^+\Psi_2(\tilde{r}_{f,S}, Le_2) + Le_2\{[(g_0^+)^2 + 2\Lambda_R]^{1/2}(\hat{a}_{T,1}^+ / g_0^+)\Psi_2(\tilde{r}_{f,S}, 1) + \hat{\theta}_1\} \quad (68c)$$

$$\hat{\theta}_1\Big|_{\bar{\xi}\to-\infty} = -\hat{T}_1^-\Big|_{\tilde{r}_{f,S}}[(g_0^-)^2 + 2\Lambda_R]^{1/2} / g_0^-, \qquad (d\hat{\theta}_1/d\bar{\xi})\Big|_{\bar{\xi}\to-\infty} = 0 \quad (69a,b)$$

$$\hat{\theta}_1\Big|_{\bar{\xi}\to\infty} = -\hat{T}_1^+\Big|_{\tilde{r}_{f,S}}[(g_0^+)^2 + 2\Lambda_R]^{1/2} / g_0^+, \qquad (d\hat{\theta}_1/d\bar{\xi})\Big|_{\bar{\xi}\to\infty} = 0 \quad (70a,b)$$

Rescaling

The parameters q_1, ν_1, ν_2, W_1, W_2 and $Y_{1,0}$ used in nondimensionalization all depend on the type of flame under observation. In order that the results from various flames can be compared, the nondimensionalized terms were rescaled using flame A (later defined) as the reference flame. For flame A, $q_1 = q_F$, $\nu_1 = \nu_F$, $\nu_2 = \nu_O$, $W_1 = W_F$, $W_2 = W_O$ and $Y_{1,0} = Y_1$. The rescaled nondimensionalized terms, notated with a bar superscript, are listed below and substituted for their respective former nondimensionalized counterparts.

$$\bar{Y}_1 = Y_1, \quad \bar{Y}_2 = Y_2\frac{\nu_F W_F}{\nu_O W_O}, \quad \bar{T} = \frac{c_{p,g}T}{q_F}, \quad \bar{E}_K = \frac{c_{p,g}E_K}{q_F}, \quad \bar{E}_R = \frac{c_{p,g}E_R}{q_F}, \quad \bar{\delta} = \frac{\bar{T}_{ad}^2}{\bar{E}_R}, \quad \bar{\varepsilon} = \frac{\bar{T}_{ad}^2}{\bar{E}_K},$$

$$\bar{D}a_K = \frac{B_K\nu_O W_O\rho_g^2 c_{p,g}r_b^2}{\lambda_g}, \quad \bar{D}a_R = \frac{4\sigma\kappa B_R c_{p,g}r_b^2}{q_F\lambda_g}, \quad \bar{\Lambda}_R = \bar{\delta}\bar{D}a_R\exp(-\tilde{E}_R/\tilde{T}_{ad}), \quad \bar{\Lambda}_K = \bar{\varepsilon}^3\bar{D}a_K\exp(-\tilde{E}_K/\tilde{T}_{ad})$$

Note: $\tilde{E}_R/\tilde{T}_{ad} = \bar{E}_R/\bar{T}_{ad}$ and $\tilde{E}_K/\tilde{T}_{ad} = \bar{E}_K/\bar{T}_{ad}$

Therefore,

$$\tilde{T} = \overline{T}\,\frac{q_F}{q_1 Y_{1,0}} \quad , \quad \tilde{Y}_2 = \frac{\overline{Y}_2}{Y_{1,0}}\frac{\nu_O W_O}{\nu_F W_F}\frac{\nu_1 W_1}{\nu_2 W_2} \quad , \quad \Lambda_R = [\,q_F/(q_1 Y_{1,0})\,]^2(\overline{T}_f/\overline{T}_{ad})^2 \exp[\,\overline{E}_R(\overline{T}_{ad}^{-1} - \overline{T}_f^{-1})\,]\overline{\Lambda}_R$$

The perturbed problem and the steady state problem were simultaneously solved numerically since steady state values $d^2\theta_{1,S}/d\xi^2$, $\phi_{1,1,S}$ and $\phi_{2,1,S}$ were needed to solve the perturbed problem.

Numerical Program

A program was developed in the Python programming language to solve the system of equations (65) and (68) subject to boundary conditions given in equations (66), (67), (69), and (70). The steady state problem given in (65) subject to boundary conditions (66) and (67), was first solved as follows. An initial value for the unknown constant $a_{1,1}^{+}$ was guessed as $a_{1,1}^{+} = 0$ and the program was run using the fourth-order Runge Kutta method from $\overline{\xi} = \infty$ approximated by $\overline{\xi} = 50$, to $-\infty$ approximated by a constant value value for $d\theta/d\overline{\xi}$ from one step to the next. A caveat was included for negative values of $(\overline{\theta} - \overline{\xi})$ or $(\overline{\theta} + \overline{\xi})$ representing negative concentration, whereby the program automatically exited from the Runge Kutta loop and continued with the following incremental value of $a_{1,1}^{+}$. Resulting boundary conditions at $-\infty$ were compared with (68b) in search of small error defined by $-1-(d\overline{\theta}/d\overline{\xi})\big|_{\overline{\xi}\to-\infty}$. This process was performed iteratively for successive incremental values of $a_{1,1}^{+}$ until the program produced a change in the sign of error, at which point successive iterations were performed to zoom in on the correct value for $a_{1,1}^{+}$ determined by small error. If no solution was found, the flame was considered to be extinguished. Steady state values from $\overline{\xi} = -\infty$ to $\overline{\xi} = \infty$ were saved in arrays for use in the perturbed equation. The perturbed problem given in (68) subject to boundary conditions (69) and (70) was then solved using a similar method. An initial value for the unknown constant $\hat{a}_{T,1}^{+}$ was guessed as $\hat{a}_{T,1}^{+} = 0$, and the fourth-order Runge Kutta method was run from $\overline{\xi} = \infty$ to $\overline{\xi} = -\infty$ as in the steady state program, using boundary conditions (69) and (70) in search of small error defined by $0-(d\hat{\theta}/d\overline{\xi})\big|_{\overline{\xi}\to-\infty}$. For flames B and C as described in the following chapter, the program was adjusted to perform iterations from $\overline{\xi} = -\infty$ to $\overline{\xi} = \infty$ to avoid numerical issues particular to the stiffness of the equation for these flame types.

CHAPTER IV. RESULTS AND DISCUSSION

Numerical calculations were performed using the following values representing the burning of ethylene in air with outer burner radius size used in experiments conducted by Sunderland et al. [22].

$$T_0 = T_\infty = 298\,K, \quad q_F = 47160\,J/g, \quad c_p = 1.3232\,J/(g \cdot K), \quad \lambda_g = 0.0012043\,W/(cm \cdot K),$$

$$E_K = 24000\,K, \quad E_R = 8000\,K, \quad \nu_F = 1, \quad \nu_O = 3, \quad W_F = 28\,g/mole, \quad W_O = 32\,g/mole, \quad r_b = 0.3175\,cm$$

Following the experimental model of Sunderland et al. [22], four different flames from stoichiometric mixtures of ethylene and air varying in flame structure and convection direction were analyzed. The stoichiometric reaction between ethylene and air is $C_2H_4 + (3O_2 + 11.286N_2) \Rightarrow 2CO_2 + 2H_2O + 11.286N_2$. Flame structure was varied by exchanging the inert gas between the oxidizer and the fuel such that the diluted fuel was represented by a mixture of ethylene and nitrogen extracted from the air. Convection direction was varied by interchanging the injected and quiescent gases. The four flames had the same stoichiometry and the adiabatic flame temperature remained constant between flames. They are as follows:

Flame A: fuel (C_2H_4) issuing into air $(79\%\,N_2$ and $21\%\,O_2)$

Flame B: diluted fuel $(8.1\%\,C_2H_4$ and $91.9\%\,N_2)$ issuing into the oxidizer (O_2)

Flame C: air issuing into fuel

Flame D: oxidizer issuing into diluted fuel

The following discussion uses results obtained from the steady state problem for Flame A only. The results for the other three flames are qualitatively similar and can be found in [25]. The curves in Figures 4, 5 and 7 can be separated into upper and lower portions from the minimum values of $\bar{\Lambda}_K$ (Fig. 4) or m (Figs. 5, 7) to maximum values (where applicable). The upper portions of these curves display analytical solutions that do not exist in reality whereas the bottom portions are representative of physical reality.

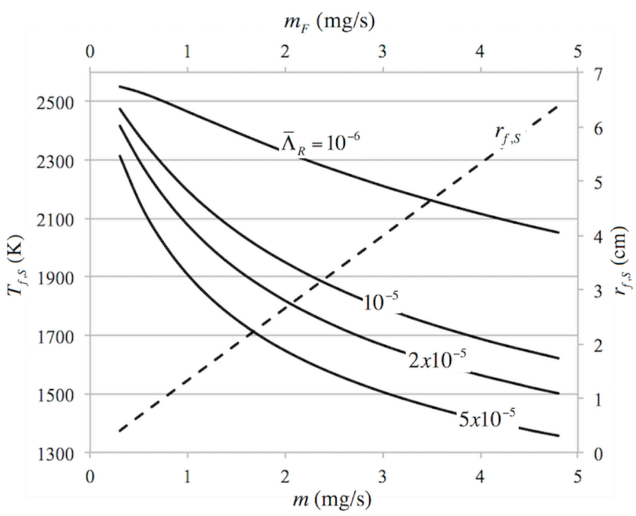


Figure 3: Steady state flame temperature and radius plotted against specified reduced radiative Damköhler numbers, mass flow rate and fuel consumption rate

Section 1: Temperature Distribution

Steady state flame temperature $T_{f,S}$ and flame radius $r_{f,S}$ plotted against mass flow rate of species 1 issuing from the burner and fuel consumption rate for varying values of $\bar{\Lambda}_R$ is shown in Figure 3. Results reveal that for fixed m, as the radiation intensity (and therefore $\bar{\Lambda}_R$) decreases, the flame temperature increases, approaching the adiabatic flame temperature. Additionally, m is shown to be directly proportional to the flame radius. From the analytical portion of this study, a qualitative analysis of the effects resulting from mass flow rate variations demonstrates that the flame radius is directly proportional to the mass flow rate and flame volume. The following qualitative analysis references (63c) and equations for mass flow rate and sphere surface area.

$$\tilde{m} = \frac{\tilde{r}_{f,S} \ln(1+\tilde{Y}_{2,\infty})}{Le_2} \Rightarrow \tilde{m} \sim \tilde{r}_{f,S} \quad ; \quad m = \rho_g u_f 4\pi(r_f^2) \Rightarrow r_f \sim 1/u_f \quad ; \quad V_f = \varepsilon 4\pi r_f^2 \Rightarrow r_f \sim V_f$$

Figure 3 also reveals that for fixed $\bar{\Lambda}_R$, as m increases, flame temperature decreases. This is due to the direct proportionality between mass flow rate and flame volume. Radiative heat loss is directly related to flame volume since radiation is a volumetric effect. A qualitative analysis is shown below.

$$m\uparrow \Rightarrow r_f\uparrow \Rightarrow V_f\uparrow \Rightarrow \text{radiative heat loss}\uparrow \Rightarrow T_f\downarrow$$

The steady state flame temperature remains constant at the adiabatic temperature when $\bar{\Lambda}_R = 0$.

Section 2: Kinetic Extinction

The curves in Figure 4 display the relationship between $\bar{\Lambda}_R$, $\bar{\Lambda}_K$ and reactant 1 leakage past the flame region, $\bar{Y}_{1,L}$. For fixed $\bar{\Lambda}_R$ and fixed fuel consumption rate $m_F = 2$ mg/s, there exists a minimum $\bar{\Lambda}_K$ below which a solution does not exist and steady burning is not possible, representing the kinetic extinction limit. As the radiation intensity increases, increased radiative heat losses cause the flame temperature to drop, thereby promoting kinetic extinction and resulting in an increased kinetic extinction limit. As

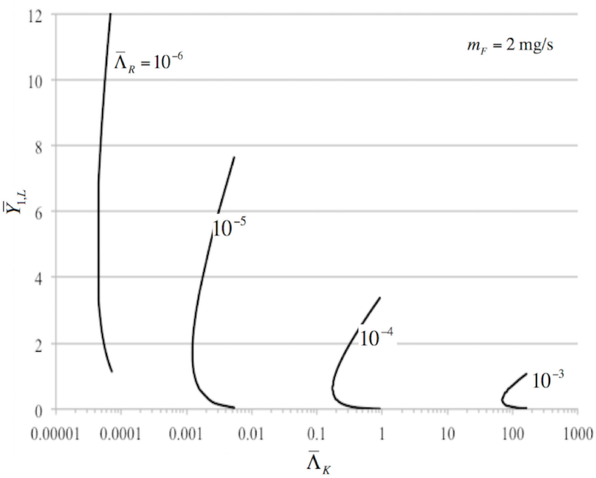


Figure 4: Reactant 1 leakage plotted against reduced kinetic Damköhler number and reduced radiative Damköhler numbers with fixed fuel consumption rate

$\bar{\Lambda}_K$ increases, reactant leakage decreases monotonically along the lower portion of the C-curve. The limit of an adiabatic flame $(\bar{\Lambda}_R = 0)$ corresponds to the smallest kinetic extinction limit for fixed m_F.

Section 3: Kinetic and Radiative Extinction

Reactant leakage is plotted against mass flow rate for fixed $\bar{\Lambda}_R$ and varying $\bar{\Lambda}_K$ in Figure 5. The lower portion of the curves display that for fixed m and fixed radiation intensity, a decrease in $\bar{\Lambda}_K$ leads to an increase in reactant leakage with a minimum $\bar{\Lambda}_K$ below which steady burning is not possible. For fixed $\bar{\Lambda}_K$, a decrease in m leads to increased reactant leakage and there exists a critical minimum m that corresponds to kinetic extinction. A qualitative analysis is shown below.

$$m \downarrow \Rightarrow r_f \downarrow \Rightarrow u_f \uparrow \Rightarrow \text{residence time} \downarrow (Y_{1,L} \uparrow)$$
$$\Rightarrow T_f \downarrow \Rightarrow \text{ leading to kinetic extinction}$$

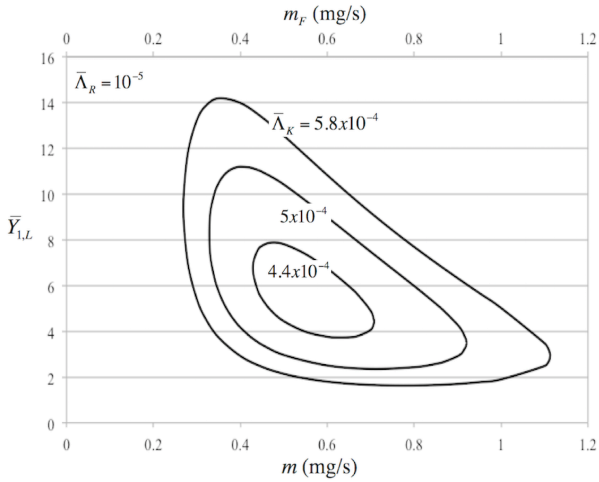


Figure 5: Reactant 1 leakage plotted against specified reduced kinetic Damköhler numbers, mass flow rate, and fuel consumption rate

It is known from previous analytical studies that reactant leakage monotonically decreases with increasing m when $\bar{\Lambda}_R = 0$ [5-7]. However, for fixed $\bar{\Lambda}_K$ values, the curves in Figure 5 display an increase in reactant leakage and flame extinction at critical maximum mass flow rates. This type of extinction is not present when there are no radiative losses and is therefore referred to as radiative extinction. Recall that Figure 3 displayed a decrease in flame temperature due to radiative heat loss as m increases given fixed non-zero radiation intensity. The combined results yield the following qualitative summary.

$$m \downarrow \Rightarrow r_f \downarrow \Rightarrow \begin{cases} u_f \uparrow \Rightarrow \text{residence time} \downarrow (Y_{1,L} \uparrow) \Rightarrow T_f \downarrow \Rightarrow \text{ leading to kinetic extinction} \\ V_f \downarrow \Rightarrow \text{radiative heat loss} \downarrow \Rightarrow T_f \uparrow \Rightarrow Y_{1,L} \downarrow \Rightarrow \text{harder to extinguish} \end{cases}$$

$$m \uparrow \Rightarrow r_f \uparrow \Rightarrow \begin{cases} u_f \downarrow \Rightarrow \text{residence time} \uparrow (Y_{1,L} \downarrow) \Rightarrow T_f \uparrow \Rightarrow \text{ harder to extinguish} \\ V_f \uparrow \Rightarrow \text{radiative heat loss} \uparrow \Rightarrow T_f \downarrow \Rightarrow Y_{1,L} \uparrow \Rightarrow \text{ leading to radiative extinction} \end{cases}$$

As m decreases, the flow velocity at the flame increases. The resulting decreased residence time yields an increase in reactant leakage and a decrease in flame temperature. For fixed $\bar{\Lambda}_R$, kinetic extinction is promoted when m decreases due to the decreased residence time, whereas an increase in m corresponds to an increase in radiative heat loss since radiation is a volumetric effect. Radiative heat loss dominates the effect of high residence time at high mass flow rates, leading to radiative extinction. Similarly, it can be shown for fixed $\bar{\Lambda}_K$ and varying $\bar{\Lambda}_R$, there exists a maximum $\bar{\Lambda}_R$ above which steady burning is not possible, indicating a flammability limit for the specified $\bar{\Lambda}_K$.

Section 4: Flammable Range

As discussed in Section 5, there exists a minimum $\bar{\Lambda}_K$ below which steady state burning is not possible for fixed $\bar{\Lambda}_R$. This minimum $\bar{\Lambda}_K$ represents the kinetic extinction limit for the specified $\bar{\Lambda}_R$. The kinetic extinction limits ($\bar{\Lambda}_{KE}$) for fixed $\bar{\Lambda}_R$ values are plotted against mass flow rate and fuel consumption rate in Figure 6. Steady burning is not possible in the regions below the curves. As m increases, increased flame volume results in greater radiative heat losses, promoting kinetic extinction. An increase in radiation intensity (and therefore $\bar{\Lambda}_R$) also results in greater radiative heat losses, and there is an increase in the kinetic extinction limits resulting from the corresponding drop in flame temperature.

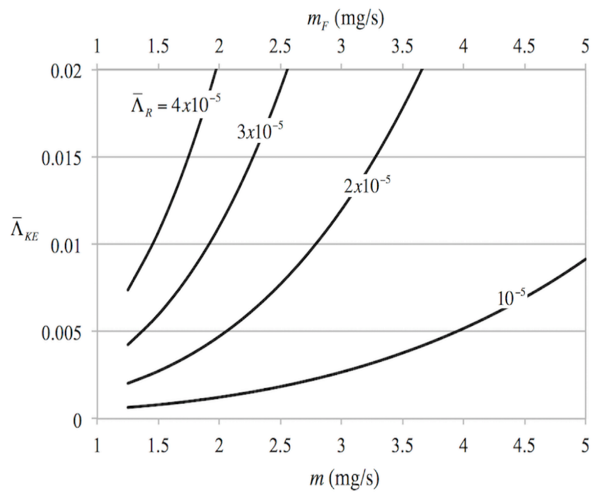
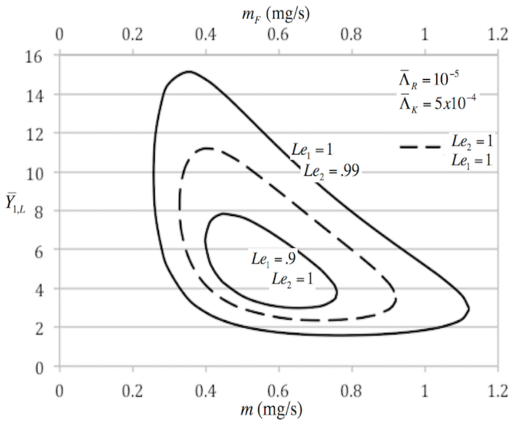


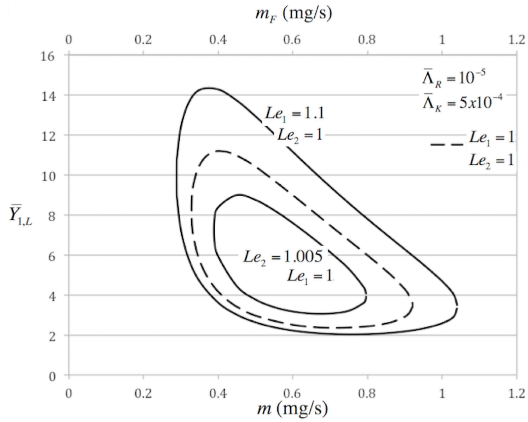
Figure 6: Kinetic extinction limit plotted against specified reduced radiative Damköhler numbers, mass flow rate and fuel consumption rate

Section 5: Lewis Numbers

A parametric study on the Lewis number was performed. Reactant 1 leakage was plotted against mass flow rate for fixed $\bar{\Lambda}_K$, fixed $\bar{\Lambda}_R$ and varying Le_1 and Le_2, shown in Figure 7. The Lewis number represents the ratio of thermal diffusivity to mass diffusivity. The dotted lines indicate the reference condition with both Lewis numbers at unity. Figure 7a displays conditions when mass diffusivity is greater than thermal diffusivity $(Le_i < 1)$. When $Le_2 < 1$, there is a higher amount of reactant 2 in the flame region, which causes the flame radius to decrease as it searches for more of reactant 1. The relationship between the flame radius and Le_2 is shown in (63c). As previously discussed, a decrease in flame radius leads to increases in both u_f and $Y_{1,L}$, causing a drop in flame temperature. However, Le_2 is inversely related to flame temperature, as can be qualitatively shown using (63) and (64). Therefore, a decrease in



(a)



(b)

Figure 7: Reactant 1 leakage plotted against mass flow rate and fuel consumption rate with fixed reduced radiative Damköhler number and varying Lewis numbers

Le_2 yields a direct increase in T_f that dominates the effect of a smaller flame radius. On the other hand, a decrease in Le_1 does not affect the flame radius. Stronger mass diffusion of reactant 1 causes reactant 1 to pass through the reaction region more quickly. This causes an increase in reactant leakage that makes the flame weaker, and flame extinction is promoted.

$$Le_2 \downarrow \Rightarrow \left\{ \begin{array}{l} D_2 \uparrow \Rightarrow r_f \downarrow \Rightarrow u_f \uparrow \Rightarrow Y_{1,L} \uparrow \Rightarrow T_f \downarrow \\ T_f \uparrow \text{ (dominant effect)} \end{array} \right\} \Rightarrow T_f \uparrow \quad \text{(stronger flame)}$$

$$Le_1 \downarrow \Rightarrow D_1 \uparrow \Rightarrow Y_{1,L} \uparrow \Rightarrow T_f \downarrow \quad \text{(weaker flame)}$$

Figure 7b shows that the opposite is true for each of the reactants when their respective Lewis numbers are greater than unity. Additionally, flame extinction is significantly more sensitive to variations in Le_2 than Le_1. This is because the reactants are $O(\varepsilon)$ quantities in the reaction region and a decrease in Le_1 only causes an $O(\varepsilon)$ drop in flame temperature, whereas temperature is directly affected by changes in Le_2. Therefore, since chemical reaction is modeled by high activation energy E_K, flame strength is more sensitive to changes in Le_2 than Le_1.

Section 6: Flame Stability

The trivial solution was the only solution found solving the final system of equations (65 and 68) in the stability analysis. This indicates that a neutral stability solution does not exist, so the flame is either absolutely stable or absolutely unstable. Since the steady state solution and other numerical and experimental papers [22, 25] show flame existence, it is expected that the flame is absolutely stable.

CHAPTER V. CONCLUDING REMARKS

The stability analysis using non-unity Lewis numbers performed on a spherical diffusion flame with linear perturbation due to wrinkling produced the trivial solution, indicating that a neutral stability solution does not exist. Therefore the flame is expected to be absolutely stable since the steady state solution showed flame existence. Numerical results in the steady state condition shown in sections 2, 3 and 4 of chapter IV are qualitatively the same as, but quantitatively different from those obtained by Wang et al. [29]. This was expected due to a nondimensionalization error in the former study. However, results in section 1 of chapter IV were unaffected by the nondimensionalization error and were identical in both studies. At low mass flow rates of the species issuing from the burner, kinetic extinction due to reactant leakage was observed. Increasing radiative heat losses promoted kinetic extinction. Increased mass flow rates resulted in increased residence times as well as increased radiative heat losses. At high mass flow rates, high radiative heat loss dominated high residence time and radiative extinction was observed. A parametric study on the Lewis number revealed that decreases in Le_2 and increases in Le_1 produced a stronger flame. Flame strength is more sensitive to changes in Le_2 than to changes in Le_1 due to high reaction activation energy.

APPENDIX A. ADDITIONAL EXPRESSIONS

$$A_{T2} = \frac{\Psi_2(1,1)}{\Psi_1^*(1,1)\Psi_2(1,1) - \Psi_1(1,1)\Psi_2^*(1,1)} \cdot$$

$$\left\{ \tilde{\lambda} \frac{A_{T1}[\Psi_1(1,\tilde{\lambda}^{-1})\Psi_2^*(1,\tilde{\lambda}^{-1}) - \Psi_1^*(1,\tilde{\lambda}^{-1})\Psi_2(1,\tilde{\lambda}^{-1})] + [\Psi_1^*(1,\tilde{\lambda}^{-1})\Psi_2(\tilde{r}_i,\tilde{\lambda}^{-1}) - \Psi_1(\tilde{r}_i,\tilde{\lambda}^{-1})\Psi_2^*(1,\tilde{\lambda}^{-1})]}{\Psi_1(1,\tilde{\lambda}^{-1})\Psi_2(\tilde{r}_i,\tilde{\lambda}^{-1}) - \Psi_1(\tilde{r}_i,\tilde{\lambda}^{-1})\Psi_2(1,\tilde{\lambda}^{-1})} - \frac{\Psi_2^*(1,1)}{\Psi_2(1,1)} \right\}$$

$$A_{T1} = \frac{\tilde{\lambda}[\Psi_1(\tilde{r}_i,1) + (-1)^n \Psi_2(\tilde{r}_i,1)][\Psi_1^*(\tilde{r}_i,\tilde{\lambda}^{-1})\Psi_2(\tilde{r}_i,\tilde{\lambda}^{-1}) - \Psi_1(\tilde{r}_i,\tilde{\lambda}^{-1})\Psi_2^*(\tilde{r}_i,\tilde{\lambda}^{-1})]}{\begin{array}{l}\{[\Psi_1^*(\tilde{r}_i,1) + (-1)^n \Psi_2^*(\tilde{r}_i,1)][\Psi_1(1,\tilde{\lambda}^{-1})\Psi_2(\tilde{r}_i,\tilde{\lambda}^{-1}) - \Psi_1(\tilde{r}_i,\tilde{\lambda}^{-1})\Psi_2(1,\tilde{\lambda}^{-1})] \\ \quad - \tilde{\lambda}[\Psi_1(\tilde{r}_i,1) + (-1)^n \Psi_2(\tilde{r}_i,1)][\Psi_1(1,\tilde{\lambda}^{-1})\Psi_2^*(\tilde{r}_i,\tilde{\lambda}^{-1}) - \Psi_1^*(\tilde{r}_i,\tilde{\lambda}^{-1})\Psi_2(1,\tilde{\lambda}^{-1})]\}\end{array}}$$

$$\Theta_{S,1}^- = \ell\text{n}\langle 1 - \{\Lambda_R / [2(g_0^-)^2]\}\exp(g_1^- + g_0^-\zeta)\rangle^2 - g_0^-\zeta - g_1^-$$

$$\Theta_{S,2}^- = (\tilde{T}_{b,S,A}/g_0^-)(d\Theta_{S,1}^-/d\zeta) = -\tilde{T}_{b,S,A}[2(g_0^-)^2 + \Lambda_R \exp(g_1^- + g_0^-\zeta)]/[2(g_0^-)^2 - \Lambda_R \exp(g_1^- + g_0^-\zeta)]$$

$$\Theta_{S,1}^+ = \ell\text{n}\langle 1 - \{\Lambda_R / [2(g_0^+)^2]\}\exp(g_1^+ - g_0^+\zeta)\rangle^2 + g_0^+\zeta - g_1^+ = 2\,\ell\text{n}\langle 1 - \{\Lambda_R / [2(g_0^+)^2]\}\exp(g_1^+ - g_0^+\zeta)\rangle + g_0^+\zeta - g_1^+$$

$$\Theta_{S,2}^+ = -(a_{T,A}^+/g_0^+)(d\Theta_{S,1}^+/d\zeta) = -a_{T,A}^+[2(g_0^+)^2 + \Lambda_R \exp(g_1^+ - g_0^+\zeta)]/[2(g_0^+)^2 - \Lambda_R \exp(g_1^+ - g_0^+\zeta)]$$

$$\tilde{\Theta}_1^- = \Theta_{S,1}^- - \bar{\delta} P_n \langle (\hat{T}_B^- + \hat{T}_A^-\zeta) + 2[\hat{T}_B^- - (2/g_0^-)\hat{T}_A^- + \hat{T}_A^-\zeta]/\{[2(g_0^-)^2/\Lambda_R]\exp[-(g_1^- + g_0^-\zeta)] - 1\}\rangle + O(\bar{\delta}^2)$$

$$\tilde{\Theta}_2^- = \Theta_{S,2}^- \left\langle 1 + \bar{\delta} P_n \left\{ \hat{T}_C^- + \frac{4(g_0^-)^2 \Lambda_R \exp(g_1^- + g_0^-\zeta)[\hat{T}_B^- - (2/g_0^-)\hat{T}_A^- + \hat{T}_A^-\zeta]}{[2(g_0^-)^2 + \Lambda_R \exp(g_1^- + g_0^-\zeta)][2(g_0^-)^2 - \Lambda_R \exp(g_1^- + g_0^-\zeta)]} \right\} + O(\bar{\delta}^2) \right\rangle$$

$$\tilde{\Theta}_1^+ = \Theta_{S,1}^+ - \bar{\delta} P_n \langle (\hat{T}_B^+ + \hat{T}_A^+\zeta) + 2[\hat{T}_B^+ + (2/g_0^+)\hat{T}_A^+ + \hat{T}_A^+\zeta]/\{[2(g_0^+)^2/\Lambda_R]\exp(g_0^+\zeta - g_1^+) - 1\}\rangle + O(\bar{\delta}^2)$$

$$\tilde{\Theta}_2^+ = \Theta_{S,2}^+ \left\langle 1 + \bar{\delta} P_n \left\{ \hat{T}_C^+ + \frac{4(g_0^+)^2 \Lambda_R \exp(g_1^+ - g_0^+\zeta)[\hat{T}_B^+ + (2/g_0^+)\hat{T}_A^+ + \hat{T}_A^+\zeta]}{[2(g_0^+)^2 + \Lambda_R \exp(g_1^+ - g_0^+\zeta)][2(g_0^+)^2 - \Lambda_R \exp(g_1^+ - g_0^+\zeta)]} \right\} + O(\bar{\delta}^2) \right\rangle$$

$$\begin{aligned}\tilde{\Theta}_3^\pm = \hat{g}_3^\pm \pm \left(\frac{\tilde{T}_E^\pm}{\hat{g}_0^\pm} - \frac{8}{3}\hat{g}_2^\pm \hat{g}_0^\pm \right) & \left\{ 2 + \frac{\sqrt{2\Lambda_R \exp(-\tilde{\Theta}_1^\pm) + (\hat{g}_0^\pm)^2}}{\hat{g}_0^\pm} \left\langle \ell\text{n}\left[\frac{2(\hat{g}_0^\pm)^2}{\Lambda_R} \frac{\sqrt{2\Lambda_R \exp(-\tilde{\Theta}_1^\pm) + (\hat{g}_0^\pm)^2} - \hat{g}_0^\pm}{\sqrt{2\Lambda_R \exp(-\tilde{\Theta}_1^\pm) + (\hat{g}_0^\pm)^2} + \hat{g}_0^\pm} \right] - 2 \right\rangle \right\} \\ & \pm \frac{\hat{g}_2^\pm}{2} \sqrt{2\Lambda_R \exp(-\tilde{\Theta}_1^\pm) + (\hat{g}_0^\pm)^2} \left\{ \left\langle \ell\text{n}\left[\frac{2(\hat{g}_0^\pm)^2}{\Lambda_R} \frac{\sqrt{2\Lambda_R \exp(-\tilde{\Theta}_1^\pm) + (\hat{g}_0^\pm)^2} - \hat{g}_0^\pm}{\sqrt{2\Lambda_R \exp(-\tilde{\Theta}_1^\pm) + (\hat{g}_0^\pm)^2} + \hat{g}_0^\pm} \right] \right\rangle^2 - (\tilde{\Theta}_1^\pm)^2 - \frac{4}{3}\tilde{\Theta}_1^\pm \right\} \\ & \pm 2\hat{g}_2^\pm \hat{g}_0^\pm \left\{ \ell\text{n}\left[\frac{2(\hat{g}_0^\pm)^2}{\Lambda_R} \frac{\sqrt{2\Lambda_R \exp(-\tilde{\Theta}_1^\pm) + (\hat{g}_0^\pm)^2} - \hat{g}_0^\pm}{\sqrt{2\Lambda_R \exp(-\tilde{\Theta}_1^\pm) + (\hat{g}_0^\pm)^2} + \hat{g}_0^\pm} \right] \right\} - \frac{\sqrt{2\Lambda_R \exp(-\tilde{\Theta}_1^\pm) + (\hat{g}_0^\pm)^2}}{\hat{g}_0^\pm} \left(\tilde{T}_D^\pm + \hat{g}_3^\pm \pm \frac{\hat{g}_1^\pm}{\hat{g}_0^\pm}\tilde{T}_E^\pm \right)\end{aligned}$$

$$g_1^- = -2\,\ell\text{n}\{[\langle 1 + [2\Lambda_R/(g_0^-)^2]\rangle^{1/2} + 1]/2\} \quad , \quad g_2^- = [2/(3\tilde{T}_f)](\tilde{T}_{b,S,A}/g_0^-)$$

$$g_1^+ = -2\,\ell\text{n}\{[\langle 1 + [2\Lambda_R/(g_0^+)^2]\rangle^{1/2} + 1]/2\} \quad , \quad g_2^+ = -[2/(3\tilde{T}_f)](a_{T,A}^+/g_0^+)$$

$$\hat{g}_0^- = g_0^- + \bar{\delta} P_n \hat{T}_A^- + O(\bar{\delta}^2) = g_0^-[1 + \bar{\delta} P_n(\hat{T}_A^-/g_0^-) + O(\bar{\delta}^2)] \quad , \quad \hat{g}_1^- = g_1^- + \bar{\delta} P_n \hat{T}_B^- + O(\bar{\delta}^2)$$

$$\hat{g}_2^- = g_2^-\{1 + \bar{\delta} P_n[\hat{T}_C^- - (\hat{T}_A^-/g_0^-)] + O(\bar{\delta}^2)\}$$

$$\hat{g}_0^+ = g_0^+ - \bar{\delta}\, P_n \hat{T}_A^+ + O(\bar{\delta}^2) = g_0^+[1 - \bar{\delta}\, P_n(\hat{T}_A^+ / g_0^+) + O(\bar{\delta}^2)] \quad , \quad \hat{g}_1^+ = g_1^+ + \bar{\delta}\, P_n \hat{T}_B^+ + O(\bar{\delta}^2)$$

$$\hat{g}_2^+ = g_2^+\{1 + \bar{\delta}\, P_n[\hat{T}_C^+ + (\hat{T}_A^+ / g_0^+)] + O(\bar{\delta}^2)\} \quad , \quad \hat{g}_3^\pm = g_3^+\langle 1 + \bar{\delta}\, P_n\{\hat{T}_C^+ + (\hat{T}_A^+ / g_0^+) - 2\hat{r}_{f,0}[(m - 2\tilde{r}_{f,S})^{-1} + \tilde{r}_{f,S}^{-1}]\} + O(\bar{\delta}^2)\rangle$$

$$\hat{T}_A^- = (d\hat{T}_0^- / d\tilde{r})_{\tilde{r}_{f,S}} + \hat{r}_{f,0}\, g_0^-[(\tilde{m} / \tilde{r}_{f,S}^2) - (2 / \tilde{r}_{f,S})] \quad , \quad \hat{T}_B^- = \hat{T}_2^-(\tilde{r}_{f,S}) + \hat{r}_{f,0}\, g_1^-(\tilde{m} / \tilde{r}_{f,S}^2)$$

$$\hat{T}_C^- = \{[\hat{T}_1^-(\tilde{r}_{f,S}) + \hat{r}_{f,1}\, g_0^-] / \tilde{T}_{b,S,A}\} + \hat{r}_{f,0}(\tilde{m} / \tilde{r}_{f,S}^2)$$

$$\tilde{T}_D^- = \tilde{T}_{S,3}^-(\tilde{r}_{f,S}) + \bar{\delta}\, P_n[\hat{T}_3^-(\tilde{r}_{f,S}) + \hat{r}_{f,0}(d\tilde{T}_{S,3}^- / d\tilde{r})_{\tilde{r}_{f,S}} + \hat{r}_{f,1}\, g_1^-(\tilde{m} / \tilde{r}_{f,S}^2)] + O(\bar{\delta}^2)$$

$$\tilde{T}_E^- = \tilde{T}_{b,S,A}(\tilde{m} / \tilde{r}_{f,S}^2) + \bar{\delta}\, P_n\{(d\hat{T}_1^- / d\tilde{r})_{\tilde{r}_{f,S}} + \hat{r}_{f,0}(d^2\tilde{T}_{S,1}^- / d\tilde{r}^2)_{\tilde{r}_{f,S}} + \hat{r}_{f,1}\, g_0^-[(\tilde{m} / \tilde{r}_{f,S}^2) - (2 / \tilde{r}_{f,S})]\} + O(\bar{\delta}^2)$$

$$\hat{T}_A^+ = (d\hat{T}_0^+ / d\tilde{r})_{\tilde{r}_{f,S}} - \hat{r}_{f,0}\, g_0^+[(\tilde{m} / \tilde{r}_{f,S}^2) - (2 / \tilde{r}_{f,S})] \quad , \quad \hat{T}_B^+ = \hat{T}_2^+(\tilde{r}_{f,S}) - \hat{r}_{f,0}\, g_1^+(\tilde{m} / \tilde{r}_{f,S}^2) / [\exp(\tilde{m} / \tilde{r}_{f,S}) - 1]$$

$$\hat{T}_C^+ = [\hat{T}_1^+(\tilde{r}_{f,S}) - a_{T,1}^+\, \hat{r}_{f,0}(\tilde{m} / \tilde{r}_{f,S}^2)\exp(-\tilde{m} / \tilde{r}_{f,S}) - \hat{r}_{f,1}\, g_0^+] / a_{T,A}^+$$

$$\tilde{T}_D^+ = \tilde{T}_{S,3}^+(\tilde{r}_{f,S}) + \bar{\delta}\, P_n\{[\hat{T}_3^+(\tilde{r}_{f,S}) + \hat{r}_{f,0}(d\tilde{T}_{S,3}^+ / d\tilde{r})_{\tilde{r}_{f,S}} - \hat{r}_{f,1}\, g_1^+(\tilde{m} / \tilde{r}_{f,S}^2) / [\exp(\tilde{m} / \tilde{r}_{f,S}) - 1]\} + O(\bar{\delta}^2)$$

$$\tilde{T}_E^+ = -a_{T,1}^+(\tilde{m} / \tilde{r}_{f,S}^2)\exp(-\tilde{m} / \tilde{r}_{f,S}) + \bar{\delta}\, P_n\{(d\hat{T}_1^+ / d\tilde{r})_{\tilde{r}_{f,S}} + \hat{r}_{f,0}(d^2\tilde{T}_{S,1}^+ / d\tilde{r}^2)_{\tilde{r}_{f,S}} - \hat{r}_{f,1}\, g_0^+[(\tilde{m} / \tilde{r}_{f,S}^2) - (2 / \tilde{r}_{f,S})]\} + O(\bar{\delta}^2)$$

$$A = \{[(g_0^-)^2 + 2\Lambda_R]^{1/2} - [(g_0^+)^2 + 2\Lambda_R]^{1/2}\} / 2 = [(g_0^-)^2 + 2\Lambda_R]^{1/2} - [\tilde{m} / (2\tilde{r}_{f,S}^2)] = [\tilde{m} / (2\tilde{r}_{f,S}^2)] - [(g_0^+)^2 + 2\Lambda_R]^{1/2}$$

$$B = (a_{1,1}^+ / 2)\{Le_{2,0}^{-1}[\tilde{Y}_{2,\infty} / (1 + \tilde{Y}_{2,\infty})] + Le_{1,0}^{-1}[1 - \exp(-Le_{1,0}\,\tilde{m} / \tilde{r}_{f,S})]\} - (Le_{2,1} / Le_{2,0})[\tilde{m} / (2\tilde{r}_{f,S})] - a_{1,1}^+[1 - \exp(-\tilde{m} / \tilde{r}_{f,S})]$$

$$D = [(g_0^-)^2 + 2\Lambda_R]^{1/2} - A = [(g_0^-)^2 + 2\Lambda_R]^{1/2} - \{[(g_0^-)^2 + 2\Lambda_R]^{1/2} - [\tilde{m} / (2\tilde{r}_{f,S}^2)]\} = \tilde{m} / (2\tilde{r}_{f,S}^2)$$

$$G = (a_{1,1}^+ / 2)\{Le_{2,0}^{-1}[\tilde{Y}_{2,\infty} / (1 + \tilde{Y}_{2,\infty})] - Le_{1,0}^{-1}[1 - \exp(-Le_{1,0}\,\tilde{m} / \tilde{r}_{f,S})]\} - (Le_{2,1} / Le_{2,0})[\tilde{m} / (2\tilde{r}_{f,S})]$$

$$\gamma = 1 - (2\tilde{r}_{f,S}^2 / \tilde{m})[(g_0^-)^2 + 2\Lambda_R]^{1/2} = (2\tilde{r}_{f,S}^2 / \tilde{m})[(g_0^+)^2 + 2\Lambda_R]^{1/2} - 1$$

$$\alpha = \Lambda_K(2\tilde{r}_{f,S}^2 / \tilde{m})^2\, Le_{1,0}\, Le_{2,0}\exp(\gamma G + B)$$

STEADY STATE BASIC SOLUTION

(A) Conservation Equations Boundary Conditions and Interface Conditions

For the steady state basic flame: $\partial/\partial\tilde{t} = \partial/\partial\tilde{z} = 0$, $\partial/\partial\tilde{r} = d/d\tilde{r}$, $\tilde{T} = \tilde{T}_S$, $\tilde{Y}_1 = \tilde{Y}_{1,S}$, $\tilde{Y}_2 = \tilde{Y}_{2,S}$

(a) Core region $(0 < \tilde{r} < \tilde{r}_i)$ $\tilde{Y}_{1,S} = 1$; $\tilde{Y}_{2,S} = 0$

$$\frac{\tilde{m}}{\tilde{r}^2}\frac{d\tilde{T}_S}{d\tilde{r}} - \frac{1}{\tilde{r}^2}\frac{d}{d\tilde{r}}\left(\tilde{r}^2\frac{d\tilde{T}_S}{d\tilde{r}}\right) = \frac{1}{\tilde{r}^2}\frac{d}{d\tilde{r}}\left(\tilde{m}\tilde{T}_S - \tilde{r}^2\frac{d\tilde{T}_S}{d\tilde{r}}\right) = 0 \quad \text{or} \quad \tilde{m}\tilde{T}_S - \tilde{r}^2\frac{d\tilde{T}_S}{d\tilde{r}} = \text{constant}$$

(b) Within the porous burner $(\tilde{r}_i < \tilde{r} < 1)$ $\tilde{Y}_{1,S} = 1$; $\tilde{Y}_{2,S} = 0$

$$\frac{\tilde{m}}{\tilde{r}^2}\frac{d\tilde{T}_S}{d\tilde{r}} - \frac{\tilde{\lambda}}{\tilde{r}^2}\frac{d}{d\tilde{r}}\left(\tilde{r}^2\frac{d\tilde{T}_S}{d\tilde{r}}\right) = \frac{1}{\tilde{r}^2}\frac{d}{d\tilde{r}}\left(\tilde{m}\tilde{T}_S - \tilde{\lambda}\tilde{r}^2\frac{d\tilde{T}_S}{d\tilde{r}}\right) = 0 \quad \text{or} \quad \tilde{m}\tilde{T}_S - \tilde{\lambda}\tilde{r}^2\frac{d\tilde{T}_S}{d\tilde{r}} = \text{constant}$$

(c) Gas region external of the burner $(1 < \tilde{r} < \infty)$

$$\frac{\tilde{m}}{\tilde{r}^2}\frac{d\tilde{T}_S}{d\tilde{r}} - \frac{1}{\tilde{r}^2}\frac{d}{d\tilde{r}}\left(\tilde{r}^2\frac{d\tilde{T}_S}{d\tilde{r}}\right) = \frac{1}{\tilde{r}^2}\frac{d}{d\tilde{r}}\left(\tilde{m}\tilde{T}_S - \tilde{r}^2\frac{d\tilde{T}_S}{d\tilde{r}}\right) = Da_K\,\tilde{Y}_{1,S}\tilde{Y}_{2,S}\exp(-\tilde{E}_K/\tilde{T}_S) - Da_R\exp(-\tilde{E}_R/\tilde{T}_S)$$

$$\frac{\tilde{m}}{\tilde{r}^2}\frac{d\tilde{Y}_{1,S}}{d\tilde{r}} - \frac{1}{Le_1\tilde{r}^2}\frac{d}{d\tilde{r}}\left(\tilde{r}^2\frac{d\tilde{Y}_{1,S}}{d\tilde{r}}\right) = \frac{1}{\tilde{r}^2}\frac{d}{d\tilde{r}}\left(\tilde{m}\tilde{Y}_{1,S} - \frac{\tilde{r}^2}{Le_1}\frac{d\tilde{Y}_{1,S}}{d\tilde{r}}\right) = -Da_K\,\tilde{Y}_{1,S}\tilde{Y}_{2,S}\exp(-\tilde{E}_K/\tilde{T}_S)$$

$$\frac{\tilde{m}}{\tilde{r}^2}\frac{d\tilde{Y}_{2,S}}{d\tilde{r}} - \frac{1}{Le_2\tilde{r}^2}\frac{d}{d\tilde{r}}\left(\tilde{r}^2\frac{d\tilde{Y}_{2,S}}{d\tilde{r}}\right) = \frac{1}{\tilde{r}^2}\frac{d}{d\tilde{r}}\left(\tilde{m}\tilde{Y}_{2,S} - \frac{\tilde{r}^2}{Le_2}\frac{d\tilde{Y}_{2,S}}{d\tilde{r}}\right) = -Da_K\,\tilde{Y}_{1,S}\tilde{Y}_{2,S}\exp(-\tilde{E}_K/\tilde{T}_S)$$

(d) Boundary and interface conditions

$\tilde{r} = 0$: $\tilde{T}_S = \tilde{T}_0$

$\tilde{r} = \tilde{r}_i$: $\tilde{T}_S = \tilde{T}_{i,S}$ (to be determined), $(d\tilde{T}_S/d\tilde{r})_{\tilde{r}_i^-} = \tilde{\lambda}(d\tilde{T}_S/d\tilde{r})_{\tilde{r}_i^+}$

$\tilde{r} = 1$: $\tilde{T}_S = \tilde{T}_{b,S}$ (to be determined), $\tilde{\lambda}(d\tilde{T}_S/d\tilde{r})_{1^-} = (d\tilde{T}_S/d\tilde{r})_{1^+}$

$$\tilde{m}\tilde{Y}_{1,S} - \frac{1}{Le_1}\frac{d\tilde{Y}_{1,S}}{d\tilde{r}} = \tilde{m}, \qquad \tilde{m}\tilde{Y}_{2,S} - \frac{1}{Le_2}\frac{d\tilde{Y}_{2,S}}{d\tilde{r}} = 0$$

$\tilde{r} \to \infty$: $\tilde{T}_S \to \tilde{T}_\infty$, $\tilde{Y}_{1,S} \to 0$, $\tilde{Y}_{2,S} \to \tilde{Y}_{2,\infty}$

(B) Solution of Temperature in the Core and Burner Regions

(a) Core region $(0 < \tilde{r} < \tilde{r}_i)$

$\tilde{m}\tilde{T}_S - \tilde{r}^2(d\tilde{T}_S/d\tilde{r}) = \tilde{m}c_1 = \text{constant}$; $\tilde{r} = 0$: $\tilde{T}_S = \tilde{T}_0$; $\tilde{r} = \tilde{r}_i$: $\tilde{T}_S = \tilde{T}_{i,S}$

$\tilde{m}(\tilde{T}_S - c_1) = \tilde{r}^2(d\tilde{T}_S/d\tilde{r}) = \tilde{r}^2[d(\tilde{T}_S - c_1)/d\tilde{r}]$ or $d(\tilde{T}_S - c_1)/(\tilde{T}_S - c_1) = \tilde{m}(d\tilde{r}/\tilde{r}^2)$

$\ell n(\tilde{T}_S - c_1) = -(\tilde{m}/\tilde{r}) + c_2'$ or $\tilde{T}_S = c_1 + c_2\exp(-\tilde{m}/\tilde{r})$

$\tilde{r} = 0$: $\tilde{T}_S = \tilde{T}_0$ $\therefore$ $c_1 = \tilde{T}_0$ Note: $\exp(-\tilde{m}/\tilde{r}) = \exp(-\tilde{m}/0) = \exp(-\infty) \to 0$

$\tilde{r} = \tilde{r}_i$: $\tilde{T}_S = \tilde{T}_{i,S}$ $\therefore$ $\tilde{T}_{i,S} = \tilde{T}_0 + c_2\exp(-\tilde{m}/\tilde{r}_i)$ or $c_2 = (\tilde{T}_{i,S} - \tilde{T}_0)\exp(\tilde{m}/\tilde{r}_i)$

$\tilde{T}_S = \tilde{T}_0 + (\tilde{T}_{i,S} - \tilde{T}_0)\exp[\tilde{m}(\tilde{r}_i^{-1} - \tilde{r}^{-1})]$

(b) Within the porous burner $(\tilde{r}_i < \tilde{r} < 1)$

$\tilde{m}\tilde{T}_S - \tilde{\lambda}\tilde{r}^2(d\tilde{T}_S/d\tilde{r}) = \tilde{m}c_1 = \text{constant;}$ $\tilde{r} = \tilde{r}_i$: $\tilde{T}_S = \tilde{T}_{i,S}$; $\tilde{r} = 1$: $\tilde{T}_S = \tilde{T}_{b,S}$

$\tilde{m}(\tilde{T}_S - c_1) = \tilde{\lambda}\tilde{r}^2(d\tilde{T}_S/d\tilde{r}) = \tilde{\lambda}\tilde{r}^2[d(\tilde{T}_S - c_1)/d\tilde{r}]$ or $d(\tilde{T}_S - c_1)/(\tilde{T}_S - c_1) = (\tilde{m}/\tilde{\lambda})(d\tilde{r}/\tilde{r}^2)$

$\ell n(\tilde{T}_S - c_1) = -\tilde{m}/(\tilde{\lambda}\tilde{r}) + c_2'$ or $\tilde{T}_S = c_1 + c_2\exp[-\tilde{m}/(\tilde{\lambda}\tilde{r})]$

$\tilde{r} = \tilde{r}_i$: $\tilde{T}_S = \tilde{T}_{i,S}$ $\therefore$ $\tilde{T}_{i,S} = c_1 + c_2\exp[-\tilde{m}/(\tilde{\lambda}\tilde{r}_i)]$; $\tilde{r} = 1$: $\tilde{T}_S = \tilde{T}_{b,S}$ $\therefore$ $\tilde{T}_{b,S} = c_1 + c_2\exp(-\tilde{m}/\tilde{\lambda})$

$$\Rightarrow \quad c_1 = \tilde{T}_{i,S} - (\tilde{T}_{b,S} - \tilde{T}_{i,S})\frac{\exp[-\tilde{m}/(\tilde{\lambda}\tilde{r}_i)]}{\exp(-\tilde{m}/\tilde{\lambda}) - \exp[-\tilde{m}/(\tilde{\lambda}\tilde{r}_i)]} \quad ; \quad c_2 = \frac{\tilde{T}_{b,S} - \tilde{T}_{i,S}}{\exp(-\tilde{m}/\tilde{\lambda}) - \exp[-\tilde{m}/(\tilde{\lambda}\tilde{r}_i)]}$$

$\tilde{T}_S = \tilde{T}_{i,S} + (\tilde{T}_{b,S} - \tilde{T}_{i,S})\{\exp[-\tilde{m}/(\tilde{\lambda}\tilde{r})] - \exp[-\tilde{m}/(\tilde{\lambda}\tilde{r}_i)]\}/\{\exp(-\tilde{m}/\tilde{\lambda}) - \exp[-\tilde{m}/(\tilde{\lambda}\tilde{r}_i)]\}$

(c) $\tilde{r} = \tilde{r}_i$: $(d\tilde{T}_S/d\tilde{r})_{\tilde{r}_i^-} = \tilde{\lambda}(d\tilde{T}_S/d\tilde{r})_{\tilde{r}_i^+}$

$$\{(\tilde{T}_{i,S}-\tilde{T}_0)(\tilde{m}/\tilde{r}^2)\exp[\tilde{m}(\tilde{r}_i^{-1}-\tilde{r}^{-1})]\}_{\tilde{r}=\tilde{r}_i}=\left\{\frac{\tilde{\lambda}(\tilde{T}_{b,S}-\tilde{T}_{i,S})[\tilde{m}/(\tilde{\lambda}\tilde{r}^2)]\exp[-\tilde{m}/(\tilde{\lambda}\tilde{r})]}{\exp(-\tilde{m}/\tilde{\lambda})-\exp[-\tilde{m}/(\tilde{\lambda}\tilde{r}_i)]}\right\}_{\tilde{r}=\tilde{r}_i}$$

$$(\tilde{T}_{i,S}-\tilde{T}_0)(\tilde{m}/\tilde{r}_i^2)\exp[\tilde{m}(\tilde{r}_i^{-1}-\tilde{r}_i^{-1})]=(\tilde{T}_{b,S}-\tilde{T}_{i,S})(\tilde{m}/\tilde{r}_i^2)\exp[-\tilde{m}/(\tilde{\lambda}\tilde{r}_i)]/\{\exp(-\tilde{m}/\tilde{\lambda})-\exp[-\tilde{m}/(\tilde{\lambda}\tilde{r}_i)]\}$$

$$(\tilde{T}_{i,S}-\tilde{T}_0)\{\exp(-\tilde{m}/\tilde{\lambda})-\exp[-\tilde{m}/(\tilde{\lambda}\tilde{r}_i)]\}=(\tilde{T}_{b,S}-\tilde{T}_{i,S})\exp[-\tilde{m}/(\tilde{\lambda}\tilde{r}_i)]$$

$$\tilde{T}_{i,S}\exp(-\tilde{m}/\tilde{\lambda})=\tilde{T}_0\{\exp(-\tilde{m}/\tilde{\lambda})-\exp[-\tilde{m}/(\tilde{\lambda}\tilde{r}_i)]\}+\tilde{T}_{b,S}\exp[-\tilde{m}/(\tilde{\lambda}\tilde{r}_i)]$$

$$\tilde{T}_{i,S}=\tilde{T}_0\{1-\exp[(\tilde{m}/\tilde{\lambda})(1-\tilde{r}_i^{-1})]\}+\tilde{T}_{b,S}\exp[(\tilde{m}/\tilde{\lambda})(1-\tilde{r}_i^{-1})]=\tilde{T}_0+(\tilde{T}_{b,S}-\tilde{T}_0)\exp[(\tilde{m}/\tilde{\lambda})(1-\tilde{r}_i^{-1})]$$

Core region: $\tilde{T}_S=\tilde{T}_0+(\tilde{T}_{b,S}-\tilde{T}_0)\exp[(\tilde{m}/\tilde{\lambda})(1-\tilde{r}_i^{-1})]\exp[\tilde{m}(\tilde{r}_i^{-1}-\tilde{r}^{-1})]$

Burner Region: $\tilde{T}_S=\tilde{T}_0+(\tilde{T}_{b,S}-\tilde{T}_0)\exp[(\tilde{m}/\tilde{\lambda})(1-\tilde{r}^{-1})]$

(C) Outer Solutions in the Gas Region

In the outer regions, chemical reaction and radiation are negligible because of the low gas temperature.

$$\frac{d}{d\tilde{r}}\left(\tilde{m}\tilde{T}_S^{\pm}-\tilde{r}^2\frac{d\tilde{T}_S^{\pm}}{d\tilde{r}}\right)=\frac{d}{d\tilde{r}}\left(\tilde{m}\tilde{Y}_{1,S}^{\pm}-\frac{\tilde{r}^2}{Le_1}\frac{d\tilde{Y}_{1,S}^{\pm}}{d\tilde{r}}\right)=\frac{d}{d\tilde{r}}\left(\tilde{m}\tilde{Y}_{2,S}^{\pm}-\frac{\tilde{r}^2}{Le_2}\frac{d\tilde{Y}_{2,S}^{\pm}}{d\tilde{r}}\right)=0$$

$\tilde{r}=1$: $\tilde{T}_S^-=\tilde{T}_{b,S}$, $\tilde{\lambda}(d\tilde{T}_S/d\tilde{r})_{1^-}=(d\tilde{T}_S/d\tilde{r})_{1^+}$, $\tilde{m}\tilde{Y}_{1,S}^- -(1/Le_1)(d\tilde{Y}_{1,S}^-/d\tilde{r})=\tilde{m}$,

$\tilde{m}\tilde{Y}_{2,S}^- -(1/Le_2)(d\tilde{Y}_{2,S}^-/d\tilde{r})=0$

$\tilde{r}\to\infty$: $\tilde{T}_S^+\to\tilde{T}_\infty$, $\tilde{Y}_{1,S}^+\to 0$, $\tilde{Y}_{2,S}^+\to\tilde{Y}_{2,\infty}$

– solutions between the burner and the flame ; + solutions out side of the flame

(1) $\frac{d}{d\tilde{r}}\left(\tilde{m}\tilde{Y}_{1,S}^{\pm}-\frac{\tilde{r}^2}{Le_1}\frac{d\tilde{Y}_{1,S}^{\pm}}{d\tilde{r}}\right)=0$ or $\Rightarrow$ $\tilde{m}\tilde{Y}_{1,S}^{\pm}-\frac{\tilde{r}^2}{Le_1}\frac{d\tilde{Y}_{1,S}^{\pm}}{d\tilde{r}}=\tilde{m}c_1^{\pm}=$ constant

$Le_1\tilde{m}(\tilde{Y}_{1,S}^{\pm}-c_1^{\pm})=\tilde{r}^2(d\tilde{Y}_{1,S}^{\pm}/d\tilde{r})=\tilde{r}^2[d(\tilde{Y}_{1,S}^{\pm}-c_1^{\pm})/d\tilde{r}]$ or $d(\tilde{Y}_{1,S}^{\pm}-c_1^{\pm})/(\tilde{Y}_{1,S}^{\pm}-c_1^{\pm})=Le_1\tilde{m}(d\tilde{r}/\tilde{r}^2)$

$\ell n(\tilde{Y}_{1,S}^{\pm}-c_1^{\pm})=-(Le_1\tilde{m}/\tilde{r})+\tilde{c}_2^{\pm}$ or $\tilde{Y}_{1,S}^{\pm}=c_1^{\pm}+c_2^{\pm}\exp(-Le_1\tilde{m}/\tilde{r})$

$\tilde{Y}_{1,S}^{\pm}=[\tilde{Y}_{1,S,0}^{\pm}+\varepsilon\tilde{Y}_{1,S,1}^{\pm}+O(\varepsilon^2)]+\delta[\tilde{Y}_{1,S,2}^{\pm}+O(\varepsilon)]+O(\delta^2)$; $c_1^{\pm}=[c_{1,0}^{\pm}+\varepsilon c_{1,1}^{\pm}+O(\varepsilon^2)]+\delta[c_{1,2}^{\pm}+O(\varepsilon)]+O(\delta^2)$

(Le_1 is independent of radiation)

$$\{\tilde{m}(\tilde{Y}_{1,S,0}^{\pm}+\varepsilon\tilde{Y}_{1,S,1}^{\pm})-\tilde{r}^2(1/Le_1)[(d\tilde{Y}_{1,S,0}^{\pm}/d\tilde{r})+\varepsilon(d\tilde{Y}_{1,S,1}^{\pm}/d\tilde{r})]+\cdots\}+O(\delta)$$
$$=\tilde{m}(c_{1,0}^{\pm}+\varepsilon c_{1,1}^{\pm}+\cdots)+O(\delta)$$

$(\tilde{r}^2/Le_1)(d\tilde{Y}_{1,S,0}^{\pm}/d\tilde{r})-\tilde{m}\tilde{Y}_{1,S,0}^{\pm}=-\tilde{m}c_{1,0}^{\pm}$; $(\tilde{r}^2/Le_1)(d^2\tilde{Y}_{1,S,0}^{\pm}/d\tilde{r}^2)-[\tilde{m}-(2\tilde{r}/Le_1)](d\tilde{Y}_{1,S,0}^{\pm}/d\tilde{r})=0$

$$(\tilde{r}^2/Le_1)(d^3\tilde{Y}_{1,S,0}^{\pm}/d\tilde{r}^3)-[\tilde{m}-(4\tilde{r}/Le_1)](d^2\tilde{Y}_{1,S,0}^{\pm}/d\tilde{r}^2)+(2/Le_1)](d\tilde{Y}_{1,S,0}^{\pm}/d\tilde{r})=0$$

$$(\tilde{r}^2/Le_1)(d^3\tilde{Y}_{1,S,0}^{\pm}/d\tilde{r}^3)-[\tilde{m}-(2\tilde{r}/Le_1)](d^2\tilde{Y}_{1,S,0}^{\pm}/d\tilde{r}^2)+[(2\tilde{m}/\tilde{r})-(2/Le_1)](d\tilde{Y}_{1,S,0}^{\pm}/d\tilde{r})=0$$

$$(1/Le_1)(d\tilde{Y}_{1,S,1}^{\pm}/d\tilde{r})-(\tilde{m}/\tilde{r}^2)\tilde{Y}_{1,S,1}^{\pm}=-c_{1,1}^{\pm}\tilde{m}/\tilde{r}^2$$

$$(1/Le_1)(d^2\tilde{Y}_{1,S,1}^{\pm}/d\tilde{r}^2)-(\tilde{m}/\tilde{r}^2)(d\tilde{Y}_{1,S,1}^{\pm}/d\tilde{r})+(2\tilde{m}/\tilde{r}^3)\tilde{Y}_{1,S,1}^{\pm}=2c_{1,1}^{\pm}\tilde{m}/\tilde{r}^3$$

At $\tilde{r}=1$: $\tilde{m}\tilde{Y}_{1,S}^- -(1/Le_1)(d\tilde{Y}_{1,S}^-/d\tilde{r})=\tilde{m}$ $\Rightarrow$ $c_1^-=1$ $\therefore$ $c_{1,0}^-=1$, $c_{1,1}^-=0$ and $\tilde{Y}_{1,S}^-=1+c_2^-\exp(-Le_1\tilde{m}/\tilde{r})$

$$(1/Le_1)(d\tilde{Y}_{1,S,1}^-/d\tilde{r})-(\tilde{m}/\tilde{r}^2)\tilde{Y}_{1,S,1}^-=0$$

$$(1/Le_1)(d^2\tilde{Y}_{1,S,1}^-/d\tilde{r}^2)-(\tilde{m}/\tilde{r}^2)(d\tilde{Y}_{1,S,1}^-/d\tilde{r})+(2\tilde{m}/\tilde{r}^3)\tilde{Y}_{1,S,1}^-=0$$

Expand c_2^- as: $c_2^-=-\{[a_{1,0}^-+\varepsilon a_{1,1}^-+O(\varepsilon^2)]+\delta[a_{1,2}^-+O(\varepsilon)]+O(\delta^2)\}$

$$\tilde{Y}_{1,S}^-=1+c_2^-\exp(-Le_1\tilde{m}/\tilde{r})=1-\{[a_{1,0}^-+\varepsilon a_{1,1}^-+O(\varepsilon^2)]+\delta[a_{1,2}^-+O(\varepsilon)]+O(\delta^2)\}\exp(-Le_1\tilde{m}/\tilde{r})$$
$$=1-\{[a_{1,0}^-+\varepsilon a_{1,1}^-+O(\varepsilon^2)]+\delta[a_{1,2}^-+O(\varepsilon)]+O(\delta^2)\}[1-O(\varepsilon)]\exp(-Le_1\tilde{m}/\tilde{r})$$
$$=\{[1-a_{1,0}^-\exp(-Le_1\tilde{m}/\tilde{r})]-\varepsilon a_{1,1}^-\exp(-Le_1\tilde{m}/\tilde{r})+O(\varepsilon^2)\}-\delta[a_{1,2}^-\exp(-Le_1\tilde{m}/\tilde{r})+O(\varepsilon)]+O(\delta^2)$$
$$=[\tilde{Y}_{1,S,0}^-+\varepsilon\tilde{Y}_{1,S,1}^-+O(\varepsilon^2)]+\delta[\tilde{Y}_{1,S,2}^-+O(\varepsilon)]+O(\delta^2)$$

$$\therefore \quad \tilde{Y}^-_{1,S,0} = 1 - a^-_{1,0}\exp(-Le_1\tilde{m}/\tilde{r}) \quad \Rightarrow \quad d\tilde{Y}^-_{1,S,0}/d\tilde{r} = -a^-_{1,0}(Le_1\tilde{m}/\tilde{r}^2)\exp(-Le_1\tilde{m}/\tilde{r})$$

$$\tilde{Y}^-_{1,S,1} = -a^-_{1,1}\exp(-Le_1\tilde{m}/\tilde{r}) \quad ; \quad \tilde{Y}^-_{1,S,2} = -a^-_{1,2}\exp(-Le_1\tilde{m}/\tilde{r})$$

As $\tilde{r}\to\infty$: $\tilde{Y}^+_{1,S}\to 0 \quad \Rightarrow \quad c^+_1 + c^+_2 = 0$ or $c^+_2 = -c^+_1 \quad \therefore \quad \tilde{Y}^+_{1,S} = c^+_1[1-\exp(-Le_1\tilde{m}/\tilde{r})]$

Let $c = [c^+_{1,0} + \varepsilon c^+_{1,1} + O(\varepsilon^2)] + \delta[c^+_{1,2} + O(\varepsilon)] + O(\delta^2) = [a^+_{1,0} + \varepsilon a^+_{1,1} + O(\varepsilon^2)] + \delta[a^+_{1,2} + O(\varepsilon)] + O(\delta^2)$

$$\tilde{Y}^+_{1,S} = c^+_1[1-\exp(-Le_1\tilde{m}/\tilde{r})] = \{[a^+_{1,0} + \varepsilon a^+_{1,1} + O(\varepsilon^2)] + \delta[a^+_{1,2} + O(\varepsilon)] + O(\delta^2)\}[1-\exp(-Le_1\tilde{m}/\tilde{r})]$$

$$= \{[a^+_{1,0} + \varepsilon a^+_{1,1} + O(\varepsilon^2)] + \delta[a^+_{1,2} + O(\varepsilon)] + O(\delta^2)\}\{1-[1-O(\varepsilon)]\exp(-Le_1\tilde{m}/\tilde{r})\}$$

$$= \{a^+_{1,0}[1-\exp(-Le_1\tilde{m}/\tilde{r})] + \varepsilon\langle a^+_{1,1}[1-\exp(-Le_1\tilde{m}/\tilde{r})]\rangle + O(\varepsilon^2)\}$$

$$+\delta\{[a^+_{1,2}[1-\exp(-Le_1\tilde{m}/\tilde{r})] + O(\varepsilon)\} + O(\delta^2)$$

$$= [\tilde{Y}^+_{1,S,0} + \varepsilon\tilde{Y}^+_{1,S,1} + O(\varepsilon^2)] + \delta[\tilde{Y}^+_{1,S,2} + O(\varepsilon)] + O(\delta^2)$$

$$\therefore \quad \tilde{Y}^+_{1,S,0} = a^+_{1,0}[1-\exp(-Le_1\tilde{m}/\tilde{r})] \quad ; \quad \tilde{Y}^+_{1,S,2} = a^+_{1,2}[1-\exp(-Le_1\tilde{m}/\tilde{r})]$$

$$\tilde{Y}^+_{1,S,1} = a^+_{1,1}[1-\exp(-Le_1\tilde{m}/\tilde{r})]$$

(2) $\dfrac{d}{d\tilde{r}}\left(\tilde{m}\tilde{Y}^\pm_{2,S} - \dfrac{\tilde{r}^2}{Le_2}\dfrac{d\tilde{Y}^\pm_{2,S}}{d\tilde{r}}\right) = 0 \quad \Rightarrow \quad \tilde{m}\tilde{Y}^\pm_{2,S} - \dfrac{\tilde{r}^2}{Le_2}\dfrac{d\tilde{Y}^\pm_{2,S}}{d\tilde{r}} = \tilde{m}c^\pm_1 = \text{constant}$

$Le_2\tilde{m}(\tilde{Y}^\pm_{2,S} - c^\pm_1) = \tilde{r}^2(d\tilde{Y}^\pm_{2,S}/d\tilde{r}) = \tilde{r}^2[d(\tilde{Y}^\pm_{2,S} - c^\pm_1)/d\tilde{r}]$ or $d(\tilde{Y}^\pm_{2,S} - c^\pm_1)/(\tilde{Y}^\pm_{2,S} - c^\pm_1) = Le_2\tilde{m}(d\tilde{r}/\tilde{r}^2)$

$\ell n(\tilde{Y}^\pm_{2,S} - c^\pm_1) = -(Le_2\tilde{m}/\tilde{r}) + \tilde{c}^\pm_2$ or $\tilde{Y}^\pm_{2,S} = c^\pm_1 + c^\pm_2\exp(-Le_2\tilde{m}/\tilde{r})$

$\tilde{Y}^\pm_{2,S} = [\tilde{Y}^\pm_{2,S,0} + \varepsilon\tilde{Y}^\pm_{2,S,1} + O(\varepsilon^2)] + \delta[\tilde{Y}^\pm_{2,S,2} + O(\varepsilon)] + O(\delta^2)$; $c^\pm_1 = [c^\pm_{1,0} + \varepsilon c^\pm_{1,1} + O(\varepsilon^2)] + \delta[c^\pm_{1,2} + O(\varepsilon)] + O(\delta^2)$

(Le_2 is independent of radiation)

$$\tilde{m}\{(\tilde{Y}^\pm_{2,S,0} + \varepsilon\tilde{Y}^\pm_{2,S,1}) - \tilde{r}^2(1/Le_2)[(d\tilde{Y}^\pm_{2,S,0}/d\tilde{r}) + \varepsilon(d\tilde{Y}^\pm_{2,S,1}/d\tilde{r})] + ..\} + O(\delta) = \tilde{m}(c^\pm_{1,0} + \varepsilon c^\pm_{1,1} + ..) + O(\delta)$$

$$(\tilde{r}^2/Le_2)(d\tilde{Y}^\pm_{2,S,0}/d\tilde{r}) - \tilde{m}\tilde{Y}^\pm_{2,S,0} = -\tilde{m}c^\pm_{1,0} \quad ; \quad (\tilde{r}^2/Le_2)(d^2\tilde{Y}^\pm_{2,S,0}/d\tilde{r}^2) - [\tilde{m} - (2\tilde{r}/Le_2)](d\tilde{Y}^\pm_{2,S,0}/d\tilde{r}) = 0$$

$$(\tilde{r}^2/Le_2)(d^3\tilde{Y}^\pm_{2,S,0}/d\tilde{r}^3) - [\tilde{m} - (4\tilde{r}/Le_2)](d^2\tilde{Y}^\pm_{2,S,0}/d\tilde{r}^2) + (2/Le_2)](d\tilde{Y}^\pm_{2,S,0}/d\tilde{r}) = 0$$

$$(\tilde{r}^2/Le_2)(d^3\tilde{Y}^\pm_{2,S,0}/d\tilde{r}^3) - [\tilde{m} - (2\tilde{r}/Le_2)](d^2\tilde{Y}^\pm_{2,S,0}/d\tilde{r}^2) + [(2\tilde{m}/\tilde{r}) - (2/Le_2)](d\tilde{Y}^\pm_{2,S,0}/d\tilde{r}) = 0$$

$$(1/Le_2)(d\tilde{Y}^\pm_{2,S,1}/d\tilde{r}) - (\tilde{m}/\tilde{r}^2)\tilde{Y}^\pm_{2,S,1} = -c^\pm_{1,1}\tilde{m}/\tilde{r}^2$$

$$(1/Le_2)(d^2\tilde{Y}^\pm_{2,S,1}/d\tilde{r}^2) - (\tilde{m}/\tilde{r}^2)(d\tilde{Y}^\pm_{2,S,1}/d\tilde{r}) + (2\tilde{m}/\tilde{r}^3)\tilde{Y}^\pm_{2,S,1} = 2c^\pm_{1,1}\tilde{m}/\tilde{r}^3$$

At $\tilde{r} = 1$: $\tilde{m}\tilde{Y}^-_{2,S} - (1/Le_2)(d\tilde{Y}^-_{2,S}/d\tilde{r}) = 0 \quad \Rightarrow \quad c^-_1 = 0 \quad \therefore \quad c^-_{1,1} = 0$ and $\tilde{Y}^-_{2,S} = c^-_2\exp(-Le_2\tilde{m}/\tilde{r})$

$$(1/Le_2)(d\tilde{Y}^-_{2,S,1}/d\tilde{r}) - (\tilde{m}/\tilde{r}^2)\tilde{Y}^-_{2,S,1} = 0$$

$$(1/Le_2)(d^2\tilde{Y}^-_{2,S,1}/d\tilde{r}^2) - (\tilde{m}/\tilde{r}^2)(d\tilde{Y}^-_{2,S,1}/d\tilde{r}) + (2\tilde{m}/\tilde{r}^3)\tilde{Y}^-_{2,S,1} = 0$$

Expand c^-_2 as: $c^-_2 = [a^-_{2,0} + \varepsilon a^-_{2,1} + O(\varepsilon^2)] + \delta[a^-_{2,2} + O(\varepsilon)] + O(\delta^2)$

$$\tilde{Y}^-_{2,S} = c^-_2\exp(-Le_2\tilde{m}/\tilde{r}) = \{[a^-_{2,0} + \varepsilon a^-_{2,1} + O(\varepsilon^2)] + \delta[a^-_{2,2} + O(\varepsilon)] + O(\delta^2)\}\exp(-Le_2\tilde{m}/\tilde{r})$$

$$= \{[a^-_{2,0} + \varepsilon a^-_{2,1} + O(\varepsilon^2)] + \delta[a^-_{2,2} + O(\varepsilon)] + O(\delta^2)\}[1 + O(\varepsilon^2)]\exp(-Le_2\tilde{m}/\tilde{r})$$

$$= \{a^-_{2,0}\exp(-Le_2\tilde{m}/\tilde{r}) + \varepsilon a^-_{2,1}\exp(-Le_2\tilde{m}/\tilde{r}) + O(\varepsilon^2)\} + \delta[a^-_{2,2}\exp(-Le_2\tilde{m}/\tilde{r}) + O(\varepsilon)] + O(\delta^2)$$

$$= [\tilde{Y}^-_{2,S,0} + \varepsilon\tilde{Y}^-_{2,S,1} + O(\varepsilon^2)] + \delta[\tilde{Y}^-_{2,S,2} + \varepsilon\tilde{Y}^-_{2,S,3} + O(\varepsilon^2)] + O(\delta^2)$$

$$\therefore \tilde{Y}^-_{2,S,0} = a^-_{2,0}\exp(-Le_2\tilde{m}/\tilde{r}) \quad ; \quad \tilde{Y}^-_{2,S,1} = a^-_{2,1}\exp(-Le_2\tilde{m}/\tilde{r}) \quad ; \quad \tilde{Y}^-_{2,S,2} = a^-_{2,2}\exp(-Le_2\tilde{m}/\tilde{r})$$

As $\tilde{r}\to\infty$: $\tilde{Y}^+_{2,S}\to\tilde{Y}_{2,\infty} \quad \Rightarrow \quad c^+_1 + c^+_2 = \tilde{Y}_{2,\infty}$ or $c^+_1 = \tilde{Y}_{2,\infty} - c^+_2 \quad \therefore \quad \tilde{Y}^+_{2,S} = \tilde{Y}_{2,\infty} - c^+_2[1-\exp(-Le_2\tilde{m}/\tilde{r})]$

Expand c^+_2 as: $c^+_2 = [a^+_{2,0} + \varepsilon a^+_{2,1} + O(\varepsilon^2)] + \delta[a^+_{2,2} + O(\varepsilon)] + O(\delta^2)$

$$c^+_1 = [c^+_{1,0} + \varepsilon c^+_{1,1} + O(\varepsilon^2)] + O(\delta) = \tilde{Y}_{2,\infty} - c^+_2 = \tilde{Y}_{2,\infty} - [a^+_{2,0} + \varepsilon a^+_{2,1} + O(\varepsilon^2)] + O(\delta) \quad \therefore \quad c^+_{1,1} = -a^+_{2,1}$$

$$(1/Le_2)(d\tilde{Y}^+_{2,S,1}/d\tilde{r}) - (\tilde{m}/\tilde{r}^2)\tilde{Y}^\pm_{2,S,1} = a^+_{2,1}\tilde{m}/\tilde{r}^2$$

$$(1/Le_2)(d^2\tilde{Y}^+_{2,S,1}/d\tilde{r}^2)-(\tilde{m}/\tilde{r}^2)(d\tilde{Y}^+_{2,S,1}/d\tilde{r})+(2\tilde{m}/\tilde{r}^3)\tilde{Y}^+_{2,S,1}=-2a^+_{2,1}\tilde{m}/\tilde{r}^3$$

$$\tilde{Y}^+_{2,S}=\tilde{Y}_{2,\infty}-c^+_2[1-\exp(-Le_2\tilde{m}/\tilde{r})]=\tilde{Y}_{2,\infty}-\{[a^+_{2,0}+\varepsilon a^+_{2,1}+O(\varepsilon^2)]+\delta[a^+_{2,2}+O(\varepsilon)]+O(\delta^2)\}[1-\exp(-Le_2\tilde{m}/\tilde{r})]$$

$$=\tilde{Y}_{2,\infty}-\{[a^+_{2,0}+\varepsilon a^+_{2,1}+O(\varepsilon^2)]+\delta[a^+_{2,2}+O(\varepsilon)]+O(\delta^2)\}\{1-[1+O(\varepsilon^2)]\exp(-Le_2\tilde{m}/\tilde{r})\}$$

$$=\tilde{Y}_{2,\infty}-\langle a^+_{2,0}[1-\exp(-Le_2\tilde{m}/\tilde{r})]+\varepsilon\{a^+_{2,1}[1-\exp(-Le_2\tilde{m}/\tilde{r})]\}+O(\varepsilon^2)\rangle$$

$$-\delta\{a^+_{2,2}[1-\exp(-Le_2\tilde{m}/\tilde{r})]+O(\varepsilon)\}+O(\delta^2)$$

$$=[\tilde{Y}^+_{2,S,0}+\varepsilon\tilde{Y}^+_{2,S,1}+O(\varepsilon^2)]+\delta[\tilde{Y}^+_{2,S,2}+O(\varepsilon)]+O(\delta^2)$$

$$\therefore\quad \tilde{Y}^+_{2,S,0}=\tilde{Y}_{2,\infty}-a^+_{2,0}[1-\exp(-Le_2\tilde{m}/\tilde{r})]\quad\Rightarrow\quad d\tilde{Y}^+_{2,S,0}/d\tilde{r}=a^+_{2,0}(Le_2\tilde{m}/\tilde{r}^2)\exp(-Le_2\tilde{m}/\tilde{r})$$

$$\tilde{Y}^+_{2,S,1}=-a^+_{2,1}[1-\exp(-Le_2\tilde{m}/\tilde{r})]\quad;\quad \tilde{Y}^+_{2,S,2}=-a^+_{2,2}[1-\exp(-Le_2\tilde{m}/\tilde{r})]$$

(3) $$\frac{d}{d\tilde{r}}\left(\tilde{m}\tilde{T}^\pm_S-\tilde{r}^2\frac{d\tilde{T}^\pm_S}{d\tilde{r}}\right)=0\quad\Rightarrow\quad \tilde{m}\tilde{T}^\pm_S-\tilde{r}^2\frac{d\tilde{T}^\pm_S}{d\tilde{r}}=\tilde{m}c^\pm_1=\text{constant}$$

$$\tilde{m}(\tilde{T}^\pm_S-c^\pm_1)=\tilde{r}^2(d\tilde{T}^\pm_S/d\tilde{r})=\tilde{r}^2[d(\tilde{T}^\pm_S-c^\pm_1)/d\tilde{r}]\quad\text{or}\quad d(\tilde{T}^\pm_S-c^\pm_1)/(\tilde{T}^\pm_S-c^\pm_1)=\tilde{m}(d\tilde{r}/\tilde{r}^2)$$

$$\Rightarrow\quad \ell n(\tilde{T}^\pm_S-c^\pm_1)=-(\tilde{m}/\tilde{r})+\tilde{c}^\pm_2\quad\text{or}\quad \tilde{T}^\pm_S=c^\pm_1+c^\pm_2\exp(-\tilde{m}/\tilde{r})$$

At $\tilde{r}=1$: $\tilde{\lambda}(d\tilde{T}_S/d\tilde{r})_{1^-}=(d\tilde{T}_S/d\tilde{r})_{1^+}$

$$\{\tilde{\lambda}(\tilde{T}_{b,S}-\tilde{T}_0)[\tilde{m}/(\tilde{\lambda}\tilde{r}^2)]\exp[(\tilde{m}/\tilde{\lambda})(1-\tilde{r}^{-1})]\}_{\tilde{r}=1}=[c(\tilde{m}/\tilde{r}^2)\exp(-\tilde{m}/\tilde{r})]_{\tilde{r}=1}$$

$$\therefore\quad(\tilde{T}_{b,S}-\tilde{T}_0)\tilde{m}=c^-_2\tilde{m}\exp(-\tilde{m})\quad\text{or}\quad c^-_2=(\tilde{T}_{b,S}-\tilde{T}_0)\exp(\tilde{m})\quad\Rightarrow\quad \tilde{T}^-_S=c^-_1+(\tilde{T}_{b,S}-\tilde{T}_0)\exp[\tilde{m}(1-\tilde{r}^{-1})]$$

$$\tilde{T}^-_S=\tilde{T}_{b,S}\quad\therefore\quad \tilde{T}_{b,S}=c^-_1+\{(\tilde{T}_{b,S}-\tilde{T}_0)\exp[\tilde{m}(1-\tilde{r}^{-1})]\}_{\tilde{r}=1}=c^-_1+(\tilde{T}_{b,S}-\tilde{T}_0)\quad\text{or}\quad c^-_1=\tilde{T}_0$$

$$\Rightarrow\quad \tilde{T}^-_S=\tilde{T}_0+(\tilde{T}_{b,S}-\tilde{T}_0)\exp[\tilde{m}(1-\tilde{r}^{-1})]$$

As $\tilde{r}\to\infty$: $\tilde{T}^+_S\to\tilde{T}_\infty\quad\Rightarrow\quad c^+_1+c^+_2=\tilde{T}_\infty$ or $c^+_1=\tilde{T}_\infty-c^+_2\quad\therefore\quad \tilde{T}^+_S=\tilde{T}_\infty-c^+_2[1-\exp(-\tilde{m}/\tilde{r})]$

Expand $\tilde{T}_{b,S}$ and c^+_2 as

$$\tilde{T}_{b,S}=[\tilde{T}_{b,S,0}+\varepsilon\tilde{T}_{b,S,1}+O(\varepsilon^2)]+\delta[\tilde{T}_{b,S,2}+O(\varepsilon)]+O(\delta^2)$$

$$c^+_2=-\{[a^+_{T,0}+\varepsilon a^+_{T,1}+O(\varepsilon^2)]+\delta[a^+_{T,2}+O(\varepsilon)]+O(\delta^2)\}$$

$$\tilde{T}^-_S=\tilde{T}_0+(\tilde{T}_{b,S}-\tilde{T}_0)\exp[\tilde{m}(1-\tilde{r}^{-1})]=\tilde{T}_0+\{[\langle\tilde{T}_{b,S,0}+\varepsilon\tilde{T}_{b,S,1}+O(\varepsilon^2)\rangle+\delta\langle\tilde{T}_{b,S,2}+O(\varepsilon)\rangle+O(\delta^2)]-\tilde{T}_0\}\exp[\tilde{m}(1-\tilde{r}^{-1})]$$

$$=[\tilde{T}^-_{S,0}+\varepsilon\tilde{T}^-_{S,1}+O(\varepsilon^2)]+\delta[\tilde{T}^-_{S,2}+O(\varepsilon)]+O(\delta^2)$$

$$\therefore\quad \tilde{T}^-_{S,0}=\tilde{T}_0+(\tilde{T}_{b,S,0}-\tilde{T}_0)\exp[\tilde{m}(1-\tilde{r}^{-1})]\quad\Rightarrow\quad d\tilde{T}^-_{S,0}/d\tilde{r}=(\tilde{T}_{b,S,0}-\tilde{T}_0)(\tilde{m}/\tilde{r}^2)\exp[\tilde{m}(1-\tilde{r}^{-1})]$$

$$\tilde{T}^-_{S,1}=\tilde{T}_{b,S,1}\exp[\tilde{m}(1-\tilde{r}^{-1})]\quad\Rightarrow\quad d\tilde{T}^-_{S,1}/d\tilde{r}=\tilde{T}_{b,S,1}(\tilde{m}/\tilde{r}^2)\exp[\tilde{m}(1-\tilde{r}^{-1})]$$

$$\tilde{T}^-_{S,2}=\tilde{T}_{b,S,2}\exp[\tilde{m}(1-\tilde{r}^{-1})]$$

$$\tilde{T}^+_S=\tilde{T}_\infty-c^+_2[1-\exp(-\tilde{m}/\tilde{r})]=\tilde{T}_\infty+\{[a^+_{T,0}+\varepsilon a^+_{T,1}+O(\varepsilon^2)]+\delta[a^+_{T,2}+O(\varepsilon)]+O(\delta^2)\}[1-\exp(-\tilde{m}/\tilde{r})]$$

$$=[\tilde{T}^+_{S,0}+\varepsilon\tilde{T}^+_{S,1}+O(\varepsilon^2)]+\delta[\tilde{T}^+_{S,2}+O(\varepsilon)]+O(\delta^2)$$

$$\therefore\quad \tilde{T}^+_{S,0}=\tilde{T}_\infty+a^+_{T,0}[1-\exp(-\tilde{m}/\tilde{r})]\quad\Rightarrow\quad d\tilde{T}^+_{S,0}/d\tilde{r}=-a^+_{T,0}(\tilde{m}/\tilde{r}^2)\exp(-\tilde{m}/\tilde{r})$$

$$\tilde{T}^+_{S,1}=a^+_{T,1}[1-\exp(-\tilde{m}/\tilde{r})]\quad\Rightarrow\quad d\tilde{T}^+_{S,1}/d\tilde{r}=-a^+_{T,1}(\tilde{m}/\tilde{r}^2)\exp(-\tilde{m}/\tilde{r})$$

$$\tilde{T}^+_{S,2}=a^+_{T,2}[1-\exp(-\tilde{m}/\tilde{r})]$$

(4) Summary

$\tilde{T}_S^- = [\tilde{T}_{S,0}^- + \varepsilon \tilde{T}_{S,1}^- + O(\varepsilon^2)] + \delta[\tilde{T}_{S,2}^- + O(\varepsilon)] + O(\delta^2)$

$\tilde{T}_{S,0}^- = \tilde{T}_0 + (\tilde{T}_{b,S,0} - \tilde{T}_0)\exp[\tilde{m}(1-\tilde{r}^{-1})]$, $d\tilde{T}_{S,0}^- / dr = (\tilde{T}_{b,S,0} - \tilde{T}_0)(\tilde{m}/\tilde{r}^2)\exp[\tilde{m}(1-\tilde{r}^{-1})]$

$\tilde{T}_{S,1}^- = \tilde{T}_{b,S,1}\exp[\tilde{m}(1-\tilde{r}^{-1})]$, $d\tilde{T}_{S,1}^- / d\tilde{r} = \tilde{T}_{b,S,1}(\tilde{m}/\tilde{r}^2)\exp[\tilde{m}(1-\tilde{r}^{-1})]$

$\tilde{T}_{S,2}^- = \tilde{T}_{b,S,2}\exp[\tilde{m}(1-\tilde{r}^{-1})]$

$\tilde{T}_S^+ = [\tilde{T}_{S,0}^+ + \varepsilon \tilde{T}_{S,1}^+ + O(\varepsilon^2)] + \delta[\tilde{T}_{S,2}^+ + O(\varepsilon)] + O(\delta^2)$

$\tilde{T}_{S,0}^+ = \tilde{T}_\infty + a_{T,0}^+[1-\exp(-\tilde{m}/\tilde{r})]$, $d\tilde{T}_{S,0}^+ / d\tilde{r} = -a_{T,0}^+(\tilde{m}/\tilde{r}^2)\exp(-\tilde{m}/\tilde{r})$

$\tilde{T}_{S,1}^+ = a_{T,1}^+[1-\exp(-\tilde{m}/\tilde{r})]$, $d\tilde{T}_{S,1}^+ / d\tilde{r} = -a_{T,1}^+(\tilde{m}/\tilde{r}^2)\exp(-\tilde{m}/\tilde{r})$

$\tilde{T}_{S,2}^+ = a_{T,2}^+[1-\exp(-\tilde{m}/\tilde{r})]$

$\tilde{Y}_{1,S}^- = [\tilde{Y}_{1,S,0}^- + \varepsilon \tilde{Y}_{1,S,1}^- + O(\varepsilon^2)] + \delta[\tilde{Y}_{1,S,2}^- + O(\varepsilon)] + O(\delta^2)$

$\tilde{Y}_{1,S,0}^- = 1 - a_{1,0}^-\exp(-Le_1\tilde{m}/\tilde{r})$, $d\tilde{Y}_{1,S,0}^- / d\tilde{r} = -a_{1,0}^-(Le_1\tilde{m}/\tilde{r}^2)\exp(-Le_1\tilde{m}/\tilde{r})$

$\tilde{Y}_{1,S,1}^- = -a_{1,1}^-\exp(-Le_1\tilde{m}/\tilde{r})$, $\tilde{Y}_{1,S,2}^- = -a_{1,2}^-\exp(-Le_1\tilde{m}/\tilde{r})$

$(\tilde{r}^2/Le_1)(d^2\tilde{Y}_{1,S,0}^-/d\tilde{r}^2) - [\tilde{m} - (2\tilde{r}/Le_1)](d\tilde{Y}_{1,S,0}^-/d\tilde{r}) = 0$

$(\tilde{r}^2/Le_1)(d^3\tilde{Y}_{1,S,0}^-/d\tilde{r}^3) - [\tilde{m} - (2\tilde{r}/Le_1)](d^2\tilde{Y}_{1,S,0}^-/d\tilde{r}^2) + [(2\tilde{m}/\tilde{r}) - (2/Le_1)](d\tilde{Y}_{1,S,0}^-/d\tilde{r}) = 0$

$(1/Le_1)(d\tilde{Y}_{1,S,1}^-/d\tilde{r}) - (\tilde{m}/\tilde{r}^2)\tilde{Y}_{1,S,1}^- = 0$

$(1/Le_1)(d^2\tilde{Y}_{1,S,1}^-/d\tilde{r}^2) - (\tilde{m}/\tilde{r}^2)(d\tilde{Y}_{1,S,1}^-/d\tilde{r}) + (2\tilde{m}/\tilde{r}^3)\tilde{Y}_{1,S,1}^- = 0$

$\tilde{Y}_{1,S}^+ = [\tilde{Y}_{1,S,0}^+ + \varepsilon \tilde{Y}_{1,S,1}^+ + O(\varepsilon^2)] + \delta[\tilde{Y}_{1,S,2}^+ + O(\varepsilon)] + O(\delta^2)$

$\tilde{Y}_{1,S,0}^+ = a_{1,0}^+[1-\exp(-Le_1\tilde{m}/\tilde{r})]$

$\tilde{Y}_{1,S,1}^+ = a_{1,1}^+[1-\exp(-Le_1\tilde{m}/\tilde{r})]$

$\tilde{Y}_{1,S,2}^+ = a_{1,2}^+[1-\exp(-Le_1\tilde{m}/\tilde{r})]$

$(1/Le_1)(d\tilde{Y}_{1,S,1}^+/d\tilde{r}) - (\tilde{m}/\tilde{r}^2)\tilde{Y}_{1,S,1}^+ = -\tilde{m}a_{1,1}^+/\tilde{r}^2$

$(1/Le_1)(d^2\tilde{Y}_{1,S,1}^+/d\tilde{r}^2) - (\tilde{m}/\tilde{r}^2)(d\tilde{Y}_{1,S,1}^+/d\tilde{r}) + (2\tilde{m}/\tilde{r}^3)\tilde{Y}_{1,S,1}^+ = 2\tilde{m}a_{1,1}^+/\tilde{r}^3$

$\tilde{Y}_{2,S}^- = [\tilde{Y}_{2,S,0}^- + \varepsilon \tilde{Y}_{2,S,1}^- + O(\varepsilon^2)] + \delta[\tilde{Y}_{2,S,2}^- + O(\varepsilon)] + O(\delta^2)$

$\tilde{Y}_{2,S,0}^- = a_{2,0}^-\exp(-Le_2\tilde{m}/\tilde{r})$

$\tilde{Y}_{2,S,1}^- = a_{2,1}^-\exp(-Le_2\tilde{m}/\tilde{r})$

$\tilde{Y}_{2,S,2}^- = a_{2,2}^-\exp(-Le_2\tilde{m}/\tilde{r})$

$(1/Le_2)(d\tilde{Y}_{2,S,1}^-/d\tilde{r}) - (\tilde{m}/\tilde{r}^2)\tilde{Y}_{2,S,1}^- = 0$

$(1/Le_2)(d^2\tilde{Y}_{2,S,1}^-/d\tilde{r}^2) - (\tilde{m}/\tilde{r}^2)(d\tilde{Y}_{2,S,1}^-/d\tilde{r}) + (2\tilde{m}/\tilde{r}^3)\tilde{Y}_{2,S,1}^- = 0$

$\tilde{Y}_{2,S}^+ = [\tilde{Y}_{2,S,0}^+ + \varepsilon \tilde{Y}_{2,S,1}^+ + O(\varepsilon^2)] + \delta[\tilde{Y}_{2,S,2}^+ + O(\varepsilon)] + O(\delta^2)$

$\tilde{Y}_{2,S,0}^+ = \tilde{Y}_{2,\infty} - a_{2,0}^+[1-\exp(-Le_2\tilde{m}/\tilde{r})]$, $d\tilde{Y}_{2,S,0}^+/d\tilde{r} = a_{2,0}^+(Le_2\tilde{m}/\tilde{r}^2)\exp(-Le_2\tilde{m}/\tilde{r})$

$\tilde{Y}_{2,S,1}^+ = -a_{2,1}^+[1-\exp(-Le_2\tilde{m}/\tilde{r})]$, $\tilde{Y}_{2,S,2}^+ = -a_{2,2}^+[1-\exp(-Le_2\tilde{m}/\tilde{r})]$

$(\tilde{r}^2/Le_2)(d^2\tilde{Y}_{2,S,0}^+/d\tilde{r}^2) - [\tilde{m} - (2\tilde{r}/Le_2)](d\tilde{Y}_{2,S,0}^+/d\tilde{r}) = 0$

$(\tilde{r}^2/Le_2)(d^3\tilde{Y}_{2,S,0}^+/d\tilde{r}^3) - [\tilde{m} - (2\tilde{r}/Le_2)](d^2\tilde{Y}_{2,S,0}^+/d\tilde{r}^2) + [(2\tilde{m}/\tilde{r}) - (2/Le_2)](d\tilde{Y}_{2,S,0}^+/d\tilde{r}) = 0$

$(1/Le_2)(d\tilde{Y}_{2,S,1}^+/d\tilde{r}) - (\tilde{m}/\tilde{r}^2)\tilde{Y}_{2,S,1}^+ = a_{2,1}^+\tilde{m}/\tilde{r}^2$

$(1/Le_2)(d^2\tilde{Y}_{2,S,1}^+/d\tilde{r}^2) - (\tilde{m}/\tilde{r}^2)(d\tilde{Y}_{2,S,1}^+/d\tilde{r}) + (2\tilde{m}/\tilde{r}^3)\tilde{Y}_{2,S,1}^+ = -2a_{2,1}^+\tilde{m}/\tilde{r}^3$

Note: the outer solutions of $\tilde{Y}_{1,S}^{\pm}$ and $\tilde{Y}_{2,S}^{\pm}$ are applicable to the Radiation Regions.

