INFLATION OF FINITE LATTICES ALONG ALL-OR-NOTHING SETS

A DISSERTATION SUBMITTED TO THE GRADUATE DIVISION OF THE
UNIVERSITY OF HAWAII AT MĀNOA IN PARTIAL FULFILLMENT OF THE
REQUIREMENTS FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

IN

MATHEMATICS

MAY 2015

By

Tristan Holmes

Dissertation Committee:

J.B. Nation, Chairperson

Ralph Freese

Bjørn Kjos-Hanssen

Michelle Manes

Anders Høst Madsen

Dedicated to dad, John Eric Holmes.

ACKNOWLEDGMENTS

First and foremost I would like to thank my doctoral adviser J.B. Nation. Without his guidance, patience, and constructive criticism this project would never have been possible. I would also like to thank Ralph Freese and Bill Lampe who, along with J.B., introduced me to the fascinating world of lattice theory and universal algebra. My thanks go to all faculty, staff, and students at the University of Hawai'i at Mānoa who have made my studies not only possible, but very enjoyable.

My thanks go to all the teachers and professors I have had over the years who helped me reach this point. In particular I would like to thank Mrs. Taylor, Ms. Meskimen, and Mr. Zarazza. The time I spent learning with them still inspires me today. I thank my friends and family for their continued love and support, in all aspects of life. I could never have done it without you.

ABSTRACT

We introduce a new generalization of Alan Day's doubling construction. For ordered sets $\mathcal{L}$ and $\mathcal{K}$ and a subset $E \subseteq \leq_{\mathcal{L}}$ we define the ordered set $\mathcal{L} \star_E \mathcal{K}$ arising from inflation of $\mathcal{L}$ along E by $\mathcal{K}$. Under the restriction that $\mathcal{L}$ and $\mathcal{K}$ are lattices and $\mathcal{K}$ has maximum and minimum elements, we find those subsets $E \subseteq \leq_{\mathcal{L}}$ such that the ordered set $\mathcal{L} \star_E \mathcal{K}$ is a lattice. Finite lattices that can be constructed in this way are classified in terms of their congruence lattices.

A finite lattice is binary cut-through codable if and only if there exists a $0 - 1$ spanning chain $\{\theta_i : 0 \leq i \leq n\}$ in $Con(\mathcal{L})$ such that the cardinality of the largest block of θ_i/θ_{i-1} is two for every i with $1 \leq i \leq n$. These are exactly the lattices that can be obtained from the one element lattice using a sequence of inflations by the two element lattice. The structure of binary cut-through codable lattices is studied, and those lattices that generate varieties that are binary cut-through codable are characterized.

TABLE OF CONTENTS

Acknowledgments	iii
Abstract	iv
List of Figures	vii
List of Symbols	viii
Preface	x
1 Day’s Doubling Construction	1
1.1 Basic concepts of lattice theory	1
1.2 Day’s Doubling: Intervals	2
1.3 Upper and Lower Bounded Lattices	3
1.4 Congruence Normal Lattices	9
2 Constructing Finite Lattices by Inflation	14
2.1 Motivating Examples	14
2.2 Inflation of Partially Ordered Sets	15
3 Inflation and Congruences	21
3.1 Introduction	21
3.2 Classification of Inflatable Lattices	21
3.3 Consequences of the Classification Theorem	25
4 Binary Cut-Through Codable Lattices	29
4.1 Introduction	29
4.2 Cut-Through Codability	29
4.3 Lattice Varieties and Cut-through Codes	33
4.4 A Summary of Known Classes of Inflatable Lattices	35
5 Open Problems	37

Appendix	39
Bibliography	42

LIST OF FIGURES

1.1	Doublings of lower pseudo-intervals	7
1.2	The lattice $\mathcal{M}_3$	8
1.3	$\mathcal{L}[C]$ is neither lower nor upper bounded.	9
1.4	For no $k \in \omega$ is $F_k(\mathcal{L}) = JI(\mathcal{L})$	10
2.1	A doubling that fails to produce a lattice.	14
2.2	Reppe's lattice.	14
2.3	A, so far, unexplained doubling.	15
2.4	The edge sets E (left), and F (right).	16
2.5	A “twisted” doubling (left), and doubling (right).	17
2.6	Inflation by the ordered set $\mathcal{M}_3$	17
3.1	A lattice that is not inflatable.	21
3.2	A lattice $\mathcal{L}$ that fails condition iii	23
3.3	An illustration of the proof that E is an all-or-nothing set.	24
3.4	The congruence normal lattice $\mathcal{L}$	26
3.5	The lattice $\mathcal{L} \star_E \mathbf{2}$ (left) and $Con(\mathcal{L} \star_E \mathbf{2})$ (right).	26
3.6	An image of $\mathcal{L} \star_E \mathbf{2}$ that is not inflatable.	27
3.7	A lattice $\mathcal{L}$ disproving the converse of Theorem 3.4.	28
4.1	The construction of the binary cut-through code.	30
4.2	A binary cut-through code (left), and inflation sequence (right).	30
4.3	The lattice $\mathcal{L}$ (left), and its image $\mathcal{L}/\Phi_p$ (right).	32
4.4	The lattice $\mathcal{L}$ (left), and $Con(\mathcal{L})$ (right).	36
4.5	The lattice $\mathcal{L}$ (left), and $Con(\mathcal{L})$ (right).	36

