WEAK AMENABILITY IS STABLE UNDER GRAPH PRODUCTS

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ABSTRACT

Weak amenability of discrete groups was introduced by Haagerup and co-authors in the 1980's. It is an approximation property known to be stable under direct products and free products. In this thesis we show that graph products of weakly amenable discrete groups are weakly amenable (with Cowling-Haagerup constant 1). Along the way we construct a wall space associated to the word length structure of a graph product and also give a method for extending completely bounded functions on discrete groups to a completely bounded function on their graph product.

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CHAPTER 1

INTRODUCTION

Amenability has been widely studied since the early twentieth century, finding applications across many fields of mathematics. The idea was first studied by von Neumann and the term amenable was later coined by Day. An amenable group is one that has an invariant mean; in other words there is a way to uniformly average a bounded function over the entire (infinite) group. This can also be seen as the ability to approximate the group by finite subsets, as is shown by the existence of Følner sequences: if a group is amenable then one can find a sequence of finite subsets such that the limit of averages over the sequence is a uniform average over the whole group. Interest in other finite approximation properties like this have lead to a variety of generalizations of amenability, including a-T-menability (introduced by Gromov [12], see [6] for a useful reference) and weak amenability (studied by Haagerup and various co-authors [10], [8], [14]).

Weak amenability and a-T-menability are closely linked, as can be seen in the following characterization. An approximate identity of a group G is a sequence of functions $\{f_n\}$ from G to $\mathbb{C}$ such that $f_n \rightarrow 1$ pointwise as n goes to infinity. A group is amenable if there is an approximate identity consisting of finitely supported positive definite functions, a-T-menable if there is an approximate identity consisting of positive definite functions which vanish at infinity, and is weakly amenable if there is an approximate identity consisting of finitely supported functions which are uniformly completely bounded with cb-norm approaching one (we always assume with Cowling-Haagerup constant one). Note that a positive definite function f is completely bounded with cb-norm $\|f\|_{\text{cb}} = f(e)$, and if a group is amenable then it is both a-T-menable and weakly amenable.

Furthermore, there are many examples of groups which are non-amenable but both weakly amenable and a-T-menable, such as free products of amenable groups with finite amalgam, Coxeter groups, and groups acting on $\text{CAT}(0)$ -cube complexes. These similarities led to the conjecture that

the two properties were equivalent, but it was shown that a-T-menability is stable under wreath products (De Cornulier, Stalder, and Valette [7]) but weak amenability is not (Ozawa and Popa [17]). In this paper we study weak amenability.

It was first shown by deCanierre and Haagerup [10] that the free group on 2 generators, the classic non-amenable group, is weakly amenable. This came from their study on the weak amenability of rank one Lie groups. Later, Bozejko and Picardello [2] showed free products (with finite amalgam) of amenable groups are weakly amenable, following the treatment of Haagerup in his seminal paper [14] on the C^* -algebra of the free group. In particular, they presented a cb-norm bound on the characteristic function of the n -sphere for a tree. The stability of weak amenability under free products was finally proved by Ricard and Xu [20] in 2006. In 2008, Mizuta extended Bozejko and Picardello's technique to groups acting properly on finite dimensional $CAT(0)$ -cube complexes, reproving a result of Guentner and Higson [13].

A generalization of free products is the graph product: given a simplicial graph Γ and a set of discrete groups $\{G_v\}$ indexed by the vertices of Γ , there is the associated graph product $G(\Gamma)$, which is the free product of the G_v 's with the relation that for each edge $[vw]$ of Γ , elements of G_v and G_w commute with each other. The reduced word length structure of a graph product can be modelled with a $CAT(0)$ -cube complex, as we demonstrate in this paper. In 2013, Antolin and Dreesen [1] showed that a-T-menability is stable under graph products. This, combined with Mizuta's and Ricard and Xu's results, strongly indicated that weak amenability should be stable under graph products as well.

In this paper we prove the following:

Theorem. *The graph product of weakly amenable groups (with Cowling-Haagerup constant 1) is weakly amenable.*

This paper is organized as follows. Chapter 2 covers the required background material. Chapter 3 describes a wall space on which a given graph product acts. From this, using the correspondence between $CAT(0)$ cube complexes and wall spaces, we obtain a $CAT(0)$ cube complex that describes

the word length structure of the graph product. In Chapter 4, we treat the extension of positive definite functions on groups to a positive definite function on their free product. For this we use an infinite tensor product technique, similar to the one used by Chen, Dadarlat, Guenter, and Yu [5] to show that free products with finite amalgam preserve uniform embeddability. We then modify this construction to extend completely bounded functions to a free product. This motivates the proof of the main theorem in Chapter 5, where we generalize the extension to graph products and construct a completely bounded approximate identity from such extensions. These results contribute to my doctoral thesis at the University of Hawai‘i at Mānoa.

Finally, I would like to thank my advisor Erik Guenter for all of his help and support in my pursuit of mathematics.

CHAPTER 2

BACKGROUND

In this section we introduce the concepts necessary to understand this thesis.

2.1 Groups

A group is a set G along with a associative binary operation $\cdot : G \times G \rightarrow G$, called multiplication, such that there is an unique identity element, denoted e , and for every $g \in G$, there is an element denoted g^{-1} such that $g^{-1} \cdot g = g \cdot g^{-1} = e$, called the inverse of G .

If a subset H of a group G is a group with the inherited multiplication from G , then we say H is a subgroup. H is normal if $ghg^{-1} \in H$ for all $g \in G$ and $h \in H$ and we denote by $\langle\langle H \rangle\rangle$ the smallest normal subgroup which contains H . Given two elements $g, h \in G$, we say that g and h belong to the same coset of H if $h^{-1}g$ is in H . Each coset is an equivalence class of group elements and is written tH , for some $t \in G$. When H is normal, these equivalence classes form the quotient group G/H .

Let S be a set and define $S^{-1} := \{s^{-1} | s \in S\}$ to be the set of formal inverses of S , which is isomorphic to S . We call a sequence of elements in S , such as $(s_1, s_2, \dots, s_n)$ a word in S , and a subword is a contiguous subsequence of a word. A word in $S \cup S^{-1}$ is called reduced if it contains no subwords of the form (s, s^{-1}) or (s^{-1}, s) for some $s \in S$, and every word of $S \cup S^{-1}$ can be reduced to a unique reduced word by deleting these subwords.

The set of all reduced words of $S \cup S^{-1}$ form a group $\mathbb{F}(S)$, called the free group on S . The identity element is the empty word, multiplication is concatenation followed by reduction, and the inverse of a reduced word $(s_1, \dots, s_n)$ is $(s_n^{-1}, \dots, s_1^{-1})$. The elements of S are called the generators of $\mathbb{F}(S)$.

Given a set R of elements of $\mathbb{F}(S)$, we can define a new group $G := \mathbb{F}(S)/\langle\langle R \rangle\rangle$. We write $G = \langle S|R \rangle$, where the right hand side is called a presentation of G . If S and R are finite sets, then G is called finitely presented. All groups in this paper are assumed to be finitely presented.

Given two groups, say $G_i = \langle S_i|R_i \rangle$, for $i = 1, 2$, the free product of the G_i , denoted $G_1 * G_2$, is the group presented by $\langle S_1 \cup S_2|R_1 \cup R_2 \rangle$. Elements of the free product are alternating words in G_1 and G_2 .

The direct product of the G_i , denoted $G_1 \oplus G_2$, is defined by the presentation $\langle S_1 \cup S_2|R_1 \cup R_2 \cup [G_1, G_2] \rangle$, where $[G_1, G_2] := \{g_1 g_2 g_1^{-1} g_2^{-1} \mid \text{for all } g_1 \in G_1, g_2 \in G_2\}$ is called the commutator of the two groups. Elements of the direct product are ordered pairs of elements, one from G_1 , one from G_2 .

Given a finitely presented group $G = \langle S|R \rangle$, we may construct a discrete metric space whose group of symmetries is G . This metric space is called the Cayley graph of G , written $\text{Cay}(G)$, and is defined as follows. The vertices of $\text{Cay}(G)$ are the group elements of G and two vertices x and y are connected by an edge if as group elements x and y differ by a generator of G . Said in another way, x is adjacent to y if $x^{-1}y \in S \cup S^{-1}$.

The Cayley graph comes equipped with a natural graph metric, with each edge being of unit length. For $x, y \in \text{Cay}(G)$, this metric, called the word metric, corresponds to length of the shortest word in $S \cup S^{-1}$ representing $x^{-1}y$. The group G acts on $\text{Cay}(G)$ by left multiplication: an element $g \in G$, sends $x \in \text{Cay}(G)$ to $gx \in \text{Cay}(G)$. This action is an isometry, as it preserves the word distance between two points.

2.2 Hilbert spaces and tensor products

Given a vector space V , an inner product on V is a map $\langle \cdot, \cdot \rangle : V \times V \rightarrow \mathbb{C}$ such that the following hold for all $x, y, z \in V$ and $\alpha \in \mathbb{C}$:

- $\langle x, y \rangle = \overline{\langle y, x \rangle}$,

- $\langle \alpha x + z, y \rangle = \alpha \langle x, y \rangle + \langle z, y \rangle$,
- $\langle x, x \rangle \geq 0$, and $\langle x, x \rangle = 0$ if and only if $x = 0$.

An inner product induces a norm on V by $\|x\| := \sqrt{\langle x, x \rangle}$ for all $x \in V$. A vector space with an inner product which also is complete with respect to this norm is called a Hilbert space.

The direct product of two Hilbert spaces, $\mathcal{H}_1 \oplus \mathcal{H}_2$ is simply the collection of ordered pairs (ξ_1, ξ_2) , with $\xi_i \in \mathcal{H}_i$. This forms a Hilbert space whose inner product is just the sum of the two inner products:

$$\langle (\xi_1, \xi_2), (\zeta_1, \zeta_2) \rangle_{\mathcal{H}_1 \oplus \mathcal{H}_2} := \langle \xi_1, \zeta_1 \rangle_{\mathcal{H}_1} + \langle \xi_2, \zeta_2 \rangle_{\mathcal{H}_2}.$$

Recall that a simple tensor of two vector spaces V_1 and V_2 is an ordered pair denoted by $x_1 \otimes x_2$ for $x_1 \in V_1$ and $x_2 \in V_2$. The tensor product $V_1 \otimes V_2$ of V_1 and V_2 is the collection of linear combinations of simple tensors, subject to the following conditions:

- $x_1 \otimes x_2 + x_1 \otimes y_2 = x_1 \otimes (x_2 + y_2)$
- $x_1 \otimes x_2 + y_1 \otimes x_2 = (x_1 + y_1) \otimes x_2$
- $c(x_1 \otimes x_2) = cx_1 \otimes x_2 = x_1 \otimes cx_2$

If V_1 and V_2 have inner products, then their tensor product has a natural inner product defined by

$$\langle x_1 \otimes x_2, y_1 \otimes y_2 \rangle_{V_1 \otimes V_2} := \langle x_1, y_1 \rangle_{V_1} \langle x_2, y_2 \rangle_{V_2}$$

for simple tensors $x_1 \otimes x_2$ and $y_1 \otimes y_2$ and which extends to all tensors by linearity. The tensor product of two Hilbert spaces, $\mathcal{H}_1 \otimes \mathcal{H}_2$, is the tensor product of the underlying vector spaces of $\mathcal{H}_1$ and $\mathcal{H}_2$, completed with the norm given by the inner product above.

For most of Chapters 4 and 5 we use a tensor product of Hilbert spaces over an infinite index set. Let us recall how to construct such an object. Suppose T in an infinite set and for every $t \in T$, we have a Hilbert space $\mathcal{H}_t$ and an unit vector ω_t of $\mathcal{H}_t$ called the *vacuum vector* of $\mathcal{H}_t$. Given a

finite subset $S \subset T$, we can construct the standard tensor product over S , denoted $\bigotimes_{t \in S} \mathcal{H}_t$. We write a simple tensor ξ as $\xi_{t_1} \otimes \xi_{t_2} \otimes \cdots \otimes \xi_{t_n}$, where $t_i \in S$ and ξ_{t_i} is the t_i component of ξ . For another finite subset R of T such that $R \subset S$ there is an inclusion of $\bigotimes_{t \in R} \mathcal{H}_t$ into $\bigotimes_{t \in S} \mathcal{H}_t$ with $\bigotimes_{t \in R} \xi_t \mapsto \bigotimes_{t \in R} \xi_t \otimes \bigotimes_{t \in S \setminus R} \omega_t$. The inner product of any two finite tensors $\langle \bigotimes_{t \in S} \xi_t, \bigotimes_{t \in S'} \xi_s \rangle$ can then be evaluated in $\bigotimes_{t \in S \cup S'} \mathcal{H}_t$.

Define $\widehat{\mathcal{H}} := \bigotimes_{t \in T} \mathcal{H}_t$ as the completed direct limit of tensor products over finite subsets of T . The span of finite tensors is dense in $\widehat{\mathcal{H}}$ and in this paper we shall only be concerned with simple finite tensors. It is useful to think of such tensors as infinite tensors with only finitely many components not equal to ω_t , hence the term vacuum vectors.

Given a group G , a natural Hilbert space is $\ell^2(G)$, the square-summable functions from G to $\mathbb{C}$. An element of $\ell^2(G)$ may be written as $\sum_{g \in G} a_g \delta_g$, where $a_g \in \mathbb{C}$, $\sum_{g \in G} |a_g|^2 < \infty$, and $\delta_g : G \rightarrow \mathbb{C}$ is the Dirac function at g , which is 1 at g and 0 everywhere else. The inner product of two elements $\sum_{g \in G} a_g \delta_g$ and $\sum_{g \in G} b_g \delta_g$ of $\ell^2(G)$, is defined to be

$$\langle \sum_{g \in G} a_g \delta_g, \sum_{g \in G} b_g \delta_g \rangle := \sum_{g \in G} a_g \bar{b}_g.$$

2.3 C^* -algebras

For a Hilbert space $\mathcal{H}$, an operator is a map $T : \mathcal{H} \rightarrow \mathcal{H}$, with operator norm

$\|T\| := \sup_{\xi \in \mathcal{H}, \|\xi\|=1} \|T\xi\|$. An operator is bounded if $\|T\| < \infty$ and $\mathbb{B}(\mathcal{H})$, the set of all bounded operators on $\mathcal{H}$, is a normed linear space. The adjoint T^* of an operator T is the unique operator such that $\langle T\xi, \zeta \rangle = \langle \xi, T^*\zeta \rangle$ for all $\xi, \zeta \in \mathcal{H}$. An operator T is called unitary if $TT^* = T^*T = I$, where I is the identity operator on $\mathcal{H}$, and T is positive if $\langle T\xi, \xi \rangle \geq 0$ for all $\xi \in \mathcal{H}$.