(D) Expansion of Energy Equation in the Radiation Regions

$$\frac{1}{\tilde r^2}\frac{d}{d\tilde r}\left(\tilde m \tilde T_S - \tilde r^2 \frac{d\tilde T_S}{d\tilde r}\right) = \frac{\tilde m - 2\tilde r}{\tilde r^2}\frac{d\tilde T_S}{d\tilde r} - \frac{d^2\tilde T_S}{d\tilde r^2} = -Da_R \exp(-\tilde E_R/\tilde T_S)$$

Define stretched variable: $\zeta = (\tilde r - \tilde r_{f,S})/\delta$ or $\tilde r = \tilde r_{f,S} + \delta\zeta \Rightarrow d\tilde r = \delta d\zeta$

$$\tilde r^2 = (\tilde r_{f,S} + \delta\zeta)^2 = \tilde r_{f,S}^2 + \delta(2\tilde r_{f,S}\zeta) + O(\delta^2) = \tilde r_{f,S}^2\left[1 + \delta(2\zeta/\tilde r_{f,S}) + O(\delta^2)\right]$$

Define inner expansions and the small expansion parameter:

$$\tilde T_S^{\pm} = [\tilde T_f - \varepsilon\Theta_{S,2}^{\pm} + O(\varepsilon^2)] - \delta[\Theta_{S,1}^{\pm} + \varepsilon\Theta_{S,3}^{\pm} + O(\varepsilon^2)] + O(\delta^2)$$

$$1/\tilde T_S^{\pm} = 1/\{[\tilde T_f - \varepsilon\Theta_{S,2}^{\pm} + O(\varepsilon^2)] - \delta[\Theta_{S,1}^{\pm} + \varepsilon\Theta_{S,3}^{\pm} + O(\varepsilon^2)] + O(\delta^2)\}$$

$$= [\tilde T_f - \varepsilon\Theta_{S,2}^{\pm} + O(\varepsilon^2)]^{-1}\left\langle 1 - \delta\{[\Theta_{S,1}^{\pm} + \varepsilon\Theta_{S,3}^{\pm} + O(\varepsilon^2)]/[\tilde T_f - \varepsilon\Theta_{S,2}^{\pm} + O(\varepsilon^2)]\} + O(\delta^2)\right\rangle^{-1}$$

$$= \frac{1}{\tilde T_f[1 - \varepsilon(\Theta_{S,2}^{\pm}/\tilde T_f) + O(\varepsilon^2)]}\frac{1}{1 - \delta\langle[\Theta_{S,1}^{\pm} + \varepsilon\Theta_{S,3}^{\pm} + O(\varepsilon^2)]/\{\tilde T_f[1 - \varepsilon(\Theta_{S,2}^{\pm}/\tilde T_f) + O(\varepsilon^2)]\}\rangle + O(\delta^2)}$$

$$= \tilde T_f^{-1}[1 + \varepsilon(\Theta_{S,2}^{\pm}/\tilde T_f) + O(\varepsilon^2)]\{1 - (\delta/\tilde T_f)[\Theta_{S,1}^{\pm} + \varepsilon\Theta_{S,3}^{\pm} + O(\varepsilon^2)][1 + \varepsilon(\Theta_{S,2}^{\pm}/\tilde T_f) + O(\varepsilon^2)] + O(\delta^2)\}^{-1}$$

$$= (1/\tilde T_f)[1 + \varepsilon(\Theta_{S,2}^{\pm}/\tilde T_f) + O(\varepsilon^2)]\{1 + (\delta/\tilde T_f)[\Theta_{S,1}^{\pm} + \varepsilon\Theta_{S,3}^{\pm} + O(\varepsilon^2)][1 + \varepsilon(\Theta_{S,2}^{\pm}/\tilde T_f) + O(\varepsilon^2)] + O(\delta^2)\}$$

$$= (1/\tilde T_f)[1 + \varepsilon(\Theta_{S,2}^{\pm}/\tilde T_f) + O(\varepsilon^2)]\{1 + (\delta/\tilde T_f)[\Theta_{S,1}^{\pm} + \varepsilon(\Theta_{S,3}^{\pm} + \Theta_{S,1}^{\pm}\Theta_{S,2}^{\pm}/\tilde T_f) + O(\varepsilon^2)] + O(\delta^2)\}$$

$$= (1/\tilde T_f)\{[1 + \varepsilon(\Theta_{S,2}^{\pm}/\tilde T_f) + O(\varepsilon^2)] + (\delta/\tilde T_f)[\Theta_{S,1}^{\pm} + \varepsilon(\Theta_{S,3}^{\pm} + \Theta_{S,1}^{\pm}\Theta_{S,2}^{\pm}/\tilde T_f) + O(\varepsilon^2)][1 + \varepsilon(\Theta_{S,2}^{\pm}/\tilde T_f) + O(\varepsilon^2)] + O(\delta^2)\}$$

$$= (1/\tilde T_f)\{[1 + \varepsilon(\Theta_{S,2}^{\pm}/\tilde T_f) + O(\varepsilon^2)] + (\delta/\tilde T_f)[\Theta_{S,1}^{\pm} + \varepsilon\langle\Theta_{S,3}^{\pm} + (2\Theta_{S,1}^{\pm}\Theta_{S,2}^{\pm}/\tilde T_f)\rangle + O(\varepsilon^2)] + O(\delta^2)\}$$

Define small expansion parameter: $\delta = \tilde T_f^2/\tilde E_R$

$$\exp(-\tilde E_R/\tilde T_S^{\pm}) = \exp\langle -(\tilde E_R/\tilde T_f)\{[1 + \varepsilon(\Theta_{S,2}^{\pm}/\tilde T_f) + O(\varepsilon^2)] + (\delta/\tilde T_f)[\Theta_{S,1}^{\pm} + \varepsilon\langle\Theta_{S,3}^{\pm} + (2\Theta_{S,1}^{\pm}\Theta_{S,2}^{\pm}/\tilde T_f)\rangle + O(\varepsilon^2)] + O(\delta^2)\}\rangle$$

$$= \exp(-\tilde E_R/\tilde T_f)\exp\{-\varepsilon(\tilde E_R/\tilde T_f^2)[\Theta_{S,2}^{\pm} + O(\varepsilon)]\}\exp\langle -\delta(\tilde E_R/\tilde T_f^2)\{\Theta_{S,1}^{\pm} + \varepsilon[\Theta_{S,3}^{\pm} + (2\Theta_{S,1}^{\pm}\Theta_{S,2}^{\pm}/\tilde T_f)] + O(\varepsilon^2)\} + O(\delta^2)\rangle$$

$$= \exp(-\tilde E_R/\tilde T_f)\exp\{-(\varepsilon/\delta)[\Theta_{S,2}^{\pm} + O(\varepsilon)]\}\exp\langle -\{\Theta_{S,1}^{\pm} + \varepsilon[\Theta_{S,3}^{\pm} + (2\Theta_{S,1}^{\pm}\Theta_{S,2}^{\pm}/\tilde T_f)] + O(\varepsilon^2)\} + O(\delta)\rangle$$

$$= \exp(-\tilde E_R/\tilde T_f)\{1 - (\varepsilon/\delta)[\Theta_{S,2}^{\pm} + O(\varepsilon)] + \ldots\}\exp(-\Theta_{S,1}^{\pm})\exp\langle -\varepsilon\{[\Theta_{S,3}^{\pm} + (2\Theta_{S,1}^{\pm}\Theta_{S,2}^{\pm}/\tilde T_f)] + O(\varepsilon)\}\rangle\exp[O(\delta)]$$

$$= \exp(-\tilde E_R/\tilde T_f)\exp(-\Theta_{S,1}^{\pm})\{1 - (\varepsilon/\delta)[\Theta_{S,2}^{\pm} + O(\varepsilon)] + \cdots\}\{1 - \varepsilon[\Theta_{S,3}^{\pm} + (2\Theta_{S,1}^{\pm}\Theta_{S,2}^{\pm}/\tilde T_f)] + O(\varepsilon^2)\}[1 + O(\delta)]$$

$$= \exp(-\tilde E_R/\tilde T_f)\exp(-\Theta_{S,1}^{\pm})\langle\{1 - (\varepsilon/\delta)[\Theta_{S,2}^{\pm} + O(\varepsilon)] - \varepsilon[\Theta_{S,3}^{\pm} + (2\Theta_{S,1}^{\pm}\Theta_{S,2}^{\pm}/\tilde T_f)] + \cdots\} + O(\delta)\rangle$$

Energy conservation equation: $\frac{\tilde m - 2\tilde r}{\tilde r^2}\frac{d\tilde T_S}{d\tilde r} - \frac{d^2\tilde T_S}{d\tilde r^2} = -Da_R\exp(-\tilde E_R/\tilde T_S)$

$$\frac{\tilde m - 2\tilde r_{f,S} + O(\delta)}{\tilde r_{f,S}^2 + O(\delta)}\frac{d\{[\tilde T_f - \varepsilon\Theta_{S,2}^{\pm} + O(\varepsilon^2)] + O(\delta)\}}{\delta d\zeta} - \frac{d^2\{[\tilde T_f - \varepsilon\Theta_{S,2}^{\pm} + O(\varepsilon^2)] - \delta[\Theta_{S,1}^{\pm} + \varepsilon\Theta_{S,3}^{\pm} + O(\varepsilon^2)] + O(\delta^2)\}}{\delta^2 d\zeta^2}$$

$$= -Da_R\exp(-\tilde E_R/\tilde T_{f,S})\exp(-\Theta_{S,1}^{\pm})\langle\{1 - (\varepsilon/\delta)[\Theta_{S,2}^{\pm} + O(\varepsilon)] - \varepsilon[\Theta_{S,3}^{\pm} + (2\Theta_{S,1}^{\pm}\Theta_{S,2}^{\pm}/\tilde T_f)] + \cdots\} + O(\delta)\rangle$$

$$-\left[\frac{\tilde m - 2\tilde r_{f,S}}{\tilde r_{f,S}^2} + O(\delta)\right]\left\{\varepsilon\left[\frac{d\Theta_{S,2}^{\pm}}{d\zeta} + O(\varepsilon)\right] + O(\delta)\right\} + \left\{\left[\frac{d^2\Theta_{S,1}^{\pm}}{d\zeta^2} + \varepsilon\frac{d^2\Theta_{S,3}^{\pm}}{d\zeta^2} + O(\varepsilon^2)\right] + \frac{\varepsilon}{\delta}\left[\frac{d^2\Theta_{S,2}^{\pm}}{d\zeta^2} + O(\varepsilon)\right] + O(\delta)\right\}$$

$$= -\delta Da_R\exp(-\tilde E_R/\tilde T_f)\exp(-\Theta_{S,1}^{\pm})\{1 - (\varepsilon/\delta)\Theta_{S,2}^{\pm} - \varepsilon[\Theta_{S,3}^{\pm} + (2\Theta_{S,1}^{\pm}\Theta_{S,2}^{\pm}/\tilde T_f)] + \cdots\}$$

Define $\Lambda_R = \delta Da_R\exp(-\tilde E_R/\tilde T_f)$ Then

$$d^2\Theta_{S,1}^{\pm}/d\zeta^2 = -\Lambda_R\exp(-\Theta_{S,1}^{\pm}) \quad ; \quad d^2\Theta_{S,2}^{\pm}/d\zeta^2 = \Lambda_R\Theta_{S,2}^{\pm}\exp(-\Theta_{S,1}^{\pm})$$

$$\frac{d^2\Theta_{S,3}^{\pm}}{d\zeta^2} - \frac{\tilde m - 2\tilde r_{f,S}}{\tilde r_{f,S}^2}\frac{d\Theta_{S,2}^{\pm}}{d\zeta} = \Lambda_R[\Theta_{S,3}^{\pm} + (2\Theta_{S,1}^{\pm}\Theta_{S,2}^{\pm}/\tilde T_f)]\exp(-\Theta_{S,1}^{\pm})$$

(E) Matching the Outer Solutions with the Solutions in the Radiation Region

In the common regions between the outer and radiation active regions, $\tilde{r} = \tilde{r}_{f,S} + \delta\zeta$

$$\tilde{T}_S^{\pm} = [\tilde{T}_{S,0}^{\pm} + \varepsilon\tilde{T}_{S,1}^{\pm} + O(\varepsilon^2)] + \delta[\tilde{T}_{S,2}^{\pm} + \varepsilon\tilde{T}_{S,3}^{\pm} + O(\varepsilon^2)] + O(\delta^2)$$

$$= \{[\tilde{T}_{S,0}^{\pm}(\tilde{r}_{f,S}) + (d\tilde{T}_{S,0}^{\pm}/d\tilde{r})_{\tilde{r}_{f,S}}(\delta\zeta) + O(\delta^2)] + \varepsilon[\tilde{T}_{S,1}^{\pm}(\tilde{r}_{f,S}) + (d\tilde{T}_{S,1}^{\pm}/d\tilde{r})_{\tilde{r}_{f,S}}(\delta\zeta) + O(\delta^2)] + O(\varepsilon^2)\}$$

$$+ \delta\{[\tilde{T}_{S,2}^{\pm}(\tilde{r}_{f,S}) + O(\delta)] + \varepsilon[\tilde{T}_{S,3}^{\pm}(\tilde{r}_{f,S}) + O(\delta)] + O(\varepsilon^2)\} + O(\delta^2)$$

$$= \{[\tilde{T}_f - \varepsilon\Theta_{S,2}^{\pm} + O(\varepsilon^2)] - \delta[\Theta_{S,1}^{\pm} + \varepsilon\Theta_{S,3}^{\pm} + O(\varepsilon^2)] + O(\delta^2)\}_{\zeta\to\pm\infty}$$

(1) $\zeta \to -\infty$ Define $g_0^- = (\tilde{T}_f - \tilde{T}_0)(\tilde{m}/\tilde{r}_{f,S}^2) > 0$, $g_1^- = \tilde{T}_{b,S,2}\exp[\tilde{m}(1-\tilde{r}_{f,S}^{-1})]$, $\tilde{T}_{b,S,A} = \tilde{T}_{b,S,1}\exp[\tilde{m}(1-\tilde{r}_{f,S}^{-1})]$

$\tilde{T}_{S,0}^-(\tilde{r}_{f,S}) = \tilde{T}_f$: $\tilde{T}_f = \tilde{T}_0 + (\tilde{T}_{b,S,0} - \tilde{T}_0)\exp[\tilde{m}(1-\tilde{r}_{f,S}^{-1})]$ or $\tilde{T}_{b,S,0} = \tilde{T}_0 + \{(\tilde{T}_f - \tilde{T}_0)/\exp[\tilde{m}(1-\tilde{r}_{f,S}^{-1})]\}$

$$\Theta_{S,1}^-(\zeta\to-\infty) = -\tilde{T}_{S,2}^-(\tilde{r}_{f,S}) - (d\tilde{T}_{S,0}^-/d\tilde{r})_{\tilde{r}_{f,S}}\zeta = -\tilde{T}_{b,S,2}\exp[\tilde{m}(1-\tilde{r}_{f,S}^{-1})] - (\tilde{T}_{b,S,0} - \tilde{T}_0)(\tilde{m}/\tilde{r}_{f,S}^2)\exp[\tilde{m}(1-\tilde{r}_{f,S}^{-1})]\zeta$$

$$= -\tilde{T}_{b,S,2}\exp[\tilde{m}(1-\tilde{r}_{f,S}^{-1})] - (\tilde{T}_f - \tilde{T}_0)(\tilde{m}/\tilde{r}_{f,S}^2)\zeta = -g_1^- - g_0^-\zeta \to \infty$$

$$(d\Theta_{S,1}^-/d\zeta)_{\zeta\to-\infty} = -(\tilde{T}_f - \tilde{T}_0)(\tilde{m}/\tilde{r}_{f,S}^2) = -g_0^-$$

$$\Theta_{S,2}^-(\zeta\to-\infty) = -\tilde{T}_{S,1}^-(\tilde{r}_{f,S}) = -\tilde{T}_{b,S,1}\exp[\tilde{m}(1-\tilde{r}_{f,S}^{-1})] = -\tilde{T}_{b,S,A} \quad ; \quad (d\Theta_{S,2}^-/d\zeta)_{\zeta\to-\infty} = 0$$

$$\Theta_{S,3}^-(\zeta\to-\infty) = -\tilde{T}_{S,3}^-(\tilde{r}_{f,S}) - (d\tilde{T}_{S,1}^-/d\tilde{r})_{\tilde{r}_{f,S}}\zeta = -\tilde{T}_{S,3}^-(\tilde{r}_{f,S}) - \tilde{T}_{b,S,1}(\tilde{m}/\tilde{r}_{f,S}^2)\exp[\tilde{m}(1-\tilde{r}_{f,S}^{-1})]\zeta$$

$$(d\Theta_{S,3}^-/d\zeta)_{\zeta\to-\infty} = -\tilde{T}_{b,S,1}(\tilde{m}/\tilde{r}_{f,S}^2)\exp[\tilde{m}(1-\tilde{r}_{f,S}^{-1})] = -\tilde{T}_{b,S,A}(\tilde{m}/\tilde{r}_{f,S}^2)$$

(2) $\zeta \to \infty$ Define $g_0^+ = (\tilde{T}_f - \tilde{T}_\infty)(\tilde{m}/\tilde{r}_{f,S}^2)/[\exp(\tilde{m}/\tilde{r}_{f,S}) - 1] > 0$, $g_1^+ = a_{T,2}^+[1 - \exp(-\tilde{m}/\tilde{r}_{f,S})]$,

$a_{T,A}^+ = a_{T,1}^+[1 - \exp(-\tilde{m}/\tilde{r}_{f,S})]$

$\tilde{T}_{S,0}^+(\tilde{r}_{f,S}) = \tilde{T}_f$: $\tilde{T}_f = \tilde{T}_\infty + a_{T,0}^+[1 - \exp(-\tilde{m}/\tilde{r}_{f,S})]$ or $a_{T,0}^+ = (\tilde{T}_f - \tilde{T}_\infty)/[1 - \exp(-\tilde{m}/\tilde{r}_{f,S})]$

$$\Theta_{S,1}^+(\zeta\to\infty) = -\tilde{T}_{S,2}^+(\tilde{r}_{f,S}) - (d\tilde{T}_{S,0}^+/d\tilde{r})_{\tilde{r}_{f,S}}\zeta = -a_{T,2}^+[1 - \exp(-\tilde{m}/\tilde{r}_{f,S})] + a_{T,0}^+(\tilde{m}/\tilde{r}_{f,S}^2)\exp(-\tilde{m}/\tilde{r}_{f,S})\zeta$$

$$= -a_{T,2}^+[1 - \exp(-\tilde{m}/\tilde{r}_{f,S})] + \{(\tilde{T}_f - \tilde{T}_\infty)(\tilde{m}/\tilde{r}_{f,S}^2)\exp(-\tilde{m}/\tilde{r}_{f,S})/[1 - \exp(-\tilde{m}/\tilde{r}_{f,S})]\}\zeta = -g_1^+ + g_0^+\zeta \to \infty$$

$$(d\Theta_{S,1}^+/d\zeta)_{\zeta\to\infty} = (\tilde{T}_f - \tilde{T}_\infty)(\tilde{m}/\tilde{r}_{f,S}^2)\exp(-\tilde{m}/\tilde{r}_{f,S})/[1 - \exp(-\tilde{m}/\tilde{r}_{f,S})] = g_0^+$$

$$\Theta_{S,2}^+(\zeta\to\infty) = -\tilde{T}_{S,1}^+(\tilde{r}_{f,S}) = -a_{T,1}^+[1 - \exp(-\tilde{m}/\tilde{r}_{f,S})] = -a_{T,A}^+ \quad ; \quad (d\Theta_{S,2}^+/d\zeta)_{\zeta\to\infty} = 0$$

$$\Theta_{S,3}^+(\zeta\to\infty) = -\tilde{T}_{S,3}^+(\tilde{r}_{f,S}) - (d\tilde{T}_{S,1}^+/d\tilde{r})_{\tilde{r}_{f,S}}\zeta = -\tilde{T}_{S,3}^+(\tilde{r}_{f,S}) + a_{T,1}^+(\tilde{m}/\tilde{r}_{f,S}^2)\exp(-\tilde{m}/\tilde{r}_{f,S})\zeta$$

$$(d\Theta_{S,3}^+/d\zeta)_{\zeta\to\infty} = a_{T,1}^+(\tilde{m}/\tilde{r}_{f,S}^2)\exp(-\tilde{m}/\tilde{r}_{f,S})$$

(F) Solution of Energy Equation in the Radiation Regions

(1) $d^2\Theta_{S,1}^{\pm}/d\zeta^2 = -\Lambda_R\exp(-\Theta_{S,1}^{\pm})$ $\quad\therefore\quad$ $d\Theta_{S,1}^{\pm}(d^2\Theta_{S,1}^{\pm}/d\zeta^2) = -\Lambda_R\exp(-\Theta_{S,1}^{\pm})d\Theta_{S,1}^{\pm}$

$d(d\Theta_{S,1}^{\pm}/d\zeta)^2 = 2\Lambda_R d[\exp(-\Theta_{S,1}^{\pm})]$ or $(d\Theta_{S,1}^{\pm}/d\zeta)^2 = 2\Lambda_R\exp(-\Theta_{S,1}^{\pm}) + (c_1^{\pm})^2$

$d\Theta_{S,1}^{\pm}/d\zeta = \pm[2\Lambda_R\exp(-\Theta_{S,1}^{\pm}) + (c_1^{\pm})^2]^{1/2}$ [From (E), we know that $d\Theta_{S,1}^-/d\zeta < 0$ and $d\Theta_{S,1}^+/d\zeta > 0$]

Let $w^{\pm} = \exp(-\Theta_{S,1}^{\pm})$ then $dw^{\pm}/d\zeta = -\exp(-\Theta_{S,1}^{\pm})(d\Theta_{S,1}^{\pm}/d\zeta) = -w^{\pm}(d\Theta_{S,1}^{\pm}/d\zeta)$ or

$d\Theta_{S,1}^{\pm}/d\zeta = -(w^{\pm})^{-1}(dw/d\zeta)$

$$\Rightarrow \quad -(w^{\pm})^{-1}(dw^{\pm}/d\zeta) = \pm[2\Lambda_R w + (c_1^{\pm})^2]^{1/2} \quad \text{or} \quad dw^{\pm}/\{w^{\pm}[2\Lambda_R w + (c_1^{\pm})^2]^{1/2}\} = \mp d\zeta$$

$$\frac{1}{|c_1^{\pm}|}\ell\text{n}\left[\frac{[2\Lambda_R w^{\pm} + (c_1^{\pm})^2]^{1/2} - |c_1^{\pm}|}{[2\Lambda_R w^{\pm} + (c_1^{\pm})^2]^{1/2} + |c_1^{\pm}|}\right] = \tilde{c}_2^{\pm} \mp \zeta \quad \text{or} \quad \frac{[2\Lambda_R w^{\pm} + (c_1^{\pm})^2]^{1/2} - |c_1^{\pm}|}{[2\Lambda_R w^{\pm} + (c_1^{\pm})^2]^{1/2} + |c_1^{\pm}|} = c_2^{\pm}\exp(\mp|c_1^{\pm}|\zeta) \quad , \quad c_2^{\pm} > 0$$

$$c_2^{\pm}\exp(\mp|c_1^{\pm}|\zeta) = \frac{\{[2\Lambda_R w^{\pm} + (c_1^{\pm})^2]^{1/2} - |c_1^{\pm}|\}^2}{\{[2\Lambda_R w^{\pm} + (c_1^{\pm})^2]^{1/2} + |c_1^{\pm}|\}\{[2\Lambda_R w^{\pm} + (c_1^{\pm})^2]^{1/2} - |c_1^{\pm}|\}}$$

$$= \frac{2\{\Lambda_R w^{\pm} + (c_1^{\pm})^2 - |c_1^{\pm}|[2\Lambda_R w^{\pm} + (c_1^{\pm})^2]^{1/2}\}}{2\Lambda_R w^{\pm}}$$

$$c_2^{\pm}\exp(\mp|c_1^{\pm}|\zeta)(\Lambda_R w^{\pm}) = \Lambda_R w^{\pm} + (c_1^{\pm})^2 - |c_1^{\pm}|[2\Lambda_R w^{\pm} + (c_1^{\pm})^2]^{1/2}$$

$$|c_1^\pm|[2\Lambda_R w^\pm + (c_1^\pm)^2]^{1/2} = (c_1^\pm)^2 + [1 - c_2^\pm \exp(\mp|c_1^\pm|\zeta)](\Lambda_R w^\pm)$$

$$(c_1^\pm)^2[2\Lambda_R w^\pm + (c_1^\pm)^2] = (c_1^\pm)^4 + 2(c_1^\pm)^2[1 - c_2^\pm \exp(\mp|c_1^\pm|\zeta)](\Lambda_R w^\pm) + [1 - c_2^\pm \exp(\mp|c_1^\pm|\zeta)]^2(\Lambda_R w^\pm)^2$$

$$[1 - c_2^\pm \exp(\mp|c_1^\pm|\zeta)]^2(\Lambda_R w^\pm)^2 - 2(c_1^\pm)^2 c_2^\pm \exp(\mp|c_1^\pm|\zeta)(\Lambda_R w^\pm) = 0$$

Since $\Lambda_R w^\pm \neq 0$, $[1 - c_2^\pm \exp(\mp|c_1^\pm|\zeta)]^2(\Lambda_R w^\pm) - 2(c_1^\pm)^2 c_2^\pm \exp(\mp|c_1^\pm|\zeta) = 0$

$$w^\pm = [2(c_1^\pm)^2/\Lambda_R]c_2^\pm \exp(\mp|c_1^\pm|\zeta)/[1 - c_2^\pm \exp(\mp|c_1^\pm|\zeta)]^2 = \exp(-\Theta_{S,1}^\pm)$$

$$\exp(\Theta_{S,1}^\pm) = [1 - c_2^\pm \exp(\mp|c_1^\pm|\zeta)]^2/\{[2(c_1^\pm)^2/\Lambda_R]c_2^\pm \exp(\mp|c_1^\pm|\zeta)\}$$

$$\Theta_{S,1}^\pm = \ell n[1 - c_2^\pm \exp(\mp|c_1^\pm|\zeta)]^2 - \ell n\{[2(c_1^\pm)^2/\Lambda_R]c_2^\pm\} \pm |c_1^\pm|\zeta$$

$$\Theta_{S,1}^- = \ell n[1 - c_2^- \exp(|c_1^-|\zeta)]^2 - \ell n\{[2(c_1^-)^2/\Lambda_R]c_2^-\} - |c_1^-|\zeta$$

$\zeta \to -\infty$: $\Theta_{S,1}^- \to -g_1^- - g_0^-\zeta$ $\therefore$ $|c_1^-| = g_0^-$

$$\exp(|c_1^-|\zeta) = \exp(g_0^-\zeta) \to \exp(-\infty) \to 0, \quad \ell n[1 - c_2^- \exp(|c_1^-|\zeta)]^2 \to \ell n 1 = 0$$

$$g_1^- = \ell n\{[2(c_1^-)^2/\Lambda_R]c_2^-\} = \ell n\{[2(g_0^-)^2/\Lambda_R]c_2^-\} \quad \text{or} \quad c_2^- = \{\Lambda_R/[2(g_0^-)^2]\}\exp(g_1^-)$$

$$\Rightarrow \quad (d\Theta_{S,1}^-/d\zeta)^2 = 2\Lambda_R \exp(-\Theta_{S,1}^-) + (g_0^-)^2 \quad \text{or} \quad d\Theta_{S,1}^-/d\zeta = -[2\Lambda_R \exp(-\Theta_{S,1}^-) + (g_0^-)^2]^{1/2}$$

$$\Theta_{S,1}^- = \ell n\{1 - [\Lambda_R/\langle 2(g_0^-)^2\rangle]\exp(g_1^- + g_0^-\zeta)\}^2 - g_0^-\zeta - g_1^-$$

$$\exp(-\Theta_{S,1}^-) = \exp(g_1^- + g_0^-\zeta)/\{1 - [\Lambda_R/\langle 2(g_0^-)^2\rangle]\exp(g_1^- + g_0^-\zeta)\}^2$$

$$\frac{d\Theta_{S,1}^-}{d\zeta} = \frac{2\{1 - [\Lambda_R/\langle 2(g_0^-)^2\rangle]\exp(g_1^- + g_0^-\zeta)\}\{-g_0^-[\Lambda_R/\langle 2(g_0^-)^2\rangle]\exp(g_1^- + g_0^-\zeta)\}}{\{1 - [\Lambda_R/\langle 2(g_0^-)^2\rangle]\exp(g_1^- + g_0^-\zeta)\}^2} - g_0^-$$

$$= -2g_0^- \frac{\{\Lambda_R/[2(g_0^-)^2]\}\exp(g_1^- + g_0^-\zeta)}{1 - \{\Lambda_R/[2(g_0^-)^2]\}\exp(g_1^- + g_0^-\zeta)} - g_0^- = -g_0^- \frac{2(g_0^-)^2 + \Lambda_R \exp(g_1^- + g_0^-\zeta)}{2(g_0^-)^2 - \Lambda_R \exp(g_1^- + g_0^-\zeta)}$$

$$\Theta_{S,1}^+ = \ell n[1 - c_2^+ \exp(-|c_1^+|\zeta)]^2 - \ell n\{[2(c_1^+)^2/\Lambda_R]c_2^+\} + |c_1^+|\zeta$$

$\zeta \to \infty$: $\Theta_{S,1}^+ \to -g_1^+ + g_0^+\zeta$ $\therefore$ $|c_1^+| = g_0^+$

$$\exp(-|c_1^+|\zeta) = \exp(-g_0^+\zeta) \to \exp(-\infty) \to 0, \quad \ell n[1 - c_2^+ \exp(-|c_1^+|\zeta)]^2 \to \ell n 1 = 0$$

$$g_1^+ = \ell n\{[2(c_1^+)^2/\Lambda_R]c_2^+\} = \ell n\{[2(g_0^+)^2/\Lambda_R]c_2^+\} \quad \text{or} \quad c_2^+ = \{\Lambda_R/[2(g_0^+)^2]\}\exp(g_1^+)$$

$$\Rightarrow \quad (d\Theta_{S,1}^+/d\zeta)^2 = 2\Lambda_R \exp(-\Theta_{S,1}^+) + (g_0^+)^2 \quad \text{or} \quad d\Theta_{S,1}^+/d\zeta = [2\Lambda_R \exp(-\Theta_{S,1}^+) + (g_0^+)^2]^{1/2}$$

$$\Theta_{S,1}^+ = \ell n\{1 - [\Lambda_R/\langle 2(g_0^+)^2\rangle]\exp(g_1^+ - g_0^+\zeta)\}^2 + g_0^+\zeta - g_1^+$$

$$\exp(-\Theta_{S,1}^+) = \exp(g_1^+ - g_0^+\zeta)/\{1 - [\Lambda_R/\langle 2(g_0^+)^2\rangle]\exp(g_1^+ - g_0^+\zeta)\}^2$$

$$\frac{d\Theta_{S,1}^+}{d\zeta} = \frac{2\langle 1 - \{\Lambda_R/[2(g_0^+)^2]\}\exp(g_1^+ - g_0^+\zeta)\rangle\langle g_0^+\{\Lambda_R/[2(g_0^+)^2]\}\exp(g_1^+ - g_0^+\zeta)\rangle}{\langle 1 - \{\Lambda_R/[2(g_0^+)^2]\}\exp(g_1^+ - g_0^+\zeta)\rangle^2} + g_0^+$$

$$= 2g_0^+ \frac{\{\Lambda_R/[2(g_0^+)^2]\}\exp(g_1^+ - g_0^+\zeta)}{1 - \{\Lambda_R/[2(g_0^+)^2]\exp(g_1^+ - g_0^+\zeta)} + g_0^+ = g_0^+ \frac{2(g_0^+)^2 + \Lambda_R \exp(g_1^+ - g_0^+\zeta)}{2(g_0^+)^2 - \Lambda_R \exp(g_1^+ - g_0^+\zeta)}$$

(2) $d^2\Theta_{S,2}^\pm/d\zeta^2 = \Lambda_R \Theta_{S,2}^\pm \exp(-\Theta_{S,1}^\pm)$ $\therefore$ $d\Theta_{S,1}^\pm(d^2\Theta_{S,2}^\pm/d\zeta^2) = \Lambda_R \Theta_{S,2}^\pm \exp(-\Theta_{S,1}^\pm) d\Theta_{S,1}^\pm$

$$d\left(\frac{d\Theta_{S,1}^\pm}{d\zeta}\frac{d\Theta_{S,2}^\pm}{d\zeta}\right) - d\Theta_{S,2}^\pm \frac{d^2\Theta_{S,1}^\pm}{d\zeta^2} = \Theta_{S,2}^\pm d[-\Lambda_R \exp(-\Theta_{S,1}^\pm)] = \Theta_{S,2}^\pm d\left(\frac{d^2\Theta_{S,1}^\pm}{d\zeta^2}\right)$$

$$d\left(\frac{d\Theta_{S,1}^\pm}{d\zeta}\frac{d\Theta_{S,2}^\pm}{d\zeta}\right) = \Theta_{S,2}^\pm d\left(\frac{d^2\Theta_{S,1}^\pm}{d\zeta^2}\right) + d\Theta_{S,2}^\pm \frac{d^2\Theta_{S,1}^\pm}{d\zeta^2} = d\left(\Theta_{S,2}^\pm \frac{d^2\Theta_{S,1}^\pm}{d\zeta^2}\right) \quad \text{or} \quad \frac{d\Theta_{S,1}^\pm}{d\zeta}\frac{d\Theta_{S,2}^\pm}{d\zeta} = c^\pm + \Theta_{S,2}^\pm \frac{d^2\Theta_{S,1}^\pm}{d\zeta^2}$$

$\zeta \to \pm\infty$: $d\Theta_{S,2}^\pm/d\zeta \to 0$, $\Theta_{S,2}^\pm \to$ constant, $d^2\Theta_{S,1}^\pm/d\zeta^2 \to 0$, $d\Theta_{S,1}^\pm/d\zeta \to$ constant $\therefore$ $c^\pm = 0$

$$\frac{d\Theta_{S,1}^\pm}{d\zeta}\frac{d\Theta_{S,2}^\pm}{d\zeta} = \Theta_{S,2}^\pm \frac{d^2\Theta_{S,1}^\pm}{d\zeta^2} = \Theta_{S,2}^\pm \frac{d(d\Theta_{S,1}^\pm/d\zeta)}{d\zeta} \quad \therefore \quad \frac{d\Theta_{S,2}^\pm}{\Theta_{S,2}^\pm} = \frac{d(d\Theta_{S,1}^\pm/d\zeta)}{d\Theta_{S,1}^\pm/d\zeta}$$

$$\ell n\,\Theta_{S,2}^{\pm}=\ell n\left(\frac{d\,\Theta_{S,1}^{\pm}}{d\,\zeta}\right)+\tilde{c}^{\pm}\qquad\text{or}\qquad\Theta_{S,2}^{\pm}=c^{\pm}\frac{d\,\Theta_{S,1}^{\pm}}{d\,\zeta}=\pm c^{\pm}\,g_0^{\pm}\,\frac{2(g_0^{\pm})^2+\Lambda_R\exp(g_1^{\pm}\mp g_0^{\pm}\zeta)}{2(g_0^{\pm})^2-\Lambda_R\exp(g_1^{\pm}\mp g_0^{\pm}\zeta)}$$

$$\zeta\to-\infty:\quad\Theta_{S,2}^{-}\to-\tilde{T}_{b,S,A}\,,\quad d\,\Theta_{S,1}^{-}/d\,\zeta\to-g_0^{-}\qquad\therefore\quad c^{-}\,g_0^{-}=\tilde{T}_{b,S,A}$$

$$\Theta_{S,2}^{-}=(\tilde{T}_{b,S,A}/g_0^{-})(d\,\Theta_{S,1}^{-}/d\,\zeta)=-\tilde{T}_{b,S,A}[\,2(g_0^{-})^2+\Lambda_R\exp(g_1^{-}+g_0^{-}\zeta)]/[\,2(g_0^{-})^2-\Lambda_R\exp(g_1^{-}+g_0^{-}\zeta)]$$

$$\zeta\to\infty:\quad\Theta_{S,2}^{+}\to-a_{T,A}^{+}\,,\quad d\,\Theta_{S,1}^{+}/d\,\zeta\to g_0^{+}\qquad\therefore\quad c^{+}\,g_0^{+}=-a_{T,A}^{+}$$

$$\Theta_{S,2}^{+}=-(a_{T,A}^{+}/g_0^{+})(\,d\,\Theta_{S,1}^{+}/d\,\zeta)=-a_{T,A}^{+}[\,2(g_0^{+})^2+\Lambda_R\exp(g_1^{+}-g_0^{+}\zeta)]/[\,2(g_0^{+})^2-\Lambda_R\exp(g_1^{+}-g_0^{+}\zeta)]$$

(3) $$\frac{d^2\,\Theta_{S,3}^{\pm}}{d\,\zeta^2}-\frac{\tilde{m}-2\tilde{r}_{f,S}}{\tilde{r}_{f,S}^2}\frac{d\,\Theta_{S,2}^{\pm}}{d\,\zeta}=\Lambda_R[\,\Theta_{S,3}^{\pm}+(2\,\Theta_{S,1}^{\pm}\,\Theta_{S,2}^{\pm}/\tilde{T}_f\,)]\exp(-\Theta_{S,1}^{\pm})$$

$$=\Lambda_R\,\Theta_{S,3}^{\pm}\exp(-\Theta_{S,1}^{\pm})+(2/\tilde{T}_f\,)\Theta_{S,1}^{\pm}[\,\Lambda_R\,\Theta_{S,2}^{\pm}\exp(-\Theta_{S,1}^{\pm})]=\Lambda_R\,\Theta_{S,3}^{\pm}\exp(-\Theta_{S,1}^{\pm})+(2/\tilde{T}_f\,)\Theta_{S,1}^{\pm}(d^2\,\Theta_{S,2}^{\pm}/d\,\zeta^2)$$

Related integrations

$$\int\frac{d\,x}{x(x+p^2)^{1/2}}=\int\frac{d\,(x+p^2)}{x(x+p^2)^{1/2}}=\int\frac{d(x+p^2)/(x+p^2)^{1/2}}{x}=\int\frac{2d(x+p^2)^{1/2}}{x+p^2-p^2}=2\int\frac{d\,(x+p^2)^{1/2}}{[(x+p^2)^{1/2}-p][(x+p^2)^{1/2}+p]}$$

$$=2\int\frac{1}{2\,p}\left[\frac{1}{(x+p^2)^{1/2}-p}-\frac{1}{(x+p^2)^{1/2}+p}\right]d\,(x+p^2)^{1/2}=\frac{1}{p}\left[\int\frac{d\,(x+p^2)^{1/2}}{(x+p^2)^{1/2}-p}-\int\frac{d\,(x+p^2)^{1/2}}{[(x+p^2)^{1/2}+p]}\right]$$

$$=\frac{1}{p}\left\{\int\frac{d[(x+p^2)^{1/2}-p]}{(x+p^2)^{1/2}-p}-\int\frac{d[(x+p^2)^{1/2}+p]}{[(x+p^2)^{1/2}+p]}\right\}=\frac{1}{p}\{\ell n[(x+p^2)^{1/2}-p]-\ell n[(x+p^2)^{1/2}+p]\}$$

$$=\frac{1}{p}\,\ell n\frac{(x+p^2)^{1/2}-p}{(x+p^2)^{1/2}+p}$$

$$\int\frac{(x+p^2)^{1/2}\,d\,x}{x}=\int\frac{(x+p^2)^{1/2}(x+p^2)^{1/2}\,d\,x}{x(x+p^2)^{1/2}}=\int\frac{(x+p^2)\,d\,x}{x(x+p^2)^{1/2}}=\int\frac{x\,d\,x}{x(x+p^2)^{1/2}}+\int\frac{p^2\,d\,x}{x(x+p^2)^{1/2}}$$

$$=\int\frac{d(x+p^2)}{(x+p^2)^{1/2}}+p^2\int\frac{d\,x}{x(x+p^2)^{1/2}}=2(x+p^2)^{1/2}+p^2\left[\frac{1}{p}\,\ell n\frac{(x+p^2)^{1/2}-p}{(x+p^2)^{1/2}+p}\right]=2(x+p^2)^{1/2}+p\,\ell n\frac{(x+p^2)^{1/2}-p}{(x+p^2)^{1/2}+p}$$

$$\int\frac{(x+p^2)^{3/2}\,d\,x}{x}=\int\frac{(x+p^2)(x+p^2)^{1/2}\,d\,x}{x}=\int\frac{x\,(x+p^2)^{1/2}\,d\,x}{x}+\int\frac{p^2\,(x+p^2)^{1/2}\,d\,x}{x}$$

$$=\int(x+p^2)^{1/2}\,d\,(x+p^2)+p^2\int\frac{(x+p^2)^{1/2}\,d\,x}{x}=\frac{2}{3}(x+p^2)^{3/2}+p^2\left[2\,(x+p^2)^{1/2}+p\,\ell n\frac{(x+p^2)^{1/2}-p}{(x+p^2)^{1/2}+p}\right]$$

$$=\frac{2}{3}(x+p^2)^{3/2}+2\,p^2\,(x+p^2)^{1/2}+p^3\,\ell n\frac{(x+p^2)^{1/2}-p}{(x+p^2)^{1/2}+p}$$

$$\int\frac{d\,x}{x(x+p^2)}=\int\frac{1}{p^2}\left(\frac{1}{x}-\frac{1}{x+p^2}\right)dx=\frac{1}{p^2}\int\left(\frac{dx}{x}-\frac{d\,x}{x+p^2}\right)=\frac{1}{p^2}\left[\int\frac{dx}{x}-\int\frac{d(x+p^2)}{x+p^2}\right]=\frac{1}{p^2}[\ell n x-\ell n(x+p^2)]=\frac{1}{p^2}\,\ell n\frac{x}{x+p^2}$$

$$\int\frac{d\,x}{x(x+p^2)^{3/2}}=\int\frac{1}{p^2}\,\frac{(x+p^2-x)dx}{x(x+p^2)^{3/2}}=\frac{1}{p^2}\int\left[\frac{x+p^2}{x(x+p^2)^{3/2}}-\frac{x}{x(x+p^2)^{3/2}}\right]dx=\frac{1}{p^2}\left[\int\frac{d\,x}{x(x+p^2)^{1/2}}-\int\frac{d(x+p^2)}{(x+p^2)^{3/2}}\right]$$

$$=\frac{1}{p^2}\left[\frac{1}{p}\,\ell n\frac{(x+p^2)^{1/2}-p}{(x+p^2)^{1/2}+p}+\frac{2}{(x+p^2)^{1/2}}\right]$$

$$\int\ell n\frac{(x+p^2)^{1/2}-p}{(x+p^2)^{1/2}+p}\,\frac{d\,x}{x(x+p^2)^{1/2}}=\int\ell n\frac{(x+p^2)^{1/2}-p}{(x+p^2)^{1/2}+p}\,\frac{1}{p}\,d\left[\ell n\frac{(x+p^2)^{1/2}-p}{(x+p^2)^{1/2}+p}\right]=\frac{1}{2\,p}\left[\ell n\frac{(x+p^2)^{1/2}-p}{(x+p^2)^{1/2}+p}\right]^2$$

$$\int\frac{d\,x}{x(x+p^2)^{3/2}}\,\ell n\frac{(x+p^2)^{1/2}-p}{(x+p^2)^{1/2}+p}$$ integration by parts: $\int u\,d\,v=u\,v-\int v\,d\,u$

$$u=\ell n\frac{(x+p^2)^{1/2}-p}{(x+p^2)^{1/2}+p}\;\Rightarrow\;du=\frac{p\,d\,x}{x(x+p^2)^{1/2}}\quad;\quad dv=\frac{d\,x}{x(x+p^2)^{3/2}}\;\Rightarrow\;v=\frac{1}{p^2}\left[\frac{1}{p}\,\ell n\frac{(x+p^2)^{1/2}-p}{(x+p^2)^{1/2}+p}+\frac{2}{(x+p^2)^{1/2}}\right]$$

$$\int \frac{dx}{x(x+p^2)^{3/2}} \ell n \frac{(x+p^2)^{1/2}-p}{(x+p^2)^{1/2}+p}$$

$$= \frac{1}{p^2}\left\{ \frac{1}{p}\left[\ell n \frac{(x+p^2)^{1/2}-p}{(x+p^2)^{1/2}+p} \right]^2 + \frac{2}{(x+p^2)^{1/2}} \ell n \frac{(x+p^2)^{1/2}-p}{(x+p^2)^{1/2}+p} - \int \frac{dx}{x(x+p^2)^{1/2}} \ell n \frac{(x+p^2)^{1/2}-p}{(x+p^2)^{1/2}+p} - p \int \frac{2\,dx}{x(x+p^2)} \right\}$$

$$= \frac{1}{p^2}\left\{ \frac{1}{p}\left[\ell n \frac{(x+p^2)^{1/2}-p}{(x+p^2)^{1/2}+p} \right]^2 + \frac{2}{(x+p^2)^{1/2}} \ell n \frac{(x+p^2)^{1/2}-p}{(x+p^2)^{1/2}+p} - \frac{1}{2p}\left[\ell n \frac{(x+p^2)^{1/2}-p}{(x+p^2)^{1/2}+p} \right]^2 - \frac{2}{p} \ell n \frac{x}{x+p^2} \right\}$$

$$= \frac{1}{p^3}\left\{ \frac{1}{2}\left[\ell n \frac{(x+p^2)^{1/2}-p}{(x+p^2)^{1/2}+p} \right]^2 + \frac{2p}{(x+p^2)^{1/2}} \ell n \frac{(x+p^2)^{1/2}-p}{(x+p^2)^{1/2}+p} - 2\,\ell n \frac{x}{x+p^2} \right\}$$

Back to the problem.

Define $g_2^- = [2/(3\tilde{T}_f)](\tilde{T}_{b,S,A}/g_0^-)$, $g_2^+ = -[2/(3\tilde{T}_f)](a_{T,A}^+/g_0^+)$

$g_3^- = [(\tilde{m}-2\tilde{r}_{f,S})/\tilde{r}_{f,S}^2](\tilde{T}_{b,S,A}/g_0^-)$, $g_3^+ = -[(\tilde{m}-2\tilde{r}_{f,S})/\tilde{r}_{f,S}^2](a_{T,A}^+/g_0^+)$

then $[(\tilde{m}-2\tilde{r}_{f,S})/\tilde{r}_{f,S}^2]\Theta_{S,2}^\pm = g_3^\pm (d\Theta_{S,1}^\pm/d\zeta)$ $\therefore$ $[(\tilde{m}-2\tilde{r}_{f,S})/\tilde{r}_{f,S}^2](d\Theta_{S,2}^\pm/d\zeta) = g_3^\pm (d^2\Theta_{S,1}^\pm/d\zeta^2)$

$(2/\tilde{T}_f)\Theta_{S,2}^\pm = 3g_2^\pm (d\Theta_{S,1}^\pm/d\zeta)$ $\therefore$ $(2/\tilde{T}_f)(d^2\Theta_{S,2}^\pm/d\zeta^2) = 3g_2^\pm (d^3\Theta_{S,1}^\pm/d\zeta^3)$

$$\frac{d^2\Theta_{S,3}^\pm}{d\zeta^2} - g_3^\pm \frac{d^2\Theta_{S,1}^\pm}{d\zeta^2} = \Lambda_R \Theta_{S,3}^\pm \exp(-\Theta_{S,1}^\pm) + 3g_2^\pm \Theta_{S,1}^\pm \frac{d^3\Theta_{S,1}^\pm}{d\zeta^3} = \Lambda_R \Theta_{S,3}^\pm \exp(-\Theta_{S,1}^\pm) + 3g_2^\pm \Theta_{S,1}^\pm \frac{d}{d\zeta}\left(\frac{d^2\Theta_{S,1}^\pm}{d\zeta^2}\right)$$

$$\frac{d^2\Theta_{S,3}^\pm}{d\zeta^2} - g_3^\pm[-\Lambda_R \exp(-\Theta_{S,1}^\pm)] = \Lambda_R \Theta_{S,3}^\pm \exp(-\Theta_{S,1}^\pm) + 3g_2^\pm \Theta_{S,1}^\pm \frac{d}{d\Theta_{S,1}^\pm}\left(\frac{d^2\Theta_{S,1}^\pm}{d\zeta^2}\right)\frac{d\Theta_{S,1}^\pm}{d\zeta}$$

$$(d^2\Theta_{S,3}^\pm/d\zeta^2) + (\Theta_{S,3}^\pm - g_3^\pm)[-\Lambda_R \exp(-\Theta_{S,1}^\pm)] = 3g_2^\pm \Theta_{S,1}^\pm \{d[-\Lambda_R \exp(-\Theta_{S,1}^\pm)]/d\Theta_{S,1}^\pm\}(d\Theta_{S,1}^\pm/d\zeta)$$

Let $\bar{\Theta}_{S,3}^\pm = \Theta_{S,3}^\pm - g_3^\pm$ then $d\bar{\Theta}_{S,3}^\pm/d\zeta = d\Theta_{S,3}^\pm/d\zeta$, $d^2\bar{\Theta}_{S,3}^\pm/d\zeta^2 = d^2\Theta_{S,3}^\pm/d\zeta^2$

$$(d^2\bar{\Theta}_{S,3}^\pm/d\zeta^2) + \bar{\Theta}_{S,3}^\pm[-\Lambda_R \exp(-\Theta_{S,1}^\pm)] = \pm 3g_2^\pm \Theta_{S,1}^\pm \Lambda_R \exp(-\Theta_{S,1}^\pm)\sqrt{2\Lambda_R \exp(-\Theta_{S,1}^\pm) + (g_0^\pm)^2}$$

$$d\Theta_{S,1}^\pm (d^2\bar{\Theta}_{S,3}^\pm/d\zeta^2) + \bar{\Theta}_{S,3}^\pm[-\Lambda_R \exp(-\Theta_{S,1}^\pm)]d\Theta_{S,1}^\pm = \pm 3g_2^\pm \Theta_{S,1}^\pm \Lambda_R \exp(-\Theta_{S,1}^\pm)\sqrt{2\Lambda_R \exp(-\Theta_{S,1}^\pm) + (g_0^\pm)^2}\, d\Theta_{S,1}^\pm$$

$$d\left(\frac{d\Theta_{S,1}^\pm}{d\zeta}\frac{d\bar{\Theta}_{S,3}^\pm}{d\zeta}\right) - \frac{d^2\Theta_{S,1}^\pm}{d\zeta^2} d\bar{\Theta}_{S,3}^\pm + \bar{\Theta}_{S,3}^\pm d[\Lambda_R \exp(-\Theta_{S,1}^\pm)] = \pm 3g_2^\pm \Theta_{S,1}^\pm \Lambda_R \exp(-\Theta_{S,1}^\pm)\sqrt{2\Lambda_R \exp(-\Theta_{S,1}^\pm) + (g_0^\pm)^2}\, d\Theta_{S,1}^\pm$$

$$d\left(\frac{d\Theta_{S,1}^\pm}{d\zeta}\frac{d\bar{\Theta}_{S,3}^\pm}{d\zeta}\right) - \frac{d^2\Theta_{S,1}^\pm}{d\zeta^2} d\bar{\Theta}_{S,3}^\pm - \bar{\Theta}_{S,3}^\pm d\left(\frac{d^2\Theta_{S,1}^\pm}{d\zeta^2}\right) = d\left(\frac{d\Theta_{S,1}^\pm}{d\zeta}\frac{d\bar{\Theta}_{S,3}^\pm}{d\zeta}\right) - d\left(\frac{d^2\Theta_{S,1}^\pm}{d\zeta^2}\bar{\Theta}_{S,3}^\pm\right)$$

$$= \pm 3g_2^\pm \Theta_{S,1}^\pm \Lambda_R \exp(-\Theta_{S,1}^\pm)\sqrt{2\Lambda_R \exp(-\Theta_{S,1}^\pm) + (g_0^\pm)^2}\, d\Theta_{S,1}^\pm$$

$$(d\Theta_{S,1}^\pm/d\zeta)(d\bar{\Theta}_{S,3}^\pm/d\zeta) - (d^2\Theta_{S,1}^\pm/d\zeta^2)\bar{\Theta}_{S,3}^\pm = c_1^\pm \pm 3g_2^\pm \int \Theta_{S,1}^\pm \Lambda_R \exp(-\Theta_{S,1}^\pm)\sqrt{2\Lambda_R \exp(-\Theta_{S,1}^\pm) + (g_0^\pm)^2}\, d\Theta_{S,1}^\pm$$

Integrate the term $\int \Theta_{S,1}^\pm \Lambda_R \exp(-\Theta_{S,1}^\pm)\sqrt{2\Lambda_R \exp(-\Theta_{S,1}^\pm) + (g_0^\pm)^2}\, d\Theta_{S,1}^\pm$

$$\int \Theta_{S,1}^\pm \Lambda_R \exp(-\Theta_{S,1}^\pm)\sqrt{2\Lambda_R \exp(-\Theta_{S,1}^\pm) + (g_0^\pm)^2}\, d\Theta_{S,1}^\pm = \int \Theta_{S,1}^\pm [2\Lambda_R \exp(-\Theta_{S,1}^\pm) + (g_0^\pm)^2]^{1/2}[\Lambda_R \exp(-\Theta_{S,1}^\pm) d\Theta_{S,1}^\pm]$$

$$= -(1/2)\int \Theta_{S,1}^\pm [2\Lambda_R \exp(-\Theta_{S,1}^\pm) + (g_0^\pm)^2]^{1/2} d[2\Lambda_R \exp(-\Theta_{S,1}^\pm)]$$

$$= -(1/2)\int \Theta_{S,1}^\pm [2\Lambda_R \exp(-\Theta_{S,1}^\pm) + (g_0^\pm)^2]^{1/2} d[2\Lambda_R \exp(-\Theta_{S,1}^\pm) + (g_0^\pm)^2]$$

$$= -(1/2)\int \Theta_{S,1}^\pm d\{(2/3)[2\Lambda_R \exp(-\Theta_{S,1}^\pm) + (g_0^\pm)^2]^{3/2}\}$$

$$= -(1/3)\int \{d[\Theta_{S,1}^\pm \langle 2\Lambda_R \exp(-\Theta_{S,1}^\pm) + (g_0^\pm)^2\rangle^{3/2}] - \langle 2\Lambda_R \exp(-\Theta_{S,1}^\pm) + (g_0^\pm)^2\rangle^{3/2} d\Theta_{S,1}^\pm\}$$

$$=-(1/3)\left\{\int d[\Theta_{S,1}^{\pm}\langle 2\Lambda_R\exp(-\Theta_{S,1}^{\pm})+(g_0^{\pm})^2\rangle^{3/2}]-\int[2\Lambda_R\exp(-\Theta_{S,1}^{\pm})+(g_0^{\pm})^2]^{3/2}d\Theta_{S,1}^{\pm}\right\}$$

$$=-\frac{1}{3}\left\{\Theta_{S,1}^{\pm}[2\Lambda_R\exp(-\Theta_{S,1}^{\pm})+(g_0^{\pm})^2]^{3/2}-\int\frac{[2\Lambda_R\exp(-\Theta_{S,1}^{\pm})+(g_0^{\pm})^2]^{3/2}}{2\Lambda_R\exp(-\Theta_{S,1}^{\pm})}[2\Lambda_R\exp(-\Theta_{S,1}^{\pm})d\Theta_{S,1}^{\pm}]\right\}$$

$$=-\frac{1}{3}\left\{\Theta_{S,1}^{\pm}[2\Lambda_R\exp(-\Theta_{S,1}^{\pm})+(g_0^{\pm})^2]^{3/2}+\int\frac{[2\Lambda_R\exp(-\Theta_{S,1}^{\pm})+(g_0^{\pm})^2]^{3/2}\,d[2\Lambda_R\exp(-\Theta_{S,1}^{\pm})]}{2\Lambda_R\exp(-\Theta_{S,1}^{\pm})}\right\}$$

$$=-\frac{1}{3}\left\{\Theta_{S,1}^{\pm}[2\Lambda_R\exp(-\Theta_{S,1}^{\pm})+(g_0^{\pm})^2]^{3/2}+\frac{2}{3}[2\Lambda_R\exp(-\Theta_{S,1}^{\pm})+(g_0^{\pm})^2]^{3/2}+2(g_0^{\pm})^2[2\Lambda_R\exp(-\Theta_{S,1}^{\pm})+(g_0^{\pm})^2]^{1/2}\right.$$

$$\left.+(g_0^{\pm})^3\,\ell\text{n}\frac{[2\Lambda_R\exp(-\Theta_{S,1}^{\pm})+(g_0^{\pm})^2]^{1/2}-g_0^{\pm}}{[2\Lambda_R\exp(-\Theta_{S,1}^{\pm})+(g_0^{\pm})^2]^{1/2}+g_0^{\pm}}\right\}\qquad \text{*Let } x=2\Lambda_R\exp(-\Theta_{S,1}^{\pm}),\quad p=g_0^{\pm}$$

$$=-\frac{1}{3}\left\{(\Theta_{S,1}^{\pm}+\frac{2}{3})[2\Lambda_R\exp(-\Theta_{S,1}^{\pm})+(g_0^{\pm})^2]^{3/2}+2(g_0^{\pm})^2[2\Lambda_R\exp(-\Theta_{S,1}^{\pm})+(g_0^{\pm})^2]^{1/2}\right.$$

$$\left.+(g_0^{\pm})^3\,\ell\text{n}\frac{[2\Lambda_R\exp(-\Theta_{S,1}^{\pm})+(g_0^{\pm})^2]^{1/2}-g_0^{\pm}}{[2\Lambda_R\exp(-\Theta_{S,1}^{\pm})+(g_0^{\pm})^2]^{1/2}+g_0^{\pm}}\right\}$$

$$(d\Theta_{S,1}^{\pm}/d\zeta)(d\bar{\Theta}_{S,3}^{\pm}/d\zeta)-(d^2\Theta_{S,1}^{\pm}/d\zeta^2)\bar{\Theta}_{S,3}^{\pm}$$

$$=c_1^{\pm}\mp g_2^{\pm}\Big\{[\Theta_{S,1}^{\pm}+(2/3)][2\Lambda_R\exp(-\Theta_{S,1}^{\pm})+(g_0^{\pm})^2]^{3/2}+2(g_0^{\pm})^2\sqrt{2\Lambda_R\exp(-\Theta_{S,1}^{\pm})+(g_0^{\pm})^2}$$

$$+(g_0^{\pm})^3\,\ell\text{n}\Big([\sqrt{2\Lambda_R\exp(-\Theta_{S,1}^{\pm})+(g_0^{\pm})^2}-(g_0^{\pm})]/[\sqrt{2\Lambda_R\exp(-\Theta_{S,1}^{\pm})+(g_0^{\pm})^2}+(g_0^{\pm})]\Big)\Big\}$$

$$\frac{d\Theta_{S,1}^{\pm}}{d\zeta}\frac{d\bar{\Theta}_{S,3}^{\pm}}{d\zeta}-\frac{d^2\Theta_{S,1}^{\pm}}{d\zeta^2}\bar{\Theta}_{S,3}^{\pm}=\frac{d\Theta_{S,1}^{\pm}}{d\zeta}\frac{d\bar{\Theta}_{S,3}^{\pm}}{d\Theta_{S,1}^{\pm}}\frac{d\Theta_{S,1}^{\pm}}{d\zeta}-\frac{d^2\Theta_{S,1}^{\pm}}{d\zeta^2}\bar{\Theta}_{S,3}^{\pm}=\left(\frac{d\Theta_{S,1}^{\pm}}{d\zeta}\right)^2\frac{d\bar{\Theta}_{S,3}^{\pm}}{d\Theta_{S,1}^{\pm}}-\frac{d^2\Theta_{S,1}^{\pm}}{d\zeta^2}\bar{\Theta}_{S,3}^{\pm}$$

$$=[2\Lambda_R\exp(-\Theta_{S,1}^{\pm})+(g_0^{\pm})^2](d\bar{\Theta}_{S,3}^{\pm}/d\Theta_{S,1}^{\pm})+[\Lambda_R\exp(-\Theta_{S,1}^{\pm})]\bar{\Theta}_{S,3}^{\pm}$$

Divide the equation by $[2\Lambda_R\exp(-\Theta_{S,1}^{\pm})+(g_0^{\pm})^2]$

$$(d\bar{\Theta}_{S,3}^{\pm}/d\Theta_{S,1}^{\pm})+\{[\Lambda_R\exp(-\Theta_{S,1}^{\pm})]/[2\Lambda_R\exp(-\Theta_{S,1}^{\pm})+(g_0^{\pm})^2]\}\bar{\Theta}_{S,3}^{\pm}$$

$$=\frac{c_1^{\pm}}{2\Lambda_R\exp(-\Theta_{S,1}^{\pm})+(g_0^{\pm})^2}\mp g_2^{\pm}\left\{\left(\Theta_{S,1}^{\pm}+\frac{2}{3}\right)\sqrt{2\Lambda_R\exp(-\Theta_{S,1}^{\pm})+(g_0^{\pm})^2}+\frac{2(g_0^{\pm})^2}{\sqrt{2\Lambda_R\exp(-\Theta_{S,1}^{\pm})+(g_0^{\pm})^2}}\right.$$

$$\left.+\frac{(g_0^{\pm})^3}{2\Lambda_R\exp(-\Theta_{S,1}^{\pm})+(g_0^{\pm})^2}\ell\text{n}\frac{\sqrt{2\Lambda_R\exp(-\Theta_{S,1}^{\pm})+(g_0^{\pm})^2}-g_0^{\pm}}{\sqrt{2\Lambda_R\exp(-\Theta_{S,1}^{\pm})+(g_0^{\pm})^2}+g_0^{\pm}}\right\}$$

For $(dy/dx)+p(x)y=q(x)$, the general solution is $y=\exp[-\int p(x)dx]\{c+\int q(x)\exp[\int p(x)dx]dx\}$

For this equation: $y=\bar{\Theta}_{S,3}^{\pm}$, $x=\Theta_{S,1}^{\pm}$, $c=c_2^{\pm}$, $p(\Theta_{S,1}^{\pm})=[\Lambda_R\exp(-\Theta_{S,1}^{\pm})]/[2\Lambda_R\exp(-\Theta_{S,1}^{\pm})+(g_0^{\pm})^2]$

$$\int p(\Theta_{S,1}^{\pm})d\Theta_{S,1}^{\pm}=\int\frac{\Lambda_R\exp(-\Theta_{S,1}^{\pm})}{2\Lambda_R\exp(-\Theta_{S,1}^{\pm})+(g_0^{\pm})^2}d\Theta_{S,1}^{\pm}=-\frac{1}{2}\int\frac{d[2\Lambda_R\exp(-\Theta_{S,1}^{\pm})]}{2\Lambda_R\exp(-\Theta_{S,1}^{\pm})+(g_0^{\pm})^2}$$

$$=-\frac{1}{2}\int\frac{d[2\Lambda_R\exp(-\Theta_{S,1}^{\pm})+(g_0^{\pm})^2]}{2\Lambda_R\exp(-\Theta_{S,1}^{\pm})+(g_0^{\pm})^2}=-(1/2)\ell n[2\Lambda_R\exp(-\Theta_{S,1}^{\pm})+(g_0^{\pm})^2]$$

$$=-\ell n[2\Lambda_R\exp(-\Theta_{S,1}^{\pm})+(g_0^{\pm})^2]^{1/2}=\ell n[2\Lambda_R\exp(-\Theta_{S,1}^{\pm})+(g_0^{\pm})^2]^{-1/2}$$

$$\exp\Big[\int p(\Theta_{S,1}^{\pm})d\Theta_{S,1}^{\pm}\Big]=\exp\{\ell n[2\Lambda_R\exp(-\Theta_{S,1}^{\pm})+(g_0^{\pm})^2]^{-1/2}\}=[2\Lambda_R\exp(-\Theta_{S,1}^{\pm})+(g_0^{\pm})^2]^{-1/2}$$

$$\exp\Big[-\int p(\Theta_{S,1}^{\pm})d\Theta_{S,1}^{\pm}\Big]=\exp\{\ell n[2\Lambda_R\exp(-\Theta_{S,1}^{\pm})+(g_0^{\pm})^2]^{1/2}\}=[2\Lambda_R\exp(-\Theta_{S,1}^{\pm})+(g_0^{\pm})^2]^{1/2}$$

$$\int q(\Theta_{S,1}^{\pm})\exp\Big[\int p(\Theta_{S,1}^{\pm})\,d\Theta_{S,1}^{\pm}\Big]d\Theta_{S,1}^{\pm}=\int q(\Theta_{S,1}^{\pm})[2\Lambda_R\exp(-\Theta_{S,1}^{\pm})+(g_0^{\pm})^2]^{-1/2}\,d\Theta_{S,1}^{\pm}$$

$$=c_1^{\pm}\int\frac{d\Theta_{S,1}^{\pm}}{[2\Lambda_R\exp(-\Theta_{S,1}^{\pm})+(g_0^{\pm})^2]^{3/2}}\mp g_2^{\pm}\int\left(\Theta_{S,1}^{\pm}+\frac{2}{3}\right)d\Theta_{S,1}^{\pm}\mp 2\,g_2^{\pm}(g_0^{\pm})^2\int\frac{d\Theta_{S,1}^{\pm}}{2\Lambda_R\exp(-\Theta_{S,1}^{\pm})+(g_0^{\pm})^2}$$

$$\mp g_2^{\pm}(g_0^{\pm})^3\int\frac{1}{[2\Lambda_R\exp(-\Theta_{S,1}^{\pm})+(g_0^{\pm})^2]^{3/2}}\,\ell\mathrm{n}\frac{[2\Lambda_R\exp(-\Theta_{S,1}^{\pm})+(g_0^{\pm})^2]^{1/2}-g_0^{\pm}}{[2\Lambda_R\exp(-\Theta_{S,1}^{\pm})+(g_0^{\pm})^2]^{1/2}+g_0^{\pm}}\,d\Theta_{S,1}^{\pm}$$

$$\int\frac{d\Theta_{S,1}^{\pm}}{2\Lambda_R\exp(-\Theta_{S,1}^{\pm})+(g_0^{\pm})^2}=-\int\frac{-2\Lambda_R\exp(-\Theta_{S,1}^{\pm})\,d\Theta_{S,1}^{\pm}}{2\Lambda_R\exp(-\Theta_{S,1}^{\pm})[2\Lambda_R\exp(-\Theta_{S,1}^{\pm})+(g_0^{\pm})^2]}$$

$$=-\int\frac{d[2\Lambda_R\exp(-\Theta_{S,1}^{\pm})]}{2\Lambda_R\exp(-\Theta_{S,1}^{\pm})[2\Lambda_R\exp(-\Theta_{S,1}^{\pm})+(g_0^{\pm})^2]}=-\frac{1}{(g_0^{\pm})^2}\,\ell\mathrm{n}\frac{2\Lambda_R\exp(-\Theta_{S,1}^{\pm})}{2\Lambda_R\exp(-\Theta_{S,1}^{\pm})+(g_0^{\pm})^2}$$

Note: Let $x=2\Lambda_R\exp(-\Theta_{S,1}^{\pm})$, $p=g_0^{\pm}$

$$\int\frac{d\Theta_{S,1}^{\pm}}{[2\Lambda_R\exp(-\Theta_{S,1}^{\pm})+(g_0^{\pm})^2]^{3/2}}=-\int\frac{-2\Lambda_R\exp(-\Theta_{S,1}^{\pm})\,d\Theta_{S,1}^{\pm}}{2\Lambda_R\exp(-\Theta_{S,1}^{\pm})[2\Lambda_R\exp(-\Theta_{S,1}^{\pm})+(g_0^{\pm})^2]^{3/2}}$$

$$=-\int\frac{d[2\Lambda_R\exp(-\Theta_{S,1}^{\pm})]}{2\Lambda_R\exp(-\Theta_{S,1}^{\pm})[2\Lambda_R\exp(-\Theta_{S,1}^{\pm})+(g_0^{\pm})^2]^{3/2}}$$

Note: Let $x=2\Lambda_R\exp(-\Theta_{S,1}^{\pm})$, $p=g_0^{\pm}$

$$=-\frac{1}{(g_0^{\pm})^2}\left\{\frac{1}{g_0^{\pm}}\,\ell\mathrm{n}\frac{[2\Lambda_R\exp(-\Theta_{S,1}^{\pm})+(g_0^{\pm})^2]^{1/2}-g_0^{\pm}}{[2\Lambda_R\exp(-\Theta_{S,1}^{\pm})+(g_0^{\pm})^2]^{1/2}+g_0^{\pm}}+\frac{2}{[2\Lambda_R\exp(-\Theta_{S,1}^{\pm})+(g_0^{\pm})^2]^{1/2}}\right\}$$

$$\int\frac{1}{[2\Lambda_R\exp(-\Theta_{S,1}^{\pm})+(g_0^{\pm})^2]^{3/2}}\,\ell\mathrm{n}\frac{[2\Lambda_R\exp(-\Theta_{S,1}^{\pm})+(g_0^{\pm})^2]^{1/2}-g_0^{\pm}}{[2\Lambda_R\exp(-\Theta_{S,1}^{\pm})+(g_0^{\pm})^2]^{1/2}+g_0^{\pm}}\,d\Theta_{S,1}^{\pm}$$

$$=\int\ell\mathrm{n}\frac{[2\Lambda_R\exp(-\Theta_{S,1}^{\pm})+(g_0^{\pm})^2]^{1/2}-g_0^{\pm}}{[2\Lambda_R\exp(-\Theta_{S,1}^{\pm})+(g_0^{\pm})^2]^{1/2}+g_0^{\pm}}\,\frac{2\Lambda_R\exp(-\Theta_{S,1}^{\pm})\,d\Theta_{S,1}^{\pm}}{2\Lambda_R\exp(-\Theta_{S,1}^{\pm})[2\Lambda_R\exp(-\Theta_{S,1}^{\pm})+(g_0^{\pm})^2]^{3/2}}$$

$$=-\int\ell\mathrm{n}\frac{[2\Lambda_R\exp(-\Theta_{S,1}^{\pm})+(g_0^{\pm})^2]^{1/2}-g_0^{\pm}}{[2\Lambda_R\exp(-\Theta_{S,1}^{\pm})+(g_0^{\pm})^2]^{1/2}+g_0^{\pm}}\,\frac{d[2\Lambda_R\exp(-\Theta_{S,1}^{\pm})]}{2\Lambda_R\exp(-\Theta_{S,1}^{\pm})[2\Lambda_R\exp(-\Theta_{S,1}^{\pm})+(g_0^{\pm})^2]^{3/2}}$$

$$=-\frac{1}{(g_0^{\pm})^3}\left\{\frac{1}{2}\left[\ell\mathrm{n}\frac{[2\Lambda_R\exp(-\Theta_{S,1}^{\pm})+(g_0^{\pm})^2]^{1/2}-g_0^{\pm}}{[2\Lambda_R\exp(-\Theta_{S,1}^{\pm})+(g_0^{\pm})^2]^{1/2}+g_0^{\pm}}\right]^2-2\,\ell\mathrm{n}\frac{2\Lambda_R\exp(-\Theta_{S,1}^{\pm})}{2\Lambda_R\exp(-\Theta_{S,1}^{\pm})+(g_0^{\pm})^2}\right.$$

$$\left.+\frac{2g_0^{\pm}}{[2\Lambda_R\exp(-\Theta_{S,1}^{\pm})+(g_0^{\pm})^2]^{1/2}}\,\ell\mathrm{n}\frac{[2\Lambda_R\exp(-\Theta_{S,1}^{\pm})+(g_0^{\pm})^2]^{1/2}-g_0^{\pm}}{[2\Lambda_R\exp(-\Theta_{S,1}^{\pm})+(g_0^{\pm})^2]^{1/2}+g_0^{\pm}}\right\}$$

$$\bar{\Theta}_{S,3}^{\pm}=\sqrt{2\Lambda_R\exp(-\Theta_{S,1}^{\pm})+(g_0^{\pm})^2}\left\{c_2^{\pm}\mp g_2^{\pm}\left[\frac{(\Theta_{S,1}^{\pm})^2}{2}+\frac{2}{3}\Theta_{S,1}^{\pm}\right]\pm 2\,g_2^{\pm}\,\ell\mathrm{n}\frac{2\Lambda_R\exp(-\Theta_{S,1}^{\pm})}{2\Lambda_R\exp(-\Theta_{S,1}^{\pm})+(g_0^{\pm})^2}\right.$$

$$-\frac{c_1^{\pm}}{(g_0^{\pm})^2}\left[\frac{1}{g_0^{\pm}}\,\ell\mathrm{n}\frac{\sqrt{2\Lambda_R\exp(-\Theta_{S,1}^{\pm})+(g_0^{\pm})^2}-g_0^{\pm}}{\sqrt{2\Lambda_R\exp(-\Theta_{S,1}^{\pm})+(g_0^{\pm})^2}+g_0^{\pm}}+\frac{2}{\sqrt{2\Lambda_R\exp(-\Theta_{S,1}^{\pm})+(g_0^{\pm})^2}}\right]$$

$$\pm g_2^{\pm}\left[\frac{1}{2}\left[\ell\mathrm{n}\frac{\sqrt{2\Lambda_R\exp(-\Theta_{S,1}^{\pm})+(g_0^{\pm})^2}-g_0^{\pm}}{\sqrt{2\Lambda_R\exp(-\Theta_{S,1}^{\pm})+(g_0^{\pm})^2}+g_0^{\pm}}\right]^2-2\,\ell\mathrm{n}\frac{2\Lambda_R\exp(-\Theta_{S,1}^{\pm})}{2\Lambda_R\exp(-\Theta_{S,1}^{\pm})+(g_0^{\pm})^2}\right.$$

$$\left.\left.+\frac{2g_0^{\pm}}{\sqrt{2\Lambda_R\exp(-\Theta_{S,1}^{\pm})+(g_0^{\pm})^2}}\,\ell\mathrm{n}\frac{\sqrt{2\Lambda_R\exp(-\Theta_{S,1}^{\pm})+(g_0^{\pm})^2}-g_0^{\pm}}{\sqrt{2\Lambda_R\exp(-\Theta_{S,1}^{\pm})+(g_0^{\pm})^2}+g_0^{\pm}}\right]\right\}$$

$$=-\frac{c_1^{\pm}}{(g_0^{\pm})^2}\left\{2+\frac{\sqrt{2\Lambda_R\exp(-\Theta_{S,1}^{\pm})+(g_0^{\pm})^2}}{g_0^{\pm}}\,\ell\mathrm{n}\frac{\sqrt{2\Lambda_R\exp(-\Theta_{S,1}^{\pm})+(g_0^{\pm})^2}-g_0^{\pm}}{\sqrt{2\Lambda_R\exp(-\Theta_{S,1}^{\pm})+(g_0^{\pm})^2}+g_0^{\pm}}\right\}$$

$$\pm\frac{g_2^\pm}{2}\sqrt{2\Lambda_R\exp(-\Theta_{S,1}^\pm)+(g_0^\pm)^2}\left[\ell\text{n}\frac{\sqrt{2\Lambda_R\exp(-\Theta_{S,1}^\pm)+(g_0^\pm)^2}-g_0^\pm}{\sqrt{2\Lambda_R\exp(-\Theta_{S,1}^\pm)+(g_0^\pm)^2}+g_0^\pm}\right]^2$$

$$\pm 2g_2^\pm g_0^\pm\,\ell\text{n}\frac{\sqrt{2\Lambda_R\exp(-\Theta_{S,1}^\pm)+(g_0^\pm)^2}-g_0^\pm}{\sqrt{2\Lambda_R\exp(-\Theta_{S,1}^\pm)+(g_0^\pm)^2}+g_0^\pm}+\sqrt{2\Lambda_R\exp(-\Theta_{S,1}^\pm)+(g_0^\pm)^2}\left\{c_2^\pm\mp g_2^\pm\left[\frac{(\Theta_{S,1}^\pm)^2}{2}+\frac{2}{3}\Theta_{S,1}^\pm\right]\right\}$$

As $\zeta\to\pm\infty$: $\Theta_{S,1}^\pm\to\pm g_0^\pm\zeta-g_1^\pm\to\infty$, $\exp(-\Theta_{S,1}^\pm)\to 0$, $\Theta_{S,3}^\pm\to-\tilde{T}_{S,3}^\pm(\tilde{r}_{f,S})-(d\tilde{T}_{S,1}^\pm/d\tilde{r})_{\tilde{r}_{f,S}}\zeta$

$$\sqrt{2\Lambda_R\exp(-\Theta_{S,1}^\pm)+(g_0^\pm)^2}=\langle(g_0^\pm)^2\{1+[2\Lambda_R/(g_0^\pm)^2]\exp(-\Theta_{S,1}^\pm)\}\rangle^{1/2}=[(g_0^\pm)^2]^{1/2}\{1+[2\Lambda_R/(g_0^\pm)^2]\exp(-\Theta_{S,1}^\pm)\}^{1/2}$$

$$=\left|g_0^\pm\right|\{1+(1/2)[2\Lambda_R/(g_0^\pm)^2]\exp(-\Theta_{S,1}^\pm)+\cdots\}=g_0^\pm\{1+[\Lambda_R/(g_0^\pm)^2]\exp(-\Theta_{S,1}^\pm)+\cdots\}$$

$$=g_0^\pm+(\Lambda_R/g_0^\pm)\exp(-\Theta_{S,1}^\pm)+\cdots=g_0^\pm+\cdots$$ (note: $g_0^\pm$ are defined to be non-negative)

$$\ell\text{n}\frac{\sqrt{2\Lambda_R\exp(-\Theta_{S,1}^\pm)+(g_0^\pm)^2}-g_0^\pm}{\sqrt{2\Lambda_R\exp(-\Theta_{S,1}^\pm)+(g_0^\pm)^2}+g_0^\pm}=\ell\text{n}\frac{[g_0^\pm+(\Lambda_R/g_0^\pm)\exp(-\Theta_{S,1}^\pm)+\cdots]-g_0^\pm}{[g_0^\pm+(\Lambda_R/g_0^\pm)\exp(-\Theta_{S,1}^\pm)+\cdots]+g_0^\pm}=\ell\text{n}\frac{(\Lambda_R/g_0^\pm)\exp(-\Theta_{S,1}^\pm)+\cdots}{2g_0^\pm+\cdots}$$

$$=\ell\text{n}\{\Lambda_R/[2(g_0^\pm)^2]\}+\ell\text{n}[\exp(-\Theta_{S,1}^\pm)+\cdots]=\ell\text{n}\{\Lambda_R/[2(g_0^\pm)^2]\}-\Theta_{S,1}^\pm+\cdots$$

$$-\tilde{T}_{S,3}^\pm(\tilde{r}_{f,S})-(d\tilde{T}_{S,1}^\pm/d\tilde{r})_{\tilde{r}_{f,S}}\zeta-g_3^\pm$$

$$=-[c_1^\pm/(g_0^\pm)^2]\langle 2+\ell\text{n}\{\Lambda_R/[2(g_0^\pm)^2]\}-\Theta_{S,1}^\pm\rangle\pm(g_2^\pm/2)g_0^\pm\langle\ell\text{n}\{\Lambda_R/[2(g_0^\pm)^2]\}-\Theta_{S,1}^\pm\rangle^2$$

$$\pm 2g_2^\pm g_0^\pm\langle\ell\text{n}\{\Lambda_R/[2(g_0^\pm)^2]\}-\Theta_{S,1}^\pm\rangle+g_0^\pm\langle c_2^\pm\mp g_2^\pm\{[(\Theta_{S,1}^\pm)^2/2]+(2/3)\Theta_{S,1}^\pm\}\rangle$$

$$=\left\{\frac{c_1^\pm}{(g_0^\pm)^2}\mp g_2^\pm g_0^\pm\left[\frac{8}{3}+\ell\text{n}\frac{\Lambda_R}{2(g_0^\pm)^2}\right]\right\}\Theta_{S,1}^\pm+c_2^\pm g_0^\pm-\frac{c_1^\pm}{(g_0^\pm)^2}\left[2+\ell\text{n}\frac{\Lambda_R}{2(g_0^\pm)^2}\right]\pm g_2^\pm g_0^\pm\left[\frac{1}{2}\ell\text{n}\frac{\Lambda_R}{2(g_0^\pm)^2}+2\right]\ell\text{n}\frac{\Lambda_R}{2(g_0^\pm)^2}$$

$$=\left\{\frac{c_1^\pm}{(g_0^\pm)^2}\mp g_2^\pm g_0^\pm\left[\frac{8}{3}+\ell\text{n}\frac{\Lambda_R}{2(g_0^\pm)^2}\right]\right\}(\pm g_0^\pm\zeta-g_1^\pm)+c_2^\pm g_0^\pm-\frac{c_1^\pm}{(g_0^\pm)^2}\left[2+\ell\text{n}\frac{\Lambda_R}{2(g_0^\pm)^2}\right]\pm g_2^\pm g_0^\pm\left[\frac{1}{2}\ell\text{n}\frac{\Lambda_R}{2(g_0^\pm)^2}+2\right]\ell\text{n}\frac{\Lambda_R}{2(g_0^\pm)^2}$$

$$-\left(\frac{d\tilde{T}_{S,1}^\pm}{d\tilde{r}}\right)_{\tilde{r}_{f,S}}=\left\{\pm\frac{c_1^\pm}{(g_0^\pm)^2}-g_2^\pm g_0^\pm\left[\frac{8}{3}+\ell\text{n}\frac{\Lambda_R}{2(g_0^\pm)^2}\right]\right\}g_0^\pm\qquad\therefore\qquad\frac{c_1^\pm}{(g_0^\pm)^2}=\pm g_2^\pm g_0^\pm\left[\frac{8}{3}+\ell\text{n}\frac{\Lambda_R}{2(g_0^\pm)^2}\right]\mp\frac{1}{g_0^\pm}\left(\frac{d\tilde{T}_{S,1}^\pm}{d\tilde{r}}\right)_{\tilde{r}_{f,S}}$$

$$-\tilde{T}_{S,3}^\pm(\tilde{r}_{f,S})-g_3^\pm=-g_1^\pm\left\{\frac{c_1^\pm}{(g_0^\pm)^2}\mp g_2^\pm g_0^\pm\left[\frac{8}{3}+\ell\text{n}\frac{\Lambda_R}{2(g_0^\pm)^2}\right]\right\}+c_2^\pm g_0^\pm-\frac{c_1^\pm}{(g_0^\pm)^2}\left[2+\ell\text{n}\frac{\Lambda_R}{2(g_0^\pm)^2}\right]\pm g_2^\pm g_0^\pm\left[\frac{1}{2}\ell\text{n}\frac{\Lambda_R}{2(g_0^\pm)^2}+2\right]\ell\text{n}\frac{\Lambda_R}{2(g_0^\pm)^2}$$

$$-\tilde{T}_{S,3}^\pm(\tilde{r}_{f,S})-g_3^\pm=\pm\frac{g_1^\pm}{g_0^\pm}\left(\frac{d\tilde{T}_{S,1}^\pm}{d\tilde{r}}\right)_{\tilde{r}_{f,S}}+c_2^\pm g_0^\pm-\left\{\pm g_2^\pm g_0^\pm\left[\frac{8}{3}+\ell\text{n}\frac{\Lambda_R}{2(g_0^\pm)^2}\right]\mp\frac{1}{g_0^\pm}\left(\frac{d\tilde{T}_{S,1}^\pm}{d\tilde{r}}\right)_{\tilde{r}_{f,S}}\right\}\left[2+\ell\text{n}\frac{\Lambda_R}{2(g_0^\pm)^2}\right]$$

$$\pm g_2^\pm g_0^\pm\left[\frac{1}{2}\ell\text{n}\frac{\Lambda_R}{2(g_0^\pm)^2}+2\right]\ell\text{n}\frac{\Lambda_R}{2(g_0^\pm)^2}$$

$$c_2^\pm g_0^\pm=-\tilde{T}_{S,3}^\pm(\tilde{r}_{f,S})-g_3^\pm\mp\frac{g_1^\pm}{g_0^\pm}\left(\frac{d\tilde{T}_{S,1}^\pm}{d\tilde{r}}\right)_{\tilde{r}_{f,S}}\pm\left\{g_2^\pm g_0^\pm\left[\frac{8}{3}+\ell\text{n}\frac{\Lambda_R}{2(g_0^\pm)^2}\right]-\frac{1}{g_0^\pm}\left(\frac{d\tilde{T}_{S,1}^\pm}{d\tilde{r}}\right)_{\tilde{r}_{f,S}}\right\}\left[2+\ell\text{n}\frac{\Lambda_R}{2(g_0^\pm)^2}\right]$$

$$\mp g_2^\pm g_0^\pm\left[\frac{1}{2}\ell\text{n}\frac{\Lambda_R}{2(g_0^\pm)^2}+2\right]\ell\text{n}\frac{\Lambda_R}{2(g_0^\pm)^2}$$

$$c_2^\pm=-\frac{\tilde{T}_{S,3}^\pm(\tilde{r}_{f,S})+g_3^\pm}{g_0^\pm}\mp\frac{1}{(g_0^\pm)^2}\left(\frac{d\tilde{T}_{S,1}^\pm}{d\tilde{r}}\right)_{\tilde{r}_{f,S}}\left[g_1^\pm+2+\ell\text{n}\frac{\Lambda_R}{2(g_0^\pm)^2}\right]\pm g_2^\pm\left\{\frac{16}{3}+\frac{8}{3}\ell\text{n}\frac{\Lambda_R}{2(g_0^\pm)^2}+\frac{1}{2}\left[\ell\text{n}\frac{\Lambda_R}{2(g_0^\pm)^2}\right]^2\right\}$$

$$\bar{\Theta}_{S,3}^\pm=\mp\left\{g_2^\pm g_0^\pm\left[\frac{8}{3}+\ell\text{n}\frac{\Lambda_R}{2(g_0^\pm)^2}\right]-\frac{1}{g_0^\pm}\left(\frac{d\tilde{T}_{S,1}^\pm}{d\tilde{r}}\right)_{\tilde{r}_{f,S}}\right\}\left\{2+\frac{\sqrt{2\Lambda_R\exp(-\Theta_{S,1}^\pm)+(g_0^\pm)^2}}{g_0^\pm}\ell\text{n}\frac{\sqrt{2\Lambda_R\exp(-\Theta_{S,1}^\pm)+(g_0^\pm)^2}-g_0^\pm}{\sqrt{2\Lambda_R\exp(-\Theta_{S,1}^\pm)+(g_0^\pm)^2}+g_0^\pm}\right\}$$

$$\pm\frac{g_2^\pm}{2}\sqrt{2\Lambda_R\exp(-\Theta_{S,1}^\pm)+(g_0^\pm)^2}\left[\ell n\frac{\sqrt{2\Lambda_R\exp(-\Theta_{S,1}^\pm)+(g_0^\pm)^2}-g_0^\pm}{\sqrt{2\Lambda_R\exp(-\Theta_{S,1}^\pm)+(g_0^\pm)^2}+g_0^\pm}\right]^2\pm 2g_2^\pm g_0^\pm\ell n\frac{\sqrt{2\Lambda_R\exp(-\Theta_{S,1}^\pm)+(g_0^\pm)^2}-g_0^\pm}{\sqrt{2\Lambda_R\exp(-\Theta_{S,1}^\pm)+(g_0^\pm)^2}+g_0^\pm}$$

$$+\sqrt{2\Lambda_R\exp(-\Theta_{S,1}^\pm)+(g_0^\pm)^2}\times$$

$$\left\{-\frac{\tilde{T}_{S,3}^\pm(\tilde{r}_{f,S})+g_3^\pm}{g_0^\pm}\mp\frac{1}{(g_0^\pm)^2}\left(\frac{d\tilde{T}_{S,1}^\pm}{d\tilde{r}}\right)_{\tilde{r}_{f,S}}\left[g_1^\pm+2+\ell n\frac{\Lambda_R}{2(g_0^\pm)^2}\right]\pm g_2^\pm\left[\frac{16}{3}+\frac{8}{3}\ell n\frac{\Lambda_R}{2(g_0^\pm)^2}+\frac{1}{2}\left(\ell n\frac{\Lambda_R}{2(g_0^\pm)^2}\right)^2\right]\mp g_2^\pm\left[\frac{(\Theta_{S,1}^\pm)^2}{2}+\frac{2}{3}\Theta_{S,1}^\pm\right]\right\}$$

$$=\mp\left[\frac{8}{3}g_2^\pm g_0^\pm-\frac{1}{g_0^\pm}\left(\frac{d\tilde{T}_{S,1}^\pm}{d\tilde{r}}\right)_{\tilde{r}_{f,S}}\right]\left\{2+\frac{\sqrt{2\Lambda_R\exp(-\Theta_{S,1}^\pm)+(g_0^\pm)^2}}{g_0^\pm}\left[\ell n\frac{\sqrt{2\Lambda_R\exp(-\Theta_{S,1}^\pm)+(g_0^\pm)^2}-g_0^\pm}{\sqrt{2\Lambda_R\exp(-\Theta_{S,1}^\pm)+(g_0^\pm)^2}+g_0^\pm}-\ell n\frac{\Lambda_R}{2(g_0^\pm)^2}\right]\right\}$$

$$\mp\left[\frac{8}{3}g_2^\pm g_0^\pm-\frac{1}{g_0^\pm}\left(\frac{d\tilde{T}_{S,1}^\pm}{d\tilde{r}}\right)_{\tilde{r}_{f,S}}\right]\ell n\frac{\Lambda_R}{2(g_0^\pm)^2}\frac{\sqrt{2\Lambda_R\exp(-\Theta_{S,1}^\pm)+(g_0^\pm)^2}}{g_0^\pm}$$

$$\mp 2g_2^\pm g_0^\pm\ell n\frac{\Lambda_R}{2(g_0^\pm)^2}\mp g_2^\pm\ell n\frac{\Lambda_R}{2(g_0^\pm)^2}\sqrt{2\Lambda_R\exp(-\Theta_{S,1}^\pm)+(g_0^\pm)^2}\,\ell n\frac{\sqrt{2\Lambda_R\exp(-\Theta_{S,1}^\pm)+(g_0^\pm)^2}-g_0^\pm}{\sqrt{2\Lambda_R\exp(-\Theta_{S,1}^\pm)+(g_0^\pm)^2}+g_0^\pm}$$

$$\pm\frac{g_2^\pm}{2}\sqrt{2\Lambda_R\exp(-\Theta_{S,1}^\pm)+(g_0^\pm)^2}\left[\ell n\frac{\sqrt{2\Lambda_R\exp(-\Theta_{S,1}^\pm)+(g_0^\pm)^2}-g_0^\pm}{\sqrt{2\Lambda_R\exp(-\Theta_{S,1}^\pm)+(g_0^\pm)^2}+g_0^\pm}-\ell n\frac{\Lambda_R}{2(g_0^\pm)^2}\right]^2$$

$$\pm g_2^\pm\sqrt{2\Lambda_R\exp(-\Theta_{S,1}^\pm)+(g_0^\pm)^2}\left\{\ell n\frac{\Lambda_R}{2(g_0^\pm)^2}\ell n\frac{\sqrt{2\Lambda_R\exp(-\Theta_{S,1}^\pm)+(g_0^\pm)^2}-g_0^\pm}{\sqrt{2\Lambda_R\exp(-\Theta_{S,1}^\pm)+(g_0^\pm)^2}+g_0^\pm}-\frac{1}{2}\left[\ell n\frac{\Lambda_R}{2(g_0^\pm)^2}\right]^2\right\}$$

$$\pm 2g_2^\pm g_0^\pm\left\{\ell n\frac{\sqrt{2\Lambda_R\exp(-\Theta_{S,1}^\pm)+(g_0^\pm)^2}-g_0^\pm}{\sqrt{2\Lambda_R\exp(-\Theta_{S,1}^\pm)+(g_0^\pm)^2}+g_0^\pm}-\ell n\frac{\Lambda_R}{2(g_0^\pm)^2}\right\}\pm 2g_2^\pm g_0^\pm\ell n\frac{\Lambda_R}{2(g_0^\pm)^2}$$

$$-\frac{\tilde{T}_{S,3}^\pm(\tilde{r}_{f,S})+g_3^\pm}{g_0^\pm}\sqrt{2\Lambda_R\exp(-\Theta_{S,1}^\pm)+(g_0^\pm)^2}\mp\frac{g_1^\pm}{g_0^\pm}\left(\frac{d\tilde{T}_{S,1}^\pm}{d\tilde{r}}\right)_{\tilde{r}_{f,S}}\frac{\sqrt{2\Lambda_R\exp(-\Theta_{S,1}^\pm)+(g_0^\pm)^2}}{g_0^\pm}$$

$$\mp\frac{2}{g_0^\pm}\left(\frac{d\tilde{T}_{S,1}^\pm}{d\tilde{r}}\right)_{\tilde{r}_{f,S}}\frac{\sqrt{2\Lambda_R\exp(-\Theta_{S,1}^\pm)+(g_0^\pm)^2}}{g_0^\pm}\mp\frac{1}{g_0^\pm}\left(\frac{d\tilde{T}_{S,1}^\pm}{d\tilde{r}}\right)_{\tilde{r}_{f,S}}\ell n\frac{\Lambda_R}{2(g_0^\pm)^2}\frac{\sqrt{2\Lambda_R\exp(-\Theta_{S,1}^\pm)+(g_0^\pm)^2}}{g_0^\pm}$$

$$\pm\frac{16}{3}g_2^\pm g_0^\pm\frac{\sqrt{2\Lambda_R\exp(-\Theta_{S,1}^\pm)+(g_0^\pm)^2}}{g_0^\pm}\pm\frac{8}{3}g_2^\pm g_0^\pm\ell n\frac{\Lambda_R}{2(g_0^\pm)^2}\frac{[\sqrt{2\Lambda_R\exp(-\Theta_{S,1}^\pm)+(g_0^\pm)^2}}{g_0^\pm}$$

$$\pm\frac{g_2^\pm}{2}\left[\ell n\frac{\Lambda_R}{2(g_0^\pm)^2}\right]^2\sqrt{2\Lambda_R\exp(-\Theta_{S,1}^\pm)+(g_0^\pm)^2}\mp g_2^\pm\left[\frac{(\Theta_{S,1}^\pm)^2}{2}+\frac{2}{3}\Theta_{S,1}^\pm\right]\sqrt{2\Lambda_R\exp(-\Theta_{S,1}^\pm)+(g_0^\pm)^2}$$

$$\Theta_{S,3}^\pm=\mp\left[\frac{8}{3}g_2^\pm g_0^\pm-\frac{1}{g_0^\pm}\left(\frac{d\tilde{T}_{S,1}^\pm}{d\tilde{r}}\right)_{\tilde{r}_{f,S}}\right]\left\{2+\frac{\sqrt{2\Lambda_R\exp(-\Theta_{S,1}^\pm)+(g_0^\pm)^2}}{g_0^\pm}\left\langle\ell n\left[\frac{2(g_0^\pm)^2}{\Lambda_R}\frac{\sqrt{2\Lambda_R\exp(-\Theta_{S,1}^\pm)+(g_0^\pm)^2}-g_0^\pm}{\sqrt{2\Lambda_R\exp(-\Theta_{S,1}^\pm)+(g_0^\pm)^2}+g_0^\pm}\right]-2\right\rangle\right\}$$

$$\pm\frac{g_2^\pm}{2}\sqrt{2\Lambda_R\exp(-\Theta_{S,1}^\pm)+(g_0^\pm)^2}\left\{\left\langle\ell n\left[\frac{2(g_0^\pm)^2}{\Lambda_R}\frac{\sqrt{2\Lambda_R\exp(-\Theta_{S,1}^\pm)+(g_0^\pm)^2}-g_0^\pm}{\sqrt{2\Lambda_R\exp(-\Theta_{S,1}^\pm)+(g_0^\pm)^2}+g_0^\pm}\right]\right\rangle^2-(\Theta_{S,1}^\pm)^2\right\}$$

$$\pm 2g_2^\pm g_0^\pm\left\{\ell n\left[\frac{2(g_0^\pm)^2}{\Lambda_R}\frac{\sqrt{2\Lambda_R\exp(-\Theta_{S,1}^\pm)+(g_0^\pm)^2}-g_0^\pm}{\sqrt{2\Lambda_R\exp(-\Theta_{S,1}^\pm)+(g_0^\pm)^2}+g_0^\pm}\right]\right\}-\frac{\tilde{T}_{S,3}^\pm(\tilde{r}_{f,S})+g_3^\pm}{g_0^\pm}\sqrt{2\Lambda_R\exp(-\Theta_{S,1}^\pm)+(g_0^\pm)^2}+g_3^\pm$$

$$\mp \frac{g_1^\pm}{g_0^\pm}\left(\frac{d\tilde{T}_{S,1}^\pm}{d\tilde{r}}\right)_{\tilde{r}_{f,S}} \frac{\sqrt{2\Lambda_R \exp(-\Theta_{S,1}^\pm)+(g_0^\pm)^2}}{g_0^\pm} \mp \frac{2}{3} g_2^\pm \Theta_{S,1}^\pm \sqrt{2\Lambda_R \exp(-\Theta_{S,1}^\pm)+(g_0^\pm)^2}$$

$$d\Theta_{S,3}^\pm / d\zeta = d\bar{\Theta}_{S,3}^\pm / d\zeta = (d\bar{\Theta}_{S,3}^\pm / d\Theta_{S,1}^\pm)(d\Theta_{S,1}^\pm / d\zeta) = \pm\sqrt{2\Lambda_R \exp(-\Theta_{S,1}^\pm)+(g_0^\pm)^2}\,(d\bar{\Theta}_{S,3}^\pm / d\Theta_{S,1}^\pm)$$

$$\frac{d\sqrt{2\Lambda_R \exp(-\Theta_{S,1}^\pm)+(g_0^\pm)^2}}{d\Theta_{S,1}^\pm} = \frac{1}{2}\frac{-2\Lambda_R \exp(-\Theta_{S,1}^\pm)}{\sqrt{2\Lambda_R \exp(-\Theta_{S,1}^\pm)+(g_0^\pm)^2}} = -\frac{\Lambda_R \exp(-\Theta_{S,1}^\pm)}{\sqrt{2\Lambda_R \exp(-\Theta_{S,1}^\pm)+(g_0^\pm)^2}}$$

$$\frac{d}{d\Theta_{S,1}^\pm}\left\{\ell\text{n}\left[\frac{2(g_0^\pm)^2}{\Lambda_R}\frac{\sqrt{2\Lambda_R \exp(-\Theta_{S,1}^\pm)+(g_0^\pm)^2}-g_0^\pm}{\sqrt{2\Lambda_R \exp(-\Theta_{S,1}^\pm)+(g_0^\pm)^2}+g_0^\pm}\right]\right\}$$

$$= \{d\,\ell\text{n}[\sqrt{2\Lambda_R \exp(-\Theta_{S,1}^\pm)+(g_0^\pm)^2}-g_0^\pm]/d\Theta_{S,1}^\pm\} - \{d\,\ell\text{n}[\sqrt{2\Lambda_R \exp(-\Theta_{S,1}^\pm)+(g_0^\pm)^2}+g_0^\pm]/d\Theta_{S,1}^\pm\}$$

$$= (1/2)[-2\Lambda_R \exp(-\Theta_{S,1}^\pm)]/\{\sqrt{2\Lambda_R \exp(-\Theta_{S,1}^\pm)+(g_0^\pm)^2}\,[\sqrt{2\Lambda_R \exp(-\Theta_{S,1}^\pm)+(g_0^\pm)^2}-g_0^\pm]\}$$

$$-(1/2)[-2\Lambda_R \exp(-\Theta_{S,1}^\pm)]/\{\sqrt{2\Lambda_R \exp(-\Theta_{S,1}^\pm)+(g_0^\pm)^2}\,[\sqrt{2\Lambda_R \exp(-\Theta_{S,1}^\pm)+(g_0^\pm)^2}+g_0^\pm]\}$$

$$= -\frac{\Lambda_R \exp(-\Theta_{S,1}^\pm)}{\sqrt{2\Lambda_R \exp(-\Theta_{S,1}^\pm)+(g_0^\pm)^2}}\left[\frac{1}{\sqrt{2\Lambda_R \exp(-\Theta_{S,1}^\pm)+(g_0^\pm)^2}-g_0^\pm} - \frac{1}{\sqrt{2\Lambda_R \exp(-\Theta_{S,1}^\pm)+(g_0^\pm)^2}+g_0^\pm}\right]$$

$$= -\frac{\Lambda_R \exp(-\Theta_{S,1}^\pm)}{\sqrt{2\Lambda_R \exp(-\Theta_{S,1}^\pm)+(g_0^\pm)^2}}\frac{[\sqrt{2\Lambda_R \exp(-\Theta_{S,1}^\pm)+(g_0^\pm)^2}+g_0^\pm]-[\sqrt{2\Lambda_R \exp(-\Theta_{S,1}^\pm)+(g_0^\pm)^2}-g_0^\pm]}{[\sqrt{2\Lambda_R \exp(-\Theta_{S,1}^\pm)+(g_0^\pm)^2}-g_0^\pm][\sqrt{2\Lambda_R \exp(-\Theta_{S,1}^\pm)+(g_0^\pm)^2}+g_0^\pm]}$$

$$= -\frac{\Lambda_R \exp(-\Theta_{S,1}^\pm)}{\sqrt{2\Lambda_R \exp(-\Theta_{S,1}^\pm)+(g_0^\pm)^2}}\frac{2g_0^\pm}{2\Lambda_R \exp(-\Theta_{S,1}^\pm)} = -\frac{g_0^\pm}{\sqrt{2\Lambda_R \exp(-\Theta_{S,1}^\pm)+(g_0^\pm)^2}}$$

$$\frac{d\Theta_{S,3}^\pm}{d\Theta_{S,1}^\pm} = \pm\left[\frac{8}{3}g_2^\pm g_0^\pm - \frac{1}{g_0^\pm}\left(\frac{d\tilde{T}_{S,1}^\pm}{d\tilde{r}}\right)_{\tilde{r}_{f,S}}\right]\left\{\frac{\Lambda_R \exp(-\Theta_{S,1}^\pm)}{g_0^\pm\sqrt{2\Lambda_R \exp(-\Theta_{S,1}^\pm)+(g_0^\pm)^2}}\left\langle \ell\text{n}\left[\frac{2(g_0^\pm)^2}{\Lambda_R}\frac{\sqrt{2\Lambda_R \exp(-\Theta_{S,1}^\pm)+(g_0^\pm)^2}-g_0^\pm}{\sqrt{2\Lambda_R \exp(-\Theta_{S,1}^\pm)+(g_0^\pm)^2}+g_0^\pm}\right]-2\right\rangle\right\}$$

$$\pm\left[\frac{8}{3}g_2^\pm g_0^\pm - \frac{1}{g_0^\pm}\left(\frac{d\tilde{T}_{S,1}^\pm}{d\tilde{r}}\right)_{\tilde{r}_{f,S}}\right]\left\{\frac{\sqrt{2\Lambda_R \exp(-\Theta_{S,1}^\pm)+(g_0^\pm)^2}}{g_0^\pm}\frac{g_0^\pm}{\sqrt{2\Lambda_R \exp(-\Theta_{S,1}^\pm)+(g_0^\pm)^2}}\right\}$$

$$\mp\frac{g_2^\pm}{2}\frac{\Lambda_R \exp(-\Theta_{S,1}^\pm)}{\sqrt{2\Lambda_R \exp(-\Theta_{S,1}^\pm)+(g_0^\pm)^2}}\left\{\left\langle \ell\text{n}\left[\frac{2(g_0^\pm)^2}{\Lambda_R}\frac{\sqrt{2\Lambda_R \exp(-\Theta_{S,1}^\pm)+(g_0^\pm)^2}-g_0^\pm}{\sqrt{2\Lambda_R \exp(-\Theta_{S,1}^\pm)+(g_0^\pm)^2}+g_0^\pm}\right]\right\rangle^2 - (\Theta_{S,1}^\pm)^2\right\}$$

$$\mp\frac{g_2^\pm}{2}\sqrt{2\Lambda_R \exp(-\Theta_{S,1}^\pm)+(g_0^\pm)^2}\;\ell\text{n}\left[\frac{2(g_0^\pm)^2}{\Lambda_R}\frac{\sqrt{2\Lambda_R \exp(-\Theta_{S,1}^\pm)+(g_0^\pm)^2}-g_0^\pm}{\sqrt{2\Lambda_R \exp(-\Theta_{S,1}^\pm)+(g_0^\pm)^2}+g_0^\pm}\right]\frac{2g_0^\pm}{\sqrt{2\Lambda_R \exp(-\Theta_{S,1}^\pm)+(g_0^\pm)^2}}$$

$$\mp\frac{g_2^\pm}{2}\sqrt{2\Lambda_R \exp(-\Theta_{S,1}^\pm)+(g_0^\pm)^2}\,(2\Theta_{S,1}^\pm) \mp 2g_2^\pm g_0^\pm\frac{g_0^\pm}{\sqrt{2\Lambda_R \exp(-\Theta_{S,1}^\pm)+(g_0^\pm)^2}}$$

$$+\frac{\tilde{T}_{S,3}^\pm(\tilde{r}_{f,S})+g_3^\pm}{g_0^\pm}\frac{\Lambda_R \exp(-\Theta_{S,1}^\pm)}{\sqrt{2\Lambda_R \exp(-\Theta_{S,1}^\pm)+(g_0^\pm)^2}} \pm \frac{g_1^\pm}{g_0^\pm}\left(\frac{d\tilde{T}_{S,1}^\pm}{d\tilde{r}}\right)_{\tilde{r}_{f,S}}\frac{\Lambda_R \exp(-\Theta_{S,1}^\pm)}{g_0^\pm\sqrt{2\Lambda_R \exp(-\Theta_{S,1}^\pm)+(g_0^\pm)^2}}$$

$$\mp\frac{2}{3}g_2^\pm\sqrt{2\Lambda_R \exp(-\Theta_{S,1}^\pm)+(g_0^\pm)^2} \pm \frac{2}{3}g_2^\pm\Theta_{S,1}^\pm\frac{\Lambda_R \exp(-\Theta_{S,1}^\pm)}{\sqrt{2\Lambda_R \exp(-\Theta_{S,1}^\pm)+(g_0^\pm)^2}}$$

$$= \pm\left[\frac{8}{3}g_2^\pm g_0^\pm - \frac{1}{g_0^\pm}\left(\frac{d\tilde{T}_{S,1}^\pm}{d\tilde{r}}\right)_{\tilde{r}_{f,S}}\right]\left\{1+\frac{\Lambda_R \exp(-\Theta_{S,1}^\pm)}{g_0^\pm\sqrt{2\Lambda_R \exp(-\Theta_{S,1}^\pm)+(g_0^\pm)^2}}\left\langle \ell\text{n}\left[\frac{2(g_0^\pm)^2}{\Lambda_R}\frac{\sqrt{2\Lambda_R \exp(-\Theta_{S,1}^\pm)+(g_0^\pm)^2}-g_0^\pm}{\sqrt{2\Lambda_R \exp(-\Theta_{S,1}^\pm)+(g_0^\pm)^2}+g_0^\pm}\right]-2\right\rangle\right\}$$

$$\mp\frac{g_2^\pm}{2}\frac{\Lambda_R\exp(-\Theta_{S,1}^\pm)}{\sqrt{2\Lambda_R\exp(-\Theta_{S,1}^\pm)+(g_0^\pm)^2}}\left\{\left\langle \ell\text{n}\left[\frac{2(g_0^\pm)^2}{\Lambda_R}\frac{\sqrt{2\Lambda_R\exp(-\Theta_{S,1}^\pm)+(g_0^\pm)^2}-g_0^\pm}{\sqrt{2\Lambda_R\exp(-\Theta_{S,1}^\pm)+(g_0^\pm)^2}+g_0^\pm}\right]\right\rangle^2-(\Theta_{S,1}^\pm)^2\right\}$$

$$\mp g_2^\pm g_0^\pm\,\ell\text{n}\left[\frac{2(g_0^\pm)^2}{\Lambda_R}\frac{\sqrt{2\Lambda_R\exp(-\Theta_{S,1}^\pm)+(g_0^\pm)^2}-g_0^\pm}{\sqrt{2\Lambda_R\exp(-\Theta_{S,1}^\pm)+(g_0^\pm)^2}+g_0^\pm}\right]\mp g_2^\pm\sqrt{2\Lambda_R\exp(-\Theta_{S,1}^\pm)+(g_0^\pm)^2}\left(\Theta_{S,1}^\pm+\frac{2}{3}\right)$$

$$+\left[\frac{\tilde{T}_{S,3}^\pm(\tilde{r}_{f,S})+g_3^\pm}{g_0^\pm}\pm\frac{g_1^\pm}{(g_0^\pm)^2}\left(\frac{d\tilde{T}_{S,1}^\pm}{d\tilde{r}}\right)_{\tilde{r}_{f,S}}\pm\frac{2}{3}g_2^\pm\Theta_{S,1}^\pm\right]\frac{\Lambda_R\exp(-\Theta_{S,1}^\pm)}{\sqrt{2\Lambda_R\exp(-\Theta_{S,1}^\pm)+(g_0^\pm)^2}}\mp\frac{2g_2^\pm(g_0^\pm)^2}{\sqrt{2\Lambda_R\exp(-\Theta_{S,1}^\pm)+(g_0^\pm)^2}}$$

$$d\Theta_{S,3}^\pm/d\zeta=\pm\sqrt{2\Lambda_R\exp(-\Theta_{S,1}^\pm)+(g_0^\pm)^2}\,(d\bar{\Theta}_{S,3}^\pm/d\Theta_{S,1}^\pm)$$

$$=\left[\frac{8}{3}g_2^\pm-\frac{1}{(g_0^\pm)^2}\left(\frac{d\tilde{T}_{S,1}^\pm}{d\tilde{r}}\right)_{\tilde{r}_{f,S}}\right]\left\{g_0^\pm\sqrt{2\Lambda_R\exp(-\Theta_{S,1}^\pm)+(g_0^\pm)^2}+\Lambda_R\exp(-\Theta_{S,1}^\pm)\left\langle\ell\text{n}\left[\frac{2(g_0^\pm)^2}{\Lambda_R}\frac{\sqrt{2\Lambda_R\exp(-\Theta_{S,1}^\pm)+(g_0^\pm)^2}-g_0^\pm}{\sqrt{2\Lambda_R\exp(-\Theta_{S,1}^\pm)+(g_0^\pm)^2}+g_0^\pm}\right]-2\right\rangle\right\}$$

$$-\frac{g_2^\pm}{2}\Lambda_R\exp(-\Theta_{S,1}^\pm)\left\{\left\langle\ell\text{n}\left[\frac{2(g_0^\pm)^2}{\Lambda_R}\frac{\sqrt{2\Lambda_R\exp(-\Theta_{S,1}^\pm)+(g_0^\pm)^2}-g_0^\pm}{\sqrt{2\Lambda_R\exp(-\Theta_{S,1}^\pm)+(g_0^\pm)^2}+g_0^\pm}\right]\right\rangle^2-(\Theta_{S,1}^\pm)^2+\frac{8}{3}\Theta_{S,1}^\pm+\frac{8}{3}\right\}$$

$$-g_2^\pm g_0^\pm\sqrt{2\Lambda_R\exp(-\Theta_{S,1}^\pm)+(g_0^\pm)^2}\,\ell\text{n}\left[\frac{2(g_0^\pm)^2}{\Lambda_R}\frac{\sqrt{2\Lambda_R\exp(-\Theta_{S,1}^\pm)+(g_0^\pm)^2}-g_0^\pm}{\sqrt{2\Lambda_R\exp(-\Theta_{S,1}^\pm)+(g_0^\pm)^2}+g_0^\pm}\right]-g_2^\pm(g_0^\pm)^2\left(\Theta_{S,1}^\pm+\frac{8}{3}\right)$$

$$+\left[\pm\frac{\tilde{T}_{S,3}^\pm(\tilde{r}_{f,S})+g_3^\pm}{g_0^\pm}+\frac{g_1^\pm}{(g_0^\pm)^2}\left(\frac{d\tilde{T}_{S,1}^\pm}{d\tilde{r}}\right)_{\tilde{r}_{f,S}}\right]\Lambda_R\exp(-\Theta_{S,1}^\pm)$$