A (concrete) C^* -algebra is a self-adjoint norm-closed subalgebra of the bounded operators on some Hilbert space. If $\mathbb{1}$, the identity operator, is an element of a C^* -algebra, we call it unital. In this paper all C^* -algebras are unital. Given a group G , a representation of G is a homomorphism

π from G into the bounded operators on a Hilbert space. It is called unitary if the image of G under π is contained in the unitary operators. The image of a unitary representation generates a self-adjoint subalgebra of $\mathbb{B}(\mathcal{H})$ and its closure generates a C^* -algebra.

One of the most natural representations of G is as follows: for each element $g \in G$, we may associate an operator $\lambda_g : \ell^2(G) \rightarrow \ell^2(G)$ which sends δ_h to δ_{gh} . The adjoint of λ_g is $\lambda_{g^{-1}}$ and hence λ_g is unitary. The homomorphism λ which sends g to λ_g defines a unitary representation of G on $\mathbb{B}(\ell^2(G))$ called the left regular representation. The C^* -algebra generated by λ is called the reduced C^* -algebra of G , denoted $C_r^*(G)$. Elements of $C_r^*(G)$ may be written in the form $\sum_{g \in G} \alpha_g \lambda_g$, where $\alpha_g \in \mathbb{C}$. This is the primary C^* -algebra we shall use in this paper.

2.4 Operator norms and weak amenability

If C is a C^* -algebra, then $M_n(C)$, the $n \times n$ matrices with entries in C , is also a C^* -algebra which gives it a unique norm. Any (bounded, linear) operator f on C extends to an operator f^n on $M_n(C)$, acting entrywise as f . An operator f is *completely bounded* (*cb*) if all of the f^n 's are bounded uniformly. As the norms are increasing with n , we can define the *cb-norm* of f : $\|f\|_{cb} := \lim_n \|f^n\|$. The operator f is *completely positive* if f^n is positive (as an operator on $M_n(C)$) for all n . A completely positive operator f is completely bounded with $\|f\|_{cb} = f(\mathbf{1})$. For more details, see Paulsen's book [18].

Given a group G and a finitely supported map $f : G \rightarrow \mathbb{C}$, we can naturally extend f to a multiplication operator on $C_r^*(G)$ by $\sum \alpha_g \lambda_g \mapsto \sum \alpha_g f(g) \lambda_g$. If f is completely bounded as a multiplication operator, then we say that f , as a function, is completely bounded.

Recall that a function f is *positive definite* if and only if for all finite sets of complex numbers $\{c_i\}$ and group elements $\{g_i\}$, $\sum_{i,j} c_i \bar{c}_j f(g_j^{-1} g_i) \geq 0$. This implies that, as a multiplication operator on $C_r^*(G)$, f is completely positive. Hence, a positive definite function f is completely bounded with $\|f\|_{cb} = f(\mathbf{e})$.

For most of the paper we shall use an alternate characterization of completely bounded functions, (see Theorem 8.3 and Remark 8.4 in [19]):

Proposition 2.4.1. *If a function $f : G \rightarrow \mathbb{C}$ is completely bounded then there is a Hilbert space $\mathcal{H}$ and maps $\alpha, \beta : G \rightarrow \mathcal{H}$ with $\|\alpha\| \|\beta\| = \|f\|_{cb}$ (where $\|\alpha\| = \sup_{g \in G} \|\alpha(g)\|$), such that $f(h^{-1}g) = \langle \alpha(g), \beta(h) \rangle$ for all $g, h \in G$. Conversely, given a function of the form $f(h^{-1}g) = \langle \alpha(g), \beta(h) \rangle$, then f is completely bounded with $\|f\|_{cb} \leq \|\alpha\| \|\beta\|$.*

If f is positive definite, then we may take $\alpha = \beta$ and $\|\alpha\|^2 = f(e)$. Conversely, given a function of the form $f(h^{-1}g) = \langle \alpha(g), \alpha(h) \rangle$, then f is positive definite. Note that this paragraph is just the GNS-construction.

Recall that a *kernel* on a group is a map $\phi : G \times G \rightarrow \mathbb{C}$ and is *G-invariant* if $\phi(kg, kh) = \phi(g, h)$ for any $g, h, k \in G$. For any function $f : G \rightarrow \mathbb{C}$, there is an associated G-invariant kernel $\phi : G \times G \rightarrow \mathbb{C}$, where $\phi(g, h) := f(h^{-1}g)$. The above proposition characterizes completely boundedness and positive definiteness in terms of the structure of the kernel of a group function. In this paper we often construct a kernel in the above form, then prove that it is G-invariant to show that the function $f(h^{-1}g) := \phi(g, h)$ is well defined.

A group G is *amenable* if there exists a net of finitely supported positive definite functions $\{f_n\}$ such that $f_n \rightarrow \mathbf{1}$ pointwise. A group G is *weakly amenable* if there exists a net of finitely supported completely bounded functions $\{f_n\}$ such that $f_n \rightarrow \mathbf{1}$ pointwise and $\limsup \|f_n\|_{cb} \leq c < \infty$. The *Cowling-Haagerup constant* $\Lambda_{cb}(G)$ is the smallest such c , and in this paper we shall always assume $c = 1$. Note, the main construction relies on the fact that a completely bounded function of norm close to one is not very different from a positive definite function, and may not work with Cowling-Haagerup constant bounded away from one.

2.5 Graphs and graph products

Graph products were first studied by E. Green in her thesis [11] as a generalization of several familiar classes of groups including graph groups, free products, direct products, and right-angled Coxeter groups. In this section we establish basic notation for graph products and recall several needed results.

A *graph* in this paper will be a *finite simplicial graph*. In other words a graph Γ is a finite set of vertices and undirected edges between those vertices, without loops or multiple edges. A typical vertex of Γ will be labeled as $v \in \Gamma$, and the (unique) edge between vertices v and w of Γ will be denoted $[vw] \in \Gamma$. We will show in Corollary 5.0.10 that the finiteness assumption is not needed, but we will assume it to simplify our proofs.

Two vertices are *adjacent* if they share an edge. Since Γ is finite, there is a maximal number κ of mutually adjacent vertices. The *link* of a vertex v is the set $\text{Lk}(v)$ of vertices which are adjacent to v in Γ . The *star* of a vertex v is the set $\text{St}(v)$ of vertices adjacent to v and v itself.

Given a graph Γ and set of groups $\{G_v\}$ labeling the vertices $v \in \Gamma$, called *vertex groups*, we may define their graph product. Informally, their graph product is the free product of the vertex groups, where two vertex groups commute if their vertices share an edge in Γ . Formally, we have the following: the *graph product of $\{G_v\}_{v \in \Gamma}$ over Γ* is

$$G(\Gamma) := \ast_{v \in \Gamma} G_v / \langle\langle [G_v, G_w], [vw] \in \Gamma \rangle\rangle.$$

In particular, a graph product over a totally disconnected graph (one with no edges) is a free product of the vertex groups. A graph product over a complete graph (one in which every pair of vertices are adjacent) is a direct product of the vertex groups. Graph products thus act as combinatorial interpolations between the direct product and the free product of a set of groups. Further, a graph product is a right-angled Artin group if all of the vertex groups are infinite cyclic, and is a right-angled Coxeter group when the vertex groups are isomorphic to $\mathbb{Z}_2$.

In a graph product $G(\Gamma)$, we denote by $G(\text{St}(v))$ the subgroup generated by $\bigcup_{w \in \text{St}(v)} G_w$. Likewise $G(\text{Lk}(v))$ is the subgroup generated by $\bigcup_{w \in \text{Lk}(v)} G_w$. Note that elements of $G(\text{Lk}(v))$ commute with elements of G_v . We denote by $G_v^e := G_v \setminus \{e\}$, the set of non-identity elements of G_v .

Given a graph product $G(\Gamma)$, a *formal word* in the vertex groups is a sequence of the form $(g_1, g_2, \dots, g_n)$, where $g_i \in G(i)$ (we write $G(i)$ for G_{v_i} to avoid cluttered notation). We call each g_i a *syllable* and define the *syllable length* of a formal word as the number of syllables in the word.

Each formal word represents some element of $G(\Gamma)$ via multiplication. For instance, the formal word $(g_1, g_2, \dots, g_n)$ represents $g_1 g_2 \dots g_n$. There are three operations we can perform on this word that will not change the group element it represents. First, if v_i and v_{i+1} are adjacent vertices, then g_i and g_{i+1} commute and we may perform a *syllable shuffle*: sending $(g_1, \dots, g_i, g_{i+1}, \dots, g_n)$ to $(g_1, \dots, g_{i+1}, g_i, \dots, g_n)$. If two adjacent syllables of a word are in the same vertex group, then as group elements the two syllables will multiply, and we may amalgamate the two syllables into a single syllable, reducing the length of the word by 1. Last, if one of the syllables is the identity, we may remove it, also reducing the length of the word by 1. A *reduced word* is a word that cannot be made shorter through syllable shufflings and cancellation/amalgamation.

If $g \in G(\Gamma)$, we may write it as a product: $g = h_1 h_2 \dots h_m$, $h_i \in G(i)$. We can then reduce the formal word $(h_1, \dots, h_m)$ to some reduced word $(g_1, \dots, g_n)$ such that $g = g_1 g_2 \dots g_n$. Green showed in her thesis that this process generates an unique family of reduced words representing g , and that any two reduced words representing g differ from each other by a finite number of syllable shuffles. This implies that the set of syllables representing g is unique. We write $g \equiv g_1 g_2 \dots g_n$ when $(g_1, g_2, \dots, g_n)$ is a reduced word representing g and we define the *reduced word length* $|g|_r$ as the syllable length of any (and all) of the reduced words representing g .

Remark 2.5.1. *An important observation is that if $(g_1, \dots, g_n)$ is a reduced word with two syllables g_i and g_j from the same vertex group G_v , then there must be a g_q with $i < q < j$ such*

that $g_q \notin G(\text{St}(v))$ (Equivalently, v_q is not adjacent to v). Otherwise, as g_i and g_j commute with anything in $G(\text{St}(v))$ by definition, we could shuffle g_i next to g_j and reduce the word further.

Given two elements of $G(\Gamma)$ we will often need to know the explicit form of a reduced word representing their product.

Lemma 2.5.2. *Suppose g and h are elements of $G(\Gamma)$, where $m = |h|_r$ and $n = |g|_r$. Then there exists natural numbers $q \leq \min(m, n)$, $p \leq \kappa$ and reduced words $g_1 \dots g_n$ and $h_1 \dots h_m$ representing g and h respectively, such that $g_i = h_i$ for $1 \leq i \leq q$, g_i and h_i are in the same vertex groups for $q < i \leq q + p$, and*

$$h^{-1}g \equiv h_m^{-1} \dots h_{q+p+1}^{-1} (h_{q+1}^{-1} g_{q+1}) \dots (h_{q+p}^{-1} g_{q+p}) g_{q+p+1} \dots g_n$$

for some $0 \leq q \leq \min(m, n)$ and $p \leq \kappa$. Furthermore, for each $1 \leq j \leq p$, h_{q+j} and g_{q+j} are in $G(\text{St}(q + i))$ for all $1 \leq i \leq p$.

Proof. Suppose g and h are elements of $G(\Gamma)$, $m = |h|_r$ and $n = |g|_r$. Let us find a reduced word representative of $h^{-1}g$ by concatenating the reduced words of h^{-1} and g and then reducing by syllable shuffling, cancelation, and amalgamation.

First, there is a maximal collection of syllables which cancel. Suppose g_i cancels when reducing $h^{-1}g$ and there is a g_j with $j < i$ which does not cancel. If $G(i) = G(j)$, then from Remark 2.5.1, there must be a $g_p \notin G(\text{St}(v))$ between g_i and g_j . But g_j blocks g_p from cancelling, which in turn blocks g_i from cancelling, a contradiction. So necessarily $G(i) \neq G(j)$. Now if g_j was not in $G(\text{Lk}(i))$ then g_j would block g_i from shuffling adjacent to its inverse and cancelling, a contradiction to our hypothesis. Hence g_j is in $G(\text{Lk}(i))$ and g_i commutes with g_j . Hence each syllable which cancels must be able to shuffle in its word to a point where every syllable to its left will also cancel. From this we may assume, after rearranging, that there is a natural number $0 \leq q \leq \min(m, n)$ such that $g_i = h_i$ for all $i \leq q$. In other words we assume g and h begin with the same reduced word $k \equiv g_1 \dots g_q$, which may be the empty word.

Next, some syllables may multiply non-trivially. Say $g_i \neq h_j$, but they are from the same vertex group G_v and we are able to shuffle them next to each other when reducing $h^{-1}g$. Then g_i and h_j must be able to shuffle to the beginning of the reduced words $k^{-1}g \equiv g_{q+1} \dots g_n$ and $k^{-1}h \equiv h_{q+1} \dots h_m$ respectively. If there is another such pair $g_r \neq h_s \in G_w$ which can be amalgamated, then either g_i and h_s or g_r and h_j must shuffle, implying that G_v and G_w commute. Hence all such syllables must mutually commute and there can be at most κ pairs of syllables which multiply non-trivially. After shuffling, we may write $h^{-1}g$ as the product

$$h^{-1}g \equiv h_m^{-1} \dots h_{q+p+1}^{-1} (h_{q+1}^{-1} g_{q+1}) \dots (h_{q+p}^{-1} g_{q+p}) g_{q+p+1} \dots g_n$$

for some $0 \leq q \leq \min(m, n)$ and $p \leq \kappa$. To summarize, there are reduced words $g_1 \dots g_n$ and $h_1 \dots h_m$ representing g and h respectively such that $g_i = h_i$ for all $i \leq q$ and the syllables h_{q+j} and g_{q+j} are in the same vertex group for $1 \leq j \leq p$. Further, h_{q+j} and g_{q+j} are in $G(\text{St}(q+i))$ for all $1 \leq i \leq p$. $\square$

We say g *ends with* G_v if there is a reduced word representing g such that the last syllable is in G_v . Likewise, g *begins with* G_v if the first syllable of some reduced word representing g is in G_v . Note that if $g \equiv g_1 \dots g_v g_w$, and g_v and g_w commute (v and w are adjacent), then g ends with both G_v and G_w .