From (C-4): $(d\tilde{T}_{S,1}^-/d\tilde{r})_{\tilde{r}_{f,S}}=\tilde{T}_{b,S,A}(\tilde{m}/\tilde{r}_{f,S}^2)$; $(d\tilde{T}_{S,1}^+/d\tilde{r})_{\tilde{r}_{f,S}}=-a_{T,1}^+(\tilde{m}/\tilde{r}_{f,S}^2)\exp(-\tilde{m}/\tilde{r}_{f,S})$

(5) Summary

$$\Theta_{S,1}^-=\ell\text{n}\{1-[\Lambda_R/\langle 2(g_0^-)^2\rangle]\exp(g_1^-+g_0^-\zeta)\}^2-g_0^-\zeta-g_1^-$$

$$\exp(-\Theta_{S,1}^-)=\exp(g_1^-+g_0^-\zeta)/\{1-[\Lambda_R/\langle 2(g_0^-)^2\rangle]\exp(g_1^-+g_0^-\zeta)\}^2$$

$$d\Theta_{S,1}^-/d\zeta=-[2\Lambda_R\exp(-\Theta_{S,1}^-)+(g_0^-)^2]^{1/2}=-g_0^-[2(g_0^-)^2+\Lambda_R\exp(g_1^-+g_0^-\zeta)]/[2(g_0^-)^2-\Lambda_R\exp(g_1^-+g_0^-\zeta)]$$

$$\Theta_{S,2}^-=(\tilde{T}_{b,S,A}/g_0^-)(d\Theta_{S,1}^-/d\zeta)=-\tilde{T}_{b,S,A}[2(g_0^-)^2+\Lambda_R\exp(g_1^-+g_0^-\zeta)]/[2(g_0^-)^2-\Lambda_R\exp(g_1^-+g_0^-\zeta)]$$

$$g_0^-=(\tilde{T}_f-\tilde{T}_0)(\tilde{m}/\tilde{r}_{f,S}^2),\quad g_1^-=\tilde{T}_{b,S,2}\exp[\tilde{m}(1-\tilde{r}_{f,S}^{-1})],\quad \tilde{T}_{b,S,A}=\tilde{T}_{b,S,1}\exp[\tilde{m}(1-\tilde{r}_{f,S}^{-1})]$$

$$g_2^-=[2/(3\tilde{T}_f)](\tilde{T}_{b,S,A}/g_0^-),\quad g_3^-=[(\tilde{m}-2\tilde{r}_{f,S})/\tilde{r}_{f,S}^2](\tilde{T}_{b,S,A}/g_0^-)$$

$$\Theta_{S,1}^+=\ell\text{n}\{1-[\Lambda_R/\langle 2(g_0^+)^2\rangle]\exp(g_1^+-g_0^+\zeta)\}^2+g_0^+\zeta-g_1^+$$

$$\exp(-\Theta_{S,1}^+)=\exp(g_1^+-g_0^+\zeta)/\{1-[\Lambda_R/\langle 2(g_0^+)^2\rangle]\exp(g_1^+-g_0^+\zeta)\}^2$$

$$d\Theta_{S,1}^+/d\zeta=[2\Lambda_R\exp(-\Theta_{S,1}^+)+(g_0^+)^2]^{1/2}=g_0^+[2(g_0^+)^2+\Lambda_R\exp(g_1^+-g_0^+\zeta)]/[2(g_0^+)^2-\Lambda_R\exp(g_1^+-g_0^+\zeta)]$$

$$\Theta_{S,2}^+=-(a_{T,A}^+/g_0^+)(d\Theta_{S,1}^+/d\zeta)=-a_{T,A}^+[2(g_0^+)^2+\Lambda_R\exp(g_1^+-g_0^+\zeta)]/[2(g_0^+)^2-\Lambda_R\exp(g_1^+-g_0^+\zeta)]$$

$$g_0^+=(\tilde{T}_f-\tilde{T}_\infty)(\tilde{m}/\tilde{r}_{f,S}^2)/[\exp(\tilde{m}/\tilde{r}_{f,S})-1],\quad g_1^+=a_{T,2}^+[1-\exp(-\tilde{m}/\tilde{r}_{f,S})]$$

$$g_2^+=-[2/(3\tilde{T}_f)](a_{T,A}^+/g_0^+),\quad g_3^+=-[(\tilde{m}-2\tilde{r}_{f,S})/\tilde{r}_{f,S}^2](a_{T,A}^+/g_0^+)\quad;\quad a_{T,A}^+=a_{T,1}^+[1-\exp(-\tilde{m}/\tilde{r}_{f,S})]$$

$$\Theta_{S,3}^\pm=\mp\left[\frac{8}{3}g_2^\pm g_0^\pm-\frac{1}{g_0^\pm}\left(\frac{d\tilde{T}_{S,1}^\pm}{d\tilde{r}}\right)_{\tilde{r}_{f,S}}\right]\left\{2+\frac{\sqrt{2\Lambda_R\exp(-\Theta_{S,1}^\pm)+(g_0^\pm)^2}}{g_0^\pm}\left\langle\ell\text{n}\left[\frac{2(g_0^\pm)^2}{\Lambda_R}\frac{\sqrt{2\Lambda_R\exp(-\Theta_{S,1}^\pm)+(g_0^\pm)^2}-g_0^\pm}{\sqrt{2\Lambda_R\exp(-\Theta_{S,1}^\pm)+(g_0^\pm)^2}+g_0^\pm}\right]-2\right\rangle\right\}$$

$$\pm\frac{g_2^\pm}{2}\sqrt{2\Lambda_R\exp(-\Theta_{S,1}^\pm)+(g_0^\pm)^2}\left\{\left\langle \ell\mathrm{n}\left[\frac{2(g_0^\pm)^2}{\Lambda_R}\frac{\sqrt{2\Lambda_R\exp(-\Theta_{S,1}^\pm)+(g_0^\pm)^2}-g_0^\pm}{\sqrt{2\Lambda_R\exp(-\Theta_{S,1}^\pm)+(g_0^\pm)^2}+g_0^\pm}\right]\right\rangle^2-(\Theta_{S,1}^\pm)^2\right\}$$

$$\pm 2g_2^\pm g_0^\pm\left\{\ell\mathrm{n}\left[\frac{2(g_0^\pm)^2}{\Lambda_R}\frac{\sqrt{2\Lambda_R\exp(-\Theta_{S,1}^\pm)+(g_0^\pm)^2}-g_0^\pm}{\sqrt{2\Lambda_R\exp(-\Theta_{S,1}^\pm)+(g_0^\pm)^2}+g_0^\pm}\right]\right\}-\frac{\tilde{T}_{S,3}^\pm(\tilde{r}_{f,S})+g_3^\pm}{g_0^\pm}\sqrt{2\Lambda_R\exp(-\Theta_{S,1}^\pm)+(g_0^\pm)^2}+g_3^\pm$$

$$\mp\frac{g_1^\pm}{g_0^\pm}\left(\frac{d\tilde{T}_{S,1}^\pm}{d\tilde{r}}\right)_{\tilde{r}_{f,S}}\frac{\sqrt{2\Lambda_R\exp(-\Theta_{S,1}^\pm)+(g_0^\pm)^2}}{g_0^\pm}\mp\frac{2}{3}g_2^\pm\Theta_{S,1}^\pm\sqrt{2\Lambda_R\exp(-\Theta_{S,1}^\pm)+(g_0^\pm)^2}$$

$$\frac{d\Theta_{S,3}^\pm}{d\zeta}=\left[\frac{8}{3}g_2^\pm-\frac{1}{(g_0^\pm)^2}\left(\frac{d\tilde{T}_{S,1}^\pm}{d\tilde{r}}\right)_{\tilde{r}_{f,S}}\right]\left\{g_0^\pm\sqrt{2\Lambda_R\exp(-\Theta_{S,1}^\pm)+(g_0^\pm)^2}+\Lambda_R\exp(-\Theta_{S,1}^\pm)\left\langle\ell\mathrm{n}\left[\frac{2(g_0^\pm)^2}{\Lambda_R}\frac{\sqrt{2\Lambda_R\exp(-\Theta_{S,1}^\pm)+(g_0^\pm)^2}-g_0^\pm}{\sqrt{2\Lambda_R\exp(-\Theta_{S,1}^\pm)+(g_0^\pm)^2}+g_0^\pm}\right]-2\right\rangle\right\}$$

$$-\frac{g_2^\pm}{2}\Lambda_R\exp(-\Theta_{S,1}^\pm)\left\{\left\langle\ell\mathrm{n}\left[\frac{2(g_0^\pm)^2}{\Lambda_R}\frac{\sqrt{2\Lambda_R\exp(-\Theta_{S,1}^\pm)+(g_0^\pm)^2}-g_0^\pm}{\sqrt{2\Lambda_R\exp(-\Theta_{S,1}^\pm)+(g_0^\pm)^2}+g_0^\pm}\right]\right\rangle^2-(\Theta_{S,1}^\pm)^2\right\}$$

$$-g_2^\pm g_0^\pm\sqrt{2\Lambda_R\exp(-\Theta_{S,1}^\pm)+(g_0^\pm)^2}\,\ell\mathrm{n}\left[\frac{2(g_0^\pm)^2}{\Lambda_R}\frac{\sqrt{2\Lambda_R\exp(-\Theta_{S,1}^\pm)+(g_0^\pm)^2}-g_0^\pm}{\sqrt{2\Lambda_R\exp(-\Theta_{S,1}^\pm)+(g_0^\pm)^2}+g_0^\pm}\right]$$

$$-g_2^\pm[2\Lambda_R\exp(-\Theta_{S,1}^\pm)+(g_0^\pm)^2]\left(\Theta_{S,1}^\pm+\frac{2}{3}\right)$$

$$+\left[\pm\frac{\tilde{T}_{S,3}^\pm(\tilde{r}_{f,S})+g_3^\pm}{g_0^\pm}+\frac{g_1^\pm}{(g_0^\pm)^2}\left(\frac{d\tilde{T}_{S,1}^\pm}{d\tilde{r}}\right)_{\tilde{r}_{f,S}}+\frac{2}{3}g_2^\pm\Theta_{S,1}^\pm\right]\Lambda_R\exp(-\Theta_{S,1}^\pm)-2g_2^\pm(g_0^\pm)^2$$

(G) Inner Expansions

Define stretched variable: $\xi=(\tilde{r}-\tilde{r}_{f,S})/\varepsilon$ or $\tilde{r}=\tilde{r}_{f,S}+\varepsilon\xi$ $\Rightarrow$ $d\tilde{r}=\varepsilon d\xi$

$$\tilde{r}^2=(\tilde{r}_{f,S}+\varepsilon\xi)^2=\tilde{r}_{f,S}^2+\varepsilon(2\tilde{r}_{f,S}\xi)+O(\varepsilon^2)=\tilde{r}_{f,S}^2[1+\varepsilon(2\xi/\tilde{r}_{f,S})+O(\varepsilon^2)]$$

$$1/\tilde{r}=1/(\tilde{r}_{f,S}+\varepsilon\xi)=1/\{\tilde{r}_{f,S}[1+\varepsilon(\xi/\tilde{r}_{f,S})]\}=(1/\tilde{r}_{f,S})[1+\varepsilon(\xi/\tilde{r}_{f,S})]^{-1}=(1/\tilde{r}_{f,S})[1-\varepsilon(\xi/\tilde{r}_{f,S})+O(\varepsilon^2)]$$

$$1/\tilde{r}^2=(1/\tilde{r}_{f,S})^2[1-\varepsilon(\xi/\tilde{r}_{f,S})+O(\varepsilon^2)]^2=(1/\tilde{r}_{f,S})^2[1-\varepsilon(2\xi/\tilde{r}_{f,S})+O(\varepsilon^2)]$$

Define inner expansions and the small expansion parameter:

$$\tilde{T}_S=[\tilde{T}_f-\varepsilon\theta_1-\varepsilon^2\theta_2+O(\varepsilon^3)]+O(\delta)+O(\varepsilon/\delta)$$

$$\tilde{Y}_{1,S}=[\varepsilon\phi_{1,1}+\varepsilon^2\phi_{1,2}+O(\varepsilon^3)]+O(\delta)+O(\varepsilon/\delta)\quad ;\quad \tilde{Y}_{2,S}=[\varepsilon\phi_{2,1}+\varepsilon^2\phi_{2,2}+O(\varepsilon^3)]+O(\delta)+O(\varepsilon/\delta)$$

Define small expansion parameter: $\varepsilon=\tilde{T}_f^2/\tilde{E}_K$

$$\exp(-\tilde{E}/\tilde{T}_S)=\exp\{-\tilde{E}/[\tilde{T}_f-\varepsilon(\theta_1+\cdots)+O(\varepsilon^2)]\}=\exp\langle-\tilde{E}/\{\tilde{T}_f[1-\varepsilon(\theta_1/\tilde{T}_f)+\cdots]\}\rangle$$

$$=\exp\{-(\tilde{E}/\tilde{T}_f)[1+\varepsilon(\theta_1/\tilde{T}_f)+\cdots]\}=\exp(-\tilde{E}/\tilde{T}_f)\exp[-\varepsilon(\tilde{E}/\tilde{T}_f^2)\theta_1+\cdots]$$

$$\exp(-\tilde{E}_K/\tilde{T}_S)=\exp(-\tilde{E}_K/\tilde{T}_f)\exp[-\varepsilon(\tilde{E}_K/\tilde{T}_f^2)\theta_1+\cdots]=\exp(-\tilde{E}_K/\tilde{T}_f)\exp(-\theta_1)+\cdots$$

$$\exp(-\tilde{E}_R/\tilde{T}_S)=\exp(-\tilde{E}_R/\tilde{T}_f)\exp[-\varepsilon(\tilde{E}_R/\tilde{T}_f^2)\theta_1+\cdots]=\exp(-\tilde{E}_R/\tilde{T}_f)\exp[-(\varepsilon/\delta)\theta_1+\cdots]$$

$$=\exp(-\tilde{E}_R/\tilde{T}_f)[1-(\varepsilon/\delta)\theta_1+\cdots]$$

Define reduced Damköhler number: $\Lambda_K=\varepsilon^3 Da_K\exp(-\tilde{E}_K/\tilde{T}_f)$

(1) $$\frac{1}{\tilde{r}^2}\frac{d}{d\tilde{r}}\left(\tilde{m}\tilde{T}_S-\tilde{r}^2\frac{d\tilde{T}_S}{d\tilde{r}}\right)=Da_K\tilde{Y}_{1,S}\tilde{Y}_{2,S}\exp(-\tilde{E}_K/\tilde{T}_S)-Da_R\exp(-\tilde{E}_R/\tilde{T}_S)$$

$$\frac{1+O(\varepsilon)}{\tilde{r}_{f,S}^2}\frac{d}{\varepsilon d\xi}\left\{\tilde{m}[\tilde{T}_f+O(\varepsilon)]-\tilde{r}_{f,S}^2[1+O(\varepsilon)]\frac{d[\tilde{T}_f-\varepsilon\theta_1+O(\varepsilon^2)+\cdots]}{\varepsilon d\xi}\right\}$$

$$=Da_K(\varepsilon\phi_{1,1}+\cdots)(\varepsilon\phi_{2,1}+\cdots)\{\exp(-\tilde{E}_K/\tilde{T}_f)\exp(-\theta_1)+\cdots\}-Da_R\exp(-\tilde{E}_R/\tilde{T}_f)[1+O(\varepsilon/\delta)]$$

$$=\{\varepsilon^2 Da_K\exp(-\tilde{E}_K/\tilde{T}_f)\phi_{1,1}\phi_{2,1}\exp(-\theta_1)+\cdots\}-[Da_R\exp(-\tilde{E}_R/\tilde{T}_f)+\cdots]$$

$$=\varepsilon^{-1}\{[\Lambda_K\phi_{1,1}\phi_{2,1}\exp(-\theta_1)+\cdots]-(\varepsilon/\delta)(\Lambda_R+\cdots)\}$$

The leading order term is $d^2\theta_1/d\xi^2=\Lambda_K\phi_{1,1}\phi_{2,1}\exp(-\theta_1)$

(2) $$\frac{1}{\tilde{r}^2}\frac{d}{d\tilde{r}}\left(\tilde{m}\tilde{Y}_{1,S}-\frac{\tilde{r}^2}{Le_1}\frac{d\tilde{Y}_{1,S}}{d\tilde{r}}\right)=-Da_K\tilde{Y}_{1,S}\tilde{Y}_{2,S}\exp(-\tilde{E}_K/\tilde{T}_S)$$

$$\frac{1}{\tilde{r}^2}\frac{d}{d\tilde{r}}\left(\tilde{m}\tilde{Y}_{2,S}-\frac{\tilde{r}^2}{Le_2}\frac{d\tilde{Y}_{2,S}}{d\tilde{r}}\right)=-Da_K\tilde{Y}_{1,S}\tilde{Y}_{2,S}\exp(-\tilde{E}_K/\tilde{T}_S)$$

$$\frac{1}{\tilde{r}^2}\frac{d}{d\tilde{r}}\left(\tilde{m}\tilde{Y}_{2,S}-\frac{\tilde{r}^2}{Le_2}\frac{d\tilde{Y}_{2,S}}{d\tilde{r}}\right)-\frac{1}{\tilde{r}^2}\frac{d}{d\tilde{r}}\left(\tilde{m}\tilde{Y}_{1,S}-\frac{\tilde{r}^2}{Le_1}\frac{d\tilde{Y}_{1,S}}{d\tilde{r}}\right)=0$$

$$\frac{d}{d\tilde{r}}\left(\tilde{m}\tilde{Y}_{2,S}-\frac{\tilde{r}^2}{Le_2}\frac{d\tilde{Y}_{2,S}}{d\tilde{r}}\right)-\frac{d}{d\tilde{r}}\left(\tilde{m}\tilde{Y}_{1,S}-\frac{\tilde{r}^2}{Le_1}\frac{d\tilde{Y}_{1,S}}{d\tilde{r}}\right)=0 \quad \text{or} \quad \left(\tilde{m}\tilde{Y}_{2,S}-\frac{\tilde{r}^2}{Le_2}\frac{d\tilde{Y}_{2,S}}{d\tilde{r}}\right)-\left(\tilde{m}\tilde{Y}_{1,S}-\frac{\tilde{r}^2}{Le_1}\frac{d\tilde{Y}_{1,S}}{d\tilde{r}}\right)=constant$$

$$\tilde{m}[\varepsilon\phi_{2,1}+O(\varepsilon^2)+\cdots]-\tilde{m}[\varepsilon\phi_{1,1}+O(\varepsilon^2)+\cdots]$$

$$-[\tilde{r}_{f,S}^2+\varepsilon(2\tilde{r}_{f,S}\xi)+O(\varepsilon^2)]\left[\frac{1}{Le_2}+O(\varepsilon^2)\right]\frac{d[\varepsilon\phi_{2,1}+\varepsilon^2\phi_{2,2}+O(\varepsilon^3)+\cdots]}{\varepsilon d\xi}$$

$$+[\tilde{r}_{f,S}^2+\varepsilon(2\tilde{r}_{f,S}\xi)+O(\varepsilon^2)]\left[\frac{1}{Le_1}+O(\varepsilon^2)\right]\frac{d[\varepsilon\phi_{1,1}+\varepsilon^2\phi_{1,2}+O(\varepsilon^3)+\cdots]}{\varepsilon d\xi}=c$$

Keeping the two leading order terms

$$\frac{\tilde{r}_{f,S}^2}{Le_1}\frac{d\phi_{1,1}}{d\xi}-\frac{\tilde{r}_{f,S}^2}{Le_2}\frac{d\phi_{2,1}}{d\xi}=\tilde{c}_1 \quad \text{or} \quad \frac{1}{Le_1}\frac{d\phi_{1,1}}{d\xi}-\frac{1}{Le_2}\frac{d\phi_{2,1}}{d\xi}=c_1 \quad \therefore \quad (\phi_{1,1}/Le_1)-(\phi_{2,1}/Le_2)=c_1\xi+c_2$$

$$\left[\frac{\tilde{r}_{f,S}^2}{Le_1}\frac{d\phi_{1,2}}{d\xi}+\left(\frac{2\tilde{r}_{f,S}}{Le_1}\xi\right)\frac{d\phi_{1,1}}{d\xi}-\tilde{m}\phi_{1,1}\right]-\left[\frac{\tilde{r}_{f,S}^2}{Le_2}\frac{d\phi_{2,2}}{d\xi}+\left(\frac{2\tilde{r}_{f,S}}{Le_2}\xi\right)\frac{d\phi_{2,1}}{d\xi}-\tilde{m}\phi_{2,1}\right]=\tilde{c}_3$$

or $$\left[\frac{1}{Le_1}\frac{d\phi_{1,2}}{d\xi}+\left(\frac{2}{Le_1\tilde{r}_{f,S}}\xi\right)\frac{d\phi_{1,1}}{d\xi}-\frac{\tilde{m}}{\tilde{r}_{f,S}^2}\phi_{1,1}\right]-\left[\frac{1}{Le_2}\frac{d\phi_{2,2}}{d\xi}+\left(\frac{2}{Le_2\tilde{r}_{f,S}}\xi\right)\frac{d\phi_{2,1}}{d\xi}-\frac{\tilde{m}}{\tilde{r}_{f,S}^2}\phi_{2,1}\right]=c_3$$

(3) $$\frac{1}{\tilde{r}^2}\frac{d}{d\tilde{r}}\left(\tilde{m}\tilde{T}_S-\tilde{r}^2\frac{d\tilde{T}_S}{d\tilde{r}}\right)=Da_K\tilde{Y}_{1,S}\tilde{Y}_{2,S}\exp(-\tilde{E}_K/\tilde{T}_S)-Da_R\exp(-\tilde{E}_R/\tilde{T}_S)$$

$$\frac{1}{\tilde{r}^2}\frac{d}{d\tilde{r}}\left(\tilde{m}\tilde{Y}_{1,S}-\frac{\tilde{r}^2}{Le_1}\frac{d\tilde{Y}_{1,S}}{d\tilde{r}}\right)=-Da_K\tilde{Y}_{1,S}\tilde{Y}_{2,S}\exp(-\tilde{E}_K/\tilde{T}_S)$$

$$\frac{d}{d\tilde{r}}\left(\tilde{m}\tilde{T}_S-\tilde{r}^2\frac{d\tilde{T}_S}{d\tilde{r}}\right)+\frac{d}{d\tilde{r}}\left(\tilde{m}\tilde{Y}_{1,S}-\frac{\tilde{r}^2}{Le_1}\frac{d\tilde{Y}_{1,S}}{d\tilde{r}}\right)=-Da_R\tilde{r}^2\exp(-\tilde{E}_R/\tilde{T}_S)$$

$$\frac{d}{\varepsilon d\xi}\left\{\tilde{m}[\tilde{T}_f-\varepsilon(\theta_1+\cdots)+O(\varepsilon^2)]-[\tilde{r}_{f,S}^2+\varepsilon(2\tilde{r}_{f,S}\xi)+O(\varepsilon^2)]\frac{d[\tilde{T}_f-\varepsilon\theta_1-\varepsilon^2\theta_2+O(\varepsilon^3)+\cdots]}{\varepsilon d\xi}\right\}$$

$$+\frac{d}{\varepsilon d\xi}\left\{\tilde{m}[\varepsilon\phi_{1,1}+O(\varepsilon^2)+\cdots]-[\tilde{r}_{f,S}^2+\varepsilon(2\tilde{r}_{f,S}\xi)+O(\varepsilon^2)]\left[\frac{1}{Le_1}+O(\varepsilon^2)\right]\frac{d[\varepsilon\phi_{1,1}+\varepsilon^2\phi_{1,2}+O(\varepsilon^3)+\cdots]}{\varepsilon d\xi}\right\}$$

$$=-\varepsilon^{-1}(\varepsilon/\delta)[\tilde{r}_{f,S}^2+O(\varepsilon)](\Lambda_R+\cdots)$$

Keeping the two leading order terms

$$\tilde r_{f,S}^2\frac{d^2\theta_1}{d\xi^2}-\frac{\tilde r_{f,S}^2}{Le_1}\frac{d^2\phi_{1,1}}{d\xi^2}=0 \quad \text{or} \quad \frac{d^2\theta_1}{d\xi^2}-\frac{1}{Le_1}\frac{d^2\phi_{1,1}}{d\xi^2}=0 \quad \therefore \quad \theta_1-(\phi_{1,1}/Le_1)=c_4\xi+c_5$$

$$\frac{d}{d\xi}\left(\tilde r_{f,S}^2\frac{d\theta_2}{d\xi}+2\tilde r_{f,S}\,\xi\frac{d\theta_1}{d\xi}-\tilde m\theta_1\right)-\frac{d}{d\xi}\left[\frac{\tilde r_{f,S}^2}{Le_1}\frac{d\phi_{1,2}}{d\xi}+\left(\frac{2\tilde r_{f,S}}{Le_1}\xi\right)\frac{d\phi_{1,1}}{d\xi}-\tilde m\phi_{1,1}\right]=0$$

$$\text{or} \quad \left(\frac{d\theta_2}{d\xi}+\frac{2\xi}{\tilde r_{f,S}}\frac{d\theta_1}{d\xi}-\frac{\tilde m}{\tilde r_{f,S}^2}\theta_1\right)-\left[\frac{1}{Le_1}\frac{d\phi_{1,2}}{d\xi}+\left(\frac{2}{Le_1\tilde r_{f,S}}\xi\right)\frac{d\phi_{1,1}}{d\xi}-\frac{\tilde m}{\tilde r_{f,S}^2}\phi_{1,1}\right]=c_6$$

(4) Summary, the structure equations in the inner, chemically reactive region are

$$d^2\theta_1/d\xi^2=\Lambda_K\phi_{1,1}\phi_{2,1}\exp(-\theta_1) \quad ; \quad \Lambda_K=\varepsilon^3 Da_K\exp(-\tilde E_K/\tilde T_{f,S})$$

$$(\phi_{1,1}/Le_1)-(\phi_{2,1}/Le_2)=c_1\xi+c_2 \quad ; \quad \theta_1-(\phi_{1,1}/Le_1)=c_4\xi+c_5$$

$$\left[\frac{1}{Le_1}\frac{d\phi_{1,2}}{d\xi}+\left(\frac{2}{Le_1\tilde r_{f,S}}\xi\right)\frac{d\phi_{1,1}}{d\xi}-\frac{\tilde m}{\tilde r_{f,S}^2}\phi_{1,1}\right]-\left[\frac{1}{Le_2}\frac{d\phi_{2,2}}{d\xi}+\left(\frac{2}{Le_2\tilde r_{f,S}}\xi\right)\frac{d\phi_{2,1}}{d\xi}-\frac{\tilde m}{\tilde r_{f,S}^2}\phi_{2,1}\right]=c_3$$

$$\left(\frac{d\theta_2}{d\xi}+\frac{2\xi}{\tilde r_{f,S}}\frac{d\theta_1}{d\xi}-\frac{\tilde m}{\tilde r_{f,S}^2}\theta_1\right)-\left[\frac{1}{Le_1}\frac{d\phi_{1,2}}{d\xi}-\left(-\frac{2}{Le_1\tilde r_{f,S}}\xi\right)\frac{d\phi_{1,1}}{d\xi}-\frac{\tilde m}{\tilde r_{f,S}^2}\phi_{1,1}\right]=c_6$$

(H) Matching

Species equations: Between the inner and outer solutions

Energy equation: Between the inner solution and the solution in the radiation region

In the common regions between the outer and inner regions, $\tilde r=\tilde r_{f,S}+\varepsilon\xi$

Since $\tilde r=\tilde r_{f,S}+\delta\zeta$, we also have $\tilde r=\tilde r_{f,S}+\delta\zeta=\tilde r_{f,S}+\varepsilon\xi$ and hence $\zeta=(\varepsilon/\delta)\xi$

$$\begin{aligned}\tilde Y_{i,S}^{\pm}&=[\tilde Y_{i,S,0}^{\pm}+\varepsilon\tilde Y_{i,S,1}^{\pm}+O(\varepsilon^2)]+\delta[\tilde Y_{i,S,2}^{\pm}+O(\varepsilon)]+O(\delta^2)\\
&=\{[\tilde Y_{i,S,0}^{\pm}(\tilde r_{f,S})+(d\tilde Y_{i,S,0}^{\pm}/d\tilde r)_{\tilde r_{f,S}}(\varepsilon\xi)+(d^2\tilde Y_{i,S,0}^{\pm}/d\tilde r^2)_{\tilde r_{f,S}}(\varepsilon^2\xi^2/2)+O(\varepsilon^3)]\\
&\quad+\varepsilon[\tilde Y_{i,S,1}^{\pm}(\tilde r_{f,S})+(d\tilde Y_{i,S,1}^{\pm}/d\tilde r)_{\tilde r_{f,S}}(\varepsilon\xi)+O(\varepsilon^2)]+O(\varepsilon^2)\}+\delta\{[\tilde Y_{i,S,2}^{\pm}(\tilde r_{f,S})+O(\varepsilon)]+O(\varepsilon)\}+O(\delta^2)\\
&=[\varepsilon\phi_{i,1}+\varepsilon^2\phi_{i,2}+O(\varepsilon^3)]+O(\delta)+O(\varepsilon/\delta)\Big|_{\xi\to\pm\infty}\end{aligned}$$

$$\Rightarrow \quad \tilde Y_{i,S,0}^{\pm}(\tilde r_{f,S})=0 \quad ; \quad \tilde Y_{i,S,2}^{\pm}(\tilde r_{f,S})=0 \quad ; \quad \phi_{i,1}(\xi\to\pm\infty)=\tilde Y_{i,S,1}^{\pm}(\tilde r_{f,S})+(d\tilde Y_{i,S,0}^{\pm}/d\tilde r)_{\tilde r_{f,S}}\xi$$

$$(d\phi_{i,2}/d\xi)_{\xi\to\pm\infty}=(d\tilde Y_{i,S,1}^{\pm}/d\tilde r)_{\tilde r_{f,S}}+(d^2\tilde Y_{i,S,0}^{\pm}/d\tilde r^2)_{\tilde r_{f,S}}\xi$$

$$\begin{aligned}\tilde T_S^{\pm}&=[\tilde T_f-\varepsilon\Theta_{S,2}^{\pm}+O(\varepsilon^2)]-\delta[\Theta_{S,1}^{\pm}+\varepsilon\Theta_{S,3}^{\pm}+O(\varepsilon^2)]-\delta^2[\Theta_{S,4}^{\pm}+O(\varepsilon)]+O(\delta^3)\\
&=\{\tilde T_f-\varepsilon[\Theta_{S,2}^{\pm}(\zeta=0)+O(\varepsilon/\delta)]+O(\varepsilon^2)\}-\delta\{[\Theta_{S,1}^{\pm}(\zeta=0)+(d\Theta_{S,1}^{\pm}/d\zeta)_{\zeta=0}\langle(\varepsilon/\delta)\xi\rangle+O(\varepsilon/\delta)^2]\\
&\quad+\varepsilon[\Theta_{S,3}^{\pm}(\zeta=0)+(d\Theta_{S,3}^{\pm}/d\zeta)_{\zeta=0}\langle(\varepsilon/\delta)\xi\rangle+O(\varepsilon/\delta)^2]+O(\varepsilon^2)\}\\
&\quad-\delta^2\{[\Theta_{S,4}^{\pm}(\zeta=0)+(d\Theta_{S,4}^{\pm}/d\zeta)_{\zeta=0}\langle(\varepsilon/\delta)\xi\rangle+(d^2\Theta_{S,4}^{\pm}/d\zeta^2)_{\zeta=0}\langle(\varepsilon/\delta)^2\xi^2/2\rangle+O(\varepsilon/\delta)^3]+O(\varepsilon)\}+O(\delta^3)\\
&=[\tilde T_{f,S}-\varepsilon\theta_1-\varepsilon^2\theta_2+O(\varepsilon^3)]+O(\delta)+O(\varepsilon/\delta)\Big|_{\xi\to\pm\infty}\end{aligned}$$

$$\Rightarrow \quad \Theta_{S,1}^{\pm}(\zeta=0)=0 \quad ; \quad \theta_1(\xi\to\pm\infty)=\Theta_{S,2}^{\pm}(\zeta=0)+(d\Theta_{S,1}^{\pm}/d\zeta)_{\zeta=0}\xi$$

$$(d\theta_2/d\xi)_{\xi\to\pm\infty}=(d\Theta_{S,3}^{\pm}/d\zeta)_{\zeta=0}+(d^2\Theta_{S,4}^{\pm}/d\zeta^2)_{\zeta=0}\xi$$

(1) $\tilde Y_{1,S,0}^{-}(\tilde r_{f,S})=1-a_{1,0}^{-}\exp(-Le_1\tilde m/\tilde r_{f,S})=0 \quad \therefore \quad a_{1,0}^{-}\exp(-Le_1\tilde m/\tilde r_{f,S})=1$ or $a_{1,0}^{-}=\exp(Le_1\tilde m/\tilde r_{f,S})$

$$\tilde Y_{1,S,2}^{-}(\tilde r_{f,S})=-a_{1,2}^{-}\exp(-Le_1\tilde m/\tilde r_{f,S})=0 \quad \therefore \quad a_{1,2}^{-}=0 \quad \Rightarrow \quad \tilde Y_{1,S,2}^{-}=0$$

$$\begin{aligned}\phi_{1,1}(\xi\to-\infty)&=\tilde Y_{1,S,1}^{-}(\tilde r_{f,S})+(d\tilde Y_{1,S,0}^{-}/d\tilde r)_{\tilde r_{f,S}}\xi\\
&=-a_{1,1}^{-}\exp(-Le_1\tilde m/\tilde r_{f,S})-a_{1,0}^{-}(Le_1\tilde m/\tilde r_{f,S}^2)\exp(-Le_1\tilde m/\tilde r_{f,S})\xi\\
&=-a_{1,1}^{-}\exp(-Le_1\tilde m/\tilde r_{f,S})-(Le_1\tilde m/\tilde r_{f,S}^2)\xi\end{aligned}$$

$$(d\phi_{1,1}/d\xi)_{\xi\to-\infty}=(d\tilde{Y}^{-}_{1,S,0}/d\tilde{r})_{\tilde{r}_{f,S}}=-Le_1\tilde{m}/\tilde{r}^2_{f,S} \quad ; \quad (d\phi_{1,2}/d\xi)_{\xi\to-\infty}=(d\tilde{Y}^{-}_{1,S,1}/d\tilde{r})_{\tilde{r}_{f,S}}+(d^2\tilde{Y}^{-}_{1,S,0}/d\tilde{r}^2)_{\tilde{r}_{f,S}}\xi$$

$$\left[\frac{1}{Le_1}\frac{d\phi_{1,2}}{d\xi}+\left(\frac{2}{Le_1\tilde{r}_{f,S}}\xi\right)\frac{d\phi_{1,1}}{d\xi}-\frac{\tilde{m}}{\tilde{r}^2_{f,S}}\phi_{1,1}\right]_{\xi\to-\infty}$$

$$=(1/Le_1)[(d\tilde{Y}^{-}_{1,S,1}/d\tilde{r})_{\tilde{r}_{f,S}}+(d^2\tilde{Y}^{-}_{1,S,0}/d\tilde{r}^2)_{\tilde{r}_{f,S}}\xi]+[2/(Le_1\tilde{r}_{f,S})\xi](d\tilde{Y}^{-}_{1,S,0}/d\tilde{r})_{\tilde{r}_{f,S}}$$

$$-(\tilde{m}/\tilde{r}^2_{f,S})[\tilde{Y}^{-}_{1,S,1}(\tilde{r}_{f,S})+(d\tilde{Y}^{-}_{1,S,0}/d\tilde{r})_{\tilde{r}_{f,S}}\xi]$$

$$=[(1/Le_1)(d\tilde{Y}^{-}_{1,S,1}/d\tilde{r})-(\tilde{m}/\tilde{r}^2)\tilde{Y}^{-}_{1,S,1}]_{\tilde{r}_{f,S}}$$

$$-\langle\{[\tilde{m}-(2\tilde{r}/Le_1)](d\tilde{Y}^{-}_{1,S,0}/d\tilde{r})-(\tilde{r}^2/Le_1)(d^2\tilde{Y}^{-}_{1,S,0}/d\tilde{r}^2)\}/\tilde{r}^2\rangle_{\tilde{r}_{f,S}}\xi=0$$

(2) $\tilde{Y}^{+}_{1,S,0}(\tilde{r}_{f,S})=a^{+}_{1,0}[1-\exp(-Le_1\tilde{m}/\tilde{r}_{f,S})]=0 \quad \therefore \quad a^{+}_{1,0}=0$ and $\tilde{Y}^{+}_{1,S,0}=0$

$$(1/Le_1)(d\tilde{Y}^{+}_{1,S,1}/d\tilde{r})-(\tilde{m}/\tilde{r}^2)\tilde{Y}^{+}_{1,S,1}=-\tilde{m}a^{+}_{1,1}/\tilde{r}^2$$

$$(1/Le_1)(d^2\tilde{Y}^{+}_{1,S,1}/d\tilde{r}^2)-(\tilde{m}/\tilde{r}^2)(d\tilde{Y}^{+}_{1,S,1}/d\tilde{r})+(2\tilde{m}/\tilde{r}^3)\tilde{Y}^{+}_{1,S,1}=2\tilde{m}a^{+}_{1,1}/\tilde{r}^3$$

$\tilde{Y}^{+}_{1,S,2}(\tilde{r}_{f,S})=a^{+}_{1,2}[1-\exp(-Le_1\tilde{m}/\tilde{r}_{f,S})]=0 \quad \therefore \quad a^{+}_{1,2}=0$ and $\tilde{Y}^{+}_{1,S,2}=0$

$$\phi_{1,1}(\xi\to\infty)=\tilde{Y}^{+}_{1,S,1}(\tilde{r}_{f,S})+(d\tilde{Y}^{+}_{1,S,0}/d\tilde{r})_{\tilde{r}_{f,S}}\xi=\tilde{Y}^{+}_{1,S,1}(\tilde{r}_{f,S})$$

$$=a^{+}_{1,1}[1-\exp(-Le_1\tilde{m}/\tilde{r}_{f,S})]=a^{+}_{1,1}[1-\exp(-Le_1\tilde{m}/\tilde{r}_{f,S})]$$

$$(d\phi_{1,1}/d\xi)_{\xi\to\infty}=0$$

$$(d\phi_{1,2}/d\xi)_{\xi\to\infty}=(d\tilde{Y}^{+}_{1,S,1}/d\tilde{r})_{\tilde{r}_{f,S}}+(d^2\tilde{Y}^{+}_{1,S,0}/d\tilde{r}^2)_{\tilde{r}_{f,S}}\xi=(d\tilde{Y}^{+}_{1,S,1}/d\tilde{r})_{\tilde{r}_{f,S}}$$

$$\left[\frac{1}{Le_1}\frac{d\phi_{1,2}}{d\xi}+\left(\frac{2}{Le_1\tilde{r}_{f,S}}\xi\right)\frac{d\phi_{1,1}}{d\xi}-\frac{\tilde{m}}{\tilde{r}^2_{f,S}}\phi_{1,1}\right]_{\xi\to\infty}=[(1/Le_1)(d\tilde{Y}^{+}_{1,S,1}/d\tilde{r})-0-(\tilde{m}/\tilde{r}^2)\tilde{Y}^{+}_{1,S,1}]_{\tilde{r}_{f,S}}=-a^{+}_{1,1}(\tilde{m}/\tilde{r}^2_{f,S})$$

(3) $\tilde{Y}^{-}_{2,S,0}(\tilde{r}_{f,S})=a^{-}_{2,0}\exp(-Le_2\tilde{m}/\tilde{r}_{f,S})=0 \quad \therefore \quad a^{-}_{2,0}=0$ and $\tilde{Y}^{-}_{2,S,0}=0$

$$(1/Le_2)(d\tilde{Y}^{-}_{2,S,1}/d\tilde{r})-(\tilde{m}/\tilde{r}^2)\tilde{Y}^{-}_{2,S,1}=0 \quad ; \quad (1/Le_2)(d^2\tilde{Y}^{-}_{2,S,1}/d\tilde{r}^2)-(\tilde{m}/\tilde{r}^2)(d\tilde{Y}^{-}_{2,S,1}/d\tilde{r})+(2\tilde{m}/\tilde{r}^3)\tilde{Y}^{-}_{2,S,1}=0$$

$\tilde{Y}^{-}_{2,S,2}(\tilde{r}_{f,S})=a^{-}_{2,2}\exp(-Le_2\tilde{m}/\tilde{r}_{f,S})=0 \quad \therefore \quad a^{-}_{2,2}=0$ and $\tilde{Y}^{-}_{2,S,2}=0$

$$\phi_{2,1}(\xi\to-\infty)=\tilde{Y}^{-}_{2,S,1}(\tilde{r}_{f,S})+(d\tilde{Y}^{-}_{2,S,0}/d\tilde{r})_{\tilde{r}_{f,S}}\xi=\tilde{Y}^{-}_{2,S,1}(\tilde{r}_{f,S})=a^{-}_{2,1}\exp(-Le_2\tilde{m}/\tilde{r}_{f,S}) \quad ; \quad (d\phi_{2,1}/d\xi)_{\xi\to-\infty}=0$$

$$(d\phi_{2,2}/d\xi)_{\xi\to-\infty}=(d\tilde{Y}^{-}_{2,S,1}/d\tilde{r})_{\tilde{r}_{f,S}}+(d^2\tilde{Y}^{-}_{2,S,0}/d\tilde{r}^2)_{\tilde{r}_{f,S}}\xi=(d\tilde{Y}^{-}_{2,S,1}/d\tilde{r})_{\tilde{r}_{f,S}}=a^{-}_{2,1}(Le_2\tilde{m}/\tilde{r}^2_{f,S})\exp(-Le_2\tilde{m}/\tilde{r}_{f,S})$$

$$\left[\frac{1}{Le_2}\frac{d\phi_{2,2}}{d\xi}+\left(\frac{2}{Le_2\tilde{r}_{f,S}}\xi\right)\frac{d\phi_{2,1}}{d\xi}-\frac{\tilde{m}}{\tilde{r}^2_{f,S}}\phi_{2,1}\right]_{\xi\to-\infty}=a^{-}_{2,1}\frac{\tilde{m}}{\tilde{r}^2_{f,S}}\exp\left(-\frac{Le_2\tilde{m}}{\tilde{r}_{f,S}}\right)-0-\frac{\tilde{m}}{\tilde{r}^2_{f,S}}a^{-}_{2,1}\exp\left(-\frac{Le_2\tilde{m}}{\tilde{r}_{f,S}}\right)=0$$

(4) $\tilde{Y}^{+}_{2,S,0}(\tilde{r}_{f,S})=\tilde{Y}_{2,\infty}-a^{+}_{2,0}[1-\exp(-Le_2\tilde{m}/\tilde{r}_{f,S})]=0$

$a^{+}_{2,0}[1-\exp(-Le_2\tilde{m}/\tilde{r}_{f,S})]=\tilde{Y}_{2,\infty}$ or $a^{+}_{2,0}=\tilde{Y}_{2,\infty}/[1-\exp(-Le_2\tilde{m}/\tilde{r}_{f,S})]$

$\tilde{Y}^{+}_{2,S,2}(\tilde{r}_{f,S})=-a^{+}_{2,2}[1-\exp(-Le_2\tilde{m}/\tilde{r}_{f,S})]=0 \quad \therefore \quad a^{+}_{2,2}=0$ and $\tilde{Y}^{+}_{2,S,2}=0$

$$\phi_{2,1}(\xi\to\infty)=\tilde{Y}^{+}_{2,S,1}(\tilde{r}_{f,S})+(d\tilde{Y}^{+}_{2,S,0}/d\tilde{r})_{\tilde{r}_{f,S}}\xi$$

$$=-a^{+}_{2,1}[1-\exp(-Le_2\tilde{m}/\tilde{r}_{f,S})]+a^{+}_{2,0}(Le_2\tilde{m}/\tilde{r}^2_{f,S})\exp(-Le_2\tilde{m}/\tilde{r}_{f,S})\xi$$

$$=-a^{+}_{2,1}[1-\exp(-Le_2\tilde{m}/\tilde{r}_{f,S})]+(a^{+}_{2,0}\tilde{m}/\tilde{r}_{f,S})\exp(-Le_2\tilde{m}/\tilde{r}_{f,S})(Le_2/\tilde{r}_{f,S})\xi$$

$$=-a^{+}_{2,1}[1-\exp(-Le_2\tilde{m}/\tilde{r}_{f,S})]+\{\tilde{Y}_{2,\infty}(\tilde{m}/\tilde{r}_{f,S})\exp(-Le_2\tilde{m}/\tilde{r}_{f,S})/[1-\exp(-Le_2\tilde{m}/\tilde{r}_{f,S})]\}(Le_2/\tilde{r}_{f,S})\xi$$

$$=-a^{+}_{2,1}[1-\exp(-Le_2\tilde{m}/\tilde{r}_{f,S})]+\{\tilde{Y}_{2,\infty}(\tilde{m}/\tilde{r}_{f,S})/[\exp(Le_2\tilde{m}/\tilde{r}_{f,S})-1]\}(Le_2/\tilde{r}_{f,S})\xi$$

$$(d\phi_{2,1}/d\xi)_{\xi\to\infty}=a^{+}_{2,0}(Le_2\tilde{m}/\tilde{r}^2_{f,S})\exp(-Le_2\tilde{m}/\tilde{r}_{f,S})$$

$$=\tilde{Y}_{2,\infty}(Le_2\tilde{m}/\tilde{r}^2_{f,S})\exp(-Le_2\tilde{m}/\tilde{r}_{f,S})/[1-\exp(-Le_2\tilde{m}/\tilde{r}_{f,S})]\}=\tilde{Y}_{2,\infty}(Le_2\tilde{m}/\tilde{r}^2_{f,S})/[\exp(Le_2\tilde{m}/\tilde{r}_{f,S})-1]\}$$

$$(d\phi_{2,2}/d\xi)_{\xi\to\infty}=(d\tilde{Y}^{+}_{2,S,1}/d\tilde{r})_{\tilde{r}_{f,S}}+(d^2\tilde{Y}^{+}_{2,S,0}/d\tilde{r}^2)_{\tilde{r}_{f,S}}\xi$$

$$=a_{2,1}^{+}(Le_2\,\tilde m/\tilde r_{f,S}^2)\exp(-Le_2\,\tilde m/\tilde r_{f,S})+a_{2,0}^{+}[(Le_2\,\tilde m/\tilde r_{f,S}^2)^2-(2\,Le_2\,\tilde m/\tilde r_{f,S}^3)]\exp(-Le_2\,\tilde m/\tilde r_{f,S})\xi$$

$$\left[\frac{1}{Le_2}\frac{d\phi_{2,2}}{d\xi}+\left(\frac{2}{Le_2\,\tilde r_{f,S}}\xi\right)\frac{d\phi_{2,1}}{d\xi}-\frac{\tilde m}{\tilde r_{f,S}^2}\phi_{2,1}\right]_{\xi\to\infty}$$

$$=a_{2,1}^{+}(\tilde m/\tilde r_{f,S}^2)\exp(-Le_2\,\tilde m/\tilde r_{f,S})$$

$$+a_{2,0}^{+}[Le_2(\tilde m/\tilde r_{f,S}^2)^2-(2\tilde m/\tilde r_{f,S}^3)]\exp(-Le_2\,\tilde m/\tilde r_{f,S})\xi+a_{2,0}^{+}(2/\tilde r_{f,S})\xi\,(\tilde m/\tilde r_{f,S}^2)\exp(-Le_2\,\tilde m/\tilde r_{f,S})$$

$$+(\tilde m/\tilde r_{f,S}^2)\{a_{2,1}^{+}[1-\exp(-Le_2\,\tilde m/\tilde r_{f,S})]-(a_{2,0}^{+}\,\tilde m/\tilde r_{f,S})\exp(-Le_2\,\tilde m/\tilde r_{f,S})\,(Le_2/\tilde r_{f,S})\xi\,\}$$

$$=a_{2,1}^{+}(\tilde m/\tilde r_{f,S}^2)$$

(5) $\Theta_{S,1}^{\pm}(\zeta=0)=\ell\text{n}\langle 1-\{\Lambda_R/[2(g_0^{\pm})^2]\}\exp(g_1^{\pm})\rangle^2-g_1^{\pm}=0 \quad \therefore \quad \ell\text{n}\langle 1-\{\Lambda_R/[2(g_0^{\pm})^2]\}\exp(g_1^{\pm})\rangle^2=g_1^{\pm}$

$$\exp(g_1^{\pm})=\langle 1-\{\Lambda_R/[2(g_0^{\pm})^2]\}\exp(g_1^{\pm})\rangle^2=1-[\Lambda_R/(g_0^{\pm})^2]\exp(g_1^{\pm})+\{\Lambda_R/[2(g_0^{\pm})^2]\}^2\,[\exp(g_1^{\pm})]^2$$

$$\{\Lambda_R/[2(g_0^{\pm})^2]\}^2[\exp(g_1^{\pm})]^2-\{1+[\Lambda_R/(g_0^{\pm})^2]\}\exp(g_1^{\pm})+1=0$$

$$\exp(g_1^{\pm})=\langle\{1+[\Lambda_R/(g_0^{\pm})^2]\}\pm[\{1+[\Lambda_R/(g_0^{\pm})^2]\}^2-4\{\Lambda_R/[2(g_0^{\pm})^2]\}^2]^{1/2}\rangle/\{2\{\Lambda_R/[2(g_0^{\pm})^2]\}^2\rangle$$

$$=\langle 1+[\Lambda_R/(g_0^{\pm})^2]\pm[1+\{2\Lambda_R/(g_0^{\pm})^2]\}^{1/2}\rangle/\langle 2\{\Lambda_R/[2(g_0^{\pm})^2]\}^2\rangle$$

$$=2\langle 1+[\Lambda_R/(g_0^{\pm})^2]\pm\{1+[2\Lambda_R/(g_0^{\pm})^2]\}^{1/2}\rangle/[\Lambda_R/(g_0^{\pm})^2]^2$$

$$=[(g_0^{\pm})^2/\Lambda_R]^2\langle 2+2[\Lambda_R/(g_0^{\pm})^2]\pm 2\{1+[2\Lambda_R/(g_0^{\pm})^2]\}^{1/2}\rangle$$

$$=[(g_0^{\pm})^2/\Lambda_R]^2\langle 1\pm 2\{1+[2\Lambda_R/(g_0^{\pm})^2]\}^{1/2}+\{1+[2\Lambda_R/(g_0^{\pm})^2]\}\rangle$$

$$=[(g_0^{\pm})^2/\Lambda_R]^2\langle 1\pm[1+\{2\Lambda_R/(g_0^{\pm})^2]\}^{1/2}\rangle^2=\langle[(g_0^{\pm})^2/\Lambda_R][1\pm\{1+[2\Lambda_R/(g_0^{\pm})^2]\}^{1/2}]\rangle^2$$

In the non-radiative limit, $\Lambda_R\to 0$, we have $g_1^{\pm}\to 0$. Thus

$$\exp(g_1^{\pm})=\langle[(g_0^{\pm})^2/\Lambda_R][1-\{1+[2\Lambda_R/(g_0^{\pm})^2]\}^{1/2}]\rangle^2=\langle[(g_0^{\pm})^2/\Lambda_R][\{1+[2\Lambda_R/(g_0^{\pm})^2]\}^{1/2}-1]\rangle^2$$

$$=\left[\frac{(g_0^{\pm})^2}{\Lambda_R}\left(\sqrt{1+\frac{2\Lambda_R}{(g_0^{\pm})^2}}-1\right)\right]^2=\left\{\frac{(g_0^{\pm})^2}{\Lambda_R}\frac{\langle\{1+[2\Lambda_R/(g_0^{\pm})^2]\}^{1/2}-1\rangle\langle\{1+[2\Lambda_R/(g_0^{\pm})^2]\}^{1/2}+1\rangle}{\{1+[2\Lambda_R/(g_0^{\pm})^2]\}^{1/2}+1}\right\}^2$$

$$=\left\{\frac{(g_0^{\pm})^2}{\Lambda_R}\frac{1+[2\Lambda_R/(g_0^{\pm})^2]-1^2}{\{1+[2\Lambda_R/(g_0^{\pm})^2]\}^{1/2}+1}\right\}^2=\left\{\frac{(g_0^{\pm})^2}{\Lambda_R}\frac{2\Lambda_R/(g_0^{\pm})^2}{\{1+[2\Lambda_R/(g_0^{\pm})^2]\}^{1/2}+1}\right\}^2=\left\{\frac{2}{\{1+[2\Lambda_R/(g_0^{\pm})^2]\}^{1/2}+1}\right\}^2$$

$$g_1^{\pm}=\ell\text{n}\langle 2/[\{1+[2\Lambda_R/(g_0^{\pm})^2]\}^{1/2}+1]\rangle^2=-2\,\ell\text{n}\langle[\{1+[2\Lambda_R/(g_0^{\pm})^2]\}^{1/2}+1]/2\rangle$$

Since $g_1^{-}=\tilde T_{b,S,2}\exp[\tilde m(1-\tilde r_{f,S}^{-1})]$: $\tilde T_{b,S,2}=-2\,\ell\text{n}\{[\langle 1+[2\Lambda_R/(g_0^{-})^2]\rangle^{1/2}+1]/2\}\exp[\tilde m(\tilde r_{f,S}^{-1}-1)]$

Since $g_1^{+}=a_{T,2}^{+}[1-\exp(-\tilde m/\tilde r_{f,S})]$: $a_{T,2}^{+}=-2\,\ell\text{n}\langle[\{1+[2\Lambda_R/(g_0^{+})^2]\}^{1/2}+1]/2\rangle/[1-\exp(-\tilde m/\tilde r_{f,S})]$

(6) $\theta_1(\xi\to-\infty)=\Theta_{S,2}^{-}(\zeta=0)+(d\Theta_{S,1}^{-}/d\zeta)_{\zeta=0}\,\xi$

From (F-2): $\Theta_{S,2}^{-}=(\tilde T_{b,S,A}/g_0^{-})(d\Theta_{S,1}^{-}/d\zeta)$; From (F-1): $d\Theta_{S,1}^{-}/d\zeta=-[(g_0^{-})^2+2\Lambda_R\exp(-\Theta_{S,1}^{-})]^{1/2}$

When $\zeta=0$: $\Theta_{S,1}^{-}=0 \quad \therefore \quad \exp(-\Theta_{S,1}^{-})=1$, $(d\Theta_{S,1}^{-}/d\zeta)_{\zeta=0}=-[(g_0^{-})^2+2\Lambda_R]^{1/2}$

$$\theta_1(\xi\to-\infty)=[(\tilde T_{b,S,A}/g_0^{-})+\xi](d\Theta_{S,1}^{-}/d\zeta)_{\zeta=0}=-[(\tilde T_{b,S,A}/g_0^{-})+\xi][(g_0^{-})^2+2\Lambda_R]^{1/2}$$

$$(d\theta_1/d\xi)_{\xi\to-\infty}=-[(g_0^{-})+2\Lambda_R]^{1/2}$$

$\theta_1(\xi\to\infty)=\Theta_{S,2}^{+}(\zeta=0)+(d\Theta_{S,1}^{+}/d\zeta)_{\zeta=0}\,\xi$

From (F-4): $\Theta_{S,2}^{+}=-(a_{T,A}^{+}/g_0^{+})(d\Theta_{S,1}^{+}/d\zeta)$; From (F-3): $d\Theta_{S,1}^{+}/d\zeta=[(g_0^{+})^2+2\Lambda_R\exp(-\Theta_{S,1}^{+})]^{1/2}$

When $\zeta=0$: $\Theta_{S,1}^{+}=0 \quad \therefore \quad \exp(-\Theta_{S,1}^{+})=1$, $(d\Theta_{S,1}^{+}/d\zeta)_{\zeta=0}=[(g_0^{+})^2+2\Lambda_R]^{1/2}$

$$\theta_1(\xi\to\infty)=[-(a_{T,A}^{+}/g_0^{+})+\xi](d\Theta_{S,1}^{-}/d\zeta)_{\zeta=0}=-[(a_{T,A}^{+}/g_0^{+})-\xi][(g_0^{+})^2+2\Lambda_R]^{1/2}$$

$$(d\theta_1/d\xi)_{\xi\to\infty}=[(g_0^{+})^2+2\Lambda_R]^{1/2}$$

(7) $\langle\{1+[2\Lambda_R/(g_0^{\pm})^2]\}^{1/2}+1\rangle^2=\{1+[2\Lambda_R/(g_0^{\pm})^2]\}+2\{1+[2\Lambda_R/(g_0^{\pm})^2]\}^{1/2}+1=2\langle 1+[\Lambda_R/(g_0^{\pm})^2]+\{1+[2\Lambda_R/(g_0^{\pm})^2]\}^{1/2}\rangle$

$$\Lambda_R\exp(g_1^{\pm})=4\Lambda_R/\langle\{1+[2\Lambda_R/(g_0^{\pm})^2]\}^{1/2}+1\rangle^2=4(g_0^{\pm})^2[\Lambda_R/(g_0^{\pm})^2]/\langle\{1+[2\Lambda_R/(g_0^{\pm})^2]\}^{1/2}+1\rangle^2$$

$$2(g_0^{\pm})^2+\Lambda_R\exp(g_1^{\pm})=2(g_0^{\pm})^2+4\Lambda_R/\langle\{1+[2\Lambda_R/(g_0^{\pm})^2]\}^{1/2}+1\rangle^2$$

$$=4(g_0^{\pm})^2\langle 1+[2\Lambda_R/(g_0^{\pm})^2]+\{1+[2\Lambda_R/(g_0^{\pm})^2]\}^{1/2}\rangle/\langle\{1+[2\Lambda_R/(g_0^{\pm})^2]\}^{1/2}+1\rangle^2$$

$$=4(g_0^{\pm})^2\{1+[2\Lambda_R/(g_0^{\pm})^2]\}^{1/2}\langle\{1+[2\Lambda_R/(g_0^{\pm})^2]\}^{1/2}+1\rangle/\langle\{1+[2\Lambda_R/(g_0^{\pm})^2]\}^{1/2}+1\rangle^2$$

$$=4(g_0^{\pm})^2\{1+[2\Lambda_R/(g_0^{\pm})^2]\}^{1/2}/\langle\{1+[2\Lambda_R/(g_0^{\pm})^2]\}^{1/2}+1\rangle$$

$$2(g_0^{\pm})^2-\Lambda_R\exp(g_1^{\pm})=2(g_0^{\pm})^2-4\Lambda_R/\langle\{1+[2\Lambda_R/(g_0^{\pm})^2]\}^{1/2}+1\rangle^2$$

$$=4(g_0^{\pm})^2\langle 1+\{1+[2\Lambda_R/(g_0^{\pm})^2]\}^{1/2}\rangle/\langle\{1+[2\Lambda_R/(g_0^{\pm})^2]\}^{1/2}+1\rangle^2=4(g_0^{\pm})^2/\langle\{1+[2\Lambda_R/(g_0^{\pm})^2]\}^{1/2}+1\rangle$$

$$\frac{[2(g_0^{\pm})^2+\Lambda_R\exp(g_1^{\pm})]}{[2(g_0^{\pm})^2-\Lambda_R\exp(g_1^{\pm})]}=\frac{4(g_0^{\pm})^2\{1+[2\Lambda_R/(g_0^{\pm})^2]\}^{1/2}/\langle\{1+[2\Lambda_R/(g_0^{\pm})^2]\}^{1/2}+1\rangle}{4(g_0^{\pm})^2/\langle\{1+[2\Lambda_R/(g_0^{\pm})^2]\}^{1/2}+1\rangle}=\{1+[2\Lambda_R/(g_0^{\pm})^2]\}^{1/2}$$

$$[2(g_0^{\pm})^2+\Lambda_R\exp(g_1^{\pm})][2(g_0^{\pm})^2-\Lambda_R\exp(g_1^{\pm})]=\frac{4(g_0^{\pm})^2\{1+[2\Lambda_R/(g_0^{\pm})^2]\}^{1/2}}{\{1+[2\Lambda_R/(g_0^{\pm})^2]\}^{1/2}+1}\frac{4(g_0^{\pm})^2}{\{1+[2\Lambda_R/(g_0^{\pm})^2]\}^{1/2}+1}$$

$$=16(g_0^{\pm})^4\{1+[2\Lambda_R/(g_0^{\pm})^2]\}^{1/2}/\langle\{1+[2\Lambda_R/(g_0^{\pm})^2]\}^{1/2}+1\rangle^2=4(g_0^{\pm})^4\{1+[2\Lambda_R/(g_0^{\pm})^2]\}^{1/2}\exp(g_1^{\pm})$$

$$4(g_0^{\pm})^2\Lambda_R\exp(g_1^{\pm})/\{[2(g_0^{\pm})^2+\Lambda_R\exp(g_1^{\pm})][2(g_0^{\pm})^2-\Lambda_R\exp(g_1^{\pm})]\}=[\Lambda_R/(g_0^{\pm})^2]/\{1+[2\Lambda_R/(g_0^{\pm})^2]\}^{1/2}$$

(8) Summary

$\Theta_{S,1}^{-}(\zeta=0)=\Theta_{S,1}^{+}(\zeta=0)=0$

$g_1^{-}=-2\,\ell n\{[\langle 1+[2\Lambda_R/(g_0^{-})^2]\rangle^{1/2}+1]/2\}$; $\exp(g_1^{-})=\langle 2/[\{1+[2\Lambda_R/(g_0^{-})^2]\}^{1/2}+1]\rangle^2$

$2(g_0^{\pm})^2+\Lambda_R\exp(g_1^{\pm})=4(g_0^{\pm})^2\{1+[2\Lambda_R/(g_0^{\pm})^2]\}^{1/2}/\langle\{1+[2\Lambda_R/(g_0^{\pm})^2]\}^{1/2}+1\rangle$

$2(g_0^{\pm})^2-\Lambda_R\exp(g_1^{\pm})=4(g_0^{\pm})^2/\langle\{1+[2\Lambda_R/(g_0^{\pm})^2]\}^{1/2}+1\rangle$

$[2(g_0^{\pm})^2+\Lambda_R\exp(g_1^{\pm})]/[2(g_0^{\pm})^2-\Lambda_R\exp(g_1^{\pm})]=\{1+[2\Lambda_R/(g_0^{\pm})^2]\}^{1/2}$

$4(g_0^{\pm})^2\Lambda_R\exp(g_1^{\pm})/\{[2(g_0^{\pm})^2+\Lambda_R\exp(g_1^{\pm})][2(g_0^{\pm})^2-\Lambda_R\exp(g_1^{\pm})]\}=[\Lambda_R/(g_0^{\pm})^2]/\{1+[2\Lambda_R/(g_0^{\pm})^2]\}^{1/2}$

$\tilde{T}_{b,S,2}=-2\,\ell n\{[\langle 1+[2\Lambda_R/(g_0^{-})^2]\rangle^{1/2}+1]/2\}\exp[\tilde{m}(\tilde{r}_{f,S}^{-1}-1)]$; $\tilde{T}_{b,S,A}=\tilde{T}_{b,S,1}\exp[\tilde{m}(1-\tilde{r}_{f,S}^{-1})]$

$\theta_1(\xi\to-\infty)=-[(\tilde{T}_{b,S,A}/g_0^{-})+\xi][(g_0^{-})^2+2\Lambda_R]^{1/2}\to\infty$; $(d\theta_1/d\xi)_{\xi\to-\infty}=-[(g_0^{-})^2+2\Lambda_R]^{1/2}$

$(d\theta_2/d\xi)_{\xi\to-\infty}=(d\Theta_{S,3}^{-}/d\zeta)_{\zeta=0}+(d^2\Theta_{S,4}^{-}/d\zeta^2)_{\zeta=0}\,\xi$

$a_{T,2}^{+}=-2\,\ell n\langle[\{1+[2\Lambda_R/(g_0^{+})^2]\}^{1/2}+1]/2\rangle/[1-\exp(-\tilde{m}/\tilde{r}_{f,S})]$; $a_{T,A}^{+}=a_{T,1}^{+}[1-\exp(-\tilde{m}/\tilde{r}_{f,S})]$

$\theta_1(\xi\to\infty)=-[(a_{T,A}^{+}/g_0^{+})-\xi][(g_0^{+})^2+2\Lambda_R]^{1/2}\to\infty$; $(d\theta_1/d\xi)_{\xi\to\infty}=[(g_0^{+})^2+2\Lambda_R]^{1/2}$

$(d\theta_2/d\xi)_{\xi\to\infty}=(d\Theta_{S,3}^{+}/d\zeta)_{\zeta=0}+(d^2\Theta_{S,4}^{+}/d\zeta^2)_{\zeta=0}\,\xi$

$a_{1,0}^{-}=\exp(Le_1\tilde{m}/\tilde{r}_{f,S})$, $a_{1,0}^{+}=0$, $\tilde{Y}_{1,S,0}^{+}=0$

$\phi_{1,1}(\xi\to-\infty)=-a_{1,1}^{-}\exp(-Le_1\tilde{m}/\tilde{r}_{f,S})-(Le_1\tilde{m}/\tilde{r}_{f,S}^2)\xi$; $(d\phi_{1,1}/d\xi)_{\xi\to-\infty}=-(Le_1\tilde{m}/\tilde{r}_{f,S}^2)$

$\phi_{1,1}(\xi\to\infty)=a_{1,1}^{+}[1-\exp(-Le_1\tilde{m}/\tilde{r}_{f,S})]$; $(d\phi_{1,1}/d\xi)_{\xi\to\infty}=0$

$$\left[\frac{1}{Le_1}\frac{d\varphi_{1,2}}{d\xi}+\left(\frac{2}{Le_1\tilde{r}_{f,S}}\xi\right)\frac{d\varphi_{1,1}}{d\xi}-\frac{\tilde{m}}{\tilde{r}_{f,S}^2}\varphi_{1,1}\right]_{\xi\to-\infty}=\left[\frac{1}{Le_2}\frac{d\varphi_{2,2}}{d\xi}+\left(\frac{2}{Le_2\tilde{r}_{f,S}}\xi\right)\frac{d\varphi_{2,1}}{d\xi}-\frac{\tilde{m}}{\tilde{r}_{f,S}^2}\varphi_{2,1}\right]_{\xi\to-\infty}=0$$

$$\left[\frac{1}{Le_1}\frac{d\phi_{1,2}}{d\xi}+\left(\frac{2}{Le_1\tilde{r}_{f,S}}\xi\right)\frac{d\phi_{1,1}}{d\xi}-\frac{\tilde{m}}{\tilde{r}_{f,S}^2}\phi_{1,1}\right]_{\xi\to\infty}=-a_{1,1}^{+}\frac{\tilde{m}}{\tilde{r}_{f,S}^2}$$

$a_{2,0}^{-}=0$, $\tilde{Y}_{2,S,0}^{-}=0$, $a_{2,0}^{+}=\tilde{Y}_{2,\infty}/[1-\exp(-Le_2\tilde{m}/\tilde{r}_{f,S})]$

$\phi_{2,1}(\xi\to-\infty)=a_{2,1}^{-}\exp(-Le_2\tilde{m}/\tilde{r}_{f,S})$; $(d\phi_{2,1}/d\xi)_{\xi\to-\infty}=0$

$$\phi_{2,1}(\xi\to\infty)=-a_{2,1}^{+}[1-\exp(-Le_2\tilde{m}/\tilde{r}_{f,S})]+\tilde{Y}_{2,\infty}\frac{\tilde{m}}{\tilde{r}_{f,S}}\frac{\exp(-Le_2\tilde{m}/\tilde{r}_{f,S})}{1-\exp(-Le_2\tilde{m}/\tilde{r}_{f,S})}\left(\frac{Le_2}{\tilde{r}_{f,S}}\xi\right)$$

$(d\phi_{2,1}/d\xi)_{\xi\to\infty}=\tilde{Y}_{2,\infty}(Le_2\tilde{m}/\tilde{r}_{f,S}^2)\exp(-Le_2\tilde{m}/\tilde{r}_{f,S})/[1-\exp(-Le_2\tilde{m}/\tilde{r}_{f,S})]\}$

$$\left[\frac{1}{Le_2}\frac{d\phi_{2,2}}{d\xi}+\left(\frac{2}{Le_2\tilde{r}_{f,S}}\xi\right)\frac{d\phi_{2,1}}{d\xi}-\frac{\tilde{m}}{\tilde{r}_{f,S}^2}\phi_{2,1}\right]_{\xi\to\infty}=a_{2,1}^{+}\frac{\tilde{m}}{\tilde{r}_{f,S}^2}$$

(I) Inner Solutions

(1) $(\phi_{1,1}/Le_1)-(\phi_{2,1}/Le_2)=c_1\xi+c_2$

$\xi\to-\infty$: $\phi_{1,1}\to-a_{1,1}^{-}\exp(-Le_1\tilde{m}/\tilde{r}_{f,S})-(Le_1\tilde{m}/\tilde{r}_{f,S}^2)\xi$, $\phi_{2,1}\to a_{2,1}^{-}\exp(-Le_{2,0}\tilde{m}/\tilde{r}_{f,S})$

$\therefore$ $c_1=-\tilde{m}/\tilde{r}_{f,S}^2$, $c_2=-(a_{1,1}^{-}/Le_1)\exp(-Le_1\tilde{m}/\tilde{r}_{f,S})-(a_{2,1}^{-}/Le_2)\exp(-Le_2\tilde{m}/\tilde{r}_{f,S})$

$\xi\to\infty$: $\phi_{1,1}\to a_{1,1}^{+}[1-\exp(-Le_1\tilde{m}/\tilde{r}_{f,S})]$

$\phi_{2,1}\to-a_{2,1}^{+}[1-\exp(-Le_2\tilde{m}/\tilde{r}_{f,S})]+\{\tilde{Y}_{2,\infty}(\tilde{m}/\tilde{r}_{f,S})/[\exp(Le_2\tilde{m}/\tilde{r}_{f,S})-1]\}(Le_2/\tilde{r}_{f,S})\xi$

$\therefore$ $c_1=-\tilde{Y}_{2,\infty}(\tilde{m}/\tilde{r}_{f,S}^2)/[\exp(Le_2\tilde{m}/\tilde{r}_{f,S})-1]$

$$c_2=\frac{a_{1,1}^{+}}{Le_1}\left[1-\exp\left(-\frac{Le_1\tilde{m}}{\tilde{r}_{f,S}}\right)\right]+\frac{a_{2,1}^{+}}{Le_2}\left[1-\exp\left(-\frac{Le_2\tilde{m}}{\tilde{r}_{f,S}}\right)\right]$$

$$\Rightarrow \quad \tilde{Y}_{2,\infty}/[\exp(Le_2\,\tilde{m}/\tilde{r}_{f,S})-1]=1 \quad \text{or} \quad \exp(Le_2\,\tilde{m}/\tilde{r}_{f,S})=1+\tilde{Y}_{2,\infty} \quad \therefore \quad \tilde{r}_{f,S}=Le_2\,\tilde{m}/\ell\text{n}(1+\tilde{Y}_{2,\infty})$$

$$1-\exp(-Le_2\,\tilde{m}/\tilde{r}_{f,S})=\tilde{Y}_{2,\infty}/(1+\tilde{Y}_{2,\infty})\,, \quad a^{+}_{2,0}=\tilde{Y}_{2,\infty}/[1-\exp(-Le_2\,\tilde{m}/\tilde{r}_{f,S})]=1+\tilde{Y}_{2,\infty}$$

$$\frac{a^{+}_{1,1}}{Le_1}\left[1-\exp\left(-\frac{Le_1\,\tilde{m}}{\tilde{r}_{f,S}}\right)\right]+\frac{a^{+}_{2,1}}{Le_2}\frac{\tilde{Y}_{2,\infty}}{1+\tilde{Y}_{2,\infty}}=-\frac{a^{-}_{1,1}}{Le_1}\exp\left(-\frac{Le_1\,\tilde{m}}{\tilde{r}_{f,S}}\right)-\frac{a^{-}_{2,1}}{Le_2}\frac{1}{1+\tilde{Y}_{2,\infty}}$$

$$\frac{a^{+}_{1,1}-a^{-}_{1,1}}{Le_1}\exp\left(-\frac{Le_1\,\tilde{m}}{\tilde{r}_{f,S}}\right)+\frac{a^{+}_{2,1}-a^{-}_{2,1}}{Le_2}\frac{1}{1+\tilde{Y}_{2,\infty}}=\frac{a^{+}_{1,1}}{Le_1}+\frac{a^{+}_{2,1}}{Le_2}$$

$$\frac{\phi_{1,1}}{Le_1}-\frac{\phi_{2,1}}{Le_2}=\frac{a^{+}_{1,1}}{Le_1}\left[1-\exp\left(-\frac{Le_1\,\tilde{m}}{\tilde{r}_{f,S}}\right)\right]+\frac{a^{+}_{2,1}}{Le_2}\frac{\tilde{Y}_{2,\infty}}{1+\tilde{Y}_{2,\infty}}-\frac{\tilde{m}}{\tilde{r}^2_{f,S}}\xi$$

(2) $$\left[\frac{1}{Le_1}\frac{d\phi_{1,2}}{d\xi}-\left(-\frac{2}{Le_1\,\tilde{r}_{f,S}}\xi\right)\frac{d\phi_{1,1}}{d\xi}-\frac{\tilde{m}}{\tilde{r}^2_{f,S}}\phi_{1,1}\right]-\left[\frac{1}{Le_2}\frac{d\phi_{2,2}}{d\xi}-\left(-\frac{2}{Le_2\,\tilde{r}_{f,S}}\xi\right)\frac{d\phi_{2,1}}{d\xi}-\frac{\tilde{m}}{\tilde{r}^2_{f,S}}\phi_{2,1}\right]=c_3$$

$$\xi\to-\infty: \quad \frac{1}{Le_1}\frac{d\phi_{1,2}}{d\xi}+\left(\frac{2}{Le_1\tilde{r}_{f,S}}\xi\right)\frac{d\phi_{1,1}}{d\xi}-\frac{\tilde{m}}{\tilde{r}^2_{f,S}}\phi_{1,1}\to 0 \quad ; \quad \frac{1}{Le_2}\frac{d\phi_{2,2}}{d\xi}+\left(\frac{2}{Le_2\tilde{r}_{f,S}}\xi\right)\frac{d\phi_{2,1}}{d\xi}-\frac{\tilde{m}}{\tilde{r}^2_{f,S}}\phi_{2,1}\to 0$$

$$\therefore \quad c_3=0$$

$$\xi\to\infty: \quad \frac{1}{Le_1}\frac{d\phi_{1,2}}{d\xi}+\left(\frac{2}{Le_1\tilde{r}_{f,S}}\xi\right)\frac{d\phi_{1,1}}{d\xi}-\frac{\tilde{m}}{\tilde{r}^2_{f,S}}\phi_{1,1}\to -a^{+}_{1,1}\frac{\tilde{m}}{\tilde{r}^2_{f,S}} \quad ; \quad \frac{1}{Le_2}\frac{d\phi_{2,2}}{d\xi}+\left(\frac{2}{Le_2\tilde{r}_{f,S}}\xi\right)\frac{d\phi_{2,1}}{d\xi}-\frac{\tilde{m}}{\tilde{r}^2_{f,S}}\phi_{2,1}\to a^{+}_{2,1}\frac{\tilde{m}}{\tilde{r}^2_{f,S}}$$

$$\therefore \quad -(a^{+}_{1,1}+a^{+}_{2,1})(\tilde{m}/\tilde{r}^2_{f,S})=0 \quad \text{or} \quad a^{+}_{2,1}=-a^{+}_{1,1}$$

$$(1/Le_2)(d\tilde{Y}^{+}_{2,S,1}/d\tilde{r})-(\tilde{m}/\tilde{r}^2)\tilde{Y}^{+}_{2,S,1}=a^{+}_{2,1}\,\tilde{m}/\tilde{r}^2=-a^{+}_{1,1}\,\tilde{m}/\tilde{r}^2$$

$$(1/Le_2)(d^2\tilde{Y}^{+}_{2,S,1}/d\tilde{r}^2)-(\tilde{m}/\tilde{r}^2)(d\tilde{Y}^{+}_{2,S,1}/d\tilde{r})+(2\tilde{m}/\tilde{r}^3)\tilde{Y}^{+}_{2,S,1}=-2a^{+}_{2,1}\tilde{m}/\tilde{r}^3=2a^{+}_{1,1}\tilde{m}/\tilde{r}^3$$

From (1): $$\frac{a^{+}_{2,1}-a^{-}_{2,1}}{Le_2}\exp\left(-\frac{Le_2\,\tilde{m}}{\tilde{r}_{f,S}}\right)+\frac{a^{+}_{2,1}-a^{-}_{2,1}}{Le_2}\frac{1}{1+\tilde{Y}_{2,\infty}}=\frac{a^{+}_{1,1}}{Le_1}+\frac{a^{+}_{2,1}}{Le_2}$$

$$\frac{a^{+}_{1,1}+a^{-}_{2,1}}{Le_2}\frac{1}{1+\tilde{Y}_{2,\infty}}=\frac{a^{+}_{1,1}-a^{-}_{1,1}}{Le_1}\exp\left(-\frac{Le_1\,\tilde{m}}{\tilde{r}_{f,S}}\right)-a^{+}_{1,1}\left(\frac{1}{Le_1}-\frac{1}{Le_2}\right)$$

$$a^{-}_{2,1}=(1+\tilde{Y}_{2,\infty})\left[(a^{+}_{1,1}-a^{-}_{1,1})\frac{Le_2}{Le_1}\exp\left(-\frac{Le_1\,\tilde{m}}{\tilde{r}_{f,S}}\right)+a^{+}_{1,1}\left(1-\frac{Le_2}{Le_1}\right)\right]-a^{+}_{1,1}$$

$$\frac{a^{-}_{2,1}}{1+\tilde{Y}_{2,\infty}}=(a^{+}_{1,1}-a^{-}_{1,1})\frac{Le_2}{Le_1}\exp\left(-\frac{Le_1\,\tilde{m}}{\tilde{r}_{f,S}}\right)+a^{+}_{1,1}\left(1-\frac{Le_2}{Le_1}\right)-\frac{a^{+}_{1,1}}{1+\tilde{Y}_{2,\infty}}$$

$$\frac{a^{-}_{2,1}}{1+\tilde{Y}_{2,\infty}}=a^{+}_{1,1}\frac{\tilde{Y}_{2,\infty}}{1+\tilde{Y}_{2,\infty}}-\frac{Le_2}{Le_1}\left\{a^{-}_{1,1}\exp\left(-\frac{Le_1\,\tilde{m}}{\tilde{r}_{f,S}}\right)+a^{+}_{1,1}\left[1-\exp\left(-\frac{Le_1\,\tilde{m}}{\tilde{r}_{f,S}}\right)\right]\right\}$$

(3) $\theta_1-(\phi_{1,1}/Le_1)=c_4\,\xi+c_5$

$$\xi\to-\infty: \quad \theta_1\to-[(\tilde{T}_{b,S,A}/g^{-}_0)+\xi][(g^{-}_0)^2+2\Lambda_R]^{1/2} \quad ; \quad \phi_{1,1}\to-a^{-}_{1,1}\exp(-Le_1\tilde{m}/\tilde{r}_{f,S})-(Le_1\tilde{m}/\tilde{r}^2_{f,S})\xi$$

$$\therefore \quad c_4=-[(g^{-}_0)^2+2\Lambda_R]^{1/2}+(\tilde{m}/\tilde{r}^2_{f,S})$$

$$c_5=-(\tilde{T}_{b,S,A}/g^{-}_0)[(g^{-}_0)^2+2\Lambda_R]^{1/2}+(a^{-}_{1,1}/Le_1)\exp(-Le_1\,\tilde{m}/\tilde{r}_{f,S})$$

$$\xi\to\infty: \quad \theta_1\to-[(a^{+}_{T,A}/g^{+}_0)-\xi][(g^{+}_0)^2+2\Lambda_R]^{1/2}\,, \quad \phi_{1,1}=a^{+}_{1,1}[1-\exp(-Le_1\,\tilde{m}/\tilde{r}_{f,S})]$$

$$\therefore \quad c_4=[(g^{+}_0)^2+2\Lambda_R]^{1/2}\,, \quad c_5=-(a^{+}_{T,A}/g^{+}_0)[(g^{+}_0)^2+2\Lambda_R]^{1/2}-(a^{+}_{1,1}/Le_1)[1-\exp(-Le_1\,\tilde{m}/\tilde{r}_{f,S})]$$

$$\Rightarrow \quad -[(g^{-}_0)^2+2\Lambda_R]^{1/2}+(\tilde{m}/\tilde{r}^2_{f,S})=[(g^{+}_0)^2+2\Lambda_R]^{1/2} \quad \text{or} \quad [(g^{-}_0)^2+2\Lambda_R]^{1/2}+[(g^{+}_0)^2+2\Lambda_R]^{1/2}=\tilde{m}/\tilde{r}^2_{f,S}$$

$$(a_{T,A}^{+}/g_0^{+})[(g_0^{+})^2+2\Lambda_R]^{1/2}-(\tilde{T}_{b,S,A}/g_0^{-})[(g_0^{-})^2+2\Lambda_R]^{1/2}$$
$$+Le_1^{-1}\{a_{1,1}^{-}\exp(-Le_1\tilde{m}/\tilde{r}_{f,S})+a_{1,1}^{+}[1-\exp(-Le_1\tilde{m}/\tilde{r}_{f,S})]\}=0$$

$$\theta_1-(\phi_{1,1}/Le_1)=-[(a_{T,A}^{+}/g_0^{+})-\xi][(g_0^{+})^2+2\Lambda_R]^{1/2}-(a_{1,1}^{+}/Le_1)[1-\exp(-Le_1\tilde{m}/\tilde{r}_{f,S})]$$

or $\phi_{1,1}=a_{1,1}^{+}[1-\exp(-Le_1\tilde{m}/\tilde{r}_{f,S})]+Le_1\{[(a_{T,A}^{+}/g_0^{+})-\xi][(g_0^{+})^2+2\Lambda_R]^{1/2}+\theta_1\}$

From (1): $\dfrac{\phi_{1,1}}{Le_1}-\dfrac{\phi_{2,1}}{Le_2}=\dfrac{a_{1,1}^{+}}{Le_1}\left[1-\exp\left(-\dfrac{Le_1\tilde{m}}{\tilde{r}_{f,S}}\right)\right]+\dfrac{a_{2,1}^{+}}{Le_2}\dfrac{\tilde{Y}_{2,\infty}}{(1+\tilde{Y}_{2,\infty})}-\dfrac{\tilde{m}}{\tilde{r}_{f,S}^2}\xi$

$$\theta_1-\frac{\phi_{2,1}}{Le_2}=-\left(\frac{a_{T,A}^{+}}{g_0^{+}}-\xi\right)[(g_0^{+})^2+2\Lambda_R]^{1/2}+\frac{a_{2,1}^{+}}{Le_2}\frac{\tilde{Y}_{2,\infty}}{1+\tilde{Y}_{2,\infty}}-\frac{\tilde{m}}{\tilde{r}_{f,S}^2}\xi$$
$$=-\frac{a_{T,A}^{+}}{g_0^{+}}[(g_0^{+})^2+2\Lambda_R]^{1/2}-\frac{a_{1,1}^{+}}{Le_2}\frac{\tilde{Y}_{2,\infty}}{1+\tilde{Y}_{2,\infty}}-[(g_0^{-})^2+2\Lambda_R]^{1/2}\xi$$

$$\phi_{2,1}=a_{1,1}^{+}[\tilde{Y}_{2,\infty}/(1+\tilde{Y}_{2,\infty})]+Le_2\{(a_{T,A}^{+}/g_0^{+})[(g_0^{+})^2+2\Lambda_R]^{1/2}+[(g_0^{-})^2+2\Lambda_R]^{1/2}\xi+\theta_1\}$$

(4) $\left(\dfrac{d\theta_2}{d\xi}+\dfrac{2\xi}{\tilde{r}_{f,S}}\dfrac{d\theta_1}{d\xi}-\dfrac{\tilde{m}}{\tilde{r}_{f,S}^2}\theta_1\right)-\left[\dfrac{1}{Le_1}\dfrac{d\phi_{1,2}}{d\xi}+\left(\dfrac{2}{Le_1\tilde{r}_{f,S}}\xi\right)\dfrac{d\phi_{1,1}}{d\xi}-\dfrac{\tilde{m}}{\tilde{r}_{f,S}^2}\phi_{1,1}\right]=c_6$

$\xi\to-\infty$: $\theta_1\to-[(\tilde{T}_{b,S,A}/g_0^{-})+\xi][(g_0^{-})^2+2\Lambda_R]^{1/2}$, $(d\theta_1/d\xi)\to-[(g_0^{-})^2+2\Lambda_R]^{1/2}$

$(d\theta_2/d\xi)\to(d\Theta_{S,3}^{-}/d\zeta)_{\zeta=0}+(d^2\Theta_{S,4}^{-}/d\zeta^2)_{\zeta=0}\xi$, $\dfrac{1}{Le_1}\dfrac{d\phi_{1,2}}{d\xi}+\left(\dfrac{2}{Le_1\tilde{r}_{f,S}}\xi\right)\dfrac{d\phi_{1,1}}{d\xi}-\dfrac{\tilde{m}}{\tilde{r}_{f,S}^2}\phi_{1,1}\to0$

$\therefore$ $(d^2\Theta_{S,4}^{-}/d\zeta^2)_{\zeta=0}-\tilde{r}_{f,S}^{-1}[2-(\tilde{m}/\tilde{r}_{f,S})][(g_0^{-})^2+2\Lambda_R]^{1/2}=0$

$c_6=(d\Theta_{S,3}^{-}/d\zeta)_{\zeta=0}+(\tilde{m}/\tilde{r}_{f,S}^2)(\tilde{T}_{b,S,A}/g_0^{-})[(g_0^{-})^2+2\Lambda_R]^{1/2}$

$\xi\to\infty$: $d\theta_2/d\xi\to(d\Theta_{S,3}^{+}/d\zeta)_{\zeta=0}+(d^2\Theta_{S,4}^{+}/d\zeta^2)_{\zeta=0}\xi$, $d\theta_1/d\xi\to[(g_0^{+})^2+2\Lambda_R]^{1/2}$

$\theta_1\to-[(a_{T,A}^{+}/g_0^{+})-\xi][(g_0^{+})^2+2\Lambda_R]^{1/2}$, $\dfrac{1}{Le_1}\dfrac{d\phi_{1,2}}{d\xi}+\left(\dfrac{2}{Le_1\tilde{r}_{f,S}}\xi\right)\dfrac{d\phi_{1,1}}{d\xi}-\dfrac{\tilde{m}}{\tilde{r}_{f,S}^2}\phi_{1,1}\to-a_{1,1}^{+}\dfrac{\tilde{m}}{\tilde{r}_{f,S}^2}$

$\therefore$ $(d^2\Theta_{S,4}^{+}/d\zeta^2)_{\zeta=0}+\tilde{r}_{f,S}^{-1}[2-(\tilde{m}/\tilde{r}_{f,S})][(g_0^{+})^2+2\Lambda_R]^{1/2}=0$

$c_6=(d\Theta_{S,3}^{+}/d\zeta)_{\zeta=0}+(\tilde{m}/\tilde{r}_{f,S}^2)(a_{T,A}^{+}/g_0^{+})[(g_0^{+})^2+2\Lambda_R]^{1/2}+a_{1,1}^{+}(\tilde{m}/\tilde{r}_{f,S}^2)$

$\Rightarrow$ $(d\Theta_{S,3}^{-}/d\zeta)_{\zeta=0}-(d\Theta_{S,3}^{+}/d\zeta)_{\zeta=0}=(\tilde{m}/\tilde{r}_{f,S}^2)\{(a_{T,A}^{+}/g_0^{+})[(g_0^{+})^2+2\Lambda_R]^{1/2}-(\tilde{T}_{b,S,A}/g_0^{-})[(g_0^{-})^2+2\Lambda_R]^{1/2}+a_{1,1}^{+}\}$

From (H-5): $\Theta_{S,1}^{-}(\zeta=0)=\Theta_{S,1}^{+}(\zeta=0)=0$ $\therefore$ $\exp[-\Theta_{S,1}^{-}(\zeta=0)]=\exp[-\Theta_{S,1}^{+}(\zeta=0)]=1$

$$\left(\frac{d\Theta_{S,3}^{-}}{d\zeta}\right)_{\zeta=0}-\left(\frac{d\Theta_{S,3}^{+}}{d\zeta}\right)_{\zeta=0}=\left[\frac{8}{3}g_2^{-}-\frac{1}{(g_0^{-})^2}\left(\frac{d\tilde{T}_{S,1}^{-}}{d\tilde{r}}\right)_{\tilde{r}_{f,S}}\right]\left\{g_0^{-}\sqrt{(g_0^{-})^2+2\Lambda_R}+\Lambda_R\left\langle\ell\text{n}\left[\frac{2(g_0^{-})^2}{\Lambda_R}\frac{\sqrt{(g_0^{-})^2+2\Lambda_R}-g_0^{-}}{\sqrt{(g_0^{-})^2+2\Lambda_R}+g_0^{-}}\right]-2\right\rangle\right\}$$
$$-\frac{g_2^{-}}{2}\Lambda_R\left\{\ell\text{n}\left[\frac{2(g_0^{-})^2}{\Lambda_R}\frac{\sqrt{(g_0^{-})^2+2\Lambda_R}-g_0^{-}}{\sqrt{(g_0^{-})^2+2\Lambda_R}+g_0^{-}}\right]\right\}^2-g_2^{-}g_0^{-}\sqrt{(g_0^{-})^2+2\Lambda_R}\,\ell\text{n}\left[\frac{2(g_0^{-})^2}{\Lambda_R}\frac{\sqrt{(g_0^{-})^2+2\Lambda_R}-g_0^{-}}{\sqrt{(g_0^{-})^2+2\Lambda_R}+g_0^{-}}\right]$$
$$-\frac{2}{3}g_2^{-}[(g_0^{-})^2+2\Lambda_R]+\Lambda_R\left[-\frac{\tilde{T}_{S,3}^{-}(\tilde{r}_{f,S})+g_3^{-}}{g_0^{-}}+\frac{g_1^{-}}{(g_0^{-})^2}\left(\frac{d\tilde{T}_{S,1}^{-}}{d\tilde{r}}\right)_{\tilde{r}_{f,S}}\right]-2g_2^{-}(g_0^{-})^2$$
$$-\left[\frac{8}{3}g_2^{+}-\frac{1}{(g_0^{+})^2}\left(\frac{d\tilde{T}_{S,1}^{+}}{d\tilde{r}}\right)_{\tilde{r}_{f,S}}\right]\left\{g_0^{+}\sqrt{(g_0^{+})^2+2\Lambda_R}+\Lambda_R\left\langle\ell\text{n}\left[\frac{2(g_0^{+})^2}{\Lambda_R}\frac{\sqrt{(g_0^{+})^2+2\Lambda_R}-g_0^{+}}{\sqrt{(g_0^{+})^2+2\Lambda_R}+g_0^{+}}\right]-2\right\rangle\right\}$$
$$+\frac{g_2^{+}}{2}\Lambda_R\left\{\ell\text{n}\left[\frac{2(g_0^{+})^2}{\Lambda_R}\frac{\sqrt{(g_0^{+})^2+2\Lambda_R}-g_0^{+}}{\sqrt{(g_0^{+})^2+2\Lambda_R}+g_0^{+}}\right]\right\}^2+g_2^{+}g_0^{+}\sqrt{(g_0^{+})^2+2\Lambda_R}\,\ell\text{n}\left[\frac{2(g_0^{+})^2}{\Lambda_R}\frac{\sqrt{(g_0^{+})^2+2\Lambda_R}-g_0^{+}}{\sqrt{(g_0^{+})^2+2\Lambda_R}+g_0^{+}}\right]$$

$$+\frac{2}{3}g_2^+[(g_0^+)^2+2\Lambda_R]-\Lambda_R\left[\frac{\tilde{T}_{S,3}^+(\tilde{r}_{f,S})+g_3^+}{g_0^+}+\frac{g_1^+}{(g_0^+)^2}\left(\frac{d\tilde{T}_{S,1}^+}{d\tilde{r}}\right)_{\tilde{r}_{f,S}}\right]+2g_2^+(g_0^+)^2$$

In the limit of $\Lambda_R \to 0$:

$$\sqrt{(g_0^\pm)^2+2\Lambda_R}=g_0^\pm\{1+2[\Lambda_R/(g_0^\pm)^2]\}^{1/2}=g_0^\pm\{1+[\Lambda_R/(g_0^\pm)^2]+O(\Lambda_R^2)\}=g_0^\pm+(\Lambda_R/g_0^\pm)+O(\Lambda_R^2)$$

$$\ell\text{n}\left[\frac{2(g_0^\pm)^2}{\Lambda_R}\frac{\sqrt{(g_0^\pm)^2+2\Lambda_R}-g_0^\pm}{\sqrt{(g_0^\pm)^2+2\Lambda_R}+g_0^\pm}\right]\to\ell\text{n}\left[\frac{2(g_0^\pm)^2}{\Lambda_R}\frac{g_0^\pm+(\Lambda_R/g_0^\pm)+O(\Lambda_R^2)-g_0^\pm}{g_0^\pm+O(\Lambda_R)+g_0^\pm}\right]=\ell\text{n}\left[\frac{2g_0^\pm+O(\Lambda_R)}{2g_0^\pm+O(\Lambda_R)}\right]\to\ell\text{n}1=0$$

$$(d\Theta_{S,3}^-/d\zeta)_{\zeta=0}-(d\Theta_{S,3}^+/d\zeta)_{\zeta=0}=(d\tilde{T}_{S,1}^+/d\tilde{r})_{\tilde{r}_{f,S}}[\sqrt{(g_0^+)^2+2\Lambda_R}/g_0^+]-(d\tilde{T}_{S,1}^-/d\tilde{r})_{\tilde{r}_{f,S}}[\sqrt{(g_0^-)^2+2\Lambda_R}/g_0^-]$$

$$=-(a_{T,1}^+/g_0^+)(\tilde{m}/\tilde{r}_{f,S}^2)\exp(-\tilde{m}/\tilde{r}_{f,S})[(g_0^+)^2+2\Lambda_R]^{1/2}-(\tilde{T}_{b,S,A}/g_0^-)(\tilde{m}/\tilde{r}_{f,S}^2)[(g_0^-)^2+2\Lambda_R]^{1/2}$$

$$=-(\tilde{m}/\tilde{r}_{f,S}^2)\{(a_{T,1}^+/g_0^+)\exp(-\tilde{m}/\tilde{r}_{f,S})[(g_0^+)^2+2\Lambda_R]^{1/2}+(\tilde{T}_{b,S,A}/g_0^-)[(g_0^-)^2+2\Lambda_R]^{1/2}\}$$

Since $\tilde{m}/\tilde{r}_{f,S}^2\neq 0$

$$(a_{T,1}^+/g_0^+)[1-\exp(-\tilde{m}/\tilde{r}_{f,S})][(g_0^+)^2+2\Lambda_R]^{1/2}-(\tilde{T}_{b,S,A}/g_0^-)[(g_0^-)^2+2\Lambda_R]^{1/2}+a_{1,1}^+$$

$$=-(a_{T,1}^+/g_0^+)\exp(-\tilde{m}/\tilde{r}_{f,S})[(g_0^+)^2+2\Lambda_R]^{1/2}-(\tilde{T}_{b,S,A}/g_0^-)[(g_0^-)^2+2\Lambda_R]^{1/2}$$

or $\quad (a_{T,1}^+/g_0^+)[(g_0^+)^2+2\Lambda_R]^{1/2}=-a_{1,1}^+$

From (E)–(2): $a_{T,A}^+=a_{T,1}^+[1-\exp(-\tilde{m}/\tilde{r}_{f,S})]$

$$(a_{T,A}^+/g_0^+)[(g_0^+)^2+2\Lambda_R]^{1/2}=(a_{T,1}^+/g_0^+)[(g_0^+)^2+2\Lambda_R]^{1/2}[1-\exp(-\tilde{m}/\tilde{r}_{f,S})]=-a_{1,1}^+[1-\exp(-\tilde{m}/\tilde{r}_{f,S})]$$

From (3):

$$(a_{T,A}^+/g_0^+)[(g_0^+)^2+2\Lambda_R]^{1/2}-(\tilde{T}_{b,S,A}/g_0^-)[(g_0^-)^2+2\Lambda_R]^{1/2}$$
$$+Le_1^{-1}\{a_{1,1}^-\exp(-Le_1\tilde{m}/\tilde{r}_{f,S})+a_{1,1}^+[1-\exp(-Le_1\tilde{m}/\tilde{r}_{f,S})]\}=0$$

$$-a_{1,1}^+[1-\exp(-\tilde{m}/\tilde{r}_{f,S})]-(\tilde{T}_{b,S,A}/g_0^-)[(g_0^-)^2+2\Lambda_R]^{1/2}$$
$$+Le_1^{-1}\{a_{1,1}^-\exp(-Le_1\tilde{m}/\tilde{r}_{f,S})+a_{1,1}^+[1-\exp(-Le_1\tilde{m}/\tilde{r}_{f,S})]\}=0$$

$$a_{1,1}^-\exp(-Le_1\tilde{m}/\tilde{r}_{f,S})+a_{1,1}^+[1-\exp(-Le_1\tilde{m}/\tilde{r}_{f,S})]=Le_1\{(\tilde{T}_{b,S,A}/g_0^-)[(g_0^-)^2+2\Lambda_R]^{1/2}+a_{1,1}^+[1-\exp(-\tilde{m}/\tilde{r}_{f,S})]\}$$

$$a_{1,1}^-\exp(-Le_1\tilde{m}/\tilde{r}_{f,S})=Le_1\{(\tilde{T}_{b,S,A}/g_0^-)[(g_0^-)^2+2\Lambda_R]^{1/2}+a_{1,1}^+[1-\exp(-\tilde{m}/\tilde{r}_{f,S})]\}-a_{1,1}^+[1-\exp(-Le_1\tilde{m}/\tilde{r}_{f,S})]$$

From (2) : $\displaystyle \frac{a_{2,1}^-}{1+\tilde{Y}_{2,\infty}}=a_{1,1}^+\frac{\tilde{Y}_{2,\infty}}{1+\tilde{Y}_{2,\infty}}-\frac{Le_2}{Le_1}\left\{a_{1,1}^-\exp\left(-\frac{Le_1\tilde{m}}{\tilde{r}_{f,S}}\right)+a_{1,1}^+\left[1-\exp\left(-\frac{Le_1\tilde{m}}{\tilde{r}_{f,S}}\right)\right]\right\}$

$$a_{2,1}^-/(1+\tilde{Y}_{2,\infty})=a_{1,1}^+\tilde{Y}_{2,\infty}/(1+\tilde{Y}_{2,\infty})-Le_2\{(\tilde{T}_{b,S,A}/g_0^-)[(g_0^-)^2+2\Lambda_R]^{1/2}+a_{1,1}^+[1-\exp(-\tilde{m}/\tilde{r}_{f,S})]\}$$

(5) Summary

$\tilde{r}_{f,S}=Le_2\tilde{m}/\ell\text{n}(1+\tilde{Y}_{2,\infty})\quad$ or $\quad\exp(Le_2\tilde{m}/\tilde{r}_{f,S})=1+\tilde{Y}_{2,\infty}$

$[(g_0^-)^2+2\Lambda_R]^{1/2}+[(g_0^+)^2+2\Lambda_R]^{1/2}=\tilde{m}/\tilde{r}_{f,S}^2\qquad$ (determines $\tilde{T}_f$ iteratively)

$(a_{T,1}^+/g_0^+)[(g_0^+)^2+2\Lambda_R]^{1/2}=-a_{1,1}^+\quad;\quad a_{2,1}^+=-a_{1,1}^+$

$a_{1,1}^-\exp(-Le_1\tilde{m}/\tilde{r}_{f,S})=Le_1\{(\tilde{T}_{b,S,A}/g_0^-)[(g_0^-)^2+2\Lambda_R]^{1/2}+a_{1,1}^+[1-\exp(-\tilde{m}/\tilde{r}_{f,S})]\}-a_{1,1}^+[1-\exp(-Le_1\tilde{m}/\tilde{r}_{f,S})]$

$a_{2,1}^-/(1+\tilde{Y}_{2,\infty})=a_{1,1}^+\tilde{Y}_{2,\infty}/(1+\tilde{Y}_{2,\infty})-Le_2\{(\tilde{T}_{b,S,A}/g_0^-)[(g_0^-)^2+2\Lambda_R]^{1/2}+a_{1,1}^+[1-\exp(-\tilde{m}/\tilde{r}_{f,S})]\}$

The structure equation is

$d^2\theta_1/d\xi^2=\Lambda_K\phi_{1,1}\phi_{2,1}\exp(-\theta_1)\qquad$ where $\qquad\Lambda_K=\varepsilon^3 Da_K\exp(-\tilde{E}_K/\tilde{T}_{f,S})$

$\theta_1(\xi\to-\infty)=-[(\tilde{T}_{b,S,A}/g_0^-)+\xi][(g_0^-)^2+2\Lambda_R]^{1/2},\quad(d\theta_1/d\xi)_{\xi\to-\infty}=-[(g_0^-)^2+2\Lambda_R]^{1/2}$

$\tilde{T}_{b,S,A}=\tilde{T}_{b,S,1}\exp[\tilde{m}(1-\tilde{r}_{f,S}^{-1})]$

$\theta_1(\xi\to\infty)\to-[(a_{T,A}^+/g_0^+)-\xi][(g_0^+)^2+2\Lambda_R]^{1/2},\quad(d\theta_1/d\xi)_{\xi\to\infty}=[(g_0^+)^2+2\Lambda_R]^{1/2}$

$$\phi_{1,1} = a_{1,1}^{+}[1-\exp(-Le_1\tilde{m}/\tilde{r}_{f,S})] + Le_1\{[(a_{T,A}^{+}/g_0^{+})-\xi][(g_0^{+})^2+2\Lambda_R]^{1/2}+\theta_1\}$$

$$\phi_{2,1} = a_{1,1}^{+}[\tilde{Y}_{2,\infty}/(1+\tilde{Y}_{2,\infty})] + Le_2\{(a_{T,A}^{+}/g_0^{+})[(g_0^{+})^2+2\Lambda_R]^{1/2}+[(g_0^{-})^2+2\Lambda_R]^{1/2}\xi+\theta_1\}$$

(J) Adiabatic Limit

In the adiabatic limit, $Da_R = \Lambda_R = 0$, $\tilde{T}_f = \tilde{T}_f^0$

$[(g_0^{-})^2+2\Lambda_R]^{1/2}+[(g_0^{+})^2+2\Lambda_R]^{1/2} = \tilde{m}/\tilde{r}_{f,S}^2$ is reduced to

$$\tilde{m}/\tilde{r}_{f,S}^2 = g_0^{-}+g_0^{+} = (\tilde{T}_f^0-\tilde{T}_0)(\tilde{m}/\tilde{r}_{f,S}^2)+(\tilde{T}_f^0-\tilde{T}_\infty)(\tilde{m}/\tilde{r}_{f,S}^2)/[\exp(\tilde{m}/\tilde{r}_{f,S})-1]$$

or $\tilde{T}_f^0 = 1+\tilde{T}_0-(1+\tilde{T}_0-\tilde{T}_\infty)(1+\tilde{Y}_{2,\infty})^{-1/Le_2}$

In the limit of $Le_2 = 1$, we have the adiabatic flame temperature: $\tilde{T}_{ad} = 1+\tilde{T}_0-(1+\tilde{T}_0-\tilde{T}_\infty)(1+\tilde{Y}_{2,\infty})^{-1}$

(K) Summary

$\delta = \tilde{T}_f^2/\tilde{E}_R$, $\varepsilon = \tilde{T}_f^2/\tilde{E}_K$

(1) Bulk flame behavior

$\tilde{T}_f^0 = 1+\tilde{T}_0\exp(-\tilde{m}/\tilde{r}_{f,S}) = 1+\tilde{T}_0-(1+\tilde{T}_0-\tilde{T}_\infty)(1+\tilde{Y}_{2,\infty})^{-1/Le_2}$; $\tilde{T}_{ad} = 1+\tilde{T}_0-(1+\tilde{T}_0-\tilde{T}_\infty)(1+\tilde{Y}_{2,\infty})^{-1}$

$\tilde{r}_{f,S} = Le_2\tilde{m}/\ell n(1+\tilde{Y}_{2,\infty})$ or $\exp(-Le_2\tilde{m}/\tilde{r}_{f,S}) = 1/(1+\tilde{Y}_{2,\infty})$

$[(g_0^{-})^2+2\Lambda_R]^{1/2}+[(g_0^{+})^2+2\Lambda_R]^{1/2} = \tilde{m}/\tilde{r}_{f,S}^2$; $\Lambda_R = \delta Da_R\exp(-\tilde{E}_R/\tilde{T}_f)$ (determines $\tilde{T}_f$ iteratively)

$g_0^{-} = (\tilde{T}_f-\tilde{T}_0)(\tilde{m}/\tilde{r}_{f,S}^2)$, $g_0^{+} = (\tilde{T}_f-\tilde{T}_\infty)(\tilde{m}/\tilde{r}_{f,S}^2)/[\exp(\tilde{m}/\tilde{r}_{f,S})-1]$

$\tilde{T}_{b,S,A} = \tilde{T}_{b,S,1}\exp[\tilde{m}(1-\tilde{r}_{f,S}^{-1})]$, $a_{T,A}^{+} = a_{T,1}^{+}[1-\exp(-\tilde{m}/\tilde{r}_{f,S})]$

(2) Structure equation

$d^2\theta_1/d\xi^2 = \Lambda_K\phi_{1,1}\phi_{2,1}\exp(-\theta_1)$, $\Lambda_K = \varepsilon^3 Da_K\exp(-\tilde{E}_K/\tilde{T}_f)$

$\theta_1(\xi\to-\infty) = -[(\tilde{T}_{b,S,A}/g_0^{-})+\xi][(g_0^{-})^2+2\Lambda_R]^{1/2}$, $(d\theta_1/d\xi)_{\xi\to-\infty} = -[(g_0^{-})^2+2\Lambda_R]^{1/2}$

$\theta_1(\xi\to\infty)\to-[(a_{T,A}^{+}/g_0^{+})-\xi][(g_0^{+})^2+2\Lambda_R]^{1/2}$, $(d\theta_1/d\xi)_{\xi\to\infty} = [(g_0^{+})^2+2\Lambda_R]^{1/2}$

$$\phi_{1,1} = a_{1,1}^{+}[1-\exp(-Le_1\tilde{m}/\tilde{r}_{f,S})] + Le_1\{[(a_{T,A}^{+}/g_0^{+})-\xi][(g_0^{+})^2+2\Lambda_R]^{1/2}+\theta_1\}$$

$$\phi_{2,1} = a_{1,1}^{+}[\tilde{Y}_{2,\infty}/(1+\tilde{Y}_{2,\infty})] + Le_2\{(a_{T,A}^{+}/g_0^{+})[(g_0^{+})^2+2\Lambda_R]^{1/2}+[(g_0^{-})^2+2\Lambda_R]^{1/2}\xi+\theta_1\}$$

(3) Solution in the core and burner regions $\tilde{Y}_{1,S} = 1$; $\tilde{Y}_{2,S} = 0$

Core Region: $\tilde{T}_S = \tilde{T}_0+(\tilde{T}_{b,S}-\tilde{T}_0)\exp[(\tilde{m}/\tilde{\lambda})(1-\tilde{r}_i^{-1})]\exp[\tilde{m}(\tilde{r}_i^{-1}-\tilde{r}^{-1})]$

Burner Region: $\tilde{T}_S = \tilde{T}_0+(\tilde{T}_{b,S}-\tilde{T}_0)\exp[(\tilde{m}/\tilde{\lambda})(1-\tilde{r}^{-1})]$

(4) Outer solution of $\tilde{Y}_{1,S}$ and $\tilde{Y}_{2,S}$

$\tilde{Y}_{1,S}^{-} = [\tilde{Y}_{1,S,0}^{-}+O(\delta^2)]+\varepsilon[\tilde{Y}_{1,S,1}^{-}+O(\delta)]+O(\varepsilon^2)$

$\tilde{Y}_{1,S,0}^{-} = 1-\exp[Le_1\tilde{m}(\tilde{r}_{f,S}^{-1}-\tilde{r}^{-1})]$; $\tilde{Y}_{1,S,1}^{-} = -a_{1,1}^{-}\exp(-Le_1\tilde{m}/\tilde{r})$

$d\tilde{Y}_{1,S,0}^{-}/d\tilde{r} = -(Le_1\tilde{m}/\tilde{r}^2)\exp[Le_1\tilde{m}(\tilde{r}_{f,S}^{-1}-\tilde{r}^{-1})]$; $(d\tilde{Y}_{1,S,0}^{-}/d\tilde{r})_{\tilde{r}_{f,S}} = -(Le_1\tilde{m}/\tilde{r}_{f,S}^2)$

$d\tilde{Y}_{1,S,1}^{-}/d\tilde{r} = -(\tilde{m}/\tilde{r}^2)a_{1,1}^{-}Le_1\exp(-Le_1\tilde{m}/\tilde{r})$

$(\tilde{r}^2/Le_1)(d^2\tilde{Y}_{1,S,0}^{-}/d\tilde{r}^2)-[\tilde{m}-(2\tilde{r}/Le_1)](d\tilde{Y}_{1,S,0}^{-}/d\tilde{r}) = 0$

$(\tilde{r}^2/Le_1)(d^3\tilde{Y}_{1,S,0}^{-}/d\tilde{r}^3)-[\tilde{m}-(2\tilde{r}/Le_1)](d^2\tilde{Y}_{1,S,0}^{-}/d\tilde{r}^2)+[(2\tilde{m}/\tilde{r})-(2/Le_1)](d\tilde{Y}_{1,S,0}^{-}/d\tilde{r}) = 0$

$(1/Le_1)(d\tilde{Y}_{1,S,1}^{-}/d\tilde{r})-(\tilde{m}/\tilde{r}^2)\tilde{Y}_{1,S,1}^{-} = 0$

$(1/Le_1)(d^2\tilde{Y}_{1,S,1}^{-}/d\tilde{r}^2)-(\tilde{m}/\tilde{r}^2)(d\tilde{Y}_{1,S,1}^{-}/d\tilde{r})+(2\tilde{m}/\tilde{r}^3)\tilde{Y}_{1,S,1}^{-} = 0$

$\tilde{Y}_{1,S}^{+} = \varepsilon[\tilde{Y}_{1,S,1}^{+}+O(\delta)]+O(\varepsilon^2)$; $\tilde{Y}_{1,S,1}^{+} = a_{1,1}^{+}[1-\exp(-Le_1\tilde{m}/\tilde{r})]$; $\tilde{Y}_{1,S,1}^{+}(\tilde{r}_{f,S}) = a_{1,1}^{+}[1-(1+\tilde{Y}_{2,\infty})^{-Le_1/Le_2}]$

$d\tilde{Y}_{1,S,1}^{+}/d\tilde{r} = -a_{1,1}^{+}(Le_1\tilde{m}/\tilde{r}^2)\exp(-Le_1\tilde{m}/\tilde{r})$; $(d\tilde{Y}_{1,S,1}^{+}/d\tilde{r})_{\tilde{r}_{f,S}} = -a_{1,1}^{+}(Le_1\tilde{m}/\tilde{r}_{f,S}^2)\exp(-Le_1\tilde{m}/\tilde{r}_{f,S})$

$(1/Le_1)(d\tilde{Y}_{1,S,1}^{+}/d\tilde{r})-(\tilde{m}/\tilde{r}^2)\tilde{Y}_{1,S,1}^{+} = -\tilde{m}a_{1,1}^{+}/\tilde{r}^2$; $(1/Le_1)(d^2\tilde{Y}_{1,S,1}^{+}/d\tilde{r}^2)-(\tilde{m}/\tilde{r}^2)(d\tilde{Y}_{1,S,1}^{+}/d\tilde{r})+(2\tilde{m}/\tilde{r}^3)\tilde{Y}_{1,S,1}^{+} = 2\tilde{m}a_{1,1}^{+}/\tilde{r}^3$

$$\tilde{Y}_{2,S}^{-} = \varepsilon[\tilde{Y}_{2,S,1}^{-} + O(\delta)] + O(\varepsilon^2) \quad ; \quad \tilde{Y}_{2,S,1}^{-} = a_{2,1}^{-}\exp(-Le_2\,\tilde{m}/\tilde{r}) \quad ; \quad \tilde{Y}_{2,S,1}^{-}(\tilde{r}_{f,S}) = a_{2,1}^{-}/(1+\tilde{Y}_{2,\infty})$$

$$d\tilde{Y}_{2,S,1}^{-}/d\tilde{r} = a_{2,1}^{-}(Le_2\,\tilde{m}/\tilde{r}^2)\exp(-Le_2\,\tilde{m}/\tilde{r}) \quad ; \quad (d\tilde{Y}_{2,S,1}^{-}/d\tilde{r})_{\tilde{r}_{f,S}} = a_{2,1}^{-}(Le_2\,\tilde{m}/\tilde{r}_{f,S}^2)/(1+\tilde{Y}_{2,\infty})$$

$$(1/Le_2)(d\tilde{Y}_{2,S,1}^{-}/d\tilde{r}) - (\tilde{m}/\tilde{r}^2)\tilde{Y}_{2,S,1}^{-} = 0 \quad ; \quad (1/Le_2)(d^2\tilde{Y}_{2,S,1}^{-}/d\tilde{r}^2) - (\tilde{m}/\tilde{r}^2)(d\tilde{Y}_{2,S,1}^{-}/d\tilde{r}) + (2\tilde{m}/\tilde{r}^3)\tilde{Y}_{2,S,1}^{-} = 0$$

$$\tilde{Y}_{2,S}^{+} = [\tilde{Y}_{2,S,0}^{+} + O(\delta^2)] + \varepsilon[\tilde{Y}_{2,S,1}^{+} + O(\delta)] + O(\varepsilon^2)$$

$$\tilde{Y}_{2,S,0}^{+} = \tilde{Y}_{2,\infty} - (1+\tilde{Y}_{2,\infty})[1-\exp(-Le_2\,\tilde{m}/\tilde{r})] = (1+\tilde{Y}_{2,\infty})\exp(-Le_2\,\tilde{m}/\tilde{r}) - 1$$

$$d\tilde{Y}_{2,S,0}^{+}/d\tilde{r} = (1+\tilde{Y}_{2,\infty})(Le_2\,\tilde{m}/\tilde{r}^2)\exp(-Le_2\,\tilde{m}/\tilde{r}) \quad ; \quad (d\tilde{Y}_{2,S,0}^{+}/d\tilde{r})_{\tilde{r}_{f,S}} = (Le_2\,\tilde{m}/\tilde{r}_{f,S}^2)$$

$$\tilde{Y}_{2,S,1}^{+} = a_{1,1}^{+}[1-\exp(-Le_2\,\tilde{m}/\tilde{r})] \qquad (a_{2,1}^{+} = -a_{1,1}^{+})$$

$$d\tilde{Y}_{2,S,1}^{+}/d\tilde{r} = -(\tilde{m}/\tilde{r}^2)a_{1,1}^{+}\,Le_2\exp(-Le_{2,0}\,\tilde{m}/\tilde{r})$$

$$(\tilde{r}^2/Le_2)(d^2\tilde{Y}_{2,S,0}^{+}/d\tilde{r}^2) - [\tilde{m} - (2\tilde{r}/Le_2)](d\tilde{Y}_{2,S,0}^{+}/d\tilde{r}) = 0$$

$$(\tilde{r}^2/Le_2)(d^3\tilde{Y}_{2,S,0}^{+}/d\tilde{r}^3) - [\tilde{m} - (2\tilde{r}/Le_2)](d^2\tilde{Y}_{2,S,0}^{+}/d\tilde{r}^2) + [(2\tilde{m}/\tilde{r}) - (2/Le_2)](d\tilde{Y}_{2,S,0}^{+}/d\tilde{r}) = 0$$

$$(1/Le_2)(d\tilde{Y}_{2,S,1}^{+}/d\tilde{r}) - (\tilde{m}/\tilde{r}^2)\tilde{Y}_{2,S,1}^{+} = -a_{1,1}^{+}\,\tilde{m}/\tilde{r}^2$$

$$(1/Le_2)(d^2\tilde{Y}_{2,S,1}^{+}/d\tilde{r}^2) - (\tilde{m}/\tilde{r}^2)(d\tilde{Y}_{2,S,1}^{+}/d\tilde{r}) + (2\tilde{m}/\tilde{r}^3)\tilde{Y}_{2,S,1}^{+} = 2a_{1,1}^{+}\,\tilde{m}/\tilde{r}^3$$

$$(a_{T,1}^{+}/g_0^{+})[(g_0^{+})^2 + 2\Lambda_R]^{1/2} = -a_{1,1}^{+}$$

$$a_{1,1}^{-}\exp(-Le_1\tilde{m}/\tilde{r}_{f,S}) = Le_1\{(\tilde{T}_{b,S,A}/g_0^{-})[(g_0^{-})^2 + 2\Lambda_R]^{1/2} + a_{1,1}^{+}[1-\exp(-\tilde{m}/\tilde{r}_{f,S})]\} - a_{1,1}^{+}[1-\exp(-Le_1\tilde{m}/\tilde{r}_{f,S})]$$

$$a_{2,1}^{-}/(1+\tilde{Y}_{2,\infty}) = a_{1,1}^{+}\,\tilde{Y}_{2,\infty}/(1+\tilde{Y}_{2,\infty}) - Le_2\{(\tilde{T}_{b,S,A}/g_0^{-})[(g_0^{-})^2 + 2\Lambda_R]^{1/2} + a_{1,1}^{+}[1-\exp(-\tilde{m}/\tilde{r}_{f,S})]\}$$

$$a_{2,1}^{+} = -a_{1,1}^{+}$$

(5) Temperature at the burner exit

$$\tilde{T}_{b,S} = [\tilde{T}_{b,S,0} + \varepsilon\tilde{T}_{b,S,1} + O(\varepsilon^2)] + \delta[\tilde{T}_{b,S,2} + O(\varepsilon)] + O(\delta^2)$$

$$\tilde{T}_{b,S,0} = \tilde{T}_0 + \{(\tilde{T}_f - \tilde{T}_0)/\exp[\tilde{m}(1-\tilde{r}_{f,S}^{-1})]\}$$

$$\tilde{T}_{b,S,2} = g_1^{-}/\exp[\tilde{m}(1-\tilde{r}_{f,S}^{-1})] = -2\,\ell\text{n}\langle[\{1+[2\Lambda_R/(g_0^{-})^2]\}^{1/2}+1]/2\rangle/\exp[\tilde{m}(1-\tilde{r}_{f,S}^{-1})]$$

$$g_0^{-} = (\tilde{T}_f - \tilde{T}_0)(\tilde{m}/\tilde{r}_{f,S}^2) \quad ; \quad g_1^{-} = -2\,\ell\text{n}\{[\langle 1+[2\Lambda_R/(g_0^{-})^2]\rangle^{1/2}+1]/2\}$$

$$g_0^{+} = (\tilde{T}_f - \tilde{T}_\infty)(\tilde{m}/\tilde{r}_{f,S}^2)/[\exp(\tilde{m}/\tilde{r}_{f,S}) - 1] \quad ; \quad g_1^{+} = -2\,\ell\text{n}\{[\langle 1+[2\Lambda_R/(g_0^{+})^2]\rangle^{1/2}+1]/2\}$$

(6) Outer solution of $\tilde{T}_S$

$$\tilde{T}_S^{-} = [\tilde{T}_{S,0}^{-} + \varepsilon\tilde{T}_{S,1}^{-} + O(\varepsilon^2)] + \delta[\tilde{T}_{S,2}^{-} + O(\varepsilon)] + O(\delta^2)$$

$$\tilde{T}_{S,0}^{-} = \tilde{T}_0 + (\tilde{T}_{b,S,0} - \tilde{T}_0)\exp[\tilde{m}(1-\tilde{r}^{-1})] = \tilde{T}_0 + (\tilde{T}_f - \tilde{T}_0)\exp[\tilde{m}(\tilde{r}_{f,S}^{-1} - \tilde{r}^{-1})]$$

$$d\tilde{T}_{S,0}^{-}/d\tilde{r} = (\tilde{T}_f - \tilde{T}_0)(\tilde{m}/\tilde{r}^2)\exp[\tilde{m}(\tilde{r}_{f,S}^{-1} - \tilde{r}^{-1})]$$

$$d^2\tilde{T}_{S,0}^{-}/d\tilde{r}^2 = (\tilde{T}_f - \tilde{T}_0)(\tilde{m}/\tilde{r}^2)[(\tilde{m}/\tilde{r}^2) - (2/\tilde{r})]\exp[\tilde{m}(\tilde{r}_{f,S}^{-1} - \tilde{r}^{-1})]$$

$$\tilde{T}_{S,1}^{-} = \tilde{T}_{b,S,1}\exp[\tilde{m}(1-\tilde{r}^{-1})] = \tilde{T}_{b,S,A}\exp[\tilde{m}(\tilde{r}_{f,S}^{-1} - \tilde{r}^{-1})]$$

$$d\tilde{T}_{S,1}^{-}/d\tilde{r} = \tilde{T}_{b,S,1}(\tilde{m}/\tilde{r}^2)[\tilde{m}(1-\tilde{r}^{-1})] = \tilde{T}_{b,S,A}(\tilde{m}/\tilde{r}^2)[\tilde{m}(\tilde{r}_{f,S}^{-1} - \tilde{r}^{-1})]$$

$$\tilde{T}_{S,2}^{-} = g_1^{-}\exp[\tilde{m}(\tilde{r}_{f,S}^{-1} - \tilde{r}^{-1})] = -2\,\ell\text{n}\{[\langle 1+[2\Lambda_R/(g_0^{-})^2]\rangle^{1/2}+1]/2\}\exp[\tilde{m}(\tilde{r}_{f,S}^{-1} - \tilde{r}^{-1})]$$

$$d\tilde{T}_{S,2}^{-}/d\tilde{r} = g_1^{-}(\tilde{m}/\tilde{r}^2)\exp[\tilde{m}(\tilde{r}_{f,S}^{-1} - \tilde{r}^{-1})]$$

$$\tilde{T}_S^{+} = [\tilde{T}_{S,0}^{+} + \varepsilon\tilde{T}_{S,1}^{+} + O(\varepsilon^2)] + \delta[\tilde{T}_{S,2}^{+} + O(\varepsilon)] + O(\delta^2)$$

$$\tilde{T}_{S,0}^{+} = \tilde{T}_\infty + (\tilde{T}_f - \tilde{T}_\infty)[1-\exp(-\tilde{m}/\tilde{r})]/[1-\exp(-\tilde{m}/\tilde{r}_{f,S})]$$

$$d\tilde{T}_{S,0}^{+}/d\tilde{r} = -(\tilde{T}_f - \tilde{T}_\infty)(\tilde{m}/\tilde{r}^2)\exp(-\tilde{m}/\tilde{r})/[1-\exp(-\tilde{m}/\tilde{r}_{f,S})]$$

$$d^2\tilde{T}_{S,0}^{+}/d\tilde{r}^2 = -(\tilde{T}_f - \tilde{T}_\infty)(\tilde{m}/\tilde{r}^2)[(\tilde{m}/\tilde{r}^2) - (2/\tilde{r})]\exp(-\tilde{m}/\tilde{r})/[1-\exp(-\tilde{m}/\tilde{r}_{f,S})]$$

$$\tilde{T}_{S,1}^{+} = a_{T,1}^{+}[1-\exp(-\tilde{m}/\tilde{r})]$$

$$d\tilde{T}^{+}_{S,1}/d\tilde{r} = -a^{+}_{T,1}(\tilde{m}/\tilde{r}^2)\exp(-\tilde{m}/\tilde{r})$$

$$\tilde{T}^{+}_{S,2} = g^{+}_{1}[1-\exp(-\tilde{m}/\tilde{r})]/[1-\exp(-\tilde{m}/\tilde{r}_{f,S})]$$

$$d\tilde{T}^{+}_{S,2}/d\tilde{r} = -g^{+}_{1}(\tilde{m}/\tilde{r}^2)\exp(-\tilde{m}/\tilde{r})/[1-\exp(-\tilde{m}/\tilde{r}_{f,S})]$$

(7) Solution of $\tilde{T}_S$ in the radiation regions

$$\tilde{T}^{-}_{S} = [\tilde{T}_f - \varepsilon\Theta^{-}_{S,2} + O(\varepsilon^2)] - \delta[\Theta^{-}_{S,1} + \varepsilon\Theta^{-}_{S,3} + O(\varepsilon^2)] + O(\delta^2)$$

$$\Theta^{-}_{S,1} = \ell n\{1 - [\Lambda_R/\langle 2(g^{-}_{0})^2\rangle]\exp(g^{-}_{1} + g^{-}_{0}\zeta)\}^2 - g^{-}_{0}\zeta - g^{-}_{1}$$

$$d\Theta^{-}_{S,1}/d\zeta = -[2\Lambda_R\exp(-\Theta^{-}_{S,1}) + (g^{-}_{0})^2]^{1/2} = -g^{-}_{0}[2(g^{-}_{0})^2 + \Lambda_R\exp(g^{-}_{1} + g^{-}_{0}\zeta)]/[2(g^{-}_{0})^2 - \Lambda_R\exp(g^{-}_{1} + g^{-}_{0}\zeta)]$$

$$\Theta^{-}_{S,2} = (\tilde{T}_{b,S,A}/g^{-}_{0})(d\Theta^{-}_{S,1}/d\zeta) = -\tilde{T}_{b,S,A}[2(g^{-}_{0})^2 + \Lambda_R\exp(g^{-}_{1} + g^{-}_{0}\zeta)]/[2(g^{-}_{0})^2 - \Lambda_R\exp(g^{-}_{1} + g^{-}_{0}\zeta)]$$

$$\tilde{T}^{+}_{S} = [\tilde{T}_f - \varepsilon\Theta^{+}_{S,2} + O(\varepsilon^2)] - \delta[\Theta^{+}_{S,1} + \varepsilon\Theta^{+}_{S,3} + O(\varepsilon^2)] + O(\delta^2)$$

$$\Theta^{+}_{S,1} = \ell n\langle 1 - \{\Lambda_R/[2(g^{+}_{0})^2]\}\exp(g^{+}_{1} - g^{+}_{0}\zeta)\rangle^2 + g^{+}_{0}\zeta - g^{+}_{1}$$

$$d\Theta^{+}_{S,1}/d\zeta = [2\Lambda_R\exp(-\Theta^{+}_{S,1}) + (g^{+}_{0})^2]^{1/2} = g^{+}_{0}[2(g^{+}_{0})^2 + \Lambda_R\exp(g^{+}_{1} - g^{+}_{0}\zeta)]/[2(g^{+}_{0})^2 - \Lambda_R\exp(g^{+}_{1} - g^{+}_{0}\zeta)]$$

$$\Theta^{+}_{S,2} = -(a^{+}_{T,A}/g^{+}_{0})(d\Theta^{+}_{S,1}/d\zeta) = -a^{+}_{T,A}[2(g^{+}_{0})^2 + \Lambda_R\exp(g^{+}_{1} - g^{+}_{0}\zeta)]/[2(g^{+}_{0})^2 - \Lambda_R\exp(g^{+}_{1} - g^{+}_{0}\zeta)]$$

$$g^{\pm}_{1} = -2\,\ell n\{[\langle 1 + [2\Lambda_R/(g^{\pm}_{0})^2]\rangle^{1/2} + 1]/2\} \quad ; \quad \exp(g^{\pm}_{1}) = 4/\langle\{1 + [2\Lambda_R/(g^{\pm}_{0})^2]\}^{1/2} + 1\rangle^2$$

$$2(g^{\pm}_{0})^2 + \Lambda_R\exp(g^{\pm}_{1}) = 4(g^{\pm}_{0})^2\{1 + [2\Lambda_R/(g^{\pm}_{0})^2]\}^{1/2}/\langle\{1 + [2\Lambda_R/(g^{\pm}_{0})^2]\}^{1/2} + 1\rangle$$

$$2(g^{\pm}_{0})^2 - \Lambda_R\exp(g^{\pm}_{1}) = 4(g^{\pm}_{0})^2/\langle\{1 + [2\Lambda_R/(g^{\pm}_{0})^2]\}^{1/2} + 1\rangle$$

$$[2(g^{\pm}_{0})^2 + \Lambda_R\exp(g^{\pm}_{1})]/[2(g^{\pm}_{0})^2 - \Lambda_R\exp(g^{\pm}_{1})] = \{1 + [2\Lambda_R/(g^{\pm}_{0})^2]\}^{1/2}$$

$$4(g^{\pm}_{0})^2\Lambda_R\exp(g^{\pm}_{1})/\{[2(g^{\pm}_{0})^2 + \Lambda_R\exp(g^{\pm}_{1})][2(g^{\pm}_{0})^2 - \Lambda_R\exp(g^{\pm}_{1})]\} = [\Lambda_R/(g^{\pm}_{0})^2]/\{1 + [2\Lambda_R/(g^{\pm}_{0})^2]\}^{1/2}$$

(L) Conversion of the Structure Equation into Liñán's Form

$$d^2\theta_1/d\xi^2 = \Lambda_K\phi_{1,1}\phi_{2,1}\exp(-\theta_1) \quad \text{where} \quad \Lambda_K = \varepsilon^3 Da_K\exp(-\tilde{E}_K/\tilde{T}_f)$$

$$\theta_1(\xi\to-\infty) = -[(\tilde{T}_{b,S,A}/g^{-}_{0}) + \xi][(g^{-}_{0})^2 + 2\Lambda_R]^{1/2}$$

$$\theta_1(\xi\to\infty) \to -[(a^{+}_{T,A}/g^{+}_{0}) - \xi][(g^{+}_{0})^2 + 2\Lambda_R]^{1/2}$$

$$\phi_{1,1} = a^{+}_{1,1}[1 - \exp(-Le_1\tilde{m}/\tilde{r}_{f,S})] + Le_1\{[(a^{+}_{T,A}/g^{+}_{0}) - \xi][(g^{+}_{0})^2 + 2\Lambda_R]^{1/2} + \theta_1\}$$

$$\phi_{2,1} = a^{+}_{1,1}[\tilde{Y}_{2,\infty}/(1 + \tilde{Y}_{2,\infty})] + Le_2\{(a^{+}_{T,A}/g^{+}_{0})[(g^{+}_{0})^2 + 2\Lambda_R]^{1/2} + [(g^{-}_{0})^2 + 2\Lambda_R]^{1/2}\xi + \theta_1\}$$

$$[(g^{-}_{0})^2 + 2\Lambda_R]^{1/2} + [(g^{+}_{0})^2 + 2\Lambda_R]^{1/2} = \tilde{m}/\tilde{r}^2_{f,S} \quad ; \quad \Lambda_R = \delta Da_R\exp(-\tilde{E}_R/\tilde{T}_f)$$

$$g^{-}_{0} = (\tilde{T}_f - \tilde{T}_0)(\tilde{m}/\tilde{r}^2_{f,S}) \quad ; \quad g^{-}_{1} = -2\,\ell n\langle[\{1 + [2\Lambda_R/(g^{-}_{0})^2]\}^{1/2} + 1]/2\rangle$$

$$g^{+}_{0} = (\tilde{T}_f - \tilde{T}_\infty)(\tilde{m}/\tilde{r}^2_{f,S})/[\exp(\tilde{m}/\tilde{r}_{f,S}) - 1] \quad ; \quad g^{+}_{1} = -2\,\ell n\langle[\{1 + [2\Lambda_R/(g^{+}_{0})^2]\}^{1/2} + 1]/2\rangle$$

$$a^{+}_{1,1} = -(a^{+}_{T,1}/g^{+}_{0})[(g^{+}_{0})^2 + 2\Lambda_R]^{1/2}$$

Define $\bar{\theta} = \alpha^{1/3}(\theta_1 + A\xi + B)$; $\bar{\xi} = \alpha^{1/3}(D\xi + G)$, $D > 0$

Then $\xi = (\alpha^{-1/3}\bar{\xi} - G)/D$; As $\xi \to \pm\infty$, $\bar{\xi} \to \pm\infty$

$$\theta_1 = \alpha^{-1/3}\bar{\theta} - A\xi - B = \alpha^{-1/3}\bar{\theta} - (A/D)(\alpha^{-1/3}\bar{\xi} - G) - B$$

(1) $\theta_1(\xi\to-\infty) = -[(\tilde{T}_{b,S,A}/g^{-}_{0}) + \xi][(g^{-}_{0})^2 + 2\Lambda_R]^{1/2}$

$$\alpha^{-1/3}[\bar{\theta} - (A/D)\bar{\xi}]_{\bar{\xi}\to-\infty} + [(AG/D) - B] = -\{(\tilde{T}_{b,S,A}/g^{-}_{0}) + [(\alpha^{-1/3}\bar{\xi} - G)/D]\}[(g^{-}_{0})^2 + 2\Lambda_R]^{1/2}$$

$$\alpha^{-1/3}\{\bar{\theta} - [\{A - [(g^{-}_{0})^2 + 2\Lambda_R]^{1/2}\}/D]\bar{\xi}\}_{\bar{\xi}\to-\infty} = -(\tilde{T}_{b,S,A}/g^{-}_{0})[(g^{-}_{0})^2 + 2\Lambda_R]^{1/2} - \langle G\{A - [(g^{-}_{0})^2 + 2\Lambda_R]^{1/2}\}/D\rangle + B$$

Want: $\{A - [(g^{-}_{0})^2 + 2\Lambda_R]^{1/2}\}/D = -1$ or $D = [(g^{-}_{0})^2 + 2\Lambda_R]^{1/2} - A$

$$\Rightarrow \quad \alpha^{-1/3}(\bar{\theta} + \bar{\xi})_{\bar{\xi}\to-\infty} = -(\tilde{T}_{b,S,A}/g^{-}_{0})[(g^{-}_{0})^2 + 2\Lambda_R]^{1/2} + G + B$$

$$\bar{\theta}(\bar{\xi}\to-\infty) = -\alpha^{1/3}\{(\tilde{T}_{b,S,A}/g^{-}_{0})[(g^{-}_{0})^2 + 2\Lambda_R]^{1/2} - G - B\} - \bar{\xi} \quad , \quad (d\bar{\theta}/d\bar{\xi})_{\bar{\xi}\to-\infty} = -1$$

(2) $\theta_1(\xi \to \infty) = -\{(a_{T,A}^+ / g_0^+) - \xi\}[(g_0^+)^2 + 2\Lambda_R]^{1/2}$

$\alpha^{-1/3}[\bar{\theta} - (A/D)\bar{\xi}]_{\bar{\xi}\to\infty} + [(AG/D) - B] = -\{(a_{T,A}^+ / g_0^+) - [(\alpha^{-1/3}\bar{\xi} - G)/D]\}[(g_0^+)^2 + 2\Lambda_R]^{1/2}$

$\alpha^{-1/3}\{\bar{\theta} - [\langle A + [(g_0^+)^2 + 2\Lambda_R]^{1/2}\rangle / D]\bar{\xi}\}_{\bar{\xi}\to\infty} = -(a_{T,A}^+ / g_0^+)[(g_0^+)^2 + 2\Lambda_R]^{1/2} - \langle G\{A + [(g_0^+)^2 + 2\Lambda_R]^{1/2}\}/D\rangle + B$

Want: $\{A + [(g_0^+)^2 + 2\Lambda_R]^{1/2}\}/D = 1$ or $D = A + [(g_0^+)^2 + 2\Lambda_R]^{1/2}$

From (1): $D = [(g_0^-)^2 + 2\Lambda_R]^{1/2} - A$ $\quad\therefore\quad$ $A = \{[(g_0^-)^2 + 2\Lambda_R]^{1/2} - [(g_0^+)^2 + 2\Lambda_R]^{1/2}\}/2$

Thus: $\alpha^{-1/3}(\bar{\theta} - \bar{\xi})_{\bar{\xi}\to\infty} = -(a_{T,A}^+ / g_0^+)[(g_0^+)^2 + 2\Lambda_R]^{1/2} - G + B$

$\bar{\theta}(\bar{\xi} \to \infty) = -\alpha^{1/3}\{(a_{T,A}^+ / g_0^+)[(g_0^+)^2 + 2\Lambda_R]^{1/2} + G - B\} + \bar{\xi}$, $\quad (d\bar{\theta}/d\bar{\xi})_{\bar{\xi}\to\infty} = 1$

Since $[(g_0^-)^2 + 2\Lambda_R]^{1/2} + [(g_0^+)^2 + 2\Lambda_R]^{1/2} = \tilde{m}/\tilde{r}_{f,S}^2$

$A = \{[(g_0^-)^2 + 2\Lambda_R]^{1/2} - [(g_0^+)^2 + 2\Lambda_R]^{1/2}\}/2 = [(g_0^-)^2 + 2\Lambda_R]^{1/2} - [\tilde{m}/(2\tilde{r}_{f,S}^2)] = [\tilde{m}/(2\tilde{r}_{f,S}^2)] - [(g_0^+)^2 + 2\Lambda_R]^{1/2}$

$D = [(g_0^-)^2 + 2\Lambda_R]^{1/2} - A = [(g_0^-)^2 + 2\Lambda_R]^{1/2} - \{[(g_0^-)^2 + 2\Lambda_R]^{1/2} - [\tilde{m}/(2\tilde{r}_{f,S}^2)]\} = \tilde{m}/(2\tilde{r}_{f,S}^2)$

Define $\gamma = -A/D = 1 - (2\tilde{r}_{f,S}^2/\tilde{m})[(g_0^-)^2 + 2\Lambda_R]^{1/2} = (2\tilde{r}_{f,S}^2/\tilde{m})[(g_0^+)^2 + 2\Lambda_R]^{1/2} - 1$

$\theta_1 = \alpha^{-1/3}\bar{\theta} - (A/D)(\alpha^{-1/3}\bar{\xi} - G) - B = \alpha^{-1/3}\bar{\theta} + \gamma(\alpha^{-1/3}\bar{\xi} - G) - B = \alpha^{-1/3}(\bar{\theta} + \gamma\bar{\xi}) - \gamma G - B$

$\xi = (\alpha^{-1/3}\bar{\xi} - G)/D = (2\tilde{r}_{f,S}^2/\tilde{m})(\alpha^{-1/3}\bar{\xi} - G)$ or $\bar{\xi} = \alpha^{1/3}[\tilde{m}/(2\tilde{r}_{f,S}^2)]\xi + G$ $\Rightarrow$ $d\bar{\xi}/d\xi = \alpha^{1/3}[\tilde{m}/(2\tilde{r}_{f,S}^2)]$

(3) $\phi_{1,1} = a_{1,1}^+[1 - \exp(-Le_1\tilde{m}/\tilde{r}_{f,S})] + Le_1\{[(a_{T,A}^+ / g_0^+) - \xi][(g_0^+)^2 + 2\Lambda_R]^{1/2} + \theta_1\}$

$= a_{1,1}^+[1 - \exp(-Le_1\tilde{m}/\tilde{r}_{f,S})] + Le_1(a_{T,A}^+ / g_0^+)[(g_0^+)^2 + 2\Lambda_R]^{1/2} + Le_1\{\theta_1 - [(g_0^+)^2 + 2\Lambda_R]^{1/2}\xi\}$

$= a_{1,1}^+[1 - \exp(-Le_1\tilde{m}/\tilde{r}_{f,S})] + Le_1(a_{T,A}^+ / g_0^+)[(g_0^+)^2 + 2\Lambda_R]^{1/2}$

$+ Le_1\{\alpha^{-1/3}\bar{\theta} - (\alpha^{-1/3}\bar{\xi} - G)\langle(2\tilde{r}_{f,S}^2/\tilde{m})[(g_0^-)^2 + 2\Lambda_R]^{1/2} - 1\rangle - B - [(g_0^+)^2 + 2\Lambda_R]^{1/2}(2\tilde{r}_{f,S}^2/\tilde{m})(\alpha^{-1/3}\bar{\xi} - G)\}$

$= a_{1,1}^+[1 - \exp(-Le_1\tilde{m}/\tilde{r}_{f,S})] + Le_1(a_{T,A}^+ / g_0^+)[(g_0^+)^2 + 2\Lambda_R]^{1/2}$

$+ Le_1\{\alpha^{-1/3}\bar{\theta} - (\alpha^{-1/3}\bar{\xi} - G)[(2\tilde{r}_{f,S}^2/\tilde{m})\langle[(g_0^-)^2 + 2\Lambda_R]^{1/2} + [(g_0^+)^2 + 2\Lambda_R]^{1/2}\rangle - 1] - B\}$

Since $(2\tilde{r}_{f,S}^2/\tilde{m})\{[(g_0^-)^2 + 2\Lambda_R]^{1/2} + [(g_0^+)^2 + 2\Lambda_R]^{1/2}\} - 1 = (2\tilde{r}_{f,S}^2/\tilde{m})(\tilde{m}/\tilde{r}_{f,S}^2) - 1 = 1$

$\phi_{1,1} = a_{1,1}^+[1 - \exp(-Le_1\tilde{m}/\tilde{r}_{f,S})] + Le_1(a_{T,A}^+ / g_0^+)[(g_0^+)^2 + 2\Lambda_R]^{1/2} + Le_1\{\alpha^{-1/3}\bar{\theta} - (\alpha^{-1/3}\bar{\xi} - G) - B\}$

$= a_{1,1}^+[1 - \exp(-Le_1\tilde{m}/\tilde{r}_{f,S})] + Le_1(a_{T,A}^+ / g_0^+)[(g_0^+)^2 + 2\Lambda_R]^{1/2} + Le_1[\alpha^{-1/3}(\theta - \bar{\xi}) + G - B]$

Want: $a_{1,1}^+[1 - \exp(-Le_1\tilde{m}/\tilde{r}_{f,S})] + Le_1\{(a_{T,A}^+ / g_0^+)[(g_0^+)^2 + 2\Lambda_R]^{1/2} + G - B\} = 0$

$\Rightarrow$ $(a_{T,A}^+ / g_0^+)[(g_0^+)^2 + 2\Lambda_R]^{1/2} + G - B = -(a_{1,1}^+ / Le_1)[1 - \exp(-Le_1\tilde{m}/\tilde{r}_{f,S})]$

$B = (a_{1,1}^+ / Le_1)[1 - \exp(-Le_1\tilde{m}/\tilde{r}_{f,S})] + (a_{T,A}^+ / g_0^+)[(g_0^+)^2 + 2\Lambda_R]^{1/2} + G$

Then: $\phi_{1,1} = Le_1\alpha^{-1/3}(\bar{\theta} - \bar{\xi})$

$\bar{\theta}(\bar{\xi} \to \infty) = -\alpha^{1/3}\{(a_{T,A}^+ / g_0^+)[(g_0^+)^2 + 2\Lambda_R]^{1/2} + G - B\} + \bar{\xi} = \alpha^{1/3}(a_{1,1}^+ / Le_1)[1 - \exp(-Le_1\tilde{m}/\tilde{r}_{f,S})] + \bar{\xi}$

(4) $\phi_{2,1} = a_{1,1}^+[\tilde{Y}_{2,\infty}/(1 + \tilde{Y}_{2,\infty})] + Le_2\{(a_{T,A}^+ / g_0^+)[(g_0^+)^2 + 2\Lambda_R]^{1/2} + [(g_0^-)^2 + 2\Lambda_R]^{1/2}\xi + \theta_1\}$

$= a_{1,1}^+[\tilde{Y}_{2,\infty}/(1 + \tilde{Y}_{2,\infty})] + Le_2(a_{T,A}^+ / g_0^+)[(g_0^+)^2 + 2\Lambda_R]^{1/2} + Le_2\{\theta_1 + [(g_0^-)^2 + 2\Lambda_R]^{1/2}\xi\}$

$= a_{1,1}^+[\tilde{Y}_{2,\infty}/(1 + \tilde{Y}_{2,\infty})] + Le_2(a_{T,A}^+ / g_0^+)[(g_0^+)^2 + 2\Lambda_R]^{1/2}$

$+ Le_2\{\alpha^{-1/3}\bar{\theta} - (\alpha^{-1/3}\bar{\xi} - G)\{(2\tilde{r}_{f,S}^2/\tilde{m})[(g_0^-)^2 + 2\Lambda_R]^{1/2} - 1\} - B + [(g_0^-)^2 + 2\Lambda_R]^{1/2}(2\tilde{r}_{f,S}^2/\tilde{m})(\alpha^{-1/3}\bar{\xi} - G)\}$

$= a_{1,1}^+[\tilde{Y}_{2,\infty}/(1 + \tilde{Y}_{2,\infty})] + Le_2(a_{T,A}^+ / g_0^+)[(g_0^+)^2 + 2\Lambda_R]^{1/2} + Le_2[\alpha^{-1/3}(\bar{\theta} + \bar{\xi}) - G - B]$

Want: $a_{1,1}^+[\tilde{Y}_{2,\infty}/(1 + \tilde{Y}_{2,\infty})] + Le_2\{(a_{T,A}^+ / g_0^+)[(g_0^+)^2 + 2\Lambda_R]^{1/2} - G - B\} = 0$

$\Rightarrow$ $G + B = (a_{1,1}^+ / Le_2)[\tilde{Y}_{2,\infty}/(1 + \tilde{Y}_{2,\infty})] + (a_{T,A}^+ / g_0^+)[(g_0^+)^2 + 2\Lambda_R]^{1/2}$

Since $a_{1,1}^+ = -(a_{T,1}^+ / g_0^+)[(g_0^+)^2 + 2\Lambda_R]^{1/2}$

$G + B = (a_{1,1}^+ / Le_2)[\tilde{Y}_{2,\infty}/(1 + \tilde{Y}_{2,\infty})] - a_{1,1}^+[1 - \exp(-\tilde{m}/\tilde{r}_{f,S})]$

or $B=(a_{1,1}^{+}/Le_2)[\tilde{Y}_{2,\infty}/(1+\tilde{Y}_{2,\infty})]-a_{1,1}^{+}[1-\exp(-\tilde{m}/\tilde{r}_{f,S})]-G$

From (3): $B=(a_{1,1}^{+}/Le_1)[1-\exp(-Le_1\tilde{m}/\tilde{r}_{f,S})]+(a_{T,A}^{+}/g_0^{+})[(g_0^{+})^2+2\Lambda_R]^{1/2}+G$

$=(a_{1,1}^{+}/Le_1)[1-\exp(-Le_1\tilde{m}/\tilde{r}_{f,S})]-a_{1,1}^{+}[1-\exp(-\tilde{m}/\tilde{r}_{f,S})]+G$

$\therefore\ G=(a_{1,1}^{+}/2)\{Le_2^{-1}[\tilde{Y}_{2,\infty}/(1+\tilde{Y}_{2,\infty})]-Le_1^{-1}[1-\exp(-Le_1\tilde{m}/\tilde{r}_{f,S})]\}$

$B=(a_{1,1}^{+}/2)\{Le_2^{-1}[\tilde{Y}_{2,\infty}/(1+\tilde{Y}_{2,\infty})]+Le_1^{-1}[1-\exp(-Le_1\tilde{m}/\tilde{r}_{f,S})]\}-a_{1,1}^{+}[1-\exp(-\tilde{m}/\tilde{r}_{f,S})]$

Then: $\phi_{2,1}=Le_2\,\alpha^{-1/3}(\overline{\theta}+\overline{\xi})$

(5) $d^2\theta_1/d\xi^2=\Lambda_K\,\phi_{1,1}\,\phi_{2,1}\exp(-\theta_1)$; $d\overline{\xi}/d\xi=\alpha^{1/3}[\tilde{m}/(2\tilde{r}_{f,S}^2)]$

$\theta_1=\alpha^{-1/3}(\overline{\theta}+\gamma\overline{\xi})-\gamma G-B$; $\phi_{1,1}=Le_1\,\alpha^{-1/3}(\overline{\theta}-\overline{\xi})$; $\phi_{2,1}=Le_2\,\alpha^{-1/3}(\overline{\theta}+\overline{\xi})$

$d\theta_1/d\xi=(d\theta_1/d\overline{\xi})(d\overline{\xi}/d\xi)=\alpha^{-1/3}[(d\overline{\theta}/d\overline{\xi})+\gamma]\{\alpha^{1/3}[\tilde{m}/(2\tilde{r}_{f,S}^2)]\}=[\tilde{m}/(2\tilde{r}_{f,S}^2)][(d\overline{\theta}/d\overline{\xi})+\gamma]$

$d^2\theta_1/d\xi^2=d(d\theta_1/d\xi)/d\xi=[d(d\theta_1/d\xi)/d\overline{\xi}](d\overline{\xi}/d\xi)=\{[\tilde{m}/(2\tilde{r}_{fS}^2)](d^2\overline{\theta}/d\overline{\xi}^2)\}\{\alpha^{1/3}[\tilde{m}/(2\tilde{r}_{fS}^2)]\}$

$=\alpha^{1/3}[\tilde{m}/(2\tilde{r}_{f,S}^2)]^2(d^2\overline{\theta}/d\overline{\xi}^2)$

$\alpha^{1/3}[\tilde{m}/(2\tilde{r}_{f,S}^2)]^2(d^2\overline{\theta}/d\overline{\xi}^2)=\Lambda_K[Le_F\,\alpha^{-1/3}(\overline{\theta}-\overline{\xi})][Le_O\,\alpha^{-1/3}(\overline{\theta}+\overline{\xi})]\exp[-\alpha^{-1/3}(\overline{\theta}+\gamma\overline{\xi})+\gamma G+B]$

$d^2\overline{\theta}/d\overline{\xi}^2=\alpha^{-1}\Lambda_K(2\tilde{r}_{f,S}^2/\tilde{m})^2Le_{1,0}\,Le_2\exp(\gamma G+B)(\overline{\theta}-\overline{\xi})(\overline{\theta}+\overline{\xi})\exp[-\alpha^{-1/3}(\overline{\theta}+\gamma\overline{\xi})]$

Want $\alpha^{-1}\Lambda_K(2\tilde{r}_{f,S}^2/\tilde{m})^2Le_1\,Le_2\exp(\gamma G+B)=1$ or $\alpha=\Lambda_K(2\tilde{r}_{f,S}^2/\tilde{m})^2Le_1\,Le_2\exp(\gamma G+B)$

Then: $d^2\overline{\theta}/d\overline{\xi}^2=(\overline{\theta}-\overline{\xi})(\overline{\theta}+\overline{\xi})\exp[-\alpha^{-1/3}(\overline{\theta}+\gamma\overline{\xi})]$

(6) Summary

$d^2\overline{\theta}/d\overline{\xi}^2=(\overline{\theta}-\overline{\xi})(\overline{\theta}+\overline{\xi})\exp[-\alpha^{-1/3}(\overline{\theta}+\gamma\overline{\xi})]$

$\overline{\theta}(\overline{\xi}\to-\infty)=-\alpha^{1/3}\{(\tilde{T}_{b,S,A}/g_0^{-})[(g_0^{-})^2+2\Lambda_R]^{1/2}+a_{1,1}^{+}[1-\exp(-\tilde{m}/\tilde{r}_{f,S})]-(a_{1,1}^{+}/Le_2)[\tilde{Y}_{2,\infty}/(1+\tilde{Y}_{2,\infty})]\}-\overline{\xi}$

$(d\overline{\theta}/d\overline{\xi})_{\overline{\xi}\to-\infty}=-1$

$\overline{\theta}(\overline{\xi}\to\infty)=\alpha^{1/3}(a_{1,1}^{+}/Le_1)[1-\exp(-Le_1\tilde{m}/\tilde{r}_{f,S})]+\overline{\xi}$

$(d\overline{\theta}/d\overline{\xi})_{\overline{\xi}\to\infty}=1$

$\alpha=\Lambda_K(2\tilde{r}_{f,S}^2/\tilde{m})^2Le_1\,Le_2\exp(\gamma G+B)$

$\gamma=1-(2\tilde{r}_{f,S}^2/\tilde{m})[(g_0^{-})^2+2\Lambda_R]^{1/2}=(2\tilde{r}_{f,S}^2/\tilde{m})[(g_0^{+})^2+2\Lambda_R]^{1/2}-1$

$G=(a_{1,1}^{+}/2)\{Le_2^{-1}[\tilde{Y}_{2,\infty}/(1+\tilde{Y}_{2,\infty})]-Le_1^{-1}[1-\exp(-Le_1\tilde{m}/\tilde{r}_{f,S})]\}$

$B=(a_{1,1}^{+}/2)\{Le_2^{-1}[\tilde{Y}_{2,\infty}/(1+\tilde{Y}_{2,\infty})]+Le_1^{-1}[1-\exp(-Le_1\tilde{m}/\tilde{r}_{f,S})]\}-a_{1,1}^{+}[1-\exp(-\tilde{m}/\tilde{r}_{f,S})]$

$a_{1,1}^{+}=-(a_{T,1}^{+}/g_0^{+})[(g_0^{+})^2+2\Lambda_R]^{1/2}$

$[(g_0^{-})^2+2\Lambda_R]^{1/2}+[(g_0^{+})^2+2\Lambda_R]^{1/2}=\tilde{m}/\tilde{r}_{f,S}^2$; $g_0^{-}=(\tilde{T}_f-\tilde{T}_0)(\tilde{m}/\tilde{r}_{f,S}^2)$, $g_0^{+}=(\tilde{T}_f-\tilde{T}_\infty)(\tilde{m}/\tilde{r}_{f,S}^2)/[\exp(\tilde{m}/\tilde{r}_{f,S})-1]$

$\tilde{r}_{f,S}=Le_2\,\tilde{m}/\ell\text{n}(1+\tilde{Y}_{2,\infty})$

$\tilde{T}_f^0=1+\tilde{T}_0-(1+\tilde{T}_0-\tilde{T}_\infty)(1+\tilde{Y}_{2,\infty})^{-1/Le_{2,0}}$; $\tilde{T}_{ad}=1+\tilde{T}_0-(1+\tilde{T}_0-\tilde{T}_\infty)(1+\tilde{Y}_{2,\infty})^{-1}$

$\Lambda_R=\delta Da_R\exp(-\tilde{E}_R/\tilde{T}_f)$, $\Lambda_K=\varepsilon^3Da_K\exp(-\tilde{E}_K/\tilde{T}_f)$; $\delta=\tilde{T}_f^2/\tilde{E}_R$, $\varepsilon=\tilde{T}_f^2/\tilde{E}_K$

$\tilde{E}_K=c_{p,g}E_K/(q_1Y_{1,0})$, $\tilde{E}_R=c_{p,g}E_R/(q_1Y_{1,0})$, $(4\sigma r_b/\lambda_g)(q_FY_{F,0}/c_{p,g})^3\tilde{\kappa}(\tilde{T}^4-\tilde{T}_\infty^4)\approx Da_R\exp(-\tilde{E}_R/\tilde{T})$

$Da_K=\nu_2W_2B_K\rho_g^2c_{p,g}r_b^2Y_{1,0}/\lambda_g$, $Da_R=4\sigma r_b^2c_{p,g}\kappa B_R/(\lambda_gq_1Y_{1,0})$

(M) Rescaling

Define $\overline{Y}_1=Y_1$, $\overline{Y}_2=\dfrac{\nu_FW_F}{\nu_OW_O}Y_2$, $\overline{T}=c_{p,g}T/q_F$, $\overline{E}_R=c_{p,g}E_R/q_F$, $\overline{E}_K=c_{p,g}E_K/q_F$

$\overline{\delta}=\overline{T}_{ad}^2/\overline{E}_R$, $\overline{\varepsilon}=\overline{T}_{ad}^2/\overline{E}_K$, $\overline{Da_R}=4\sigma r_b^2c_{p,g}\kappa B_R/(\lambda_gq_F)$, $\overline{Da_K}=\nu_OW_OB_K\rho_g^2c_{p,g}r_b^2/\lambda_g$

$\overline{\Lambda}_R=\overline{\delta}\,\overline{Da_R}\exp(-\tilde{E}_R/\tilde{T}_{ad})$, $\overline{\Lambda}_K=\overline{\varepsilon}^3\,\overline{Da_K}\exp(-\tilde{E}_K/\tilde{T}_{ad})$ note: $\tilde{E}_R/\tilde{T}_{ad}=\overline{E}_R/\overline{T}_{ad}$, $\tilde{E}_K/\tilde{T}_{ad}=\overline{E}_K/\overline{T}_{ad}$

(1) $\tilde{T} = c_{p,g}T/(q_1 Y_{1,0}) = [q_F/(q_1 Y_{1,0})](c_{p,g}T/q_F) = [q_F/(q_1 Y_{1,0})]\overline{T}$;

Similarly: $\tilde{E}_R = [q_F/(q_1 Y_{1,0})]\overline{E}_R$, $\tilde{E}_K = [q_F/(q_1 Y_{1,0})]\overline{E}_K$

(2) $\tilde{Y}_1 = Y_1/Y_{1,0} = \overline{Y}_1/Y_{1,0}$; $\tilde{Y}_2 = \dfrac{\nu_1 W_1}{\nu_2 W_2}\dfrac{Y_2}{Y_{1,0}} = \left(\dfrac{\nu_1 W_1}{\nu_2 W_2}\dfrac{\nu_O W_O}{\nu_F W_F}\right)\dfrac{\nu_F W_F}{\nu_O W_O}\dfrac{Y_2}{Y_{1,0}} = \left(\dfrac{\nu_1 W_1}{\nu_2 W_2}\dfrac{\nu_O W_O}{\nu_F W_F}\right)\dfrac{\overline{Y}_2}{Y_{1,0}}$

(3) $\delta = \dfrac{\tilde{T}_f^2}{\tilde{E}_R} = \dfrac{\tilde{T}_f^2}{\tilde{T}_{ad}^2}\dfrac{\tilde{T}_{ad}^2}{\overline{T}_{ad}^2}\dfrac{\overline{T}_{ad}^2}{\overline{E}_R}\dfrac{\overline{E}_R}{\tilde{E}_R} = \dfrac{\tilde{T}_f^2}{\tilde{T}_{ad}^2}\dfrac{\{[q_F/(q_1 Y_{1,0})]\overline{T}_{ad}\}^2}{\overline{T}_{ad}^2}\overline{\delta}\dfrac{\overline{E}_R}{[q_F/(q_1 Y_{1,0})]\overline{E}_R} = \dfrac{q_F}{q_1 Y_{1,0}}\dfrac{\tilde{T}_f^2}{\tilde{T}_{ad}^2}\overline{\delta}$

Similarly : $\varepsilon = \tilde{T}_f^2/\tilde{E}_K = [q_F/(q_1 Y_{1,0})](\tilde{T}_f/\tilde{T}_{ad})^2\overline{\varepsilon}$

(4) $Da_R = 4\sigma r_b^2 c_{p,g}\kappa B_R/(\lambda_g q_1 Y_{1,0}) = [q_F/(q_1 Y_{1,0})][4\sigma r_b^2 c_{p,g}\kappa B_R/(\lambda_g q_F)] = [q_F/(q_1 Y_{1,0})]\overline{Da}_R$

(5) $Da_K = \nu_2 W_2 B_K \rho_g^2 c_{p,g} r_b^2 Y_{1,0}/\lambda_g = (\nu_O W_O B_K \rho_g^2 c_{p,g} r_b^2/\lambda_g)[(\nu_2 W_2/\nu_O W_O)Y_{1,0}] = [(\nu_2 W_2/\nu_O W_O)Y_{1,0}]\overline{Da}_K$

(6) $\Lambda_R = \delta Da_R \exp(-\tilde{E}_R/\tilde{T}_f)$

$= \{[q_F/(q_1 Y_{1,0})](\tilde{T}_f/\tilde{T}_{ad})^2\overline{\delta}\}\{[q_F/(q_1 Y_{1,0})]\overline{Da}_R\}[\exp(-\tilde{E}_R/\tilde{T}_{ad})\exp(\tilde{E}_R/\tilde{T}_{ad})\exp(-\tilde{E}_R/\tilde{T}_f)]$

$= [q_F/(q_1 Y_{1,0})]^2(\tilde{T}_f/\tilde{T}_{ad})^2\exp[\tilde{E}_R(\tilde{T}_{ad}^{-1} - \tilde{T}_f^{-1})]\overline{\delta}\,\overline{Da}_R\exp(-\tilde{E}_R/\tilde{T}_{ad})$

$= [q_F/(q_1 Y_{1,0})]^2(\tilde{T}_f/\tilde{T}_{ad})^2\exp[\tilde{E}_R(\tilde{T}_{ad}^{-1} - \tilde{T}_f^{-1})]\overline{\Lambda}_R$

(7) $\Lambda_K = \varepsilon^3 Da_K \exp(-\tilde{E}_K/\tilde{T}_f)$

$= \{[q_F/(q_1 Y_{1,0})](\tilde{T}_f/\tilde{T}_{ad})^2\overline{\varepsilon}\}^3\{[(\nu_2 W_2/\nu_O W_O)Y_{1,0}]\overline{Da}_K\}[\exp(-\tilde{E}_K/\tilde{T}_{ad})\exp(\tilde{E}_K/\tilde{T}_{ad})\exp(-\tilde{E}_K/\tilde{T}_f)]$

$= [q_F/(q_1 Y_{1,0})]^3[(\nu_2 W_2/\nu_O W_O)Y_{1,0}](\tilde{T}_f/\tilde{T}_{ad})^6\exp[\tilde{E}_K(\tilde{T}_{ad}^{-1} - \tilde{T}_f^{-1})]\overline{\varepsilon}^3\,\overline{Da}_K\exp(-\tilde{E}_K/\tilde{T}_{ad})$

$= [q_F/(q_1 Y_{1,0})]^3[(\nu_2 W_2/\nu_O W_O)Y_{1,0}](\tilde{T}_f/\tilde{T}_{ad})^6\exp[\tilde{E}_K(\tilde{T}_{ad}^{-1} - \tilde{T}_f^{-1})]\overline{\Lambda}_K$

(8) $\overline{\theta}(\overline{\xi} \to -\infty) = -\alpha^{1/3}\{(\tilde{T}_{b,S,A}/g_0^-)[(g_0^-)^2 + 2\Lambda_R]^{1/2} + a_{1,1}^+[1 - \exp(-\tilde{m}/\tilde{r}_{f,S})] - (a_{1,1}^+/Le_2)[\tilde{Y}_{2,\infty}/(1+\tilde{Y}_{2,\infty})]\} - \overline{\xi}$

$-(\overline{\theta} + \overline{\xi})_{-\infty}/\alpha^{1/3} = (\tilde{T}_{b,S,A}/g_0^-)[(g_0^-)^2 + 2\Lambda_R]^{1/2} + a_{1,1}^+[1 - \exp(-\tilde{m}/\tilde{r}_{f,S})] - (a_{1,1}^+/Le_2)[\tilde{Y}_{2,\infty}/(1+\tilde{Y}_{2,\infty})]$

$(\tilde{T}_{b,S,A}/g_0^-)[(g_0^-)^2 + 2\Lambda_R]^{1/2} + a_{1,1}^+[1 - \exp(-\tilde{m}/\tilde{r}_{f,S})] = (a_{1,1}^+/Le_{2,0})[\tilde{Y}_{2,\infty}/(1+\tilde{Y}_{2,\infty})] - [(\overline{\theta} + \overline{\xi})_{-\infty}/\alpha^{1/3}]$

$a_{1,1}^+[\tilde{Y}_{2,\infty}/(1+\tilde{Y}_{2,\infty})] - Le_2\{(\tilde{T}_{b,S,A}/g_0^-)[(g_0^-)^2 + 2\Lambda_R]^{1/2} + a_{1,1}^+[1 - \exp(-\tilde{m}/\tilde{r}_{f,S})]\} = Le_2[(\overline{\theta} + \overline{\xi})_{-\infty}/\alpha^{1/3}]$

$\tilde{T}_{b,S,A} = \{(a_{1,1}^+/Le_2)[\tilde{Y}_{2,\infty}/(1+\tilde{Y}_{2,\infty})] - a_{1,1}^+[1 - \exp(-\tilde{m}/\tilde{r}_{f,S})] - [(\overline{\theta} + \overline{\xi})_{-\infty}/\alpha^{1/3}]\}g_0^-/[(g_0^-)^2 + 2\Lambda_R]^{1/2}$

(9) $a_{2,1}^-/(1+\tilde{Y}_{2,\infty}) = a_{1,1}^+\tilde{Y}_{2,\infty}/(1+\tilde{Y}_{2,\infty}) - Le_2\{(\tilde{T}_{b,S,A}/g_0^-)[(g_0^-)^2 + 2\Lambda_R]^{1/2} + a_{1,1}^+[1 - \exp(-\tilde{m}/\tilde{r}_{f,S})]\}$

$= Le_2[(\overline{\theta} + \overline{\xi})_{-\infty}/\alpha^{1/3}]$

(10) $\overline{\theta}(\overline{\xi} \to \infty) = \alpha^{1/3}(a_{1,1}^+/Le_1)[1 - \exp(-Le_1\tilde{m}/\tilde{r}_{f,S})] + \overline{\xi}$

$(\overline{\theta} - \overline{\xi})_\infty = \alpha^{1/3}(a_{1,1}^+/Le_1)[1 - \exp(-Le_1\tilde{m}/\tilde{r}_{f,S})]$

$a_{1,1}^+ = Le_1[(\overline{\theta} - \overline{\xi})_\infty/\alpha^{1/3}]/[1 - \exp(-Le_1\tilde{m}/\tilde{r}_{f,S})] = Le_1[(\overline{\theta} - \overline{\xi})_\infty/\alpha^{1/3}]/[1 - (1+\tilde{Y}_{2,\infty})^{-Le_1/Le_2}]$

(11) Leakage of the burner reactant : $\overline{Y}_{1,S}^+(\tilde{r}_{f,S}) = \overline{\varepsilon}\,\overline{Y}_{1,S,1}^+ + \cdots$

$\tilde{Y}_{1,S}^+(\tilde{r}_{f,S}) = \varepsilon\tilde{Y}_{1,S,1}^+(\tilde{r}_{f,S}) + \cdots \quad \therefore \quad \overline{Y}_{1,S}^+(\tilde{r}_{f,S})/Y_{1,0} = \{[q_F/(q_1 Y_{1,0})](\tilde{T}_f/\tilde{T}_{ad})^2\overline{\varepsilon}\}\tilde{Y}_{1,S,1}^+(\tilde{r}_{f,S}) + \cdots$

$\overline{Y}_{1,S}^+(\tilde{r}_{f,S}) = \overline{\varepsilon}(q_F/q_1)(\tilde{T}_f/\tilde{T}_{ad})^2\tilde{Y}_{1,S,1}^+(\tilde{r}_{f,S}) + \cdots = \overline{\varepsilon}(q_F/q_1)(\tilde{T}_f/\tilde{T}_{ad})^2 a_{1,1}^+[1 - (1+\tilde{Y}_{2,\infty})^{-Le_1/Le_2}] + \cdots = \overline{\varepsilon}\,\overline{Y}_{1,S,1}^+ + \cdots$

$\overline{Y}_{1,S,1}^+ = (q_F/q_1)](\tilde{T}_f/\tilde{T}_{ad})^2 a_{1,1}^+[1 - (1+\tilde{Y}_{2,\infty})^{-Le_1/Le_2}] = (q_F/q_1)](\tilde{T}_f/\tilde{T}_{ad})^2 Le_1[(\overline{\theta} - \overline{\xi})_\infty/\alpha^{1/3}]$

(12) Leakage of the ambient reactant : $\overline{Y}_{2,S}^-(\tilde{r}_{f,S}) = \overline{\varepsilon}\,\overline{Y}_{2,S,1}^- + \cdots$

$\tilde{Y}_{2,S}^-(\tilde{r}_{f,S}) = \varepsilon\tilde{Y}_{2,S,1}^-(\tilde{r}_{f,S}) + \cdots \quad \therefore \quad \left(\dfrac{\nu_1 W_1}{\nu_2 W_2}\dfrac{\nu_O W_O}{\nu_F W_F}\right)\dfrac{\overline{Y}_{2,S}^-(\tilde{r}_{f,S})}{Y_{1,0}} = \{[q_F/(q_1 Y_{1,0})](\tilde{T}_f/\tilde{T}_{ad})^2\overline{\varepsilon}\}\tilde{Y}_{2,S,1}^-(\tilde{r}_{f,S}) + \cdots$

$\overline{Y}_{2,S}^-(\tilde{r}_{f,S}) = \left(\dfrac{\nu_2 W_2}{\nu_1 W_1}\dfrac{\nu_F W_F}{\nu_O W_O}\right)\dfrac{q_F}{q_1}(\tilde{T}_f/\tilde{T}_{ad})^2\overline{\varepsilon}\,\tilde{Y}_{2,S,1}^-(\tilde{r}_{f,S}) + \cdots = \overline{\varepsilon}\left(\dfrac{\nu_2 W_2}{\nu_1 W_1}\dfrac{\nu_F W_F}{\nu_O W_O}\right)\dfrac{q_F}{q_1}(\tilde{T}_f/\tilde{T}_{ad})^2\dfrac{a_{2,1}^-}{1+\tilde{Y}_{2,\infty}} + \cdots = \overline{\varepsilon}\,\overline{Y}_{2,S,1}^- + \cdots$

$\overline{Y}_{2,S,1}^- = [\nu_2 W_2/(\nu_1 W_1)][\nu_F W_F/(\nu_O W_O)](q_F/q_1)(\tilde{T}_f/\tilde{T}_{ad})^2 Le_2[(\overline{\theta} + \overline{\xi})_{-\infty}/\alpha^{1/3}]$

(13) Burner temperature correction: $\overline{T}_{b,S} = \overline{T}_{b,S,0} + \overline{\varepsilon}\,\overline{T}_{b,S,1} + \cdots$; $\tilde{T}_{b,S,1} = \tilde{T}_{b,S,A} \exp[\tilde{m}(\tilde{r}_{f,S}^{-1} - 1)]$

$$\tilde{T}_{b,S} = \tilde{T}_{b,S,0} + \varepsilon \tilde{T}_{b,S,1} + \cdots \qquad \therefore \qquad [q_F/(q_1 Y_{1,0})]\overline{T}_{b,S} = [q_F/(q_1 Y_{1,0})]\overline{T}_{b,S,0} + [q_F/(q_1 Y_{1,0})](\tilde{T}_f/\tilde{T}_{ad})^2 \overline{\varepsilon}\,\tilde{T}_{b,S,1} + \cdots$$

$$\overline{T}_{b,S} = \overline{T}_{b,S,0} + \overline{\varepsilon}(\tilde{T}_f/\tilde{T}_{ad})^2 \tilde{T}_{b,S,1} + \cdots = \overline{T}_{b,S,0} + \overline{\varepsilon}\,\overline{T}_{b,S,1} + \cdots$$

$$\overline{T}_{b,S,1} = (\tilde{T}_f/\tilde{T}_{ad})^2 \tilde{T}_{b,S,1} = (\tilde{T}_f/\tilde{T}_{ad})^2 \tilde{T}_{b,S,A} \exp[\tilde{m}(\tilde{r}_{f,S}^{-1} - 1)]$$

(N) Determine $\tilde{T}_f$ by the Newton iteration method

Let $F(\tilde{T}_f) = [(g_0^-)^2 + 2\Lambda_R]^{1/2} + [(g_0^+)^2 + 2\Lambda_R]^{1/2} - \tilde{m}/\tilde{r}_{f,S}^2$; Want $F(\tilde{T}_f) = 0$

$$g_0^- = (\tilde{T}_f - \tilde{T}_0)(\tilde{m}/\tilde{r}_{f,S}^2), \qquad g_0^+ = (\tilde{T}_f - \tilde{T}_\infty)(\tilde{m}/\tilde{r}_{f,S}^2)/[\exp(\tilde{m}/\tilde{r}_{f,S}) - 1], \qquad \Lambda_R = (\tilde{T}_f/\tilde{T}_{ad})^2 \exp[\tilde{E}_R(\tilde{T}_{ad}^{-1} - \tilde{T}_f^{-1})]\overline{\Lambda}_R$$

$$d g_0^-/d\tilde{T}_f = \tilde{m}/\tilde{r}_{f,S}^2, \qquad d g_0^+/d\tilde{T}_f = (\tilde{m}/\tilde{r}_{f,S}^2)/[\exp(\tilde{m}/\tilde{r}_{f,S}) - 1]$$

$$d\Lambda_R/d\tilde{T}_f = [q_F/(q_1 Y_{1,0})]^2 \{2(\tilde{T}_f/\tilde{T}_{ad}^2)\exp[\tilde{E}_R(\tilde{T}_{ad}^{-1} - \tilde{T}_f^{-1})] + (\tilde{T}_f/\tilde{T}_{ad})^2(\tilde{E}_R/\tilde{T}_f^2)\exp[\tilde{E}_R(\tilde{T}_{ad}^{-1} - \tilde{T}_f^{-1})]\}\overline{\Lambda}_R$$

$$= [(2/\tilde{T}_f) + (\tilde{E}_R/\tilde{T}_f^2)][q_F/(q_1 Y_{1,0})]^2 (\tilde{T}_f/\tilde{T}_{ad})^2 \exp[\tilde{E}_R(\tilde{T}_{ad}^{-1} - \tilde{T}_f^{-1})]\overline{\Lambda}_R = \Lambda_R[2 + (\tilde{E}_R/\tilde{T}_f)]/\tilde{T}_f$$

$$\frac{dF}{d\tilde{T}_f} = \frac{1}{2}\frac{1}{[(g_0^-)^2 + 2\Lambda_R]^{1/2}}\left(2g_0^-\frac{dg_0^-}{d\tilde{T}_f} + 2\frac{d\Lambda_R}{d\tilde{T}_f}\right) + \frac{1}{2}\frac{1}{[(g_0^+)^2 + 2\Lambda_R]^{1/2}}\left(2g_0^+\frac{dg_0^+}{d\tilde{T}_f} + 2\frac{d\Lambda_R}{d\tilde{T}_f}\right)$$

$$= [(g_0^-)^2 + 2\Lambda_R]^{-1/2}[g_0^-(\tilde{m}/\tilde{r}_{f,S}^2) + (d\Lambda_R/d\tilde{T}_f)][(g_0^+)^2 + 2\Lambda_R]^{-1/2}\langle\{g_0^+(\tilde{m}/\tilde{r}_{f,S}^2)/[\exp(\tilde{m}/\tilde{r}_{f,S}) - 1]\} + (d\Lambda_R/d\tilde{T}_f)\rangle$$

Initial guess: $\tilde{T}_{f,1}$ (e.g. $\tilde{T}_{ad} - 0.01$ in the program)

Algorithm: $\tilde{T}_{f,i+1} = \tilde{T}_{f,i} - [F(\tilde{T}_{f,i})/(dF/d\tilde{T}_f)_{\tilde{T}_{f,i}}]$

Solution is found when $\tilde{T}_{f,i+1} = \tilde{T}_{f,i}$

APPENDIX C. STABILITY ANALYSIS

(A) Conservation Equations and Boundary Conditions

(a) Core region ($0<\tilde{r}<\tilde{r}_i$)

$$\tilde{Y}_1=1,\qquad \tilde{Y}_2=0,\qquad \frac{\partial\tilde{T}}{\partial\tilde{t}}+\frac{\tilde{m}}{\tilde{r}^2}\frac{\partial\tilde{T}}{\partial\tilde{r}}-\frac{1}{\tilde{r}^2}\left\{\frac{\partial}{\partial\tilde{r}}\left(\tilde{r}^2\frac{\partial\tilde{T}}{\partial\tilde{r}}\right)+\frac{\partial}{\partial\tilde{z}}\left[(1-\tilde{z}^2)\frac{\partial\tilde{T}}{\partial\tilde{z}}\right]\right\}=0$$

(b) Within the porous burner ($\tilde{r}_i<\tilde{r}<1$)

$$\tilde{Y}_1=1,\qquad \tilde{Y}_2=0,\qquad \alpha\frac{\partial\tilde{T}}{\partial\tilde{t}}+\frac{\tilde{m}}{\tilde{r}^2}\frac{\partial\tilde{T}}{\partial\tilde{r}}-\frac{\tilde{\lambda}}{\tilde{r}^2}\left\{\frac{\partial}{\partial\tilde{r}}\left(\tilde{r}^2\frac{\partial\tilde{T}}{\partial\tilde{r}}\right)+\frac{\partial}{\partial\tilde{z}}\left[(1-\tilde{z}^2)\frac{\partial\tilde{T}}{\partial\tilde{z}}\right]\right\}=0$$

(c) Gas region external of the burner ($1<\tilde{r}<\infty$)

$$\frac{\partial\tilde{T}}{\partial\tilde{t}}+\frac{\tilde{m}}{\tilde{r}^2}\frac{\partial\tilde{T}}{\partial\tilde{r}}-\frac{1}{\tilde{r}^2}\left\{\frac{\partial}{\partial\tilde{r}}\left(\tilde{r}^2\frac{\partial\tilde{T}}{\partial\tilde{r}}\right)+\frac{\partial}{\partial\tilde{z}}\left[(1-\tilde{z}^2)\frac{\partial\tilde{T}}{\partial\tilde{z}}\right]\right\}=Da_K\tilde{Y}_1\tilde{Y}_2\exp(-\tilde{E}_K/\tilde{T})-Da_R\exp(-\tilde{E}_R/\tilde{T})$$

$$\frac{\partial\tilde{Y}_1}{\partial\tilde{t}}+\frac{\tilde{m}}{\tilde{r}^2}\frac{\partial\tilde{Y}_1}{\partial\tilde{r}}-\frac{1}{Le_1\tilde{r}^2}\left\{\frac{\partial}{\partial\tilde{r}}\left(\tilde{r}^2\frac{\partial\tilde{Y}_1}{\partial\tilde{r}}\right)+\frac{\partial}{\partial\tilde{z}}\left[(1-\tilde{z}^2)\frac{\partial\tilde{Y}_1}{\partial\tilde{z}}\right]\right\}=-Da_K\tilde{Y}_1\tilde{Y}_2\exp(-\tilde{E}_K/\tilde{T})$$

$$\frac{\partial\tilde{Y}_2}{\partial\tilde{t}}+\frac{\tilde{m}}{\tilde{r}^2}\frac{\partial\tilde{Y}_2}{\partial\tilde{r}}-\frac{1}{Le_2\tilde{r}^2}\left\{\frac{\partial}{\partial\tilde{r}}\left(\tilde{r}^2\frac{\partial\tilde{Y}_2}{\partial\tilde{r}}\right)+\frac{\partial}{\partial\tilde{z}}\left[(1-\tilde{z}^2)\frac{\partial\tilde{Y}_2}{\partial\tilde{z}}\right]\right\}=-Da_K\tilde{Y}_1\tilde{Y}_2\exp(-\tilde{E}_K/\tilde{T})$$

(d) Boundary and interface conditions

$\tilde{r}=0$: $\tilde{T}=\tilde{T}_0$

$\tilde{r}=\tilde{r}_i$: $\tilde{T}=\tilde{T}_i$ (To be determined), $(\partial\tilde{T}/\partial\tilde{r})_{\tilde{r}_i^-}=\tilde{\lambda}(\partial\tilde{T}/\partial\tilde{r})_{\tilde{r}_i^+}$

$\tilde{r}=1$: $\tilde{T}=\tilde{T}_b$ (To be determined), $\tilde{\lambda}(\partial\tilde{T}/\partial\tilde{r})_{1^-}=(\partial\tilde{T}/\partial\tilde{r})_{1^+}$, $\tilde{m}\tilde{Y}_1-\dfrac{1}{Le_1}\dfrac{\partial\tilde{Y}_1}{\partial\tilde{r}}=\tilde{m}$, $\tilde{m}\tilde{Y}_2-\dfrac{1}{Le_2}\dfrac{\partial\tilde{Y}_2}{\partial\tilde{r}}=0$

$\tilde{r}\to\infty$: $\tilde{T}\to\tilde{T}_\infty$, $\tilde{Y}_1\to 0$, $\tilde{Y}_2\to\tilde{Y}_{2,\infty}$

Note : $-1\le\tilde{z}\le 1$

(B) Conservation Equations and Boundary Conditions of the Disturbance Field

(I) Introduce a periodic $O(\bar{\delta})$ small disturbance with $\varepsilon\ll\bar{\delta}\ll 1$ to the basic solutions

$$\tilde{T}(\tilde{r},\tilde{z},\tilde{t})=\tilde{T}_S(\tilde{r})+\bar{\delta}\tilde{T}'(\tilde{r},\tilde{z})e^{\tilde{\omega}\tilde{t}}+O(\bar{\delta}^2)$$

$$\tilde{Y}_1(\tilde{r},\tilde{z},\tilde{t})=\tilde{Y}_{1,S}(\tilde{r})+\bar{\delta}\tilde{Y}_1'(\tilde{r},\tilde{z})e^{\tilde{\omega}\tilde{t}}+O(\bar{\delta}^2)$$

$$\tilde{Y}_2(\tilde{r},\tilde{z},\tilde{t})=\tilde{Y}_{2,S}(\tilde{r})+\bar{\delta}\tilde{Y}_2'(\tilde{r},\tilde{z})e^{\tilde{\omega}\tilde{t}}+O(\bar{\delta}^2)$$

$$\exp(-\tilde{E}_i/\tilde{T})=\exp\{-\tilde{E}_i/[\tilde{T}_S+\bar{\delta}\tilde{T}'e^{\tilde{\omega}\tilde{t}}+O(\delta^2)]\}=\exp\{-\tilde{E}_i/\langle\tilde{T}_S[1+\bar{\delta}(\tilde{T}'e^{\tilde{\omega}\tilde{t}}/\tilde{T}_S)+O(\bar{\delta}^2)]\rangle\}$$

$$=\exp\left\{-\frac{\tilde{E}_i[1-\bar{\delta}(\tilde{T}'e^{\tilde{\omega}\tilde{t}}/\tilde{T}_S)+O(\delta^2)]}{\tilde{T}_S}\right\}=\exp\left(-\frac{\tilde{E}_i}{\tilde{T}_S}\right)\exp\left[\delta\frac{\tilde{E}_i}{\tilde{T}_S^2}\tilde{T}'e^{\tilde{\omega}\tilde{t}}+O(\delta^2)\right]$$

$$=\left[1+\delta\frac{\tilde{E}_i}{\tilde{T}_S^2}\tilde{T}'e^{\tilde{\omega}\tilde{t}}+O(\bar{\delta}^2)\right]\exp\left(-\frac{\tilde{E}_i}{\tilde{T}_S}\right)$$

(1) Core region $\dfrac{\partial\tilde{T}}{\partial\tilde{t}}+\dfrac{\tilde{m}}{\tilde{r}^2}\dfrac{\partial\tilde{T}}{\partial\tilde{r}}-\dfrac{1}{\tilde{r}^2}\left\{\dfrac{\partial}{\partial\tilde{r}}\left(\tilde{r}^2\dfrac{\partial\tilde{T}}{\partial\tilde{r}}\right)+\dfrac{\partial}{\partial\tilde{z}}\left[(1-\tilde{z}^2)\dfrac{\partial\tilde{T}}{\partial\tilde{z}}\right]\right\}=0$, $\tilde{Y}_1=1$, $\tilde{Y}_2=0$

$$\tilde{Y}_1(\tilde{r},\tilde{z},\tilde{t})=\tilde{Y}_{1,S}(\tilde{r})+\bar{\delta}\tilde{Y}_1'(\tilde{r},\tilde{z})e^{\tilde{\omega}\tilde{t}}+O(\bar{\delta}^2)=1+\bar{\delta}\tilde{Y}_1'(\tilde{r},\tilde{z})e^{\tilde{\omega}\tilde{t}}+O(\bar{\delta}^2)=1\quad\Rightarrow\quad\tilde{Y}_1'=0$$

$$\tilde{Y}_2(\tilde{r},\tilde{z},\tilde{t})=\tilde{Y}_{2,S}(\tilde{r})+\bar{\delta}\tilde{Y}_2'(\tilde{r},\tilde{z})e^{\tilde{\omega}\tilde{t}}+O(\bar{\delta}^2)=0+\bar{\delta}\tilde{Y}_O'(\tilde{r},\tilde{z})e^{\tilde{\omega}\tilde{t}}+O(\bar{\delta}^2)=0\quad\Rightarrow\quad\tilde{Y}_2'=0$$

$$\bar{\delta}\omega\tilde{T}'e^{\omega\tilde{t}}+\frac{\tilde{m}}{\tilde{r}^2}\left(\frac{d\tilde{T}_S}{d\tilde{r}}+\bar{\delta}\frac{\partial\tilde{T}'}{\partial\tilde{r}}e^{\omega\tilde{t}}\right)-\frac{1}{\tilde{r}^2}\left\{\frac{\partial}{\partial\tilde{r}}\left[\tilde{r}^2\left(\frac{d\tilde{T}_S}{d\tilde{r}}+\bar{\delta}\frac{\partial\tilde{T}'}{\partial\tilde{r}}e^{\omega\tilde{t}}\right)\right]+\bar{\delta}\frac{\partial}{\partial\tilde{z}}\left[(1-\tilde{z}^2)\frac{\partial\tilde{T}'}{\partial\tilde{z}}e^{\omega\tilde{t}}\right]\right\}+O(\bar{\delta}^2)=0$$

Since $\frac{\tilde{m}}{\tilde{r}^2}\frac{d\tilde{T}_S}{d\tilde{r}}-\frac{1}{\tilde{r}^2}\frac{d}{d\tilde{r}}\left(\tilde{r}^2\frac{d\tilde{T}_S}{d\tilde{r}}\right)=0$, we have $\tilde{\omega}\tilde{T}'+\frac{\tilde{m}}{\tilde{r}^2}\frac{\partial\tilde{T}'}{\partial\tilde{r}}-\frac{1}{\tilde{r}^2}\left\{\frac{\partial}{\partial\tilde{r}}\left(\tilde{r}^2\frac{\partial\tilde{T}'}{\partial\tilde{r}}\right)+\frac{\partial}{\partial\tilde{z}}\left[(1-\tilde{z}^2)\frac{\partial\tilde{T}'}{\partial\tilde{z}}\right]\right\}=0$

(2) Within the burner $\alpha\frac{\partial\tilde{T}}{\partial\tilde{t}}+\frac{\tilde{m}}{\tilde{r}^2}\frac{\partial\tilde{T}}{\partial\tilde{r}}-\frac{\tilde{\lambda}}{\tilde{r}^2}\left\{\frac{\partial}{\partial\tilde{r}}\left(\tilde{r}^2\frac{\partial\tilde{T}}{\partial\tilde{r}}\right)+\frac{\partial}{\partial\tilde{z}}\left[(1-\tilde{z}^2)\frac{\partial\tilde{T}}{\partial\tilde{z}}\right]\right\}=0$, $\tilde{Y}_1=1$, $\tilde{Y}_2=0$

Similar to (1) : $\tilde{Y}_1'=0$, $\tilde{Y}_2'=0$

$$\bar{\delta}\,\omega\alpha\tilde{T}'e^{\tilde{\omega}\tilde{t}}+\frac{\tilde{m}}{\tilde{r}^2}\left(\frac{d\tilde{T}_S}{d\tilde{r}}+\bar{\delta}\frac{\partial\tilde{T}'}{\partial\tilde{r}}e^{\tilde{\omega}\tilde{t}}\right)-\frac{\tilde{\lambda}}{\tilde{r}^2}\left\{\frac{\partial}{\partial\tilde{r}}\left[\tilde{r}^2\left(\frac{d\tilde{T}_S}{d\tilde{r}}+\bar{\delta}\frac{\partial\tilde{T}'}{\partial\tilde{r}}e^{\tilde{\omega}\tilde{t}}\right)\right]+\bar{\delta}\frac{\partial}{\partial\tilde{z}}\left[(1-\tilde{z}^2)\frac{\partial\tilde{T}'}{\partial\tilde{z}}e^{\tilde{\omega}\tilde{t}}\right]\right\}+O(\bar{\delta}^2)=0$$

Since $\frac{\tilde{m}}{\tilde{r}^2}\frac{d\tilde{T}_S}{d\tilde{r}}-\frac{\tilde{\lambda}}{\tilde{r}^2}\frac{d}{d\tilde{r}}\left(\tilde{r}^2\frac{d\tilde{T}_S}{d\tilde{r}}\right)=0$, we have $\omega\alpha\tilde{T}'+\frac{\tilde{m}}{\tilde{r}^2}\frac{\partial\tilde{T}'}{\partial\tilde{r}}-\frac{\tilde{\lambda}}{\tilde{r}^2}\left\{\frac{\partial}{\partial\tilde{r}}\left(\tilde{r}^2\frac{\partial\tilde{T}'}{\partial\tilde{r}}\right)+\frac{\partial}{\partial\tilde{z}}\left[(1-\tilde{z}^2)\frac{\partial\tilde{T}'}{\partial\tilde{z}}\right]\right\}=0$

(3) $\frac{\partial\tilde{T}}{\partial\tilde{t}}+\frac{\tilde{m}}{\tilde{r}^2}\frac{\partial\tilde{T}}{\partial\tilde{r}}-\frac{1}{\tilde{r}^2}\left\{\frac{\partial}{\partial\tilde{r}}\left(\tilde{r}^2\frac{\partial\tilde{T}}{\partial\tilde{r}}\right)+\frac{\partial}{\partial\tilde{z}}\left[(1-\tilde{z}^2)\frac{\partial\tilde{T}}{\partial\tilde{z}}\right]\right\}=Da_K\,\tilde{Y}_1\tilde{Y}_2\exp(-\tilde{E}_K/\tilde{T})-Da_R\exp(-\tilde{E}_R/\tilde{T})$

$$\bar{\delta}\,\omega\tilde{T}'e^{\tilde{\omega}\tilde{t}}+\frac{\tilde{m}}{\tilde{r}^2}\left(\frac{d\tilde{T}_S}{d\tilde{r}}+\bar{\delta}\frac{\partial\tilde{T}'}{\partial\tilde{r}}e^{\tilde{\omega}\tilde{t}}\right)-\frac{1}{\tilde{r}^2}\left\{\frac{\partial}{\partial\tilde{r}}\left[\tilde{r}^2\left(\frac{d\tilde{T}_S}{d\tilde{r}}+\bar{\delta}\frac{\partial\tilde{T}'}{\partial\tilde{r}}e^{\tilde{\omega}\tilde{t}}\right)\right]+\bar{\delta}\frac{\partial}{\partial\tilde{z}}\left[(1-\tilde{z}^2)\frac{\partial\tilde{T}'}{\partial\tilde{z}}e^{\tilde{\omega}\tilde{t}}\right]\right\}+O(\bar{\delta}^2)$$

$$=Da_K[\tilde{Y}_{1,S}+\bar{\delta}\tilde{Y}_1'e^{\tilde{\omega}\tilde{t}}+O(\bar{\delta}^2)][\tilde{Y}_{2,S}+\bar{\delta}\tilde{Y}_2'e^{\tilde{\omega}\tilde{t}}+O(\bar{\delta}^2)][1+\bar{\delta}(\tilde{E}_K/\tilde{T}_S^2)\tilde{T}'e^{\tilde{\omega}\tilde{t}}+O(\bar{\delta}^2)]\exp(-\tilde{E}_K/\tilde{T}_S)$$
$$-Da_R[1+\bar{\delta}(\tilde{E}_R/\tilde{T}_S^2)\tilde{T}'e^{\tilde{\omega}\tilde{t}}+O(\bar{\delta}^2)]\exp(-\tilde{E}_R/\tilde{T}_S)$$
$$=Da_K\,\tilde{Y}_{1,S}\tilde{Y}_{2,S}\exp(-\tilde{E}_K/\tilde{T}_S)[1+\bar{\delta}(\tilde{Y}_1'/\tilde{Y}_{1,S})e^{\tilde{\omega}\tilde{t}}+O(\bar{\delta}^2)][1+\bar{\delta}(\tilde{Y}_2'/\tilde{Y}_{2,S})e^{\tilde{\omega}\tilde{t}}+O(\bar{\delta}^2)]$$
$$\cdot[1+\bar{\delta}(\tilde{E}_K/\tilde{T}_S^2)\tilde{T}'e^{\tilde{\omega}\tilde{t}}+O(\bar{\delta}^2)]-Da_R[1+\bar{\delta}(\tilde{E}_R/\tilde{T}_S^2)\tilde{T}'e^{\tilde{\omega}\tilde{t}}+O(\bar{\delta}^2)]\exp(-\tilde{E}_R/\tilde{T}_S)$$
$$=Da_K\,\tilde{Y}_{1,S}\tilde{Y}_{2,S}\exp(-\tilde{E}_K/\tilde{T}_S)\{1+\bar{\delta}[(\tilde{Y}_1'/\tilde{Y}_{1,S})+(\tilde{Y}_2'/\tilde{Y}_{2,S})+(\tilde{E}_K/\tilde{T}_S^2)\tilde{T}']e^{\tilde{\omega}\tilde{t}}+O(\bar{\delta}^2)\}$$
$$-Da_R[1+\bar{\delta}(\tilde{E}_R/\tilde{T}_S^2)\tilde{T}'e^{\tilde{\omega}\tilde{t}}+O(\bar{\delta}^2)]\exp(-\tilde{E}_R/\tilde{T}_S)$$