Lemma 2.5.3. *Suppose that $g, h \in G(\Gamma)$, $v \in \Gamma$, and h does not end with $G(\text{Lk}(v))$. If h^{-1} begins with G_v but $h^{-1}g$ does not begin with G_v , then there is a reduced word $g_1 \dots g_n$ representing g such that $h \equiv g_1 \dots g_m$ and $g_m \in G_v$ for some $m \leq n$. Similarly, if $h^{-1}g$ begins with G_v but h^{-1} does not begin with G_v , then there is a reduced word $g_1 \dots g_n$ representing g such that $h \equiv g_1 \dots g_m$ and $g_{m+1} \in G_v$.*

Proof. Suppose $g, h \in G(\Gamma)$, $v \in \Gamma$, and h does not end with $G(\text{Lk}(v))$. From the previous lemma, we may find reduced words $g \equiv g_1 \dots g_n$ and $h \equiv h_1 \dots h_m$ such that $g_1 \dots g_q = h_1 \dots h_q$ and

$$h^{-1}g \equiv h_m^{-1} \dots h_{q+p+1}^{-1} (h_{q+1}^{-1}g_{q+1}) \dots (h_{q+p}^{-1}g_{q+p}) g_{q+p+1} \dots g_n$$

for some $q \leq \min(m, n)$ and $p \leq \kappa$.

Let us start with the first assertion and assume that h^{-1} begins with G_v but $h^{-1}g$ does not begin with G_v . To show that h is an initial subword of g , we need to prove that $q = m$. In other words that all syllables of h will cancel with syllables of g . Since h^{-1} begins with G_v there is a syllable $h_i^{-1} \in G_v$ of h^{-1} which can shuffle to the beginning of h^{-1} and so $h_j^{-1} \in G(\text{Lk}(v))$ for all $i < j \leq m$. This implies that h ends with $G(\text{Lk}(v))$, a contradiction to our hypothesis, unless $i = m$. Since $h^{-1}g$ does not start with G_v , h_i^{-1} must cancel in $h^{-1}g$, so $m = i \leq q \leq \min(m, n)$. Hence $q = m$, $p = 0$, and $h \equiv g_1 \dots g_q$. Further, since h ends with G_v , we can assume $i = q = m$ and thus have $g_m \in G_v$.

For the second assertion, suppose $h^{-1}g$ begins with G_v and h^{-1} does not begin with G_v . Again we wish to show that $q = m$, so suppose not and $q < m$. Then h_m^{-1} is either a syllable or part of an amalgamated syllable of $h^{-1}g$. Since $h^{-1}g$ begins with G_v , there is some syllable $z \in G_v$ of $h^{-1}g$ which can shuffle to the beginning of $h^{-1}g$. If z is h_m^{-1} , then h^{-1} begins with G_v , a contradiction. If not, then z must shuffle past h_m^{-1} and h ends with $G(\text{Lk}(v))$, a contradiction. Hence h_m^{-1} cannot be part of a syllable of $h^{-1}g$, so $q = m$, $p = 0$, and $h = g_1 \dots g_q$. Also, since $h^{-1}g \equiv g_{q+1} \dots g_n$ and begins with G_v , we may assume $g_{m+1} \in G_v$. $\square$

The *d-tail* of an element $g \in G(\Gamma)$ is the set of syllables which are in the last (rightmost) d syllables of some reduced word representing g . The d -tails of g and h contain all of the information in $h^{-1}g$ when $|h^{-1}g|_r = d$:

Lemma 2.5.4. *Suppose that g and h are elements of $G(\Gamma)$ and $d = |h^{-1}g|_r$. Then each syllable of $h^{-1}g$ is either a syllable from the d -tail of g or h , or an amalgamation of syllables from the d -tails of g and h .*

Proof. Let g and h be elements of $G(\Gamma)$ and define $d := |h^{-1}g|_r$. If all the syllables of g and h are in their respective d -tails, then the lemma is trivially proved. As the argument is symmetric in g and h , we may assume that g has syllables outside its d -tail. We now show that all syllables of g outside the d -tail of g must cancel with a syllable of h in the product $h^{-1}g$.

Let $g \equiv g_1 \dots g_n$, and g_i be a syllable not in the d -tail of g . Let S be the set of syllables to the right of g_i in g that cannot be shuffled to the left of g_i . Note that S is independent of the reduced word representing g , and by hypothesis there are at least d syllables in S . Hence after shuffling and relabeling we may assume $g_j \in S$ for all $j > i$ and $i < n - d$.

Now suppose g_i does not cancel with a syllable of h in the reduced word representing $h^{-1}g$. Then at least one of the syllables g_j in S must cancel with a syllable h_p of h , otherwise $|h^{-1}g|_r$ would be greater than d . This implies that h_p must shuffle past $g_i \dots g_{j-1}$, which happens if and only if g_j can shuffle past $g_i \dots g_{j-1}$. In particular, g_j must shuffle to the left of g_i , but this is a contradiction, as g_j was supposed to be in S . Thus g_i must cancel with some syllable of h when reducing $h^{-1}g$. $\square$

In the case of a free product, the d -tail of g is just the last d syllables of g . For general graph products there may be more than d syllables in the d -tail and we have the following bound on its size:

Lemma 2.5.5. *Let κ be the maximal number of mutually adjacent vertices in Γ . Then the maximum number of syllables in the d -tail of any group element is $d\kappa$.*

Proof. Let $g \in G(\Gamma)$ and $n = |g|_r$. We prove this by induction on $1 < d \leq n$. For the base case, the 1-tail of g is just the set of final syllables of g , which must all commute with each other. Hence the largest the 1-tail could be is κ .

Now let d be a natural number between 1 and n and assume for induction that the $(d - 1)$ -tail of g has cardinality at most $(d - 1)\kappa$. Split the d -tail of g into two parts: the $(d - 1)$ -tail of g and the d -tail $\setminus (d - 1)$ -tail of g .

Suppose g_i is in the d -tail $\setminus (d - 1)$ -tail of g . Then we may find a reduced word representing g in which g_i is the $(n - d)^{\text{th}}$ syllable, and cannot be shuffled further towards the end of the word. If g_j is any other syllable in the d -tail $\setminus (d - 1)$ -tail of g , then g_j is somewhere to the left of g_i and would have to shuffle past g_i in order to rewrite the word with g_j as the $(n - d)^{\text{th}}$ syllable. Hence g_j must be in $G(\text{St}(i))$. This is true of any syllable in the d -tail $\setminus (d - 1)$ -tail, so all such syllables commute, and there can be at most κ of them. Combining this with the induction hypothesis implies that the cardinality of the d -tail of any group element is at most $d\kappa$. $\square$

CHAPTER 3

THE WALL SPACE

For the main construction we need bounds on the characteristic function of words with reduced word length d in a graph product. To do this we find a metric space that reflects the intergroup geometry of the vertex groups of a graph product. We then use properties of this metric space to supply us with bounds. For a free product of groups, this space is a tree. For a direct product of n groups, it is an n -dimensional cube, as every word has at most n syllables. For a general graph product this intergroup geometry is a CAT(0) cube complex—cubes where groups commute and trees where they do not.

A *CAT(0) cube complex* is a metric space of non-positive curvature made by gluing cubes of various dimensions together according to a few constraints. CAT(0) cube complexes are known for their nice combinatorial properties and are intimately related to median algebras and partially ordered with complement (poc) sets. In this paper, we shall realize our CAT(0) cube complex by constructing its associated wall space, a geometric poc set. For a detailed treatment of CAT(0) cube complexes, see Bridson and Haefliger [3]. For their connection to median algebras, poc sets, and wall spaces, see Roller [21] or Chatterji and Niblo [4].

First, some background: a *wall* on a set X is a partition of X into a half-space w and its complement w^c . We will refer to a wall (w, w^c) by its preferred half-space w . A wall w *separates* two points in X if one point lies in the half-space w and the other in w^c . A *wall space* is a space X and a collection W of walls on X such that, for any $x, y \in X$, finitely many walls in W separate x and y . The *wall distance* $d_W(x, y)$ is defined as the number of walls separating x and y . Two walls w and u *cross* (symbolically $w \perp u$) if their four possible intersections $w \cap u$, $w^c \cap u$, $w \cap u^c$, and $w^c \cap u^c$ are all nonempty. Obviously containment of one half-space in another ($w \subset u^c$ or $w \subset u$) implies the two walls do not cross, and we call the walls *parallel* ($w \parallel u$).

Chatterji and Niblo [4] and Nica [16], both based on a construction of Sageev [23], have demonstrated that every wall space generates a cube complex. Let us paraphrase the result we need (Theorem 3 from [4]):

Theorem 3.0.6. *Let G be a discrete group acting on a space with walls Y and let W denote the set of walls. Then there exists an action of G on a $CAT(0)$ cube complex X such that $d_X(gx, x) = d_W(gw, w)$ for all $x \in X$ and $w \in W$. If the maximum number of mutually crossing walls in W is finite, then X is finite dimensional.*

We now construct a set of walls on $G(\Gamma)$ such that the wall distance $d_W(g, h)$ is proportional to the reduced word length $|h^{-1}g|_r$. The $CAT(0)$ cube complex generated by this wallspace is a generalization of Ruane and Witzel's cube complex for graph products of finitely generated abelian groups [22]. It is also a natural extension of the cube complex known for Coxeter groups as found in Davis [9]. For a completely disconnected graph, this cube complex is (the rooted) barycentrically subdivided Bass-Serre tree of the free product of the vertex groups.

For each $v \in \Gamma$, define

$$\mathbb{A}_v := \{g \in G(\Gamma) \mid \exists g_v \in G_v \text{ such that } |g_v g|_r < |g|_r\}.$$

This is the set of elements of $G(\Gamma)$ which begin with G_v : if $g \in \mathbb{A}_v$ then there exists a $g_v \in G_v$ such that $|g_v g|_r < |g|_r$ and g_v must cancel with a syllable of g in any reduced word representing $g_v g$. This means g_v^{-1} is a syllable of g and can be shuffled to the beginning of g , hence g begins with G_v . On other hand, if $g \equiv g_1 \dots g_n$, and $g_1 \in G_v$, then $|g_1^{-1} g|_r = |g_2 \dots g_n|_r < |g|_r$ and g is in $\mathbb{A}_v$. Similarly $\mathbb{A}_v^c$, the complement of $\mathbb{A}_v$, is the set of elements of $G(\Gamma)$ that do not begin with G_v :

$$\mathbb{A}_v^c := \{g \in G(\Gamma) \mid \forall g_v \in G_v, \quad |g_v g|_r \geq |g|_r\}.$$

The pair $(\mathbb{A}_v, \mathbb{A}_v^c)$ form a wall on $G(\Gamma)$. Further, using the natural left multiplication action of $G(\Gamma)$ on subsets of $G(\Gamma)$, we have for each $k \in G(\Gamma)$ and $v \in \Gamma$ a wall $(k\mathbb{A}_v, (k\mathbb{A}_v)^c)$ whose half-spaces are

$$k\mathbb{A}_v = \{kg \in G(\Gamma) \mid \exists g_v \in G_v, |g_v g|_r < |g|_r\} = \{g \in G(\Gamma) \mid \exists g_v \in G_v, |g_v k^{-1} g|_r < |k^{-1} g|_r\}$$

and

$$(k\mathbb{A}_v)^c = k\mathbb{A}_v^c = \{g \in G(\Gamma) \mid \forall g_v \in G_v, |g_v k^{-1} g|_r \geq |k^{-1} g|_r\}.$$

Our set of walls $\mathbb{W}$ is the collection of all walls of this form. I.e.,

$$\mathbb{W} := \{k\mathbb{A}_v, \forall k \in G(\Gamma), \forall v \in \Gamma\}.$$

Note that we unambiguously refer to the wall $(k\mathbb{A}_v, k\mathbb{A}_v^c)$ by the preferred half-space $k\mathbb{A}_v$. This is because the action of $G(\Gamma)$ on $\mathbb{W}$ sends preferred half spaces to preferred half spaces:

Lemma 3.0.7. *For any $k \in G(\Gamma)$, and $v, w \in \Gamma$, $k\mathbb{A}_v \neq \mathbb{A}_w^c$.*

Proof. Suppose $k \equiv k_1 \dots k_n$ is an element of $G(\Gamma)$ and v and w are vertices of Γ . First suppose that k ends with G_v , so without loss of generality $k_n \in G_v$. Then, for any $g_w \in G_w$ which is not a syllable of k , we have $|k_n k^{-1} g_w|_r = |k_{n-1}^{-1} \dots k_1^{-1} g_w|_r < |k^{-1} g_w|_r$, and hence $g_w \in k\mathbb{A}_v$. But g_w is not in $\mathbb{A}_w^c$, since g_w starts with G_w . Hence $k\mathbb{A}_v \not\subset \mathbb{A}_w^c$.

Now suppose that k does not end with G_v . Then e is in $\mathbb{A}_w^c$ but not in $k\mathbb{A}_v$.

In either case we have that $k\mathbb{A}_v \neq \mathbb{A}_w^c$ and the lemma is proved. $\square$

The following proposition describes some of the structure of $\mathbb{W}$.

Proposition 3.0.8. *For $\mathbb{W}$ as above, $k \in G(\Gamma)$, and $v, w \in \Gamma$, we have the following:*

- (i) $k\mathbb{A}_v$ and $h\mathbb{A}_w$ cross if and only if $h^{-1}k\mathbb{A}_v$ and $\mathbb{A}_w$ cross.

(ii) Suppose v and w are adjacent. Then $k\mathbb{A}_v$ and $k\mathbb{A}_w$ cross and for all $h_w \in G_w$ we have

$$kh_w\mathbb{A}_v = k\mathbb{A}_v.$$

(iii) Given a wall $k\mathbb{A}_v \in \mathbb{W}$, there is an unique minimal (with respect to $|\cdot|_r$) element $h \in G(\Gamma)$

$$\text{such that } h\mathbb{A}_v = k\mathbb{A}_v.$$

(iv) If v and w are non-adjacent or equal and k is minimal as in (iii), then $k\mathbb{A}_v$ is parallel to $\mathbb{A}_w$.

Proof. Part (i) follows from invariance built into the walls: for any wall $k\mathbb{A}_v$, an element g is in $k\mathbb{A}_v$ if and only if $h^{-1}g \in h^{-1}k\mathbb{A}_v$. Hence, given another wall $h\mathbb{A}_w$, an element g is in $k\mathbb{A}_v \cap h\mathbb{A}_w$ if and only if $h^{-1}g \in h^{-1}k\mathbb{A}_v \cap \mathbb{A}_w$. Likewise the other three intersections $k\mathbb{A}_v \cap h\mathbb{A}_w^c$, $k\mathbb{A}_v^c \cap h\mathbb{A}_w$, and $k\mathbb{A}_v^c \cap h\mathbb{A}_w^c$ are non-empty if and only if $h^{-1}k\mathbb{A}_v \cap \mathbb{A}_w^c$, $h^{-1}k\mathbb{A}_v^c \cap \mathbb{A}_w$, and $h^{-1}k\mathbb{A}_v^c \cap \mathbb{A}_w^c$ are non-empty respectively. Thus if $k\mathbb{A}_v$ and $h\mathbb{A}_w$ cross, the eight intersections above are all non-empty and $h^{-1}k\mathbb{A}_v$ and $\mathbb{A}_w$ cross.

For (ii), suppose that $k\mathbb{A}_v$ is a wall and v and w are adjacent vertices of Γ . We first prove that $k\mathbb{A}_v$ and $k\mathbb{A}_w$ cross. Note for any $g_v \in G_v$ and $g_w \in G_w$ we have the following containments:

$$k^{-1} \in k\mathbb{A}_v^c \cap k\mathbb{A}_w^c, \quad k^{-1}g_v \in k\mathbb{A}_v \cap k\mathbb{A}_w^c, \quad k^{-1}g_w \in k\mathbb{A}_v^c \cap k\mathbb{A}_w, \quad \text{and } k^{-1}g_v g_w \in k\mathbb{A}_v \cap k\mathbb{A}_w.$$

As these four intersections are all non-empty, by definition $k\mathbb{A}_v$ and $k\mathbb{A}_w$ cross.

For the second part, suppose $h_w \in G_w$. Since h_w commutes with elements of G_v , the group element $h_w^{-1}k^{-1}g$ begins with G_v if and only if $k^{-1}g$ begins with G_v . This implies that $kh_w\mathbb{A}_v = k\mathbb{A}_v$.

To prove part (iii), let $k\mathbb{A}_v$ be a wall. If k ends with $G(\text{Lk}(v))$, then we may write $k \equiv k_1 \dots k_n$ where $k_n \in G_w$ and v and w are adjacent. By part (ii) we see that $k\mathbb{A}_v = k_1 \dots k_n\mathbb{A}_v = k_1 \dots k_{n-1}\mathbb{A}_v$. We may repeat this process until we have an $h = k_1 \dots k_j$ which no longer ends with $G(\text{Lk}(v))$ and $k\mathbb{A}_v = h\mathbb{A}_v$.

Next suppose $h\mathbb{A}_v$ is a wall such that $h\mathbb{A}_v = \mathbb{A}_v$. This imposes a strong constraint on h : for any $g \in G(\Gamma)$, g begins with G_v if and only if $h^{-1}g$ begins with G_v . In particular, e does not begin with G_v , so $h^{-1}e = h^{-1}$ must not begin with G_v . Let g_v be any element of G_v . Then g_v begins with G_v which implies that $h^{-1}g_v$ must begin with G_v . But h^{-1} does not begin with G_v , hence g_v must be able to shuffle to the beginning of h^{-1} . In other words, $h^{-1}g_v = g_vh^{-1}$ and every syllable of h must be in $G(\text{Lk}(v))$.

Finally, given two equivalent walls $k\mathbb{A}_v = h\mathbb{A}_v$, both reduced as above, we have that $h^{-1}k\mathbb{A}_v = \mathbb{A}_v$, so every syllable of $h^{-1}k$ must be in $G(\text{Lk}(v))$. By Lemma 2.5.2, we may find reduced words $k \equiv k_1 \dots k_n$ and $h \equiv h_1 \dots h_m$ such that $k_1 \dots k_q = h_1 \dots h_q$ and

$$h^{-1}k \equiv h_m^{-1} \dots h_{q+p+1}^{-1} (h_{q+1}^{-1} k_{q+1}) \dots (h_{q+p}^{-1} k_{q+p}) k_{q+p+1} \dots k_n$$

for some $q \leq \min(m, n)$ and $p \leq \kappa$.

Since reduction implies that h does not end with $G(\text{Lk}(v))$, $q \geq m$. Similarly, since k does not end with $G(\text{Lk}(v))$, $q \geq n$. Since $q \leq \min(m, n)$, we have $q = m = n$, and $h^{-1}k = e$. Hence if k and h are representatives of the same wall and reduced as above, then $k = h$.

For part (iv), suppose that v and w are non-adjacent or equal and $k\mathbb{A}_v$ is a wall. We wish to show $k\mathbb{A}_v \parallel \mathbb{A}_w$. If k is not a reduced as in part (iii), then we could reduce it to $h\mathbb{A}_v$, and $h\mathbb{A}_v \parallel \mathbb{A}_w$ if and only if $k\mathbb{A}_v \parallel \mathbb{A}_w$. So without loss of generality, k does not end with $G(\text{Lk}(v))$.

If $k = e$, then if $v = w$, $\mathbb{A}_v$ and $\mathbb{A}_w$ are parallel since they are equal. If v and w are not adjacent, then $\mathbb{A}_v \cap \mathbb{A}_w$ is empty, as no word can begin with both G_v and G_w . We now assume $0 < m = |k|_r$.

First suppose k ends with G_v and $g \in k\mathbb{A}_v^c$. Then $k^{-1}g$ does not begin with G_v and by Lemma 2.5.3, $g \equiv g_1 \dots g_n$ such that $k \equiv g_1 \dots g_m$ and $g_m \in G_v$. In other words, g must begin with k . If k begins with G_w , then g must begin with G_w and $k\mathbb{A}_v^c \cap \mathbb{A}_w^c$ must be empty. If k does not begin with G_w but g does, then some syllable g_i with $i > m$ is in G_w and $g_j \in G(\text{Lk}(w))$ for all $j < i$.

But when v and w are non-adjacent, g_m is not in $G(\text{Lk}(w))$, a contradiction. When $v = w$, by Remark 2.5.1, there is some $m < j < i$ such that $g_j \notin G(\text{St}(w))$, again a contradiction. Hence g cannot begin with G_w and $k\mathbb{A}_v^c \cap \mathbb{A}_w$ must be empty.

Now suppose k does not end with G_v and $g \in k\mathbb{A}_v$. Then $k^{-1}g$ begins with G_v and by Lemma 2.5.3 g must begin with $k \equiv g_1 \dots g_m$ and $g_{m+1} \in G_v$. Following from the arguments in the previous paragraph, but replacing g_m with g_{m+1} , we see that if k begins with G_w , then g begins with G_w , and $k\mathbb{A}_v \cap \mathbb{A}_w^c$ must be empty. If k does not begin with G_w , then g cannot begin with G_w and $k\mathbb{A}_v \cap \mathbb{A}_w$ must be empty.

As these cases exhaust all the possibilities for k , we have shown that $k\mathbb{A}_v$ is parallel to $\mathbb{A}_w$ whenever v and w are not adjacent or $v = w$. $\square$

We call the unique minimal wall representative h of Proposition 3.0.8 (iii) a *reduced wall representative*.

Corollary 3.0.9. *There is an uniform bound on the size of a set of mutually crossing walls of $\mathbb{W}$.*

Proof. Recall that κ is the maximum number of mutually adjacent vertices of Γ . Combining Proposition 3.0.8 (i) and (iv), we see that two walls cross only if they have adjacent vertices. Since a maximum of κ vertices are mutually adjacent, at most κ walls mutually cross. $\square$

Lemma 3.0.10. *For $\mathbb{W}$ as above, the wall distance between e and $g \in G(\Gamma)$ is $d_W(e, g) = 2|g|_r$.*

Proof. The case when $g = e$ is trivial, so we assume $g \neq e$ and our task is to count the number of walls which separate e and g . Let $g \equiv g_1 g_2 \dots g_m$ and suppose k is a reduced wall representative such that $k\mathbb{A}_v$ separates g and e . Recall that $k\mathbb{A}_v$ separates e and g if either e is in $k\mathbb{A}_v$ and g is in $k\mathbb{A}_v^c$ or g is in $k\mathbb{A}_v$ and e is in $k\mathbb{A}_v^c$.

If e is in $k\mathbb{A}_v$ and g is in $k\mathbb{A}_v^c$ then k^{-1} begins with G_v and $k^{-1}g$ does not. By Lemma 2.5.3, $k \equiv g_1 \dots g_n$ and $g_n \in G_v$ for some $n \leq m$. If g is in $k\mathbb{A}_v$ and e is in $k\mathbb{A}_v^c$, then k^{-1} does not begin with G_v and $k^{-1}g$ does begin with G_v . Appealing again to Lemma 2.5.3, k must have the

form $k \equiv g_1 \dots g_n$ and $g_{n+1} \in G_v$ for some $n < m$. From these two statements we see that for each syllable g_i of g , with $v \in \Gamma$ such that $g_i \in G_v$, the walls $g_1 \dots g_i \mathbb{A}_v$ and $g_1 \dots g_{i-1} \mathbb{A}_v$ separate e and g and are the only walls to do so. Hence $d_W(e, g) = 2|g|_r$. $\square$

Lemma 3.0.11. *The wall distance is $G(\Gamma)$ -invariant: $d_W(h, hg) = d_W(e, g)$ for all $g, h \in G(\Gamma)$.*

Proof. This follows from the construction of the walls: For any g and $h \in G(\Gamma)$, $k\mathbb{A}_v$ separates g and e if and only if $hk\mathbb{A}_v$ separates hg and h , since $g \in k\mathbb{A}_v$ if and only if $hg \in hk\mathbb{A}_v$. $\square$

We may now apply Theorem 3.0.6 to $\mathbb{W}$ to realize a finite dimensional CAT(0) cube complex X on which $G(\Gamma)$ acts, with $d_X(gx, hx) = 2|h^{-1}g|_r$ for any $x \in X$. We shall return to this distance function in Chapter 5.

CHAPTER 4

EXTENDING POSITIVE DEFINITE FUNCTIONS TO FREE PRODUCTS

In this section we shall prove two special cases of the main theorem: if two discrete groups A and B are either amenable or weakly amenable, then $A * B$ is weakly amenable. In doing so we shall develop the machinery used in the final proof. The case for amenable factors was originally proved by Bozejko and Picardello [2] in 1993. In 2006 Ricard and Xu [20] proved the weakly amenable case in the more general context of free products of C^* -algebras. The latter proof was analytic in nature. In this paper we give a constructive proof which is generalizable to graph products. The key step in this construction is the extension of positive definite or completely bounded (with cb -norm close to 1) functions from the factor groups to their free product.

Let us motivate this extension. Consider $\mathbb{Z} * \mathbb{Z} = \mathbb{F}_2$, the free group on two generators. By Schoenberg's theorem, $f_{\mathbb{Z}}(n) := e^{-|n|}$ is a positive definite function on $\mathbb{Z}$. If we wish to extend $f_{\mathbb{Z}}$ to a function on the free group, there is a natural choice: the function $F(g) := e^{-\|g\|}$. It is positive definite (Haagerup gives a proof in [14]), the restriction of F to either factor group is $f_{\mathbb{Z}}$, and for a word $g = a^{p_1} b^{q_1} \dots a^{p_n} b^{q_n}$ in $\mathbb{F}_2$,

$$\begin{aligned} F(g) &= e^{-\|g\|} = e^{-(\sum_{i=1}^n |p_i| + |q_i|)} \\ &= e^{-|p_1|} e^{-|q_1|} \dots e^{-|p_n|} e^{-|q_n|} \\ &= f_{\mathbb{Z}}(p_1) f_{\mathbb{Z}}(q_1) \dots f_{\mathbb{Z}}(p_n) f_{\mathbb{Z}}(q_n). \end{aligned}$$

We see that $F(g)$ is just the product of $f_{\mathbb{Z}}$ applied to each syllable (under the isomorphism $a^n \mapsto n$). For discrete groups other than $\mathbb{Z}$ we would like something similar.

4.1 The positive definite case

Let $G = A * B$ and let f_A and f_B be positive definite functions on the factor groups with $f_A(e) = 1 = f_B(e)$. Recall that since f_A is positive definite, we may represent it as $f_A(a^{-1}a') = \langle \alpha_A(a'), \alpha_A(a) \rangle$, where $\mathcal{H}_A$ is a Hilbert space and α_A is a map of A into $\mathcal{H}_A$ with $\|\alpha_A(a)\| = 1$ for all $a \in A$. We have a similar Hilbert space-valued map α_B associated with f_B . Guided by our example above, for $g = a_1 b_1 \dots a_n b_n$, we would like to define our extension as

$$\begin{aligned} \langle \alpha(g), \alpha(e) \rangle &\stackrel{?}{=} F(g) := f_A(a_1) f_B(b_1) \dots f_A(a_n) f_B(b_n) \\ &= \langle \alpha_A(a_1), \alpha_A(e) \rangle \langle \alpha_B(b_1), \alpha_B(e) \rangle \dots \langle \alpha_A(a_n), \alpha_A(e) \rangle \langle \alpha_B(b_n), \alpha_B(e) \rangle \end{aligned}$$

The questionable equality would characterize F as positive definite. We can realize such an equality if the Hilbert space in which we take the inner product on the left hand-side is a tensor product of the Hilbert spaces used on the right-hand side. This product must be over some nice index set that isolates syllables of words in G .

Let T be the set of left cosets of A and B ($T \cong G/A \sqcup G/B$). For each $t \in T$, define the Hilbert space $\mathcal{H}_t$ and its vacuum vector ω_t as either $\mathcal{H}_A$ and $\alpha_A(e)$ or $\mathcal{H}_B$ and $\alpha_B(e)$, depending on if t is a left coset of A or B , respectively. Let $\widehat{\mathcal{H}} := \bigotimes_T \mathcal{H}_t$ be the infinite tensor product of the $\mathcal{H}_t$ over T as described in Section 2.2.