Since $\frac{\tilde{m}}{\tilde{r}^2}\frac{d\tilde{T}_S}{d\tilde{r}}-\frac{1}{\tilde{r}^2}\frac{d}{d\tilde{r}}\left(\tilde{r}^2\frac{d\tilde{T}_S}{d\tilde{r}}\right)=Da_K\,\tilde{Y}_{1,S}\tilde{Y}_{2,S}\exp(-\tilde{E}_K/\tilde{T}_S)-Da_R\exp(-\tilde{E}_R/\tilde{T}_S)$, we have

$$\omega\tilde{T}'+\frac{\tilde{m}}{\tilde{r}^2}\frac{\partial\tilde{T}'}{\partial\tilde{r}}-\frac{1}{\tilde{r}^2}\left\{\frac{\partial}{\partial\tilde{r}}\left(\tilde{r}^2\frac{\partial\tilde{T}'}{\partial\tilde{r}}\right)+\frac{\partial}{\partial\tilde{z}}\left[(1-\tilde{z}^2)\frac{\partial\tilde{T}'}{\partial\tilde{z}}\right]\right\}=Da_K\tilde{Y}_{1,S}\tilde{Y}_{2,S}\exp\left(-\frac{\tilde{E}_K}{\tilde{T}_S}\right)\left(\frac{\tilde{Y}_1'}{\tilde{Y}_{1,S}}+\frac{\tilde{Y}_2'}{\tilde{Y}_{2,S}}+\frac{\tilde{E}_K}{\tilde{T}_S^2}\tilde{T}'\right)-Da_R\exp\left(-\frac{\tilde{E}_R}{\tilde{T}_S}\right)\frac{\tilde{E}_R}{\tilde{T}_S^2}\tilde{T}'$$

(4) $\frac{\partial\tilde{Y}_1}{\partial\tilde{t}}+\frac{\tilde{m}}{\tilde{r}^2}\frac{\partial\tilde{Y}_1}{\partial\tilde{r}}-\frac{1}{Le_1\tilde{r}^2}\left\{\frac{\partial}{\partial\tilde{r}}\left(\tilde{r}^2\frac{\partial\tilde{Y}_1}{\partial\tilde{r}}\right)+\frac{\partial}{\partial\tilde{z}}\left[(1-\tilde{z}^2)\frac{\partial\tilde{Y}_1}{\partial\tilde{z}}\right]\right\}=-Da_K\,\tilde{Y}_1\tilde{Y}_2\exp(-\tilde{E}_K/\tilde{T})$

$$\bar{\delta}\,\omega\tilde{Y}_1'e^{\omega\tilde{t}}+\frac{\tilde{m}}{\tilde{r}^2}\left(\frac{d\tilde{Y}_{1,S}}{d\tilde{r}}+\bar{\delta}\frac{\partial\tilde{Y}_1'}{\partial\tilde{r}}e^{\omega\tilde{t}}\right)-\frac{1}{Le_1\tilde{r}^2}\left\{\frac{\partial}{\partial\tilde{r}}\left[\tilde{r}^2\left(\frac{d\tilde{Y}_{1,S}}{d\tilde{r}}+\bar{\delta}\frac{\partial\tilde{Y}_1'}{\partial\tilde{r}}e^{\omega\tilde{t}}\right)\right]+\bar{\delta}\frac{\partial}{\partial\tilde{z}}\left[(1-\tilde{z}^2)\frac{\partial\tilde{Y}_1'}{\partial\tilde{z}}e^{\omega\tilde{t}}\right]\right\}+O(\bar{\delta}^2)$$
$$=Da_K\,\tilde{Y}_{1,S}\tilde{Y}_{2,S}\exp(-\tilde{E}_K/\tilde{T}_S)\{1+\bar{\delta}[(\tilde{Y}_1'/\tilde{Y}_{1,S})+(\tilde{Y}_2'/\tilde{Y}_{2,S})+(\tilde{E}_K/\tilde{T}_S^2)\tilde{T}']e^{\omega\tilde{t}}+O(\bar{\delta}^2)\}$$

Since $\frac{\tilde{m}}{\tilde{r}^2}\frac{d\tilde{Y}_{1,S}}{d\tilde{r}}-\frac{1}{Le_1\tilde{r}^2}\frac{d}{d\tilde{r}}\left(\tilde{r}^2\frac{d\tilde{Y}_{1,S}}{d\tilde{r}}\right)=-Da_K\,\tilde{Y}_{1,S}\tilde{Y}_{2,S}\exp(-\tilde{E}_K/\tilde{T}_S)$, we have

$$\omega\tilde{Y}_1'+\frac{\tilde{m}}{\tilde{r}^2}\frac{\partial\tilde{Y}_1'}{\partial\tilde{r}}-\frac{1}{Le_1\tilde{r}^2}\left\{\frac{\partial}{\partial\tilde{r}}\left(\tilde{r}^2\frac{\partial\tilde{Y}_1'}{\partial\tilde{r}}\right)+\frac{\partial}{\partial\tilde{z}}\left[(1-\tilde{z}^2)\frac{\partial\tilde{Y}_1'}{\partial\tilde{z}}\right]\right\}=-Da_K\,\tilde{Y}_{1,S}\tilde{Y}_{2,S}\exp\left(-\frac{\tilde{E}_K}{\tilde{T}_S}\right)\left(\frac{\tilde{Y}_1'}{\tilde{Y}_{1,S}}+\frac{\tilde{Y}_2'}{\tilde{Y}_{2,S}}+\frac{\tilde{E}_K}{\tilde{T}_S^2}\tilde{T}'\right)$$

(5) $\frac{\partial\tilde{Y}_2}{\partial\tilde{t}}+\frac{\tilde{m}}{\tilde{r}^2}\frac{\partial\tilde{Y}_2}{\partial\tilde{r}}-\frac{1}{Le_2\tilde{r}^2}\left\{\frac{\partial}{\partial\tilde{r}}\left(\tilde{r}^2\frac{\partial\tilde{Y}_2}{\partial\tilde{r}}\right)+\frac{\partial}{\partial\tilde{z}}\left[(1-\tilde{z}^2)\frac{\partial\tilde{Y}_2}{\partial\tilde{z}}\right]\right\}=-Da_K\,\tilde{Y}_1\tilde{Y}_2\exp(-\tilde{E}_K/\tilde{T})$

Similar to (4): $\omega\tilde{Y}_2'+\frac{\tilde{m}}{\tilde{r}^2}\frac{\partial\tilde{Y}_2'}{\partial\tilde{r}}-\frac{1}{Le_2\tilde{r}^2}\left\{\frac{\partial}{\partial\tilde{r}}\left(\tilde{r}^2\frac{\partial\tilde{Y}_2'}{\partial\tilde{r}}\right)+\frac{\partial}{\partial\tilde{z}}\left[(1-\tilde{z}^2)\frac{\partial\tilde{Y}_2'}{\partial\tilde{z}}\right]\right\}=-Da_K\,\tilde{Y}_{1,S}\tilde{Y}_{2,S}\exp\left(-\frac{\tilde{E}_K}{\tilde{T}_S}\right)\left(\frac{\tilde{Y}_1'}{\tilde{Y}_{1,S}}+\frac{\tilde{Y}_2'}{\tilde{Y}_{2,S}}+\frac{\tilde{E}_K}{\tilde{T}_S^2}\tilde{T}'\right)$

(6) $\tilde{r}=0$: $\tilde{T}=\tilde{T}_0 \quad \Rightarrow \quad \tilde{T}_S+\bar{\delta}\,\tilde{T}' e^{\tilde{\omega}\tilde{t}}+O(\bar{\delta}^2)=\tilde{T}_0+\bar{\delta}\,\tilde{T}'e^{\tilde{\omega}\tilde{t}}+O(\bar{\delta}^2)=\tilde{T}_0$ or $\tilde{T}'=0$

(7) $\tilde{r}=\tilde{r}_i$: $\tilde{T}=\tilde{T}_i$, $(\partial\tilde{T}/\partial\tilde{r})_{\tilde{r}_i^-}=\tilde{\lambda}(\partial\tilde{T}/\partial\tilde{r})_{\tilde{r}_i^+} \quad \Rightarrow \quad \tilde{T}'=\tilde{T}_i'$, $(\partial\tilde{T}'/\partial\tilde{r})_{\tilde{r}_i^-}=\tilde{\lambda}(\partial\tilde{T}'/\partial\tilde{r})_{\tilde{r}_i^+}$

(8) $\tilde{r}=1$: $\tilde{T}=\tilde{T}_b$, $\tilde{\lambda}(\partial\tilde{T}/\partial\tilde{r})_{1^-}=(\partial\tilde{T}/\partial\tilde{r})_{1^+}$, $\tilde{m}\tilde{Y}_1-\dfrac{1}{Le_1}\dfrac{\partial\tilde{Y}_1}{\partial\tilde{r}}=\tilde{m}$, $\tilde{m}\tilde{Y}_2-\dfrac{1}{Le_2}\dfrac{\partial\tilde{Y}_2}{\partial\tilde{r}}=0$, $\tilde{T}'=\tilde{T}_b'$,

$\tilde{\lambda}(\partial\tilde{T}'/\partial\tilde{r})_{1^-}=(\partial\tilde{T}'/\partial\tilde{r})_{1^+}$

$$\tilde{m}(\tilde{Y}_{1,S}+\bar{\delta}\,\tilde{Y}_1'e^{\tilde{\omega}\tilde{t}})-\frac{1}{Le_1}\frac{\partial(\tilde{Y}_{1,S}+\bar{\delta}\,\tilde{Y}_1'\,e^{\tilde{\omega}\tilde{t}})}{\partial\tilde{r}}+O(\bar{\delta}^2)=\left(\tilde{m}\tilde{Y}_{1,S}-\frac{1}{Le_1}\frac{d\tilde{Y}_{1,S}}{d\tilde{r}}\right)+\bar{\delta}\left(\tilde{m}\tilde{Y}_1'-\frac{1}{Le_1}\frac{\partial\tilde{Y}_1'}{\partial\tilde{r}}\right)e^{\tilde{\omega}\tilde{t}}+O(\bar{\delta}^2)$$

$$=\tilde{m}+\bar{\delta}\left(\tilde{m}\tilde{Y}_1'-\frac{1}{Le_1}\frac{\partial\tilde{Y}_1'}{\partial\tilde{r}}\right)e^{\tilde{\omega}\tilde{t}}+O(\bar{\delta}^2)=\tilde{m}$$

$\therefore \quad \tilde{m}\tilde{Y}_1'-\dfrac{1}{Le_1}\dfrac{\partial\tilde{Y}_1'}{\partial\tilde{r}}=0$ similarly: $\tilde{m}\tilde{Y}_2'-\dfrac{1}{Le_2}\dfrac{\partial\tilde{Y}_2'}{\partial\tilde{r}}=0$

(9) $\tilde{r}\to\infty$: $\tilde{T}\to\tilde{T}_\infty$, $\tilde{Y}_1\to 0$, $\tilde{Y}_2\to\tilde{Y}_{2,\infty} \quad \Rightarrow \quad \tilde{T}'\to 0$, $\tilde{Y}_1'\to 0$, $\tilde{Y}_2'\to 0$

(10) Summary

Core region ($0<\tilde{r}<\tilde{r}_i$)

$$\tilde{\omega}\tilde{T}'+\frac{\tilde{m}}{\tilde{r}^2}\frac{\partial\tilde{T}'}{\partial\tilde{r}}-\frac{1}{\tilde{r}^2}\left\{\frac{\partial}{\partial\tilde{r}}\left(\tilde{r}^2\frac{\partial\tilde{T}'}{\partial\tilde{r}}\right)+\frac{\partial}{\partial\tilde{z}}\left[(1-\tilde{z}^2)\frac{\partial\tilde{T}'}{\partial\tilde{z}}\right]\right\}=0,\quad \tilde{Y}_1'=0,\quad \tilde{Y}_2'=0$$

Within the porous burner ($\tilde{r}_i<\tilde{r}<1$)

$$\tilde{\omega}\,\alpha\tilde{T}'+\frac{\tilde{m}}{\tilde{r}^2}\frac{\partial\tilde{T}'}{\partial\tilde{r}}-\frac{\tilde{\lambda}}{\tilde{r}^2}\left\{\frac{\partial}{\partial\tilde{r}}\left(\tilde{r}^2\frac{\partial\tilde{T}'}{\partial\tilde{r}}\right)+\frac{\partial}{\partial\tilde{z}}\left[(1-\tilde{z}^2)\frac{\partial\tilde{T}'}{\partial\tilde{z}}\right]\right\}=0,\quad \tilde{Y}_1'=0,\quad \tilde{Y}_2'=0$$

Gas region external of the burner ($1<\tilde{r}<\infty$)

$$\tilde{\omega}\,\tilde{T}'+\frac{\tilde{m}}{\tilde{r}^2}\frac{\partial\tilde{T}'}{\partial\tilde{r}}-\frac{1}{\tilde{r}^2}\left\{\frac{\partial}{\partial\tilde{r}}\left(\tilde{r}^2\frac{\partial\tilde{T}'}{\partial\tilde{r}}\right)+\frac{\partial}{\partial\tilde{z}}\left[(1-\tilde{z}^2)\frac{\partial\tilde{T}'}{\partial\tilde{z}}\right]\right\}=Da_K\,\tilde{Y}_{1,S}\,\tilde{Y}_{2,S}\exp\left(-\frac{\tilde{E}_K}{\tilde{T}_S}\right)\left(\frac{\tilde{Y}_1'}{\tilde{Y}_{1,S}}+\frac{\tilde{Y}_2'}{\tilde{Y}_{2,S}}+\frac{\tilde{E}_K}{\tilde{T}_S^2}\tilde{T}'\right)-Da_R\exp\left(-\frac{\tilde{E}_R}{\tilde{T}_S}\right)\frac{\tilde{E}_R}{\tilde{T}_S^2}\tilde{T}'$$

$$\tilde{\omega}\,\tilde{Y}_1'+\frac{\tilde{m}}{\tilde{r}^2}\frac{\partial\tilde{Y}_1'}{\partial\tilde{r}}-\frac{1}{Le_1\tilde{r}^2}\left\{\frac{\partial}{\partial\tilde{r}}\left(\tilde{r}^2\frac{\partial\tilde{Y}_1'}{\partial\tilde{r}}\right)+\frac{\partial}{\partial\tilde{z}}\left[(1-\tilde{z}^2)\frac{\partial\tilde{Y}_1'}{\partial\tilde{z}}\right]\right\}=-Da_K\,\tilde{Y}_{1,S}\,\tilde{Y}_{2,S}\exp\left(-\frac{\tilde{E}_K}{\tilde{T}_S}\right)\left(\frac{\tilde{Y}_1'}{\tilde{Y}_{1,S}}+\frac{\tilde{Y}_2'}{\tilde{Y}_{2,S}}+\frac{\tilde{E}_K}{\tilde{T}_S^2}\tilde{T}'\right)$$

$$\tilde{\omega}\,\tilde{Y}_2'+\frac{\tilde{m}}{\tilde{r}^2}\frac{\partial\tilde{Y}_2'}{\partial\tilde{r}}-\frac{1}{Le_2\tilde{r}^2}\left\{\frac{\partial}{\partial\tilde{r}}\left(\tilde{r}^2\frac{\partial\tilde{Y}_2'}{\partial\tilde{r}}\right)+\frac{\partial}{\partial\tilde{z}}\left[(1-\tilde{z}^2)\frac{\partial\tilde{Y}_2'}{\partial\tilde{z}}\right]\right\}=-Da_K\,\tilde{Y}_{1,S}\,\tilde{Y}_{2,S}\exp\left(-\frac{\tilde{E}_K}{\tilde{T}_S}\right)\left(\frac{\tilde{Y}_1'}{\tilde{Y}_{1,S}}+\frac{\tilde{Y}_2'}{\tilde{Y}_{2,S}}+\frac{\tilde{E}_K}{\tilde{T}_S^2}\tilde{T}'\right)$$

Boundary and interface conditions

$\tilde{r}=0$: $\tilde{T}'=0$

$\tilde{r}=\tilde{r}_i$: $\tilde{T}'=\tilde{T}_i'$, $(\partial\tilde{T}'/\partial\tilde{r})_{\tilde{r}_i^-}=\tilde{\lambda}(\partial\tilde{T}'/\partial\tilde{r})_{\tilde{r}_i^+}$

$\tilde{r}=1$: $\tilde{T}'=\tilde{T}_b'$, $\tilde{\lambda}(\partial\tilde{T}'/\partial\tilde{r})_{1^-}=(\partial\tilde{T}'/\partial\tilde{r})_{1^+}$, $\tilde{m}\tilde{Y}_1'-\dfrac{1}{Le_1}\dfrac{\partial\tilde{Y}_1'}{\partial\tilde{r}}=0$, $\tilde{m}\tilde{Y}_2'-\dfrac{1}{Le_2}\dfrac{\partial\tilde{Y}_2'}{\partial\tilde{r}}=0$

$\tilde{r}\to\infty$: $\tilde{T}'\to 0$, $\tilde{Y}_1'\to 0$, $\tilde{Y}_2'\to 0$

(C) Determine the function of the disturbance in the transverse (θ or $\tilde{z}$) direction

Use the $\tilde{Y}_1'$ equation (without reaction) as the indication

$$\tilde{\omega}\,\tilde{Y}_1'+\frac{\tilde{m}}{\tilde{r}^2}\frac{\partial\tilde{Y}_1'}{\partial\tilde{r}}-\frac{1}{Le_1\tilde{r}^2}\left\{\frac{\partial}{\partial\tilde{r}}\left(\tilde{r}^2\frac{\partial\tilde{Y}_1'}{\partial\tilde{r}}\right)+\frac{\partial}{\partial\tilde{z}}\left[(1-\tilde{z}^2)\frac{\partial\tilde{Y}_1'}{\partial\tilde{z}}\right]\right\}=0$$

or $\quad \tilde{\omega}\, Le_1\, \tilde{r}^2\, \tilde{Y}_1' + \tilde{m}\, Le_1 \dfrac{\partial \tilde{Y}_1'}{\partial \tilde{r}} - \dfrac{\partial}{\partial \tilde{r}}\left(\tilde{r}^2 \dfrac{\partial \tilde{Y}_1'}{\partial \tilde{r}}\right) - \dfrac{\partial}{\partial \tilde{z}}\left[(1-\tilde{z}^2)\dfrac{\partial \tilde{Y}_1'}{\partial \tilde{z}}\right] = 0$

Solve by separation of variables: let $\tilde{Y}_1'(\tilde{r},\tilde{z}) = \hat{Y}_1(\tilde{r})A(\tilde{z})$ then

$$\tilde{\omega} Le_1 \tilde{r}^2 \hat{Y}_1 A + \tilde{m} Le_1 A \frac{d\hat{Y}_1}{d\tilde{r}} - A\frac{d}{d\tilde{r}}\left(\tilde{r}^2 \frac{d\hat{Y}_1}{d\tilde{r}}\right) - \hat{Y}_1 \frac{d}{d\tilde{z}}\left[(1-\tilde{z}^2)\frac{dA}{d\tilde{z}}\right] = 0$$

Divide the equation by $\hat{Y}_1 A$ and re-arrange: $\tilde{\omega} Le_1 \tilde{r}^2 + \tilde{m} Le_1 \dfrac{1}{\hat{Y}_1}\dfrac{d\hat{Y}_1}{d\tilde{r}} - \dfrac{1}{\hat{Y}_1}\dfrac{d}{d\tilde{r}}\left(\tilde{r}^2 \dfrac{d\hat{Y}_1}{d\tilde{r}}\right) = \dfrac{1}{A}\dfrac{d}{d\tilde{z}}\left[(1-\tilde{z}^2)\dfrac{dA}{d\tilde{z}}\right]$

Since the LHS is a function of $\tilde{r}$ only and the RHS is a function of $\tilde{z}$ only, the only possibility to satisfy the equation is that they both equal to the same constant. Let the constant be $-c$, then

$$\tilde{\omega} Le_1 \tilde{r}^2 + \tilde{m} Le_1 \frac{1}{\hat{Y}_1}\frac{d\hat{Y}_1}{d\tilde{r}} - \frac{1}{\hat{Y}_1}\frac{d}{d\tilde{r}}\left(\tilde{r}^2 \frac{d\hat{Y}_1}{d\tilde{r}}\right) = \frac{1}{A}\frac{d}{d\tilde{z}}\left[(1-\tilde{z}^2)\frac{dA}{d\tilde{z}}\right] = -c$$

The equation that determines A is $\dfrac{1}{A}\dfrac{d}{d\tilde{z}}\left[(1-\tilde{z}^2)\dfrac{dA}{d\tilde{z}}\right] = -c$ or $\dfrac{d}{d\tilde{z}}\left[(1-\tilde{z}^2)\dfrac{dA}{d\tilde{z}}\right] + cA = 0 \quad , \quad -1 \le \tilde{z} \le 1$

This is the Legendre's equation. To have a non-trivial solution bounded on $-1 \le \tilde{z} \le 1$, $c = n(n+1)$ where n is any nonnegative integer. The solution of this equation is then $A = P_n(\tilde{z})$ where P_n is the n^{th} Legendre polynomial. Based on this result, we can further split the disturbance variables to

$$\tilde{T}'(\tilde{r},\tilde{z}) = \hat{T}(\tilde{r})P_n(\tilde{z}); \quad \tilde{Y}_1'(\tilde{r},\tilde{z}) = \hat{Y}_1(\tilde{r})P_n(\tilde{z}), \quad \tilde{Y}_2'(\tilde{r},\tilde{z}) = \hat{Y}_2(\tilde{r})P_n(\tilde{z})$$

Perturbed flame location: $\tilde{r}_f(\tilde{z},\tilde{t}) = \tilde{r}_{f,S} + \overline{\delta}\,\hat{r}_f\, P_n(\tilde{z})e^{\tilde{\omega}\tilde{t}}$; $\quad \hat{r}_f = \hat{r}_{f,0} + \varepsilon \hat{r}_{f,1} + O(\varepsilon^2)$

Define $\tilde{r}_{f,0}(\tilde{z},\tilde{t}) = \tilde{r}_{f,S} + \overline{\delta}\,\hat{r}_{f,0}\,P_n(\tilde{z})e^{\tilde{\omega}\tilde{t}}$ then $\tilde{r}_f(\tilde{z},\tilde{t}) = \tilde{r}_{f,0} + \overline{\delta}\,\varepsilon\,\hat{r}_{f,1}\,P_n(\tilde{z})e^{\tilde{\omega}\tilde{t}}$

(1) Core region ($0 < \tilde{r} < \tilde{r}_i$): $\tilde{Y}_1' = 0$, $\tilde{Y}_2' = 0$ $\Rightarrow$ $\hat{Y}_1 = 0$, $\hat{Y}_2 = 0$

$$\tilde{\omega}\tilde{T}' + \frac{\tilde{m}}{\tilde{r}^2}\frac{\partial \tilde{T}'}{\partial \tilde{r}} - \frac{1}{\tilde{r}^2}\left\{\frac{\partial}{\partial \tilde{r}}\left(\tilde{r}^2 \frac{\partial \tilde{T}'}{\partial \tilde{r}}\right) + \frac{\partial}{\partial \tilde{z}}\left[(1-\tilde{z}^2)\frac{\partial \tilde{T}'}{\partial \tilde{z}}\right]\right\} = 0 \quad \therefore$$

$$\tilde{\omega}\hat{T}P_n + \frac{\tilde{m}}{\tilde{r}^2}\frac{d\hat{T}}{d\tilde{r}}P_n - \frac{1}{\tilde{r}^2}\left\{\frac{d}{d\tilde{r}}\left(\tilde{r}^2 \frac{d\hat{T}}{d\tilde{r}}\right)P_n + \hat{T}\frac{d}{d\tilde{z}}\left[(1-\tilde{z}^2)\frac{dP_n}{d\tilde{z}}\right]\right\} = 0$$

Since $\dfrac{d}{d\tilde{z}}\left[(1-\tilde{z}^2)\dfrac{dP_n}{d\tilde{z}}\right] + n(n+1)P_n = 0$: $\quad \tilde{\omega}\hat{T}P_n + \dfrac{\tilde{m}}{\tilde{r}^2}\dfrac{d\hat{T}}{d\tilde{r}}P_n - \dfrac{1}{\tilde{r}^2}\left[\dfrac{d}{d\tilde{r}}\left(\tilde{r}^2\dfrac{d\hat{T}}{d\tilde{r}}\right)P_n - n(n+1)P_n\hat{T}\right] = 0$

$$\frac{d}{d\tilde{r}}\left(\tilde{r}^2\frac{d\hat{T}}{d\tilde{r}}\right) - \tilde{m}\frac{d\hat{T}}{d\tilde{r}} - [\tilde{\omega}\tilde{r}^2 + n(n+1)]\hat{T} = 0 \quad \text{or} \quad \tilde{r}^2\frac{d^2\hat{T}}{d\tilde{r}^2} + (2\tilde{r} - \tilde{m})\frac{d\hat{T}}{d\tilde{r}} - [\tilde{\omega}\tilde{r}^2 + n(n+1)]\hat{T} = 0$$

For this equation, $\tilde{r} = 0$ is an irregular singular point.

(2) Within the porous burner ($\tilde{r}_i < \tilde{r} < 1$): $\tilde{Y}_1' = 0$, $\tilde{Y}_2' = 0$ $\Rightarrow$ $\hat{Y}_1 = 0$, $\hat{Y}_2 = 0$

$$\tilde{\omega}\alpha\tilde{T}' + \frac{\tilde{m}}{\tilde{r}^2}\frac{\partial \tilde{T}'}{\partial \tilde{r}} - \frac{\tilde{\lambda}}{\tilde{r}^2}\left\{\frac{\partial}{\partial \tilde{r}}\left(\tilde{r}^2\frac{\partial \tilde{T}'}{\partial \tilde{r}}\right) + \frac{\partial}{\partial \tilde{z}}\left[(1-\tilde{z}^2)\frac{\partial \tilde{T}'}{\partial \tilde{z}}\right]\right\} = 0 \quad \text{or}$$

$$\tilde{\omega}\frac{\alpha}{\tilde{\lambda}}\tilde{T}' + \frac{\tilde{m}}{\tilde{\lambda}\tilde{r}^2}\frac{\partial \tilde{T}'}{\partial \tilde{r}} - \frac{1}{\tilde{r}^2}\left\{\frac{\partial}{\partial \tilde{r}}\left(\tilde{r}^2\frac{\partial \tilde{T}'}{\partial \tilde{r}}\right) + \frac{\partial}{\partial \tilde{z}}\left[(1-\tilde{z}^2)\frac{\partial \tilde{T}'}{\partial \tilde{z}}\right]\right\} = 0$$

Comparing the equation with that of (1), we have

$$\frac{d}{d\tilde{r}}\left(\tilde{r}^2\frac{d\hat{T}}{d\tilde{r}}\right) - \frac{\tilde{m}}{\tilde{\lambda}}\frac{d\hat{T}}{d\tilde{r}} - \left[\tilde{\omega}\frac{\alpha}{\tilde{\lambda}}\tilde{r}^2 + n(n+1)\right]\hat{T} = 0 \quad \text{or} \quad \tilde{r}^2\frac{d^2\hat{T}}{d\tilde{r}^2} + \left(2\tilde{r} - \frac{\tilde{m}}{\tilde{\lambda}}\right)\frac{d\hat{T}}{d\tilde{r}} - \left[\tilde{\omega}\frac{\alpha}{\tilde{\lambda}}\tilde{r}^2 + n(n+1)\right]\hat{T} = 0$$

(3) $\tilde{\omega}\tilde{T}' + \dfrac{\tilde{m}}{\tilde{r}^2}\dfrac{\partial \tilde{T}'}{\partial \tilde{r}} - \dfrac{1}{\tilde{r}^2}\left\{\dfrac{\partial}{\partial \tilde{r}}\left(\tilde{r}^2\dfrac{\partial \tilde{T}'}{\partial \tilde{r}}\right) + \dfrac{\partial}{\partial \tilde{z}}\left[(1-\tilde{z}^2)\dfrac{\partial \tilde{T}'}{\partial \tilde{z}}\right]\right\} = Da_K\,\tilde{Y}_{1,S}\,\tilde{Y}_{2,S}\exp\left(-\dfrac{\tilde{E}_K}{\tilde{T}_S}\right)\left(\dfrac{\tilde{Y}_1'}{\tilde{Y}_{1,S}} + \dfrac{\tilde{Y}_2'}{\tilde{Y}_{2,S}} + \dfrac{\tilde{E}_K}{\tilde{T}_S^2}\tilde{T}'\right) - Da_R\exp\left(-\dfrac{\tilde{E}_R}{\tilde{T}_S}\right)\dfrac{\tilde{E}_R}{\tilde{T}_S^2}\tilde{T}'$

The LHS of this equation is the same as that of (1).

$$\left\{\tilde\omega\,\hat T+\frac{\tilde m}{\tilde r^2}\frac{d\hat T}{d\tilde r}-\frac{1}{\tilde r^2}\left[\frac{d}{d\tilde r}\left(\tilde r^2\frac{d\hat T}{d\tilde r}\right)-n(n+1)\hat T\right]\right\}P_n$$

$$=Da_K\,\tilde Y_{1,S}\tilde Y_{2,S}\exp\left(-\frac{\tilde E_K}{\tilde T_S}\right)\left(\frac{\hat Y_1}{\tilde Y_{1,S}}+\frac{\hat Y_2}{\tilde Y_{2,S}}+\frac{\tilde E_K}{\tilde T_S^2}\hat T\right)P_n-Da_R\exp\left(-\frac{\tilde E_R}{\tilde T_S}\right)\frac{\tilde E_R}{\tilde T_S^2}\hat T\,P_n$$

$$\frac{d}{d\tilde r}\left(\tilde r^2\frac{d\hat T}{d\tilde r}\right)-\tilde m\frac{d\hat T}{d\tilde r}-[\tilde\omega\tilde r^2+n(n+1)]\hat T=\tilde r^2\left[Da_R\exp\left(-\frac{\tilde E_R}{\tilde T_S}\right)\frac{\tilde E_R}{\tilde T_S^2}\hat T-Da_K\tilde Y_{1,S}\tilde Y_{2,S}\exp\left(-\frac{\tilde E_K}{\tilde T_S}\right)\left(\frac{\hat Y_1}{\tilde Y_{1,S}}+\frac{\hat Y_2}{\tilde Y_{2,S}}+\frac{\tilde E_K}{\tilde T_S^2}\hat T\right)\right]$$

(4) $$\tilde\omega\,\tilde Y_1'+\frac{\tilde m}{\tilde r^2}\frac{\partial\tilde Y_1'}{\partial\tilde r}-\frac{1}{Le_1\tilde r^2}\left\{\frac{\partial}{\partial\tilde r}\left(\tilde r^2\frac{\partial\tilde Y_1'}{\partial\tilde r}\right)+\frac{\partial}{\partial\tilde z}\left[(1-\tilde z^2)\frac{\partial\tilde Y_1'}{\partial\tilde z}\right]\right\}=-Da_K\,\tilde Y_{1,S}\tilde Y_{2,S}\exp\left(-\frac{\tilde E_K}{\tilde T_S}\right)\left(\frac{\tilde Y_1'}{\tilde Y_{1,S}}+\frac{\tilde Y_2'}{\tilde Y_{2,S}}+\frac{\tilde E_K}{\tilde T_S^2}\tilde T'\right)$$

$$\tilde\omega\,Le_1\tilde r^2\hat Y_1P_n+Le_1\tilde m\frac{d\hat Y_1}{d\tilde r}P_n-\left\{P_n\frac{d}{d\tilde r}\left(\tilde r^2\frac{d\hat Y_1}{d\tilde r}\right)+\hat Y_1\frac{d}{d\tilde z}\left[(1-\tilde z^2)\frac{dP_n}{d\tilde z}\right]\right\}$$

$$=-Le_1\tilde r^2\,Da_K\,\tilde Y_{1,S}\tilde Y_{2,S}\exp\left(-\frac{\tilde E_K}{\tilde T_S}\right)\left(\frac{\hat Y_1}{\tilde Y_{1,S}}+\frac{\hat Y_2}{\tilde Y_{2,S}}+\frac{\tilde E_K}{\tilde T_S^2}\hat T\right)P_n$$

$$\left[P_n\frac{d}{d\tilde r}\left(\tilde r^2\frac{d\hat Y_1}{d\tilde r}\right)-n(n+1)P_n\hat Y_1\right]-Le_1\tilde m\frac{d\hat Y_1}{d\tilde r}P_n-\tilde\omega\,Le_1\tilde r^2\hat Y_1P_n=Le_1\tilde r^2Da_K\tilde Y_{1,S}\tilde Y_{2,S}\exp\left(-\frac{\tilde E_K}{\tilde T_S}\right)\left(\frac{\hat Y_1}{\tilde Y_{1,S}}+\frac{\hat Y_2}{\tilde Y_{2,S}}+\frac{\tilde E_K}{\tilde T_S^2}\hat T\right)P_n$$

$$\frac{d}{d\tilde r}\left(\tilde r^2\frac{d\hat Y_1}{d\tilde r}\right)-Le_1\tilde m\frac{d\hat Y_1}{d\tilde r}-[\tilde\omega\,Le_1\tilde r^2+n(n+1)]\hat Y_1=Le_1\tilde r^2\,Da_K\,\tilde Y_{1,S}\tilde Y_{2,S}\exp\left(-\frac{\tilde E_K}{\tilde T_S}\right)\left(\frac{\hat Y_1}{\tilde Y_{1,S}}+\frac{\hat Y_2}{\tilde Y_{2,S}}+\frac{\tilde E_K}{\tilde T_S^2}\hat T\right)$$

(5) $$\tilde\omega\,\tilde Y_2'+\frac{\tilde m}{\tilde r^2}\frac{\partial\tilde Y_2'}{\partial\tilde r}-\frac{1}{Le_2\tilde r^2}\left\{\frac{\partial}{\partial\tilde r}\left(\tilde r^2\frac{\partial\tilde Y_2'}{\partial\tilde r}\right)+\frac{\partial}{\partial\tilde z}\left[(1-\tilde z^2)\frac{\partial\tilde Y_2'}{\partial\tilde z}\right]\right\}=-Da_K\,\tilde Y_{1,S}\tilde Y_{2,S}\exp\left(-\frac{\tilde E_K}{\tilde T_S}\right)\left(\frac{\tilde Y_1'}{\tilde Y_{1,S}}+\frac{\tilde Y_2'}{\tilde Y_{2,S}}+\frac{\tilde E_K}{\tilde T_S^2}\tilde T'\right)$$

Similar to (4): $$\frac{d}{d\tilde r}\left(\tilde r^2\frac{d\hat Y_2}{d\tilde r}\right)-Le_2\tilde m\frac{d\hat Y_2}{d\tilde r}-[\omega Le_2\tilde r^2+n(n+1)]\hat Y_2=Le_2\tilde r^2\,Da_K\,\tilde Y_{1,S}\tilde Y_{2,S}\exp\left(-\frac{\tilde E_K}{\tilde T_S}\right)\left(\frac{\hat Y_1}{\tilde Y_{1,S}}+\frac{\hat Y_2}{\tilde Y_{2,S}}+\frac{\tilde E_K}{\tilde T_S^2}\hat T\right)$$

(6) $\tilde r=0$: $\quad\tilde T'=0\quad\Rightarrow\quad\hat T(\tilde r=0)P_n(\tilde z)=0$ or $\hat T(\tilde r=0)=0$ (P_n is independent of $\tilde r$)

(7) $\tilde r=\tilde r_i$: $\quad\tilde T'=\tilde T_i'$, $\quad(\partial\tilde T'/\partial\tilde r)_{\tilde r_i^-}=\tilde\lambda(\partial\tilde T'/\partial\tilde r)_{\tilde r_i^+}$

$\hat T(\tilde r=\tilde r_i)P_n(\tilde z)=\hat T_iP_n(\tilde z)$ or $\hat T(\tilde r=\tilde r_i)=\hat T_i$

$(d\hat T/d\tilde r)_{\tilde r_i^-}P_n=\tilde\lambda(d\hat T/d\tilde r)_{\tilde r_i^+}P_n$ or $(d\hat T/d\tilde r)_{\tilde r_i^-}=\tilde\lambda(d\hat T/d\tilde r)_{\tilde r_i^+}$

(8) $\tilde r=1$: $\quad\tilde T'=\tilde T_b'$, $\quad\tilde\lambda(\partial\tilde T'/\partial\tilde r)_{1^-}=(\partial\tilde T'/\partial\tilde r)_{1^+}$, $\quad\tilde m\tilde Y_1'-\frac{1}{Le_1}\frac{\partial\tilde Y_1'}{\partial\tilde r}=0$, $\quad\tilde m\tilde Y_2'-\frac{1}{Le_2}\frac{\partial\tilde Y_2'}{\partial\tilde r}=0$

$\hat T(\tilde r=1)P_n(\tilde z)=\hat T_bP_n(\tilde z)$ or $\hat T(\tilde r=1)=\hat T_b$

$\tilde\lambda(d\hat T/d\tilde r)_{1^-}P_n=(d\hat T/d\tilde r)_{1^+}P_n$ or $\tilde\lambda(d\hat T/d\tilde r)_{1^-}=(d\hat T/d\tilde r)_{1^+}$

$\tilde m\hat Y_1P_n-\frac{1}{Le_1}\frac{d\hat Y_1}{d\tilde r}P_n=0$ or $\tilde m\hat Y_1-\frac{1}{Le_1}\frac{d\hat Y_1}{d\tilde r}=0$ Similarly $\tilde m\hat Y_2-\frac{1}{Le_2}\frac{d\hat Y_2}{d\tilde r}=0$

(9) $\tilde r\to\infty$: $\quad\tilde T'\to0$, $\quad\tilde Y_1'\to0$, $\quad\tilde Y_2'\to0\quad\Rightarrow\quad\hat T\to0$, $\quad\hat Y_1\to0$, $\quad\hat Y_2\to0$

(10) Summary

Core region ($0<\tilde r<\tilde r_i$)

$$\frac{d}{d\tilde r}\left(\tilde r^2\frac{d\hat T}{d\tilde r}\right)-\tilde m\frac{d\hat T}{d\tilde r}-[\tilde\omega\,\tilde r^2+n(n+1)]\hat T=\tilde r^2\frac{d^2\hat T}{d\tilde r^2}+(2\tilde r-\tilde m)\frac{d\hat T}{d\tilde r}-[\tilde\omega\,\tilde r^2+n(n+1)]\hat T=0;\quad\hat Y_1=0;\quad\hat Y_2=0$$

Within the porous burner ($\tilde r_i<\tilde r<1$)

$$\frac{d}{d\tilde{r}}\left(\tilde{r}^2\frac{d\hat{T}}{d\tilde{r}}\right)-\frac{\tilde{m}}{\tilde{\lambda}}\frac{d\hat{T}}{d\tilde{r}}-\left[\tilde{\omega}\,\frac{\alpha}{\tilde{\lambda}}\tilde{r}^2+n(n+1)\right]\hat{T}=\tilde{r}^2\frac{d^2\hat{T}}{d\tilde{r}^2}+\left(2\tilde{r}-\frac{\tilde{m}}{\tilde{\lambda}}\right)\frac{d\hat{T}}{d\tilde{r}}-\left[\tilde{\omega}\,\frac{\alpha}{\tilde{\lambda}}\tilde{r}^2+n(n+1)\right]\hat{T}=0\ \ ;\ \ \hat{Y}_1=0\ \ ;\ \ \hat{Y}_2=0$$

Gas region external of the burner ($1<\tilde{r}<\infty$)

$$\frac{d}{d\tilde{r}}\left(\tilde{r}^2\frac{d\hat{T}}{d\tilde{r}}\right)-\tilde{m}\frac{d\hat{T}}{d\tilde{r}}-[\tilde{\omega}\tilde{r}^2+n(n+1)]\hat{T}=\tilde{r}^2\left[Da_R\exp\left(-\frac{\tilde{E}_R}{\tilde{T}_S}\right)\frac{\tilde{E}_R}{\tilde{T}_S^2}\hat{T}-Da_K\tilde{Y}_{1,S}\tilde{Y}_{2,S}\exp\left(-\frac{\tilde{E}_K}{\tilde{T}_S}\right)\left(\frac{\hat{Y}_1}{\tilde{Y}_{1,S}}+\frac{\hat{Y}_2}{\tilde{Y}_{2,S}}+\frac{\tilde{E}_K}{\tilde{T}_S^2}\hat{T}\right)\right]$$

$$\frac{d}{d\tilde{r}}\left(\tilde{r}^2\frac{d\hat{Y}_1}{d\tilde{r}}\right)-Le_1\,\tilde{m}\frac{d\hat{Y}_1}{d\tilde{r}}-[\tilde{\omega}\,Le_1\tilde{r}^2+n(n+1)]\hat{Y}_1=Le_1\tilde{r}^2\,Da_K\,\tilde{Y}_{1,S}\,\tilde{Y}_{2,S}\exp\left(-\frac{\tilde{E}_K}{\tilde{T}_S}\right)\left(\frac{\hat{Y}_1}{\tilde{Y}_{1,S}}+\frac{\hat{Y}_2}{\tilde{Y}_{2,S}}+\frac{\tilde{E}_K}{\tilde{T}_S^2}\hat{T}\right)$$

$$\frac{d}{d\tilde{r}}\left(\tilde{r}^2\frac{d\hat{Y}_2}{d\tilde{r}}\right)-Le_2\,\tilde{m}\frac{d\hat{Y}_2}{d\tilde{r}}-[\tilde{\omega}\,Le_2\tilde{r}^2+n(n+1)]\hat{Y}_2=Le_2\tilde{r}^2\,Da_K\,\tilde{Y}_{1,S}\,\tilde{Y}_{2,S}\exp\left(-\frac{\tilde{E}_K}{\tilde{T}_S}\right)\left(\frac{\hat{Y}_1}{\tilde{Y}_{1,S}}+\frac{\hat{Y}_2}{\tilde{Y}_{2,S}}+\frac{\tilde{E}_K}{\tilde{T}_S^2}\hat{T}\right)$$

Boundary and interface conditions

$\tilde{r}=0:\quad \hat{T}=0$

$\tilde{r}=\tilde{r}_i:\quad \hat{T}=\hat{T}_i\,,\quad (d\hat{T}/d\tilde{r})_{\tilde{r}_i^-}=\tilde{\lambda}(d\hat{T}/d\tilde{r})_{\tilde{r}_i^+}$

$\tilde{r}=1:\quad \hat{T}=\hat{T}_b\,,\quad \tilde{\lambda}(d\hat{T}/d\tilde{r})_{1^-}=(d\hat{T}/d\tilde{r})_{1^+}\,,\quad \tilde{m}\hat{Y}_1-\frac{1}{Le_1}\frac{d\hat{Y}_1}{d\tilde{r}}=0\,,\quad \tilde{m}\hat{Y}_2-\frac{1}{Le_2}\frac{d\hat{Y}_2}{d\tilde{r}}=0$

$\tilde{r}\to\infty:\quad \hat{T}\to 0\,,\quad \hat{Y}_1\to 0\,,\quad \hat{Y}_2\to 0$

Perturbed flame location : $\tilde{r}_f(\tilde{z},\tilde{t})=\tilde{r}_{f,S}+\overline{\delta}\,\hat{r}_f=\tilde{r}_{f,S}+\overline{\delta}[\hat{r}_{f,0}+\varepsilon\,\hat{r}_{f,1}P_n(\tilde{z})e^{\tilde{\omega}\tilde{t}}+O(\varepsilon^2)]=\tilde{r}_{f,0}+\overline{\delta}[\varepsilon\,\hat{r}_{f,1}P_n(\tilde{z})e^{\tilde{\omega}\tilde{t}}+O(\varepsilon^2)]$

$\hat{r}_f=\hat{r}_{f,0}+\varepsilon\,\hat{r}_{f,1}+O(\varepsilon^2)\,;\quad \tilde{r}_{f,0}=\tilde{r}_{f,S}+\overline{\delta}\,\hat{r}_{f,0}\,P_n(\tilde{z})e^{\tilde{\omega}\tilde{t}}$

Cellular Instability

To study the cellular instability, we let $\omega = 0$ and the system is reduced to

$$\tilde{T}(\tilde{r},\tilde{z}) = \tilde{T}_S(\tilde{r}) + \bar{\delta}\, P_n(\tilde{z})\hat{T}(\tilde{r}) \quad ; \quad \tilde{Y}_i(\tilde{r},\tilde{z}) = \tilde{Y}_{i,S}(\tilde{r}) + \bar{\delta}\, P_n(\tilde{z})\hat{Y}_1(\tilde{r}) \quad , \quad i = 1, 2$$

$$\tilde{r}_f(\tilde{z}) = \tilde{r}_{f,S} + \bar{\delta}\,\hat{r}_f\, P_n(\tilde{z}) = \tilde{r}_{f,0} + \bar{\delta}[\varepsilon\hat{r}_{f,1} P_n(\tilde{z}) + O(\varepsilon^2)] \quad ; \quad \hat{r}_f = \hat{r}_{f,0} + \varepsilon\hat{r}_{f,1} + O(\varepsilon^2)$$

Core region ($0 < \tilde{r} < \tilde{r}_i$): $\dfrac{d}{d\tilde{r}}\left(\tilde{r}^2\dfrac{d\hat{T}}{d\tilde{r}}\right) - \tilde{m}\dfrac{d\hat{T}}{d\tilde{r}} - n(n+1)\hat{T} = 0 \quad ; \quad \hat{Y}_1 = 0 \quad ; \quad \hat{Y}_2 = 0$

Within the porous burner ($\tilde{r}_i < \tilde{r} < 1$): $\dfrac{d}{d\tilde{r}}\left(\tilde{r}^2\dfrac{d\hat{T}}{d\tilde{r}}\right) - \dfrac{\tilde{m}}{\tilde{\lambda}}\dfrac{d\hat{T}}{d\tilde{r}} - n(n+1)\hat{T} = 0 \quad ; \quad \hat{Y}_1 = 0 \quad ; \quad \hat{Y}_2 = 0$

Gas region external of the burner ($1 < \tilde{r} < \infty$)

$$\frac{d}{d\tilde{r}}\left(\tilde{r}^2\frac{d\hat{T}}{d\tilde{r}}\right) - \tilde{m}\frac{d\hat{T}}{d\tilde{r}} - n(n+1)\hat{T} = \tilde{r}^2\left[Da_R\exp\left(-\frac{\tilde{E}_R}{\tilde{T}_S}\right)\frac{\tilde{E}_R}{\tilde{T}_S^2}\hat{T} - Da_K\,\tilde{Y}_{1,S}\,\tilde{Y}_{2,S}\exp\left(-\frac{\tilde{E}_K}{\tilde{T}_S}\right)\left(\frac{\hat{Y}_1}{\tilde{Y}_{1,S}} + \frac{\hat{Y}_2}{\tilde{Y}_{2,S}} + \frac{\tilde{E}_K}{\tilde{T}_S^2}\hat{T}\right)\right]$$

$$\frac{d}{d\tilde{r}}\left(\tilde{r}^2\frac{d\hat{Y}_1}{d\tilde{r}}\right) - Le_1\,\tilde{m}\frac{d\hat{Y}_1}{d\tilde{r}} - n(n+1)\hat{Y}_1 = Le_1\,\tilde{r}^2\,Da_K\,\tilde{Y}_{1,S}\,\tilde{Y}_{2,S}\exp\left(-\frac{\tilde{E}_K}{\tilde{T}_S}\right)\left(\frac{\hat{Y}_1}{\tilde{Y}_{1,S}} + \frac{\hat{Y}_2}{\tilde{Y}_{2,S}} + \frac{\tilde{E}_K}{\tilde{T}_S^2}\hat{T}\right)$$

$$\frac{d}{d\tilde{r}}\left(\tilde{r}^2\frac{d\hat{Y}_2}{d\tilde{r}}\right) - Le_2\,\tilde{m}\frac{d\hat{Y}_2}{d\tilde{r}} - n(n+1)\hat{Y}_2 = Le_2\,\tilde{r}^2\,Da_K\,\tilde{Y}_{1,S}\,\tilde{Y}_{2,S}\exp\left(-\frac{\tilde{E}_K}{\tilde{T}_S}\right)\left(\frac{\hat{Y}_1}{\tilde{Y}_{1,S}} + \frac{\hat{Y}_2}{\tilde{Y}_{2,S}} + \frac{\tilde{E}_K}{\tilde{T}_S^2}\hat{T}\right)$$

Boundary and interface conditions

$\tilde{r} = 0$: $\quad \hat{T} = 0$;

$\tilde{r} = \tilde{r}_i$: $\quad \hat{T} = \hat{T}_i$, $\quad (d\hat{T}/d\tilde{r})_{\tilde{r}_i^-} = \tilde{\lambda}(d\hat{T}/d\tilde{r})_{\tilde{r}_i^+}$

$\tilde{r} = 1$: $\quad \hat{T} = \hat{T}_b$, $\quad \tilde{\lambda}(d\hat{T}/d\tilde{r})_{1^-} = (d\hat{T}/d\tilde{r})_{1^+}$, $\quad \tilde{m}\hat{Y}_1 - Le_1^{-1}(d\hat{Y}_1/d\tilde{r}) = 0$, $\quad \tilde{m}\hat{Y}_2 - Le_2^{-1}(d\hat{Y}_2/d\tilde{r}) = 0$

$\tilde{r} \to \infty$: $\quad \hat{T} \to 0$, $\quad \hat{Y}_1 \to 0$, $\quad \hat{Y}_2 \to 0$

(A) General solutions of chemically inert equations

From Appendix D, for $\dfrac{d}{d\tilde{r}}\left(\tilde{r}^2\dfrac{d\hat{Y}}{d\tilde{r}}\right) - p\tilde{m}\dfrac{d\hat{Y}}{d\tilde{r}} - n(n+1)\hat{Y} = 0$, the general solution is $\hat{Y} = c_1\Psi_1 + c_2\Psi_2$

$$\Psi_1(\tilde{r},p) = 1 + \sum_{k=1}^{n}\left\{\prod_{i=1}^{k}[i(i-1) - n(n+1)]\right\}\frac{\tilde{r}^k}{(k!)(p\tilde{m})^k}$$

$$\Psi_2(\tilde{r},p) = e^{-p\tilde{m}/\tilde{r}}\left\{1 + \sum_{k=1}^{n}\left[\prod_{i=1}^{k}[n(n+1) - i(i-1)]\right]\frac{\tilde{r}^k}{(k!)(p\tilde{m})^k}\right\} - (-1)^n\,\Psi_1(\tilde{r},p)$$

Differentiation of the functions

$$\Psi_1^*(\tilde{r},p) = \frac{d\Psi_1}{d\tilde{r}} = \sum_{k=1}^{n}\left\{\prod_{i=1}^{k}[i(i-1) - n(n+1)]\right\}\frac{k\,\tilde{r}^{k-1}}{(k!)(p\tilde{m})^k};$$

$$\Psi_2^*(\tilde{r},p) = d\Psi_2/d\tilde{r} = e^{-p\tilde{m}/\tilde{r}}\frac{p\tilde{m}}{\tilde{r}^2}\left\{1 + \sum_{k=1}^{n}\left[\prod_{i=1}^{k}[n(n+1) - i(i-1)]\right]\frac{\tilde{r}^k}{(k!)(p\tilde{m})^k}\right\}$$

$$+ e^{-p\tilde{m}/\tilde{r}}\left\{\sum_{k=1}^{n}\left[\prod_{i=1}^{k}[n(n+1) - i(i-1)]\right]\frac{k\,\tilde{r}^{k-1}}{(k!)(p\tilde{m})^k}\right\} - (-1)^n\frac{d\Psi_1}{d\tilde{r}}$$

$$= e^{-p\tilde{m}/\tilde{r}}\left\{\frac{p\tilde{m}}{\tilde{r}^2} + \sum_{k=1}^{n}\left[\prod_{i=1}^{k}[n(n+1) - i(i-1)]\right]\frac{(k\tilde{r} + p\tilde{m})\tilde{r}^{k-2}}{(k!)(p\tilde{m})^k}\right\} - (-1)^n\,\Psi_1^*(\tilde{r},p)$$

At $\tilde{r} = 0$: $\quad \Psi_1(0,p) = 1$, $\quad \Psi_2(0,p) = (-1)^{n-1}$

As $\tilde{r} \to \infty$: $\quad |\Psi_1| \to \infty$, $\quad \Psi_2 \to 0$

(B) Core region $(0<\tilde{r}<\tilde{r}_i)$

$$\frac{d}{d\tilde{r}}\left(\tilde{r}^2\frac{d\hat{T}}{d\tilde{r}}\right)-\tilde{m}\frac{d\hat{T}}{d\tilde{r}}-n(n+1)\hat{T}=0$$

From (A) : $p=1$ and $\hat{T}=c_1\Psi_1(\tilde{r},1)+c_2\Psi_2(\tilde{r},1)$

At $\tilde{r}=0$: $\hat{T}=0$ $\Psi_1=1$, $\Psi_2=(-1)^{n-1}=-(-1)^n$

$c_1-(-1)^n c_2=0$ or $c_2=(-1)^n c_1$ $\Rightarrow$ $\hat{T}=c_1[\Psi_1(\tilde{r},1)+(-1)^n\Psi_2(\tilde{r},1)]$

At $\tilde{r}=\tilde{r}_i$: $\hat{T}=\hat{T}_i$ $\therefore$ $c_1[\Psi_1(\tilde{r}_i,1)+(-1)^n\Psi_2(\tilde{r}_i,1)]=\hat{T}_i$ or $c_1=\hat{T}_i/[\Psi_1(\tilde{r}_i,1)+(-1)^n\Psi_2(\tilde{r}_i,1)]$

$$\hat{T}=\hat{T}_i\frac{\Psi_1(\tilde{r},1)+(-1)^n\Psi_2(\tilde{r},1)}{\Psi_1(\tilde{r}_i,1)+(-1)^n\Psi_2(\tilde{r}_i,1)}=\hat{T}_i e^{\tilde{m}(\tilde{r}_i^{-1}-\tilde{r}^{-1})}\left\{1+\sum_{k=1}^{n}\left[\prod_{i=1}^{k}[n(n+1)-i(i-1)]\right]\frac{\tilde{r}^k}{(k!)\tilde{m}^k}\right\}$$
$$\cdot\left\{1+\sum_{k=1}^{n}\left[\prod_{i=1}^{k}[n(n+1)-i(i-1)]\right]\frac{\tilde{r}_i^k}{(k!)\tilde{m}^k}\right\}^{-1}$$

$$\frac{d\hat{T}}{d\tilde{r}}=\hat{T}_i e^{\tilde{m}(\tilde{r}_i^{-1}-\tilde{r}^{-1})}\left\{\frac{\tilde{m}}{\tilde{r}^2}+\sum_{k=1}^{n}\left[\prod_{i=1}^{k}[n(n+1)-i(i-1)]\right]\frac{(k\tilde{r}+\tilde{m})\tilde{r}^{k-2}}{(k!)\tilde{m}^k}\right\}\cdot\left\{1+\sum_{k=1}^{n}\left[\prod_{i=1}^{k}[n(n+1)-i(i-1)]\right]\frac{\tilde{r}_i^k}{(k!)\tilde{m}^k}\right\}^{-1}$$

(C) Within the porous burner $(\tilde{r}_i<\tilde{r}<1)$

$$\frac{d}{d\tilde{r}}\left(\tilde{r}^2\frac{d\hat{T}}{d\tilde{r}}\right)-\frac{\tilde{m}}{\tilde{\lambda}}\frac{d\hat{T}}{d\tilde{r}}-n(n+1)\hat{T}=0$$

From (A): $p=\tilde{\lambda}^{-1}$ and $\hat{T}=c_1\Psi_1(\tilde{r},\tilde{\lambda}^{-1})+c_2\Psi_2(\tilde{r},\tilde{\lambda}^{-1})$

At $\tilde{r}=\tilde{r}_i$: $\hat{T}=\hat{T}_i$ $\therefore$ $c_1\Psi_1(\tilde{r}_i,\tilde{\lambda}^{-1})+c_2\Psi_2(\tilde{r}_i,\tilde{\lambda}^{-1})=\hat{T}_i$ or $c_2=[\hat{T}_i-c_1\Psi_1(\tilde{r}_i,\tilde{\lambda}^{-1})]/\Psi_2(\tilde{r}_i,\tilde{\lambda}^{-1})$

$$\hat{T}=c_1\Psi_1(\tilde{r},\tilde{\lambda}^{-1})+[\hat{T}_i-c_1\Psi_1(\tilde{r}_i,\tilde{\lambda}^{-1})]\Psi_2(\tilde{r},\tilde{\lambda}^{-1})/\Psi_2(\tilde{r}_i,\tilde{\lambda}^{-1})$$
$$=\{\hat{T}_i\Psi_2(\tilde{r},\tilde{\lambda}^{-1})+c_1[\Psi_1(\tilde{r},\tilde{\lambda}^{-1})\Psi_2(\tilde{r}_i,\tilde{\lambda}^{-1})-\Psi_1(\tilde{r}_i,\tilde{\lambda}^{-1})\Psi_2(\tilde{r},\tilde{\lambda}^{-1})]\}/\Psi_2(\tilde{r}_i,\tilde{\lambda}^{-1})$$

At $\tilde{r}=1$: $\hat{T}=\hat{T}_b$ $\therefore$ $\{\hat{T}_i\Psi_2(1,\tilde{\lambda}^{-1})+c_1[\Psi_1(1,\tilde{\lambda}^{-1})\Psi_2(\tilde{r}_i,\tilde{\lambda}^{-1})-\Psi_1(\tilde{r}_i,\tilde{\lambda}^{-1})\Psi_2(1,\tilde{\lambda}^{-1})]\}/\Psi_2(\tilde{r}_i,\tilde{\lambda}^{-1})=\hat{T}_b$

$$c_1=[\hat{T}_b\Psi_2(\tilde{r}_i,\tilde{\lambda}^{-1})-\hat{T}_i\Psi_2(1,\tilde{\lambda}^{-1})]/[\Psi_1(1,\tilde{\lambda}^{-1})\Psi_2(\tilde{r}_i,\tilde{\lambda}^{-1})-\Psi_1(\tilde{r}_i,\tilde{\lambda}^{-1})\Psi_2(1,\tilde{\lambda}^{-1})]$$

$$\hat{T}=\hat{T}_i\frac{\Psi_2(\tilde{r},\tilde{\lambda}^{-1})}{\Psi_2(\tilde{r}_i,\tilde{\lambda}^{-1})}+\frac{\hat{T}_b\Psi_2(\tilde{r}_i,\tilde{\lambda}^{-1})-\hat{T}_i\Psi_2(1,\tilde{\lambda}^{-1})}{\Psi_1(1,\tilde{\lambda}^{-1})\Psi_2(\tilde{r}_i,\tilde{\lambda}^{-1})-\Psi_1(\tilde{r}_i,\tilde{\lambda}^{-1})\Psi_2(1,\tilde{\lambda}^{-1})}\frac{\Psi_1(\tilde{r},\tilde{\lambda}^{-1})\Psi_2(\tilde{r}_i,\tilde{\lambda}^{-1})-\Psi_1(\tilde{r}_i,\tilde{\lambda}^{-1})\Psi_2(\tilde{r},\tilde{\lambda}^{-1})}{\Psi_2(\tilde{r}_i,\tilde{\lambda}^{-1})}$$
$$=\frac{\hat{T}_i[\Psi_1(1,\tilde{\lambda}^{-1})\Psi_2(\tilde{r},\tilde{\lambda}^{-1})-\Psi_1(\tilde{r},\tilde{\lambda}^{-1})\Psi_2(1,\tilde{\lambda}^{-1})]+\hat{T}_b[\Psi_1(\tilde{r},\tilde{\lambda}^{-1})\Psi_2(\tilde{r}_i,\tilde{\lambda}^{-1})-\Psi_1(\tilde{r}_i,\tilde{\lambda}^{-1})\Psi_2(\tilde{r},\tilde{\lambda}^{-1})]}{\Psi_1(1,\tilde{\lambda}^{-1})\Psi_2(\tilde{r}_i,\tilde{\lambda}^{-1})-\Psi_1(\tilde{r}_i,\tilde{\lambda}^{-1})\Psi_2(1,\tilde{\lambda}^{-1})}$$

$$\frac{d\hat{T}}{d\tilde{r}}=\frac{\hat{T}_i[\Psi_1(1,\tilde{\lambda}^{-1})(d\Psi_2/d\tilde{r})-(d\Psi_1/d\tilde{r})\Psi_2(1,\tilde{\lambda}^{-1})]+\hat{T}_b[(d\Psi_1/d\tilde{r})\Psi_2(\tilde{r}_i,\tilde{\lambda}^{-1})-\Psi_1(\tilde{r}_i,\tilde{\lambda}^{-1})(d\Psi_2/d\tilde{r})]}{\Psi_1(1,\tilde{\lambda}^{-1})\Psi_2(\tilde{r}_i,\tilde{\lambda}^{-1})-\Psi_1(\tilde{r}_i,\tilde{\lambda}^{-1})\Psi_2(1,\tilde{\lambda}^{-1})}$$
$$=\frac{\hat{T}_i[\Psi_1(1,\tilde{\lambda}^{-1})\Psi_2^*(\tilde{r},\tilde{\lambda}^{-1})-\Psi_1^*(\tilde{r},\tilde{\lambda}^{-1})\Psi_2(1,\tilde{\lambda}^{-1})]+\hat{T}_b[\Psi_1^*(\tilde{r},\tilde{\lambda}^{-1})\Psi_2(\tilde{r}_i,\tilde{\lambda}^{-1})-\Psi_1(\tilde{r}_i,\tilde{\lambda}^{-1})\Psi_2^*(\tilde{r},\tilde{\lambda}^{-1})]}{\Psi_1(1,\tilde{\lambda}^{-1})\Psi_2(\tilde{r}_i,\tilde{\lambda}^{-1})-\Psi_1(\tilde{r}_i,\tilde{\lambda}^{-1})\Psi_2(1,\tilde{\lambda}^{-1})}$$

At $\tilde{r}=\tilde{r}_i$: $(d\hat{T}/d\tilde{r})_{\tilde{r}_i^-}=\tilde{\lambda}(d\hat{T}/d\tilde{r})_{\tilde{r}_i^+}$

$$\frac{\hat{T}_i}{\tilde{\lambda}}\frac{\Psi_1^*(\tilde{r}_i,1)+(-1)^n\Psi_2^*(\tilde{r}_i,1)}{\Psi_1(\tilde{r}_i,1)+(-1)^n\Psi_2(\tilde{r}_i,1)}$$
$$=\frac{\hat{T}_i[\Psi_1(1,\tilde{\lambda}^{-1})\Psi_2^*(\tilde{r}_i,\tilde{\lambda}^{-1})-\Psi_1^*(\tilde{r}_i,\tilde{\lambda}^{-1})\Psi_2(1,\tilde{\lambda}^{-1})]+\hat{T}_b[\Psi_1^*(\tilde{r}_i,\tilde{\lambda}^{-1})\Psi_2(\tilde{r}_i,\tilde{\lambda}^{-1})-\Psi_1(\tilde{r}_i,\tilde{\lambda}^{-1})\Psi_2^*(\tilde{r}_i,\tilde{\lambda}^{-1})]}{\Psi_1(1,\tilde{\lambda}^{-1})\Psi_2(\tilde{r}_i,\tilde{\lambda}^{-1})-\Psi_1(\tilde{r}_i,\tilde{\lambda}^{-1})\Psi_2(1,\tilde{\lambda}^{-1})}$$

$$\hat{T}_i=\hat{T}_b\frac{\tilde{\lambda}[\Psi_1(\tilde{r}_i,1)+(-1)^n\Psi_2(\tilde{r}_i,1)][\Psi_1^*(\tilde{r}_i,\tilde{\lambda}^{-1})\Psi_2(\tilde{r}_i,\tilde{\lambda}^{-1})-\Psi_1(\tilde{r}_i,\tilde{\lambda}^{-1})\Psi_2^*(\tilde{r}_i,\tilde{\lambda}^{-1})]}{\{[\Psi_1^*(\tilde{r}_i,1)+(-1)^n\Psi_2^*(\tilde{r}_i,1)][\Psi_1(1,\tilde{\lambda}^{-1})\Psi_2(\tilde{r}_i,\tilde{\lambda}^{-1})-\Psi_1(\tilde{r}_i,\tilde{\lambda}^{-1})\Psi_2(1,\tilde{\lambda}^{-1})]\atop -\tilde{\lambda}[\Psi_1(\tilde{r}_i,1)+(-1)^n\Psi_2(\tilde{r}_i,1)][\Psi_1(1,\tilde{\lambda}^{-1})\Psi_2^*(\tilde{r}_i,\tilde{\lambda}^{-1})-\Psi_1^*(\tilde{r}_i,\tilde{\lambda}^{-1})\Psi_2(1,\tilde{\lambda}^{-1})]\}}=\hat{T}_b A_{T1}$$

Thus:

$$\hat{T}=\hat{T}_b\frac{A_{T1}[\Psi_1(1,\tilde{\lambda}^{-1})\Psi_2(\tilde{r},\tilde{\lambda}^{-1})-\Psi_1(\tilde{r},\tilde{\lambda}^{-1})\Psi_2(1,\tilde{\lambda}^{-1})]+[\Psi_1(\tilde{r},\tilde{\lambda}^{-1})\Psi_2(\tilde{r}_i,\tilde{\lambda}^{-1})-\Psi_1(\tilde{r}_i,\tilde{\lambda}^{-1})\Psi_2(\tilde{r},\tilde{\lambda}^{-1})]}{\Psi_1(1,\tilde{\lambda}^{-1})\Psi_2(\tilde{r}_i,\tilde{\lambda}^{-1})-\Psi_1(\tilde{r}_i,\tilde{\lambda}^{-1})\Psi_2(1,\tilde{\lambda}^{-1})}$$

$$\frac{d\hat{T}}{d\tilde{r}}=\hat{T}_b\frac{A_{T1}[\Psi_1(1,\tilde{\lambda}^{-1})\Psi_2^*(\tilde{r},\tilde{\lambda}^{-1})-\Psi_1^*(\tilde{r},\tilde{\lambda}^{-1})\Psi_2(1,\tilde{\lambda}^{-1})]+[\Psi_1^*(\tilde{r},\tilde{\lambda}^{-1})\Psi_2(\tilde{r}_i,\tilde{\lambda}^{-1})-\Psi_1(\tilde{r}_i,\tilde{\lambda}^{-1})\Psi_2^*(\tilde{r},\tilde{\lambda}^{-1})]}{\Psi_1(1,\tilde{\lambda}^{-1})\Psi_2(\tilde{r}_i,\tilde{\lambda}^{-1})-\Psi_1(\tilde{r}_i,\tilde{\lambda}^{-1})\Psi_2(1,\tilde{\lambda}^{-1})}$$

(D) Outer solutions in the gas region

(1) $\{d[\tilde{r}^2(d\hat{Y}_1^{\pm}/d\tilde{r})]/d\tilde{r}\}-Le_1\tilde{m}(d\hat{Y}_1^{\pm}/d\tilde{r})-n(n+1)\hat{Y}_1^{\pm}=0$

From (A) : $p=Le_1$ and $\hat{Y}_1^{\pm}=\hat{c}_1^{\pm}\Psi_1(\tilde{r},Le_1)+\hat{c}_2^{\pm}\Psi_2(\tilde{r},Le_1)$

At $\tilde{r}=1$: $\tilde{m}\hat{Y}_1^{-}-Le_1^{-1}(d\hat{Y}_1^{-}/d\tilde{r})=0$ $\quad\therefore\quad$ $\tilde{m}Le_1[\hat{c}_1^{-}\Psi_1(1,Le_1)+\hat{c}_2^{-}\Psi_2(1,Le_1)]-[\hat{c}_1^{-}\Psi_1^*(1,Le_1)+\hat{c}_2^{-}\Psi_2^*(1,Le_1)]=0$

$$\Rightarrow\quad \hat{c}_2^{-}=-\hat{c}_1^{-}\frac{\tilde{m}Le_1\Psi_1(1,Le_1)-\Psi_1^*(1,Le_1)}{\tilde{m}Le_1\Psi_2(1,Le_1)-\Psi_2^*(1,Le_1)}\quad\text{and}\ \hat{Y}_1^{-}=\hat{c}_1^{-}\left[\Psi_1(\tilde{r},Le_1)-\frac{\tilde{m}Le_1\Psi_1(1,Le_1)-\Psi_1^*(1,Le_1)}{\tilde{m}Le_1\Psi_2(1,Le_1)-\Psi_2^*(1,Le_1)}\Psi_2(\tilde{r},Le_1)\right]$$

Expand $\hat{Y}_1^{-}=[\hat{Y}_{1,0}^{-}+\varepsilon\hat{Y}_{1,1}^{-}+O(\varepsilon^2)]+\delta[\hat{Y}_{1,2}^{-}+O(\varepsilon)]+O(\delta^2)$; $\hat{c}_1^{-}=[\hat{a}_{1,0}^{-}+\varepsilon\hat{a}_{1,1}^{-}+O(\varepsilon^2)]+\delta[\hat{a}_{1,2}^{-}+O(\varepsilon)]+O(\delta^2)$

$$\{d[\tilde{r}^2(d\hat{Y}_{1,0}^{-}/d\tilde{r})]/d\tilde{r}\}-Le_1\tilde{m}(d\hat{Y}_{1,0}^{-}/d\tilde{r})=\tilde{r}^2(d^2\hat{Y}_{1,0}^{-}/d\tilde{r}^2)-(Le_1\tilde{m}-2\tilde{r})(d\hat{Y}_{1,0}^{-}/d\tilde{r})=n(n+1)\hat{Y}_{1,0}^{-}$$

$$\hat{Y}_{1,0}^{-}=\hat{a}_{1,0}^{-}\left\{\Psi_1(\tilde{r},Le_1)-\frac{[\tilde{m}Le_1\Psi_1(1,Le_1)-\Psi_1^*(1,Le_1)]}{[\tilde{m}Le_1\Psi_2(1,Le_1)-\Psi_2^*(1,Le_1)]}\Psi_2(\tilde{r},Le_1)\right\}$$

$$\hat{Y}_{1,1}^{-}=\hat{a}_{1,1}^{-}\left\{\Psi_1(\tilde{r},Le_1)-\frac{[\tilde{m}Le_1\Psi_1(1,Le_1)-\Psi_1^*(1,Le_1)]}{[\tilde{m}Le_1\Psi_2(1,Le_1)-\Psi_2^*(1,Le_1)]}\Psi_2(\tilde{r},Le_1)\right\}$$

(3) $\frac{d}{d\tilde{r}}\left(\tilde{r}^2\frac{d\hat{T}^{\pm}}{d\tilde{r}}\right)-\tilde{m}\frac{d\hat{T}^{\pm}}{d\tilde{r}}-n(n+1)\hat{T}^{\pm}=0$ From (A) : $p=1$ and $\hat{T}^{\pm}=c_1^{\pm}\Psi_1(\tilde{r},1)+c_2^{\pm}\Psi_2(\tilde{r},1)$

As $\tilde{r}\to\infty$: $\hat{T}^{+}\to 0$, $|\Psi_1|\to\infty$, $\Psi_2\to 0$ $\quad\therefore\quad$ $c_1^{+}=0$ and $\hat{T}^{+}=c_2^{+}\Psi_2(\tilde{r},1)$

Expand $\hat{T}^{+}=[\hat{T}_0^{+}+\varepsilon\hat{T}_1^{+}+O(\varepsilon^2)]+\delta[\hat{T}_2^{+}+O(\varepsilon)]+O(\delta^2)$; $c_2^{+}=[\hat{a}_{T,0}^{+}+\varepsilon\hat{a}_{T,1}^{+}+O(\varepsilon^2)]+\delta[\hat{a}_{T,2}^{+}+O(\varepsilon)]+O(\delta^2)$

$$[\hat{T}_0^{+}+\varepsilon\hat{T}_1^{+}+O(\varepsilon^2)]+\delta[\hat{T}_2^{+}+O(\varepsilon)]+O(\delta^2)=\{[\hat{a}_{T,0}^{+}+\varepsilon\hat{a}_{T,1}^{+}+O(\varepsilon^2)]+\delta[\hat{a}_{T,2}^{+}+O(\varepsilon)]+O(\delta^2)\}\Psi_2(\tilde{r},1)$$

$\therefore\quad \hat{T}_0^{+}=\hat{a}_{T,0}^{+}\Psi_2(\tilde{r},1)$; $\hat{T}_1^{+}=\hat{a}_{T,1}^{+}\Psi_2(\tilde{r},1)$; $\hat{T}_2^{+}=\hat{a}_{T,2}^{+}\Psi_2(\tilde{r},1)$

At $\tilde{r}=1$: $\hat{T}^{-}=\hat{T}_b$ $\quad\therefore\quad$ $\hat{T}_b=c_1^{-}\Psi_1(1,1)+c_2^{-}\Psi_2(1,1)$ or $c_2^{-}=[\hat{T}_b-c_1^{-}\Psi_1(1,1)]/\Psi_2(1,1)$

$$\hat{T}^{-}=c_1^{-}\Psi_1(\tilde{r},1)+c_2^{-}\Psi_2(\tilde{r},1)=c_1^{-}\Psi_1(\tilde{r},1)+[\hat{T}_b-c_1^{-}\Psi_1(1,1)]\Psi_2(\tilde{r},1)/\Psi_2(1,1)$$

$$\tilde{\lambda}(d\hat{T}/d\tilde{r})_{1^-}=(d\hat{T}/d\tilde{r})_{1^+}$$

$$\hat{T}_b\tilde{\lambda}\frac{A_{T1}[\Psi_1(1,\tilde{\lambda}^{-1})\Psi_2^*(1,\tilde{\lambda}^{-1})-\Psi_1^*(1,\tilde{\lambda}^{-1})\Psi_2(1,\tilde{\lambda}^{-1})]+[\Psi_1^*(1,\tilde{\lambda}^{-1})\Psi_2(\tilde{r}_i,\tilde{\lambda}^{-1})-\Psi_1(\tilde{r}_i,\tilde{\lambda}^{-1})\Psi_2^*(1,\tilde{\lambda}^{-1})]}{\Psi_1(1,\tilde{\lambda}^{-1})\Psi_2(\tilde{r}_i,\tilde{\lambda}^{-1})-\Psi_1(\tilde{r}_i,\tilde{\lambda}^{-1})\Psi_2(1,\tilde{\lambda}^{-1})}$$

$$=c_1^{-}\Psi_1^*(1,1)+[\hat{T}_b-c_1^{-}\Psi_1(1,1)]\Psi_2^*(1,1)/\Psi_2(1,1)$$

$$c_1^{-}=\frac{\hat{T}_b\Psi_2(1,1)}{\Psi_1^*(1,1)\Psi_2(1,1)-\Psi_1(1,1)\Psi_2^*(1,1)}\cdot\left\{\tilde{\lambda}\frac{A_{T1}[\Psi_1(1,\tilde{\lambda}^{-1})\Psi_2^*(1,\tilde{\lambda}^{-1})-\Psi_1^*(1,\tilde{\lambda}^{-1})\Psi_2(1,\tilde{\lambda}^{-1})]+[\Psi_1^*(1,\tilde{\lambda}^{-1})\Psi_2(\tilde{r}_i,\tilde{\lambda}^{-1})-\Psi_1(\tilde{r}_i,\tilde{\lambda}^{-1})\Psi_2^*(1,\tilde{\lambda}^{-1})]}{\Psi_1(1,\tilde{\lambda}^{-1})\Psi_2(\tilde{r}_i,\tilde{\lambda}^{-1})-\Psi_1(\tilde{r}_i,\tilde{\lambda}^{-1})\Psi_2(1,\tilde{\lambda}^{-1})}-\frac{\Psi_2^*(1,1)}{\Psi_2(1,1)}\right\}$$

$$=\hat{T}_bA_{T2}$$

$$\hat{T}^{-}=\hat{T}_b\{A_{T2}\Psi_1(\tilde{r},1)+[1-A_{T2}\Psi_1(1,1)]\Psi_2(\tilde{r},1)/\Psi_2(1,1)\}$$

Expand $\hat{T}^{-}=[\hat{T}_0^{-}+\varepsilon\hat{T}_1^{-}+O(\varepsilon^2)]+\delta[\hat{T}_2^{-}+O(\varepsilon)]+O(\delta^2)$; $\hat{T}_b=[\hat{T}_{b,0}+\varepsilon\hat{T}_{b,1}+O(\varepsilon^2)]+\delta[\hat{T}_{b,2}+O(\varepsilon)]+O(\delta^2)$

$$\hat{T}_j^{-}=\hat{T}_{b,j}\{A_{T2}\Psi_1(\tilde{r},1)+[1-A_{T2}\Psi_1(1,1)]\Psi_2(\tilde{r},1)/\Psi_2(1,1)\}$$

(4) Summary

$$\hat{T}^- = [\hat{T}_0^- + \varepsilon \hat{T}_1^- + O(\varepsilon^2)] + \delta[\hat{T}_2^- + O(\varepsilon)] + O(\delta^2)$$

$$\hat{T}_j^- = \hat{T}_{b,j}\{A_{T2}\Psi_1(\tilde{r},1) + [1 - A_{T2}\Psi_1(1,1)]\Psi_2(\tilde{r},1)/\Psi_2(1,1)\}$$

$$A_{T2} = \frac{\Psi_2(1,1)}{\Psi_1^*(1,1)\Psi_2(1,1) - \Psi_1(1,1)\Psi_2^*(1,1)} \cdot \left\{\tilde{\lambda}\frac{A_{T1}[\Psi_1(1,\tilde{\lambda}^{-1})\Psi_2^*(1,\tilde{\lambda}^{-1}) - \Psi_1^*(1,\tilde{\lambda}^{-1})\Psi_2(1,\tilde{\lambda}^{-1})] + [\Psi_1^*(1,\tilde{\lambda}^{-1})\Psi_2(\tilde{r}_i,\tilde{\lambda}^{-1}) - \Psi_1(\tilde{r}_i,\tilde{\lambda}^{-1})\Psi_2^*(1,\tilde{\lambda}^{-1})]}{\Psi_1(1,\tilde{\lambda}^{-1})\Psi_2(\tilde{r}_i,\tilde{\lambda}^{-1}) - \Psi_1(\tilde{r}_i,\tilde{\lambda}^{-1})\Psi_2(1,\tilde{\lambda}^{-1})} - \frac{\Psi_2^*(1,1)}{\Psi_2(1,1)}\right\}$$

$$A_{T1} = \frac{\tilde{\lambda}[\Psi_1(\tilde{r}_i,1) + (-1)^n\Psi_2(\tilde{r}_i,1)][\Psi_1^*(\tilde{r}_i,\tilde{\lambda}^{-1})\Psi_2(\tilde{r}_i,\tilde{\lambda}^{-1}) - \Psi_1(\tilde{r}_i,\tilde{\lambda}^{-1})\Psi_2^*(\tilde{r}_i,\tilde{\lambda}^{-1})]}{\begin{array}{l}\{[\Psi_1^*(\tilde{r}_i,1) + (-1)^n\Psi_2^*(\tilde{r}_i,1)][\Psi_1(1,\tilde{\lambda}^{-1})\Psi_2(\tilde{r}_i,\tilde{\lambda}^{-1}) - \Psi_1(\tilde{r}_i,\tilde{\lambda}^{-1})\Psi_2(1,\tilde{\lambda}^{-1})] \\ \quad - \tilde{\lambda}[\Psi_1(\tilde{r}_i,1) + (-1)^n\Psi_2(\tilde{r}_i,1)][\Psi_1(1,\tilde{\lambda}^{-1})\Psi_2^*(\tilde{r}_i,\tilde{\lambda}^{-1}) - \Psi_1^*(\tilde{r}_i,\tilde{\lambda}^{-1})\Psi_2(1,\tilde{\lambda}^{-1})]\}\end{array}}$$

$$\hat{T}^+ = [\hat{T}_0^+ + \varepsilon\hat{T}_1^+ + O(\varepsilon^2)] + \delta[\hat{T}_2^+ + O(\varepsilon)] + O(\delta^2) \;;\; \hat{T}_0^+ = \hat{a}_{T,0}^+\Psi_2(\tilde{r},1) \;;\; \hat{T}_1^+ = \hat{a}_{T,1}^+\Psi_2(\tilde{r},1) \;;\; \hat{T}_2^+ = \hat{a}_{T,2}^+\Psi_2(\tilde{r},1)$$

$$\hat{Y}_1^- = [\hat{Y}_{1,0}^- + \varepsilon\hat{Y}_{1,1}^- + O(\varepsilon^2)] + \delta[\hat{Y}_{1,2}^- + O(\varepsilon)] + O(\delta^2) \;;\; \tilde{r}^2(d^2\hat{Y}_{1,0}^-/d\tilde{r}^2) - (Le_1\tilde{m} - 2\tilde{r})(d\hat{Y}_{1,0}^-/d\tilde{r}) = n(n+1)\hat{Y}_{1,0}^-$$

$$\hat{Y}_{1,0}^- = \hat{a}_{1,0}^-\left\{\Psi_1(\tilde{r},Le_1) - \frac{[\tilde{m}Le_1\Psi_1(1,Le_1) - \Psi_1^*(1,Le_1)]}{[\tilde{m}Le_1\Psi_2(1,Le_1) - \Psi_2^*(1,Le_1)]}\Psi_2(\tilde{r},Le_1)\right\}$$

$$\hat{Y}_{1,1}^- = \hat{a}_{1,1}^-\left\{\Psi_1(\tilde{r},Le_1) - \frac{[\tilde{m}Le_1\Psi_1(1,Le_1) - \Psi_1^*(1,Le_1)]}{[\tilde{m}Le_1\Psi_2(1,Le_1) - \Psi_2^*(1,Le_1)]}\Psi_2(\tilde{r},Le_1)\right\}$$
$$- Le_{1,1}\hat{a}_{1,0}^-\left\{\frac{\tilde{m}\Psi_1(1,Le_1)}{\tilde{m}Le_1\Psi_2(1,Le_1) - \Psi_2^*(1,Le_1)} - \frac{[\tilde{m}Le_1\Psi_1(1,Le_1) - \Psi_1^*(1,Le_1)]\tilde{m}\Psi_2(1,Le_1)}{[\tilde{m}Le_1\Psi_2(1,Le_1) - \Psi_2^*(1,Le_1)]^2}\right\rangle\Psi_2(\tilde{r},Le_1)\Bigg\}$$

$$\hat{Y}_{1,2}^- = \hat{a}_{1,2}^-\left\{\Psi_1(\tilde{r},Le_1) - \frac{[\tilde{m}Le_1\Psi_1(1,Le_1) - \Psi_1^*(1,Le_1)]}{[\tilde{m}Le_1\Psi_2(1,Le_1) - \Psi_2^*(1,Le_1)]}\Psi_2(\tilde{r},Le_1)\right\}$$

$$\hat{Y}_1^+ = [\hat{Y}_{1,0}^+ + \varepsilon\hat{Y}_{1,1}^+ + O(\varepsilon^2)] + \delta[\hat{Y}_{1,2}^+ + O(\varepsilon)] + O(\delta^2)$$

$$\hat{Y}_{1,0}^+ = \hat{a}_{1,0}^+\Psi_2(\tilde{r},Le_1) \;;\; \hat{Y}_{1,1}^+ = \hat{a}_{1,1}^+\Psi_2(\tilde{r},Le_1) \;;\; \hat{Y}_{1,2}^+ = \hat{a}_{1,2}^+\Psi_2(\tilde{r},Le_1)$$

$$\hat{Y}_2^- = [\hat{Y}_{2,0}^- + \varepsilon\hat{Y}_{2,1}^- + O(\varepsilon^2)] + \delta[\hat{Y}_{2,2}^- + O(\varepsilon)] + O(\delta^2)$$

$$\hat{Y}_{2,0}^- = \hat{a}_{2,0}^-\left\{\Psi_1(\tilde{r},Le_2) - \frac{[\tilde{m}Le_2\Psi_1(1,Le_2) - \Psi_1^*(1,Le_2)]}{[\tilde{m}Le_2\Psi_2(1,Le_2) - \Psi_2^*(1,Le_2)]}\Psi_2(\tilde{r},Le_2)\right\}$$

$$\hat{Y}_{2,1}^- = \hat{a}_{2,1}^-\left\{\Psi_1(\tilde{r},Le_2) - \frac{[\tilde{m}Le_2\Psi_1(1,Le_2) - \Psi_1^*(1,Le_2)]}{[\tilde{m}Le_2\Psi_2(1,Le_2) - \Psi_2^*(1,Le_2)]}\Psi_2(\tilde{r},Le_2)\right\}$$

$$\hat{Y}_{2,2}^- = \hat{a}_{2,2}^-\left\{\Psi_1(\tilde{r},Le_2) - \frac{[\tilde{m}Le_2\Psi_1(1,Le_2) - \Psi_1^*(1,Le_2)]}{[\tilde{m}Le_2\Psi_2(1,Le_2) - \Psi_2^*(1,Le_2)]}\Psi_2(\tilde{r},Le_2)\right\}$$

$$\hat{Y}_2^+ = [\hat{Y}_{2,0}^+ + \varepsilon\hat{Y}_{2,1}^+ + O(\varepsilon^2)] + \delta[\hat{Y}_{2,2}^+ + O(\varepsilon)] + O(\delta^2)$$

$$\hat{Y}_{2,0}^+ = \hat{a}_{2,0}^+\Psi_2(\tilde{r},Le_2) \;;\; \hat{Y}_{2,1}^+ = \hat{a}_{2,1}^+\Psi_2(\tilde{r},Le_2) \;;\; \hat{Y}_{2,2}^+ = \hat{a}_{2,2}^+\Psi_2(\tilde{r},Le_2)$$

(E) Expansion of the Energy Equation in the Radiation regions

Define inner variable: $\zeta = (\tilde{r} - \tilde{r}_f)/\delta$ or $\tilde{r} = \tilde{r}_f + \delta\zeta = \tilde{r}_f[1 + \delta(\zeta/\tilde{r}_f)] \Rightarrow d\tilde{r} = \delta d\zeta$; $\tilde{r}_{f,0} = \tilde{r}_{f,S} + \bar{\delta}\hat{r}_{f,0}P_n(\tilde{z})$

$$1/\tilde{r} = (1/\tilde{r}_f)[1 + \delta(\zeta/\tilde{r}_f)]^{-1} = [1 - \delta(\zeta/\tilde{r}_f) + O(\delta^2)]/\tilde{r}_f \;;\; 1/\tilde{r}^2 = [1 - \delta(\zeta/\tilde{r}_f) + O(\delta^2)]^2/\tilde{r}_f^2 = [1 - \delta(2\zeta/\tilde{r}_f) + O(\delta^2)]/\tilde{r}_f^2$$

$$\tilde{r}_f = \tilde{r}_{f,S} + \bar{\delta}(\hat{r}_{f,0} + \varepsilon\hat{r}_{f,1})P_n(\tilde{z}) = \tilde{r}_{f,S}[1 + \bar{\delta}(\hat{r}_{f,0}/\tilde{r}_{f,S})P_n(\tilde{z})] + \varepsilon\bar{\delta}P_n(\tilde{z})\hat{r}_{f,1} = \tilde{r}_{f,0} + \varepsilon\bar{\delta}P_n(\tilde{z})\hat{r}_{f,1}$$

$$\tilde{r}_f^2 = \tilde{r}_{f,0}^2 + O(\varepsilon) = \tilde{r}_{f,S}^2[1 + \bar{\delta}(2\hat{r}_{f,0}/\tilde{r}_{f,S})P_n + O(\bar{\delta}^2)] + O(\varepsilon)$$

$$1/\tilde{r}_f = (1/\tilde{r}_{f,0}) + O(\varepsilon) = (1/\tilde{r}_{f,S})[1 - \bar{\delta}(\hat{r}_{f,0}/\tilde{r}_{f,S})P_n(\tilde{z}) + O(\bar{\delta}^2)] + O(\varepsilon)$$

$$1/\tilde{r}_f^2 = \{[1 - \bar{\delta}(2\hat{r}_{f,0}P_n/\tilde{r}_{f,S}) + O(\bar{\delta}^2)]/\tilde{r}_{f,S}^2\} + O(\varepsilon)$$

Define inner expansion: $\tilde{T}=[\tilde{T}_f-\varepsilon\tilde{\Theta}_2^{\pm}+O(\varepsilon^2)]-\delta[\tilde{\Theta}_1^{\pm}+\varepsilon\tilde{\Theta}_3^{\pm}+O(\varepsilon^2)]+O(\delta^2)$; $\tilde{\Theta}_j^{\pm}=\Theta_{S,j}^{\pm}+\bar{\delta}P_n(\tilde{z})\hat{\Theta}_j^{\pm}(\zeta)$

From steady state analysis: $1/\tilde{T}=(1/\tilde{T}_f)\langle[1+\varepsilon(\tilde{\Theta}_2^{\pm}/\tilde{T}_f)+O(\varepsilon^2)]+(\delta/\tilde{T}_f)\{\tilde{\Theta}_1^{\pm}+\varepsilon[\tilde{\Theta}_3^{\pm}+(2\tilde{\Theta}_1^{\pm}\tilde{\Theta}_2^{\pm}/\tilde{T}_f)]+O(\varepsilon^2)\}+O(\delta^2)\rangle$

$$\exp(-\tilde{E}_R/\tilde{T}^{\pm})=\exp(-\tilde{E}_R/\tilde{T}_f)\exp(-\tilde{\Theta}_1^{\pm})\langle\{1-(\varepsilon/\delta)[\tilde{\Theta}_2^{\pm}+O(\varepsilon)]-\varepsilon[\tilde{\Theta}_3^{\pm}+(2\tilde{\Theta}_1^{\pm}\tilde{\Theta}_2^{\pm}/\tilde{T}_f)]+\cdots\}+O(\delta)\rangle$$

$$\frac{\tilde{m}}{\tilde{r}^2}\frac{\partial\tilde{T}}{\partial\tilde{r}}-\frac{1}{\tilde{r}^2}\left\{\frac{\partial}{\partial\tilde{r}}\left(\tilde{r}^2\frac{\partial\tilde{T}}{\partial\tilde{r}}\right)+\frac{\partial}{\partial\tilde{z}}\left[(1-\tilde{z}^2)\frac{\partial\tilde{T}}{\partial\tilde{z}}\right]\right\}=-Da_R\exp(-\tilde{E}_R/\tilde{T})-\frac{\tilde{m}}{\tilde{r}_{f,0}^2+\cdots}\frac{\delta\partial[\tilde{\Theta}_1^{\pm}+(\varepsilon/\delta)\tilde{\Theta}_2^{\pm}+\varepsilon\tilde{\Theta}_3^{\pm}+\cdots]}{\delta\partial\zeta}$$

$$+\frac{1-\delta(2\zeta/\tilde{r}_{f,0})+\ldots}{\tilde{r}_{f,0}^2}\left\{\frac{\partial}{\delta\partial\zeta}\left[\tilde{r}_{f,0}^2\left[1+\delta(2\zeta/\tilde{r}_{f,0})+\ldots\right]\frac{\delta\partial[\tilde{\Theta}_1^{\pm}+(\varepsilon/\delta)\tilde{\Theta}_2^{\pm}+\varepsilon\tilde{\Theta}_3^{\pm}+\ldots]}{\delta\partial\zeta}\right]\right.+O(\delta)$$

$$=-Da_R\exp(-\tilde{E}_R/\tilde{T}_f)\exp(-\tilde{\Theta}_1^{\pm})\langle\{1-(\varepsilon/\delta)[\tilde{\Theta}_2^{\pm}+O(\varepsilon)]-\varepsilon[\tilde{\Theta}_3^{\pm}+(2\tilde{\Theta}_1^{\pm}\tilde{\Theta}_2^{\pm}/\tilde{T}_f)]+\cdots\}+O(\delta)\rangle$$

Since $\Lambda_R=\delta Da_R\exp(-\tilde{E}_R/\tilde{T}_f)$, The 3 leading order terms are :

$$\partial^2\tilde{\Theta}_1^{\pm}/\partial\zeta^2=-\Lambda_R\exp(-\tilde{\Theta}_1^{\pm})\quad;\quad\partial^2\tilde{\Theta}_2^{\pm}/\partial\zeta^2=\Lambda_R\tilde{\Theta}_2^{\pm}\exp(-\tilde{\Theta}_1^{\pm})\quad;\quad\frac{\partial^2\tilde{\Theta}_3^{\pm}}{\partial\zeta^2}-\frac{\tilde{m}-2\tilde{r}_{f,0}}{\tilde{r}_{f,0}^2}\frac{\partial\tilde{\Theta}_2^{\pm}}{\partial\zeta}=\Lambda_R\left(\tilde{\Theta}_3^{\pm}+\frac{2\tilde{\Theta}_1^{\pm}\tilde{\Theta}_2^{\pm}}{\tilde{T}_f}\right)\exp(-\tilde{\Theta}_1^{\pm})$$

(F) Matching the Outer Solutions with the Solutions in the Radiation Region

Recall: $\tilde{T}_{S,0}^{-}(\tilde{r}_{f,S})=\tilde{T}_f$, $(d\tilde{T}_{S,0}^{-}/d\tilde{r})_{\tilde{r}_{f,S}}=g_0^{-}$, $(d^2\tilde{T}_{S,0}^{-}/d\tilde{r}^2)_{\tilde{r}_{f,S}}=g_0^{-}[(\tilde{m}/\tilde{r}_{f,S}^2)-(2/\tilde{r}_{f,S})]$

$\tilde{T}_{S,1}^{-}(\tilde{r}_{f,S})=\tilde{T}_{b,S,A}$, $(d\tilde{T}_{S,1}^{-}/d\tilde{r})_{\tilde{r}_{f,S}}=\tilde{T}_{b,S,A}(\tilde{m}/\tilde{r}_{f,S}^2)$; $\tilde{T}_{S,2}^{-}(\tilde{r}_{f,S})=g_1^{-}$, $(d\tilde{T}_{S,2}^{-}/d\tilde{r})_{\tilde{r}_{f,S}}=g_1^{-}(\tilde{m}/\tilde{r}_{f,S}^2)$

$\tilde{T}_{S,0}^{+}(\tilde{r}_{f,S})=\tilde{T}_f$, $(d\tilde{T}_{S,0}^{+}/d\tilde{r})_{\tilde{r}_{f,S}}=-g_0^{+}$, $(d^2\tilde{T}_{S,0}^{+}/d\tilde{r}^2)_{\tilde{r}_{f,S}}=-g_0^{+}[(\tilde{m}/\tilde{r}_{f,S}^2)-(2/\tilde{r}_{f,S})]$

$\tilde{T}_{S,1}^{+}(\tilde{r}_{f,S})=a_{T,A}^{+}$, $(d\tilde{T}_{S,1}^{+}/d\tilde{r})_{\tilde{r}_{f,S}}=-a_{T,1}^{+}(\tilde{m}/\tilde{r}_{f,S}^2)\exp(-\tilde{m}/\tilde{r}_{f,S})$

$\tilde{T}_{S,2}^{+}(\tilde{r}_{f,S})=g_1^{+}$, $(d\tilde{T}_{S,2}^{+}/d\tilde{r})_{\tilde{r}_{f,S}}=-g_1^{+}(\tilde{m}/\tilde{r}_{f,S}^2)/[\exp(\tilde{m}/\tilde{r}_{f,S})-1]$, $g_0^{-}=(\tilde{T}_f-\tilde{T}_0)(\tilde{m}/\tilde{r}_{f,S}^2)$

$g_0^{+}=(\tilde{T}_f-\tilde{T}_\infty)(\tilde{m}/\tilde{r}_{f,S}^2)/[\exp(\tilde{m}/\tilde{r}_{f,S})-1]$, $[(g_0^{-})^2+2\Lambda_R]^{1/2}+[(g_0^{+})^2+2\Lambda_R]^{1/2}=\tilde{m}/\tilde{r}_{f,S}^2$

$g_1^{-}=-2\ell n\{[\langle 1+[2\Lambda_R/(g_0^{-})^2]\rangle^{1/2}+1]/2\}$, $g_1^{+}=-2\ell n\{[\langle 1+[2\Lambda_R/(g_0^{+})^2]\rangle^{1/2}+1]/2\}$

$g_2^{-}=[2/(3\tilde{T}_f)](\tilde{T}_{b,S,A}/g_0^{-})$, $g_2^{+}=-[2/(3\tilde{T}_f)](a_{T,A}^{+}/g_0^{+})$, $g_3^{-}=[(\tilde{m}-2\tilde{r}_{f,S})/\tilde{r}_{f,S}^2](\tilde{T}_{b,S,A}/g_0^{-})$

$g_3^{+}=-[(\tilde{m}-2\tilde{r}_{f,S})/\tilde{r}_{f,S}^2](a_{T,A}^{+}/g_0^{+})$, $\tilde{T}_{b,S,A}=\tilde{T}_{b,S,1}\exp[\tilde{m}(1-\tilde{r}_{f,S}^{-1})]$, $a_{T,A}^{+}=a_{T,1}^{+}[1-\exp(-\tilde{m}/\tilde{r}_{f,S})]$