Since G acts by left multiplication on T , it induces an unitary action π on $\widehat{\mathcal{H}}$ by permuting indices, namely $\pi_g \xi_t = \xi_{gt}$, where ξ_t is the t component of the tensor $\xi \in \widehat{\mathcal{H}}$.

From now on we shall write, for g in G , $g = g_1 g_2 \dots g_n$, where the representation is understood to be an alternating word in the factor groups A and B . For a syllable g_i , we write $G(i)$ for the factor group containing g_i and write $\alpha(g_i)$ for $\alpha_{G(i)}(g_i) \in \mathcal{H}_{G(i)}$. For such a word $g = g_1 g_2 \dots g_n$ in G , define

$$\widehat{\alpha}(g) := \alpha(g_1)_{G(1)} \otimes \alpha(g_2)_{g_1 G(2)} \otimes \alpha(g_3)_{g_1 g_2 G(3)} \otimes \dots \otimes \alpha(g_n)_{g_1 \dots g_{n-1} G(n)}.$$

This is a map from G to $\hat{\mathcal{H}}$, and the kernel defined by the inner product $\langle \hat{\alpha}(g), \hat{\alpha}(e) \rangle$ is exactly the extension we were looking for:

$$\begin{aligned} \langle \hat{\alpha}(g), \hat{\alpha}(e) \rangle &= \langle \alpha(g_1), \alpha(e) \rangle_{G(1)} \langle \alpha(g_2), \alpha(e) \rangle_{g_1 G(2)} \cdots \langle \alpha(g_n), \alpha(e) \rangle_{g_1 \dots g_{n-1} G(n)} \\ &= f_{G(1)}(g_1) f_{G(2)}(g_2) \cdots f_{G(n)}(g_n). \end{aligned}$$

Figure 4.1: The Bass-Serre tree for a free product. On the left is the path for $g = ab\hat{a}\hat{b}$, on the right is the path for $h = aba\hat{a}$, and on top is their difference $h^{-1}g$.

Our choice of T gives us a geometric interpretation of this construction. We can construct a graph X using T as the vertex set, connecting two cosets with an edge if they share an element. This implies that no two cosets of the same factor group share an edge. This graph is a tree (called the Bass-Serre tree) and each edge corresponds uniquely to a group element. There is an embedding of G into the paths of X , sending each g to the path from $G(1)$ to $g_1 \dots g_{n-1} G(n)$. Each edge along the path corresponds to a subword $g_1 \dots g_i$ of g . Given two group elements g and h , the paths of g and h overlap if and only if g and h begin with the same subword.

Notice that $\hat{\alpha}(g)$ has non-vacuum entries only for indices which are along the path of g in X , including endpoints. When we take an inner product $\langle \hat{\alpha}(g), \hat{\alpha}(h) \rangle$, we get values other than one only on indices along the paths of g and h in X . If the two paths overlap then g and h must begin with the same subword $k = g_1 \dots g_i$. For each syllable g_j of k , the $g_1 \dots g_{j-1} G(j)^{\text{th}}$ factor of the inner product $\langle \hat{\alpha}(g), \hat{\alpha}(h) \rangle$ is $\langle \alpha(g_j), \alpha(g_j) \rangle = 1$. We are left with a translated factor of the remainder which is of the form

$$\begin{aligned} \langle \pi_k \hat{\alpha}(k^{-1}g), \pi_k \hat{\alpha}(k^{-1}h) \rangle &= \langle \alpha(g_{i+1}), \alpha(h_{i+1}) \rangle \langle \alpha(g_j), \alpha(e) \rangle \cdots \langle \alpha(e), \alpha(h_j) \rangle \cdots \\ &= \langle \alpha(h_{i+1}^{-1} g_{i+1}), \alpha(e) \rangle \langle \alpha(g_j), \alpha(e) \rangle \cdots \langle \alpha(h_j^{-1}), \alpha(e) \rangle \cdots \\ &= \langle \hat{\alpha}(h^{-1}g), \hat{\alpha}(e) \rangle \end{aligned}$$

where j is larger than i . This in particular shows that the positive definite function $F(h^{-1}g) := \langle \hat{\alpha}(g), \hat{\alpha}(h) \rangle$ is well defined on G .

Note that F is not finitely supported regardless of whether or not f_A or f_B are. This is as it should be—if we could construct a finitely supported positive definite function using the techniques above, our free product would be amenable! We now have enough to prove the first special case. As it is a corollary of the main theorem, we only sketch the proof.

Theorem 4.1.1. *If A and B are amenable discrete groups, $A * B$ is weakly amenable.*

sketch. If A and B are amenable, then we have two sequences $\{f_A^n\}$ and $\{f_B^n\}$ consisting of finitely supported positive definite functions which go to the identity as n goes to infinity. We can use the above construction to make a sequence of positive definite functions $\{F^n\}$ on $A * B$. As n goes to infinity, F^n approaches the identity on each factor group, and hence on $A * B$, but is not finitely supported. In [2], Bozejko and Picardello showed that χ_d , the characteristic function of the d -sphere of a tree, is completely bounded with cb -norm growing polynomially as d goes to infinity. The growth of words in a free product is described by its Bass-Serre tree T , our index set above, hence we may induce finite support by truncation, taking $\sum_N e^{-d/r} F^n \chi_d$. The exponential tames the growth of $\|\chi_d\|_{cb}$, where r is chosen large enough that $e^{-|g|/r} F^n(g)$ is close to one on a given subset. We lose our positive definiteness but remain completely bounded with cb -norm approaching one from above as we truncate closer to infinity. We can thus construct a sequence of finitely supported completely bounded functions on G which converge to the identity, and G is weakly amenable. $\square$

4.2 The completely bounded case

Now suppose that the factor groups are only weakly amenable. Given two completely bounded functions on A and B with cb -norms less than $1 + \varepsilon$, and $f_A(e) = 1 = f_B(e)$, we would like to extend them to a completely bounded function on G . By Proposition 2.4.1, for each f we have a Hilbert space $\mathcal{H}$ with two maps α and β into $\mathcal{H}$, such that $f(h^{-1}g) = \langle \alpha(g), \beta(h) \rangle$. Note: we decorate f as f_A or f_B only if the usage is ambiguous, otherwise relying on context. The same holds for α , β , and $\mathcal{H}$. The fact that the kernel of a completely bounded function is the inner product of some Hilbert space-valued maps gives us hope that we can mimick the positive definite construction above, with the added complications that $\alpha \neq \beta$ and $\|\alpha\| \neq 1 \neq \|\beta\|$.

The first difficulty arises in the construction of the tensor product, as we no longer have an obvious choice for the vacuum vectors. As a first step, we create positive definite kernels which are close to our f s.

For each element g of each factor group, define the weighted average

$$\omega(g) := \frac{\alpha(g) + \beta(g)}{2 + \sqrt{2\varepsilon}} \text{ and } D(g) := \sqrt{\frac{1 - \|\omega(g)\|^2}{2}},$$

an additive renormalization constant associated to $\omega(g)$.

For the factor group A and any element $a \in A$, the map

$$a \mapsto \begin{cases} (\omega(a), [\frac{D(a)}{D(a)}], [\frac{0}{0}]) & \text{if } a \neq e \\ (\omega(e), [\frac{0}{0}], [\frac{D(e)}{D(e)}]) & \text{if } a = e \end{cases}$$

maps A into $\mathcal{H}_A \oplus \mathbb{C}^2 \oplus \mathbb{C}^2$ and generates the (non A -invariant) positive definite kernel

$$\psi_A(a, a') := \left\langle (\omega(a), [\frac{D(a)}{D(a)}], [\frac{0}{0}]), (\omega(a'), [\frac{D(a')}{D(a')}] , [\frac{0}{0}]) \right\rangle.$$

By the definition of $D(a)$, we have that $\psi_A(a, a) = \left\| (\omega(a), [\frac{D(a)}{D(a)}], [\frac{0}{0}]) \right\|^2 = 1$ for all $a \in A$.

We have a matching ψ_B on B as well.

Now that we have two positive definite kernels, we extend them to G as above. Let $T \cong G/A \sqcup G/B$ be the set of left cosets as in the previous section. For each $t \in T$, define $\mathcal{H}_t$ and its vacuum vector to be $\mathcal{H}$ and $(\omega(e), [\frac{0}{0}], [\frac{D(e)}{D(e)}])$ which depend on the factor group of which t is a coset. Let $\widehat{\mathcal{H}} := \bigotimes_T (\mathcal{H}_t \oplus \mathbb{C}^2 \oplus \mathbb{C}^2)$. For $g = g_1 g_2 \dots g_n$, define

$$\begin{aligned} \widehat{\omega}(g) := & (\omega(g_1), [\frac{D(g_1)}{D(g_1)}], [\frac{0}{0}])_{G(1)} \otimes (\omega(g_2), [\frac{D(g_2)}{D(g_2)}], [\frac{0}{0}])_{g_1 G(2)} \otimes \\ & \dots \otimes (\omega(g_n), [\frac{D(g_n)}{D(g_n)}], [\frac{0}{0}])_{g_1 \dots g_{n-1} G(n)}. \end{aligned}$$

This is a map from G to $\widehat{\mathcal{H}}$.

We call $\Psi(g, h) := \langle \widehat{\omega}(g), \widehat{\omega}(h) \rangle$ the *ambient positive definite kernel* associated with f_A and f_B . If ε is very small, then $\omega \approx \alpha \approx \beta$, and Ψ is very close to the map we want. Unfortunately this kernel is not G -invariant.

Recall that in the proof sketch above our final functions were of the form $\sum^n e^{-d/r} F_{\chi_d}$. This implies that instead of searching for a single extension F which is well defined on the whole group, we may search for a family of functions F_d , each well defined only for words of length d . In other words, for each d we may look for a kernel which is G -invariant on pairs of elements g and h such that $|h^{-1}g|_r = d$. By Lemma 2.5.4, if $|h^{-1}g|_r = d$ for a pair $g, h \in G$, then all but the last d syllables of g and h cancel and hence we may try to modify our map $\widehat{\omega}$ on the last d syllables to match our original maps α and β .

Proposition 4.2.1. *Suppose that f_A and f_B are two completely bounded functions on the groups A and B with cb -norms less than $1 + \varepsilon$, and $f_A(e) = 1 = f_B(e)$. Then for each natural number d there exists a completely bounded kernel ϕ_d on $A * B$ such that $\phi_d(g, e) = f_1(g_1)f_2(g_2) \cdots f_d(g_d)$ for all $g \in G$ with word length d .*

Proof. Let $G = A * B$ and d be a natural number. For each factor group we have a completely bounded function f and some Hilbert space $\mathcal{H}$ with two maps α and β into $\mathcal{H}$, such that $f(h^{-1}g) = \langle \alpha(g), \beta(h) \rangle$. Since we assume $\|f\|_{cb} < 1 + \varepsilon$, $\|\alpha\|^2$ and $\|\beta\|^2$ are bounded between $1 - \varepsilon$ and $1 + \varepsilon$. Let ω , $D(g)$, and $\widehat{\omega}$ be defined as above. For every pair of elements g and h in the same factor group, let

$$C^\alpha(g, h) := \frac{\langle \alpha(g), \beta(h) \rangle - \langle \alpha(g), \omega(h) \rangle}{D(h)} \text{ and } C^\beta(g, h) = \frac{\langle \alpha(h), \beta(g) \rangle - \langle \omega(h), \beta(g) \rangle}{D(h)}.$$

These are additive renormalization constants used to correctly fit the two embeddings together. They are all less than $8\varepsilon^{1/4}$, as we will show in Lemma 4.2.3.

Define, for $g = g_1 \dots g_n \in G$,

$$\hat{\alpha}(g) := \bigotimes_{0 < i \leq n} \begin{cases} (\alpha(g_i), [C^\alpha(g_i, g_i)], [C^\alpha(g_i, e)])_{g_1 \dots g_{i-1} G(i)} & \text{if } n - i \leq d \\ (\omega(g_i), [D(g_i)], [\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}])_{g_1 \dots g_{i-1} G(i)} & \text{otherwise} \end{cases}$$

and likewise (note the different placement of the constants),

$$\hat{\beta}(g) := \bigotimes_{0 < i \leq n} \begin{cases} (\beta(g_i), [C^\beta(g_i, g_i)], [C^\beta(g_i, e)])_{g_1 \dots g_{i-1} G(i)} & \text{if } n - i \leq d \\ (\omega(g_i), [D(g_i)], [\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}])_{g_1 \dots g_{i-1} G(i)} & \text{otherwise.} \end{cases}$$

In other words, these maps are $\varepsilon^{1/4}$ -perturbations of α or β on the last d syllables of g and is equal to $\hat{\omega}$ on the rest of g .

To uncompress the notation above, we have

$$\begin{aligned} \hat{\alpha}(g) &= (\omega(g_1), [D(g_1)], [\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}])_{G(1)} \otimes \dots \\ &\otimes (\omega(g_{j-1}), [D(g_{j-1})], [\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}])_{g_1 \dots g_{j-2} G(j-1)} \otimes (\alpha(g_j), [C^\alpha(g_j, g_j)], [C^\alpha(g_j, e)])_{g_1 \dots g_{j-1} G(j)} \otimes \\ &\dots \otimes (\alpha(g_n), [C^\alpha(g_n, g_n)], [C^\alpha(g_n, e)])_{g_1 \dots g_{n-1} G(n)}, \end{aligned}$$

where $n - j = d$.

Let us show that $\hat{\alpha}(g)$ is bounded. First notice that $\hat{\alpha}(g)$ has at most d non-unit vectors corresponding to the last d syllables of g . We have

$$\begin{aligned} \|\hat{\alpha}(g)\| &= \prod_{i=n-d}^n \|(\alpha(g_i), [C^\alpha(g_i, g_i)], [C^\alpha(g_i, e)])\| \\ &= \prod_{i=n-d}^n \sqrt{\|\alpha_i(g_i)\|^2 + |C^\alpha(g_i, g_i)|^2 + |C^\alpha(g_i, e)|^2} \\ &\leq \left(1 + \varepsilon + 128\varepsilon^{\frac{1}{2}}\right)^{\frac{d}{2}} \leq (1 + 129\sqrt{\varepsilon})^{\frac{d}{2}} \end{aligned}$$

where the penultimate inequality comes from Lemma 4.2.3 and is independent of the choice of g . We have the same inequality for $\|\widehat{\beta}(g)\|$.