In the common regions between the outer and radiation active regions, $\tilde{r}=\tilde{r}_f+\delta\zeta$, $\tilde{r}_f=\tilde{r}_{f,S}+\bar{\delta}(\hat{r}_{f,0}+\varepsilon\hat{r}_{f,1})P_n(\tilde{z})$

$$\tilde{T}^{\pm}=[\tilde{T}_0^{\pm}+\varepsilon\tilde{T}_1^{\pm}+O(\varepsilon^2)]+\delta[\tilde{T}_2^{\pm}+\varepsilon\tilde{T}_3^{\pm}+O(\varepsilon^2)]+O(\delta^2)$$

$$=\{[\tilde{T}_0^{\pm}(\tilde{r}_f)+(\partial\tilde{T}_0^{\pm}/\partial\tilde{r})_{\tilde{r}_f}(\delta\zeta)+O(\delta^2)]+\varepsilon[\tilde{T}_1^{\pm}(\tilde{r}_f)+(\partial\tilde{T}_1^{\pm}/\partial\tilde{r})_{\tilde{r}_f}(\delta\zeta)+O(\delta^2)]+O(\varepsilon^2)\}$$

$$+\delta\{[\tilde{T}_2^{\pm}(\tilde{r}_f)+O(\delta)]+\varepsilon[\tilde{T}_3^{\pm}(\tilde{r}_f)+O(\delta)]+O(\varepsilon^2)\}+O(\delta^2)$$

$$=\{[\tilde{T}_{S,0}^{\pm}(\tilde{r}_f)+\bar{\delta}P_n\hat{T}_0^{\pm}(\tilde{r}_f)]+\delta[(d\tilde{T}_{S,0}^{\pm}/d\tilde{r})_{\tilde{r}_f}+\bar{\delta}P_n(d\hat{T}_0^{\pm}/d\tilde{r})_{\tilde{r}_f}]\zeta\}$$

$$+\varepsilon\{[\tilde{T}_{S,1}^{\pm}(\tilde{r}_f)+\bar{\delta}P_n\hat{T}_1^{\pm}(\tilde{r}_f)]+\delta[(d\tilde{T}_{S,1}^{\pm}/d\tilde{r})_{\tilde{r}_f}+\bar{\delta}P_n(d\hat{T}_1^{\pm}/d\tilde{r})_{\tilde{r}_f}]\zeta\}$$

$$+\delta\{[\tilde{T}_{S,2}^{\pm}(\tilde{r}_f)+\bar{\delta}P_n\hat{T}_2^{\pm}(\tilde{r}_f)]+\varepsilon[\tilde{T}_{S,3}^{\pm}(\tilde{r}_f)+\bar{\delta}P_n\hat{T}_3^{\pm}(\tilde{r}_f)]\}+O(\delta^2,\varepsilon^2)$$

$$=\{\tilde{T}_{S,0}^{\pm}(\tilde{r}_{f,S}+\bar{\delta}\hat{r}_fP_n)+\bar{\delta}P_n\hat{T}_0^{\pm}(\tilde{r}_{f,S}+\bar{\delta}\hat{r}_fP_n)+\delta[(d\tilde{T}_{S,0}^{\pm}/d\tilde{r})_{\tilde{r}_{f,S}+\bar{\delta}\hat{r}_fP_n}+\bar{\delta}P_n(d\hat{T}_0^{\pm}/d\tilde{r})_{\tilde{r}_{f,S}+\bar{\delta}\hat{r}_fP_n}]\zeta\}$$

$$+\varepsilon\{\tilde{T}_{S,1}^{\pm}(\tilde{r}_{f,S}+\bar{\delta}\hat{r}_fP_n)+\bar{\delta}P_n\hat{T}_1^{\pm}(\tilde{r}_{f,S}+\bar{\delta}\hat{r}_fP_n)+\delta[(d\tilde{T}_{S,1}^{\pm}/d\tilde{r})_{\tilde{r}_{f,S}+\bar{\delta}\hat{r}_fP_n}+\bar{\delta}P_n(d\hat{T}_1^{\pm}/d\tilde{r})_{\tilde{r}_{f,S}+\bar{\delta}\hat{r}_fP_n}]\zeta\}$$

$$+\delta\{[\tilde{T}_{S,2}^{\pm}(\tilde{r}_{f,S}+\bar{\delta}\hat{r}_fP_n)+\bar{\delta}P_n\hat{T}_2^{\pm}(\tilde{r}_{f,S}+\bar{\delta}\hat{r}_fP_n)]+\varepsilon[\tilde{T}_{S,3}^{\pm}(\tilde{r}_{f,S}+\bar{\delta}\hat{r}_fP_n)+\bar{\delta}P_n\hat{T}_3^{\pm}(\tilde{r}_{f,S}+\bar{\delta}\hat{r}_fP_n)]\}+O(\delta^2,\varepsilon^2)$$

$$=\langle\{[\tilde{T}_{S,0}^{\pm}(\tilde{r}_{f,S})+\bar{\delta}(\hat{r}_{f,0}+\varepsilon\hat{r}_{f,1})P_n(d\tilde{T}_{S,0}^{\pm}/d\tilde{r})_{\tilde{r}_{f,S}}+O(\bar{\delta}^2)]+\bar{\delta}P_n[\hat{T}_0^{\pm}(\tilde{r}_{f,S})+O(\bar{\delta})]\}$$

$$+\delta\{[(d\tilde{T}_{S,0}^{\pm}/d\tilde{r})_{\tilde{r}_{f,S}}+\bar{\delta}(\hat{r}_{f,0}+\varepsilon\hat{r}_{f,1})P_n(d^2\tilde{T}_{S,0}^{\pm}/d\tilde{r}^2)_{\tilde{r}_{f,S}}+O(\bar{\delta}^2)]+\bar{\delta}P_n[(d\hat{T}_0^{\pm}/d\tilde{r})_{\tilde{r}_{f,S}}+O(\bar{\delta})]\}\zeta\rangle$$

$$+\varepsilon\langle\{[\tilde{T}_{S,1}^{\pm}(\tilde{r}_{f,S})+\bar{\delta}(\hat{r}_{f,0}+\varepsilon\hat{r}_{f,1})P_n(d\tilde{T}_{S,1}^{\pm}/d\tilde{r})_{\tilde{r}_{f,S}}+O(\bar{\delta}^2)]+\bar{\delta}P_n[\hat{T}_1^{\pm}(\tilde{r}_{f,S})+O(\bar{\delta})]\}$$

$$+\delta\{[(d\tilde{T}_{S,1}^{\pm}/d\tilde{r})_{\tilde{r}_{f,S}}+\bar{\delta}(\hat{r}_{f,0}+\varepsilon\hat{r}_{f,1})P_n(d^2\tilde{T}_{S,1}^{\pm}/d\tilde{r}^2)_{\tilde{r}_{f,S}}+O(\bar{\delta}^2)]+\bar{\delta}P_n[(d\hat{T}_1^{\pm}/d\tilde{r})_{\tilde{r}_{f,S}}+O(\bar{\delta})]\}\zeta\rangle$$

$$+\delta\{[\tilde{T}_{S,2}^{\pm}(\tilde{r}_{f,S})+\bar{\delta}(\hat{r}_{f,0}+\varepsilon\hat{r}_{f,1})P_n(d\tilde{T}_{S,2}^{\pm}/d\tilde{r})_{\tilde{r}_{f,S}}+O(\bar{\delta}^2)]+\bar{\delta}P_n[\hat{T}_2^{\pm}(\tilde{r}_{f,S})+O(\bar{\delta})]\}$$

$$+\varepsilon\delta\{[\tilde{T}^{\pm}_{S,3}(\tilde{r}_{f,S})+\bar{\delta}(\hat{r}_{f,0}+\varepsilon\hat{r}_{f,1})P_n(d\tilde{T}^{\pm}_{S,3}/d\tilde{r})_{\tilde{r}_{f,S}}+O(\bar{\delta}^2)]+\bar{\delta}P_n[\hat{T}^{\pm}_3(\tilde{r}_{f,S})+O(\bar{\delta})]\}+O(\delta^2,\varepsilon^2)$$

$$=\langle\{\tilde{T}_f+\bar{\delta}P_n[\hat{T}^{\pm}_0(\tilde{r}_{f,S})+\hat{r}_{f,0}(d\tilde{T}^{\pm}_{S,0}/d\tilde{r})_{\tilde{r}_{f,S}}+\varepsilon\hat{r}_{f,1}(d\tilde{T}^{\pm}_{S,0}/d\tilde{r})_{\tilde{r}_{f,S}}]\}$$

$$+\delta\{(d\tilde{T}^{\pm}_{S,0}/d\tilde{r})_{\tilde{r}_{f,S}}+\bar{\delta}P_n[(d\hat{T}^{\pm}_0/d\tilde{r})_{\tilde{r}_{f,S}}+\hat{r}_{f,0}(d^2\tilde{T}^{\pm}_{S,0}/d\tilde{r}^2)_{\tilde{r}_{f,S}}+\varepsilon\hat{r}_{f,1}(d^2\tilde{T}^{\pm}_{S,0}/d\tilde{r}^2)_{\tilde{r}_{f,S}}]\}\zeta\rangle$$

$$+\varepsilon\langle\{\tilde{T}^{\pm}_{S,1}(\tilde{r}_{f,S})+\bar{\delta}P_n[\hat{T}^{\pm}_1(\tilde{r}_{f,S})+\hat{r}_{f,0}(d\tilde{T}^{\pm}_{S,1}/d\tilde{r})_{\tilde{r}_{f,S}}+\varepsilon\hat{r}_{f,1}(d\tilde{T}^{\pm}_{S,1}/d\tilde{r})_{\tilde{r}_{f,S}}]\}$$

$$+\delta\{(d\tilde{T}^{\pm}_{S,1}/d\tilde{r})_{\tilde{r}_{f,S}}+\bar{\delta}P_n[(d\hat{T}^{\pm}_1/d\tilde{r})_{\tilde{r}_{f,S}}+\hat{r}_{f,0}(d^2\tilde{T}^{\pm}_{S,1}/d\tilde{r}^2)_{\tilde{r}_{f,S}}+\varepsilon\hat{r}_{f,1}(d^2\tilde{T}^{\pm}_{S,1}/d\tilde{r}^2)_{\tilde{r}_{f,S}}]\}\zeta\rangle$$

$$+\delta\{\tilde{T}^{\pm}_{S,2}(\tilde{r}_{f,S})+\bar{\delta}P_n[\hat{T}^{\pm}_2(\tilde{r}_{f,S})+\hat{r}_{f,0}(d\tilde{T}^{\pm}_{S,2}/d\tilde{r})_{\tilde{r}_{f,S}}+\varepsilon\hat{r}_{f,1}(d\tilde{T}^{\pm}_{S,2}/d\tilde{r})_{\tilde{r}_{f,S}}]\}$$

$$+\varepsilon\delta\{\tilde{T}^{\pm}_{S,3}(\tilde{r}_{f,S})+\bar{\delta}P_n[\hat{T}^{\pm}_3(\tilde{r}_{f,S})+\hat{r}_{f,0}(d\tilde{T}^{\pm}_{S,3}/d\tilde{r})_{\tilde{r}_{f,S}}+\varepsilon\hat{r}_{f,1}(d\tilde{T}^{\pm}_{S,3}/d\tilde{r})_{\tilde{r}_{f,S}}]\}+O(\delta^2,\varepsilon^2,\bar{\delta}^2)$$

$$=\{\tilde{T}_f+\bar{\delta}P_n[\hat{T}^{\pm}_0(\tilde{r}_{f,S})+\hat{r}_{f,0}(d\tilde{T}^{\pm}_{S,0}/d\tilde{r})_{\tilde{r}_{f,S}}]\}+\varepsilon\{\tilde{T}^{\pm}_{S,1}(\tilde{r}_{f,S})+\bar{\delta}P_n[\hat{T}^{\pm}_1(\tilde{r}_{f,S})+\hat{r}_{f,0}(d\tilde{T}^{\pm}_{S,1}/d\tilde{r})_{\tilde{r}_{f,S}}+\hat{r}_{f,1}(d\tilde{T}^{\pm}_{S,0}/d\tilde{r})_{\tilde{r}_{f,S}}]\}$$

$$+\delta\langle[\tilde{T}^{\pm}_{S,2}(\tilde{r}_{f,S})+(d\tilde{T}^{\pm}_{S,0}/d\tilde{r})_{\tilde{r}_{f,S}}\zeta]+\bar{\delta}P_n\{[\hat{T}^{\pm}_2(\tilde{r}_{f,S})+\hat{r}_{f,0}(d\tilde{T}^{\pm}_{S,2}/d\tilde{r})_{\tilde{r}_{f,S}}]+[(d\hat{T}^{\pm}_0/d\tilde{r})_{\tilde{r}_{f,S}}+\hat{r}_{f,0}(d^2\tilde{T}^{\pm}_{S,0}/d\tilde{r}^2)_{\tilde{r}_{f,S}}]\zeta\}\rangle$$

$$+\varepsilon\delta\langle[\tilde{T}^{\pm}_{S,3}(\tilde{r}_{f,S})+(d\tilde{T}^{\pm}_{S,1}/d\tilde{r})_{\tilde{r}_{f,S}}\zeta]+\bar{\delta}P_n\{[\hat{T}^{\pm}_3(\tilde{r}_{f,S})+\hat{r}_{f,0}(d\tilde{T}^{\pm}_{S,3}/d\tilde{r})_{\tilde{r}_{f,S}}+\hat{r}_{f,1}(d\tilde{T}^{\pm}_{S,2}/d\tilde{r})_{\tilde{r}_{f,S}}]$$

$$+[(d\hat{T}^{\pm}_1/d\tilde{r})_{\tilde{r}_{f,S}}+\hat{r}_{f,0}(d^2\tilde{T}^{\pm}_{S,1}/d\tilde{r}^2)_{\tilde{r}_{f,S}}+\hat{r}_{f,1}(d^2\tilde{T}^{\pm}_{S,0}/d\tilde{r}^2)_{\tilde{r}_{f,S}}]\zeta\}\rangle+O(\delta^2,\varepsilon^2,\bar{\delta}^2)$$

$$=\{[\tilde{T}_f-\varepsilon\tilde{\Theta}^{\pm}_2+O(\varepsilon^2)]-\delta[\tilde{\Theta}^{\pm}_1+\varepsilon\tilde{\Theta}^{\pm}_3+O(\varepsilon^2)]+O(\delta^2)\}_{\zeta\to\pm\infty}$$

(1) $\hat{T}^{\pm}_0(\tilde{r}_{f,S})+\hat{r}_{f,0}(d\tilde{T}^{\pm}_{S,0}/d\tilde{r})_{\tilde{r}_{f,S}}=0$ or $\hat{T}^{\pm}_0(\tilde{r}_{f,S})=-\hat{r}_{f,0}(d\tilde{T}^{\pm}_{S,0}/d\tilde{r})_{\tilde{r}_{f,S}}$

(2) $\tilde{\Theta}^{\pm}_1(\zeta\to\pm\infty)=\Theta^{\pm}_{S,1}(\zeta\to\pm\infty)+\bar{\delta}P_n\hat{\Theta}^{\pm}_1(\zeta\to\pm\infty)$

$$=-[\tilde{T}^{\pm}_{S,2}(\tilde{r}_{f,S})+(d\tilde{T}^{\pm}_{S,0}/d\tilde{r})_{\tilde{r}_{f,S}}\zeta]-\bar{\delta}P_n\{[\hat{T}^{\pm}_2(\tilde{r}_{f,S})+\hat{r}_{f,0}(d\tilde{T}^{\pm}_{S,2}/d\tilde{r})_{\tilde{r}_{f,S}}]$$

$$+[(d\hat{T}^{\pm}_0/d\tilde{r})_{\tilde{r}_{f,S}}+\hat{r}_{f,0}(d^2\tilde{T}^{\pm}_{S,0}/d\tilde{r}^2)_{\tilde{r}_{f,S}}]\zeta\}+O(\bar{\delta}^2)$$

$$(\partial\tilde{\Theta}^{\pm}_1/\partial\zeta)_{\zeta\to\pm\infty}=-(d\tilde{T}^{\pm}_{S,0}/d\tilde{r})_{\tilde{r}_{f,S}}-\bar{\delta}P_n[(d\hat{T}^{\pm}_0/d\tilde{r})_{\tilde{r}_{f,S}}+\hat{r}_{f,0}(d^2\tilde{T}^{\pm}_{S,0}/d\tilde{r}^2)_{\tilde{r}_{f,S}}]+O(\bar{\delta}^2)\ ;\quad(\partial^2\tilde{\Theta}^{\pm}_1/\partial\zeta^2)_{\zeta\to\pm\infty}=0$$

(3) $\tilde{\Theta}^{\pm}_2(\zeta\to\pm\infty)=\Theta^{\pm}_{S,2}(\zeta\to\pm\infty)+\bar{\delta}P_n\hat{\Theta}^{\pm}_2(\zeta\to\pm\infty)$

$$=-\tilde{T}^{\pm}_{S,1}(\tilde{r}_{f,S})-\bar{\delta}P_n[\hat{T}^{\pm}_1(\tilde{r}_{f,S})+\hat{r}_{f,0}(d\tilde{T}^{\pm}_{S,1}/d\tilde{r})_{\tilde{r}_{f,S}}+\hat{r}_{f,1}(d\tilde{T}^{\pm}_{S,0}/d\tilde{r})_{\tilde{r}_{f,S}}]+O(\bar{\delta}^2)$$

$$(\partial\tilde{\Theta}^{\pm}_2/\partial\zeta)_{\zeta\to\pm\infty}=0$$

(4) $\tilde{\Theta}^{\pm}_3(\zeta\to\pm\infty)=\Theta^{\pm}_{S,3}(\zeta\to\pm\infty)+\bar{\delta}P_n\hat{\Theta}^{\pm}_3(\zeta\to\pm\infty)$

$$=-[\tilde{T}^{\pm}_{S,3}(\tilde{r}_{f,S})+(d\tilde{T}^{\pm}_{S,1}/d\tilde{r})_{\tilde{r}_{f,S}}\zeta]-\bar{\delta}P_n\{[\hat{T}^{\pm}_3(\tilde{r}_{f,S})+\hat{r}_{f,0}(d\tilde{T}^{\pm}_{S,3}/d\tilde{r})_{\tilde{r}_{f,S}}+\hat{r}_{f,1}(d\tilde{T}^{\pm}_{S,2}/d\tilde{r})_{\tilde{r}_{f,S}}]$$

$$+[(d\hat{T}^{\pm}_1/d\tilde{r})_{\tilde{r}_{f,S}}+\hat{r}_{f,0}(d^2\tilde{T}^{\pm}_{S,1}/d\tilde{r}^2)_{\tilde{r}_{f,S}}+\hat{r}_{f,1}(d^2\tilde{T}^{\pm}_{S,0}/d\tilde{r}^2)_{\tilde{r}_{f,S}}]\zeta\}+O(\bar{\delta}^2)$$

$$(\partial\tilde{\Theta}^{\pm}_3/\partial\zeta)_{\zeta\to\pm\infty}=-(d\tilde{T}^{\pm}_{S,1}/d\tilde{r})_{\tilde{r}_{f,S}}-\bar{\delta}P_n[(d\hat{T}^{\pm}_1/d\tilde{r})_{\tilde{r}_{f,S}}+\hat{r}_{f,0}(d^2\tilde{T}^{\pm}_{S,1}/d\tilde{r}^2)_{\tilde{r}_{f,S}}+\hat{r}_{f,1}(d^2\tilde{T}^{\pm}_{S,0}/d\tilde{r}^2)_{\tilde{r}_{f,S}}]+O(\bar{\delta}^2)$$

(5) $\hat{T}^{-}_0(\tilde{r}_{f,S})=-\hat{r}_{f,0}(d\tilde{T}^{-}_{S,0}/d\tilde{r})_{\tilde{r}_{f,S}}=-\hat{r}_{f,0}\,g^{-}_0\ ;\quad\hat{T}^{-}_0=\hat{T}_{b,0}\{A_{T2}\Psi_1(\tilde{r},1)+[1-A_{T2}\Psi_1(1,1)]\Psi_2(\tilde{r},1)/\Psi_2(1,1)\}$

$$\hat{T}_{b,0}\{A_{T2}\Psi_1(\tilde{r}_{f,S},1)+[1-A_{T2}\Psi_1(1,1)]\Psi_2(\tilde{r}_{f,S},1)/\Psi_2(1,1)\}=-\hat{r}_{f,0}\,g^{-}_0$$

or $\hat{T}_{b,0}=-\hat{r}_{f,0}\,g^{-}_0/\{A_{T2}\Psi_1(\tilde{r}_{f,S},1)+[1-A_{T2}\Psi_1(1,1)]\Psi_2(\tilde{r}_{f,S},1)/\Psi_2(1,1)\}$

$$\hat{T}^{-}_0=-\hat{r}_{f,0}g^{-}_0\{A_{T2}\Psi_2(1,1)\Psi_1(\tilde{r},1)+[1-A_{T2}\Psi_1(1,1)]\Psi_2(\tilde{r},1)\}/\{A_{T2}\Psi_2(1,1)\Psi_1(\tilde{r}_{f,S},1)$$

$$+[1-A_{T2}\Psi_1(1,1)]\Psi_2(\tilde{r}_{f,S},1)\}$$

$$d\hat{T}^{-}_0/d\tilde{r}=-\hat{r}_{f,0}g^{-}_0\{A_{T2}\Psi_2(1,1)\Psi^{*}_1(\tilde{r},1)+[1-A_{T2}\Psi_1(1,1)]\Psi^{*}_2(\tilde{r},1)\}/\{A_{T2}\Psi_2(1,1)\Psi_1(\tilde{r}_{f,S},1)$$

$$+[1-A_{T2}\Psi_1(1,1)]\Psi_2(\tilde{r}_{f,S},1)\}$$

(6) $\hat{T}_0^+(\tilde{r}_{f,S}) = -\hat{r}_{f,0}(d\tilde{T}_{S,0}^+/d\tilde{r})_{\tilde{r}_{f,S}} = \hat{r}_{f,0}\,g_0^+ \quad ; \quad \hat{T}_0^+ = \hat{a}_{T,0}^+\,\Psi_2(\tilde{r},1)$

$\hat{a}_{T,0}^+\,\Psi_2(\tilde{r}_{f,S},1) = \hat{r}_{f,0}\,g_0^+ \quad \text{or} \quad \hat{a}_{T,0}^+ = \hat{r}_{f,0}\,g_0^+/\Psi_2(\tilde{r}_{f,S},1)$

$\hat{T}_0^+ = \hat{r}_{f,0}\,g_0^+\,\Psi_2(\tilde{r},1)/\Psi_2(\tilde{r}_{f,S},1) \quad ; \quad d\hat{T}_0^+/d\tilde{r} = \hat{r}_{f,0}\,g_0^+\,\Psi_2^*(\tilde{r},1)/\Psi_2(\tilde{r}_{f,S},1)$

(G) Solution of Energy Equation in the Radiation Regions

(1) $\partial^2\tilde{\Theta}_1^\pm/\partial\zeta^2 = -\Lambda_R\exp(-\tilde{\Theta}_1^\pm)$

$$\Theta_{S,1}^- = \ell n\langle 1-\{\Lambda_R/[2(g_0^-)^2]\}\exp(g_1^- + g_0^-\zeta)\rangle^2 - g_0^-\zeta - g_1^- = 2\ell n\langle 1-\{\Lambda_R/[2(g_0^-)^2]\}\exp(g_1^- + g_0^-\zeta)\rangle - g_0^-\zeta - g_1^-$$

$$d\Theta_{S,1}^-/d\zeta = -[2\Lambda_R\exp(-\Theta_{S,1}^-) + (g_0^-)^2]^{1/2} = -g_0^-[2(g_0^-)^2 + \Lambda_R\exp(g_1^- + g_0^-\zeta)]/[2(g_0^-)^2 - \Lambda_R\exp(g_1^- + g_0^-\zeta)]$$

$$\Theta_{S,1}^+ = \ell n\langle 1-\{\Lambda_R/[2(g_0^+)^2]\}\exp(g_1^+ - g_0^+\zeta)\rangle^2 + g_0^+\zeta - g_1^+ = 2\ell n\langle 1-\{\Lambda_R/[2(g_0^+)^2]\}\exp(g_1^+ - g_0^+\zeta)\rangle + g_0^+\zeta - g_1^+$$

$$d\Theta_{S,1}^+/d\zeta = [2\Lambda_R\exp(-\Theta_{S,1}^+) + (g_0^+)^2]^{1/2} = g_0^+[2(g_0^+)^2 + \Lambda_R\exp(g_1^+ - g_0^+\zeta)]/[2(g_0^+)^2 - \Lambda_R\exp(g_1^+ - g_0^+\zeta)]$$

$\zeta \to -\infty$

Define: $\hat{T}_A^- = (d\hat{T}_0^-/d\tilde{r})_{\tilde{r}_{f,S}} + \hat{r}_{f,0}\,g_0^-[(\tilde{m}/\tilde{r}_{f,S}^2) - (2/\tilde{r}_{f,S})] \quad ; \quad \hat{T}_B^- = \hat{T}_2^-(\tilde{r}_{f,S}) + \hat{r}_{f,0}\,g_1^-(\tilde{m}/\tilde{r}_{f,S}^2)$

$$\hat{g}_0^- = g_0^- + \bar{\delta}P_n\hat{T}_A^- + O(\bar{\delta}^2) = g_0^-[1 + \bar{\delta}P_n(\hat{T}_A^-/g_0^-) + O(\bar{\delta}^2)] \quad ; \quad \hat{g}_1^- = g_1^- + \bar{\delta}P_n\hat{T}_B^- + O(\bar{\delta}^2)$$

$$(\hat{g}_0^-)^2 = (g_0^-)^2 + \bar{\delta}P_n\,2g_0^-\hat{T}_A^- + O(\bar{\delta}^2) = (g_0^-)^2[1 + \bar{\delta}P_n(2/g_0^-)\hat{T}_A^- + O(\bar{\delta}^2)]$$

$$1/(\hat{g}_0^-)^2 = \{(g_0^-)^2[1 + \bar{\delta}P_n(2/g_0^-)\hat{T}_A^- + O(\bar{\delta}^2)]\}^{-1} = (g_0^-)^{-2}[1 - \bar{\delta}P_n(2/g_0^-)\hat{T}_A^- + O(\bar{\delta}^2)]$$

$$\exp(\hat{g}_0^-\zeta) = \exp\{[g_0^- + \bar{\delta}P_n\hat{T}_A^- + O(\bar{\delta}^2)]\zeta\} = \exp(g_0^-\zeta)\exp[\bar{\delta}P_n\hat{T}_A^-\zeta + O(\bar{\delta}^2)]$$
$$= \exp(g_0^-\zeta)[1 + \bar{\delta}P_n\hat{T}_A^-\zeta + O(\bar{\delta}^2)]$$

$$\exp(\hat{g}_1^-) = \exp[g_1^- + \bar{\delta}P_n\hat{T}_B^- + O(\bar{\delta}^2)] = \exp(g_1^-)\exp[\bar{\delta}P_n\hat{T}_B^- + O(\bar{\delta}^2)] = \exp(g_1^-)[1 + \bar{\delta}P_n\hat{T}_B^- + O(\bar{\delta}^2)]$$

$$\tilde{\Theta}_1^- \to -[\tilde{T}_{S,2}^-(\tilde{r}_{f,S}) + (d\tilde{T}_{S,0}^-/d\tilde{r})_{\tilde{r}_{f,S}}\zeta] - \bar{\delta}P_n\{[\hat{T}_2^-(\tilde{r}_{f,S}) + \hat{r}_{f,0}(d\tilde{T}_{S,2}^-/d\tilde{r})_{\tilde{r}_{f,S}}] + [(d\hat{T}_0^-/d\tilde{r})_{\tilde{r}_{f,S}} + \hat{r}_{f,0}(d^2\tilde{T}_{S,0}^-/d\tilde{r}^2)_{\tilde{r}_{f,S}}]\zeta\} + O(\bar{\delta}^2)$$
$$= -(g_1^- + g_0^-\zeta) - \bar{\delta}P_n\langle[\hat{T}_2^-(\tilde{r}_{f,S}) + \hat{r}_{f,0}\,g_1^-(\tilde{m}/\tilde{r}_{f,S}^2)] + \{(d\hat{T}_0^-/d\tilde{r})_{\tilde{r}_{f,S}} + \hat{r}_{f,0}g_0^-[(\tilde{m}/\tilde{r}_{f,S}^2) - (2/\tilde{r}_{f,S})]\}\zeta\rangle + O(\bar{\delta}^2)$$
$$= -(g_1^- + g_0^-\zeta) - \bar{\delta}P_n(\hat{T}_B^- + \hat{T}_A^-\zeta) + O(\bar{\delta}^2) = -\hat{g}_1^- - \hat{g}_0^-\zeta$$

$$\partial\tilde{\Theta}_1^-/\partial\zeta \to -g_0^- - \bar{\delta}P_n\hat{T}_A^- + O(\bar{\delta}^2) = -\hat{g}_0^-$$

Since the equation and matching conditions are similar to those of the steady state problem, we have:

$$\tilde{\Theta}_1^- = 2\ell n\{1 - [\Lambda_R/\langle 2(\hat{g}_0^-)^2\rangle]\exp(\hat{g}_1^- + \hat{g}_0^-\zeta)\} - \hat{g}_0^-\zeta - \hat{g}_1^-$$
$$= 2\ell n\langle 1 - \{\Lambda_R/[2(g_0^-)^2]\}[1 - \bar{\delta}P_n(2/g_0^-)\hat{T}_A^- + O(\bar{\delta}^2)]\exp(g_1^-)[1 + \bar{\delta}P_n\hat{T}_B^- + O(\bar{\delta}^2)]\exp(g_0^-\zeta)$$
$$\cdot[1 + \bar{\delta}P_n\hat{T}_A^-\zeta + O(\bar{\delta}^2)]\rangle - [g_0^- + \bar{\delta}P_n\hat{T}_A^- + O(\bar{\delta}^2)]\zeta - [g_1^- + \bar{\delta}P_n\hat{T}_B^- + O(\bar{\delta}^2)]$$
$$= 2\ell n\langle 1 - \{\Lambda_R/[2(g_0^-)^2]\}\exp(g_1^- + g_0^-\zeta)\{1 + \bar{\delta}P_n[\hat{T}_B^- - (2/g_0^-)\hat{T}_A^- + \hat{T}_A^-\zeta] + O(\bar{\delta}^2)\}\rangle - (g_1^- + g_0^-\zeta)$$
$$-\bar{\delta}P_n(\hat{T}_B^- + \hat{T}_A^-\zeta) + O(\bar{\delta}^2)$$
$$= 2\ell n\langle 1 - \{\Lambda_R/[2(g_0^-)^2]\}\exp(g_1^- + g_0^-\zeta)\rangle - \bar{\delta}P_n 2[\hat{T}_B^- - (2/g_0^-)\hat{T}_A^- + \hat{T}_A^-\zeta]/\{[2(g_0^-)^2/\Lambda_R]\exp[-(g_1^- + g_0^-\zeta)] - 1\}$$
$$-(g_1^- + g_0^-\zeta) - \bar{\delta}P_n(\hat{T}_B^- + \hat{T}_A^-\zeta) + O(\bar{\delta}^2)$$
$$= \Theta_{S,1}^- - \bar{\delta}P_n\langle(\hat{T}_B^- + \hat{T}_A^-\zeta) + 2[\hat{T}_B^- - (2/g_0^-)\hat{T}_A^- + \hat{T}_A^-\zeta]/\{[2(g_0^-)^2/\Lambda_R]\exp[-(g_1^- + g_0^-\zeta)] - 1\}\rangle + O(\bar{\delta}^2)$$

$$\partial\tilde{\Theta}_1^-/\partial\zeta = (d\Theta_{S,1}^-/d\zeta) - \bar{\delta}P_n\langle\hat{T}_A^- + 2\hat{T}_A^-/\{[2(g_0^-)^2/\Lambda_R]\exp[-(g_1^- + g_0^-\zeta)] - 1\}$$
$$-2[\hat{T}_B^- - (2/g_0^-)\hat{T}_A^- + \hat{T}_A^-\zeta][2(g_0^-)^2/\Lambda_R]\exp[-(g_1^- + g_0^-\zeta)](-g_0^-)/\{[2(g_0^-)^2/\Lambda_R]\exp[-(g_1^- + g_0^-\zeta)] - 1\}^2\rangle + O(\bar{\delta}^2)$$
$$= \frac{d\Theta_{S,1}^-}{d\zeta} - \bar{\delta}P_n\left\{\hat{T}_A^-\frac{2(g_0^-)^2 + \Lambda_R\exp(g_1^- + g_0^-\zeta)}{2(g_0^-)^2 - \Lambda_R\exp(g_1^- + g_0^-\zeta)} + [\hat{T}_B^- - (2/g_0^-)\hat{T}_A^- + \hat{T}_A^-\zeta]\frac{4(g_0^-)^3\Lambda_R\exp(g_1^- + g_0^-\zeta)}{[2(g_0^-)^2 - \Lambda_R\exp(g_1^- + g_0^-\zeta)]^2}\right\} + O(\bar{\delta}^2)$$
$$= \frac{d\Theta_{S,1}^-}{d\zeta}\left\langle 1 + \bar{\delta}P_n\left\{\frac{\hat{T}_A^-}{g_0^-} + \frac{4(g_0^-)^2\Lambda_R\exp(g_1^- + g_0^-\zeta)[\hat{T}_B^- - (2/g_0^-)\hat{T}_A^- + \hat{T}_A^-\zeta]}{[2(g_0^-)^2 + \Lambda_R\exp(g_1^- + g_0^-\zeta)][2(g_0^-)^2 - \Lambda_R\exp(g_1^- + g_0^-\zeta)]}\right\} + O(\bar{\delta}^2)\right\rangle$$

$\zeta \to \infty$

Define: $\hat{T}_A^+ = (d\hat{T}_0^+/d\tilde{r})_{\tilde{r}_{f,S}} - \hat{r}_{f,0}g_0^+[(\tilde{m}/\tilde{r}_{f,S}^2) - (2/\tilde{r}_{f,S})] \quad ; \quad \hat{T}_B^+ = \hat{T}_2^+(\tilde{r}_{f,S}) - \hat{r}_{f,0}\,g_1^+(\tilde{m}/\tilde{r}_{f,S}^2)/[\exp(\tilde{m}/\tilde{r}_{f,S}) - 1]$

$$\hat{g}_0^+ = g_0^+ - \bar{\delta} P_n \hat{T}_A^+ + O(\bar{\delta}^2) = g_0^+[1-\bar{\delta} P_n (\hat{T}_A^+ / g_0^+) + O(\bar{\delta}^2)] \quad ; \quad \hat{g}_1^+ = g_1^+ + \bar{\delta} P_n \hat{T}_B^+ + O(\bar{\delta}^2)$$

$$(\hat{g}_0^+)^2 = (g_0^+)^2 - \bar{\delta} P_n 2 g_0^+ \hat{T}_A^+ + O(\bar{\delta}^2) = (g_0^+)^2 [1-\bar{\delta} P_n (2/g_0^+)\hat{T}_A^+ + O(\bar{\delta}^2)]$$

$$1/(\hat{g}_0^+)^2 = \{(g_0^+)^2[1-\bar{\delta} P_n (2/g_0^+)\hat{T}_A^+ + O(\bar{\delta}^2)]\}^{-1} = (g_0^+)^{-2}[1+\bar{\delta} P_n (2/g_0^+)\hat{T}_A^+ + O(\bar{\delta}^2)]$$

$$\exp(-\hat{g}_0^+\zeta) = \exp\{[-g_0^+ + \bar{\delta} P_n \hat{T}_A^+ + O(\bar{\delta}^2)]\zeta\} = \exp(-g_0^+\zeta)\exp[\bar{\delta} P_n \hat{T}_A^+\zeta + O(\bar{\delta}^2)]$$
$$= \exp(-g_0^+\zeta)[1+\bar{\delta} P_n \hat{T}_A^+\zeta + O(\bar{\delta}^2)]$$

$$\exp(\hat{g}_1^+) = \exp[g_1^+ + \bar{\delta} P_n \hat{T}_B^+ + O(\bar{\delta}^2)] = \exp(g_1^+)\exp[\bar{\delta} P_n \hat{T}_B^+ + O(\bar{\delta}^2)] = \exp(g_1^+)[1+\bar{\delta} P_n \hat{T}_B^+ + O(\bar{\delta}^2)]$$

$$\tilde{\Theta}_1^+ \to -[\tilde{T}_{S,2}^+(\tilde{r}_{f,S}) + (d\tilde{T}_{S,0}^+/d\tilde{r})_{\tilde{r}_{f,S}}\zeta] - \bar{\delta} P_n\{[\hat{T}_2^+(\tilde{r}_{f,S}) + \hat{r}_{f,0}(d\tilde{T}_{S,2}^+/d\tilde{r})_{\tilde{r}_{f,S}}] + [(d\hat{T}_0^+/d\tilde{r})_{\tilde{r}_{f,S}} + \hat{r}_{f,0}(d^2\tilde{T}_{S,0}^+/d\tilde{r}^2)_{\tilde{r}_{f,S}}]\zeta\} + O(\bar{\delta}^2)$$
$$= -(g_1^+ - g_0^+\zeta) - \bar{\delta} P_n \langle\{\hat{T}_2^+(\tilde{r}_{f,S}) - \hat{r}_{f,0} g_1^+(\tilde{m}/\tilde{r}_{f,S}^2)/[\exp(\tilde{m}/\tilde{r}_{f,S})-1]\} + \{(d\hat{T}_0^+/d\tilde{r})_{\tilde{r}_{f,S}}$$
$$-\hat{r}_{f,0} g_0^+[(\tilde{m}/\tilde{r}_{f,S}^2) - (2/\tilde{r}_{f,S})]\}\zeta\rangle + O(\bar{\delta}^2)$$
$$= -(g_1^+ - g_0^+\zeta) - \bar{\delta} P_n (\hat{T}_B^+ + \hat{T}_A^+\zeta) + O(\bar{\delta}^2) = -\hat{g}_1^+ + \hat{g}_0^+\zeta$$

$$\partial\tilde{\Theta}_1^+/\partial\zeta \to g_0^+ - \bar{\delta} P_n \hat{T}_A^- + O(\bar{\delta}^2) = \hat{g}_0^+$$

Since the equation and matching conditions are similar to those of the steady state problem, we have:

$$\tilde{\Theta}_1^+ = \ell n\{1-[\Lambda_R/\langle 2(\hat{g}_0^+)^2\rangle]\exp(\hat{g}_1^+ - \hat{g}_0^+\zeta)\}^2 + \hat{g}_0^+\zeta - \hat{g}_1^+$$
$$= 2\ell n\langle 1-\{\Lambda_R/[2(g_0^+)^2]\}[1+\bar{\delta} P_n(2/g_0^+)\hat{T}_A^+ + O(\bar{\delta}^2)\}\exp(g_1^+)[1+\bar{\delta} P_n\hat{T}_B^+ + O(\bar{\delta}^2)]\exp(-g_0^+\zeta)$$
$$\cdot[1+\bar{\delta} P_n \hat{T}_A^+\zeta + O(\bar{\delta}^2)]\rangle + [g_0^+ - \bar{\delta} P_n \hat{T}_A^+ + O(\bar{\delta}^2)]\zeta - [g_1^+ + \bar{\delta} P_n \hat{T}_B^+ + O(\bar{\delta}^2)]$$
$$= 2\ell n\langle 1-\{\Lambda_R/[2(g_0^+)^2]\}\exp(g_1^+ - g_0^+\zeta)\{1+\bar{\delta} P_n[\hat{T}_B^+ + (2/g_0^+)\hat{T}_A^+ + \hat{T}_A^+\zeta] + O(\bar{\delta}^2)\}\rangle - (g_1^+ - g_0^+\zeta)$$
$$-\bar{\delta} P_n(\hat{T}_B^+ + \hat{T}_A^+\zeta) + O(\bar{\delta}^2)$$
$$= 2\ell n\langle 1-\{\Lambda_R/[2(g_0^+)^2]\}\exp(g_1^+ - g_0^+\zeta)\rangle - \bar{\delta} P_n 2[\hat{T}_B^+ + (2/g_0^+)\hat{T}_A^+ + \hat{T}_A^+\zeta]/\{[2(g_0^+)^2/\Lambda_R]$$
$$\cdot\exp(g_0^+\zeta - g_1^+) - 1\} + (g_0^+\zeta - g_1^+) - \bar{\delta} P_n(\hat{T}_B^+ + \hat{T}_A^+\zeta) + O(\bar{\delta}^2)$$
$$= \Theta_{S,1}^+ - \bar{\delta} P_n\langle(\hat{T}_B^+ + \hat{T}_A^+\zeta) + 2[\hat{T}_B^+ + (2/g_0^+)\hat{T}_A^+ + \hat{T}_A^+\zeta]/\{[2(g_0^+)^2/\Lambda_R]\exp(g_0^+\zeta - g_1^+) - 1\}\rangle + O(\bar{\delta}^2)$$

$$\partial\tilde{\Theta}_1^+/\partial\zeta = (d\Theta_{S,1}^+/d\zeta) - \bar{\delta} P_n\langle\hat{T}_A^+ + 2\hat{T}_A^+/\{[2(g_0^+)^2/\Lambda_R]\exp(g_0^+\zeta - g_1^+) - 1\}$$
$$-2[\hat{T}_B^+ + (2/g_0^+)\hat{T}_A^+ + \hat{T}_A^+\zeta][2(g_0^+)^2/\Lambda_R]\exp(g_0^+\zeta - g_1^-)g_0^+/\{[2(g_0^+)^2/\Lambda_R]\exp(g_0^+\zeta - g_1^+)] - 1\}^2\rangle + O(\bar{\delta}^2)$$
$$= \frac{d\Theta_{S,1}^+}{d\zeta} - \bar{\delta} P_n\left\{\hat{T}_A^+\frac{2(g_0^+)^2 + \Lambda_R\exp(g_1^+ - g_0^+\zeta)}{2(g_0^+)^2 - \Lambda_R\exp(g_1^+ - g_0^+\zeta)} - [\hat{T}_B^+ + (2/g_0^+)\hat{T}_A^+ + \hat{T}_A^+\zeta]\frac{4(g_0^+)^3\Lambda_R\exp(g_1^+ - g_0^+\zeta)}{[2(g_0^+)^2 - \Lambda_R\exp(g_1^+ - g_0^+\zeta)]^2}\right\} + O(\bar{\delta}^2)$$
$$= \frac{d\Theta_{S,1}^+}{d\zeta}\left\langle 1-\bar{\delta} P_n\left\{\frac{\hat{T}_A^+}{g_0^+} - \frac{4(g_0^+)^2\Lambda_R\exp(g_1^+ - g_0^+\zeta)[\hat{T}_B^+ + (2/g_0^-)\hat{T}_A^+ + \hat{T}_A^+\zeta]}{[2(g_0^+)^2 + \Lambda_R\exp(g_1^+ - g_0^+\zeta)][2(g_0^+)^2 - \Lambda_R\exp(g_1^+ - g_0^+\zeta)]}\right\} + O(\bar{\delta}^2)\right\rangle$$

(2) $\partial^2\tilde{\Theta}_2^\pm/\partial\zeta^2 = \Lambda_R\tilde{\Theta}_2^\pm\exp(-\tilde{\Theta}_1^\pm)$

$$\Theta_{S,2}^- = (\tilde{T}_{b,S,A}/g_0^-)(d\Theta_{S,1}^-/d\zeta) = -\tilde{T}_{b,S,A}[2(g_0^-)^2 + \Lambda_R\exp(g_1^- + g_0^-\zeta)]/[2(g_0^-)^2 - \Lambda_R\exp(g_1^- + g_0^-\zeta)]$$

$$\Theta_{S,2}^+ = -(a_{T,A}^+/g_0^+)(d\Theta_{S,1}^+/d\zeta) = -a_{T,A}^+[2(g_0^+)^2 + \Lambda_R\exp(g_1^+ - g_0^+\zeta)]/[2(g_0^+)^2 - \Lambda_R\exp(g_1^+ - g_0^+\zeta)]$$

$$\hat{T}_{b,S,A}/\hat{g}_0^- = \tilde{T}_{b,S,A}[1+\bar{\delta} P_n\hat{T}_C^- + O(\bar{\delta}^2)]/g_0^-[1+\bar{\delta} P_n(\hat{T}_A^-/g_0^-) + O(\bar{\delta}^2)]\}$$
$$= (\tilde{T}_{b,S,A}/g_0^-)\{1+\bar{\delta} P_n[\hat{T}_C^- - (\hat{T}_A^-/g_0^-)] + O(\bar{\delta}^2)\}$$

$$\hat{a}_{T,A}^+/\hat{g}_0^+ = a_{T,A}^+[1+\bar{\delta} P_n\hat{T}_C^+ + O(\bar{\delta}^2)]/\{g_0^+[1-\bar{\delta} P_n(\hat{T}_A^+/g_0^+) + O(\bar{\delta}^2)]\}$$
$$= (a_{T,A}^+/g_0^+)\{1+\bar{\delta} P_n[\hat{T}_C^+ + (\hat{T}_A^+/g_0^+)] + O(\bar{\delta}^2)\}$$

Define: $\hat{T}_C^- = \{[\hat{T}_1^-(\tilde{r}_{f,S}) + \hat{r}_{f,1} g_0^-]/\tilde{T}_{b,S,A}\} + \hat{r}_{f,0}(\tilde{m}/\tilde{r}_{f,S}^2)$;

$$\hat{T}_C^+ = [\hat{T}_1^+(\tilde{r}_{f,S}) - a_{T,1}^+\hat{r}_{f,0}(\tilde{m}/\tilde{r}_{f,S}^2)\exp(-\tilde{m}/\tilde{r}_{f,S}) - \hat{r}_{f,1} g_0^+]/a_{T,A}^+$$

$$\hat{T}_{b,S,A} = \tilde{T}_{b,S,A}[1+\bar{\delta} P_n\hat{T}_C^- + O(\bar{\delta}^2)] \quad ; \quad \hat{a}_{T,A}^+ = a_{T,A}^+[1+\bar{\delta} P_n\hat{T}_C^+ + O(\bar{\delta}^2)]$$

$\zeta \to -\infty$: $\partial\tilde{\Theta}_2^-/\partial\zeta \to 0$, $\partial\tilde{\Theta}_1^-/\partial\zeta \to -\hat{g}_0^- = -g_0^- - \bar{\delta} P_n\hat{T}_A^- + O(\bar{\delta}^2) = -g_0^-[1+\bar{\delta} P_n(\hat{T}_A^-/g_0^-) + O(\bar{\delta}^2)]$, $\partial^2\tilde{\Theta}_1^-/\partial\zeta^2 \to 0$

$$\tilde{\Theta}_2^- \to -\tilde{T}_{S,1}^-(\tilde{r}_{f,S}) - \bar{\delta} P_n [\hat{T}_1^-(\tilde{r}_{f,S}) + \hat{r}_{f,0}(d\tilde{T}_{S,1}^-/d\tilde{r})_{\tilde{r}_{f,S}} + \hat{r}_{f,1}(d\tilde{T}_{S,0}^-/d\tilde{r})_{\tilde{r}_{f,S}}] + O(\bar{\delta}^2)$$

$$= -\tilde{T}_{b,S,A} - \bar{\delta} P_n [\hat{T}_1^-(\tilde{r}_{f,S}) + \hat{r}_{f,0}\tilde{T}_{b,S,A}(\tilde{m}/\tilde{r}_{f,S}^2) + \hat{r}_{f,1} g_0^-] + O(\bar{\delta}^2)$$

$$= -\tilde{T}_{b,S,A}[1 + \bar{\delta} P_n \hat{T}_C^- + O(\bar{\delta}^2)] = -\hat{T}_{b,S,A}$$

Since the equation and matching conditions are similar to those of the steady state problem, we have:

$$\tilde{\Theta}_2^- = (\hat{T}_{b,S,A}/\hat{g}_0^-)(\partial\tilde{\Theta}_1^-/\partial\zeta)$$

$$= \frac{\tilde{T}_{b,S,A}}{g_0^-}\left[1 + \bar{\delta} P_n\left(\hat{T}_C^- - \frac{\hat{T}_A^-}{g_0^-}\right)\right]\frac{d\Theta_{S,1}^-}{d\zeta}\left\langle 1 + \bar{\delta} P_n\left\{\frac{\hat{T}_A^-}{g_0^-} + \frac{4(g_0^-)^2\Lambda_R \exp(g_1^- + g_0^-\zeta)[\hat{T}_B^- - (2/g_0^-)\hat{T}_A^- + \hat{T}_A^-\zeta]}{[2(g_0^-)^2 + \Lambda_R\exp(g_1^- + g_0^-\zeta)][2(g_0^-)^2 - \Lambda_R\exp(g_1^- + g_0^-\zeta)]}\right\}\right\rangle + O(\bar{\delta}^2)$$

$$= \Theta_{S,2}^-\left\langle 1 + \bar{\delta} P_n\left\{\hat{T}_C^- + \frac{4(g_0^-)^2\Lambda_R \exp(g_1^- + g_0^-\zeta)[\hat{T}_B^- - (2/g_0^-)\hat{T}_A^- + \hat{T}_A^-\zeta]}{[2(g_0^-)^2 + \Lambda_R\exp(g_1^- + g_0^-\zeta)][2(g_0^-)^2 - \Lambda_R\exp(g_1^- + g_0^-\zeta)]}\right\} + O(\bar{\delta}^2)\right\rangle$$

$$\partial\tilde{\Theta}_2^-/\partial\zeta = (\hat{T}_{b,S,A}/\hat{g}_0^-)(\partial^2\tilde{\Theta}_1^-/\partial\zeta^2)$$

$$\zeta \to \infty:\ \partial\tilde{\Theta}_2^+/\partial\zeta \to 0\ ,\ \partial^2\tilde{\Theta}_1^+/\partial\zeta^2 \to 0\ ,\ \partial\tilde{\Theta}_1^+/\partial\zeta \to \hat{g}_0^+ = g_0^+ - \bar{\delta}P_n\hat{T}_A^+ + O(\bar{\delta}^2) = g_0^+[1 - \bar{\delta}P_n(\hat{T}_A^+/g_0^+) + O(\bar{\delta}^2)]$$

$$\tilde{\Theta}_2^+ \to -\tilde{T}_{S,1}^+(\tilde{r}_{f,S}) - \bar{\delta} P_n [\hat{T}_1^+(\tilde{r}_{f,S}) + \hat{r}_{f,0}(d\tilde{T}_{S,1}^+/d\tilde{r})_{\tilde{r}_{f,S}} + \hat{r}_{f,1}(d\tilde{T}_{S,0}^+/d\tilde{r})_{\tilde{r}_{f,S}}] + O(\bar{\delta}^2)$$

$$= -a_{T,A}^+ - \bar{\delta}P_n[\hat{T}_1^+(\tilde{r}_{f,S}) - a_{T,1}^+\hat{r}_{f,0}(\tilde{m}/\tilde{r}_{f,S}^2)\exp(-\tilde{m}/\tilde{r}_{f,S}) - \hat{r}_{f,1}g_0^+] + O(\bar{\delta}^2)$$

$$= -a_{T,A}^+[1 + \bar{\delta}P_n\hat{T}_C^+ + O(\bar{\delta}^2)] = -\hat{a}_{T,A}^+$$

Since the equation and matching conditions are similar to those of the steady state problem, we have:

$$\tilde{\Theta}_2^+ = -(\hat{a}_{T,A}^+/\hat{g}_0^+)(d\tilde{\Theta}_1^+/d\zeta)$$

$$= \frac{-a_{T,A}^+}{g_0^+}\left[1 + \bar{\delta} P_n\left(\hat{T}_C^+ + \frac{\hat{T}_A^+}{g_0^+}\right)\right]\frac{d\Theta_{S,1}^+}{d\zeta}\left\langle 1 - \bar{\delta} P_n\left\{\frac{\hat{T}_A^+}{g_0^+} - \frac{4(g_0^+)^2\Lambda_R \exp(g_1^+ - g_0^+\zeta)[\hat{T}_B^+ + (2/g_0^+)\hat{T}_A^+ + \hat{T}_A^+\zeta]}{[2(g_0^+)^2 + \Lambda_R\exp(g_1^+ - g_0^+\zeta)][2(g_0^+)^2 - \Lambda_R\exp(g_1^+ - g_0^+\zeta)]}\right\}\right\rangle + O(\bar{\delta}^2)$$

$$= \Theta_{S,2}^+\left\langle 1 + \bar{\delta} P_n\left\{\hat{T}_C^+ + \frac{4(g_0^+)^2\Lambda_R \exp(g_1^+ - g_0^+\zeta)[\hat{T}_B^+ + (2/g_0^+)\hat{T}_A^+ + \hat{T}_A^+\zeta]}{[2(g_0^+)^2 + \Lambda_R\exp(g_1^+ - g_0^+\zeta)][2(g_0^+)^2 - \Lambda_R\exp(g_1^+ - g_0^+\zeta)]}\right\} + O(\bar{\delta}^2)\right\rangle$$

$$\partial\tilde{\Theta}_2^+/\partial\zeta = -(\hat{a}_{T,A}^+/\hat{g}_0^+)(\partial^2\tilde{\Theta}_1^+/\partial\zeta^2)$$

(3) $$\frac{\partial^2\tilde{\Theta}_3^\pm}{\partial\zeta^2} - \frac{\tilde{m} - 2\tilde{r}_{f,0}}{\tilde{r}_{f,0}^2}\frac{\partial\tilde{\Theta}_2^\pm}{\partial\zeta} = \Lambda_R\left(\tilde{\Theta}_3^\pm + \frac{2\tilde{\Theta}_1^\pm\tilde{\Theta}_2^\pm}{\tilde{T}_f}\right)\exp(-\tilde{\Theta}_1^\pm) = \Lambda_R\tilde{\Theta}_3^\pm\exp(-\tilde{\Theta}_1^\pm) + \frac{2}{\tilde{T}_f}\Lambda_R\tilde{\Theta}_1^\pm\tilde{\Theta}_2^\pm\exp(-\tilde{\Theta}_1^\pm)$$

$$\partial\tilde{\Theta}_2^-/\partial\zeta = (\hat{T}_{b,S,A}/\hat{g}_0^-)(\partial^2\tilde{\Theta}_1^-/\partial\zeta^2) \quad ; \quad \partial\tilde{\Theta}_2^+/\partial\zeta = -(\hat{a}_{T,A}^+/\hat{g}_0^+)(\partial^2\tilde{\Theta}_1^+/\partial\zeta^2)$$

$$(m - 2\tilde{r}_{f,0})/\tilde{r}_{f,0}^2 = (m - 2\tilde{r}_{f,S} - \bar{\delta}2\hat{r}_{f,0}P_n)[1 - \bar{\delta}(2\hat{r}_{f,0}P_n/\tilde{r}_{f,S}) + O(\bar{\delta}^2)]/\tilde{r}_{f,S}^2$$

$$= (m - 2\tilde{r}_{f,S})\{1 - \bar{\delta}P_n[2\hat{r}_{f,0}/(m - 2\tilde{r}_{f,S})]\}[1 - \bar{\delta}P_n(2\hat{r}_{f,0}/\tilde{r}_{f,S}) + O(\bar{\delta}^2)]/\tilde{r}_{f,S}^2$$

$$= [(m - 2\tilde{r}_{f,S})/\tilde{r}_{f,S}^2]\{1 - \bar{\delta}P_n 2\hat{r}_{f,0}[(m - 2\tilde{r}_{f,S})^{-1} + \tilde{r}_{f,S}^{-1}] + O(\bar{\delta}^2)\}$$

Define $\hat{g}_2^- = [2/(3\tilde{T}_f)](\hat{T}_{b,S,A}/\hat{g}_0^-) = [2/(3\tilde{T}_f)](\tilde{T}_{b,S,A}/g_0^-)\{1 + \bar{\delta}P_n[\hat{T}_C^- - (\hat{T}_A^-/g_0^-)] + O(\bar{\delta}^2)\}$

$$= g_2^-\{1 + \bar{\delta}P_n[\hat{T}_C^- - (\hat{T}_A^-/g_0^-)] + O(\bar{\delta}^2)\}$$

$$\hat{g}_2^+ = -[2/(3\tilde{T}_f)](\hat{a}_{T,A}^+/\hat{g}_0^+) = -[2/(3\tilde{T}_f)](a_{T,A}^+/g_0^+)\{1 + \bar{\delta}P_n[\hat{T}_C^+ + (\hat{T}_A^+/g_0^+)] + O(\bar{\delta}^2)\}$$

$$= g_2^+\{1 + \bar{\delta}P_n[\hat{T}_C^+ + (\hat{T}_A^+/g_0^+)] + O(\bar{\delta}^2)\}$$

$$\hat{g}_3^- = [(m - 2\tilde{r}_{f,0})/\tilde{r}_{f,0}^2](\hat{T}_{b,S,A}/\hat{g}_0^-)$$

$$= [(m - 2\tilde{r}_{f,S})/\tilde{r}_{f,S}^2]\{1 - \bar{\delta}P_n 2\hat{r}_{f,0}[(m - 2\tilde{r}_{f,S})^{-1} + \tilde{r}_{f,S}^{-1}] + O(\bar{\delta}^2)\}(\tilde{T}_{b,S,A}/g_0^-)\{1 + \bar{\delta}P_n[\hat{T}_C^- - (\hat{T}_A^-/g_0^-)] + O(\bar{\delta}^2)\}$$

$$= g_3^-\langle 1 + \bar{\delta}P_n\{\hat{T}_C^- - (\hat{T}_A^-/g_0^-) - 2\hat{r}_{f,0}[(m - 2\tilde{r}_{f,S})^{-1} + \tilde{r}_{f,S}^{-1}]\} + O(\bar{\delta}^2)\rangle$$

$$\hat{g}_3^+ = -[(m - 2\tilde{r}_{f,0})/\tilde{r}_{f,0}^2](\hat{a}_{T,A}^+/\hat{g}_0^+)$$

$$= -[(m - 2\tilde{r}_{f,S})/\tilde{r}_{f,S}^2]\{1 - \bar{\delta}P_n 2\hat{r}_{f,0}[(m - 2\tilde{r}_{f,S})^{-1} + \tilde{r}_{f,S}^{-1}] + O(\bar{\delta}^2)\}(a_{T,A}^+/g_0^+)\{1 + \bar{\delta}P_n[\hat{T}_C^+ + (\hat{T}_A^+/g_0^+)] + O(\bar{\delta}^2)\}$$

$$= g_3^+\langle 1 + \bar{\delta}P_n\{\hat{T}_C^+ + (\hat{T}_A^+/g_0^+) - 2\hat{r}_{f,0}[(m - 2\tilde{r}_{f,S})^{-1} + \tilde{r}_{f,S}^{-1}]\} + O(\bar{\delta}^2)\rangle$$

$$\tilde{T}_D^- = \tilde{T}_{S,3}^-(\tilde{r}_{f,S}) + \bar{\delta}P_n[\hat{T}_3^-(\tilde{r}_{f,S}) + \hat{r}_{f,0}(d\tilde{T}_{S,3}^-/d\tilde{r})_{\tilde{r}_{f,S}} + \hat{r}_{f,1}g_1^-(\tilde{m}/\tilde{r}_{f,S}^2)] + O(\bar{\delta}^2)$$

$$\tilde{T}_E^- = \tilde{T}_{b,S,A}(\tilde{m}/\tilde{r}_{f,S}^2) + \bar{\delta}P_n\{(d\hat{T}_1^-/d\tilde{r})_{\tilde{r}_{f,S}} + \hat{r}_{f,0}(d^2\tilde{T}_{S,1}^-/d\tilde{r}^2)_{\tilde{r}_{f,S}} + \hat{r}_{f,1}g_0^-[(\tilde{m}/\tilde{r}_{f,S}^2) - (2/\tilde{r}_{f,S})]\} + O(\bar{\delta}^2)$$

$$\tilde{T}_D^+ = \tilde{T}_{S,3}^+(\tilde{r}_{f,S}) + \bar{\delta}P_n\{[\hat{T}_3^+(\tilde{r}_{f,S}) + \hat{r}_{f,0}(d\tilde{T}_{S,3}^+/d\tilde{r})_{\tilde{r}_{f,S}} - \hat{r}_{f,1}g_1^+(\tilde{m}/\tilde{r}_{f,S}^2)/[\exp(\tilde{m}/\tilde{r}_{f,S}) - 1]\} + O(\bar{\delta}^2)$$

$$\tilde{T}_E^+ = -a_{T,1}^+(\tilde{m}/\tilde{r}_{f,S}^2)\exp(-\tilde{m}/\tilde{r}_{f,S}) + \bar{\delta}P_n\{(d\hat{T}_1^+/d\tilde{r})_{\tilde{r}_{f,S}} + \hat{r}_{f,0}(d^2\tilde{T}_{S,1}^+/d\tilde{r}^2)_{\tilde{r}_{f,S}} - \hat{r}_{f,1}g_0^+[(\tilde{m}/\tilde{r}_{f,S}^2) - (2/\tilde{r}_{f,S})]\} + O(\bar{\delta}^2)$$

$$\zeta \to \pm\infty \quad : \quad \tilde{\Theta}_1^{\pm} \to -\hat{g}_1^{\pm} \pm \hat{g}_0^{\pm}\zeta$$

$$\begin{aligned}
\tilde{\Theta}_3^- \to & -[\tilde{T}_{S,3}^-(\tilde{r}_{f,S}) + (d\tilde{T}_{S,1}^-/d\tilde{r})_{\tilde{r}_{f,S}}\zeta] - \bar{\delta}P_n\{[\hat{T}_3^-(\tilde{r}_{f,S}) + \hat{r}_{f,0}(d\tilde{T}_{S,3}^-/d\tilde{r})_{\tilde{r}_{f,S}} + \hat{r}_{f,1}(d\tilde{T}_{S,2}^-/d\tilde{r})_{\tilde{r}_{f,S}}] \\
& + [(d\hat{T}_1^-/d\tilde{r})_{\tilde{r}_{f,S}} + \hat{r}_{f,0}(d^2\tilde{T}_{S,1}^-/d\tilde{r}^2)_{\tilde{r}_{f,S}} + \hat{r}_{f,1}(d^2\tilde{T}_{S,0}^-/d\tilde{r}^2)_{\tilde{r}_{f,S}}]\zeta\} + O(\bar{\delta}^2) \\
= & -[\tilde{T}_{S,3}^-(\tilde{r}_{f,S}) + \tilde{T}_{b,S,A}(\tilde{m}/\tilde{r}_{f,S}^2)\zeta] - \bar{\delta}P_n\langle[\hat{T}_3^-(\tilde{r}_{f,S}) + \hat{r}_{f,0}(d\tilde{T}_{S,3}^-/d\tilde{r})_{\tilde{r}_{f,S}} + \hat{r}_{f,1}g_1^-(\tilde{m}/\tilde{r}_{f,S}^2)] \\
& + \{(d\hat{T}_1^-/d\tilde{r})_{\tilde{r}_{f,S}} + \hat{r}_{f,0}(d^2\tilde{T}_{S,1}^-/d\tilde{r}^2)_{\tilde{r}_{f,S}} + \hat{r}_{f,1}g_0^-[(\tilde{m}/\tilde{r}_{f,S}^2) - (2/\tilde{r}_{f,S})]\}\zeta\rangle + O(\bar{\delta}^2) \\
= & -\tilde{T}_D^- - \tilde{T}_E^-\zeta
\end{aligned}$$

$$\begin{aligned}
\tilde{\Theta}_3^+ \to & -[\tilde{T}_{S,3}^+(\tilde{r}_{f,S}) + (d\tilde{T}_{S,1}^+/d\tilde{r})_{\tilde{r}_{f,S}}\zeta] - \bar{\delta}P_n\{[\hat{T}_3^+(\tilde{r}_{f,S}) + \hat{r}_{f,0}(d\tilde{T}_{S,3}^+/d\tilde{r})_{\tilde{r}_{f,S}} + \hat{r}_{f,1}(d\tilde{T}_{S,2}^+/d\tilde{r})_{\tilde{r}_{f,S}}] \\
& + [(d\hat{T}_1^+/d\tilde{r})_{\tilde{r}_{f,S}} + \hat{r}_{f,0}(d^2\tilde{T}_{S,1}^+/d\tilde{r}^2)_{\tilde{r}_{f,S}} + \hat{r}_{f,1}(d^2\tilde{T}_{S,0}^+/d\tilde{r}^2)_{\tilde{r}_{f,S}}]\zeta\} + O(\bar{\delta}^2) \\
= & -[\tilde{T}_{S,3}^+(\tilde{r}_{f,S}) - a_{T,1}^+(\tilde{m}/\tilde{r}_{f,S}^2)\exp(-\tilde{m}/\tilde{r}_{f,S})\zeta] - \bar{\delta}P_n\langle\{[\hat{T}_3^+(\tilde{r}_{f,S}) + \hat{r}_{f,0}(d\tilde{T}_{S,3}^+/d\tilde{r})_{\tilde{r}_{f,S}} \\
& + \{(d\hat{T}_1^+/d\tilde{r})_{\tilde{r}_{f,S}} + \hat{r}_{f,0}(d^2\tilde{T}_{S,1}^+/d\tilde{r}^2)_{\tilde{r}_{f,S}} - \hat{r}_{f,1}g_0^+[(\tilde{m}/\tilde{r}_{f,S}^2) - (2/\tilde{r}_{f,S})]\}\zeta\rangle + O(\bar{\delta}^2) \\
= & -\tilde{T}_D^+ - \tilde{T}_E^+\zeta
\end{aligned}$$

Since the equation and matching conditions are similar to those of the steady state problem, we have :

$$\begin{aligned}
\tilde{\Theta}_3^{\pm} = & \hat{g}_3^{\pm} \pm \left(\frac{\tilde{T}_E^{\pm}}{\hat{g}_0^{\pm}} - \frac{8}{3}\hat{g}_2^{\pm}\hat{g}_0^{\pm}\right)\left\{2 + \frac{\sqrt{2\Lambda_R\exp(-\tilde{\Theta}_1^{\pm}) + (\hat{g}_0^{\pm})^2}}{\hat{g}_0^{\pm}}\left\langle \ell\text{n}\left[\frac{2(\hat{g}_0^{\pm})^2}{\Lambda_R}\frac{\sqrt{2\Lambda_R\exp(-\tilde{\Theta}_1^{\pm}) + (\hat{g}_0^{\pm})^2} - \hat{g}_0^{\pm}}{\sqrt{2\Lambda_R\exp(-\tilde{\Theta}_1^{\pm}) + (\hat{g}_0^{\pm})^2} + \hat{g}_0^{\pm}}\right] - 2\right\rangle\right\} \\
& \pm \frac{\hat{g}_2^{\pm}}{2}\sqrt{2\Lambda_R\exp(-\tilde{\Theta}_1^{\pm}) + (\hat{g}_0^{\pm})^2}\left\{\left\langle \ell\text{n}\left[\frac{2(\hat{g}_0^{\pm})^2}{\Lambda_R}\frac{\sqrt{2\Lambda_R\exp(-\tilde{\Theta}_1^{\pm}) + (\hat{g}_0^{\pm})^2} - \hat{g}_0^{\pm}}{\sqrt{2\Lambda_R\exp(-\tilde{\Theta}_1^{\pm}) + (\hat{g}_0^{\pm})^2} + \hat{g}_0^{\pm}}\right]\right\rangle^2 - (\tilde{\Theta}_1^{\pm})^2 - \frac{4}{3}\tilde{\Theta}_1^{\pm}\right\} \\
& \pm 2\hat{g}_2^{\pm}\hat{g}_0^{\pm}\left\{\ell\text{n}\left[\frac{2(\hat{g}_0^{\pm})^2}{\Lambda_R}\frac{\sqrt{2\Lambda_R\exp(-\tilde{\Theta}_1^{\pm}) + (\hat{g}_0^{\pm})^2} - \hat{g}_0^{\pm}}{\sqrt{2\Lambda_R\exp(-\tilde{\Theta}_1^{\pm}) + (\hat{g}_0^{\pm})^2} + \hat{g}_0^{\pm}}\right]\right\} - \frac{\sqrt{2\Lambda_R\exp(-\tilde{\Theta}_1^{\pm}) + (\hat{g}_0^{\pm})^2}}{\hat{g}_0^{\pm}}\left(\tilde{T}_D^{\pm} + \hat{g}_3^{\pm} \pm \frac{\hat{g}_1^{\pm}}{\hat{g}_0^{\pm}}\tilde{T}_E^{\pm}\right)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial\tilde{\Theta}_3^{\pm}}{\partial\zeta} = & \left[\frac{8}{3}\hat{g}_2^{\pm} - \frac{\tilde{T}_E^{\pm}}{(\hat{g}_0^{\pm})^2}\right]\left\{\hat{g}_0^{\pm}\sqrt{2\Lambda_R\exp(-\tilde{\Theta}_1^{\pm}) + (\hat{g}_0^{\pm})^2} + \Lambda_R\exp(-\tilde{\Theta}_1^{\pm})\left\langle \ell\text{n}\left[\frac{2(\hat{g}_0^{\pm})^2}{\Lambda_R}\frac{\sqrt{2\Lambda_R\exp(-\tilde{\Theta}_1^{\pm}) + (\hat{g}_0^{\pm})^2} - \hat{g}_0^{\pm}}{\sqrt{2\Lambda_R\exp(-\tilde{\Theta}_1^{\pm}) + (\hat{g}_0^{\pm})^2} + \hat{g}_0^{\pm}}\right] - 2\right\rangle\right\} \\
& + \Lambda_R\exp(-\tilde{\Theta}_1^{\pm})\left(\frac{\hat{g}_1^{\pm}}{(\hat{g}_0^{\pm})^2}\tilde{T}_E^{\pm} \pm \frac{\tilde{T}_D^{\pm} + \hat{g}_3^{\pm}}{\hat{g}_0^{\pm}} - \frac{\hat{g}_2^{\pm}}{2}\left\{\left\langle \ell\text{n}\left[\frac{2(\hat{g}_0^{\pm})^2}{\Lambda_R}\frac{\sqrt{2\Lambda_R\exp(-\tilde{\Theta}_1^{\pm}) + (\hat{g}_0^{\pm})^2} - \hat{g}_0^{\pm}}{\sqrt{2\Lambda_R\exp(-\tilde{\Theta}_1^{\pm}) + (\hat{g}_0^{\pm})^2} + \hat{g}_0^{\pm}}\right]\right\rangle^2 - (\tilde{\Theta}_1^{\pm})^2 + \frac{8}{3}\tilde{\Theta}_1^{\pm} + \frac{8}{3}\right\}\right) \\
& - \hat{g}_2^{\pm}\hat{g}_0^{\pm}\sqrt{2\Lambda_R\exp(-\tilde{\Theta}_1^{\pm}) + (\hat{g}_0^{\pm})^2}\,\ell\text{n}\left[\frac{2(\hat{g}_0^{\pm})^2}{\Lambda_R}\frac{\sqrt{2\Lambda_R\exp(-\tilde{\Theta}_1^{\pm}) + (\hat{g}_0^{\pm})^2} - \hat{g}_0^{\pm}}{\sqrt{2\Lambda_R\exp(-\tilde{\Theta}_1^{\pm}) + (\hat{g}_0^{\pm})^2} + \hat{g}_0^{\pm}}\right] - \hat{g}_2^{\pm}(\hat{g}_0^{\pm})^2\left(\tilde{\Theta}_1^{\pm} + \frac{8}{3}\right)
\end{aligned}$$

(4) Summary

$$\Theta_{S,1}^- = \ell n\langle 1-\{\Lambda_R/[2(g_0^-)^2]\}\exp(g_1^-+g_0^-\zeta)\rangle^2 - g_0^-\zeta - g_1^-$$

$$d\Theta_{S,1}^-/d\zeta = -[2\Lambda_R\exp(-\Theta_{S,1}^-)+(g_0^-)^2]^{1/2} = -g_0^-[2(g_0^-)^2+\Lambda_R\exp(g_1^-+g_0^-\zeta)]/[2(g_0^-)^2-\Lambda_R\exp(g_1^-+g_0^-\zeta)]$$

$$\Theta_{S,2}^- = (\tilde{T}_{b,S,A}/g_0^-)(d\Theta_{S,1}^-/d\zeta) = -\tilde{T}_{b,S,A}[2(g_0^-)^2+\Lambda_R\exp(g_1^-+g_0^-\zeta)]/[2(g_0^-)^2-\Lambda_R\exp(g_1^-+g_0^-\zeta)]$$

$$\Theta_{S,1}^+ = \ell n\langle 1-\{\Lambda_R/[2(g_0^+)^2]\}\exp(g_1^+-g_0^+\zeta)\rangle^2 + g_0^+\zeta - g_1^+ = 2\,\ell n\langle 1-\{\Lambda_R/[2(g_0^+)^2]\}\exp(g_1^+-g_0^+\zeta)\rangle + g_0^+\zeta - g_1^+$$

$$d\Theta_{S,1}^+/d\zeta = [2\Lambda_R\exp(-\Theta_{S,1}^+)+(g_0^+)^2]^{1/2} = g_0^+[2(g_0^+)^2+\Lambda_R\exp(g_1^+-g_0^+\zeta)]/[2(g_0^+)^2-\Lambda_R\exp(g_1^+-g_0^+\zeta)]$$

$$\Theta_{S,2}^+ = -(a_{T,A}^+/g_0^+)(d\Theta_{S,1}^+/d\zeta) = -a_{T,A}^+[2(g_0^+)^2+\Lambda_R\exp(g_1^+-g_0^+\zeta)]/[2(g_0^+)^2-\Lambda_R\exp(g_1^+-g_0^+\zeta)]$$

$$\hat{T}_0^- = -\hat{r}_{f,0}g_0^-\{A_{T2}\Psi_2(1,1)\Psi_1(\tilde{r},1)+[1-A_{T2}\Psi_1(1,1)]\Psi_2(\tilde{r},1)\}/\{A_{T2}\Psi_2(1,1)\Psi_1(\tilde{r}_{f,S},1)+[1-A_{T2}\Psi_1(1,1)]\Psi_2(\tilde{r}_{f,S},1)\}$$

$$d\hat{T}_0^-/d\tilde{r} = -\hat{r}_{f,0}g_0^-\{A_{T2}\Psi_2(1,1)\Psi_1^*(\tilde{r},1)+[1-A_{T2}\Psi_1(1,1)]\Psi_1^*(\tilde{r},1)\}/\{A_{T2}\Psi_2(1,1)\Psi_1(\tilde{r}_{f,S},1)+[1-A_{T2}\Psi_1(1,1)]\Psi_2(\tilde{r}_{f,S},1)\}$$

$$\hat{T}_0^+ = \hat{r}_{f,0}\,g_0^+\Psi_2(\tilde{r},1)/\Psi_2(\tilde{r}_{f,S},1) \quad ; \quad d\hat{T}_0^+/d\tilde{r} = \hat{r}_{f,0}\,g_0^+\Psi_2^*(\tilde{r},1)/\Psi_2(\tilde{r}_{f,S},1)$$

$$\tilde{\Theta}_1^- = \Theta_{S,1}^- - \bar{\delta}P_n\langle(\hat{T}_B^-+\hat{T}_A^-\zeta)+2[\hat{T}_B^- -(2/g_0^-)\hat{T}_A^-+\hat{T}_A^-\zeta]/\{[2(g_0^-)^2/\Lambda_R]\exp[-(g_1^-+g_0^-\zeta)]-1\}\rangle + O(\bar{\delta}^2)$$

$$\frac{\partial\tilde{\Theta}_1^-}{\partial\zeta} = \frac{d\Theta_{S,1}^-}{d\zeta}\left\langle 1+\bar{\delta}P_n\left\{\frac{\hat{T}_A^-}{g_0^-}+\frac{4(g_0^-)^2\Lambda_R\exp(g_1^-+g_0^-\zeta)[\hat{T}_B^- -(2/g_0^-)\hat{T}_A^-+\hat{T}_A^-\zeta]}{[2(g_0^-)^2+\Lambda_R\exp(g_1^-+g_0^-\zeta)][2(g_0^-)^2-\Lambda_R\exp(g_1^-+g_0^-\zeta)]}\right\}+O(\bar{\delta}^2)\right\rangle$$

$$\tilde{\Theta}_2^- = \Theta_{S,2}^-\left\langle 1+\bar{\delta}P_n\left\{\hat{T}_C^-+\frac{4(g_0^-)^2\Lambda_R\exp(g_1^-+g_0^-\zeta)[\hat{T}_B^- -(2/g_0^-)\hat{T}_A^-+\hat{T}_A^-\zeta]}{[2(g_0^-)^2+\Lambda_R\exp(g_1^-+g_0^-\zeta)][2(g_0^-)^2-\Lambda_R\exp(g_1^-+g_0^-\zeta)]}\right\}+O(\bar{\delta}^2)\right\rangle$$

$$\tilde{\Theta}_1^+ = \Theta_{S,1}^+ - \bar{\delta}P_n\langle(\hat{T}_B^++\hat{T}_A^+\zeta)+2[\hat{T}_B^+ +(2/g_0^+)\hat{T}_A^++\hat{T}_A^+\zeta]/\{[2(g_0^+)^2/\Lambda_R]\exp(g_0^+\zeta-g_1^+)-1\}\rangle + O(\bar{\delta}^2)$$

$$\frac{\partial\tilde{\Theta}_1^+}{\partial\zeta} = \frac{d\Theta_{S,1}^+}{d\zeta}\left\langle 1-\bar{\delta}P_n\left\{\frac{\hat{T}_A^+}{g_0^+}-\frac{4(g_0^+)^2\Lambda_R\exp(g_1^+-g_0^+\zeta)[\hat{T}_B^+ +(2/g_0^+)\hat{T}_A^++\hat{T}_A^+\zeta]}{[2(g_0^+)^2+\Lambda_R\exp(g_1^+-g_0^+\zeta)][2(g_0^+)^2-\Lambda_R\exp(g_1^+-g_0^+\zeta)]}\right\}+O(\bar{\delta}^2)\right\rangle$$

$$\tilde{\Theta}_2^+ = \Theta_{S,2}^+\left\langle 1+\bar{\delta}P_n\left\{\hat{T}_C^++\frac{4(g_0^+)^2\Lambda_R\exp(g_1^+-g_0^+\zeta)[\hat{T}_B^+ +(2/g_0^+)\hat{T}_A^++\hat{T}_A^+\zeta]}{[2(g_0^+)^2+\Lambda_R\exp(g_1^+-g_0^+\zeta)][2(g_0^+)^2-\Lambda_R\exp(g_1^+-g_0^+\zeta)]}\right\}+O(\bar{\delta}^2)\right\rangle$$

$$\begin{aligned}\tilde{\Theta}_3^\pm = \hat{g}_3^\pm \pm\left(\frac{\tilde{T}_E^\pm}{\hat{g}_0^\pm}-\frac{8}{3}\hat{g}_2^\pm\hat{g}_0^\pm\right)&\left\{2+\frac{\sqrt{2\Lambda_R\exp(-\tilde{\Theta}_1^\pm)+(\hat{g}_0^\pm)^2}}{\hat{g}_0^\pm}\left\langle \ell n\left[\frac{2(\hat{g}_0^\pm)^2}{\Lambda_R}\frac{\sqrt{2\Lambda_R\exp(-\tilde{\Theta}_1^\pm)+(\hat{g}_0^\pm)^2}-\hat{g}_0^\pm}{\sqrt{2\Lambda_R\exp(-\tilde{\Theta}_1^\pm)+(\hat{g}_0^\pm)^2}+\hat{g}_0^\pm}\right]-2\right\rangle\right\}\\
&\pm\frac{\hat{g}_2^\pm}{2}\sqrt{2\Lambda_R\exp(-\tilde{\Theta}_1^\pm)+(\hat{g}_0^\pm)^2}\left\{\left\langle \ell n\left[\frac{2(\hat{g}_0^\pm)^2}{\Lambda_R}\frac{\sqrt{2\Lambda_R\exp(-\tilde{\Theta}_1^\pm)+(\hat{g}_0^\pm)^2}-\hat{g}_0^\pm}{\sqrt{2\Lambda_R\exp(-\tilde{\Theta}_1^\pm)+(\hat{g}_0^\pm)^2}+\hat{g}_0^\pm}\right]\right\rangle^2-(\tilde{\Theta}_1^\pm)^2-\frac{4}{3}\tilde{\Theta}_1^\pm\right\}\\
&\pm 2\hat{g}_2^\pm\hat{g}_0^\pm\left\{\ell n\left[\frac{2(\hat{g}_0^\pm)^2}{\Lambda_R}\frac{\sqrt{2\Lambda_R\exp(-\tilde{\Theta}_1^\pm)+(\hat{g}_0^\pm)^2}-\hat{g}_0^\pm}{\sqrt{2\Lambda_R\exp(-\tilde{\Theta}_1^\pm)+(\hat{g}_0^\pm)^2}+\hat{g}_0^\pm}\right]\right\}-\frac{\sqrt{2\Lambda_R\exp(-\tilde{\Theta}_1^\pm)+(\hat{g}_0^\pm)^2}}{\hat{g}_0^\pm}\left(\tilde{T}_D^\pm+\hat{g}_3^\pm\pm\frac{\hat{g}_1^\pm}{\hat{g}_0^\pm}\tilde{T}_E^\pm\right)\end{aligned}$$

$$\begin{aligned}\frac{\partial\tilde{\Theta}_3^\pm}{\partial\zeta} = \left[\frac{8}{3}\hat{g}_2^\pm-\frac{\tilde{T}_E^\pm}{(\hat{g}_0^\pm)^2}\right]&\left\{\hat{g}_0^\pm\sqrt{2\Lambda_R\exp(-\tilde{\Theta}_1^\pm)+(\hat{g}_0^\pm)^2}+\Lambda_R\exp(-\tilde{\Theta}_1^\pm)\left\langle \ell n\left[\frac{2(\hat{g}_0^\pm)^2}{\Lambda_R}\frac{\sqrt{2\Lambda_R\exp(-\tilde{\Theta}_1^\pm)+(\hat{g}_0^\pm)^2}-\hat{g}_0^\pm}{\sqrt{2\Lambda_R\exp(-\tilde{\Theta}_1^\pm)+(\hat{g}_0^\pm)^2}+\hat{g}_0^\pm}\right]-2\right\rangle\right\}\\
&+\Lambda_R\exp(-\tilde{\Theta}_1^\pm)\left(\frac{\hat{g}_1^\pm}{(\hat{g}_0^\pm)^2}\tilde{T}_E^\pm\pm\frac{\tilde{T}_D^\pm+\hat{g}_3^\pm}{\hat{g}_0^\pm}-\frac{\hat{g}_2^\pm}{2}\left\{\left\langle \ell n\left[\frac{2(\hat{g}_0^\pm)^2}{\Lambda_R}\frac{\sqrt{2\Lambda_R\exp(-\tilde{\Theta}_1^\pm)+(\hat{g}_0^\pm)^2}-\hat{g}_0^\pm}{\sqrt{2\Lambda_R\exp(-\tilde{\Theta}_1^\pm)+(\hat{g}_0^\pm)^2}+\hat{g}_0^\pm}\right]\right\rangle^2-(\tilde{\Theta}_1^\pm)^2+\frac{8}{3}\tilde{\Theta}_1^\pm+\frac{8}{3}\right\}\right)\\
&-\hat{g}_2^\pm\hat{g}_0^\pm\sqrt{2\Lambda_R\exp(-\tilde{\Theta}_1^\pm)+(\hat{g}_0^\pm)^2}\,\ell n\left[\frac{2(\hat{g}_0^\pm)^2}{\Lambda_R}\frac{\sqrt{2\Lambda_R\exp(-\tilde{\Theta}_1^\pm)+(\hat{g}_0^\pm)^2}-\hat{g}_0^\pm}{\sqrt{2\Lambda_R\exp(-\tilde{\Theta}_1^\pm)+(\hat{g}_0^\pm)^2}+\hat{g}_0^\pm}\right]-\hat{g}_2^\pm(\hat{g}_0^\pm)^2\left(\tilde{\Theta}_1^\pm+\frac{8}{3}\right)\end{aligned}$$

$$\hat{g}_0^- = g_0^-+\bar{\delta}P_n\hat{T}_A^-+O(\bar{\delta}^2) = g_0^-[1+\bar{\delta}P_n(\hat{T}_A^-/g_0^-)+O(\bar{\delta}^2)] \quad ; \quad g_0^- = (\tilde{T}_f-\tilde{T}_0)(\tilde{m}/\tilde{r}_{f,S}^2)$$

$$\hat{g}_1^- = g_1^-+\bar{\delta}P_n\hat{T}_B^-+O(\bar{\delta}^2) \quad ; \quad g_1^- = -2\,\ell n\{[\langle 1+[2\Lambda_R/(g_0^-)^2]\rangle^{1/2}+1]/2\}$$

$$\hat{g}_2^- = g_2^-\{1+\bar{\delta}P_n[\hat{T}_C^- -(\hat{T}_A^-/g_0^-)]+O(\bar{\delta}^2)\} \quad ; \quad g_2^- = [2/(3\tilde{T}_f)](\tilde{T}_{b,S,A}/g_0^-)$$

$$\hat{g}_3^- = g_3^- \langle 1 + \bar{\delta} P_n \{ \hat{T}_C^- - (\hat{T}_A^- / g_0^-) - 2\hat{r}_{f,0} [(m - 2\tilde{r}_{f,S})^{-1} + \tilde{r}_{f,S}^{-1}] \} + O(\bar{\delta}^2) \rangle \quad ; \quad g_3^- = [(\tilde{m} - 2\tilde{r}_{f,S}) / \tilde{r}_{f,S}^2] (\tilde{T}_{b,S,A} / g_0^-)$$

$$\hat{T}_A^- = (d\hat{T}_0^- / d\tilde{r})_{\tilde{r}_{f,S}} + \hat{r}_{f,0}\, g_0^- [(\tilde{m}/\tilde{r}_{f,S}^2) - (2/\tilde{r}_{f,S})] \quad ; \quad \hat{T}_B^- = \hat{T}_2^- (\tilde{r}_{f,S}) + \hat{r}_{f,0}\, g_1^- (\tilde{m}/\tilde{r}_{f,S}^2)$$

$$\hat{T}_C^- = \{ [\hat{T}_1^- (\tilde{r}_{f,S}) + \hat{r}_{f,1}\, g_0^-] / \tilde{T}_{b,S,A} \} + \hat{r}_{f,0} (\tilde{m}/\tilde{r}_{f,S}^2)$$

$$\tilde{T}_D^- = \tilde{T}_{S,3}^- (\tilde{r}_{f,S}) + \bar{\delta} P_n [\hat{T}_3^- (\tilde{r}_{f,S}) + \hat{r}_{f,0} (d\tilde{T}_{S,3}^- / d\tilde{r})_{\tilde{r}_{f,S}} + \hat{r}_{f,1}\, g_1^- (\tilde{m}/\tilde{r}_{f,S}^2)] + O(\bar{\delta}^2)$$

$$\tilde{T}_E^- = \tilde{T}_{b,S,A} (\tilde{m}/\tilde{r}_{f,S}^2) + \bar{\delta} P_n \{ (d\hat{T}_1^- / d\tilde{r})_{\tilde{r}_{f,S}} + \hat{r}_{f,0} (d^2 \tilde{T}_{S,1}^- / d\tilde{r}^2)_{\tilde{r}_{f,S}} + \hat{r}_{f,1}\, g_0^- [(\tilde{m}/\tilde{r}_{f,S}^2) - (2/\tilde{r}_{f,S})] \} + O(\bar{\delta}^2)$$

$$\hat{g}_0^+ = g_0^+ - \bar{\delta} P_n \hat{T}_A^+ + O(\bar{\delta}^2) = g_0^+ [1 - \bar{\delta} P_n (\hat{T}_A^+ / g_0^+) + O(\bar{\delta}^2)] \quad ; \quad g_0^+ = (\tilde{T}_f - \tilde{T}_\infty)(\tilde{m}/\tilde{r}_{f,S}^2) / [\exp(\tilde{m}/\tilde{r}_{f,S}) - 1]$$

$$\hat{g}_1^+ = g_1^+ + \bar{\delta} P_n \hat{T}_B^+ + O(\bar{\delta}^2) \quad ; \quad g_1^+ = -2\,\ell\text{n} \{ [\langle 1 + [2\Lambda_R / (g_0^+)^2] \rangle^{1/2} + 1] / 2 \}$$

$$\hat{g}_2^+ = g_2^+ \{ 1 + \bar{\delta} P_n [\hat{T}_C^+ + (\hat{T}_A^+ / g_0^+)] + O(\bar{\delta}^2) \} \quad ; \quad g_2^+ = -[2 / (3\tilde{T}_f)] (a_{T,A}^+ / g_0^+)$$

$$\hat{g}_3^\pm = g_3^+ \langle 1 + \bar{\delta} P_n \{ \hat{T}_C^+ + (\hat{T}_A^+ / g_0^+) - 2\hat{r}_{f,0} [(m - 2\tilde{r}_{f,S})^{-1} + \tilde{r}_{f,S}^{-1}] \} + O(\bar{\delta}^2) \rangle \quad ; \quad g_3^+ = -[(\tilde{m} - 2\tilde{r}_{f,S}) / \tilde{r}_{f,S}^2] (a_{T,A}^+ / g_0^+)$$

$$\hat{T}_A^+ = (d\hat{T}_0^+ / d\tilde{r})_{\tilde{r}_{f,S}} - \hat{r}_{f,0}\, g_0^+ [(\tilde{m}/\tilde{r}_{f,S}^2) - (2/\tilde{r}_{f,S})] \quad ; \quad \hat{T}_B^+ = \hat{T}_2^+ (\tilde{r}_{f,S}) - \hat{r}_{f,0}\, g_1^+ (\tilde{m}/\tilde{r}_{f,S}^2) / [\exp(\tilde{m}/\tilde{r}_{f,S}) - 1]$$

$$\hat{T}_C^+ = [\hat{T}_1^+ (\tilde{r}_{f,S}) - a_{T,1}^+ \hat{r}_{f,0} (\tilde{m}/\tilde{r}_{f,S}^2) \exp(-\tilde{m}/\tilde{r}_{f,S}) - \hat{r}_{f,1}\, g_0^+] / a_{T,A}^+ \quad ; \quad a_{T,A}^+ = a_{T,1}^+ [1 - \exp(-\tilde{m}/\tilde{r}_{f,S})]$$

$$\tilde{T}_D^+ = \tilde{T}_{S,3}^+ (\tilde{r}_{f,S}) + \bar{\delta} P_n \{ [\hat{T}_3^+ (\tilde{r}_{f,S}) + \hat{r}_{f,0} (d\tilde{T}_{S,3}^+ / d\tilde{r})_{\tilde{r}_{f,S}} - \hat{r}_{f,1}\, g_1^+ (\tilde{m}/\tilde{r}_{f,S}^2) / [\exp(\tilde{m}/\tilde{r}_{f,S}) - 1] \} + O(\bar{\delta}^2)$$

$$\tilde{T}_E^+ = -a_{T,1}^+ (\tilde{m}/\tilde{r}_{f,S}^2) \exp(-\tilde{m}/\tilde{r}_{f,S}) + \bar{\delta} P_n \{ (d\hat{T}_1^+ / d\tilde{r})_{\tilde{r}_{f,S}} + \hat{r}_{f,0} (d^2 \tilde{T}_{S,1}^+ / d\tilde{r}^2)_{\tilde{r}_{f,S}} - \hat{r}_{f,1}\, g_0^+ [(\tilde{m}/\tilde{r}_{f,S}^2) - (2/\tilde{r}_{f,S})] \} + O(\bar{\delta}^2)$$

(H) Inner Expansions

Define stretched variable: $\xi = (\tilde{r} - \tilde{r}_{f,0})/\varepsilon \quad \therefore \quad \tilde{r} = \tilde{r}_{f,0} + \varepsilon\xi$ and $d\tilde{r} = \varepsilon d\xi \quad ; \quad \tilde{r}_{f,0} = \tilde{r}_{f,S} + \bar{\delta}\,\hat{r}_{f,0}\,P_n(\tilde{z})$

$$\tilde{r}^2 = (\tilde{r}_{f,0} + \varepsilon\xi)^2 = \tilde{r}_{f,0} + \varepsilon 2\tilde{r}_{f,0}\xi + O(\varepsilon^2) = \tilde{r}_{f,0}^2[1 + \varepsilon(2\xi/\tilde{r}_{f,0}) + O(\varepsilon^2)$$

$$1/\tilde{r}_{f,0} = 1/\{\tilde{r}_{f,S}[1 + \bar{\delta}P_n(\hat{r}_{f,0}/\tilde{r}_{f,S})]\} = (1/\tilde{r}_{f,S})[1 - \bar{\delta}P_n(\hat{r}_{f,0}/\tilde{r}_{f,S}) + O(\bar{\delta}^2)]$$

$$1/\tilde{r}_{f,0}^2 = (1/\tilde{r}_{f,S}^2)[1 - \bar{\delta}P_n(\hat{r}_{f,0}/\tilde{r}_{f,S}) + O(\bar{\delta}^2)]^2 = (1/\tilde{r}_{f,S}^2)[1 - \bar{\delta}P_n(2\hat{r}_{f,0}/\tilde{r}_{f,S}) + O(\bar{\delta}^2)]$$

Define inner expansions and the small expansion parameter:

$$\tilde{T} = [\tilde{T}_f - \varepsilon\tilde{\theta}_1 - \varepsilon^2\tilde{\theta}_2 + O(\varepsilon^3)] + O(\varepsilon/\delta) \quad ; \quad \tilde{Y}_1 = [\varepsilon\tilde{\phi}_{1,1} + \varepsilon^2\tilde{\phi}_{1,2} + O(\varepsilon^3)] \quad ; \quad \tilde{Y}_2 = [\varepsilon\tilde{\phi}_{2,1} + \varepsilon^2\tilde{\phi}_{2,2} + O(\varepsilon^3)]$$

$$\exp(-\tilde{E}/\tilde{T}_f) = \exp\{-(\tilde{E}/[\tilde{T}_f - \varepsilon\tilde{\theta}_1 - \varepsilon^2\tilde{\theta}_2 + O(\varepsilon^3)]\} = \exp\left\langle -\tilde{E}/\{\tilde{T}_f[1 - \varepsilon(\tilde{\theta}_1/\tilde{T}_f^2) - \varepsilon^2(\tilde{\theta}_2/\tilde{T}_f^2) + \cdots]\}\right\rangle$$

$$= \exp\{-(\tilde{E}/\tilde{T}_f)[1 + \varepsilon(\tilde{\theta}_1/\tilde{T}_f) + \varepsilon^2(\tilde{\theta}_2/\tilde{T}_f) + \cdots]\} = \exp(-\tilde{E}/\tilde{T}_f)\exp[-\varepsilon(\tilde{E}/\tilde{T}_f^2)\tilde{\theta}_1 - \varepsilon^2(\tilde{E}/\tilde{T}_f^2)\tilde{\theta}_2 + \cdots]$$

$$\exp(-\tilde{E}_K/\tilde{T}) = \exp(-\tilde{E}_K/\tilde{T}_f)\exp[-\varepsilon(\tilde{E}_K/\tilde{T}_f^2)\tilde{\theta}_1 - \varepsilon^2(\tilde{E}_K/\tilde{T}_f^2)\tilde{\theta}_2 + \cdots] = \exp(-\tilde{E}_K/\tilde{T}_f)\exp(-\tilde{\theta}_1)\exp(-\varepsilon\tilde{\theta}_2 + \cdots)$$

$$= \exp(-\tilde{E}_K/\tilde{T}_f)\exp(-\tilde{\theta}_1)(1 - \varepsilon\tilde{\theta}_2 + \cdots)$$

$$\exp(-\tilde{E}_R/\tilde{T}) = \exp(-\tilde{E}_R/\tilde{T}_f)\exp[-\varepsilon(\tilde{E}_R/\tilde{T}_f^2)\tilde{\theta}_1 + \ldots] = \exp(-\tilde{E}_R/\tilde{T}_f)\exp[-(\varepsilon/\delta)\tilde{\theta}_1 + \ldots] = \exp(-\tilde{E}_R/\tilde{T}_f)[1 - (\varepsilon/\delta)\tilde{\theta}_1 + \ldots]$$

(1) $$\frac{\tilde{m} - 2\tilde{r}}{\tilde{r}^2}\frac{\partial\tilde{T}}{\partial\tilde{r}} - \frac{\partial^2\tilde{T}}{\partial\tilde{r}^2} - \frac{1}{\tilde{r}^2}\frac{\partial}{\partial\tilde{z}}\left[(1 - \tilde{z}^2)\frac{\partial\tilde{T}}{\partial\tilde{z}}\right] = Da_K\tilde{Y}_1\tilde{Y}_2\exp(-\tilde{E}_K/\tilde{T}) - Da_R\exp(-\tilde{E}_R/\tilde{T})$$

$$\Lambda_K = \varepsilon^3 Da_K\exp(-\tilde{E}_K/\tilde{T}_f) \quad ; \quad \Lambda_R = \delta Da_R\exp(-\tilde{E}_R/\tilde{T}_f)$$

$$\left[\frac{\tilde{m} - 2\tilde{r}_{f,0}}{\tilde{r}_{f,0}^2} + O(\varepsilon)\right]\frac{\partial[\tilde{T}_f - \varepsilon\tilde{\theta}_1 + O(\varepsilon^2)]}{\varepsilon\partial\xi} - \frac{\partial^2[\tilde{T}_f - \varepsilon\tilde{\theta}_1 - \varepsilon^2\tilde{\theta}_2 + O(\varepsilon^3)]}{\varepsilon^2\partial\xi^2} - \left[\frac{1}{\tilde{r}_{f,0}^2} + O(\varepsilon)\right]\frac{\partial}{\partial\tilde{z}}\left\{(1 - \tilde{z}^2)\frac{\partial[\tilde{T}_f - \varepsilon\tilde{\theta}_1 + O(\varepsilon^2)]}{\partial\tilde{z}}\right\}$$

$$= Da_K(\varepsilon\tilde{\phi}_{1,1} + \varepsilon^2\tilde{\phi}_{1,2} + \ldots)(\varepsilon\tilde{\phi}_{2,1} + \varepsilon^2\tilde{\phi}_{2,2} + \ldots)[\exp(-\tilde{E}_K/\tilde{T}_f)\exp(-\tilde{\theta}_1)(1 - \varepsilon\tilde{\theta}_2 + \ldots)] - Da_R\exp(-\tilde{E}_R/\tilde{T}_f)[1 + O(\varepsilon/\delta)]$$

$$= \{\varepsilon^2 Da_K\exp(-\tilde{E}_K/\tilde{T}_f)[\tilde{\phi}_{1,1}\tilde{\phi}_{2,1} + \varepsilon(\tilde{\phi}_{1,1}\tilde{\phi}_{2,2} + \tilde{\phi}_{2,1}\tilde{\phi}_{1,2} - \tilde{\phi}_{1,1}\tilde{\phi}_{2,1}\tilde{\theta}_2) + \cdots]\exp(-\tilde{\theta}_1)\} - [Da_R\exp(-\tilde{E}_R/\tilde{T}_f) + \cdots]$$

$$= \varepsilon^{-1}\{\Lambda_K[\tilde{\phi}_{1,1}\tilde{\phi}_{2,1} + \varepsilon(\tilde{\phi}_{1,1}\tilde{\phi}_{2,2} + \tilde{\phi}_{2,1}\tilde{\phi}_{1,2} - \tilde{\phi}_{1,1}\tilde{\phi}_{2,1}\tilde{\theta}_2) + \cdots]\exp(-\tilde{\theta}_1) - (\varepsilon/\delta)(\Lambda_R + \cdots)\}$$

The two leading order terms are

$$\partial^2\tilde{\theta}_1/\partial\xi^2 = \Lambda_K\tilde{\phi}_{1,1}\tilde{\phi}_{2,1}\exp(-\tilde{\theta}_1) \quad ; \quad \partial^2\tilde{\theta}_2/\partial\xi^2 - [(\tilde{m} - 2\tilde{r}_{f,0})/\tilde{r}_{f,0}^2](\partial\tilde{\theta}_1/\partial\xi) = \Lambda_K(\tilde{\phi}_{1,1}\tilde{\phi}_{2,2} + \tilde{\phi}_{2,1}\tilde{\phi}_{1,2} - \tilde{\phi}_{1,1}\tilde{\phi}_{2,1}\tilde{\theta}_2)\exp(-\tilde{\theta}_1)$$

(2) $$\tilde{m}\frac{\partial\tilde{Y}_1}{\partial\tilde{r}} - \frac{1}{Le_1}\left\{\frac{\partial}{\partial\tilde{r}}\left(\tilde{r}^2\frac{\partial\tilde{Y}_1}{\partial\tilde{r}}\right) + \frac{\partial}{\partial\tilde{z}}\left[(1 - \tilde{z}^2)\frac{\partial\tilde{Y}_1}{\partial\tilde{z}}\right]\right\} = -\tilde{r}^2 Da_K\tilde{Y}_1\tilde{Y}_2\exp(-\tilde{E}_K/\tilde{T})$$

$$\tilde{m}\frac{\partial\tilde{Y}_2}{\partial\tilde{r}} - \frac{1}{Le_2}\left\{\frac{\partial}{\partial\tilde{r}}\left(\tilde{r}^2\frac{\partial\tilde{Y}_2}{\partial\tilde{r}}\right) + \frac{\partial}{\partial\tilde{z}}\left[(1 - \tilde{z}^2)\frac{\partial\tilde{Y}_2}{\partial\tilde{z}}\right]\right\} = -\tilde{r}^2 Da_K\tilde{Y}_1\tilde{Y}_2\exp(-\tilde{E}_K/\tilde{T})$$

$$\tilde{m}\frac{\partial\tilde{Y}_2}{\partial\tilde{r}} - \frac{1}{Le_2}\left\{\frac{\partial}{\partial\tilde{r}}\left(\tilde{r}^2\frac{\partial\tilde{Y}_2}{\partial\tilde{r}}\right) + \frac{\partial}{\partial\tilde{z}}\left[(1 - \tilde{z}^2)\frac{\partial\tilde{Y}_2}{\partial\tilde{z}}\right]\right\} - \tilde{m}\frac{\partial\tilde{Y}_1}{\partial\tilde{r}} + \frac{1}{Le_1}\left\{\frac{\partial}{\partial\tilde{r}}\left(\tilde{r}^2\frac{\partial\tilde{Y}_1}{\partial\tilde{r}}\right) + \frac{\partial}{\partial\tilde{z}}\left[(1 - \tilde{z}^2)\frac{\partial\tilde{Y}_1}{\partial\tilde{z}}\right]\right\} = 0$$

$$\tilde{m}\frac{\partial(\varepsilon\tilde{\phi}_{2,1} + \cdots)}{\varepsilon\partial\xi} - \frac{1}{Le_2}\left\{\frac{\partial}{\varepsilon\partial\xi}\left[\tilde{r}_{f,0}^2\left(1 + \varepsilon\frac{2\xi}{\tilde{r}_{f,0}} + \cdots\right)\frac{\partial(\varepsilon\tilde{\phi}_{2,1} + \varepsilon^2\tilde{\phi}_{2,2} + \cdots)}{\varepsilon\partial\xi}\right] + \frac{\partial}{\partial\tilde{z}}\left[(1 - \tilde{z}^2)\frac{\partial(\varepsilon\tilde{\phi}_{2,1} + \cdots)}{\partial\tilde{z}}\right]\right\}$$

$$-\tilde{m}\frac{\partial(\varepsilon\tilde{\phi}_{1,1} + \ldots)}{\varepsilon\partial\xi} + \frac{1}{Le_1}\left\{\frac{\partial}{\varepsilon\partial\xi}\left[\tilde{r}_{f,0}^2\left(1 + \varepsilon\frac{2\xi}{\tilde{r}_{f,0}} + \ldots\right)\frac{\partial(\varepsilon\tilde{\phi}_{1,1} + \varepsilon^2\tilde{\phi}_{1,2} + \ldots)}{\varepsilon\partial\xi}\right] + \frac{\partial}{\partial\tilde{z}}\left[(1 - \tilde{z}^2)\frac{\partial(\varepsilon\tilde{\phi}_{1,1} + \ldots)}{\partial\tilde{z}}\right]\right\} = 0$$