Our final kernel is $\phi_d(g, h) := \langle \widehat{\alpha}(g), \widehat{\beta}(h) \rangle$. This kernel is completely bounded, with

$$\|\phi_d\|_{cb} \leq \sup_{g, h \in G} \|\widehat{\alpha}(g)\| \|\widehat{\beta}(h)\| \leq \left(1 + 129\varepsilon^{\frac{1}{2}}\right)^d.$$

Further, for $g = g_1 \dots g_d$ in G ,

$$\begin{aligned} \phi_d(g, e) &= \prod_{i=1}^d \left\langle (\alpha(g_i), [{}^C \alpha_{g_i, g_i}^{\alpha}], [{}^C \alpha_{g_i, e}^{\alpha}]), (\omega(e), [\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}], [\begin{smallmatrix} D(e) \\ D(e) \end{smallmatrix}]) \right\rangle_{g_1 \dots g_{i-1} G(i)} \\ &= \prod_{i=1}^d \langle \alpha(g_i), \beta(e) \rangle = \prod_{i=1}^d f(g_i), \end{aligned}$$

where the second equality follows from calculation and the definition of $C^\alpha(g, e)$.

□

Lemma 4.2.2. *For words $g, h \in G$ with $d = |h^{-1}g|$, the kernel ϕ_d constructed in the last proposition is G -invariant: $\phi_d(g, h) = \phi_d(h^{-1}g, e)$.*

Proof. Let us show that $\phi_d(g, h) = \phi_d(h^{-1}g, e)$ whenever $|h^{-1}g|_r = d$.

Let $g = g_1 \dots g_n$ and $h = h_1 \dots h_m$. Then we may write $h^{-1}g = h_m^{-1} \dots h_q^{-1} g_q \dots g_n$ for some q (possibly 0), where h_q and g_q are in the same factor group, and $g_i = h_i$ for all $i < q$. As in the positive definite extension, $\widehat{\alpha}(g)$ and $\widehat{\beta}(h)$ have non-vacuum vectors only for indices on the paths of g and h in X , respectively. In particular, $\widehat{\alpha}(g)$ and $\widehat{\beta}(h)$ have the indices $g_1 \dots g_{i-1} G(i)$ in common for all $i \leq q$.

Suppose $g_1 \dots g_{i-1}G(i)$ is a common non-vacuum index of $\widehat{\alpha}(g)$ and $\widehat{\beta}(h)$. If $i < q$, then $g_i = h_i$ and the $g_1 \dots g_{i-1}G(i)$ factor of $\langle \widehat{\alpha}(g), \widehat{\beta}(h) \rangle$ is one of the following:

$$\begin{aligned} & \langle (\alpha(g_i), [C^\alpha(g_i, g_i)], [C^\alpha(g_i, e)]), (\beta(g_i), [C^\beta(g_i, g_i)], [C^\beta(g_i, e)]) \rangle, \\ & \langle (\alpha(g_i), [C^\alpha(g_i, g_i)], [C^\alpha(g_i, e)]), (\omega(g_i), [D(g_i)], [\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}]) \rangle, \\ & \langle (\omega(g_i), [D(g_i)], [\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}]), (\beta(g_i), [C^\beta(g_i, g_i)], [C^\beta(g_i, e)]) \rangle, \text{ or} \\ & \langle (\omega(g_i), [D(g_i)], [\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}]), (\omega(g_i), [D(g_i)], [\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}]) \rangle, \end{aligned}$$

depending on whether or not g_i is one of the last d syllables of g or h . In any case, the factor is equivalent to 1 by the choice of constants. If $i = q$ then $g_i \neq h_i$ but they are in the same factor group. and by Lemma 2.5.4 g_i and h_i are both in the last d syllables of their respective words, so the corresponding factor of the inner product is

$$\langle (\alpha(g_q), [C^\alpha(g_q, g_q)], [C^\alpha(g_q, e)]), (\beta(h_q), [C^\beta(h_q, h_q)], [C^\beta(h_q, e)]) \rangle = \langle \alpha(g_q), \beta(h_q) \rangle.$$

For all other non-vacuum indices we either have a factor of

$$\begin{aligned} & \langle (\alpha(g_i), [C^\alpha(g_i, g_i)], [C^\alpha(g_i, e)]), (\omega(e), [\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}], [D(e)]) \rangle = \langle \alpha(g_i), \beta(e) \rangle, \text{ or} \\ & \langle (\omega(e), [\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}], [D(e)]), (\beta(h_i), [C^\beta(h_i, h_i)], [C^\beta(h_i, e)]) \rangle = \langle \alpha(e), \beta(h_i) \rangle. \end{aligned}$$

Hence

$$\begin{aligned} \phi_d(g, h) &= \langle \widehat{\alpha}(g), \widehat{\beta}(h) \rangle \\ &= \langle \alpha(g_q), \beta(h_q) \rangle \prod_{i=q+1}^n \langle \alpha(g_i), \beta(e) \rangle \prod_{i=q+1}^m \langle \alpha(e), \beta(h_i) \rangle \\ &= \langle \alpha(h_q^{-1}g_q), \beta(e) \rangle \prod_{i=q+1}^n \langle \alpha(g_i), \beta(e) \rangle \prod_{i=q+1}^m \langle \alpha(h_i), \beta(e) \rangle \\ &= \langle \widehat{\alpha}(h^{-1}g), \widehat{\beta}(e) \rangle = \phi_d(h^{-1}g, e) \end{aligned}$$

and we have our desired equality. $\square$

Lemma 4.2.3. *The constants $C^\alpha(g, h)$ and $C^\beta(g, h)$ are less than $8\varepsilon^{1/4}$ for all g, h in G .*

Proof. Recall that given two Hilbert space-valued maps α and β , with $\|\alpha\|^2, \|\beta\|^2 < 1 + \varepsilon$, we defined $\omega(g) = \frac{\alpha(g) + \beta(g)}{2 + \sqrt{2\varepsilon}}$, $D(g) = \sqrt{\frac{1 - \|\omega(g)\|^2}{2}}$, $C^\alpha(g, h) = \frac{\langle \alpha(g), \beta(h) \rangle - \langle \alpha(g), \omega(h) \rangle}{D(h)}$ and $C^\beta(g, h) = \frac{\langle \alpha(h), \beta(g) \rangle - \langle \omega(h), \beta(g) \rangle}{D(h)}$ for all g and h in the same factor group.

First observe that β is close in norm to ω :

$$\begin{aligned} \|\beta(g) - \omega(g)\|^2 &= \left\| \frac{((2 + \sqrt{2\varepsilon}) - 1)\beta(g) - \alpha(g)}{2 + \sqrt{2\varepsilon}} \right\|^2 \\ &= \frac{\|(1 + \sqrt{2\varepsilon})\beta(g)\|^2 + \|\alpha(g)\|^2 - 2((2 + \sqrt{2\varepsilon}) - 1)\langle \beta(g), \alpha(g) \rangle}{(2 + \sqrt{2\varepsilon})^2} \\ &\leq \frac{(1 + \sqrt{2\varepsilon})^2(1 + \varepsilon) + (1 + \varepsilon) - (2 + 2\sqrt{2\varepsilon})}{(2 + \sqrt{2\varepsilon})^2} = \frac{2\varepsilon + 2\sqrt{2\varepsilon} + 4}{(2 + \sqrt{2\varepsilon})^2} \varepsilon \leq \varepsilon. \end{aligned}$$

We also have that $\|\omega(g)\|^2 = \frac{\|\alpha(g)\|^2 + \|\beta(g)\|^2 + 2\langle \alpha(g), \beta(g) \rangle}{(2 + \sqrt{2\varepsilon})^2} \leq \frac{4 + 2\varepsilon}{(2 + \sqrt{2\varepsilon})^2}$, which implies that $D(g)$ is bounded away from 0:

$$D(g) = \sqrt{\frac{1 - \|\omega(g)\|^2}{2}} \geq \sqrt{\frac{1}{2} - \frac{2 + \varepsilon}{(2 + \sqrt{2\varepsilon})^2}} = \sqrt{\frac{2\sqrt{2\varepsilon}}{(2 + \sqrt{2\varepsilon})^2}} \geq \frac{\varepsilon^{1/4}}{4}.$$

Putting these together, we see that

$$C^\alpha(g, h) = \frac{\langle \alpha(g), \beta(h) - \omega(h) \rangle}{D(h)} \leq \frac{\|\alpha(g)\| \|\beta(h) - \omega(h)\|}{D(h)} \leq \frac{4\sqrt{1 + \varepsilon}\sqrt{\varepsilon}}{\varepsilon^{1/4}} \leq 8\varepsilon^{1/4}.$$

We have the same bounds for $C^\beta(g, h)$, as, by an almost identical proof, $\|\alpha(g) - \omega(g)\|^2 \leq \varepsilon$. These are coarse estimates and the constant may be improved, but they are sufficient for our purposes. $\square$

We can now sketch the proof of the next theorem, inspired by Ricard and Xu's proof in [20],

Theorem 4.2.4. *If A and B are weakly amenable discrete groups, then $A * B$ is weakly amenable.*

sketch. If A and B are weakly amenable, then we have two sequences consisting of finitely supported completely bounded functions $\{f_A^n\}$ and $\{f_B^n\}$ which go to the identity and whose cb-norms go to one as n goes to infinity. For each n we have an ambient positive definite kernel Ψ^n and, by Proposition 4.2.1, a family of completely bounded functions $\{F_d^n\}_{d \in \mathbb{N}}$ on G with $F_d^n(h^{-1}g) := \phi_d(g, h)\chi_d(g, h)$ for each $d \in \mathbb{N}$. We would like to take finite sums of the form $F^{r,n,M} := \sum^M e^{-d/r} F_d^n$. We can bound $\|F^{r,n,M}\|_{cb}$ by $\|\phi_d^n - \Psi^n\|_{cb}$ and the cb-norm of a truncated Ψ^n . The first summand is controlled by $\varepsilon_n^{1/4}$ and the second approaches one as M goes to infinity, since the infinite sum $\sum^{\infty} e^{-d/r} \Psi^n \chi_d = e^{-d(\cdot, \cdot)/r} \Psi^n(\cdot, \cdot)$ is positive definite. Hence $\|F^{r,n,M}\|_{cb}$ approaches 1 as n and M approach infinity. This comparison to a positive definite kernel is the reason why we only discuss weak amenability with Cowling-Haagerup constant 1. We then notice that, as r , M , and n become sufficiently large, $F^{r,n,M}$ approaches the identity on the group and its cb-norm approaches one. Thus we may construct an approximate identity of completely bounded functions on G and G is weakly amenable. $\square$

CHAPTER 5

THE MAIN THEOREM

In this section we prove that weak amenability is stable under graph products. The first step is to generalize to graph products the above extension of completely bounded functions. The only difference between the construction for free products and the following construction for graph products is the index set over which we build the Hilbert space.

Proposition 5.0.5. *Suppose $G(\Gamma)$ is a graph product over Γ , which is a graph with at most κ mutually adjacent vertices, and, for each vertex group G_v of $G(\Gamma)$, there is a completely bounded function f_v with cb-norm less than $1 + \varepsilon$ such that $f_v(e) = 1$. Then for each natural number d there exists a completely bounded kernel ϕ_d on $G(\Gamma)$ such that $\phi_d(g, e) = f_1(g_1)f_2(g_2) \cdots f_d(g_d)$ for all $g \in G(\Gamma)$ with $|g|_r = d$.*

Proof. Suppose $G(\Gamma)$ is a graph product of vertex groups $\{G_v\}$ over a graph Γ and for each vertex group G_v we have a completely bounded function f_v with $\|f_v\|_{cb} < 1 + \varepsilon$ and $f_v(e) = 1$. Associated with f_v is a Hilbert space $\mathcal{H}_v$ and maps $\alpha_v, \beta_v : G_v \rightarrow \mathcal{H}_v$ such that $f_v(h^{-1}g) = \langle \alpha_v(g), \beta_v(h) \rangle$. Furthermore, $\|\alpha_v\|$ and $\|\beta_v\|$ are both less than $\sqrt{1 + \varepsilon}$ and $\langle \alpha_v(g), \beta_v(g) \rangle = 1$ for all $g \in G_v$. We shall again drop subscripts when not needed. As in Proposition 4.2.1, for each g and $h \in G_v$, define the map and renormalization constant

$$\omega(g) := \frac{\alpha(g) + \beta(g)}{2 + \sqrt{2\varepsilon}} \text{ and } D(g) := \sqrt{\frac{1 - \|\omega(g)\|^2}{2}},$$

and the constants

$$C^\alpha(g, h) := \frac{\langle \alpha(g), \beta(h) \rangle - \langle \alpha(g), \omega(h) \rangle}{D(h)} \text{ and } C^\beta(g, h) := \frac{\langle \alpha(h), \beta(g) \rangle - \langle \omega(h), \beta(g) \rangle}{D(h)}.$$

We now have all the data we need to construct our extension except for the index set of our tensor product. We shall use $T := \sqcup_{v \in \Gamma} G(\Gamma)/G(\text{St}(v))$, the left cosets of $G(\text{St}(v))$ for all v . When $G(\Gamma)$ is a free product, this is identical to the T defined in previous section. For each $kG(\text{St}(v))$ in T , define $\mathcal{H}_{kG(\text{St}(v))} := \mathcal{H}_v$ and use $(\omega_v(e), [\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}], [\begin{smallmatrix} D(e) \\ D(e) \end{smallmatrix}]))$ as its vacuum vector. Our Hilbert space is $\widehat{\mathcal{H}} = \bigotimes_{t \in T} (\mathcal{H}_t \oplus \mathbb{C}^2 \oplus \mathbb{C}^2)$.