The two leading order terms are

$$\frac{\partial^2}{\partial\xi^2}\left(\frac{\tilde{\phi}_{1,1}}{Le_1} - \frac{\tilde{\phi}_{2,1}}{Le_2}\right) = 0 \quad \therefore \quad \frac{\partial}{\partial\xi}\left(\frac{\tilde{\phi}_{1,1}}{Le_1} - \frac{\tilde{\phi}_{2,1}}{Le_2}\right) = \tilde{c}_1 \quad \text{and} \quad \frac{\tilde{\phi}_{1,1}}{Le_1} - \frac{\tilde{\phi}_{2,1}}{Le_2} = \tilde{c}_1\xi + \tilde{c}_2$$

$$\frac{\partial}{\partial\xi}\left[\frac{1}{Le_1}\frac{\partial\tilde{\phi}_{1,2}}{\partial\xi} + \left(\frac{2\xi}{Le_1\tilde{r}_{f,0}}\right)\frac{\partial\tilde{\phi}_{1,1}}{\partial\xi} - \frac{\tilde{m}}{\tilde{r}_{f,0}^2}\tilde{\phi}_{1,1}\right] - \frac{\partial}{\partial\xi}\left[\frac{1}{Le_2}\frac{\partial\tilde{\phi}_{2,2}}{\partial\xi} + \left(\frac{2\xi}{Le_2\tilde{r}_{f,0}}\right)\frac{\partial\tilde{\phi}_{2,1}}{\partial\xi} - \frac{\tilde{m}}{\tilde{r}_{f,0}^2}\tilde{\phi}_{2,1}\right] = 0$$

or $\left[\frac{1}{Le_1}\frac{\partial\tilde{\phi}_{1,2}}{\partial\xi}+\left(\frac{2\xi}{Le_1\,\tilde{r}_{f,0}}\right)\frac{\partial\tilde{\phi}_{1,1}}{\partial\xi}-\frac{\tilde{m}}{\tilde{r}_{f,0}^2}\tilde{\phi}_{1,1}\right]-\left[\frac{1}{Le_2}\frac{\partial\tilde{\phi}_{2,2}}{\partial\xi}+\left(\frac{2\xi}{Le_2\,\tilde{r}_{f,0}}\right)\frac{\partial\tilde{\phi}_{2,1}}{\partial\xi}-\frac{\tilde{m}}{\tilde{r}_{f,0}^2}\tilde{\phi}_{2,1}\right]=\tilde{c}_3$

(3) $\tilde{m}\frac{\partial\tilde{T}}{\partial\tilde{r}}-\frac{\partial}{\partial\tilde{r}}\left(\tilde{r}^2\frac{\partial\tilde{T}}{\partial\tilde{r}}\right)-\frac{\partial}{\partial\tilde{z}}\left[(1-\tilde{z}^2)\frac{\partial\tilde{T}}{\partial\tilde{z}}\right]=\tilde{r}^2[Da_K\,\tilde{Y}_1\tilde{Y}_2\exp(-\tilde{E}_K/\tilde{T})-Da_R\exp(-\tilde{E}_R/\tilde{T})]$

$$\tilde{m}\frac{\partial\tilde{Y}_1}{\partial\tilde{r}}-\frac{1}{Le_1}\left\{\frac{\partial}{\partial\tilde{r}}\left(\tilde{r}^2\frac{\partial\tilde{Y}_1}{\partial\tilde{r}}\right)+\frac{\partial}{\partial\tilde{z}}\left[(1-\tilde{z}^2)\frac{\partial\tilde{Y}_1}{\partial\tilde{z}}\right]\right\}=-\tilde{r}^2\,Da_K\,\tilde{Y}_1\tilde{Y}_2\exp(-\tilde{E}_K/\tilde{T})$$

$$\tilde{m}\frac{\partial\tilde{T}}{\partial\tilde{r}}-\frac{\partial}{\partial\tilde{r}}\left(\tilde{r}^2\frac{\partial\tilde{T}}{\partial\tilde{r}}\right)-\frac{\partial}{\partial\tilde{z}}\left[(1-\tilde{z}^2)\frac{\partial\tilde{T}}{\partial\tilde{z}}\right]+\tilde{m}\frac{\partial\tilde{Y}_1}{\partial\tilde{r}}-\frac{1}{Le_1}\left\{\frac{\partial}{\partial\tilde{r}}\left(\tilde{r}^2\frac{\partial\tilde{Y}_1}{\partial\tilde{r}}\right)+\frac{\partial}{\partial\tilde{z}}\left[(1-\tilde{z}^2)\frac{\partial\tilde{Y}_1}{\partial\tilde{z}}\right]\right\}=-\tilde{r}^2\,Da_R\exp(-\tilde{E}_R/\tilde{T})$$

$$\tilde{m}\frac{\partial(\tilde{T}_f-\varepsilon\tilde{\theta}_1+\cdots)}{\varepsilon\partial\xi}-\frac{\partial}{\varepsilon\partial\xi}\left[\tilde{r}_{f,0}^2\left(1+\varepsilon\frac{2\xi}{\tilde{r}_{f,0}}+\cdots\right)\frac{\partial(\tilde{T}_f-\varepsilon\tilde{\theta}_1-\varepsilon^2\tilde{\theta}_2+\cdots)}{\varepsilon\partial\xi}\right]-\frac{\partial}{\partial\tilde{z}}\left[(1-\tilde{z}^2)\frac{\partial(\tilde{T}_f-\varepsilon\tilde{\theta}_1+\cdots)}{\partial\tilde{z}}\right]$$

$$+\tilde{m}\frac{\partial(\varepsilon\tilde{\phi}_{1,1}+\ldots)}{\varepsilon\partial\xi}-\frac{1}{Le_1}\left\{\frac{\partial}{\varepsilon\partial\xi}\left[\tilde{r}_{f,0}^2\left(1+\varepsilon\frac{2\xi}{\tilde{r}_{f,0}}+\ldots\right)\frac{\partial(\varepsilon\tilde{\phi}_{1,1}+\varepsilon^2\tilde{\phi}_{1,2}+\ldots)}{\varepsilon\partial\xi}\right]+\frac{\partial}{\partial\tilde{z}}\left[(1-\tilde{z}^2)\frac{\partial(\varepsilon\tilde{\phi}_{1,1}+\ldots)}{\partial\tilde{z}}\right]\right\}$$

$$=-\varepsilon^{-1}(\varepsilon/\delta)[\tilde{r}_{f,S}^2+O(\varepsilon)](\Lambda_R+\cdots)$$

Keeping the two leading order terms

$$\frac{\partial^2}{\partial\xi^2}\left(\tilde{\theta}_1-\frac{\tilde{\phi}_{1,1}}{Le_1}\right)=0\quad\therefore\quad\frac{\partial}{\partial\xi}\left(\tilde{\theta}_1-\frac{\tilde{\phi}_{1,1}}{Le_1}\right)=\tilde{c}_4\quad\text{and}\quad\tilde{\theta}_1-\frac{\tilde{\phi}_{1,1}}{Le_1}=\tilde{c}_4\,\xi+\tilde{c}_5$$

$$\frac{\partial}{\partial\xi}\left(\frac{\partial\tilde{\theta}_2}{\partial\xi}+\frac{2\xi}{\tilde{r}_{f,0}}\frac{\partial\tilde{\theta}_1}{\partial\xi}-\frac{\tilde{m}}{\tilde{r}_{f,0}^2}\tilde{\theta}_1\right)-\frac{\partial}{\partial\xi}\left[\frac{1}{Le_1}\frac{\partial\tilde{\phi}_{1,2}}{\partial\xi}+\left(\frac{2\xi}{Le_1\,\tilde{r}_{f,0}}\right)\frac{\partial\tilde{\phi}_{1,1}}{\partial\xi}-\frac{\tilde{m}}{\tilde{r}_{f,0}^2}\tilde{\phi}_{1,1}\right]=0$$

or $\left(\frac{\partial\tilde{\theta}_2}{\partial\xi}+\frac{2\xi}{\tilde{r}_{f,0}}\frac{\partial\tilde{\theta}_1}{\partial\xi}-\frac{\tilde{m}}{\tilde{r}_{f,0}^2}\tilde{\theta}_1\right)-\left[\frac{1}{Le_1}\frac{\partial\tilde{\phi}_{1,2}}{\partial\xi}+\left(\frac{2\xi}{Le_1\,\tilde{r}_{f,0}}\right)\frac{\partial\tilde{\phi}_{1,1}}{\partial\xi}-\frac{\tilde{m}}{\tilde{r}_{f,0}^2}\tilde{\phi}_{1,1}\right]=\tilde{c}_6$

(4) Summary, the structure equations in the inner, chemically reactive region are

$$\partial^2\tilde{\theta}_1/\partial\xi^2=\Lambda_K\,\tilde{\phi}_{1,1}\,\tilde{\phi}_{2,1}\exp(-\tilde{\theta}_1)\quad;\quad\Lambda_K=\varepsilon^3\,Da_K\exp(-\tilde{E}_K/\tilde{T}_{f,S})$$

$$\frac{\tilde{\phi}_{1,1}}{Le_1}-\frac{\tilde{\phi}_{2,1}}{Le_2}=\tilde{c}_1\,\xi+\tilde{c}_2\quad;\quad\tilde{\theta}_1-\frac{\tilde{\phi}_{1,1}}{Le_1}=\tilde{c}_4\,\xi+\tilde{c}_5$$

$$\left[\frac{1}{Le_1}\frac{\partial\tilde{\phi}_{1,2}}{\partial\xi}+\left(\frac{2\xi}{Le_1\,\tilde{r}_{f,0}}\right)\frac{\partial\tilde{\phi}_{1,1}}{\partial\xi}-\frac{\tilde{m}}{\tilde{r}_{f,0}^2}\tilde{\phi}_{1,1}\right]-\left[\frac{1}{Le_2}\frac{\partial\tilde{\phi}_{2,2}}{\partial\xi}+\left(\frac{2\xi}{Le_2\,\tilde{r}_{f,0}}\right)\frac{\partial\tilde{\phi}_{2,1}}{\partial\xi}-\frac{\tilde{m}}{\tilde{r}_{f,0}^2}\tilde{\phi}_{2,1}\right]=\tilde{c}_3$$

$$\left(\frac{\partial\tilde{\theta}_2}{\partial\xi}+\frac{2\xi}{\tilde{r}_{f,0}}\frac{\partial\tilde{\theta}_1}{\partial\xi}-\frac{\tilde{m}}{\tilde{r}_{f,0}^2}\tilde{\theta}_1\right)-\left[\frac{1}{Le_1}\frac{\partial\tilde{\phi}_{1,2}}{\partial\xi}+\left(\frac{2\xi}{Le_1\,\tilde{r}_{f,0}}\right)\frac{\partial\tilde{\phi}_{1,1}}{\partial\xi}-\frac{\tilde{m}}{\tilde{r}_{f,0}^2}\tilde{\phi}_{1,1}\right]=\tilde{c}_6$$

(I) Matching

$$\tilde{Y}_{1,S,0}^-(\tilde{r}_{f,S})=\tilde{Y}_{1,S,2}^-=\tilde{Y}_{1,S,0}^+=\tilde{Y}_{1,S,2}^+=\tilde{Y}_{2,S,0}^-=\tilde{Y}_{2,S,2}^-=\tilde{Y}_{2,S,0}^+(\tilde{r}_{f,S})=\tilde{Y}_{2,S,2}^+=\hat{Y}_{1,0}^+=\hat{Y}_{2,0}^-=0$$

$$(d\tilde{Y}_{1,S,0}^-/d\tilde{r})_{\tilde{r}_{f,S}}=-(Le_1\,\tilde{m}/\tilde{r}_{f,S}^2)\quad;\quad(d^2\tilde{Y}_{1,S,0}^-/d\tilde{r}^2)_{\tilde{r}_{f,S}}=-(Le_1\,\tilde{m}/\tilde{r}_{f,S}^3)[(Le_1\,\tilde{m}/\tilde{r}_{f,S})-2]$$

$$(d\tilde{Y}_{2,S,0}^+/d\tilde{r})_{\tilde{r}_{f,S}}=(Le_2\,\tilde{m}/\tilde{r}_{f,S}^2)\quad;\quad(d^2\tilde{Y}_{2,S,0}^+/d\tilde{r}^2)_{\tilde{r}_{f,S}}=(Le_2\,\tilde{m}/\tilde{r}_{f,S}^3)[(Le_2\,\tilde{m}/\tilde{r}_{f,S})-2]$$

$$\tilde{Y}_{1,S,1}^-(\tilde{r}_{f,S})=-a_{1,1}^-\exp(-Le_1\,\tilde{m}/\tilde{r}_{f,S})\quad,\quad\tilde{Y}_{1,S,1}^+(\tilde{r}_{f,S})=a_{1,1}^+[1-\exp(-Le_1\,\tilde{m}/\tilde{r}_{f,S})]$$

$$(d\tilde{Y}_{1,S,1}^-/d\tilde{r})_{\tilde{r}_{f,S}}=-(\tilde{m}/\tilde{r}_{f,S}^2)\,a_{1,1}^-\,Le_1\exp(-Le_1\,\tilde{m}/\tilde{r}_{f,S})$$

$$(d\tilde{Y}_{1,S,1}^+/d\tilde{r})_{\tilde{r}_{f,S}}=-a_{1,1}^+(Le_1\,\tilde{m}/\tilde{r}_{f,S}^2)\exp(-Le_1\,\tilde{m}/\tilde{r}_{f,S})$$

$$(d^2\tilde{Y}_{1,S,1}^+/d\tilde{r}^2)_{r_{f,S}}=a_{1,1}^+\exp(-Le_1\,\tilde{m}/\tilde{r}_{f,S})[2-(Le_1\,\tilde{m}/\tilde{r}_{f,S})](Le_1\,\tilde{m}/\tilde{r}_{f,S}^3)$$

$$\tilde{Y}_{2,S,1}^-(\tilde{r}_{f,S})=a_{2,1}^-/(1+\tilde{Y}_{2,\infty})\quad;\quad\tilde{Y}_{2,S,1}^+(\tilde{r}_{f,S})=a_{1,1}^+[\tilde{Y}_{2,\infty}/(1+\tilde{Y}_{2,\infty})]$$

$$(d\tilde{Y}^-_{2,S,1}/d\tilde{r})_{\tilde{r}_{f,S}} = a^-_{2,1}(Le_2\,\tilde{m}/\tilde{r}^2_{f,S})/(1+\tilde{Y}_{2,\infty})$$

$$(d\tilde{Y}^+_{2,S,1}/d\tilde{r})_{\tilde{r}_{f,S}} = -(\tilde{m}/\tilde{r}^2_{f,S})\,[a^+_{1,1}\,Le_2/(1+\tilde{Y}_{2,\infty})]$$

$$(d^2\tilde{Y}^+_{2,S,1}/d\tilde{r}^2)_{\tilde{r}_{fs}} = a^+_{11}(Le_2\tilde{m}/\tilde{r}^3_{fs})\exp(-Le_2\,\tilde{m}/\tilde{r}_{fs})[2-(Le_2\tilde{m}/\tilde{r}_{fs})]$$

$$\hat{T}^-_A = (d\hat{T}^-_0/d\tilde{r})_{\tilde{r}_{f,S}} + \hat{r}_{f,0}\,g^-_0[(\tilde{m}/\tilde{r}^2_{f,S})-(2/\tilde{r}_{f,S})] \quad , \quad \hat{T}^-_B = \hat{T}^-_2(\tilde{r}_{f,S}) + \hat{r}_{f,0}\,g^-_1(\tilde{m}/\tilde{r}^2_{f,S})\,, \quad \hat{T}^-_C = \{[\hat{T}^-_1(\tilde{r}_{f,S}) + \hat{r}_{f,1}\,g^-_0]/\tilde{T}_{b,S,A}\}$$

$$\hat{T}^+_A = (d\hat{T}^+_0/d\tilde{r})_{\tilde{r}_{f,S}} - \hat{r}_{f,0}\,g^+_0[(\tilde{m}/\tilde{r}^2_{f,S})-(2/\tilde{r}_{f,S})] \quad , \quad \hat{T}^+_B = \hat{T}^+_2(\tilde{r}_{f,S}) - \hat{r}_{f,0}\,g^+_1(\tilde{m}/\tilde{r}^2_{f,S})/[\exp(\tilde{m}/\tilde{r}_{f,S})-1]$$

$$\hat{T}^+_C = [\hat{T}^+_1(\tilde{r}_{f,S}) - \hat{r}_{f,1}\,g^+_0]/a^+_{T,A}$$

Species equations: Between the inner and outer solutions

Energy equation: Between the inner solution and the solution in the radiation region

In the common regions between the outer and inner regions, $\tilde{r} = \tilde{r}_{f,0} + \varepsilon\xi$, $\tilde{r}_{f,0} = \tilde{r}_{f,S} + \bar{\delta}\,\hat{r}_{f,0}\,P_n(\tilde{z})$

Define $\hat{\xi} = \xi - \bar{\delta}\,P_n(\tilde{z})\hat{r}_{f,1}$ then $\partial\hat{\xi} = \partial\xi$

Since $\tilde{r} = \tilde{r}_f + \delta\zeta = \tilde{r}_{f,0} + \varepsilon\bar{\delta}\,P_n(\tilde{z})\hat{r}_{f,1} + \delta\zeta = \tilde{r}_{f,0} + \varepsilon\xi$, we also have $\zeta = (\varepsilon/\delta)[\xi - \bar{\delta}\,P_n(\tilde{z})\hat{r}_{f,1}] = (\varepsilon/\delta)\hat{\xi}$

$$\tilde{T}^{\pm} = [\tilde{T}_f - \varepsilon\tilde{\Theta}^{\pm}_2 + O(\varepsilon^2)] - \delta[\tilde{\Theta}^{\pm}_1 + \varepsilon\tilde{\Theta}^{\pm}_3 + O(\varepsilon^2)] - \delta^2[\tilde{\Theta}^{\pm}_4 + O(\varepsilon)] + O(\delta^3)$$

$$= \{\tilde{T}_f - \varepsilon[\tilde{\Theta}^{\pm}_2(\zeta=0) + O(\varepsilon/\delta)] + O(\varepsilon^2)\} - \delta\langle\{\tilde{\Theta}^{\pm}_1(\zeta=0) + (\partial\tilde{\Theta}^{\pm}_1/\partial\zeta)_{\zeta=0}[(\varepsilon/\delta)\hat{\xi}] + O(\varepsilon/\delta)^2\}$$

$$+\varepsilon\{\tilde{\Theta}^{\pm}_3(\zeta=0) + (\partial\tilde{\Theta}^{\pm}_3/\partial\zeta)_{\zeta=0}[(\varepsilon/\delta)\hat{\xi}] + O(\varepsilon/\delta)^2\} + O(\varepsilon^2)\rangle$$

$$-\delta^2\langle\{\tilde{\Theta}^{\pm}_4(\zeta=0) + (\partial\tilde{\Theta}^{\pm}_4/\partial\zeta)_{\zeta=0}(\varepsilon/\delta)\hat{\xi} + (\partial^2\tilde{\Theta}^{\pm}_4/\partial\zeta^2)_{\zeta=0}[(\varepsilon/\delta)^2\hat{\xi}^2/2] + O(\varepsilon/\delta)^3\} + O(\varepsilon)\rangle + O(\delta^3)$$

$$= [\tilde{T}_f - \varepsilon\tilde{\theta}_1 - \varepsilon^2\tilde{\theta}_2 + O(\varepsilon^3)] + O(\varepsilon/\delta)\Big|_{\xi\to\pm\infty}$$

$$\Rightarrow \quad \tilde{\Theta}^{\pm}_1(\zeta=0) = 0\,; \quad \tilde{\theta}_1(\xi\to\pm\infty) = \tilde{\Theta}^{\pm}_2(\zeta=0) + (\partial\tilde{\Theta}^{\pm}_1/\partial\zeta)_{\zeta=0}\hat{\xi} = \tilde{\Theta}^{\pm}_2(\zeta=0) + (\partial\tilde{\Theta}^{\pm}_1/\partial\zeta)_{\zeta=0}[\xi - \bar{\delta}\,P_n(\tilde{z})\hat{r}_{f,1}]$$

$$(\partial\tilde{\theta}_2/\partial\xi)_{\xi\to\pm\infty} = (\partial\tilde{\Theta}^{\pm}_3/\partial\zeta)_{\zeta=0} + (\partial^2\tilde{\Theta}^{\pm}_4/\partial\zeta^2)_{\zeta=0}\,\hat{\xi} = (\partial\tilde{\Theta}^{\pm}_3/\partial\zeta)_{\zeta=0} + (\partial^2\tilde{\Theta}^{\pm}_4/\partial\zeta^2)_{\zeta=0}[\xi - \bar{\delta}\,P_n(\tilde{z})\hat{r}_{f,1}]$$

$$\tilde{Y}^{\pm}_i = [\tilde{Y}^{\pm}_{i,0} + \varepsilon\tilde{Y}^{\pm}_{i,1} + O(\varepsilon^2)] + \delta[\tilde{Y}^{\pm}_{i,2} + O(\varepsilon)] + O(\delta^2)$$

$$= \{[\tilde{Y}^{\pm}_{i,0}(\tilde{r}_{f,0}) + (\partial\tilde{Y}^{\pm}_{i,0}/\partial\tilde{r})_{\tilde{r}_{f,0}}(\varepsilon\xi) + (\partial^2\tilde{Y}^{\pm}_{i,0}/\partial\tilde{r}^2)_{\tilde{r}_{f,0}}(\varepsilon^2\xi^2/2) + O(\varepsilon^3)] + \varepsilon[\tilde{Y}^{\pm}_{i,1}(\tilde{r}_{f,0}) + (\partial\tilde{Y}^{\pm}_{i,1}/\partial\tilde{r})_{\tilde{r}_{f,0}}(\varepsilon\xi) + O(\varepsilon^2)] + O(\varepsilon^2)\}$$

$$+\delta\{[\tilde{Y}^{\pm}_{i,2}(\tilde{r}_{f,0}) + O(\varepsilon)] + O(\varepsilon)\} + O(\delta^2)$$

$$= [\varepsilon\tilde{\phi}_{i,1} + \varepsilon^2\tilde{\phi}_{i,2} + O(\varepsilon^3)] + O(\delta) + O(\varepsilon/\delta)\Big|_{\xi\to\pm\infty}$$

$$\Rightarrow \quad \tilde{Y}^{\pm}_{i,0}(\tilde{r}_{f,0}) = 0 \quad ; \quad \tilde{Y}^{\pm}_{i,2}(\tilde{r}_{f,0}) = 0 \quad ; \quad \tilde{\phi}_{i,1}(\xi\to\pm\infty) = \tilde{Y}^{\pm}_{i,1}(\tilde{r}_{f,0}) + (\partial\tilde{Y}^{\pm}_{i,0}/\partial\tilde{r})_{\tilde{r}_{f,0}}\,\xi$$

$$(\partial\tilde{\phi}_{i,2}/\partial\xi)_{\xi\to\pm\infty} = (\partial\tilde{Y}^{\pm}_{i,1}/\partial\tilde{r})_{\tilde{r}_{f,0}} + (\partial^2\tilde{Y}^{\pm}_{i,0}/\partial\tilde{r}^2)_{\tilde{r}_{f,0}}\,\xi \quad ; \quad (\partial\phi_{i,1}/\partial\xi)_{\xi\to\pm\infty} = (\partial\tilde{Y}^{\pm}_{i,0}/\partial\tilde{r})_{\tilde{r}_{f,0}}$$

(1) $$\tilde{Y}^-_{1,0}(\tilde{r}_{f,0}) = \tilde{Y}^-_{1,S,0}(\tilde{r}_{f,0}) + \bar{\delta}\,P_n\hat{Y}^-_{1,0}(\tilde{r}_{f,0}) = \tilde{Y}^-_{1,S,0}(\tilde{r}_{f,S} + \bar{\delta}\,P_n\,\hat{r}_{f,0}) + \bar{\delta}\,P_n\,\hat{Y}^-_{1,0}(\tilde{r}_{f,S} + \bar{\delta}\,P_n\,\hat{r}_{f,0})$$

$$= [\tilde{Y}^-_{1,S,0}(\tilde{r}_{f,S}) + \bar{\delta}\,P_n\,\hat{r}_{f,0}(d\tilde{Y}^-_{1,S,0}/d\tilde{r})_{\tilde{r}_{f,S}} + O(\bar{\delta}^2)] + \bar{\delta}\,P_n[\hat{Y}^-_{1,0}(\tilde{r}_{f,S}) + O(\bar{\delta})]$$

$$= \tilde{Y}^-_{1,S,0}(\tilde{r}_{f,S}) + \bar{\delta}\,P_n[\hat{Y}^-_{1,0}(\tilde{r}_{f,S}) + \hat{r}_{f,0}(d\tilde{Y}^-_{1,S,0}/d\tilde{r})_{\tilde{r}_{f,S}}] + O(\bar{\delta}^2) = \bar{\delta}\,P_n[\hat{Y}^-_{1,0}(\tilde{r}_{f,S}) - \hat{r}_{f,0}(Le_{1,0}\,\tilde{m}/\tilde{r}^2_{f,S})] + O(\bar{\delta}^2) = 0$$

$$\hat{Y}^-_{1,0}(\tilde{r}_{f,S}) - \hat{r}_{f,0}(Le_1\,\tilde{m}/\tilde{r}^2_{f,S}) = 0 \quad \text{or} \quad \hat{Y}^-_{1,0}(\tilde{r}_{f,S}) = Le_1\,\hat{r}_{f,0}\,\tilde{m}/\tilde{r}^2_{f,S}$$

$$\hat{Y}^-_{1,0}(\tilde{r}_{f,S}) = \hat{a}^-_{1,0}\left\{\Psi_1(\tilde{r}_{f,S},Le_1) - \frac{[\tilde{m}\,Le_1\,\Psi_1(1,Le_1) - \Psi^*_1(1,Le_1)]}{[\tilde{m}\,Le_1\,\Psi_2(1,Le_1) - \Psi^*_2(1,Le_1)]}\Psi_2(\tilde{r}_{f,S},Le_1)\right\} = Le_1\,\hat{r}_{f,0}\,\tilde{m}/\tilde{r}^2_{f,S}$$

$$\hat{a}^-_{1,0} = Le_1\,\hat{r}_{f,0}(\tilde{m}/\tilde{r}^2_{f,S})/\left\{\Psi_1(\tilde{r}_{f,S},Le_1) - \frac{[\tilde{m}\,Le_1\,\Psi_1(1,Le_1) - \Psi^*_1(1,Le_1)]}{[\tilde{m}\,Le_{1,0}\,\Psi_2(1,Le_1) - \Psi^*_2(1,Le_1)]}\Psi_2(\tilde{r}_{f,S},Le_1)\right\}$$

$$\hat{Y}^-_{1,0} = Le_1\,\hat{r}_{f,0}\,\frac{\tilde{m}}{\tilde{r}^2_{f,S}}\,\frac{[\tilde{m}\,Le_1\,\Psi_2(1,Le_1) - \Psi^*_2(1,Le_1)]\Psi_1(\tilde{r},Le_1) - [\tilde{m}\,Le_1\,\Psi_1(1,Le_1) - \Psi^*_1(1,Le_1)]\Psi_2(\tilde{r},Le_1)}{[\tilde{m}\,Le_1\,\Psi_2(1,Le_1) - \Psi^*_2(1,Le_1)]\Psi_1(\tilde{r}_{f,S},Le_1) - [\tilde{m}\,Le_1\,\Psi_1(1,Le_1) - \Psi^*_1(1,Le_1)]\Psi_2(\tilde{r}_{f,S},Le_1)}$$

$$[\tilde{r}^2(d^2\hat{Y}^-_{1,0}/d\tilde{r}^2) - (Le_1\,\tilde{m} - 2\tilde{r})(d\hat{Y}^-_{1,0}/d\tilde{r})]_{\tilde{r}_{f,S}} = n(n+1)\hat{Y}^-_{1,0}(\tilde{r}_{f,S}) = n(n+1)Le_1\,\hat{r}_{f,0}\,\tilde{m}/\tilde{r}^2_{f,S}$$

(2) $$\tilde{Y}^+_{1,0}(\tilde{r}_{f,0}) = \tilde{Y}^+_{1,S,0}(\tilde{r}_{f,S}) + \bar{\delta}\,P_n[\hat{Y}^+_{1,0}(\tilde{r}_{f,S}) + \hat{r}_{f,0}(d\tilde{Y}^+_{1,S,0}/d\tilde{r})_{\tilde{r}_{f,S}}] + O(\bar{\delta}^2) = \bar{\delta}\,P_n\,\hat{Y}^+_{1,0}(\tilde{r}_{f,S}) + O(\bar{\delta}^2) = 0$$

$$\hat{Y}^{+}_{1,0}(\tilde{r}_{f,S})=\hat{a}^{+}_{1,0}\,\Psi_2(\tilde{r}_{f,S},Le_1)=0 \qquad \therefore \quad \hat{a}^{+}_{1,0}=0 \quad \text{and} \quad \hat{Y}^{+}_{1,0}=0$$

(3) $\tilde{Y}^{-}_{1,2}(\tilde{r}_{f,0})=\tilde{Y}^{-}_{1,S,2}(\tilde{r}_{f,S})+\bar{\delta}\,P_n[\hat{Y}^{-}_{1,2}(\tilde{r}_{f,S})+\hat{r}_{f,0}(d\tilde{Y}^{-}_{1,S,2}/d\tilde{r})_{\tilde{r}_{f,S}}]+O(\bar{\delta}^2)=\bar{\delta}\,P_n\hat{Y}^{-}_{1,2}(\tilde{r}_{f,S})+O(\bar{\delta}^2)=0$

$$\hat{Y}^{-}_{1,2}(\tilde{r}_{f,S})=\hat{a}^{-}_{1,2}\left\{\Psi_1(\tilde{r}_{f,S},Le_1)-\frac{[\tilde{m}Le_1\Psi_1(1,Le_1)-\Psi^{*}_1(1,Le_1)]}{[\tilde{m}Le_1\Psi_2(1,Le_1)-\Psi^{*}_2(1,Le_1)]}\Psi_2(\tilde{r}_{f,S},Le_1)\right\}=0 \quad \therefore \quad \hat{a}^{-}_{1,2}=0\,, \quad \hat{Y}^{-}_{1,2}=0$$

(4) $\tilde{Y}^{+}_{1,2}(\tilde{r}_{f,0})=\tilde{Y}^{+}_{1,S,2}(\tilde{r}_{f,S})+\bar{\delta}\,P_n[\hat{Y}^{+}_{1,2}(\tilde{r}_{f,S})+\hat{r}_{f,0}(d\tilde{Y}^{+}_{1,S,2}/d\tilde{r})_{\tilde{r}_{f,S}}]+O(\bar{\delta}^2)=\bar{\delta}\,P_n\hat{Y}^{+}_{1,2}(\tilde{r}_{f,S})+O(\bar{\delta}^2)=0$

$$\hat{Y}^{+}_{1,2}(\tilde{r}_{f,S})=\hat{a}^{+}_{1,2}\,\Psi_2(\tilde{r}_{f,S},Le_1)=0 \qquad \therefore \quad \hat{a}^{+}_{1,2}=0\,, \quad \hat{Y}^{+}_{1,2}=0$$

(5) $\tilde{Y}^{-}_{2,0}(\tilde{r}_{f,0})=\tilde{Y}^{-}_{2,S,0}(\tilde{r}_{f,S})+\bar{\delta}\,P_n[\hat{Y}^{-}_{2,0}(\tilde{r}_{f,S})+\hat{r}_{f,0}(d\tilde{Y}^{-}_{2,S,0}/d\tilde{r})_{\tilde{r}_{f,S}}]+O(\bar{\delta}^2)=\bar{\delta}\,P_n\hat{Y}^{-}_{2,0}(\tilde{r}_{f,S})+O(\bar{\delta}^2)=0$

$$\hat{Y}^{-}_{2,0}(\tilde{r}_{f,S})=\hat{a}^{-}_{2,0}\left\{\Psi_1(\tilde{r}_{f,S},Le_2)-\frac{[\tilde{m}Le_2\Psi_1(1,Le_2)-\Psi^{*}_1(1,Le_2)]}{[\tilde{m}Le_2\Psi_2(1,Le_2)-\Psi^{*}_2(1,Le_2)]}\Psi_2(\tilde{r}_{f,S},Le_2)\right\}=0 \qquad \therefore \quad \hat{a}^{-}_{2,0}=0\,, \quad \hat{Y}^{-}_{2,0}=0$$

(6) $\tilde{Y}^{+}_{2,0}(\tilde{r}_{f,0})=\tilde{Y}^{+}_{2,S,0}(\tilde{r}_{f,S})+\bar{\delta}P_n[\hat{Y}^{+}_{2,0}(\tilde{r}_{f,S})+\hat{r}_{f,0}(d\tilde{Y}^{+}_{2,S,0}/d\tilde{r})_{\tilde{r}_{f,S}}]+O(\bar{\delta}^2)=\bar{\delta}P_n[\hat{Y}^{+}_{2,0}(\tilde{r}_{f,S})+\hat{r}_{f,0}(Le_2\tilde{m}/\tilde{r}^2_{f,S})]+O(\bar{\delta}^2)=0$

$$\hat{Y}^{+}_{2,0}(\tilde{r}_{f,S})+\hat{r}_{f,0}(Le_2\,\tilde{m}/\tilde{r}^2_{f,S})=0 \qquad \text{or} \qquad \hat{Y}^{+}_{2,0}(\tilde{r}_{f,S})=\hat{a}^{+}_{2,0}\,\Psi_2(\tilde{r}_{f,S},Le_2)=-Le_2\,\hat{r}_{f,0}\,\tilde{m}/\tilde{r}^2_{f,S}$$

$$\hat{a}^{+}_{2,0}=-Le_2\,\hat{r}_{f,0}(\tilde{m}/\tilde{r}^2_{f,S})/\Psi_2(\tilde{r}_{f,S},Le_2) \qquad \text{and} \qquad \hat{Y}^{+}_{2,0}=-Le_2\,\hat{r}_{f,0}(\tilde{m}/\tilde{r}^2_{f,S})[\Psi_2(\tilde{r},Le_2)/\Psi_2(\tilde{r}_{f,S},Le_2)]$$

$$[\tilde{r}^2(d^2\hat{Y}^{+}_{2,0}/d\tilde{r}^2)-(Le_2\tilde{m}-2\tilde{r})(d\hat{Y}^{+}_{2,0}/d\tilde{r})]_{\tilde{r}_{f,S}}=n(n+1)\hat{Y}^{+}_{2,0}(\tilde{r}_{f,S})=-n(n+1)\,Le_2\,\hat{r}_{f,0}\,\tilde{m}/\tilde{r}^2_{f,S}$$

(7) $\tilde{Y}^{-}_{2,2}(\tilde{r}_{f,0})=\tilde{Y}^{-}_{2,S,2}(\tilde{r}_{f,S})+\bar{\delta}\,P_n[\hat{Y}^{-}_{2,2}(\tilde{r}_{f,S})+\hat{r}_{f,0}(d\tilde{Y}^{-}_{2,S,2}/d\tilde{r})_{\tilde{r}_{f,S}}]+O(\bar{\delta}^2)=\bar{\delta}\,P_n\hat{Y}^{-}_{2,2}(\tilde{r}_{f,S})+O(\bar{\delta}^2)=0$

$$\hat{Y}^{-}_{2,2}(\tilde{r}_{f,S})=\hat{a}^{-}_{2,2}\left\{\Psi_1(\tilde{r}_{f,S},Le_2)-\frac{[\tilde{m}Le_2\Psi_1(1,Le_2)-\Psi^{*}_1(1,Le_2)]}{[\tilde{m}Le_2\Psi_2(1,Le_2)-\Psi^{*}_2(1,Le_2)]}\Psi_2(\tilde{r}_{f,S},Le_2)\right\}=0 \qquad \therefore \quad \hat{a}^{-}_{2,2}=0\,, \quad \hat{Y}^{-}_{2,2}=0$$

(8) $\tilde{Y}^{+}_{2,2}(\tilde{r}_{f,0})=\tilde{Y}^{+}_{2,S,2}(\tilde{r}_{f,S})+\bar{\delta}\,P_n[\hat{Y}^{+}_{2,2}(\tilde{r}_{f,S})+\hat{r}_{f,0}(d\tilde{Y}^{+}_{2,S,2}/d\tilde{r})_{\tilde{r}_{f,S}}]+O(\bar{\delta}^2)=\bar{\delta}\,P_n\hat{Y}^{+}_{2,2}(\tilde{r}_{f,S})+O(\bar{\delta}^2)=0$

$$\hat{Y}^{+}_{2,2}(\tilde{r}_{f,S})=\hat{a}^{+}_{2,2}\,\Psi_2(\tilde{r}_{f,S},Le_2)=0 \qquad \therefore \quad \hat{a}^{+}_{2,2}=0 \quad \text{and} \quad \hat{Y}^{+}_{2,2}=0$$

(9) $\tilde{\Theta}^{-}_1(\zeta=0)=\Theta^{-}_{S,1}(\zeta=0)-\bar{\delta}\,P_n\langle\hat{T}^{-}_B+2[\hat{T}^{-}_B-(2/g^{-}_0)\hat{T}^{-}_A]/\{[2(g^{-}_0)^2/\Lambda_R]\exp(-g^{-}_1)-1\}\rangle+O(\bar{\delta}^2)=0\,; \qquad \Theta^{-}_{S,1}(\zeta=0)=0$

$$\hat{T}^{-}_B+2[\hat{T}^{-}_B-(2/g^{-}_0)\hat{T}^{-}_A]/\{[2(g^{-}_0)^2/\Lambda_R]\exp(-g^{-}_1)-1\}=0 \qquad \text{or} \qquad \hat{T}^{-}_B=4\hat{T}^{-}_A/\langle g^{-}_0\{[2(g^{-}_0)^2/\Lambda_R]\exp(-g^{-}_1)+1\}\rangle$$

Since $\hat{T}^{-}_B=\hat{T}^{-}_2(\tilde{r}_{f,S})+\hat{r}_{f,0}\,g^{-}_1(\tilde{m}/\tilde{r}^2_{f,S})$, $\hat{T}^{-}_2=\hat{T}_{b,2}\{A_{T2}\Psi_1(\tilde{r},1)+[1-A_{T2}\Psi_1(1,1)]\Psi_2(\tilde{r},1)/\Psi_2(1,1)\}$

$$\hat{T}_{b,2}\{A_{T2}\Psi_1(\tilde{r}_{f,S},1)+[1-A_{T2}\Psi_1(1,1)]\Psi_2(\tilde{r}_{f,S},1)/\Psi_2(,1)\}=4\hat{T}^{-}_A/\langle g^{-}_0\{[2(g^{-}_0)^2/\Lambda_R]\exp(-g^{-}_1)+1\}\rangle-\hat{r}_{f,0}g^{-}_1(\tilde{m}/\tilde{r}^2_{f,S})$$

$$\hat{T}^{-}_2=\left[\frac{4\hat{T}^{-}_A}{g^{-}_0\{[2(g^{-}_0)^2/\Lambda_R]\exp(-g^{-}_1)+1\}}-\hat{r}_{f,0}\,g^{-}_1\frac{\tilde{m}}{\tilde{r}^2_{f,S}}\right]\frac{A_{T2}\Psi_1(\tilde{r},1)+[1-A_{T2}\Psi_1(1,1)]\Psi_2(\tilde{r},1)/\Psi_2(1,1)}{A_{T2}\Psi_1(\tilde{r}_{f,S},1)+[1-A_{T2}\Psi_1(1,1)]\Psi_2(\tilde{r}_{f,S},1)/\Psi_2(1,1)}$$

$$\hat{T}^{-}_A=(d\hat{T}^{-}_0/d\tilde{r})_{\tilde{r}_{f,S}}+\hat{r}_{f,0}\,g^{-}_0[(\tilde{m}/\tilde{r}^2_{f,S})-(2/\tilde{r}_{f,S})]=-\hat{r}_{f,0}\,g^{-}_0\left\{\frac{A_{T2}\Psi_2(1,1)\Psi^{*}_1(\tilde{r}_{f,S},1)+[1-A_{T2}\Psi_1(1,1)]\Psi^{*}_1(\tilde{r}_{f,S},1)}{A_{T2}\Psi_2(1,1)\Psi_1(\tilde{r}_{f,S},1)+[1-A_{T2}\Psi_1(1,1)]\Psi_2(\tilde{r}_{f,S},1)}-\frac{\tilde{m}}{\tilde{r}^2_{f,S}}+\frac{2}{\tilde{r}_{f,S}}\right\}$$

$$\exp(-g^{-}_1)=\left[\langle\{1+[2\Lambda_R/(g^{-}_0)^2]\}^{1/2}+1\rangle/2\right]^2$$

(10) $\tilde{\Theta}^{+}_1(\zeta=0)=\Theta^{+}_{S,1}(\zeta=0)-\bar{\delta}\,P_n\langle\hat{T}^{+}_B+2[\hat{T}^{+}_B+(2/g^{+}_0)\hat{T}^{+}_A]/\{[2(g^{+}_0)^2/\Lambda_R]\exp(-g^{+}_1)-1\}\rangle+O(\bar{\delta}^2)=0\,; \qquad \Theta^{+}_{S,1}(\zeta=0)=0$

$$\hat{T}^{+}_B+2[\hat{T}^{+}_B+(2/g^{+}_0)\hat{T}^{+}_A]/\{[2(g^{+}_0)^2/\Lambda_R]\exp(-g^{+}_1)-1\}\rangle=0 \qquad \text{or} \qquad \hat{T}^{+}_B=-4\hat{T}^{+}_A/\langle g^{+}_0\{[2(g^{+}_0)^2/\Lambda_R]\exp(-g^{+}_1)+1\}\rangle$$

Since $\hat{T}^{+}_B=\hat{T}^{+}_2(\tilde{r}_{f,S})-\hat{r}_{f,0}\,g^{+}_1(\tilde{m}/\tilde{r}^2_{f,S})/[\exp(\tilde{m}/\tilde{r}_{f,S})-1]$, $\hat{T}^{+}_2=\hat{a}^{+}_{T,2}\,\Psi_2(\tilde{r},1)$

$$\hat{a}^{+}_{T,2}=\left[\{\hat{r}_{f,0}\,g^{+}_1(\tilde{m}/\tilde{r}^2_{f,S})/[\exp(\tilde{m}/\tilde{r}_{f,S})-1]\}-4\hat{T}^{+}_A/\langle g^{+}_0\{[2(g^{+}_0)^2/\Lambda_R]\exp(-g^{+}_1)+1\}\rangle\right]/\Psi_2(\tilde{r}_{f,S},1)$$

$$\hat{T}^{+}_2=\left[\{\hat{r}_{f,0}\,g^{+}_1(\tilde{m}/\tilde{r}^2_{f,S})/[\exp(\tilde{m}/\tilde{r}_{f,S})-1]\}-4\hat{T}^{+}_A/\langle g^{+}_0\{[2(g^{+}_0)^2/\Lambda_R]\exp(-g^{+}_1)+1\}\rangle\right][\Psi_2(\tilde{r},1)/\Psi_2(\tilde{r}_{f,S},1)]$$

$$\hat{T}^{+}_A=(d\hat{T}^{+}_0/d\tilde{r})_{\tilde{r}_{f,S}}-\hat{r}_{f,0}\,g^{+}_0[(\tilde{m}/\tilde{r}^2_{f,S})-(2/\tilde{r}_{f,S})]=\hat{r}_{f,0}\,g^{+}_0\{[\Psi^{*}_2(\tilde{r}_{f,S},1)/\Psi_2(\tilde{r}_{f,S},1)]-(\tilde{m}/\tilde{r}^2_{f,S})+(2/\tilde{r}_{f,S})\}$$

$$\exp(-g^{+}_1)=\left[\langle\{1+[2\Lambda_R/(g^{+}_0)^2]\}^{1/2}+1\rangle/2\right]^2$$

(11) $\tilde{\phi}_{1,1}(\xi\to-\infty)=\tilde{Y}^{-}_{1,1}(\tilde{r}_{f,0})+(\partial\tilde{Y}^{-}_{1,0}/\partial\tilde{r})_{\tilde{r}_{f,0}}\,\xi=[\tilde{Y}^{-}_{1,S,1}(\tilde{r}_{f,0})+\bar{\delta}\,P_n\hat{Y}^{-}_{1,1}(\tilde{r}_{f,0})]+[(d\tilde{Y}^{-}_{1,S,0}/d\tilde{r})_{\tilde{r}_{f,0}}+\bar{\delta}\,P_n(d\hat{Y}^{-}_{1,0}/d\tilde{r})_{\tilde{r}_{f,0}}]\xi+O(\bar{\delta}^2)$

$$=\{\tilde{Y}^{-}_{1,S,1}(\tilde{r}_{f,S})+\bar{\delta}\,P_n[\hat{Y}^{-}_{1,1}+\hat{r}_{f,0}(d\tilde{Y}^{-}_{1,S,1}/d\tilde{r})]_{\tilde{r}_{f,S}}\}+\{(d\tilde{Y}^{-}_{1,S,0}/d\tilde{r})_{\tilde{r}_{f,S}}+\bar{\delta}\,P_n[(d\hat{Y}^{-}_{1,0}/d\tilde{r})+\hat{r}_{f,0}(d^2\tilde{Y}^{-}_{1,S,0}/d\tilde{r}^2)]_{\tilde{r}_{f,S}}\}\xi+O(\bar{\delta}^2)$$

$$=\left[-a^{-}_{1,1}\exp(-Le_1\tilde{m}/\tilde{r}_{f,S})+\bar{\delta}P_n\{\hat{Y}^{-}_{1,1}(\tilde{r}_{f,S})-\hat{r}_{f,0}(\tilde{m}/\tilde{r}^2_{f,S})a^{-}_{1,1}Le_1\exp(-Le_1\tilde{m}/\tilde{r}_{f,S})\}\right]$$

$$+\langle-(Le_1\tilde{m}/\tilde{r}^2_{f,S})+\bar{\delta}\,P_n\{(d\hat{Y}^{-}_{1,0}/d\tilde{r})_{\tilde{r}_{f,S}}-\hat{r}_{f,0}(Le_1\tilde{m}/\tilde{r}^3_{f,S})[(Le_1\tilde{m}/\tilde{r}_{f,S})-2]\}\rangle\xi+O(\bar{\delta}^2)$$

$$(\partial\tilde{\phi}_{1,1}/\partial\xi)_{\xi\to-\infty}=(d\tilde{Y}^{-}_{1,S,0}/d\tilde{r})_{\tilde{r}_{f,S}}+\bar{\delta}\,P_n[(d\hat{Y}^{-}_{1,0}/d\tilde{r})+\hat{r}_{f,0}(d^2\tilde{Y}^{-}_{1,S,0}/d\tilde{r}^2)]_{\tilde{r}_{f,S}}+O(\bar{\delta}^2)$$

$$=-(Le_1\tilde{m}/\tilde{r}^2_{f,S})+\bar{\delta}\,P_n\{(d\hat{Y}^{-}_{1,0}/d\tilde{r})_{\tilde{r}_{f,S}}-\hat{r}_{f,0}(Le_1\tilde{m}/\tilde{r}^3_{f,S})[(Le_1\tilde{m}/\tilde{r}_{f,S})-2]\}+O(\bar{\delta}^2)$$

(12) $\tilde{\phi}_{1,1}(\xi\to\infty)=\tilde{Y}^{+}_{1,1}(\tilde{r}_{f,0})+(\partial\tilde{Y}^{+}_{1,0}/\partial\tilde{r})_{\tilde{r}_{f,0}}\xi=[\tilde{Y}^{+}_{1,S,1}(\tilde{r}_{f,0})+\bar{\delta}P_n\hat{Y}^{+}_{1,1}(\tilde{r}_{f,0})]+[(d\tilde{Y}^{+}_{1,S,0}/d\tilde{r})_{\tilde{r}_{f,0}}+\bar{\delta}P_n(d\hat{Y}^{+}_{1,0}/d\tilde{r})_{\tilde{r}_{f,0}}]\xi+O(\bar{\delta}^2)$

$$=\{\tilde{Y}^{+}_{1,S,1}(\tilde{r}_{f,S})+\bar{\delta}P_n[\hat{Y}^{+}_{1,1}+\hat{r}_{f,0}(d\tilde{Y}^{+}_{1,S,1}/d\tilde{r})]_{\tilde{r}_{f,S}}\}+\{(d\tilde{Y}^{+}_{1,S,0}/d\tilde{r})_{\tilde{r}_{f,S}}+\bar{\delta}P_n[(d\hat{Y}^{+}_{1,0}/d\tilde{r})+\hat{r}_{f,0}(d^2\tilde{Y}^{+}_{1,S,0}/d\tilde{r}^2)]_{\tilde{r}_{f,S}}\}\xi+O(\bar{\delta}^2)$$

$$=\tilde{Y}^{+}_{1,S,1}(\tilde{r}_{f,S})+\bar{\delta}\,P_n[\hat{Y}^{+}_{1,1}+\hat{r}_{f,0}(d\tilde{Y}^{+}_{1,S,1}/d\tilde{r})]_{\tilde{r}_{f,S}}+O(\bar{\delta}^2)$$

$$=a^{+}_{1,1}[1-\exp(-Le_1\tilde{m}/\tilde{r}_{f,S})]+\bar{\delta}\,P_n[\hat{Y}^{+}_{1,1}(\tilde{r}_{f,S})-a^{+}_{1,1}\hat{r}_{f,0}(Le_1\tilde{m}/\tilde{r}^2_{f,S})\exp(-Le_1\tilde{m}/\tilde{r}_{f,S})]+O(\bar{\delta}^2)$$

$$(\partial\tilde{\phi}_{1,1}/\partial\xi)_{\xi\to\infty}=0$$

(13) $\tilde{\phi}_{2,1}(\xi\to-\infty)=\tilde{Y}^{-}_{2,1}(\tilde{r}_{f,0})+(\partial\tilde{Y}^{-}_{2,0}/\partial\tilde{r})_{\tilde{r}_{f,0}}\xi=[\tilde{Y}^{-}_{2,S,1}(\tilde{r}_{f,0})+\bar{\delta}P_n\hat{Y}^{-}_{2,1}(\tilde{r}_{f,0})]+[(d\tilde{Y}^{-}_{2,S,0}/d\tilde{r})_{\tilde{r}_{f,0}}+\bar{\delta}P_n(d\hat{Y}^{-}_{2,0}/d\tilde{r})_{\tilde{r}_{f,0}}]\xi+O(\bar{\delta}^2)$

$$=\{\tilde{Y}^{-}_{2,S,1}(\tilde{r}_{f,S})+\bar{\delta}P_n[\hat{Y}^{-}_{2,1}+\hat{r}_{f,0}(d\tilde{Y}^{-}_{2,S,1}/d\tilde{r})]_{\tilde{r}_{f,S}}\}+\{(d\tilde{Y}^{-}_{2,S,0}/d\tilde{r})_{\tilde{r}_{f,S}}+\bar{\delta}P_n[(d\hat{Y}^{-}_{2,0}/d\tilde{r})+\hat{r}_{f,0}(d^2\tilde{Y}^{-}_{2,S,0}/d\tilde{r}^2)]_{\tilde{r}_{f,S}}\}\xi+O(\bar{\delta}^2)$$

$$=\tilde{Y}^{-}_{2,S,1}(\tilde{r}_{f,S})+\bar{\delta}\,P_n[\hat{Y}^{-}_{2,1}+\hat{r}_{f,0}(d\tilde{Y}^{-}_{2,S,1}/d\tilde{r})]_{\tilde{r}_{f,S}}+O(\bar{\delta}^2)$$

$$=[a^{-}_{2,1}/(1+\tilde{Y}_{2,\infty})]+\bar{\delta}\,P_n\{\hat{Y}^{-}_{2,1}(\tilde{r}_{f,S})+\hat{r}_{f,0}[a^{-}_{2,1}(Le_2\tilde{m}/\tilde{r}^2_{f,S})/(1+\tilde{Y}_{2,\infty})]\}+O(\bar{\delta}^2)$$

$$(\partial\tilde{\phi}_{2,1}/\partial\xi)_{\xi\to-\infty}=0$$

(14) $\tilde{\phi}_{2,1}(\xi\to\infty)=\tilde{Y}^{+}_{2,1}(\tilde{r}_{f,0})+(\partial\tilde{Y}^{+}_{2,0}/\partial\tilde{r})_{\tilde{r}_{f,0}}\xi$

$$=[\tilde{Y}^{+}_{2,S,1}(\tilde{r}_{f,0})+\bar{\delta}\,P_n\hat{Y}^{+}_{2,1}(\tilde{r}_{f,0})]+[(d\tilde{Y}^{+}_{2,S,0}/d\tilde{r})_{\tilde{r}_{f,0}}+\bar{\delta}\,P_n(d\hat{Y}^{+}_{2,0}/d\tilde{r})_{\tilde{r}_{f,0}}]\xi+O(\bar{\delta}^2)$$

$$=\{\tilde{Y}^{+}_{2,S,1}(\tilde{r}_{f,S})+\bar{\delta}P_n[\hat{Y}^{+}_{2,1}+\hat{r}_{f,0}(d\tilde{Y}^{+}_{2,S,1}/d\tilde{r})]_{\tilde{r}_{f,S}}\}+\{(d\tilde{Y}^{+}_{2,S,0}/d\tilde{r})_{\tilde{r}_{f,S}}+\bar{\delta}P_n[(d\hat{Y}^{+}_{2,0}/d\tilde{r})+\hat{r}_{f,0}(d^2\tilde{Y}^{+}_{2,S,0}/d\tilde{r}^2)]_{\tilde{r}_{f,S}}\}\xi+O(\bar{\delta}^2)$$

$$=\left[a^{+}_{1,1}[\tilde{Y}_{2,\infty}/(1+\tilde{Y}_{2,\infty})]+\bar{\delta}P_n\langle\hat{Y}^{+}_{2,1}(\tilde{r}_{f,S})-\hat{r}_{f,0}(\tilde{m}/\tilde{r}^2_{f,S})\{[a^{+}_{1,1}Le_2/(1+\tilde{Y}_{2,\infty})]\}\rangle\right]$$

$$+\langle(Le_2\tilde{m}/\tilde{r}^2_{f,S})+\bar{\delta}\,P_n\{(d\hat{Y}^{+}_{2,0}/d\tilde{r})_{\tilde{r}_{f,S}}+\hat{r}_{f,0}(Le_2\tilde{m}/\tilde{r}^3_{f,S})[(Le_2\tilde{m}/\tilde{r}_{f,S})-2]\}\rangle\xi+O(\bar{\delta}^2)$$

$$(\partial\tilde{\phi}_{2,1}/\partial\xi)_{\xi\to\infty}=(d\tilde{Y}^{+}_{2,S,0}/d\tilde{r})_{\tilde{r}_{f,S}}+\bar{\delta}\,P_n[(d\hat{Y}^{+}_{2,0}/d\tilde{r})+\hat{r}_{f,0}(d^2\tilde{Y}^{+}_{2,S,0}/d\tilde{r}^2)]_{\tilde{r}_{f,S}}+O(\bar{\delta}^2)$$

$$=(Le_2\tilde{m}/\tilde{r}^2_{f,S})+\bar{\delta}\,P_n\{(d\hat{Y}^{+}_{2,0}/d\tilde{r})_{\tilde{r}_{f,S}}+\hat{r}_{f,0}(Le_2\tilde{m}/\tilde{r}^3_{f,S})[(Le_2\tilde{m}/\tilde{r}_{f,S})-2]\}+O(\bar{\delta}^2)$$

(15) $(\partial\tilde{\phi}_{1,2}/\partial\xi)_{\xi\to-\infty}=(\partial\tilde{Y}^{-}_{1,1}/\partial\tilde{r})_{\tilde{r}_{f,0}}+(\partial^2\tilde{Y}^{-}_{1,0}/\partial\tilde{r}^2)_{\tilde{r}_{f,0}}\xi$

$$=[(d\tilde{Y}^{-}_{1,S,1}/d\tilde{r})_{\tilde{r}_{f,0}}+\bar{\delta}\,P_n(d\hat{Y}^{-}_{1,1}/d\tilde{r})_{\tilde{r}_{f,0}}]+[(d^2\tilde{Y}^{-}_{1,S,0}/d\tilde{r}^2)_{\tilde{r}_{f,0}}+\bar{\delta}\,P_n(d^2\hat{Y}^{-}_{1,0}/d\tilde{r}^2)_{\tilde{r}_{f,0}}]\xi$$

$$=\{[(d\tilde{Y}^{-}_{1,S,1}/d\tilde{r})_{\tilde{r}_{f,S}}+\bar{\delta}\,P_n\hat{r}_{f,0}(d^2\tilde{Y}^{-}_{1,S,1}/d\tilde{r}^2)_{\tilde{r}_{f,S}}]+\bar{\delta}\,P_n(d\hat{Y}^{-}_{1,1}/d\tilde{r})_{\tilde{r}_{f,S}}+O(\bar{\delta}^2)\}$$

$$+\{[(d^2\tilde{Y}^{-}_{1,S,0}/d\tilde{r}^2)_{\tilde{r}_{f,S}}+\bar{\delta}\,P_n\hat{r}_{f,0}(d^3\tilde{Y}^{-}_{1,S,0}/d\tilde{r}^3)_{\tilde{r}_{f,S}}]+\bar{\delta}\,P_n(d^2\hat{Y}^{-}_{1,0}/d\tilde{r}^2)_{\tilde{r}_{f,S}}+O(\bar{\delta}^2)]\}\xi$$

$$=\{(d\tilde{Y}^{-}_{1,S,1}/d\tilde{r})_{\tilde{r}_{f,S}}+\bar{\delta}\,P_n[(d\hat{Y}^{-}_{1,1}/d\tilde{r})+\hat{r}_{f,0}(d^2\tilde{Y}^{-}_{1,S,1}/d\tilde{r}^2)]_{\tilde{r}_{f,S}}\}$$

$$+\{(d^2\tilde{Y}^{-}_{1,S,0}/d\tilde{r}^2)_{\tilde{r}_{f,S}}+\bar{\delta}\,P_n[(d^2\hat{Y}^{-}_{1,0}/d\tilde{r}^2)+\hat{r}_{f,0}(d^3\tilde{Y}^{-}_{1,S,0}/d\tilde{r}^3)]_{\tilde{r}_{f,S}}\}\xi+O(\bar{\delta}^2)$$

$$\left[\frac{1}{Le_1}\frac{\partial\tilde{\phi}_{1,2}}{\partial\xi}+\left(\frac{2\xi}{Le_1\tilde{r}_{f,0}}\right)\frac{\partial\tilde{\phi}_{1,1}}{\partial\xi}-\frac{\tilde{m}}{\tilde{r}^2_{f,0}}\tilde{\phi}_{1,1}\right]_{\xi\to-\infty}$$

$$=(1/Le_1)\{(d\tilde{Y}^{-}_{1,S,1}/d\tilde{r})_{\tilde{r}_{f,S}}+\bar{\delta}\,P_n[(d\hat{Y}^{-}_{1,1}/d\tilde{r})+\hat{r}_{f,0}(d^2\tilde{Y}^{-}_{1,S,1}/d\tilde{r}^2)]_{\tilde{r}_{f,S}}\}$$

$$+(1/Le_1)\{(d^2\tilde{Y}^{-}_{1,S,0}/d\tilde{r}^2)_{\tilde{r}_{f,S}}+\bar{\delta}\,P_n[(d^2\hat{Y}^{-}_{1,0}/d\tilde{r}^2)+\hat{r}_{f,0}(d^3\tilde{Y}^{-}_{1,S,0}/d\tilde{r}^3)]_{\tilde{r}_{f,S}}\}\xi$$

$$+[2/(Le_1\tilde{r}_{f,S})][1-\bar{\delta}\,P_n(\hat{r}_{f,0}/\tilde{r}_{f,S})]\{(d\tilde{Y}^{-}_{1,S,0}/d\tilde{r})_{\tilde{r}_{f,S}}+\bar{\delta}\,P_n[(d\hat{Y}^{-}_{1,0}/d\tilde{r})+\hat{r}_{f,0}(d^2\tilde{Y}^{-}_{1,S,0}/d\tilde{r}^2)]_{\tilde{r}_{f,S}}\}\xi$$

$$-(\tilde{m}/\tilde{r}^2_{f,S})[1-\bar{\delta}\,P_n(2\hat{r}_{f,0}/\tilde{r}_{f,S})]\{\tilde{Y}^{-}_{1,S,1}(\tilde{r}_{f,S})+\bar{\delta}\,P_n[\hat{Y}^{-}_{1,1}+\hat{r}_{f,0}(d\tilde{Y}^{-}_{1,S,1}/d\tilde{r})]_{\tilde{r}_{f,S}}\}$$

$$
\begin{aligned}
&-(\tilde{m}/\tilde{r}_{f,S}^{2})[1-\bar{\delta}P_n(2\hat{r}_{f,0}/\tilde{r}_{f,S})]\{(d\tilde{Y}_{1,S,0}^{-}/d\tilde{r})_{\tilde{r}_{f,S}}+\bar{\delta}P_n[(d\hat{Y}_{1,0}^{-}/d\tilde{r})+\hat{r}_{f,0}(d^2\tilde{Y}_{1,S,0}^{-}/d\tilde{r}^2)]_{\tilde{r}_{f,S}}\}\xi+O(\bar{\delta}^2)\\
&=[(1/Le_1)(d\tilde{Y}_{1,S,1}^{-}/d\tilde{r})_{\tilde{r}_{f,S}}-(\tilde{m}/\tilde{r}^2)\tilde{Y}_{1,S,1}^{-}]_{\tilde{r}_{f,S}}\\
&\quad+\bar{\delta}P_n\{[(1/Le_1)(d\hat{Y}_{1,1}^{-}/d\tilde{r})-(\tilde{m}/\tilde{r}^2)\hat{Y}_{1,1}^{-}]_{\tilde{r}_{f,S}}\\
&\quad+\hat{r}_{f,0}[(1/Le_1)(d^2\tilde{Y}_{1,S,1}^{-}/d\tilde{r}^2)-(\tilde{m}/\tilde{r}^2)(d\tilde{Y}_{1,S,1}^{-}/d\tilde{r})+(2\tilde{m}/\tilde{r}^3)\tilde{Y}_{1,S,1}^{-}]_{\tilde{r}_{f,S}}\}\\
&\quad+\langle\{(\tilde{r}^2/Le_1)(d^2\tilde{Y}_{1,S,0}^{-}/d\tilde{r}^2)-[\tilde{m}-(2\tilde{r}/Le_1)](d\tilde{Y}_{1,S,0}^{-}/d\tilde{r})\}/\tilde{r}^2\rangle_{\tilde{r}_{f,S}}\,\xi\\
&\quad+\bar{\delta}P_n\Big\{\langle[\tilde{r}^2(d^2\hat{Y}_{1,0}^{-}/d\tilde{r}^2)-(Le_{1,0}\tilde{m}-2\tilde{r})(d\hat{Y}_{1,0}^{-}/d\tilde{r})]/(Le_{1,0}\tilde{r}^2)\rangle_{\tilde{r}_{f,S}}\\
&\quad+\hat{r}_{f,0}\langle\{(\tilde{r}^2/Le_1)(d^3\tilde{Y}_{1,S,0}^{-}/d\tilde{r}^3)-[\tilde{m}-(2\tilde{r}/Le_1)](d^2\tilde{Y}_{1,S,0}^{-}/d\tilde{r}^2)+[(2\tilde{m}/\tilde{r})-(2/Le_1)](d\tilde{Y}_{1,S,0}^{-}/d\tilde{r})\}/\tilde{r}^2\rangle_{\tilde{r}_{f,S}}\Big\}\xi+O(\bar{\delta}^2)\\
&=\bar{\delta}P_n\{[(1/Le_1)(d\hat{Y}_{1,1}^{-}/d\tilde{r})-(\tilde{m}/\tilde{r}^2)\hat{Y}_{1,1}^{-}]_{\tilde{r}_{f,S}}+\hat{r}_{f,0}[n(n+1)\tilde{m}/\tilde{r}_{f,S}^4]\xi\}+O(\bar{\delta}^2)
\end{aligned}
$$

(16)
$$
\begin{aligned}
(\partial\tilde{\phi}_{1,2}/\partial\xi)_{\xi\to\infty}&=\{(d\tilde{Y}_{1,S,1}^{+}/d\tilde{r})_{\tilde{r}_{f,S}}+\bar{\delta}P_n[(d\hat{Y}_{1,1}^{+}/d\tilde{r})+\hat{r}_{f,0}(d^2\tilde{Y}_{1,S,1}^{+}/d\tilde{r}^2)]_{\tilde{r}_{f,S}}\}\\
&\quad+\{(d^2\tilde{Y}_{1,S,0}^{+}/d\tilde{r}^2)_{\tilde{r}_{f,S}}+\bar{\delta}P_n[(d^2\hat{Y}_{1,0}^{+}/d\tilde{r}^2)+\hat{r}_{f,0}(d^3\tilde{Y}_{1,S,0}^{+}/d\tilde{r}^3)]_{\tilde{r}_{f,S}}+O(\bar{\delta}^2)\}\xi\\
&=(d\tilde{Y}_{1,S,1}^{+}/d\tilde{r})_{\tilde{r}_{f,S}}+\bar{\delta}P_n[(d\hat{Y}_{1,1}^{+}/d\tilde{r})+\hat{r}_{f,0}(d^2\tilde{Y}_{1,S,1}^{+}/d\tilde{r}^2)]_{\tilde{r}_{f,S}}+O(\bar{\delta}^2)
\end{aligned}
$$

$$
\begin{aligned}
&\left[\frac{1}{Le_1}\frac{\partial\tilde{\phi}_{1,2}}{\partial\xi}+\left(\frac{2\xi}{Le_1\tilde{r}_{f,o}}\right)\frac{\partial\tilde{\phi}_{1,1}}{\partial\xi}-\frac{\tilde{m}}{\tilde{r}_{f,o}^2}\tilde{\phi}_{1,1}\right]_{\xi\to\infty}\\
&=(1/Le_1)\{(d\tilde{Y}_{1,S,1}^{+}/d\tilde{r})_{\tilde{r}_{f,S}}+\bar{\delta}P_n[(d\hat{Y}_{1,1}^{+}/d\tilde{r})+\hat{r}_{f,0}(d^2\tilde{Y}_{1,S,1}^{+}/d\tilde{r}^2)]_{\tilde{r}_{f,S}}\}\\
&\quad-(\tilde{m}/\tilde{r}_{f,S}^2)[1-\bar{\delta}P_n(2\hat{r}_{f,0}/\tilde{r}_{f,S})]\{\tilde{Y}_{1,S,1}^{+}(\tilde{r}_{f,S})+\bar{\delta}P_n[\hat{Y}_{1,1}^{+}+\hat{r}_{f,0}(d\tilde{Y}_{1,S,1}^{+}/d\tilde{r})]_{\tilde{r}_{f,S}}\}+O(\bar{\delta}^2)\\
&=[(1/Le_1)(d\tilde{Y}_{1,S,1}^{+}/d\tilde{r})-(\tilde{m}/\tilde{r}^2)\tilde{Y}_{1,S,1}^{+}]_{\tilde{r}_{f,S}}+\bar{\delta}P_n\{[(1/Le_1)(d\hat{Y}_{1,1}^{+}/d\tilde{r})-(\tilde{m}/\tilde{r}^2)\hat{Y}_{1,1}^{+}]_{\tilde{r}_{f,S}}\\
&\quad+\hat{r}_{f,0}[(1/Le_1)(d^2\tilde{Y}_{1,S,1}^{+}/d\tilde{r}^2)-(\tilde{m}/\tilde{r}^2)(d\tilde{Y}_{1,S,1}^{+}/d\tilde{r})+(2\tilde{m}/\tilde{r}^3)\tilde{Y}_{1,S,1}^{+}]_{\tilde{r}_{f,S}}\}+O(\bar{\delta}^2)\\
&=-a_{1,1}^{+}(\tilde{m}/\tilde{r}_{f,S}^2)+\bar{\delta}P_n\{[(1/Le_1)(d\hat{Y}_{1,1}^{+}/d\tilde{r})-(\tilde{m}/\tilde{r}^2)\hat{Y}_{1,1}^{+}]_{\tilde{r}_{f,S}}+2a_{1,1}^{+}\hat{r}_{f,0}(\tilde{m}/\tilde{r}_{f,S}^3)\}+O(\bar{\delta}^2)
\end{aligned}
$$

(17)
$$
\begin{aligned}
(\partial\tilde{\phi}_{2,2}/\partial\xi)_{\xi\to-\infty}&=\{(d\tilde{Y}_{2,S,1}^{-}/d\tilde{r})_{\tilde{r}_{f,S}}+\bar{\delta}P_n[(d\hat{Y}_{2,1}^{-}/d\tilde{r})+\hat{r}_{f,0}(d^2\tilde{Y}_{2,S,1}^{-}/d\tilde{r}^2)]_{\tilde{r}_{f,S}}\}\\
&\quad+\{(d^2\tilde{Y}_{2,S,0}^{-}/d\tilde{r}^2)_{\tilde{r}_{f,S}}+\bar{\delta}P_n[(d^2\hat{Y}_{2,0}^{-}/d\tilde{r}^2)+\hat{r}_{f,0}(d^3\tilde{Y}_{2,S,0}^{-}/d\tilde{r}^3)]_{\tilde{r}_{f,S}}+O(\bar{\delta}^2)\}\xi\\
&=(d\tilde{Y}_{2,S,1}^{-}/d\tilde{r})_{\tilde{r}_{f,S}}+\bar{\delta}P_n[(d\hat{Y}_{2,1}^{-}/d\tilde{r})+\hat{r}_{f,0}(d^2\tilde{Y}_{2,S,1}^{-}/d\tilde{r}^2)]_{\tilde{r}_{f,S}}+O(\bar{\delta}^2)
\end{aligned}
$$

$$
\begin{aligned}
&\left[\frac{1}{Le_2}\frac{\partial\tilde{\phi}_{2,2}}{\partial\xi}+\left(\frac{2\xi}{Le_2\tilde{r}_{f,o}}\right)\frac{\partial\tilde{\phi}_{2,1}}{\partial\xi}-\frac{\tilde{m}}{\tilde{r}_{f,o}^2}\tilde{\phi}_{2,1}\right]_{\xi\to-\infty}\\
&=(1/Le_2)\{(d\tilde{Y}_{2,S,1}^{-}/d\tilde{r})_{\tilde{r}_{f,S}}+\bar{\delta}P_n[(d\hat{Y}_{2,1}^{-}/d\tilde{r})+\hat{r}_{f,0}(d^2\tilde{Y}_{2,S,1}^{-}/d\tilde{r}^2)]_{\tilde{r}_{f,S}}+O(\bar{\delta}^2)\}\\
&\quad-(\tilde{m}/\tilde{r}_{f,S}^2)[1-\bar{\delta}P_n(2\hat{r}_{f,o}/\tilde{r}_{f,S})]\{\tilde{Y}_{2,S,1}^{-}(\tilde{r}_{f,S})+\bar{\delta}P_n[\hat{Y}_{2,1}^{-}+\hat{r}_{f,0}(d\tilde{Y}_{2,S,1}^{-}/d\tilde{r})]_{\tilde{r}_{f,S}}+O(\bar{\delta}^2)\}\\
&=[(1/Le_2)(d\tilde{Y}_{2,S,1}^{-}/d\tilde{r})-(\tilde{m}/\tilde{r}^2)\tilde{Y}_{2,S,1}^{-}]_{\tilde{r}_{f,S}}+\bar{\delta}P_n\{[(1/Le_2)(d\hat{Y}_{2,1}^{-}/d\tilde{r})-(\tilde{m}/\tilde{r}^2)\hat{Y}_{2,1}^{-}]_{\tilde{r}_{f,S}}\\
&\quad+\hat{r}_{f,0}[(1/Le_2)(d^2\tilde{Y}_{2,S,1}^{-}/d\tilde{r}^2)-(\tilde{m}/\tilde{r}^2)(d\tilde{Y}_{2,S,1}^{-}/d\tilde{r})+(2\tilde{m}/\tilde{r}^3)\tilde{Y}_{2,S,1}^{-}]_{\tilde{r}_{f,S}}\}+O(\bar{\delta}^2)\\
&=\bar{\delta}P_n[(1/Le_2)(d\hat{Y}_{2,1}^{-}/d\tilde{r})-(\tilde{m}/\tilde{r}^2)\hat{Y}_{2,1}^{-}]_{\tilde{r}_{f,S}}+O(\bar{\delta}^2)
\end{aligned}
$$

(18)
$$
\begin{aligned}
(\partial\tilde{\phi}_{2,2}/\partial\xi)_{\xi\to\infty}&=\{(d\tilde{Y}_{2,S,1}^{+}/d\tilde{r})_{\tilde{r}_{f,S}}+\bar{\delta}P_n[(d\hat{Y}_{2,1}^{+}/d\tilde{r})+\hat{r}_{f,0}(d^2\tilde{Y}_{2,S,1}^{+}/d\tilde{r}^2)]_{\tilde{r}_{f,S}}+O(\bar{\delta}^2)\}\\
&\quad+\{(d^2\tilde{Y}_{2,S,0}^{+}/d\tilde{r}^2)_{\tilde{r}_{f,S}}+\bar{\delta}P_n[(d^2\hat{Y}_{2,0}^{+}/d\tilde{r}^2)+\hat{r}_{f,0}(d^3\tilde{Y}_{2,S,0}^{+}/d\tilde{r}^3)]_{\tilde{r}_{f,S}}+O(\bar{\delta}^2)\}\xi
\end{aligned}
$$

$$
\begin{aligned}
&\left[\frac{1}{Le_2}\frac{\partial\tilde{\phi}_{2,2}}{\partial\xi}+\left(\frac{2\xi}{Le_2\tilde{r}_{f,o}}\right)\frac{\partial\tilde{\phi}_{2,1}}{\partial\xi}-\frac{\tilde{m}}{\tilde{r}_{f,o}^2}\tilde{\phi}_{2,1}\right]_{\xi\to\infty}\\
&=[(1/Le_2)(d\tilde{Y}_{2,S,1}^{+}/d\tilde{r})_{\tilde{r}_{f,S}}-(\tilde{m}/\tilde{r}^2)\tilde{Y}_{2,S,1}^{+}]_{\tilde{r}_{f,S}}\\
&\quad+\bar{\delta}P_n\{[(1/Le_2)(d\hat{Y}_{2,1}^{+}/d\tilde{r})-(\tilde{m}/\tilde{r}^2)\hat{Y}_{2,1}^{+}]_{\tilde{r}_{f,S}}
\end{aligned}
$$

$$+\hat{r}_{f,0}[(1/Le_2)(d^2\tilde{Y}^+_{2,S,1}/d\tilde{r}^2)-(\tilde{m}/\tilde{r}^2)(d\tilde{Y}^+_{2,S,1}/d\tilde{r})+(2\tilde{m}/\tilde{r}^3)\tilde{Y}^+_{2,S,1}]_{\tilde{r}_{f,S}}\}$$

$$+\langle\{(\tilde{r}^2/Le_2)(d^2\tilde{Y}^+_{2,S,0}/d\tilde{r}^2)-[\tilde{m}-(2\tilde{r}/Le_2)](d\tilde{Y}^+_{2,S,0}/d\tilde{r})\}/\tilde{r}^2\rangle_{\tilde{r}_{f,S}}\xi$$

$$+\bar{\delta}P_n\Big\{\langle[\tilde{r}^2(d^2\hat{Y}^+_{2,0}/d\tilde{r}^2)-(Le_2\tilde{m}-2\tilde{r})(d\hat{Y}^+_{2,0}/d\tilde{r})]/(Le_2\tilde{r}^2)\rangle_{\tilde{r}_{f,S}}$$

$$+\hat{r}_{f,0}\langle\{(\tilde{r}^2/Le_2)(d^3\tilde{Y}^+_{2,S,0}/d\tilde{r}^3)-[\tilde{m}-(2\tilde{r}/Le_2)](d^2\tilde{Y}^+_{2,S,0}/d\tilde{r}^2)+[(2\tilde{m}/\tilde{r})-(2/Le_2)](d\tilde{Y}^+_{2,S,0}/d\tilde{r})\}/\tilde{r}^2\rangle_{\tilde{r}_{f,S}}\Big\}\xi+O(\bar{\delta}^2)$$

$$=-(a^+_{1,1}\tilde{m}/\tilde{r}^2_{f,S})+\bar{\delta}P_n\{[(1/Le_2)(d\hat{Y}^+_{2,1}/d\tilde{r})-(\tilde{m}/\tilde{r}^2)\hat{Y}^+_{2,1}]_{\tilde{r}_{f,S}}+\hat{r}_{f,0}(2a^+_{1,1}\tilde{m}/\tilde{r}^3_{f,S})$$

$$-\hat{r}_{f,0}[n(n+1)\tilde{m}/\tilde{r}^4_{f,S}]\xi\}+O(\bar{\delta}^2)$$

(J) Solution of the Inner Equations

(1) $(\tilde{\phi}_{1,1}/Le_1)-(\tilde{\phi}_{2,1}/Le_2)=\tilde{c}_1\xi+\tilde{c}_2$

$$\xi\to-\infty:\quad \tilde{\phi}_{2,1}\to[a^-_{2,1}/(1+\tilde{Y}_{2,\infty})]+\bar{\delta}P_n\{\hat{Y}^-_{2,1}(\tilde{r}_{f,S})+\hat{r}_{f,0}[a^-_{2,1}(Le_2\tilde{m}/\tilde{r}^2_{f,S})/(1+\tilde{Y}_{2,\infty})]\}+O(\bar{\delta}^2)$$

$$\tilde{\phi}_{1,1}\to\{-a^-_{1,1}\exp(-Le_1\tilde{m}/\tilde{r}_{f,S})-(Le_1\tilde{m}/\tilde{r}^2_{f,S})\xi\}+\bar{\delta}P_n\langle\{\hat{Y}^-_{1,1}(\tilde{r}_{f,S})-a^-_{1,1}\hat{r}_{f,0}(Le_1\tilde{m}/\tilde{r}^2_{f,S})\exp(-Le_1\tilde{m}/\tilde{r}_{f,S})\}$$

$$+\{(d\hat{Y}^-_{1,0}/d\tilde{r})_{\tilde{r}_{f,S}}-\hat{r}_{f,0}(Le_1\tilde{m}/\tilde{r}^3_{f,S})[(Le_1\tilde{m}/\tilde{r}_{f,S})-2]\}\xi\rangle+O(\bar{\delta}^2)$$

$$\tilde{c}_1=-(\tilde{m}/\tilde{r}^2_{f,S})+\bar{\delta}P_n\{[(d\hat{Y}^-_{1,0}/d\tilde{r})_{\tilde{r}_{f,S}}/Le_1]-\hat{r}_{f,0}(\tilde{m}/\tilde{r}^3_{f,S})[(Le_1\tilde{m}/\tilde{r}_{f,S})-2]\}+O(\bar{\delta}^2)$$

$$\tilde{c}_2=-(a^-_{1,1}/Le_1)\exp(-Le_1\tilde{m}/\tilde{r}_{f,S})-\{a^-_{2,1}/[Le_2(1+\tilde{Y}_{2,\infty})]\}$$

$$+\bar{\delta}P_n\{[\hat{Y}^-_{1,1}(\tilde{r}_{f,S})/Le_1]-a^-_{1,1}\hat{r}_{f,0}(\tilde{m}/\tilde{r}^2_{f,S})\exp(-Le_1\tilde{m}/\tilde{r}_{f,S})-[\hat{Y}^-_{2,1}(\tilde{r}_{f,S})/Le_2]-\hat{r}_{f,0}[a^-_{2,1}(\tilde{m}/\tilde{r}^2_{f,S})/(1+\tilde{Y}_{2,\infty})]\}+O(\bar{\delta}^2)$$

$$\xi\to\infty:\quad \tilde{\phi}_{1,1}\to a^+_{1,1}[1-\exp(-Le_1\tilde{m}/\tilde{r}_{f,S})]+\bar{\delta}P_n[\hat{Y}^+_{1,1}(\tilde{r}_{f,S})-a^+_{1,1}\hat{r}_{f,0}(Le_1\tilde{m}/\tilde{r}^2_{f,S})\exp(-Le_1\tilde{m}/\tilde{r}_{f,S})]+O(\bar{\delta}^2)$$

$$\tilde{\phi}_{2,1}\to\{a^+_{1,1}[\tilde{Y}_{2,\infty}/(1+\tilde{Y}_{2,\infty})]+(Le_2\tilde{m}/\tilde{r}^2_{f,S})\xi\}+\bar{\delta}P_n\langle\{\hat{Y}^+_{2,1}(\tilde{r}_{f,S})-[a^+_{1,1}\hat{r}_{f,0}(Le_2\tilde{m}/\tilde{r}^2_{f,S})/(1+\tilde{Y}_{2,\infty})]$$

$$+\{(d\hat{Y}^+_{2,0}/d\tilde{r})_{\tilde{r}_{f,S}}+\hat{r}_{f,0}(Le_2\tilde{m}/\tilde{r}^3_{f,S})[(Le_2\tilde{m}/\tilde{r}_{f,S})-2]\}\xi\rangle+O(\bar{\delta}^2)$$

$$\tilde{c}_1=-(\tilde{m}/\tilde{r}^2_{f,S})-\bar{\delta}P_n\{[(d\hat{Y}^+_{2,0}/d\tilde{r})_{\tilde{r}_{f,S}}/Le_2]+\hat{r}_{f,0}(\tilde{m}/\tilde{r}^3_{f,S})[(Le_2\tilde{m}/\tilde{r}_{f,S})-2]\}+O(\bar{\delta}^2)$$

$$\tilde{c}_2=(a^+_{1,1}/Le_1)[1-\exp(-Le_1\tilde{m}/\tilde{r}_{f,S})]-(a^+_{1,1}/Le_2)[\tilde{Y}_{2,\infty}/(1+\tilde{Y}_{2,\infty})]$$

$$+\bar{\delta}P_n\{[\hat{Y}^+_{1,1}(\tilde{r}_{f,S})/Le_1)]-[\hat{Y}^+_{2,1}(\tilde{r}_{f,S})/Le_2]+a^+_{1,1}\hat{r}_{f,0}(\tilde{m}/\tilde{r}^2_{f,S})[(1+\tilde{Y}_{2,\infty})^{-1}-\exp(-Le_1\tilde{m}/\tilde{r}_{f,S})]\}+O(\bar{\delta}^2)$$

(a) $[(d\hat{Y}^-_{1,0}/d\tilde{r})_{\tilde{r}_{f,S}}/Le_1]+[(d\hat{Y}^+_{2,0}/d\tilde{r})_{\tilde{r}_{f,S}}/Le_2]=\hat{r}_{f,0}(\tilde{m}^2/\tilde{r}^4_{f,S})(Le_1-Le_2)$

$$\hat{r}_{f,0}\frac{\tilde{m}}{\tilde{r}^2_{f,S}}\left\{\frac{[\tilde{m}Le_1\Psi_2(1,Le_1)-\Psi^*_2(1,Le_1)]\Psi^*_1(\tilde{r}_{f,S},Le_1)-[\tilde{m}Le_1\Psi_1(1,Le_1)-\Psi^*_1(1,Le_1)]\Psi^*_2(\tilde{r}_{f,S},Le_1)}{[\tilde{m}Le_1\Psi_2(1,Le_1)-\Psi^*_2(1,Le_1)]\Psi_1(\tilde{r}_{f,S},Le_1)-[\tilde{m}Le_1\Psi_1(1,Le_1)-\Psi^*_1(1,Le_1)]\Psi_2(\tilde{r}_{f,S},Le_1)}-\frac{\Psi^*_2(\tilde{r}_{f,S},Le_2)}{\Psi_2(\tilde{r}_{f,S},Le_2)}\right\}$$

$$=\hat{r}_{f,0}(\tilde{m}^2/\tilde{r}^4_{f,S})(Le_1-Le_2)$$

This equation is satisfied only for $\hat{r}_{f,0}=0$. Thus $\hat{T}^-_0=\hat{T}^+_0=\hat{Y}^-_{1,0}=\hat{Y}^+_{2,0}=\hat{T}^-_2=\hat{T}^+_2=\hat{T}^-_A=\hat{T}^+_A=\hat{T}^-_B=\hat{T}^+_B=0$, $\tilde{r}_{f,0}=\tilde{r}_{f,S}$

$\tilde{\Theta}^-_1=\Theta^-_{S,1}$, $\tilde{\Theta}^+_1=\Theta^+_{S,1}$, $\Theta^-_{S,2}(\zeta=0)=-(\tilde{T}_{b,S,A}/g^-_0)[(g^-_0)^2+2\Lambda_R]^{1/2}$; $\Theta^+_{S,2}(\zeta=0)=-(a^+_{T,A}/g^+_0)[(g^+_0)^2+2\Lambda_R]^{1/2}$

$\hat{g}^\pm_0=g^\pm_0$, $\hat{g}^\pm_2=g^\pm_2[1+\bar{\delta}P_n\hat{T}^\pm_C+O(\bar{\delta}^2)]$, $\hat{T}^-_C=[\hat{T}^-_1(\tilde{r}_{f,S})+\hat{r}_{f,1}g^-_0]/\tilde{T}_{b,S,A}$, $\hat{T}^+_C=[\hat{T}^+_1(\tilde{r}_{f,S})-\hat{r}_{f,1}g^+_0]/a^+_{T,A}$

(b) $[\hat{Y}^-_{1,1}(\tilde{r}_{f,S})/Le_1]-[\hat{Y}^-_{2,1}(\tilde{r}_{f,S})/Le_2]=[\hat{Y}^+_{1,1}(\tilde{r}_{f,S})/Le_1)]-[\hat{Y}^+_{2,1}(\tilde{r}_{f,S})/Le_2]$

Note: The leading order terms are satisfied in the steady state analysis.

$$\frac{\hat{a}^-_{1,1}}{Le_1}\left\{\Psi_1(\tilde{r}_{f,S},Le_1)-\frac{[\tilde{m}Le_{1,0}\Psi_1(1,Le_1)-\Psi^*_1(1,Le_1)]}{[\tilde{m}Le_{1,0}\Psi_2(1,Le_1)-\Psi^*_2(1,Le_1)]}\Psi_2(\tilde{r}_{f,S},Le_1)\right\}-\frac{\hat{a}^+_{1,1}}{Le_1}\Psi_2(\tilde{r}_{f,S},Le_1)$$

$$=\frac{\hat{a}^-_{2,1}}{Le_2}\left\{\Psi_1(\tilde{r}_{f,S},Le_2)-\frac{[\tilde{m}Le_2\Psi_1(1,Le_2)-\Psi^*_1(1,Le_2)]}{[\tilde{m}Le_2\Psi_2(1,Le_2)-\Psi^*_2(1,Le_2)]}\Psi_2(\tilde{r}_{f,S},Le_2)\right\}-\frac{\hat{a}^+_{2,1}}{Le_2}\Psi_2(\tilde{r}_{f,S},Le_2)$$

(c) $$\frac{\tilde{\phi}_{1,1}}{Le_1}-\frac{\tilde{\phi}_{2,1}}{Le_2}=\frac{a^+_{1,1}}{Le_1}\left[1-\exp\left(-\frac{Le_1\tilde{m}}{\tilde{r}_{f,S}}\right)\right]-\frac{a^+_{1,1}}{Le_2}\frac{\tilde{Y}_{2,\infty}}{1+\tilde{Y}_{2,\infty}}+\bar{\delta}P_n\left[\frac{\hat{a}^+_{1,1}}{Le_1}\Psi_2(\tilde{r}_{f,S},Le_1)-\frac{\hat{a}^+_{2,1}}{Le_2}\Psi_2(\tilde{r}_{f,S},Le_2)\right]-\frac{\tilde{m}}{\tilde{r}^2_{f,S}}\xi+O(\bar{\delta}^2)$$

(2) $\tilde{\theta}_1-(\tilde{\phi}_{1,1}/Le_1)=\tilde{c}_4\xi+\tilde{c}_5$

$$\xi \to -\infty \ : \quad \tilde{\phi}_{1,1} \to \{-a_{1,1}^{-}\exp(-Le_1\tilde{m}/\tilde{r}_{f,S}) - (Le_1\tilde{m}/\tilde{r}_{f,S}^2)\xi\} + \bar{\delta}P_n\hat{Y}_{1,1}^{-}(\tilde{r}_{f,S}) + O(\bar{\delta}^2)$$

$$\tilde{\theta}_1 \to \tilde{\Theta}_2^{-}(\zeta=0) + (d\Theta_{S,1}^{-}/d\zeta)_{\zeta=0}(\xi - \bar{\delta}P_n\hat{r}_{f,1}) = -\sqrt{(g_0^{-})^2 + 2\Lambda_R}[(\tilde{T}_{b,S,A}/g_0^{-})(1+\bar{\delta}P_n\hat{T}_C^{-}) + (\xi - \bar{\delta}P_n\hat{r}_{f,1})] + O(\bar{\delta}^2)$$

$$\tilde{c}_4 = -\sqrt{(g_0^{-})^2 + 2\Lambda_R} + (\tilde{m}/\tilde{r}_{f,S}^2) + O(\bar{\delta}^2)$$

$$\tilde{c}_5 = -\sqrt{(g_0^{-})^2 + 2\Lambda_R}[(\tilde{T}_{b,S,A}/g_0^{-})(1+\bar{\delta}P_n\hat{T}_C^{-}) - \bar{\delta}P_n\hat{r}_{f,1}] + \{[a_{1,1}^{-}\exp(-Le_1\tilde{m}/\tilde{r}_{f,S}) - \bar{\delta}P_n\hat{Y}_{1,1}^{-}(\tilde{r}_{f,S})]/Le_1\} + O(\bar{\delta}^2)$$

$$\xi \to \infty \ : \quad \tilde{\phi}_{1,1} \to a_{1,1}^{+}[1-\exp(-Le_1\tilde{m}/\tilde{r}_{f,S})] + \bar{\delta}P_n\hat{Y}_{1,1}^{+}(\tilde{r}_{f,S}) + O(\bar{\delta}^2)$$

$$\tilde{\theta}_1 \to \tilde{\Theta}_2^{+}(\zeta=0) + (d\Theta_{S,1}^{+}/d\zeta)_{\zeta=0}(\xi - \bar{\delta}P_n\hat{r}_{f,1}) = -\sqrt{(g_0^{+})^2 + 2\Lambda_R}[(a_{T,A}^{+}/g_0^{+})(1+\bar{\delta}P_n\hat{T}_C^{+}) - (\xi - \bar{\delta}P_n\hat{r}_{f,1})] + O(\bar{\delta}^2)$$

$$\tilde{c}_4 = \sqrt{(g_0^{+})^2 + 2\Lambda_R} + O(\bar{\delta}^2)$$

$$\tilde{c}_5 = -\sqrt{(g_0^{+})^2 + 2\Lambda_R}[(a_{T,A}^{+}/g_0^{+})(1+\bar{\delta}P_n\hat{T}_C^{+}) + \bar{\delta}P_n\hat{r}_{f,1}] - \langle\{a_{1,1}^{+}[1-\exp(-Le_1\tilde{m}/\tilde{r}_{f,S})] + \bar{\delta}P_n\hat{Y}_{1,1}^{+}(\tilde{r}_{f,S})\}/Le_1\rangle + O(\bar{\delta}^2)$$

(a) $-\sqrt{(g_0^{-})^2 + 2\Lambda_R} + (\tilde{m}/\tilde{r}_{f,S}^2) = \sqrt{(g_0^{+})^2 + 2\Lambda_R}$ already satisfied in the steady state analysis

(b) $$-\sqrt{(g_0^{+})^2 + 2\Lambda_R}[(a_{T,A}^{+}/g_0^{+})(1+\bar{\delta}P_n\hat{T}_C^{+}) + \hat{r}_{f,1}\bar{\delta}P_n] - (a_{1,1}^{+}/Le_1)[1-\exp(-Le_1\tilde{m}/\tilde{r}_{f,S})] - \bar{\delta}P_n[\hat{Y}_{1,1}^{+}(\tilde{r}_{f,S})/Le_1] + O(\bar{\delta}^2)$$

$$= -\sqrt{(g_0^{-})^2 + 2\Lambda_R}[(\tilde{T}_{b,S,A}/g_0^{-})(1+\bar{\delta}P_n\hat{T}_C^{-}) - \bar{\delta}P_n\hat{r}_{f,1}] + (a_{1,1}^{-}/Le_1)\exp(-Le_1\tilde{m}/\tilde{r}_{f,S}) - \bar{\delta}P_n[\hat{Y}_{1,1}^{-}(\tilde{r}_{f,S})/Le_1] + O(\bar{\delta}^2)$$

From the steady state analysis :

$$-(a_{T,A}^{+}/g_0^{+})\sqrt{(g_0^{+})^2 + 2\Lambda_R} - (a_{1,1}^{+}/Le_1)[1-\exp(-Le_1\tilde{m}/\tilde{r}_{f,S})]$$

$$= (\tilde{T}_{b,S,A}/g_0^{-})\sqrt{(g_0^{-})^2 + 2\Lambda_R} + (a_{1,1}^{-}/Le_1)\exp(-Le_1\tilde{m}/\tilde{r}_{f,S})$$

$$-(\tilde{T}_{b,S,A}/g_0^{-})\sqrt{(g_0^{-})^2 + 2\Lambda_R}\{[\hat{T}_1^{-}(\tilde{r}_{f,S}) + \hat{r}_{f,1}g_0^{-}]/\tilde{T}_{b,S,A}\} + \sqrt{(g_0^{-})^2 + 2\Lambda_R}\,\hat{r}_{f,1} - [\hat{Y}_{1,1}^{-}(\tilde{r}_{f,S})/Le_1]$$

$$= -(a_{T,A}^{+}/g_0^{+})\sqrt{(g_0^{+})^2 + 2\Lambda_R}[\hat{T}_1^{+}(\tilde{r}_{f,S}) - \hat{r}_{f,1}g_0^{+}]/a_{T,A}^{+} - \sqrt{(g_0^{+})^2 + 2\Lambda_R}\,\hat{r}_{f,1} - [\hat{Y}_{1,1}^{+}(\tilde{r}_{f,S})/Le_1]$$

$$\sqrt{(g_0^{-})^2 + 2\Lambda_R}[\hat{T}_1^{-}(\tilde{r}_{f,S})/g_0^{-}] - \sqrt{(g_0^{+})^2 + 2\Lambda_R}[\hat{T}_1^{+}(\tilde{r}_{f,S})/g_0^{+}] = [\hat{Y}_{1,1}^{+}(\tilde{r}_{f,S})/Le_1] - [\hat{Y}_{1,1}^{-}(\tilde{r}_{f,S})/Le_1]$$

$$[\sqrt{(g_0^{-})^2 + 2\Lambda_R}/g_0^{-}]\hat{T}_{b,1}\{A_{T2}\Psi_1(\tilde{r}_{f,S},1) + [1 - A_{T2}\Psi_1(1,1)]\Psi_2(\tilde{r}_{f,S},1)/\Psi_2(1,1)\} - [\sqrt{(g_0^{+})^2 + 2\Lambda_R}/g_0^{+}]\hat{a}_{T,1}^{+}\Psi_2(\tilde{r}_{f,S},1)$$

$$= \frac{\hat{a}_{1,1}^{+}}{Le_1}\Psi_2(\tilde{r}_{f,S},Le_1) - \frac{\hat{a}_{1,1}^{-}}{Le_1}\left\{\Psi_1(\tilde{r}_{f,S},Le_1) - \frac{[\tilde{m}Le_1\Psi_1(1,Le_1) - \Psi_1^{*}(1,Le_1)]}{[\tilde{m}Le_1\Psi_2(1,Le_1) - \Psi_2^{*}(1,Le_1)]}\Psi_2(\tilde{r}_{f,S},Le_1)\right\}$$

(c) $$\tilde{\theta}_1 - (\tilde{\phi}_{1,1}/Le_1) = -(a_{T,A}^{+}/g_0^{+})\sqrt{(g_0^{+})^2 + 2\Lambda_R} - (a_{1,1}^{+}/Le_1)[1-\exp(-Le_1\tilde{m}/\tilde{r}_{f,S})] + \sqrt{(g_0^{+})^2 + 2\Lambda_R}\,\xi$$

$$-\bar{\delta}P_n[\sqrt{(g_0^{+})^2 + 2\Lambda_R}(\hat{a}_{T,1}^{+}/g_0^{+})\Psi_2(\tilde{r}_{f,S},1) + (\hat{a}_{1,1}^{+}/Le_1)\Psi_2(\tilde{r}_{f,S},Le_1)] + O(\bar{\delta}^2)$$

From (1) :

$$\frac{\tilde{\phi}_{1,1}}{Le_1} - \frac{\tilde{\phi}_{2,1}}{Le_2} = \frac{a_{1,1}^{+}}{Le_1}\left[1-\exp\left(-\frac{Le_1\tilde{m}}{\tilde{r}_{f,S}}\right)\right] - \frac{a_{1,1}^{+}}{Le_2}\frac{\tilde{Y}_{2,\infty}}{(1+\tilde{Y}_{2,\infty})} + \bar{\delta}P_n\left[\frac{\hat{a}_{1,1}^{+}}{Le_1}\Psi_2(\tilde{r}_{f,S},Le_1) - \frac{\hat{a}_{2,1}^{+}}{Le_2}\Psi_2(\tilde{r}_{f,S},Le_2)\right] - \frac{\tilde{m}}{\tilde{r}_{f,S}^2}\xi + O(\bar{\delta}^2)$$

$$\tilde{\theta}_1 - (\tilde{\phi}_{2,1}/Le_2) = -(a_{1,1}^{+}/Le_2)[\tilde{Y}_{2,\infty}/(1+\tilde{Y}_{2,\infty})] - (a_{T,A}^{+}/g_0^{+})\sqrt{(g_0^{+})^2 + 2\Lambda_R} - \sqrt{(g_0^{-})^2 + 2\Lambda_R}\,\xi$$

$$-\bar{\delta}P_n[\sqrt{(g_0^{+})^2 + 2\Lambda_R}(\hat{a}_{T,1}^{+}/g_0^{+})\Psi_2(\tilde{r}_{f,S},1) + (\hat{a}_{2,1}^{+}/Le_2)\Psi_2(\tilde{r}_{f,S},Le_2) + O(\bar{\delta}^2)$$