Define for $g \equiv g_1 \dots g_n \in G(\Gamma)$,

$$\widehat{\alpha}(g) := \bigotimes_{0 < i \leq n} \begin{cases} (\alpha(g_i), [\begin{smallmatrix} C^\alpha(g_i, g_i) \\ 0 \end{smallmatrix}], [\begin{smallmatrix} C^\alpha(g_i, e) \\ 0 \end{smallmatrix}]))_{g_1 \dots g_{i-1} G(\text{St}(i))} & \text{if } g_i \text{ is in the d-tail of } g \\ (\omega(g_i), [\begin{smallmatrix} D(g_i) \\ D(g_i) \end{smallmatrix}], [\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}]))_{g_1 \dots g_{i-1} G(\text{St}(i))} & \text{otherwise,} \end{cases}$$

and

$$\widehat{\beta}(g) := \bigotimes_{0 < i \leq n} \begin{cases} (\beta(g_i), [\begin{smallmatrix} 0 \\ C^\beta(g_i, g_i) \end{smallmatrix}], [\begin{smallmatrix} 0 \\ C^\beta(g_i, e) \end{smallmatrix}]))_{g_1 \dots g_{i-1} G(\text{St}(i))} & \text{if } g_i \text{ is in the d-tail of } g \\ (\omega(g_i), [\begin{smallmatrix} D(g_i) \\ D(g_i) \end{smallmatrix}], [\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}]))_{g_1 \dots g_{i-1} G(\text{St}(i))} & \text{otherwise.} \end{cases}$$

These are maps from $G(\Gamma)$ to $\widehat{\mathcal{H}}$. Uncompressed and rearranged so that the d-tail is rightmost in the word, we have

$$\begin{aligned} \widehat{\alpha}(g) = & (\omega(g_1), [\begin{smallmatrix} D(g_1) \\ D(g_1) \end{smallmatrix}], [\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}]))_{G(\text{St}(1))} \otimes \dots \otimes (\omega(g_{j-1}), [\begin{smallmatrix} D(g_{j-1}) \\ D(g_{j-1}) \end{smallmatrix}], [\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}]))_{g_1 \dots g_{j-2} G(\text{St}(j-1))} \\ & \otimes (\alpha(g_j), [\begin{smallmatrix} C^\alpha(g_j, g_j) \\ 0 \end{smallmatrix}], [\begin{smallmatrix} C^\alpha(g_j, e) \\ 0 \end{smallmatrix}]))_{g_1 \dots g_{j-1} G(\text{St}(j))} \otimes \\ & \dots \otimes (\alpha(g_n), [\begin{smallmatrix} C^\alpha(g_n, g_n) \\ 0 \end{smallmatrix}], [\begin{smallmatrix} C^\alpha(g_n, e) \\ 0 \end{smallmatrix}]))_{g_1 \dots g_{n-1} G(\text{St}(n))} \end{aligned}$$

where $j > n - dk$ by Lemma 2.5.5. Note that the choice functions in $\widehat{\alpha}$ and $\widehat{\beta}$ are well defined, as membership in the d-tail is independent of representation. Let us show that the indexing of $\widehat{\alpha}$ and $\widehat{\beta}$ is well defined.

We shall focus on $\widehat{\alpha}$, as the same arguments hold for $\widehat{\beta}$. Suppose we have $g \in G(\Gamma)$ and two equivalent reduced words of g which differ by a single syllable shuffle, say

$$g \equiv g_1 \dots g_i g_{i+1} \dots g_n \equiv g_1 \dots g_{i+1} g_i \dots g_n$$

where $G(i)$ and $G(i+1)$ commute. Then we have two equalities:

$$\begin{aligned} g_1 \dots g_{i-1} g_i G(\text{St}(i+1)) &= g_1 \dots g_{i-1} G(\text{St}(i+1)) \text{ and} \\ g_1 \dots g_{i-1} g_{i+1} G(\text{St}(i)) &= g_1 \dots g_{i-1} G(\text{St}(i)), \end{aligned}$$

as g_i is in $G(\text{St}(i+1))$ and g_{i+1} is in $G(\text{St}(i))$. This implies that the index cosets corresponding to g_i in $\widehat{\alpha}(g_1 \dots g_i g_{i+1} \dots g_n)$ and $\widehat{\alpha}(g_1 \dots g_{i+1} g_i \dots g_n)$ are the same, and likewise for the index coset of g_{i+1} . For any other syllable g_j of g , the coset $g_1 \dots g_{j-1} G(\text{St}(j))$ is not effected by shuffling g_i and g_{i+1} , hence $\widehat{\alpha}(g_1 \dots g_i g_{i+1} \dots g_n) = \widehat{\alpha}(g_1 \dots g_{i+1} g_i \dots g_n)$. Any other reduced word representing g may be reached by a finite sequence of syllable shuffles, so we may unambiguously define $\widehat{\alpha}(g)$ (similarly $\widehat{\beta}(g)$ is well defined).

Let us show that $\widehat{\alpha}(g)$ and $\widehat{\beta}(g)$ are bounded. First, C^α and C^β are bounded by $8\varepsilon^{1/4}$ by Lemma 4.2.3. Since the only non-unit vectors of $\widehat{\alpha}(g)$ correspond to syllables in the d -tail of g , of which there are at most $d\kappa$ by Lemma 2.5.5, we have

$$\begin{aligned} \|\widehat{\alpha}(g)\| &= \prod_{\{g_i \text{ in the } d\text{-tail of } g\}} \|(\alpha(g_i), [C^\alpha(g_i, g_i)], [C^\alpha(g_i, e)])\| \\ &= \prod_{\{g_i \text{ in the } d\text{-tail of } g\}} \sqrt{\|\alpha_i(g_i)\|^2 + |C^\alpha(g_i, g_i)|^2 + |C^\alpha(g_i, e)|^2} \\ &\leq (1 + \varepsilon + 128\sqrt{\varepsilon})^{\frac{d\kappa}{2}} \leq (1 + 129\sqrt{\varepsilon})^{\frac{d\kappa}{2}} \end{aligned}$$

when $\varepsilon < 1$. We have the same inequality for $\|\widehat{\beta}(g)\|$.

The kernel

$$\phi_d(g, h) := \langle \widehat{\alpha}(g), \widehat{\beta}(h) \rangle$$

is then completely bounded with cb-norm $\|\phi_d\|_{cb} = \sup_{g, h \in G} \|\widehat{\alpha}(g)\| \|\widehat{\beta}(h)\| \leq (1 + 129\sqrt{\varepsilon})^d$.

For a word $g \equiv g_1 \dots g_d$ of length d in $G(\Gamma)$, we have

$$\begin{aligned} \phi_d(g, e) &= \prod_{i=1}^d \left\langle (\alpha(g_i), [C^\alpha(g_i, g_i)], [C^\alpha(g_i, e)]), (\omega(e), [\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}], [\begin{smallmatrix} D(e) \\ D(e) \end{smallmatrix}])) \right\rangle_{g_1 \dots g_{i-1} G(\text{St}(i))} \\ &= \prod_{i=1}^d \langle \alpha(g_i), \beta(e) \rangle = f(g_1) f(g_2) \dots f(g_d). \end{aligned}$$

where the second equality follows from calculation and the definition of $C^\alpha(g_i, g_i)$. $\square$

For the rest of the section, ϕ_d refers to the kernel constructed in Proposition 5.0.5. As in the free product case, we designed ϕ_d to be the kernel of a function restricted to words of length d :

Lemma 5.0.6. *For a pair $g, h \in G(\Gamma)$ such that $|h^{-1}g|_r = d$, we have that $\phi_d(g, h) = \phi_d(h^{-1}g, e)$.*

Proof. Let us show that $\phi_d(g, h) = \phi_d(h^{-1}g, e)$ whenever $|h^{-1}g|_r = d$.

Suppose g and h are elements of $G(\Gamma)$, and let $n = |g|_r$, $m = |h|_r$, and $d = |h^{-1}g|_r$. By Lemma 2.5.2, there are reduced representations $g \equiv g_1 \dots g_n$ and $h \equiv h_1 \dots h_m$ such that $g_1 \dots g_q = h_1 \dots h_q$ and

$$h^{-1}g \equiv h_m^{-1} \dots h_{q+p+1}^{-1} (h_{q+1}^{-1} g_{q+1}) \dots (h_{q+p}^{-1} g_{q+p}) g_{q+p+1} \dots g_n$$

for some $q \leq \min(m, n)$ and $p \leq \kappa$.

We showed in the proposition above that

$$\phi_d(h^{-1}g, e) = \prod_{i=q+p+1}^m f(h_i^{-1}) \prod_{i=q+1}^{q+p} f(h_i^{-1} g_i) \prod_{i=q+p+1}^n f(g_i).$$

. Now we analyze $\phi_d(g, h) = \langle \widehat{\alpha}(g), \widehat{\beta}(h) \rangle$. Suppose $i \leq q$, in which case $g_i = h_i$ and $g_1 \dots g_{i-1} G(\text{St}(i)) = h_1 \dots h_{i-1} G(\text{St}(i))$. Then the $g_1 \dots g_{i-1} G(\text{St}(i))$ factor of $\langle \widehat{\alpha}(g), \widehat{\beta}(h) \rangle$ is one of the following:

$$\begin{aligned} & \langle (\alpha(g_i), [C^{\alpha(g_i, g_i)}_0], [C^{\alpha(g_i, e)}_0]), (\beta(g_i), [C^{\beta(g_i, g_i)}_0], [C^{\beta(g_i, e)}_0]) \rangle, \\ & \langle (\alpha(g_i), [C^{\alpha(g_i, g_i)}_0], [C^{\alpha(g_i, e)}_0]), (\omega(g_i), [D^{(g_i)}_{D(g_i)}], [\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}]) \rangle, \\ & \langle (\omega(g_i), [D^{(g_i)}_{D(g_i)}], [\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}]), (\beta(g_i), [C^{\beta(g_i, g_i)}_0], [C^{\beta(g_i, e)}_0]) \rangle, \text{ or} \\ & \langle (\omega(g_i), [D^{(g_i)}_{D(g_i)}], [\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}]), (\omega(g_i), [D^{(g_i)}_{D(g_i)}], [\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}]) \rangle, \end{aligned}$$

depending on whether or not g_i is in the d -tails of g or h . In any case, the factor is equivalent to 1 by the choice of constants.

If $q < i \leq q + p$, then $g_i \neq h_i$, but they amalgamate in the product $h^{-1}g$. By Lemma 2.5.4, g_i and h_i are in the d -tails of their respective words. Since g_{q+j} and h_{q+j} are in $G(\text{St}(i))$ for all $1 \leq j < i$, we have

$$g_1 \dots g_q g_{q+1} \dots g_{i-1} G(\text{St}(i)) = g_1 \dots g_q G(\text{St}(i)) = h_1 \dots h_q h_{q+1} \dots h_{i-1} G(\text{St}(i))$$

which implies that the $g_1 \dots g_q G(\text{St}(i))$ factor of the inner product is

$$\langle (\alpha(g_i), [C^{\alpha(g_i, g_i)}_0], [C^{\alpha(g_i, e)}_0]), (\beta(h_i), [C^{\beta(h_i, h_i)}_0], [C^{\beta(h_i, e)}_0]) \rangle = \langle \alpha(g_i), \beta(h_i) \rangle.$$

For all other non-vacuum indices we either have a factor of

$$\begin{aligned} & \langle (\alpha(g_i), [C^{\alpha(g_i, g_i)}_0], [C^{\alpha(g_i, e)}_0]), (\omega(e), [\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}], [D^{(e)}_{D(e)}]) \rangle = \langle \alpha(g_i), \beta(e) \rangle, \text{ or} \\ & \langle (\omega(e), [\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}], [D^{(e)}_{D(e)}]), (\beta(h_i), [C^{\beta(h_i, h_i)}_0], [C^{\beta(h_i, e)}_0]) \rangle = \langle \alpha(e), \beta(h_i) \rangle. \end{aligned}$$

Hence

$$\begin{aligned}
\phi_d(g, h) &= \langle \hat{\alpha}(g), \hat{\beta}(h) \rangle \\
&= \prod_{i=q+1}^{q+p} \langle \alpha(g_i), \beta(h_i) \rangle \prod_{i=q+p+1}^n \langle \alpha(g_i), \beta(e) \rangle \prod_{i=q+p+1}^m \langle \alpha(e), \beta(h_i) \rangle \\
&= \prod_{i=q+1}^{q+p} \langle \alpha(h_i^{-1} g_i), \beta(e) \rangle \prod_{i=q+p+1}^n \langle \alpha(g_i), \beta(e) \rangle \prod_{i=q+p+1}^m \langle \alpha(h_i^{-1}), \beta(e) \rangle \\
&= \prod_{i=q+1}^{q+p} f(h_i^{-1} g_i) \prod_{i=q+p+1}^n f(g_i) \prod_{i=q+p+1}^m f(h_i^{-1}) = \phi_d(h^{-1} g, e)
\end{aligned}$$

and we have our desired equality. $\square$

For a natural number n and real number x , let $q(n, x) := 2\sqrt{n\kappa 129} (1 + 129\sqrt{x})^{n\kappa} x^{1/4}$. Clearly for a fixed n , the function $q(n, x)$ goes to 0 as x goes to 0.

Lemma 5.0.7. *Given the hypotheses of Proposition 5.0.5, there is a positive definite kernel Ψ on $G(\Gamma)$ of norm 1 such that $\|\phi_d - \Psi\|_{cb} \leq q(d, \varepsilon)$ for all natural numbers d .*

Proof. Let ω , D , and $\hat{\mathcal{H}}$ be defined as in Proposition 5.0.5.

Define, for $g \equiv g_1 \dots g_n$,

$$\hat{\omega}(g) := \bigotimes_{0 < i \leq n} (\omega(g_i), \begin{bmatrix} D(g_i) \\ D(g_i) \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix})_{g_1 \dots g_{i-1} G(\text{St}(i))},$$

By the analysis of $\hat{\alpha}$ in Prop 5.0.5, the indexing of $\hat{\omega}$ is well defined, and hence $\hat{\omega}$ is a well defined map from $G(\Gamma)$ to $\hat{\mathcal{H}}$. Also, $\|\hat{\omega}\| = 1$, since $\|(\omega(g), \begin{bmatrix} D(g) \\ D(g) \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix})\| = 1$ for all $g \in G(\Gamma)$ by the definition of $D(g)$.

Our positive definite kernel is then $\Psi(g, h) := \langle \hat{\omega}(g), \hat{\omega}(h) \rangle$. Note that this is not $G(\Gamma)$ -invariant.

Now let d be a natural number and let us show that $\|\phi_d - \Psi\|_{cb} \leq q(d, \varepsilon)$. Let $g \equiv g_1 \dots g_n$ and h be elements of G . Recall that $\phi_d(g, h) = \langle \hat{\alpha}(g), \hat{\beta}(h) \rangle$, where the maps $\hat{\alpha}(g)$ and $\hat{\beta}(g)$

differ from $\widehat{\omega}(g)$ in value only on the d -tail of g . Since the shared values are unit vectors, we have that

$$\begin{aligned}\langle \widehat{\alpha}(g), \widehat{\omega}(g) \rangle &= \prod_{\{g_i \text{ in } d\text{-tail of } g\}} \langle (\alpha(g_i), [C^{\alpha(g_i, g_i)}_0], [C^{\alpha(g_i, e)}_0]), (\omega(g_i), [D^{(g_i)}_{(g_i)}], [0]) \rangle \\ &= \prod_{\{g_i \text{ in } d\text{-tail of } g\}} \langle \alpha(g_i), \beta(g_i) \rangle = 1\end{aligned}$$

and hence

$$\|\widehat{\alpha}(g) - \widehat{\omega}(g)\|^2 = \|\widehat{\alpha}(g)\|^2 + \|\widehat{\omega}(g)\|^2 - 2\langle \widehat{\alpha}(g), \widehat{\omega}(g) \rangle = \|\widehat{\alpha}(g)\|^2 - 1.$$

The same analysis shows that $\|\widehat{\beta}(g) - \widehat{\omega}(g)\|^2 \leq \|\widehat{\beta}(g)\|^2 - 1$.

As for the kernels ϕ_d and Ψ themselves, we can decompose their difference into the sum of two completely bounded functions:

$$\begin{aligned}\phi_d(g, h) - \Psi(g, h) &= \langle \widehat{\alpha}(g), \widehat{\beta}(h) \rangle - \langle \widehat{\omega}(g), \widehat{\omega}(h) \rangle \\ &= \langle \widehat{\alpha}(g) - \widehat{\omega}(g), \widehat{\beta}(h) \rangle + \langle \widehat{\omega}(g), \widehat{\beta}(h) - \widehat{\omega}(h) \rangle.\end{aligned}$$

Hence

$$\begin{aligned}\|\phi_d - \Psi\|_{cb} &\leq \|\langle \widehat{\alpha}(\cdot) - \widehat{\omega}(\cdot), \widehat{\beta}(\cdot) \rangle\|_{cb} + \|\langle \widehat{\omega}(\cdot), \widehat{\beta}(\cdot) - \widehat{\omega}(\cdot) \rangle\|_{cb} \\ &\leq \sup_{g, h \in G} \|\widehat{\alpha}(g) - \widehat{\omega}(g)\| \|\widehat{\beta}(h)\| + \sup_{g, h \in G} \|\widehat{\omega}(g)\| \|\widehat{\beta}(h) - \widehat{\omega}(h)\| \\ &\leq \sup_{g, h \in G} (\|\widehat{\alpha}(g)\|^2 - 1)^{\frac{1}{2}} \|\widehat{\beta}(h)\| + \sup_{h \in G} (\|\widehat{\beta}(h)\|^2 - 1)^{\frac{1}{2}} \\ &\leq ((1 + 129\sqrt{\varepsilon})^{d\kappa} - 1)^{\frac{1}{2}} (1 + 129\sqrt{\varepsilon})^{\frac{d\kappa}{2}} + ((1 + 129\sqrt{\varepsilon})^{d\kappa} - 1)^{\frac{1}{2}} \\ &\leq (d\kappa 129 (1 + 129\sqrt{\varepsilon})^{d\kappa} \sqrt{\varepsilon})^{\frac{1}{2}} (1 + 129\sqrt{\varepsilon})^{\frac{d\kappa}{2}} + (d\kappa 129 (1 + 129\sqrt{\varepsilon})^{d\kappa} \sqrt{\varepsilon})^{\frac{1}{2}} \\ &\leq 2\sqrt{d\kappa 129} (1 + 129\sqrt{\varepsilon})^{d\kappa} \varepsilon^{\frac{1}{4}} = q(d, \varepsilon).\end{aligned}$$

which proves the lemma. $\square$

Now let Ξ_d be the characteristic function of the d sphere in the $\text{CAT}(0)$ -cube complex X from Chapter 3 and χ_d be the characteristic function of words of reduced length d in $G(\Gamma)$. For a pair $g, h \in G(\Gamma)$ (and a base point $x_0 \in X$), we have

$$\chi_d(h^{-1}g) = \Xi_{2d}(gx_0, hx_0) = \begin{cases} 1 & \text{if } |h^{-1}g|_r = d, \\ 0 & \text{else.} \end{cases}$$

We quote following theorem of Mizuta:

Theorem 5.0.8 ((Theorem 2 from [15])). *Let X be a finite-dimensional $\text{CAT}(0)$ cube complex and let $\chi_n = \{(x, y) \in X \times X : d_X(x, y) = n\}$ for $n \in \mathbb{Z}^+$. Then the norms of Schur multipliers of the characteristic function χ_n increase polynomially, i.e. there exists a polynomial p such that $\|\chi_n\|_{cb} \leq p(n)$.*

Hence $\|\chi_d\|_{cb} \leq p(d)$ for some polynomial p . Given a fixed $r > 0$ and finite sum of the form $\sum_{d=0}^M e^{-d/r} \chi_d$, we note that

$$\left\| \sum_{d=0}^M e^{-d/r} \chi_d \right\|_{cb} \leq \sum_{d=0}^M e^{-d/r} \|\chi_d\|_{cb} \leq \sum_{d=0}^M e^{-d/r} p(d),$$

which converges as M goes to infinity. The limit function $\sum_{d=0}^{\infty} e^{-d/r} \chi_d = e^{|\cdot|_r/r}$ is positive definite.

We now have everything necessary to prove the main theorem.

Theorem 5.0.9. *The graph product of weakly amenable groups (with Cowling-Haagerup constant 1) is weakly amenable.*

Proof. Suppose we are given a graph product $G(\Gamma)$ with weakly amenable vertex groups $\{G_v\}$, a small $\varepsilon > 0$, and a finite subset S of $G(\Gamma)$. We show that there is a completely bounded function F such that $|1 - F(g)| \leq \varepsilon$ for all $g \in S$ and $\|F\|_{cb} \leq 1 + \varepsilon$.

First we need to define some constants. Let $N = \max\{|g|_r : g \in S\}$. Choose δ such that, if you have $N + 1$ numbers $\{a_i\}$ each of which is at most δ away from 1, then $|1 - \prod_{i=1}^{N+1} a_i| < \varepsilon$. Choose r and M large enough such that $|1 - e^{-N/r}| < \delta$ and $\sum_{d=M+1}^{\infty} e^{-d/r} p(d) < \frac{\varepsilon}{2}$. Let $\mu > 0$ be small enough that $q(M, \mu) \sum_{d=0}^M e^{-d/r} p(d) \leq \frac{\varepsilon}{2}$.

For each v , let $S_v := \{g_v \in G_v \mid g_v \text{ is a syllable of some } g \in S\}$ and, by the weak amenability of G_v , choose a finitely supported completely bounded function f_v on G_v such that $\|f_v\|_{cb} \leq 1 + \mu$ and $|1 - f_v(g_v)| < \delta$ for all $g_v \in S_v$. Using Proposition 5.0.5, for each $d \in \mathbb{N}$, we can construct the completely bounded kernel ϕ_d on $G(\Gamma)$, and by Lemma 5.0.7 there exists an ambient positive definite kernel Ψ on $G(\Gamma)$ such that $\|\phi_d - \Psi\|_{cb} < q(d, \mu)$.

Our final function is then $F = \sum_{d=1}^M e^{-d/r} \phi_d \chi_d$. For any $g \equiv g_1 \dots g_n \in S$, we have that

$$F(g) = e^{-|g|_r/r} \phi_n(g, e) = e^{-|g|_r/r} f_1(g_1) f_2(g_2) \dots f_n(g_n)$$

and each of the at most $N + 1$ terms in this product is within δ of 1. Hence, by our definition of δ , $|1 - F(g)| < \varepsilon$. More generally, F is non-zero only on words with at most M syllables. On such a word, it is a product of f applied to each syllable. As each f is finitely supported, F is only non-zero on finitely many words. We also have that

$$\begin{aligned} \|F\|_{cb} &= \left\| \sum_{d=0}^M e^{-d/r} \phi_d \chi_d \right\|_{cb} \leq \left\| \sum_{d=0}^M e^{-d/r} (\phi_d - \Psi) \chi_d \right\|_{cb} + \left\| \sum_{d=0}^M e^{-d/r} \Psi \chi_d \right\|_{cb} \\ &\leq \sum_{d=0}^M e^{-d/r} \|\phi_d - \Psi\|_{cb} \|\chi_d\|_{cb} + \sum_{d=M+1}^{\infty} e^{-d/r} \|\Psi\|_{cb} \|\chi_d\|_{cb} + \left\| \sum_{d=0}^{\infty} e^{-d/r} \Psi \chi_d \right\|_{cb} \\ &\leq \sum_{d=0}^M e^{-d/r} q(d, \mu) p(d) + \sum_{d=M+1}^{\infty} e^{-d/r} p(d) + \|e^{-|\cdot|_r/r} \Psi\|_{cb} \\ &\leq q(M, \mu) \sum_{d=0}^M e^{-d/r} p(d) + \frac{\varepsilon}{2} + 1 \\ &\leq 1 + \varepsilon. \end{aligned}$$

The third line follows from Lemma 5.0.7 and the fact that $\|\Psi\|_{cb} = 1$.

Given a net of finite subsets that exhausts $G(\Gamma)$ and numbers ε_n which go to 0, we can construct a net of finitely supported completely bounded functions which approximate the identity of $G(\Gamma)$. Further, the cb-norms go to one, hence $G(\Gamma)$ is weakly amenable with Cowling-Haagerup constant one. $\square$

Corollary 5.0.10. *If Γ is not finite, then Theorem 5.0.9 still holds.*

Proof. Given an infinite graph Γ , we can decompose it as the direct limit of finite subgraphs Γ_n , and $G(\Gamma)$ is then the direct limit of $G(\Gamma_n)$. If the vertex groups of $G(\Gamma)$ are weakly amenable, then we may apply Theorem 5.0.9 to $G(\Gamma_n)$ for each n . We now have that $G(\Gamma)$ is the direct limit of weakly amenable groups and weak amenability is preserved by direct limits. Hence $G(\Gamma)$ is weakly amenable. $\square$

BIBLIOGRAPHY

- [1] Y. Antolín and D. Dreesen. The Haagerup property is stable under graph products. ArXiv e-prints, May 2013.
- [2] Marek Bożejko and Massimo A. Picardello. Weakly amenable groups and amalgamated products. Proc. Amer. Math. Soc., (4):1039–1046, 1993.
- [3] M. Bridson and A. Haefliger. Metric Spaces of Non-Positive Curvature. Springer-Verlag, Berlin, 1999.
- [4] Indira Chatterji and Graham Niblo. From wall spaces to $CAT(0)$ cube complexes. Internat. J. Algebra Comput., 15(5-6):875–885, 2005.
- [5] Xiaoman Chen, Marius Dadarlat, Erik Guentner, and Guoliang Yu. Uniform embeddability and exactness of free products. J. Funct. Anal., 205(1):168–179, 2003.
- [6] Pierre-Alain Cherix, Michael Cowling, Paul Jolissaint, Pierre Julg, and Alain Valette. Groups with the Haagerup property, volume 197 of Progress in Mathematics. Birkhäuser Verlag, Basel, 2001. Gromov’s a-T-menability.
- [7] Yves Cornuier, Yves Stalder, and Alain Valette. Proper actions of wreath products and generalizations. Trans. Amer. Math. Soc., 364(6):3159–3184, 2012.
- [8] Michael Cowling and Uffe Haagerup. Completely bounded multipliers of the Fourier algebra of a simple Lie group of real rank one. Invent. Math., 96(3):507–549, 1989.
- [9] Michael W. Davis. The geometry and topology of Coxeter groups, volume 32 of London Mathematical Society Monographs Series. Princeton University Press, Princeton, NJ, 2008.

- [10] Jean De Cannière and Uffe Haagerup. Multipliers of the Fourier algebras of some simple Lie groups and their discrete subgroups. Amer. J. Math., 107(2):455–500, 1985.
- [11] Elisabeth Ruth Green. Graph products of groups. March 1990.
- [12] Michael Gromov. Rigid transformations groups. In Géométrie différentielle (Paris, 1986), volume 33 of Travaux en Cours, pages 65–139. Hermann, Paris, 1988.
- [13] Erik Guentner and Nigel Higson. Weak amenability of $\text{CAT}(0)$ -cubical groups. Geom. Dedicata, 148:137–156, 2010.
- [14] Uffe Haagerup. An example of a nonnuclear C^* -algebra, which has the metric approximation property. Invent. Math., 50(3):279–293, 1978/79.
- [15] Naokazu Mizuta. A Bożejko-Picardello type inequality for finite-dimensional $\text{CAT}(0)$ cube complexes. J. Funct. Anal., 254(3):760–772, 2008.
- [16] Bogdan Nica. Cubulating spaces with walls. Algebr. Geom. Topol., 4:297–309 (electronic), 2004.
- [17] Narutaka Ozawa and Sorin Popa. On a class of II_1 factors with at most one Cartan subalgebra. Ann. of Math. (2), 172(1):713–749, 2010.
- [18] Vern Paulsen. Completely bounded maps and operator algebras, volume 78 of Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 2002.
- [19] Gilles Pisier. Introduction to operator space theory, volume 294 of London Mathematical Society Lecture Note Series. Cambridge University Press, Cambridge, 2003.
- [20] Éric Ricard and Quanhua Xu. Khintchine type inequalities for reduced free products and applications. J. Reine Angew. Math., 599:27–59, 2006.
- [21] M. Roller. Poc sets, median algebras and group actions. Preprint, 1998.

- [22] K. Ruane and S. Witzel. CAT(0) cubical complexes for graph products of finitely generated abelian groups. ArXiv e-prints, October 2013.
- [23] Michah Sageev. Ends of group pairs and non-positively curved cube complexes. Proc. London Math. Soc. (3), 71(3):585–617, 1995.