(3) $$\left[\frac{1}{Le_1}\frac{\partial\tilde{\phi}_{1,2}}{\partial\xi} + \left(\frac{2\xi}{Le_1\tilde{r}_{f,0}}\right)\frac{\partial\tilde{\phi}_{1,1}}{\partial\xi} - \frac{\tilde{m}}{\tilde{r}_{f,0}^2}\tilde{\phi}_{1,1}\right] - \left[\frac{1}{Le_2}\frac{\partial\tilde{\phi}_{2,2}}{\partial\xi} + \left(\frac{2\xi}{Le_2\tilde{r}_{f,0}}\right)\frac{\partial\tilde{\phi}_{2,1}}{\partial\xi} - \frac{\tilde{m}}{\tilde{r}_{f,0}^2}\tilde{\phi}_{2,1}\right] = \tilde{c}_3$$

$$\xi \to -\infty \ : \ \frac{1}{Le_1}\frac{\partial\tilde{\phi}_{1,2}}{\partial\xi} + \left(\frac{2\xi}{Le_1\tilde{r}_{f,S}}\right)\frac{\partial\tilde{\phi}_{1,1}}{\partial\xi} - \frac{\tilde{m}}{\tilde{r}_{f,S}^2}\tilde{\phi}_{1,1} \to \bar{\delta}P_n\left[\frac{1}{Le_1}\left(\frac{d\hat{Y}_{1,1}^{-}}{d\tilde{r}}\right)_{\tilde{r}_{f,S}} - \frac{\tilde{m}}{\tilde{r}_{f,S}^2}\hat{Y}_{1,1}^{-}(\tilde{r}_{f,S})\right] + O(\bar{\delta}^2)$$

$$\frac{1}{Le_2}\frac{\partial\tilde{\phi}_{2,2}}{\partial\xi} + \left(\frac{2\xi}{Le_2\tilde{r}_{f,S}}\right)\frac{\partial\tilde{\phi}_{2,1}}{\partial\xi} - \frac{\tilde{m}}{\tilde{r}_{f,S}^2}\tilde{\phi}_{2,1} \to \bar{\delta}P_n\left[\frac{1}{Le_2}\left(\frac{d\hat{Y}_{2,1}^{-}}{d\tilde{r}}\right)_{\tilde{r}_{f,S}} - \frac{\tilde{m}}{\tilde{r}_{f,S}^2}\hat{Y}_{2,1}^{-}(\tilde{r}_{f,S})\right] + O(\bar{\delta}^2)$$

$$\xi \to \infty \ : \ \frac{1}{Le_1}\frac{\partial \tilde{\phi}_{1,2}}{\partial \xi} + \left(\frac{2\xi}{Le_1 \tilde{r}_{f,S}}\right)\frac{\partial \tilde{\phi}_{1,1}}{\partial \xi} - \frac{\tilde{m}}{\tilde{r}_{f,S}^2}\tilde{\phi}_{1,1} \to -a_{1,1}^{+}\frac{\tilde{m}}{\tilde{r}_{f,S}^2} + \bar{\delta}\, P_n \left[\frac{1}{Le_1}\left(\frac{d\hat{Y}_{1,1}^{+}}{d\tilde{r}}\right)_{\tilde{r}_{f,S}} - \frac{\tilde{m}}{\tilde{r}_{f,S}^2}\hat{Y}_{1,1}^{+}(\tilde{r}_{f,S})\right] + O(\bar{\delta}^2)$$

$$\frac{1}{Le_2}\frac{\partial \tilde{\phi}_{2,2}}{\partial \xi} + \left(\frac{2\xi}{Le_{2,0} \tilde{r}_{f,S}}\right)\frac{\partial \tilde{\phi}_{2,1}}{\partial \xi} - \frac{\tilde{m}}{\tilde{r}_{f,S}^2}\tilde{\phi}_{2,1} \to -a_{1,1}^{+}\frac{\tilde{m}}{\tilde{r}_{f,S}^2} + \bar{\delta}\, P_n \left[\frac{1}{Le_2}\left(\frac{d\hat{Y}_{2,1}^{+}}{d\tilde{r}}\right)_{\tilde{r}_{f,S}} - \frac{\tilde{m}}{\tilde{r}_{f,S}^2}\hat{Y}_{2,1}^{+}(\tilde{r}_{f,S})\right] + O(\bar{\delta}^2)$$

$$\frac{1}{Le_1}\left(\frac{d\hat{Y}_{1,1}^{-}}{d\tilde{r}}\right)_{\tilde{r}_{f,S}} - \frac{1}{Le_2}\left(\frac{d\hat{Y}_{2,1}^{-}}{d\tilde{r}}\right)_{\tilde{r}_{f,S}} - \frac{\tilde{m}}{\tilde{r}_{f,S}^2}[\hat{Y}_{1,1}^{-}(\tilde{r}_{f,S}) - \hat{Y}_{2,1}^{-}(\tilde{r}_{f,S})] = \frac{1}{Le_1}\left(\frac{d\hat{Y}_{1,1}^{+}}{d\tilde{r}}\right)_{\tilde{r}_{f,S}} - \frac{1}{Le_2}\left(\frac{d\hat{Y}_{2,1}^{+}}{d\tilde{r}}\right)_{\tilde{r}_{f,S}} - \frac{\tilde{m}}{\tilde{r}_{f,S}^2}[\hat{Y}_{1,1}^{+}(\tilde{r}_{f,S}) - \hat{Y}_{2,1}^{+}(\tilde{r}_{f,S})]$$

$$\frac{1}{Le_1}\left(\frac{d\hat{Y}_{1,1}^{-}}{d\tilde{r}}\right)_{\tilde{r}_{f,S}} - \frac{1}{Le_1}\left(\frac{d\hat{Y}_{1,1}^{+}}{d\tilde{r}}\right)_{\tilde{r}_{f,S}} - \frac{\tilde{m}}{\tilde{r}_{f,S}^2}[\hat{Y}_{1,1}^{-}(\tilde{r}_{f,S}) - \hat{Y}_{1,1}^{+}(\tilde{r}_{f,S})] = \frac{1}{Le_2}\left(\frac{d\hat{Y}_{2,1}^{-}}{d\tilde{r}}\right)_{\tilde{r}_{f,S}} - \frac{1}{Le_2}\left(\frac{d\hat{Y}_{2,1}^{+}}{d\tilde{r}}\right)_{\tilde{r}_{f,S}} - \frac{\tilde{m}}{\tilde{r}_{f,S}^2}[\hat{Y}_{2,1}^{-}(\tilde{r}_{f,S}) - \hat{Y}_{2,1}^{+}(\tilde{r}_{f,S})]$$

$$\hat{a}_{1,1}^{-}\left\{\left[\frac{\Psi_1^*(\tilde{r}_{f,S}, Le_{1,0})}{Le_{1,0}} - \frac{\tilde{m}}{\tilde{r}_{f,S}^2}\Psi_1(\tilde{r}_{f,S}, Le_{1,0})\right] - \frac{[\tilde{m}\, Le_{1,0}\, \Psi_1(1, Le_{1,0}) - \Psi_1^*(1, Le_{1,0})]}{[\tilde{m}\, Le_{1,0}\, \Psi_2(1, Le_{1,0}) - \Psi_2^*(1, Le_{1,0})]}\left[\frac{\Psi_2^*(\tilde{r}_{f,S}, Le_{1,0})}{Le_{1,0}} - \frac{\tilde{m}}{\tilde{r}_{f,S}^2}\Psi_2(\tilde{r}_{f,S}, Le_{1,0})\right]\right\}$$

$$-\hat{a}_{1,1}^{+}\{[\Psi_2^*(\tilde{r}_{f,S}, Le_1)/Le_1] - (\tilde{m}/\tilde{r}_{f,S}^2)\Psi_2(\tilde{r}_{f,S}, Le_1)\}$$

$$= \hat{a}_{2,1}^{-}\left\{\left[\frac{\Psi_1^*(\tilde{r}_{f,S}, Le_2)}{Le_2} - \frac{\tilde{m}}{\tilde{r}_{f,S}^2}\Psi_1(\tilde{r}_{f,S}, Le_2)\right]\right.$$

$$\left. - \frac{[\tilde{m}\, Le_2\, \Psi_1(1, Le_2) - \Psi_1^*(1, Le_2)]}{[\tilde{m}\, Le_2\, \Psi_2(1, Le_2) - \Psi_2^*(1, Le_2)]}\left[\frac{\Psi_2^*(\tilde{r}_{f,S}, Le_2)}{Le_2} - \frac{\tilde{m}}{\tilde{r}_{f,S}^2}\Psi_2(\tilde{r}_{f,S}, Le_2)\right]\right\}$$

$$-\hat{a}_{2,1}^{+}\{[\Psi_2^*(\tilde{r}_{f,S}, Le_2)/Le_2] - (\tilde{m}/\tilde{r}_{f,S}^2)\Psi_2(\tilde{r}_{f,S}, Le_2)\}$$

(4) $$\left(\frac{\partial \tilde{\theta}_2}{\partial \xi} + \frac{2\xi}{\tilde{r}_{f,0}}\frac{\partial \tilde{\theta}_1}{\partial \xi} - \frac{\tilde{m}}{\tilde{r}_{f,0}^2}\tilde{\theta}_1\right) - \left[\frac{1}{Le_1}\frac{\partial \tilde{\phi}_{1,2}}{\partial \xi} + \left(\frac{2\xi}{Le_1 \tilde{r}_{f,0}}\right)\frac{\partial \tilde{\phi}_{1,1}}{\partial \xi} - \frac{\tilde{m}}{\tilde{r}_{f,0}^2}\tilde{\phi}_{1,1}\right] = \tilde{c}_6 \quad ; \quad \tilde{r}_{f,0} = \tilde{r}_{f,S}$$

$$\left(\frac{\partial \tilde{\Theta}_3^{\pm}}{\partial \zeta}\right)_{\zeta=0} = \left[\frac{8}{3}\hat{g}_2^{\pm} - \frac{\tilde{T}_E^{\pm}}{(g_0^{\pm})^2}\right]\left\{g_0^{\pm}\sqrt{(g_0^{\pm})^2 + 2\Lambda_R} + \Lambda_R\left\langle \ell\text{n}\left[\frac{2(g_0^{\pm})^2}{\Lambda_R}\frac{\sqrt{(g_0^{\pm})^2 + 2\Lambda_R} - g_0^{\pm}}{\sqrt{(g_0^{\pm})^2 + 2\Lambda_R} + g_0^{\pm}}\right] - 2\right\rangle\right\}$$

$$+\Lambda_R\left(\frac{\hat{g}_1^{\pm}}{(g_0^{\pm})^2}\tilde{T}_E^{\pm} \pm \frac{\tilde{T}_D^{\pm} + \hat{g}_3^{\pm}}{g_0^{\pm}} - \frac{\hat{g}_2^{\pm}}{2}\left\{\left\langle \ell\text{n}\left[\frac{2(g_0^{\pm})^2}{\Lambda_R}\frac{\sqrt{(g_0^{\pm})^2 + 2\Lambda_R} - g_0^{\pm}}{\sqrt{(g_0^{\pm})^2 + 2\Lambda_R} + g_0^{\pm}}\right]\right\rangle^2 + \frac{8}{3}\right\}\right)$$

$$-\hat{g}_2^{\pm} g_0^{\pm}\sqrt{(g_0^{\pm})^2 + 2\Lambda_R}\ \ell\text{n}\left[\frac{2(g_0^{\pm})^2}{\Lambda_R}\frac{\sqrt{(g_0^{\pm})^2 + 2\Lambda_R} - g_0^{\pm}}{\sqrt{(g_0^{\pm})^2 + 2\Lambda_R} + g_0^{\pm}}\right] - \frac{8}{3}\hat{g}_2^{\pm}(g_0^{\pm})^2$$

$$\tilde{T}_E^{-} = \tilde{T}_{b,S,A}(\tilde{m}/\tilde{r}_{f,S}^2) + \bar{\delta}\, P_n\{(d\hat{T}_1^{-}/d\tilde{r})_{\tilde{r}_{f,S}} + \hat{r}_{f,1}\, g_0^{-}[(\tilde{m}/\tilde{r}_{f,S}^2) - (2/\tilde{r}_{f,S})]\} + O(\bar{\delta}^2)$$

$$\tilde{T}_E^{+} = -a_{T,1}^{+}(\tilde{m}/\tilde{r}_{f,S}^2)\exp(-\tilde{m}/\tilde{r}_{f,S}) + \bar{\delta}\, P_n\{(d\hat{T}_1^{+}/d\tilde{r})_{\tilde{r}_{f,S}} - \hat{r}_{f,1}\, g_0^{+}[(\tilde{m}/\tilde{r}_{f,S}^2) - (2/\tilde{r}_{f,S})]\} + O(\bar{\delta}^2)$$

In the limit of $\Lambda_R \to 0$: $\sqrt{(g_0^{\pm})^2 + 2\Lambda_R} \to g_0^{\pm} + (\Lambda_R / g_0^{\pm})$,

$$\ell\text{n}\{[2(g_0^{\pm})^2/\Lambda_R][\sqrt{(g_0^{\pm})^2 + 2\Lambda_R} - g_0^{\pm}]/[\sqrt{(g_0^{\pm})^2 + 2\Lambda_R} + g_0^{\pm}\}] \to 0$$

$$\therefore \quad (\partial \tilde{\Theta}_3^{\pm}/\partial \zeta)_{\zeta=0} = -\tilde{T}_E^{\pm} \quad \Rightarrow \quad (\partial \tilde{\Theta}_3^{\pm}/\partial \zeta)_{\zeta=0} = -\tilde{T}_E^{\pm}\sqrt{(g_0^{\pm})^2 + 2\Lambda_R}\,/\,g_0^{\pm}$$

$$\xi \to -\infty \ : \ \frac{1}{Le_{1,}}\frac{\partial \tilde{\phi}_{1,2}}{\partial \xi} + \left(\frac{2\xi}{Le_1 \tilde{r}_{f,0}}\right)\frac{\partial \tilde{\phi}_{1,1}}{\partial \xi} - \frac{\tilde{m}}{\tilde{r}_{f,0}^2}\tilde{\phi}_{1,1} \to \bar{\delta}\, P_n\left[\frac{1}{Le_1}\left(\frac{d\hat{Y}_{1,1}^{-}}{d\tilde{r}}\right)_{\tilde{r}_{f,S}} - \frac{\tilde{m}}{\tilde{r}_{f,S}^2}\hat{Y}_{1,1}^{-}(\tilde{r}_{f,S})\right] + O(\bar{\delta}^2)$$

$$\tilde{\theta}_1 \to -\sqrt{(g_0^{-})^2 + 2\Lambda_R}\,[(\tilde{T}_{b,S,A}/g_0^{-})(1 + \bar{\delta}\, P_n \hat{T}_C^{-}) + (\xi - \bar{\delta}\, P_n \hat{r}_{f,1})] + O(\bar{\delta}^2) \quad ;$$

$$\partial \tilde{\theta}_1/\partial \xi \to -[(g_0^{-})^2 + 2\Lambda_R]^{1/2} + O(\bar{\delta}^2)$$

$$\partial\tilde{\theta}_2/\partial\xi \to (\partial\tilde{\Theta}_3^-/d\zeta)_{\zeta=0} + (\partial^2\tilde{\Theta}_4^-/\partial\zeta^2)_{\zeta=0}[\xi - \bar{\delta}P_n\hat{r}_{f,1}]$$

$$= -[\tilde{T}_E^-\sqrt{(g_0^-)^2+2\Lambda_R}/g_0^-] + (\partial^2\tilde{\Theta}_4^-/\partial\zeta^2)_{\zeta=0}[\xi - \bar{\delta}P_n\hat{r}_{f,1}]$$

$$\therefore\ (\partial^2\tilde{\Theta}_4^-/\partial\zeta^2)_{\zeta=0} + [(\tilde{m}/\tilde{r}_{f,S}^2) - (2/\tilde{r}_{f,S})][(g_0^-)^2+2\Lambda_R]^{1/2} = 0$$

$$\tilde{c}_6 = -[\tilde{T}_E^-\sqrt{(g_0^-)^2+2\Lambda_R}/g_0^-] - \bar{\delta}P_n\{(\partial^2\tilde{\Theta}_4^-/\partial\zeta^2)_{\zeta=0} + (\tilde{m}/\tilde{r}_{f,S}^2)[(g_0^-)^2+2\Lambda_R]^{1/2}\}\hat{r}_{f,1}$$
$$+(\tilde{m}/\tilde{r}_{f,S}^2)\sqrt{(g_0^-)^2+2\Lambda_R}(\tilde{T}_{b,S,A}/g_0^-)(1+\bar{\delta}P_n\hat{T}_C^-) - \bar{\delta}P_n[Le_{1,0}^{-1}(d\hat{Y}_{1,1}^-/d\tilde{r})_{\tilde{r}_{f,S}}$$
$$-(\tilde{m}/\tilde{r}_{f,S}^2)\hat{Y}_{1,1}^-(\tilde{r}_{f,S})] + O(\bar{\delta}^2)$$
$$= -\tilde{T}_{b,S,A}(\tilde{m}/\tilde{r}_{f,S}^2)[\sqrt{(g_0^-)^2+2\Lambda_R}/g_0^-] - \bar{\delta}P_n\{(d\hat{T}_1^-/d\tilde{r})_{\tilde{r}_{f,S}} + \hat{r}_{f,1}g_0^-[(\tilde{m}/\tilde{r}_{f,S}^2)$$
$$-\bar{\delta}P_n(2/\tilde{r}_{f,S})[(g_0^-)^2+2\Lambda_R]^{1/2}\hat{r}_{f,1} + \tilde{T}_{b,S,A}(\tilde{m}/\tilde{r}_{f,S}^2)[\sqrt{(g_0^-)^2+2\Lambda_R}/g_0^-]$$
$$-(2/\tilde{r}_{f,S})]\}[\sqrt{(g_0^-)^2+2\Lambda_R}/g_0^-] + \bar{\delta}P_n(\tilde{m}/\tilde{r}_{f,S}^2)\sqrt{(g_0^-)^2+2\Lambda_R}(\tilde{T}_{b,S,A}/g_0^-)\{[\hat{T}_1^-(\tilde{r}_{f,S}) + \hat{r}_{f,1}g_0^-]/\tilde{T}_{b,S,A}\}$$
$$-\bar{\delta}P_n[Le_1^{-1}(d\hat{Y}_{1,1}^-/d\tilde{r})_{\tilde{r}_{f,S}} - (\tilde{m}/\tilde{r}_{f,S}^2)\hat{Y}_{1,1}^-(\tilde{r}_{f,S})] + O(\bar{\delta}^2)$$
$$= -\bar{\delta}P_n[(d\hat{T}_1^-/d\tilde{r})_{\tilde{r}_{f,S}} - (\tilde{m}/\tilde{r}_{f,S}^2)\hat{T}_1^-(\tilde{r}_{f,S})][\sqrt{(g_0^-)^2+2\Lambda_R}/g_0^-] - \bar{\delta}P_n[Le_1^{-1}(d\hat{Y}_{1,1}^-/d\tilde{r})_{\tilde{r}_{f,S}}$$
$$-(\tilde{m}/\tilde{r}_{f,S}^2)\hat{Y}_{1,1}^-(\tilde{r}_{f,S})] + O(\bar{\delta}^2)$$

$$\xi\to\infty\ :\ \frac{1}{Le_1}\frac{\partial\tilde{\phi}_{1,2}}{\partial\xi} + \left(\frac{2\xi}{Le_1\tilde{r}_{f,S}}\right)\frac{\partial\tilde{\phi}_{1,1}}{\partial\xi} - \frac{\tilde{m}}{\tilde{r}_{f,S}^2}\tilde{\phi}_{1,1} \to -a_{1,1}^+\frac{\tilde{m}}{\tilde{r}_{f,S}^2} + \bar{\delta}P_n\left[\frac{1}{Le_1}\left(\frac{d\hat{Y}_{1,1}^+}{d\tilde{r}}\right)_{\tilde{r}_{f,S}} - \frac{\tilde{m}}{\tilde{r}_{f,S}^2}\hat{Y}_{1,1}^+(\tilde{r}_{f,S})\right] + O(\bar{\delta}^2)$$

$$a_{T,A}^+ = a_{T,1}^+[1-\exp(-\tilde{m}/\tilde{r}_{f,S})]\quad ;\ a_{1,1}^+ = -(a_{T,1}^+/g_0^+)\sqrt{(g_0^+)^2+2\Lambda_R}$$

$$\partial\tilde{\theta}_1/\partial\xi \to -[(g_0^-)^2+2\Lambda_R]^{1/2} + O(\bar{\delta}^2)$$

$$\xi\to\infty\ :\ \tilde{\theta}_1 \to -\sqrt{(g_0^+)^2+2\Lambda_R}[(a_{T,A}^+/g_0^+)(1+\bar{\delta}P_n\hat{T}_C^+) - (\xi - \bar{\delta}P_n\hat{r}_{f,1})] + O(\bar{\delta}^2)$$
$$= -\sqrt{(g_0^+)^2+2\Lambda_R}\langle\{[a_{T,A}^+ + \bar{\delta}P_n\hat{T}_1^+(\tilde{r}_{f,S})]/g_0^+\} - \xi\rangle + O(\bar{\delta}^2)$$

$$\partial\tilde{\theta}_1/\partial\xi \to \sqrt{(g_0^+)^2+2\Lambda_R} + O(\bar{\delta}^2)$$

From the steady state solution :

$$\partial^2\theta_1/\partial\xi^2 = \Lambda_K\phi_{1,1}\phi_{2,1}\exp(-\theta_1)$$

$$\xi\to-\infty\ :\ \theta_1\to-[(\tilde{T}_{b,S,A}/g_0^-)+\xi][(g_0^-)^2+2\Lambda_R]^{1/2}\quad ;\ d\theta_1/d\xi\to-[(g_0^-)^2+2\Lambda_R]^{1/2}$$

$$\xi\to\infty\ :\ \theta_1\to-[(a_{T,A}^+/g_0^+)-\xi][(g_0^+)^2+2\Lambda_R]^{1/2}\quad ;\ d\theta_1/d\xi\to[(g_0^+)^2+2\Lambda_R]^{1/2}$$

$$\phi_{1,1} = a_{1,1}^+[1-\exp(-Le_1\tilde{m}/\tilde{r}_{f,S})] + Le_1\{[(a_{T,A}^+/g_0^+)-\xi][(g_0^+)^2+2\Lambda_R]^{1/2}+\theta_1\}$$

$$\phi_{2,1} = a_{1,1}^+[\tilde{Y}_{2,\infty}/(1+\tilde{Y}_{2,\infty})] + Le_2\{(a_{T,A}^+/g_0^+)[(g_0^+)^2+2\Lambda_R]^{1/2} + [(g_0^-)^2+2\Lambda_R]^{1/2}\xi + \theta_1\}$$

Inner expansion : $\tilde{\theta}_1 = \theta_1 + \bar{\delta}P_n\hat{\theta}_1$; $\tilde{\phi}_{1,1} = \phi_{1,1} + \bar{\delta}P_n\hat{\phi}_{1,1}$; $\tilde{\phi}_{2,1} = \phi_{2,1} + \bar{\delta}P_n\hat{\phi}_{2,1}$

(a) $\exp(-\tilde{\theta}_1) = \exp[-(\theta_1+\bar{\delta}P_n\hat{\theta}_1)] = \exp(-\theta_1)\exp(-\bar{\delta}P_n\hat{\theta}_1) = \exp(-\theta_1)(1-\bar{\delta}P_n\hat{\theta}_1+...)$

$$\partial^2(\theta_1+\bar{\delta}P_n\hat{\theta}_1)/\partial\xi^2 = \Lambda_K(\phi_{1,1}+\bar{\delta}P_n\hat{\phi}_{1,1})(\phi_{2,1}+\bar{\delta}P_n\hat{\phi}_{2,1})\exp(-\theta_1)(1-\bar{\delta}P_n\hat{\theta}_1+...)$$
$$= \Lambda_K[\phi_{1,1}\phi_{2,1} + \bar{\delta}P_n(\phi_{1,1}\hat{\phi}_{2,1} + \phi_{2,1}\hat{\phi}_{1,1} - \phi_{1,1}\phi_{2,1}\hat{\theta}_1) + ...]\exp(-\theta_1)$$

Since $d^2\theta_1/d\xi^2 = \Lambda_K\phi_{1,1}\phi_{2,1}\exp(-\theta_1)$

$$d^2\hat{\theta}_1/d\xi^2 = \Lambda_K(\phi_{1,1}\hat{\phi}_{2,1} + \phi_{2,1}\hat{\phi}_{1,1} - \phi_{1,1}\phi_{2,1}\hat{\theta}_1)\exp(-\theta_1) = (d^2\theta_1/d\xi^2)[(\hat{\phi}_{1,1}/\phi_{1,1}) + (\hat{\phi}_{2,1}/\phi_{2,1}) - \hat{\theta}_1]$$

(b) $\xi\to-\infty\ :\ \theta_1 + \bar{\delta}P_n\hat{\theta}_1 \to -\sqrt{(g_0^-)^2+2\Lambda_R}\left\langle\{[\tilde{T}_{b,S,A} + \bar{\delta}P_n\hat{T}_1^-(\tilde{r}_{f,S})]/g_0^-\} + \xi\right\rangle + O(\bar{\delta}^2)$

Since $\theta_1 \to -[(\tilde{T}_{b,S,A}/g_0^-)+\xi][(g_0^-)^2+2\Lambda_R]^{1/2}$

$$\hat{\theta}_1 \to -\hat{T}_1^-(\tilde{r}_{f,S})\sqrt{(g_0^-)^2+2\Lambda_R}/g_0^- \quad\text{and}\quad d\hat{\theta}_1/d\xi \to 0$$

(c) $\xi \to \infty$: $\theta_1 + \bar{\delta} P_n \hat{\theta}_1 \to -\sqrt{(g_0^+)^2 + 2\Lambda_R} \left\langle \{[a_{T,A}^+ + \bar{\delta} P_n \hat{T}_1^+(\tilde{r}_{f,S})] / g_0^+\} - \xi \right\rangle + O(\bar{\delta}^2)$

Since $\theta_1 \to -[(a_{T,A}^+ / g_0^+) - \xi][(g_0^+)^2 + 2\Lambda_R]^{1/2}$

$\hat{\theta}_1 \to -\hat{T}_1^+(\tilde{r}_{f,S})\sqrt{(g_0^+)^2 + 2\Lambda_R} / g_0^+$ and $d\hat{\theta}_1 / d\xi \to 0$

(d) $$(\theta_1 + \bar{\delta} P_n \hat{\theta}_1) - [(\phi_{1,1} + \bar{\delta} P_n \hat{\phi}_{1,1}) / Le_1] = -(a_{T,A}^+ / g_0^+)\sqrt{(g_0^+)^2 + 2\Lambda_R} - (a_{1,1}^+ / Le_1)[1 - \exp(-Le_{1,0}\tilde{m} / \tilde{r}_{f,S})]$$
$$+\sqrt{(g_0^+)^2 + 2\Lambda_R}\,\xi - \bar{\delta} P_n \{\sqrt{(g_0^+)^2 + 2\Lambda_R}(\hat{a}_{T,1}^+ / g_0^+)\Psi_2(\tilde{r}_{f,S},1)$$
$$+(\hat{a}_{1,1}^+ / Le_{1,0})\Psi_2(\tilde{r}_{f,S}, Le_{1,0})\} + O(\bar{\delta}^2)$$

Since $\theta_1 - (\phi_{1,1} / Le_1) = -(a_{T,A}^+ / g_0^+)\sqrt{(g_0^+)^2 + 2\Lambda_R} - (a_{1,1}^+ / Le_1)[1 - \exp(-Le_1 \tilde{m} / \tilde{r}_{f,S})] + \sqrt{(g_0^+)^2 + 2\Lambda_R}\,\xi$

$$\hat{\theta}_1 - (\hat{\phi}_{1,1} / Le_1) = -\{\sqrt{(g_0^+)^2 + 2\Lambda_R}(\hat{a}_{T,1}^+ / g_0^+)\Psi_2(\tilde{r}_{f,S},1) + (\hat{a}_{1,1}^+ / Le_1)\Psi_2(\tilde{r}_{f,S}, Le_1)\}$$

or $\hat{\phi}_{1,1} = \hat{a}_{1,1}^+\Psi_2(\tilde{r}_{f,S}, Le_1) + Le_1[\sqrt{(g_0^+)^2 + 2\Lambda_R}(\hat{a}_{T,1}^+ / g_0^+)\Psi_2(\tilde{r}_{f,S},1) + \hat{\theta}_1]$

(e) $$(\theta_1 + \bar{\delta} P_n \hat{\theta}_1) - [(\phi_{2,1} + \bar{\delta} P_n \hat{\phi}_{2,1}) / Le_2]$$
$$= -(a_{1,1}^+ / Le_2)[\tilde{Y}_{2,\infty} / (1 + \tilde{Y}_{2,\infty})] - (a_{T,A}^+ / g_0^+)\sqrt{(g_0^+)^2 + 2\Lambda_R} - \sqrt{(g_0^-)^2 + 2\Lambda_R}\,\xi$$
$$-\bar{\delta} P_n[\sqrt{(g_0^+)^2 + 2\Lambda_R}(\hat{a}_{T,1}^+ / g_0^+)\Psi_2(\tilde{r}_{f,S},1) + (\hat{a}_{2,1}^+ / Le_2)\Psi_2(\tilde{r}_{f,S}, Le_2)] + O(\bar{\delta}^2)$$

Since

$$\theta_1 - (\phi_{2,1} / Le_2) = -(a_{1,1}^+ / Le_2)[\tilde{Y}_{2,\infty} / (1 + \tilde{Y}_{2,\infty})] - (a_{T,A}^+ / g_0^+)[(g_0^+)^2 + 2\Lambda_R]^{1/2} - [(g_0^-)^2 + 2\Lambda_R]^{1/2}\,\xi$$

$$\hat{\theta}_1 - (\hat{\phi}_{2,1} / Le_2) = -[\sqrt{(g_0^+)^2 + 2\Lambda_R}(\hat{a}_{T,1}^+ / g_0^+)\Psi_2(\tilde{r}_{f,S},1) + (\hat{a}_{2,1}^+ / Le_2)\Psi_2(\tilde{r}_{f,S}, Le_2)]$$

or $\hat{\phi}_{2,1} = \hat{a}_{2,1}^+\Psi_2(\tilde{r}_{f,S}, Le_2) + Le_2[\sqrt{(g_0^+)^2 + 2\Lambda_R}(\hat{a}_{T,1}^+ / g_0^+)\Psi_2(\tilde{r}_{f,S},1) + \hat{\theta}_1]$

(6) Summary

$$\frac{\hat{a}_{1,1}^{-}}{Le_1}\left\{ \Psi_1(\tilde{r}_{f,S},Le_1) - \frac{[\tilde{m}Le_1\Psi_1(1,Le_1)-\Psi_1^*(1,Le_1)]}{[\tilde{m}Le_1\Psi_2(1,Le_1)-\Psi_2^*(1,Le_1)]}\Psi_2(\tilde{r}_{f.S},Le_1)\right\} - \frac{\hat{a}_{1,1}^{+}}{Le_1}\Psi_2(\tilde{r}_{f,S},Le_1)$$

$$= \frac{\hat{a}_{2,1}^{-}}{Le_2}\left\{ \Psi_1(\tilde{r}_{f,S},Le_2) - \frac{[\tilde{m}Le_2\Psi_1(1,Le_2)-\Psi_1^*(1,Le_2)]}{[\tilde{m}Le_2\Psi_2(1,Le_2)-\Psi_2^*(1,Le_2)]}\Psi_2(\tilde{r}_{f,S},Le_2)\right\} - \frac{\hat{a}_{2,1}^{+}}{Le_2}\Psi_2(\tilde{r},Le_2)$$

$$\hat{a}_{1,1}^{-}\left\{\left[\frac{\Psi_1^*(\tilde{r}_{f,S},Le_1)}{Le_1} - \frac{\tilde{m}}{\tilde{r}_{f,S}^2}\Psi_1(\tilde{r}_{f,S},Le_1)\right] - \frac{[\tilde{m}Le_1\Psi_1(1,Le_1)-\Psi_1^*(1,Le_1)]}{[\tilde{m}Le_1\Psi_2(1,Le_1)-\Psi_2^*(1,Le_1)]}\left[\frac{\Psi_2^*(\tilde{r}_{f,S},Le_1)}{Le_1} - \frac{\tilde{m}}{\tilde{r}_{f,S}^2}\Psi_2(\tilde{r}_{f,S},Le_1)\right]\right\}$$

$$-\hat{a}_{1,1}^{+}\{[\Psi_2^*(\tilde{r}_{f,S},Le_1)/Le_1] - (\tilde{m}/\tilde{r}_{f,S}^2)\Psi_2(\tilde{r}_{f,S},Le_1)\}$$

$$= \hat{a}_{2,1}^{-}\left\{\left[\frac{\Psi_1^*(\tilde{r}_{f,S},Le_2)}{Le_2} - \frac{\tilde{m}}{\tilde{r}_{f,S}^2}\Psi_1(\tilde{r}_{f,S},Le_2)\right] - \frac{[\tilde{m}Le_2\Psi_1(1,Le_2)-\Psi_1^*(1,Le_2)]}{[\tilde{m}Le_2\Psi_2(1,Le_2)-\Psi_2^*(1,Le_2)]}\left[\frac{\Psi_2^*(\tilde{r}_{f,S},Le_2)}{Le_{2,0}} - \frac{\tilde{m}}{\tilde{r}_{f,S}^2}\Psi_2(\tilde{r}_{f,S},Le_2)\right]\right\}$$

$$-\hat{a}_{2,1}^{+}\{[\Psi_2^*(\tilde{r}_{f,S},Le_2)/Le_2] - (\tilde{m}/\tilde{r}_{f,S}^2)\Psi_2(\tilde{r}_{f,S},Le_2)\}$$

$$\sqrt{(g_0^-)^2+2\Lambda_R}\,(\hat{T}_{b,1}/g_0^-)\{A_{T2}\Psi_1(\tilde{r}_{f,S},1)+[1-A_{T2}\Psi_1(1,1)]\Psi_2(\tilde{r}_{f,S},1)/\Psi_2(1,1)\} - \sqrt{(g_0^+)^2+2\Lambda_R}\,(\hat{a}_{T,1}^{+}/g_0^+)\Psi_2(\tilde{r}_{f,S},1)$$

$$= \frac{\hat{a}_{1,1}^{+}}{Le_1}\Psi_2(\tilde{r}_{f,S},Le_1) - \frac{\hat{a}_{1,1}^{-}}{Le_1}\left\{\Psi_1(\tilde{r}_{f,S},Le_1) - \frac{[\tilde{m}Le_1\Psi_1(1,Le_1)-\Psi_1^*(1,Le_1)]}{[\tilde{m}Le_1\Psi_2(1,Le_1)-\Psi_2^*(1,Le_1)]}\Psi_2(\tilde{r}_{f.S},Le_1)\right\}$$

$$\hat{T}_{b,1}[\sqrt{(g_0^-)^2+2\Lambda_R}/g_0^-]\langle A_{T2}[\Psi_1^*(\tilde{r}_{f,S},1)-(\tilde{m}/\tilde{r}_{f,S}^2)\Psi_1(\tilde{r}_{f,S},1)] + \{[1-A_{T2}\Psi_1(1,1)]/\Psi_2(1,1)\}[\Psi_2^*(\tilde{r}_{f,S},1)-(\tilde{m}/\tilde{r}_{f,S}^2)\Psi_2(\tilde{r}_{f,S},1)]\rangle$$

$$-\hat{a}_{T,1}^{+}[\sqrt{(g_0^+)^2+2\Lambda_R}/g_0^+)[\Psi_2^*(\tilde{r}_{f,S},1)-(\tilde{m}/\tilde{r}_{f,S}^2)\Psi_2(\tilde{r}_{f,S},1)]$$

$$= \hat{a}_{1,1}^{+}\{[\Psi_2^*(\tilde{r}_{f,S},Le_1)/Le_1] - (\tilde{m}/\tilde{r}_{f,S}^2)\Psi_2(\tilde{r}_{f,S},Le_1)\}$$

$$-\hat{a}_{1,1}^{-}\left\{\left[\frac{\Psi_1^*(\tilde{r}_{f,S},Le_1)}{Le_1} - \frac{\tilde{m}}{\tilde{r}_{f,S}^2}\Psi_1(\tilde{r}_{f,S},Le_1)\right] - \frac{[\tilde{m}Le_1\Psi_1(1,Le_1)-\Psi_1^*(1,Le_1)]}{[\tilde{m}Le_1\Psi_2(1,Le_1)-\Psi_2^*(1,Le_1)]}\left[\frac{\Psi_2^*(\tilde{r}_{f,S},Le_1)}{Le_1} - \frac{\tilde{m}}{\tilde{r}_{f,S}^2}\Psi_2(\tilde{r}_{f,S},Le_1)\right]\right\}$$

$$\tilde{\theta}_1 - (\tilde{\phi}_{1,1}/Le_1) = -(a_{T,A}^{+}/g_0^+)\sqrt{(g_0^+)^2+2\Lambda_R} - (a_{1,1}^{+}/Le_1)[1-\exp(-Le_1\tilde{m}/\tilde{r}_{f,S})] + \sqrt{(g_0^+)^2+2\Lambda_R}\,\xi$$

$$-\bar{\delta}P_n\{\sqrt{(g_0^+)^2+2\Lambda_R}\,(\hat{a}_{T,1}^{+}/g_0^+)\Psi_2(\tilde{r}_{f,S},1) + (\hat{a}_{1,1}^{+}/Le_1)\Psi_2(\tilde{r}_{f,S},Le_1)\} + O(\bar{\delta}^2)$$

$$\tilde{\theta}_1 - (\tilde{\phi}_{2,1}/Le_2) = -(a_{1,1}^{+}/Le_2)[\tilde{Y}_{2,\infty}/(1+\tilde{Y}_{2,\infty})] - (a_{T,A}^{+}/g_0^+)\sqrt{(g_0^+)^2+2\Lambda_R} - \sqrt{(g_0^-)^2+2\Lambda_R}\,\xi$$

$$-\bar{\delta}P_n[\sqrt{(g_0^+)^2+2\Lambda_R}\,(\hat{a}_{T,1}^{+}/g_0^+)\Psi_2(\tilde{r}_{f,S},1) + (\hat{a}_{2,1}^{+}/Le_2)\Psi_2(\tilde{r}_{f,S},Le_2) + O(\bar{\delta}^2)$$

$$d^2\theta_1/d\xi^2 = \Lambda_K\phi_{1,1}\phi_{2,1}\exp(-\theta_1)$$

$$\phi_{1,1} = a_{1,1}^{+}[1-\exp(-Le_1\tilde{m}/\tilde{r}_{f,S})] + Le_1\{[(a_{T,A}^{+}/g_0^+)-\xi][(g_0^+)^2+2\Lambda_R]^{1/2}+\theta_1$$

$$\phi_{2,1} = a_{1,1}^{+}[\tilde{Y}_{2,\infty}/(1+\tilde{Y}_{2,\infty})] - Le_{2,1}(\tilde{m}/\tilde{r}_{f,S}) + Le_{2,0}\{(a_{T,A}^{+}/g_0^+)[(g_0^+)^2+2\Lambda_R]^{1/2} + [(g_0^-)^2+2\Lambda_R]^{1/2}\xi + \theta_1\}$$

$$\xi\to-\infty \ : \ \theta_1\to -[(\tilde{T}_{b,S,A}/g_0^-)+\xi][(g_0^-)^2+2\Lambda_R]^{1/2} \quad ; \quad d\theta_1/d\xi\to -[(g_0^-)^2+2\Lambda_R]^{1/2}$$

$$\xi\to\infty \ : \ \theta_1\to -[(a_{T,A}^{+}/g_0^+)-\xi][(g_0^+)^2+2\Lambda_R]^{1/2} \quad ; \quad d\theta_1/d\xi\to [(g_0^+)^2+2\Lambda_R]^{1/2}$$

$$d^2\hat{\theta}_1/d\xi^2 = \Lambda_K(\phi_{1,1}\hat{\phi}_{2,1}+\phi_{2,1}\hat{\phi}_{1,1}+\phi_{1,1}\phi_{2,1}\hat{\theta}_1)\exp(-\theta_1) = (d^2\theta_1/d\xi^2)[(\hat{\phi}_{1,1}/\phi_{1,1})+(\hat{\phi}_{2,1}/\phi_{2,1})-\hat{\theta}_1]$$

$$\hat{\phi}_{1,1} = \hat{a}_{1,1}^{+}\Psi_2(\tilde{r}_{f,S},Le_1) + Le_1[\sqrt{(g_0^+)^2+2\Lambda_R}\,(\hat{a}_{T,1}^{+}/g_0^+)\Psi_2(\tilde{r}_{f,S},1)+\hat{\theta}_1]$$

$$\hat{\phi}_{2,1} = \hat{a}_{2,1}^{+}\Psi_2(\tilde{r}_{f,S},Le_2) + Le_2[\sqrt{(g_0^+)^2+2\Lambda_R}\,(\hat{a}_{T,1}^{+}/g_0^+)\Psi_2(\tilde{r}_{f,S},1)+\hat{\theta}_1]$$

$$\xi\to-\infty \ : \ \hat{\theta}_1\to -\hat{T}_1^-(\tilde{r}_{f,S})\sqrt{(g_0^-)^2+2\Lambda_R}/g_0^- \quad ; \quad d\hat{\theta}_1/d\xi\to 0$$

$$\hat{T}_1^-(\tilde{r}_{f,S}) = \hat{T}_{b,1}\{A_{T2}\Psi_1(\tilde{r}_{f,S},1)+[1-A_{T2}\Psi_1(1,1)]\Psi_2(\tilde{r}_{f,S},1)/\Psi_2(1,1)]\}$$

$$\xi\to\infty \ : \ \hat{\theta}_1\to -\hat{T}_1^+(\tilde{r}_{f,S})\sqrt{(g_0^+)^2+2\Lambda_R}/g_0^+ \quad ; \quad d\hat{\theta}_1/d\xi\to 0 \quad ; \quad \hat{T}_1^+(\tilde{r}_{f,S}) = \hat{a}_{T,1}^{+}\Psi_2(\tilde{r}_{f,S},1)$$

From the steady state solution:

$$d^2\theta_1/d\xi^2 = \alpha^{1/3}[\tilde{m}/(2\tilde{r}_{f,S}^2)]^2(d^2\bar{\theta}/d\bar{\xi}^2)$$

$$\phi_{1,1} = Le_1(\bar{\theta}-\bar{\xi})/\alpha^{1/3} \qquad , \qquad \phi_{2,1} = Le_2(\bar{\theta}+\bar{\xi})/\alpha^{1/3}$$

$$d\bar{\xi}/d\xi = \alpha^{1/3}[\tilde{m}/(2\tilde{r}_{f,S}^2)]$$

From the stability analysis:

$$d^2\hat{\theta}_1/d\xi^2 = (d^2\theta_1/d\xi^2)[(\hat{\phi}_{1,1}/\phi_{1,1}) + (\hat{\phi}_{2,1}/\phi_{2,1}) - \hat{\theta}_1]$$

$$\hat{\phi}_{1,1} = \hat{a}_{1,1}^{+}\Psi_2(\tilde{r}_{f,S}, Le_1) + Le_1[\sqrt{(g_0^{+})^2 + 2\Lambda_R}\,(\hat{a}_{T,1}^{+}/g_0^{+})\Psi_2(\tilde{r}_{f,S},1) + \hat{\theta}_1]$$

$$\hat{\phi}_{2,1} = \hat{a}_{2,1}^{+}\Psi_2(\tilde{r}_{f,S}, Le_2) + Le_2[\sqrt{(g_0^{+})^2 + 2\Lambda_R}\,(\hat{a}_{T,1}^{+}/g_0^{+})\Psi_2(\tilde{r}_{f,S},1) + \hat{\theta}_1]$$

$$\xi \to -\infty \;:\; \hat{\theta}_1 \to -\hat{T}_1^{-}(\tilde{r}_{f,S})\sqrt{(g_0^{-})^2 + 2\Lambda_R}/g_0^{-}\} \quad ; \quad d\hat{\theta}_1/d\xi \to 0$$

$$\hat{T}_1^{-}(\tilde{r}_{f,S}) = \hat{T}_{b,1}\{A_{T2}\Psi_1(\tilde{r}_{f,S},1) + [1 - A_{T2}\Psi_1(1,1)]\Psi_2(\tilde{r}_{f,S},1)/\Psi_2(1,1)\}$$

$$\xi \to \infty \;:\; \hat{\theta}_1 \to -\hat{T}_1^{+}(\tilde{r}_{f,S})\sqrt{(g_0^{+})^2 + 2\Lambda_R}/g_0^{+} \quad ; \quad d\hat{\theta}_1/d\xi \to 0 \quad ; \quad \hat{T}_1^{+}(\tilde{r}_{f,S}) = \hat{a}_{T,1}^{+}\Psi_2(\tilde{r}_{f,S},1)$$

Transform the stability equation to Liñán's variables

$$\frac{d^2\hat{\theta}_1}{d\bar{\xi}^2}\left[\alpha^{1/3}\frac{\tilde{m}}{2\tilde{r}_{f,S}^2}\right]^2 = \frac{d^2\theta_1}{d\bar{\xi}^2}\alpha^{1/3}\left(\frac{\tilde{m}}{2\tilde{r}_{f,S}^2}\right)^2\left[\frac{\alpha^{1/3}\hat{\phi}_{1,1}}{Le_1(\bar{\theta}-\bar{\xi})} + \frac{\alpha^{1/3}\hat{\phi}_{2,1}}{Le_2(\bar{\theta}+\bar{\xi})} - \hat{\theta}_1\right]$$

$$\frac{d^2\hat{\theta}_1}{d\bar{\xi}^2} = \frac{d^2\bar{\theta}}{d\bar{\xi}^2}\left[\frac{\hat{\phi}_{1,1}}{Le_1(\bar{\theta}-\bar{\xi})} + \frac{\hat{\phi}_{2,1}}{Le_2(\bar{\theta}+\bar{\xi})} - \frac{\hat{\theta}_1}{\alpha^{1/3}}\right]$$

$$\bar{\xi} \to -\infty \;:\; \hat{\theta}_1 \to -\hat{T}_1^{-}(\tilde{r}_{f,S})\sqrt{(g_0^{-})^2 + 2\Lambda_R}/g_0^{-}\} \quad ; \quad d\hat{\theta}_1/d\bar{\xi} \to 0$$

$$\bar{\xi} \to \infty \;:\; \hat{\theta}_1 \to -\hat{T}_1^{+}(\tilde{r}_{f,S})\sqrt{(g_0^{+})^2 + 2\Lambda_R}/g_0^{+} \quad ; \quad d\hat{\theta}_1/d\bar{\xi} \to 0$$

From the steady state solution, you have $d^2\bar{\theta}/d\bar{\xi}^2$, $(\bar{\theta}-\bar{\xi})$ and $(\bar{\theta}+\bar{\xi})$

APPENDIX D. HOMOGENEOUS EQUATION SOLUTION

Differential Equation: $\frac{d}{d\tilde{r}}\left(\tilde{r}^2\frac{d\hat{Y}}{d\tilde{r}}\right) - p\tilde{m}\frac{d\hat{Y}}{d\tilde{r}} - n(n+1)\hat{Y} = 0$

Because $\tilde{r}=0$ is an irregular singular point for this equation, only one fundamental solution can be solved by the conventional power series solution.

(A) Fundamental solution that can be solved by the conventional power series solution

Let the solution be $\hat{Y} = \Psi_1$

$$\hat{Y} = \Psi_1 = 1 + \sum_{k=1}^{\infty} c_k \tilde{r}^k = 1 + c_1\tilde{r} + c_2\tilde{r}^2 + c_3\tilde{r}^3 + c_4\tilde{r}^4 + c_5\tilde{r}^5 + c_6\tilde{r}^6 + \cdots \qquad (\text{let } c_0 = 1)$$

$$\frac{d\hat{Y}}{d\tilde{r}} = \frac{d\Psi_1}{d\tilde{r}} = c_1(1) + c_2(2)\tilde{r} + c_3(3)\tilde{r}^2 + c_4(4)\tilde{r}^3 + c_5(5)\tilde{r}^4 + c_6(6)\tilde{r}^5 + \cdots$$

$$\tilde{r}^2\frac{d\hat{Y}}{d\tilde{r}} = \tilde{r}^2\frac{d\Psi_1}{d\tilde{r}} = c_1(1)\tilde{r}^2 + c_2(2)\tilde{r}^3 + c_3(3)\tilde{r}^4 + c_4(4)\tilde{r}^5 + c_5(5)\tilde{r}^6 + c_6(6)\tilde{r}^7 + \cdots$$

$$\frac{d}{d\tilde{r}}\left(\tilde{r}^2\frac{d\hat{Y}}{d\tilde{r}}\right) = \frac{d}{d\tilde{r}}\left(\tilde{r}^2\frac{d\Psi_1}{d\tilde{r}}\right) = c_1(2\cdot1)\tilde{r} + c_2(3\cdot2)\tilde{r}^2 + c_3(4\cdot3)\tilde{r}^3 + c_4(5\cdot4)\tilde{r}^4 + c_5(6\cdot5)\tilde{r}^5 + c_6(7\cdot6)\tilde{r}^6 + \cdots$$

Applying the solution into the equation, we obtain the following:

	$\tilde{r}^0$	$\tilde{r}^1$	$\tilde{r}^2$	$\tilde{r}^3$	$\tilde{r}^4$	$\tilde{r}^5$
$\frac{d}{d\tilde{r}}\left(\tilde{r}^2\frac{d\hat{Y}}{d\tilde{r}}\right)$		$(2\cdot1)c_1$	$(3\cdot2)c_2$	$(4\cdot3)c_3$	$(5\cdot4)c_4$	$(6\cdot5)c_5$
$-p\tilde{m}(d\hat{Y}/d\tilde{r})$	$-p\tilde{m}c_1$	$-2p\tilde{m}c_2$	$-3p\tilde{m}c_3$	$-4p\tilde{m}c_4$	$-5p\tilde{m}c_5$	$-6p\tilde{m}c_6$
$-n(n+1)\hat{Y}$	$-n(n+1)$	$-n(n+1)c_1$	$-n(n+1)c_2$	$-n(n+1)c_3$	$-n(n+1)c_4$	$-n(n+1)c_5$

(1) For $\tilde{r}^0$ terms: $-p\tilde{m}c_1 - n(n+1) = 0$, $c_1 = -n(n+1)/(p\tilde{m})$

(2) For all other higher order terms, a recurrence relation can be found as follows:

$$-kp\tilde{m}c_k + k(k-1)c_{k-1} - n(n+1)c_{k-1} = 0 \quad \text{or} \quad c_k = \{[k(k-1)-n(n+1)]/(kp\tilde{m})\}c_{k-1} \quad , \quad k \geq 2$$

$$c_2 = \frac{2(2-1)-n(n+1)}{2p\tilde{m}}c_1 = \frac{2(2-1)-n(n+1)}{2p\tilde{m}}\frac{-n(n+1)}{p\tilde{m}} = \frac{2(2-1)-n(n+1)}{2p\tilde{m}}\frac{1(1-1)-n(n+1)}{1p\tilde{m}}$$

$$= \frac{\prod_{i=1}^{2}[i(i-1)-n(n+1)]}{(2!)(p\tilde{m})^2}$$

Therefore, $c_2 = 0$ and $c_3 = c_4 = c_5 = c_6 = ... = 0$ when $n = 1$.

$$c_3 = \frac{3(3-1)-n(n+1)}{3p\tilde{m}}c_2 = \frac{3(3-1)-n(n+1)}{3p\tilde{m}}\frac{\prod_{i=1}^{2}[i(i-1)-n(n+1)]}{(2!)(p\tilde{m})^2} = \frac{\prod_{i=1}^{3}[i(i-1)-n(n+1)]}{(3!)(p\tilde{m})^3}$$

Therefore, $c_3 = 0$ and $c_4 = c_5 = c_6 = c_7 = ... = 0$ when $n = 2$.

$$c_4 = \frac{4(4-1)-n(n+1)}{4p\tilde{m}}c_3 = \frac{4(4-1)-n(n+1)}{4p\tilde{m}}\frac{\prod_{i=1}^{3}[i(i-1)-n(n+1)]}{(3!)(p\tilde{m})^3} = \frac{\prod_{i=1}^{4}[i(i-1)-n(n+1)]}{(4!)(p\tilde{m})^4}$$

Therefore, $c_4 = 0$ and $c_5 = c_6 = c_7 = c_8 = ... = 0$ when $n = 3$.

....

These results yield: $\hat{Y} = \Psi_1(\tilde{r},p) = 1 + \sum_{k=1}^{\infty} c_k \tilde{r}^k = 1 + \sum_{k=1}^{n}\left\{\prod_{i=1}^{k}[i(i-1)-n(n+1)]\right\}\frac{\tilde{r}^k}{(k!)(p\tilde{m})^k}$

At $\tilde{r}=0$: $\Psi_1(0,p)=1$ and as $\tilde{r}\to\infty$: $|\Psi_1(\infty,p)|\to\infty$

As $\tilde{r}\to\infty$: $|\Psi_1(\infty,p)|\to\infty \quad \therefore \quad |\Psi_1(\infty,p_0)|\to\infty$

Only one fundamental solution is found from this approach.

Expressions for Ψ_1 and $\hat{\Psi}_1$

$n=1$: $n(n+1) = 1(1+1) = 2$

$$\Psi_1 = 1 + \sum_{k=1}^{1}\left\{\prod_{i=1}^{k}[i(i-1)-2]\right\}\frac{\tilde{r}^k}{(k!)(p\tilde{m})^k} = 1 + (0-2)\frac{\tilde{r}}{p\tilde{m}} = 1 - \frac{2\tilde{r}}{p\tilde{m}} \quad ;$$

$$\hat{\Psi}_1 = \sum_{k=1}^{1}\left\{\prod_{i=1}^{k}[i(i-1)-2]\right\}\frac{\tilde{r}^k}{[(k-1)!]p^{k+1}\tilde{m}^k} = (0-2)\frac{\tilde{r}}{p\tilde{m}} = -\frac{2\tilde{r}}{p^2\tilde{m}}$$

$n=2$: $n(n+1) = 2(2+1) = 6$

$$\Psi_1 = 1 + \sum_{k=1}^{2}\left\{\prod_{i=1}^{k}[i(i-1)-6]\right\}\frac{\tilde{r}^k}{(k!)(p\tilde{m})^k} = 1 + (0-6)\frac{\tilde{r}}{p\tilde{m}} + (0-6)(2-6)\frac{\tilde{r}^2}{(2!)(p\tilde{m})^2} = 1 - \frac{6\tilde{r}}{p\tilde{m}} + \frac{12\tilde{r}^2}{(p\tilde{m})^2}$$

$$\hat{\Psi}_1 = \sum_{k=1}^{2}\left\{\prod_{i=1}^{k}[i(i-1)-6]\right\}\frac{\tilde{r}^k}{[(k-1)!]p^{k+1}\tilde{m}^k} = (0-6)\frac{\tilde{r}}{p^2\tilde{m}} + (0-6)(2-6)\frac{\tilde{r}^2}{p^3\tilde{m}^2} = \frac{6\tilde{r}}{p^2\tilde{m}} + \frac{24\tilde{r}^2}{p^3\tilde{m}^2}$$

$n=3$: $n(n+1) = 3(3+1) = 12$

$$\Psi_1 = 1 + \sum_{k=1}^{3}\left\{\prod_{i=1}^{k}[i(i-1)-12]\right\}\frac{\tilde{r}^k}{(k!)(p\tilde{m})^k} = 1 + \frac{(0-12)\tilde{r}}{p\tilde{m}} + \frac{(0-12)(2-12)\tilde{r}^2}{(2!)(p\tilde{m})^2} + \frac{(0-12)(2-12)(6-12)\tilde{r}^3}{(3!)(p\tilde{m})^3}$$

$$= 1 - \frac{12\tilde{r}}{p\tilde{m}} + \frac{60\tilde{r}^2}{(p\tilde{m})^2} - \frac{120\tilde{r}^3}{(p\tilde{m})^3}$$

$n=4$: $n(n+1) = 4(4+1) = 20$

$$\Psi_1 = 1 + \sum_{k=1}^{4}\left\{\prod_{i=1}^{k}[i(i-1)-20]\right\}\frac{\tilde{r}^k}{(k!)(p\tilde{m})^k}$$

$$= 1 + \frac{(0-20)\tilde{r}}{p\tilde{m}} + \frac{(0-20)(2-20)\tilde{r}^2}{(2!)(p\tilde{m})^2} + \frac{(0-20)(2-20)(6-20)\tilde{r}^3}{(3!)(p\tilde{m})^3} + \frac{(0-20)(2-20)(6-20)(12-20)\tilde{r}^4}{(4!)(p\tilde{m})^4}$$

$$=1-\frac{20\tilde{r}}{p\tilde{m}}+\frac{180\tilde{r}^2}{(p\tilde{m})^2}-\frac{840\tilde{r}^3}{(p\tilde{m})^3}+\frac{1680\tilde{r}^4}{(p\tilde{m})^4}$$

(B) Determine the second fundamental solution by Reduction of Order

Let $\hat{Y}=\Psi_2=\hat{Y}_2\Psi_1$ (let the second solution be Ψ_2)

$$\frac{d\hat{Y}}{d\tilde{r}}=\hat{Y}_2\frac{d\Psi_1}{d\tilde{r}}+\Psi_1\frac{d\hat{Y}_2}{d\tilde{r}}$$

$$\frac{d}{d\tilde{r}}\left(\tilde{r}^2\frac{d\hat{Y}}{d\tilde{r}}\right)=\frac{d}{d\tilde{r}}\left[\tilde{r}^2\left(\hat{Y}_2\frac{d\Psi_1}{d\tilde{r}}+\Psi_1\frac{d\hat{Y}_2}{d\tilde{r}}\right)\right]=\hat{Y}_2\frac{d}{d\tilde{r}}\left(\tilde{r}^2\frac{d\Psi_1}{d\tilde{r}}\right)+\tilde{r}^2\frac{d\Psi_1}{d\tilde{r}}\frac{d\hat{Y}_2}{d\tilde{r}}+\Psi_1\frac{d}{d\tilde{r}}\left(\tilde{r}^2\frac{d\hat{Y}_2}{d\tilde{r}}\right)+\tilde{r}^2\frac{d\hat{Y}_2}{d\tilde{r}}\frac{d\Psi_1}{d\tilde{r}}$$

$$=\hat{Y}_2\frac{d}{d\tilde{r}}\left(\tilde{r}^2\frac{d\Psi_1}{d\tilde{r}}\right)+\Psi_1\frac{d}{d\tilde{r}}\left(\tilde{r}^2\frac{d\hat{Y}_2}{d\tilde{r}}\right)+2\tilde{r}^2\frac{d\Psi_1}{d\tilde{r}}\frac{d\hat{Y}_2}{d\tilde{r}}$$

Substitute these back to the original equation

$$\hat{Y}_2\frac{d}{d\tilde{r}}\left(\tilde{r}^2\frac{d\Psi_1}{d\tilde{r}}\right)+\Psi_1\frac{d}{d\tilde{r}}\left(\tilde{r}^2\frac{d\hat{Y}_2}{d\tilde{r}}\right)+2\tilde{r}^2\frac{d\Psi_1}{d\tilde{r}}\frac{d\hat{Y}_2}{d\tilde{r}}-p\tilde{m}\left(\hat{Y}_2\frac{d\Psi_1}{d\tilde{r}}+\Psi_1\frac{d\hat{Y}_2}{d\tilde{r}}\right)-n(n+1)\hat{Y}_2\Psi_1=0$$

$$\hat{Y}_2\left[\frac{d}{d\tilde{r}}\left(\tilde{r}^2\frac{d\Psi_1}{d\tilde{r}}\right)-p\tilde{m}\frac{d\Psi_1}{d\tilde{r}}-n(n+1)\Psi_1\right]+\Psi_1\frac{d}{d\tilde{r}}\left(\tilde{r}^2\frac{d\hat{Y}_2}{d\tilde{r}}\right)+2\tilde{r}^2\frac{d\Psi_1}{d\tilde{r}}\frac{d\hat{Y}_2}{d\tilde{r}}-p\tilde{m}\Psi_1\frac{d\hat{Y}_2}{d\tilde{r}}=0$$

Because Ψ_1 is a solution of the equation, $\frac{d}{d\tilde{r}}\left(\tilde{r}^2\frac{d\Psi_1}{d\tilde{r}}\right)-p\tilde{m}\frac{d\Psi_1}{d\tilde{r}}-n(n+1)\Psi_1=0$

$$\Psi_1\frac{d}{d\tilde{r}}\left(\tilde{r}^2\frac{d\hat{Y}_2}{d\tilde{r}}\right)+2\tilde{r}^2\frac{d\Psi_1}{d\tilde{r}}\frac{d\hat{Y}_2}{d\tilde{r}}-p\tilde{m}\Psi_1\frac{d\hat{Y}_2}{d\tilde{r}}=0 \quad \text{or}$$

$$\Psi_1^2\frac{d}{d\tilde{r}}\left(\tilde{r}^2\frac{d\hat{Y}_2}{d\tilde{r}}\right)+2\tilde{r}^2\Psi_1\frac{d\Psi_1}{d\tilde{r}}\frac{d\hat{Y}_2}{d\tilde{r}}-p\tilde{m}\Psi_1^2\frac{d\hat{Y}_2}{d\tilde{r}}=0$$

$$\frac{d}{d\tilde{r}}\left(\tilde{r}^2\Psi_1^2\frac{d\hat{Y}_2}{d\tilde{r}}\right)-p\tilde{m}\Psi_1^2\frac{d\hat{Y}_2}{d\tilde{r}}=0 \quad \text{or} \quad \frac{d}{d\tilde{r}}\left(\tilde{r}^2\Psi_1^2\frac{d\hat{Y}_2}{d\tilde{r}}\right)=\frac{p\tilde{m}}{\tilde{r}^2}\left(\tilde{r}^2\Psi_1^2\frac{d\hat{Y}_2}{d\tilde{r}}\right)$$

$$\frac{d[\tilde{r}^2\Psi_1^2(d\hat{Y}_2/d\tilde{r})]}{\tilde{r}^2\Psi_1^2(d\hat{Y}_2/d\tilde{r})}=\frac{p\tilde{m}}{\tilde{r}^2}d\tilde{r} \quad \text{or} \quad \ell n\left|\tilde{r}^2\Psi_1^2\frac{d\hat{Y}_2}{d\tilde{r}}\right|=-\frac{p\tilde{m}}{\tilde{r}} \quad \Rightarrow \quad \tilde{r}^2\Psi_1^2\frac{d\hat{Y}_2}{d\tilde{r}}=\pm e^{-p\tilde{m}/\tilde{r}}$$

$$\frac{d\hat{Y}_2}{d\tilde{r}}=\pm\frac{e^{-p\tilde{m}/\tilde{r}}}{\tilde{r}^2\Psi_1^2} \quad \text{or} \quad \hat{Y}_2=c\pm\int\frac{e^{-p\tilde{m}/\tilde{r}}}{\tilde{r}^2\Psi_1^2}d\tilde{r} \quad \text{can let } c=0$$

Since Ψ_1 grows with $\tilde{r}$ (*i.e.*, $|\Psi_1|\to\infty$ as $\tilde{r}\to\infty$), we would like to have Ψ_2 decay with $\tilde{r}$. Therefore, we choose $\hat{Y}_2=\int_\infty^{\tilde{r}}\frac{e^{-p\tilde{m}/\tilde{r}}}{\tilde{r}^2\Psi_1^2}d\tilde{r}$ or $\Psi_2=\Psi_1\int_\infty^{\tilde{r}}\frac{e^{-p\tilde{m}/\tilde{r}}}{\tilde{r}^2\Psi_1^2}d\tilde{r}$ such that $\Psi_2\to 0$ as $\tilde{r}\to\infty$.

Let $z=p\tilde{m}/\tilde{r}$ then $dz=-p\tilde{m}\frac{d\tilde{r}}{\tilde{r}^2}$, As $\tilde{r}\to\infty$, $z=0$.

Thus, $\Psi_2 = \Psi_1 \int_0^z -\frac{1}{p\tilde{m}} \frac{e^{-z}}{\Psi_1{}^2} dz = -\frac{\Psi_1}{p\tilde{m}} \int_0^z \frac{e^{-z}}{\Psi_1{}^2} dz$

Since $p\tilde{m}$ is a constant, we can drop it from Ψ_2 to yield $\Psi_2 = -\Psi_1 \int_0^z \frac{e^{-z}}{\Psi_1{}^2} dz$

$$\Psi_1 = 1 + \sum_{k=1}^{n} \left\{ \prod_{i=1}^{k} [i(i-1) - n(n+1)] \right\} \frac{\tilde{r}^k}{(k!)(p\tilde{m})^k} = 1 + \sum_{k=1}^{n} \left\{ \prod_{i=1}^{k} [i(i-1) - n(n+1)] \right\} \frac{1}{(k!)} \frac{1}{z^k}$$

$$= \frac{1}{z^n} \left\{ z^n + \sum_{k=1}^{n} \left[\prod_{i=1}^{k} [i(i-1) - n(n+1)] \right] \frac{z^{n-k}}{(k!)} \right\}$$

As $\tilde{r} \to \infty$: $e^{-p\tilde{m}/\tilde{r}} = 1 - \frac{p\tilde{m}}{\tilde{r}} + \frac{(p\tilde{m})^2}{(2!)\tilde{r}^2} - \frac{(p\tilde{m})^3}{(3!)\tilde{r}^3} + \frac{(p\tilde{m})^4}{(4!)\tilde{r}^4} + \cdots = 1 - \frac{p\tilde{m}}{\tilde{r}} + \frac{(p\tilde{m})^2}{2\tilde{r}^2} - \frac{(p\tilde{m})^3}{6\tilde{r}^3} + \frac{(p\tilde{m})^4}{24\tilde{r}^4} + \cdots$

(1) $n = 1$: $\Psi_1 = 1 - \frac{2\tilde{r}}{p\tilde{m}} = 1 - \frac{2}{z} = \frac{z-2}{z}$

$$(z-2)^2 = z^2 - 4z + 4 = z^2 - 4(z-2) - 4 \quad \therefore \quad z^2 = (z-2)^2 + 4(z-2) + 4 = (z-2)^2 + 4[(z-2)+1]$$

$$\int_0^z \frac{e^{-z}}{\Psi_1{}^2} dz = \int_0^z \frac{z^2 e^{-z}}{(z-2)^2} dz = \int_0^z \frac{\{(z-2)^2 + 4[(z-2)+1]\} e^{-z}}{(z-2)^2} dz = \int_0^z e^{-z} dz + 4 \int_0^z \left[\frac{e^{-z}}{z-2} + \frac{e^{-z}}{(z-2)^2} \right] dz$$

$$= -e^{-z} \Big|_0^z - 4 \frac{e^{-z}}{z-2} \Big|_0^z = -(e^{-z} - 1) - 4\left(\frac{e^{-z}}{z-2} - \frac{1}{-2} \right) = -\left[e^{-z}\left(1 + \frac{4}{z-2} \right) + 1 \right] = -\left[e^{-z}\left(1 + \frac{4}{z\Psi_1} \right) + 1 \right]$$

$$= -\left[e^{-p\tilde{m}/\tilde{r}} \left(1 + \frac{\tilde{r}}{p\tilde{m}} \frac{4}{\Psi_1} \right) + 1 \right]$$

Using integration by parts, let $u = \frac{1}{z-2}$ and $dv = e^{-z} dz \Rightarrow du = \frac{-1}{(z-2)^2}$, $v = -e^{-z}$

$$\int \frac{e^{-z}}{z-2} dz = -\frac{e^{-z}}{z-2} - \int \frac{e^{-z}}{(z-2)^2} dz \quad \text{or} \quad \int \left[\frac{e^{-z}}{z-2} + \frac{e^{-z}}{(z-2)^2} \right] dz = -\frac{e^{-z}}{z-2}$$

$$\Psi_2 = -\Psi_1 \int_0^z \frac{e^{-z}}{\Psi_1{}^2} dz = -\Psi_1 \left\{ -\left[e^{-p\tilde{m}/\tilde{r}} \left(1 + \frac{\tilde{r}}{p\tilde{m}} \frac{4}{\Psi_1} \right) + 1 \right] \right\} = e^{-p\tilde{m}/\tilde{r}} \left(\Psi_1 + \frac{4\tilde{r}}{p\tilde{m}} \right) + \Psi_1 = e^{-p\tilde{m}/\tilde{r}} \left(1 - \frac{2\tilde{r}}{p\tilde{m}} + \frac{4\tilde{r}}{p\tilde{m}} \right) + \Psi_1$$

$$= e^{-p\tilde{m}/\tilde{r}} \left(1 + \frac{2\tilde{r}}{p\tilde{m}} \right) + \Psi_1 = e^{-p\tilde{m}/\tilde{r}} \left\{ 1 + \sum_{k=1}^{1} \left[\prod_{i=1}^{k} [n(n+1) - i(i-1)] \right] \frac{\tilde{r}^k}{(k!)(p\tilde{m})^k} \right\} - (-1)^1 \Psi_1$$

As $\tilde{r} \to \infty$: $\Psi_2 = e^{-p\tilde{m}/\tilde{r}} \left(1 + \frac{2\tilde{r}}{p\tilde{m}} \right) + \Psi_1 \to \left[1 - \frac{p\tilde{m}}{\tilde{r}} + O\left(\frac{1}{\tilde{r}^2} \right) \right] \left(1 + \frac{2\tilde{r}}{p\tilde{m}} \right) + \left(1 - \frac{2\tilde{r}}{p\tilde{m}} \right) = 1 + \frac{2\tilde{r}}{p\tilde{m}} - 2 + O\left(\frac{1}{\tilde{r}} \right) + 1 - \frac{2\tilde{r}}{p\tilde{m}}$

$= O\left(\frac{1}{\tilde{r}} \right) \to 0$ (as should be)

(2) $n = 2$: $\Psi_1 = 1 - \frac{6\tilde{r}}{p\tilde{m}} + \frac{12\tilde{r}^2}{(p\tilde{m})^2} = 1 - \frac{6}{z} + \frac{12}{z^2} = \frac{z^2 - 6z + 12}{z^2}$

$$(z^2 - 6z + 12)^2 = (z^2 - 6z + 12)(z^2 - 6z + 12) = z^2(z^2 - 6z + 12) - (6z - 12)(z^2 - 6z + 12)$$

$$= z^4 - 6z^3 + 12z^2 - 6(z - 2)(z^2 - 6z + 12)$$

$$= z^4 - 6z(z^2 - 6z + 12) - 36z^2 + 72z + 12z^2 - 6(z - 2)(z^2 - 6z + 12)$$

$$\therefore \quad z^4 = (z^2 - 6z + 12)^2 + 12[(z-1)(z^2 - 6z + 12) + 2z(z-3)]$$

$$\int_0^z \frac{e^{-z}}{\Psi_1^2} dz = \int_0^z \frac{z^4 e^{-z}}{(z^2 - 6z + 12)^2} dz = \int_0^z \frac{\{(z^2 - 6z + 12)^2 + 12[(z-1)(z^2 - 6z + 12) + 2z(z-3)]\}e^{-z}}{(z^2 - 6z + 12)^2} dz$$

$$= \int_0^z e^{-z} dz + 12 \int_0^z \left[\frac{(z-1)e^{-z}}{z^2 - 6z + 12} + \frac{2z(z-3)e^{-z}}{(z^2 - 6z + 12)^2} \right] dz = -e^{-z} \Big|_0^z - 12 \frac{ze^{-z}}{z^2 - 6z + 12} \Bigg|_0^z$$

$$= -(e^{-z} - 1) - 12 \frac{ze^{-z}}{z^2 - 6z + 12} = -\left[e^{-z} \left(1 + \frac{12z}{z^2 - 6z + 12} \right) - 1 \right] = -\left[e^{-z} \left(1 + \frac{12}{z\Psi_1} \right) - 1 \right] = -\left[e^{-p\tilde{m}/\tilde{r}} \left(1 + \frac{\tilde{r}}{p\tilde{m}} \frac{12}{\Psi_1} \right) - 1 \right]$$

Let $u = \frac{1}{z^2 - 6z + 12}$ and $dv = (z-1)e^{-z} dz \;\Rightarrow\; du = \frac{-(2z-6)}{(z^2 - 6z + 12)^2} = \frac{-2(z-3)}{(z^2 - 6z + 12)^2}$, $v = -ze^{-z}$

$$\int \frac{(z-1)e^{-z}}{z^2 - 6z + 12} dz = -\frac{ze^{-z}}{z^2 - 6z + 12} - \int \frac{2z(z-3)e^{-z}}{(z^2 - 6z + 12)^2} dz \quad \text{or}$$

$$\int \left[\frac{(z-1)e^{-z}}{z^2 - 6z + 12} + \frac{2z(z-3)e^{-z}}{(z^2 - 6z + 12)^2} \right] dz = -\frac{ze^{-z}}{z^2 - 6z + 12}$$

$$\Psi_2 = -\Psi_1 \int_0^z \frac{e^{-z}}{\Psi_1^2} dz = -\Psi_1 \left\{ -\left[e^{-p\tilde{m}/\tilde{r}} \left(1 + \frac{\tilde{r}}{p\tilde{m}} \frac{12}{\Psi_1} \right) - 1 \right] \right\} = e^{-p\tilde{m}/\tilde{r}} \left(\Psi_1 + \frac{12\tilde{r}}{p\tilde{m}} \right) - \Psi_1$$

$$= e^{-p\tilde{m}/\tilde{r}} \left[1 - \frac{6\tilde{r}}{p\tilde{m}} + \frac{12\tilde{r}^2}{(p\tilde{m})^2} + \frac{12\tilde{r}}{p\tilde{m}} \right] - \Psi_1 = e^{-p\tilde{m}/\tilde{r}} \left[1 + \frac{6\tilde{r}}{p\tilde{m}} + \frac{12\tilde{r}^2}{(p\tilde{m})^2} \right] - \Psi_1$$

$$= e^{-p\tilde{m}/\tilde{r}} \left\{ 1 + \sum_{k=1}^{2} \left[\prod_{i=1}^{k} [n(n+1) - i(i-1)] \right] \frac{\tilde{r}^k}{(k!)(p\tilde{m})^k} \right\} - (-1)^2 \Psi_1$$

As $\tilde{r} \to \infty$: $\Psi_2 = e^{-p\tilde{m}/\tilde{r}} \left[1 + \frac{6\tilde{r}}{p\tilde{m}} + \frac{12\tilde{r}^2}{(p\tilde{m})^2} \right] - \Psi_1 \to \left[1 - \frac{p\tilde{m}}{\tilde{r}} + \frac{(p\tilde{m})^2}{2\tilde{r}^2} + O\left(\frac{1}{\tilde{r}^3} \right) \right] \left[1 + \frac{6\tilde{r}}{p\tilde{m}} + \frac{12\tilde{r}^2}{(p\tilde{m})^2} \right] - \left[1 - \frac{6\tilde{r}}{p\tilde{m}} + \frac{12\tilde{r}^2}{(p\tilde{m})^2} \right]$

$$= 1 + \frac{6\tilde{r}}{p\tilde{m}} + \frac{12\tilde{r}^2}{(p\tilde{m})^2} - 6 - \frac{12\tilde{r}}{p\tilde{m}} + 6 + O\left(\frac{1}{\tilde{r}} \right) - 1 + \frac{6\tilde{r}}{p\tilde{m}} - \frac{12\tilde{r}^2}{(p\tilde{m})^2} = O\left(\frac{1}{\tilde{r}} \right) \to 0 \quad \text{(as should be)}$$

(3) $n = 3$: $\Psi_1 = 1 - \frac{12\tilde{r}}{p\tilde{m}} + \frac{60\tilde{r}^2}{(p\tilde{m})^2} - \frac{120\tilde{r}^3}{(p\tilde{m})^3} = 1 - \frac{12}{z} + \frac{60}{z^2} - \frac{120}{z^3} = \frac{z^3 - 12z^2 + 60z - 120}{z^3}$

$$(z^3-12z^2+60z-120)^2=(z^3-12z^2+60z-120)(z^3-12z^2+60z-120)$$

$$=z^3(z^3-12z^2+60z-120)-(12z^2-60z+120)(z^3-12z^2+60z-120)$$

$$=z^6-12z^5+60z^4-120z^3-12(z^2-5z+10)(z^3-12z^2+60z-120)$$

$$=z^6-12[z^5-5z^4+10z^3+(z^2-5z+10)(z^3-12z^2+60z-120)]$$

$$=z^6-12[z^2(z^3-12z^2+60z-120)+12z^4-60z^3+120z^2-5z^4+10z^3$$

$$+(z^2-4z)(z^3-12z^2+60z-120)-(z-10)(z^3-12z^2+60z-120)]$$

$$=z^6-12[(2z^2-4z)(z^3-12z^2+60z-120)+7z^4-50z^3+120z^2-(z-10)(z^3-12z^2+60z-120)]$$

$$=z^6-12[(2z^2-4z)(z^3-12z^2+60z-120)+7z^4-50z^3+120z^2-(z-10)(z^3-12z^2+60z-120)]$$

$$=z^6-12[2(z^2-2z)(z^3-12z^2+60z-120)+2z^2(3z^2-24z+60)+z^4-2z^3$$

$$-(z-10)(z^3-12z^2+60z-120)]$$

$$=z^6-12\{2[(z^2-2z)(z^3-12z^2+60z-120)+z^2(3z^2-24z+60)]+z(z^3-12z^2+60z-120)$$

$$+12z^3-60z^2+120z-2z^3-(z-10)(z^3-12z^2+60z-120)\}$$

$$=z^6-12\{2[(z^2-2z)(z^3-12z^2+60z-120)+z^2(3z^2-24z+60)]+10(z^3-6z^2+12z)$$

$$+10(z^3-12z^2+60z-120)\}$$

$$=z^6-24\{[(z^2-2z)(z^3-12z^2+60z-120)+z^2(3z^2-24z+60)]+5(z^3-12z^2+60z-120)$$

$$+5(6z^2-48z+120)+5(z^3-12z^2+60z-120)\}$$

$$=z^6-24\{[(z^2-2z)(z^3-12z^2+60z-120)+z^2(3z^2-24z+60)]+10(z^3-12z^2+60z-120)$$

$$+10(3z^2-24z+60)\}$$

$$\frac{z^6}{(z^3-12z^2+60z-120)^2}=1+24\left\{\left[\frac{z^2-2z}{z^3-12z^2+60z-120}+\frac{z^2(3z^2-24z+60)}{(z^3-12z^2+60z-120)^2}\right]\right.$$

$$\left.+10\left[\frac{1}{z^3-12z^2+60z-120}+\frac{3z^2-24z+60}{(z^3-12z^2+60z-120)^2}\right]\right\}$$

$$\int_0^z\frac{e^{-z}}{\Psi_1^2}dz=\int_0^z\frac{z^6e^{-z}}{(z^3-12z^2+60z-120)^2}dz=\int_0^z e^{-z}dz\ +24\left\{\int_0^z\left[\frac{(z^2-2z)e^{-z}}{z^3-12z^2+60z-120}+\frac{z^2(3z^2-24z+60)e^{-z}}{(z^3-12z^2+60z-120)^2}\right]dz\right.$$

$$\left.+10\int_0^z\left[\frac{e^{-z}}{z^3-12z^2+60z-120}+\frac{(3z^2-24z+60)e^{-z}}{(z^3-12z^2+60z-120)^2}\right]dz\right\}$$

$$= -e^{-z}\Big|_0^z + 24\left[-\frac{z^2 e^{-z}}{z^3-12z^2+60z-120}\Big|_0^z - 10\frac{e^{-z}}{z^3-12z^2+60z-120}\Big|_0^z\right]$$

$$= -(e^{-z}-1) - 24\left[\frac{z^2 e^{-z}}{z^3-12z^2+60z-120} + 10\left(\frac{e^{-z}}{z^3-12z^2+60z-120} - \frac{1}{-120}\right)\right]$$

$$= -\left[e^{-z}\left(1+\frac{24z^2+240}{z^3-12z^2+60z-120}\right)+1\right] = -\left[e^{-z}\left(1+\frac{24z^2+240}{z^3\Psi_1}\right)+1\right] = -\left\{\frac{e^{-p\tilde{m}/\tilde{r}}}{\Psi_1}\left[\Psi_1+\frac{24\tilde{r}}{p\tilde{m}}+\frac{240\tilde{r}^3}{(p\tilde{m})^3}\right]+1\right\}$$

Let $u = \frac{1}{z^3-12z^2+60z-120}$ and $dv = (z^2-2z)e^{-z}dz \Rightarrow du = \frac{-(3z^2-24z+60)}{(z^3-12z^2+60z-120)^2}$, $v = -z^2e^{-z}$

$$\int \frac{(z^2-2z)e^{-z}}{z^3-12z^2+60z-120}dz = -\frac{z^2e^{-z}}{z^3-12z^2+60z-120} - \int \frac{z^2(3z^2-24z+60)e^{-z}}{(z^3-12z^2+60z-120)^2}dz$$

$$\int\left[\frac{(z^2-2z)e^{-z}}{z^3-12z^2+60z-120} + \frac{z^2(3z^2-24z+60)e^{-z}}{(z^3-12z^2+60z-120)^2}\right]dz = -\frac{z^2e^{-z}}{z^3-12z^2+60z-120}$$

Let $u = \frac{1}{z^3-12z^2+60z-120}$ and $dv = e^{-z}dz \Rightarrow du = \frac{-(3z^2-24z+60)}{(z^3-12z^2+60z-120)^2}$, $v = -e^{-z}$

$$\int \frac{e^{-z}}{z^3-12z^2+60z-120}dz = -\frac{e^{-z}}{z^3-12z^2+60z-120} - \int \frac{(3z^2-24z+60)e^{-z}}{(z^3-12z^2+60z-120)^2}dz$$

$$\int\left[\frac{e^{-z}}{z^3-12z^2+60z-120} + \frac{(3z^2-24z+60)e^{-z}}{(z^3-12z^2+60z-120)^2}\right]dz = -\frac{e^{-z}}{z^3-12z^2+60z-120}$$

$$\Psi_2 = -\Psi_1\int_0^z \frac{e^{-z}}{\Psi_1^2}dz = \Psi_1\left\{\frac{e^{-p\tilde{m}/\tilde{r}}}{\Psi_1}\left[\Psi_1 + 24\frac{\tilde{r}}{p\tilde{m}} + 240\left(\frac{\tilde{r}}{p\tilde{m}}\right)^3\right]+1\right\} = e^{-p\tilde{m}/\tilde{r}}\left[\Psi_1 + \frac{24\tilde{r}}{p\tilde{m}} + \frac{240\tilde{r}^3}{(p\tilde{m})^3}\right]+\Psi_1$$

$$= e^{-p\tilde{m}/\tilde{r}}\left[1 - \frac{12\tilde{r}}{p\tilde{m}} + \frac{60\tilde{r}^2}{(p\tilde{m})^2} - \frac{120\tilde{r}^3}{(p\tilde{m})^3} + \frac{24\tilde{r}}{p\tilde{m}} + \frac{240\tilde{r}^3}{(p\tilde{m})^3}\right] + \Psi_1 = e^{-p\tilde{m}/\tilde{r}}\left[1 + \frac{12\tilde{r}}{p\tilde{m}} + \frac{60\tilde{r}^2}{(p\tilde{m})^2} + \frac{120\tilde{r}^3}{(p\tilde{m})^3}\right]+\Psi_1$$

$$= e^{-p\tilde{m}/\tilde{r}}\left\{1 + \sum_{k=1}^{3}\left[\prod_{i=1}^{k}[n(n+1)-i(i-1)]\right]\frac{\tilde{r}^k}{(k!)(p\tilde{m})^k}\right\} - (-1)^3\Psi_1$$

As $\tilde{r}\to\infty$: $\Psi_2 = e^{-p\tilde{m}/\tilde{r}}\left[1+\frac{6\tilde{r}}{p\tilde{m}}+\frac{12\tilde{r}^2}{(p\tilde{m})^2}\right]+\Psi_1$

$$\therefore\ \Psi_2 \to \left[1 - \frac{p\tilde{m}}{\tilde{r}} + \frac{(p\tilde{m})^2}{2\tilde{r}^2} - \frac{(p\tilde{m})^3}{6\tilde{r}^3} + O\left(\frac{1}{\tilde{r}^4}\right)\right]\left[1 + \frac{12\tilde{r}}{p\tilde{m}} + \frac{60\tilde{r}^2}{(p\tilde{m})^2} + \frac{120\tilde{r}^3}{(p\tilde{m})^3}\right] + \left[1 - \frac{12\tilde{r}}{p\tilde{m}} + \frac{60\tilde{r}^2}{(p\tilde{m})^2} - \frac{120\tilde{r}^3}{(p\tilde{m})^3}\right]$$

$$= 1 + \frac{12\tilde{r}}{p\tilde{m}} + \frac{60\tilde{r}^2}{(p\tilde{m})^2} + \frac{120\tilde{r}^3}{(p\tilde{m})^3} - 12 - \frac{60\tilde{r}}{p\tilde{m}} - \frac{120\tilde{r}^2}{(p\tilde{m})^2} + 30 + \frac{60\tilde{r}}{(p\tilde{m})} - 20 + O\left(\frac{1}{\tilde{r}}\right) + 1 - \frac{12\tilde{r}}{p\tilde{m}} + \frac{60\tilde{r}^2}{(p\tilde{m})^2} - \frac{120\tilde{r}^3}{(p\tilde{m})^3}$$

$$= O\left(\frac{1}{\tilde{r}}\right) \to 0 \quad \text{(as should be)}$$

(4) $n = 4$: $n\,(n + 1) = 4\,(4 + 1) = 20$

$$\Psi_1 = 1 + \sum_{k=1}^{4} \left\{ \prod_{i=1}^{k} [i(i-1) - n(n+1)] \right\} \frac{\tilde{r}^k}{(k!)(p\tilde{m})^k}$$

$$= 1 + \frac{(0-20)\tilde{r}}{p\tilde{m}} + \frac{(0-20)(2-20)\tilde{r}^2}{(2!)(p\tilde{m})^2} + \frac{(0-20)(2-20)(6-20)\tilde{r}^3}{(3!)(p\tilde{m})^3} + \frac{(0-20)(2-20)(6-20)(12-20)\tilde{r}^4}{(4!)(p\tilde{m})^4}$$

$$= 1 - \frac{20\tilde{r}}{p\tilde{m}} + \frac{180\tilde{r}^2}{(p\tilde{m})^2} - \frac{840\tilde{r}^3}{(p\tilde{m})^3} + \frac{1680\tilde{r}^4}{(p\tilde{m})^4} = 1 - \frac{20}{z} + \frac{180}{z^2} - \frac{840}{z^3} + \frac{1680}{z^4} = \frac{z^4 - 20z^3 + 180z^2 - 840z + 1680}{z^4}$$

$$(z^4 - 20z^3 + 180z^2 - 840z + 1680)^2 = (z^4 - 20z^3 + 180z^2 - 840z + 1680)(z^4 - 20z^3 + 180z^2 - 840z + 1680)$$

$$= z^4(z^4 - 20z^3 + 180z^2 - 840z + 1680) - (20z^3 - 180z^2 + 840z - 1680)(z^4 - 20z^3 + 180z^2 - 840z + 1680)$$

$$= z^8 - 20z^7 + 180z^6 - 840z^5 + 1680z^4 - 20(z^3 - 9z^2 + 42z - 84)(z^4 - 20z^3 + 180z^2 - 840z + 1680)$$

$$= z^8 - 20[z^7 - 9z^6 + 42z^5 - 84z^4 + (z^3 - 9z^2 + 42z - 84)(z^4 - 20z^3 + 180z^2 - 840z + 1680)]$$

$$= z^8 - 20[z^3(z^4 - 20z^3 + 180z^2 - 840z + 1680) + 20z^6 - 180z^5 + 840z^4 - 1680z^3 - 9z^6 + 42z^5 - 84z^4$$
$$+ (z^3 - 6z^2)(z^4 - 20z^3 + 180z^2 - 840z + 1680) - (3z^2 - 42z + 84)(z^4 - 20z^3 + 180z^2 - 840z + 1680)]$$

$$= z^8 - 20[(2z^3 - 6z^2)(z^4 - 20z^3 + 180z^2 - 840z + 1680) + 11z^6 - 138z^5 + 756z^4 - 1680z^3$$
$$- 3(z^2 - 14z + 28)(z^4 - 20z^3 + 180z^2 - 840z + 1680)]$$

$$= z^8 - 20[(2z^3 - 6z^2)(z^4 - 20z^3 + 180z^2 - 840z + 1680) + 2z^3(4z^3 - 60z^2 + 360z - 840) - 8z^6 + 120z^5$$
$$- 720z^4 + 1680z^3 + 11z^6 - 138z^5 + 756z^4 - 1680z^3$$
$$- 3(z^2 - 14z + 28)(z^4 - 20z^3 + 180z^2 - 840z + 1680)]$$

$$= z^8 - 20\{2[(z^3 - 3z^2)(z^4 - 20z^3 + 180z^2 - 840z + 1680) + z^3(4z^3 - 60z^2 + 360z - 840)] + 3z^6 - 18z^5$$
$$+ 36z^4 - 3(z^2 - 14z + 28)(z^4 - 20z^3 + 180z^2 - 840z + 1680)\}$$

$$= z^8 - 20\{2[(z^3 - 3z^2)(z^4 - 20z^3 + 180z^2 - 840z + 1680) + z^3(4z^3 - 60z^2 + 360z - 840)]$$
$$+ 3[z^2(z^4 - 20z^3 + 180z^2 - 840z + 1680) + 20z^5 - 180z^4 + 840z^3 - 1680z^2 - 6z^5 + 12z^4$$
$$- (z^2 - 14z + 28)(z^4 - 20z^3 + 180z^2 - 840z + 1680)]\}$$

$$= z^8 - 20\{2[(z^3 - 3z^2)(z^4 - 20z^3 + 180z^2 - 840z + 1680) + z^3(4z^3 - 60z^2 + 360z - 840)]$$
$$+ 3[14z^5 - 168z^4 + 840z^3 - 1680z^2 + 14(z - 2)(z^4 - 20z^3 + 180z^2 - 840z + 1680)]\}$$

$$= z^8 - 20\{2[(z^3 - 3z^2)(z^4 - 20z^3 + 180z^2 - 840z + 1680) + z^3(4z^3 - 60z^2 + 360z - 840)]$$

$$+42[z(z^4-20z^3+180z^2-840z+1680)+20z^4-180z^3+840z^2-1680z-12z^4+60z^3-120z^2$$

$$+(z-2)(z^4-20z^3+180z^2-840z+1680)]]\}$$

$$=z^8-40\{[(z^3-3z^2)(z^4-20z^3+180z^2-840z+1680)+z^3(4z^3-60z^2+360z-840)]$$

$$+21[(2z-2)(z^4-20z^3+180z^2-840z+1680)+8z^4-120z^3+720z^2-1680z]\}$$

$$\frac{z^8}{(z^4-20z^3+180z^2-840z+1680)^2}=1+40\left\{\left[\frac{z^3-3z^2}{z^4-20z^3+180z^2-840z+1680}+\frac{z^3(4z^3-60z^2+360z-840)}{(z^4-20z^3+180z^2-840z+1680)^2}\right]\right.$$

$$\left.+42\left[\frac{z-1}{z^4-20z^3+180z^2-840z+1680}+\frac{z(4z^3-60z^2+360z-840)}{(z^4-20z^3+180z^2-840z+1680)^2}\right]\right\}$$

$$\int_0^z\frac{e^{-z}}{\Psi_1{}^2}dz=\int_0^z\frac{z^8e^{-z}}{(z^4-20z^3+180z^2-840z+1680)^2}dz$$

$$=\int_0^z e^{-z}dz+40\left\{\int_0^z\left[\frac{(z^3-3z^2)e^{-z}}{z^4-20z^3+180z^2-840z+1680}+\frac{z^3(4z^3-60z^2+360z-840)e^{-z}}{(z^4-20z^3+180z^2-840z+1680)^2}\right]dz\right.$$

$$\left.+42\int_0^z\left[\frac{(z-1)e^{-z}}{z^4-20z^3+180z^2-840z+1680}+\frac{z(4z^3-60z^2+360z-840)e^{-z}}{(z^4-20z^3+180z^2-840z+1680)^2}\right]dz\right\}$$

$$=-e^{-z}\Big|_0^z+40\left[-\frac{z^3e^{-z}}{z^4-20z^3+180z^2-840z+1680}\Big|_0^z-42\frac{ze^{-z}}{z^4-20z^3+180z^2-840z+1680}\Big|_0^z\right]$$

$$=-(e^{-z}-1)-40\left[\frac{z^3e^{-z}}{z^4-20z^3+180z^2-840z+1680}+42\frac{ze^{-z}}{z^4-20z^3+180z^2-840z+1680}\right]$$

$$=-\left[e^{-z}\left(1+\frac{40z^3+1680z}{z^4-20z^3+180z^2-840z+1680}\right)-1\right]=-\left[e^{-z}\left(1+\frac{40z^2+1680}{z^3\Psi_1}\right)-1\right]$$

$$=-\left\{\frac{e^{-p\tilde{m}/\tilde{r}}}{\Psi_1}\left[\Psi_1+\frac{40\tilde{r}}{p\tilde{m}}+\frac{1680\tilde{r}^3}{(p\tilde{m})^3}\right]-1\right\}$$

Let $u=1/(z^4-20z^3+180z^2-840z+1680)$ and $dv=(z^3-3z^2)e^{-z}dz$

$$\Rightarrow\ du=-(4z^3-60z^2+360z-840)/(z^4-20z^3+180z^2-840z+1680)\ ,\ v=-z^3e^{-z}$$

$$\int\frac{(z^3-3z^2)e^{-z}}{z^4-20z^3+180z^2-840z+1680}dz=-\frac{z^3e^{-z}}{z^4-20z^3+180z^2-840z+1680}-\int\frac{z^3(4z^3-60z^2+360z-840)e^{-z}}{(z^4-20z^3+180z^2-840z+1680)^2}dz$$

$$\int\left[\frac{(z^3-3z^2)e^{-z}}{z^4-20z^3+180z^2-840z+1680}+\frac{z^3(4z^3-60z^2+360z-840)e^{-z}}{(z^4-20z^3+180z^2-840z+1680)^2}\right]dz=-\frac{z^3e^{-z}}{z^4-20z^3+180z^2-840z+1680}$$

Let $u=1/(z^4-20z^3+180z^2-840z+1680)$ and $dv=(z-1)e^{-z}dz$

$$\Rightarrow \; du=-(4z^3-60z^2+360z-840)/(z^4-20z^3+180z^2-840z+1680)\;,\; v=-ze^{-z}$$

$$\int\frac{(z-1)e^{-z}}{z^4-20z^3+180z^2-840z+1680}dz=-\frac{ze^{-z}}{z^4-20z^3+180z^2-840z+1680}-\int\frac{z(4z^3-60z^2+360z-840)e^{-z}}{(z^4-20z^3+180z^2-840z+1680)^2}dz$$

$$\int\left[\frac{(z-1)e^{-z}}{z^4-20z^3+180z^2-840z+1680}+\frac{z(4z^3-60z^2+360z-840)e^{-z}}{(z^4-20z^3+180z^2-840z+1680)^2}\right]dz=-\frac{ze^{-z}}{z^4-20z^3+180z^2-840z+1680}$$

$$\Psi_2=-\Psi_1\int_0^z\frac{e^{-z}}{\Psi_1{}^2}dz=\Psi_1\left\{\frac{e^{-p\tilde{m}/\tilde{r}}}{\Psi_1}\left[\Psi_1+\frac{40\tilde{r}}{p\tilde{m}}+\frac{1680\tilde{r}^3}{(p\tilde{m})^3}\right]-1\right\}=e^{-p\tilde{m}/\tilde{r}}\left[\Psi_1+\frac{40\tilde{r}}{p\tilde{m}}+\frac{1680\tilde{r}^3}{(p\tilde{m})^3}\right]-\Psi_1$$

$$=e^{-p\tilde{m}/\tilde{r}}\left[1-\frac{20\tilde{r}}{p\tilde{m}}+\frac{180\tilde{r}^2}{(p\tilde{m})^2}-\frac{840\tilde{r}^3}{(p\tilde{m})^3}+\frac{1680\tilde{r}^4}{(p\tilde{m})^4}+\frac{40\tilde{r}}{p\tilde{m}}+\frac{1680\tilde{r}^3}{(p\tilde{m})^3}\right]-\Psi_1$$

$$=e^{-p\tilde{m}/\tilde{r}}\left[1+\frac{20\tilde{r}}{p\tilde{m}}+\frac{180\tilde{r}^2}{(p\tilde{m})^2}+\frac{840\tilde{r}^3}{(p\tilde{m})^3}+\frac{1680\tilde{r}^4}{(p\tilde{m})^4}\right]-\Psi_1$$

$$=e^{-p\tilde{m}/\tilde{r}}\left\{1+\sum_{k=1}^{4}\left[\prod_{i=1}^{k}[n(n+1)-i(i-1)]\right]\frac{\tilde{r}^k}{(k!)(p\tilde{m})^k}\right\}-(-1)^4\Psi_1$$

As $\tilde{r}\to\infty$: $\Psi_2=e^{-p\tilde{m}/\tilde{r}}\left[1+\frac{20\tilde{r}}{p\tilde{m}}+\frac{180\tilde{r}^2}{(p\tilde{m})^2}+\frac{840\tilde{r}^3}{(p\tilde{m})^3}+\frac{1680\tilde{r}^4}{(p\tilde{m})^4}\right]-\Psi_1$

$$=\left[1-\frac{p\tilde{m}}{\tilde{r}}+\frac{(p\tilde{m})^2}{2\tilde{r}^2}-\frac{(p\tilde{m})^3}{6\tilde{r}^3}+\frac{(p\tilde{m})^4}{24\tilde{r}^4}+O\left(\frac{1}{\tilde{r}^4}\right)\right]\left[1+\frac{20\tilde{r}}{p\tilde{m}}+\frac{180\tilde{r}^2}{(p\tilde{m})^2}+\frac{840\tilde{r}^3}{(p\tilde{m})^3}+\frac{1680\tilde{r}^4}{(p\tilde{m})^4}\right]$$

$$-\left[1-\frac{20\tilde{r}}{p\tilde{m}}+\frac{180\tilde{r}^2}{(p\tilde{m})^2}-\frac{840\tilde{r}^3}{(p\tilde{m})^3}+\frac{1680\tilde{r}^4}{(p\tilde{m})^4}\right]$$

$$=1+\frac{20\tilde{r}}{p\tilde{m}}+\frac{180\tilde{r}^2}{(p\tilde{m})^2}+\frac{840\tilde{r}^3}{(p\tilde{m})^3}+\frac{1680\tilde{r}^4}{(p\tilde{m})^4}-20-\frac{180\tilde{r}}{p\tilde{m}}-\frac{840\tilde{r}^2}{(p\tilde{m})^2}-\frac{1680\tilde{r}^3}{(p\tilde{m})^3}+90+\frac{420\tilde{r}}{p\tilde{m}}+\frac{840\tilde{r}^2}{(p\tilde{m})^2}$$

$$-140-\frac{280\tilde{r}}{p\tilde{m}}+70+O\left(\frac{1}{\tilde{r}}\right)-1+\frac{20\tilde{r}}{p\tilde{m}}-\frac{180\tilde{r}^2}{(p\tilde{m})^2}+\frac{840\tilde{r}^3}{(p\tilde{m})^3}-\frac{1680\tilde{r}^4}{(p\tilde{m})^4}$$

$=O\left(\frac{1}{\tilde{r}}\right)\to 0$ (as should be)

(5) From (1) – (4), we conclude that:

$$\Psi_2(\tilde{r},p)=e^{-p\tilde{m}/\tilde{r}}\left\{1+\sum_{k=1}^{n}\left[\prod_{i=1}^{k}[n(n+1)-i(i-1)]\right]\frac{\tilde{r}^k}{(k!)(p\tilde{m})^k}\right\}-(-1)^n\Psi_1(\tilde{r},p)$$

At $\tilde{r}=0$: $\Psi_2(0,p)=-(-1)^n=(-1)^{n-1}$ and as $\tilde{r}\to\infty$: $\Psi_2(\infty,p)\to 0$

As $\tilde{r}\to\infty$: $\Psi_2(\infty,p)=0$

(C) Summary

General solution : $\hat{Y} = c_1 \Psi_1 + c_2 \Psi_2$

$$\Psi_1(\tilde{r},p) = 1 + \sum_{k=1}^{n} \left\{ \prod_{i=1}^{k} [i(i-1) - n(n+1)] \right\} \frac{\tilde{r}^k}{(k!)(p\tilde{m})^k}$$

$$\Psi_2(\tilde{r},p) = e^{-p\tilde{m}/\tilde{r}} \left\{ 1 + \sum_{k=1}^{n} \left[\prod_{i=1}^{k} [n(n+1) - i(i-1)] \right] \frac{\tilde{r}^k}{(k!)(p\tilde{m})^k} \right\} - (-1)^n \Psi_1(\tilde{r},p)$$

At $\tilde{r} = 0$: $\Psi_1(0,p) = 1$, $\hat{\Psi}_1(0,p) = 0$, $\Psi_2(0,p) = -(-1)^n = (-1)^{n-1}$, $\hat{\Psi}_2(0,p) = 0$

As $\tilde{r} \to \infty$: $|\Psi_1| \to \infty$, $|\hat{\Psi}_1| \to \infty$, $\Psi_2 \to 0$, $\hat{\Psi}_2 \to 0$

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