LIST OF SYMBOLS

- $A \subseteq B, A \subset B$: Set inclusion, proper set inclusion.
- $\mathcal{L} \cong \mathcal{K}$: Isomorphism of algebras.
- $a \vee b, \bigvee S$: Join, least upper bound.
- $a \wedge b, \bigwedge S$: Meet, greatest lower bound.
- $\mathcal{L} = \langle L, \vee, \wedge \rangle$: A lattice on the set L .
- $a \prec b$: a is covered by b .
- b/a : The interval from a to b .
- $0_{\mathcal{L}}, 1_{\mathcal{L}}$: The minimum, maximum elements of $\mathcal{L}$.
- $JI(\mathcal{L})$: The join irreducible elements of $\mathcal{L}$.
- $MI(\mathcal{L})$: The meet irreducible elements of $\mathcal{L}$.
- p_*, m^* : The unique lower, upper cover of p, m .
- $\uparrow a, \downarrow a$: The principal ideal, principal filter generated by a .
- **1, 2**: The one element lattice, two element lattice.
- $\mathcal{L}[C]$: The ordered set $\mathcal{L}$ with a convex set C doubled.
- **FL**(X): The free lattice generated by X .
- $\beta(a), \beta_h(a)$: The least pre-image of a .
- **1, 2**: The one element lattice, two element lattice.
- $\alpha(a), \alpha_h(a)$: The greatest pre-image of a .
- $A \ll B$: Refinement.
- $p D q$: Join dependency.
- $Cg(a, b)$: The congruence generated by (a, b) .
- Φ_p, Φ^m : The congruence $Cg(p, p_*), Cg(m, m^*)$.
- Ψ_p, Ψ^m : The largest congruence not containing, $(p, p_*), (m, m^*)$.
- $T^\vee, T^\wedge$: The join closure, meet closure of T .
- $Con(\mathcal{L})$: The lattice of congruences on $\mathcal{L}$.
- $rk_D(p), rk_F(p)$: The D -rank, F -rank of p .
- ω : The countable ordinal.
- $b/a \approx v/u$: The interval b/a is projective to the interval v/u .

- $\mathcal{L} \star_E \mathcal{K}$: The ordered set $\mathcal{L}$ with edges E inflated by $\mathcal{K}$.
- $\|\theta\|$: Cardinality of the largest block of θ .
- $Ct(\mathcal{L})$: Cut-through index of $\mathcal{L}$.
- $Var(\mathcal{L})$: The variety generated by $\mathcal{L}$.
- Λ : The lattice of lattice varieties ordered by inclusion.

PREFACE

Lattice theory resides at the intersection of the study of ordered sets and universal algebra. Its primary application in other areas of mathematics is to study the relationship between the lattice of congruence relations on a given algebra and the properties of that algebra. Thus lattice theory is used in some guise in disciplines such group theory and ring theory, as well as in the study of varieties of algebras.

In addition to providing excellent tools for the study of varieties of algebras, lattices provide a wealth of examples of varieties to study. Indeed, in many ways varieties of lattices are more well behaved than varieties of other types of algebras such as groups. This has led to the discovery of the structure at the bottom of Λ , the lattice of lattice varieties ordered by inclusion.

In [D 79], Alan Day showed there was a connection between finite lattices in certain varieties near the bottom of Λ , and structural properties that were preserved by his doubling construction. Specifically, finite lattices that are lower bounded are precisely those that can be constructed from the one element lattice by sequentially doubling a certain class of convex sets. Bounded homomorphisms of lattices are an important tool in the study of free lattices.

Day's construction is applicable to any convex subset of a finite lattice. In the years after the classification of finite bounded lattices, some progress was made in expanding the scope of Day's construction. Day himself classified those finite lattices that are constructible by doubling arbitrary convex sets in [D 94]. This was not quite complete at the time of Day's death, and J.B. Nation helped finish the work with [N 94]. In [N1 95] Nation observed that it was possible to construct new lattices by doubling sets that were not necessarily convex, but he did not classify such sets.

Progress toward any such characterization was not made until Heiko Reppe published his thesis [R 11]. Reppe classified those subsets of a lattice that would yield a lattice when doubled according to Day's doubling construction, and called them *municipal* sets. The present study uses a modification of Day's doubling construction to classify those finite lattices that are sublattices of lattices constructible by doubling municipal sets.

In the first chapter we present Day's doubling construction, and the classes of finite lattices that can be obtained using this construction. In Chapter 2 we introduce our modification of Day's doubling that allows us to construct additional finite lattices. We call this modified construction *inflation*. In Chapter 3 we classify those finite lattices that can be constructed using the inflation construction. We restrict our attention to inflation by the two element lattice in Chapter 4. We classify lattices that generate varieties in which every finite lattice can be constructed by inflation by the two element lattice. Finally, we present possible directions for future research in Chapter 5.

CHAPTER 1

DAY'S DOUBLING CONSTRUCTION

1.1 Basic concepts of lattice theory

We begin this chapter with a very brief overview of the basic concepts and definitions of lattice theory. For further details, the reader is referred to Nation [N2 95] and the references therein.

A *partial order* on a set X is a binary relation that satisfies

- $x \leq x$, for all $x \in X$ (reflexivity),
- $x \leq y$ and $y \leq x$ implies $x = y$, for all $x, y \in X$ (antisymmetry),
- $x \leq y$ and $y \leq x$ implies $x \leq z$, for all $x, y, z \in X$ (transitivity).

A *partially ordered set*, often *ordered set*, is a pair $P = \langle X, \leq \rangle$ where X is a set and $\leq$ is a partial order on X . The relation $\geq$ is defined by $x \geq y$ if and only if $y \leq x$. The ordered set $\langle X, \geq \rangle$ is known as the *dual* of $\langle X, \leq \rangle$. Every concept and theorem about ordered sets has a dual obtained by reversing the roles of $\leq$ and $\geq$. In proofs we often use the phrase “by duality” to express this symmetry and avoid repetitive arguments. As is standard for the notation used, $x < y$ if and only if $x \leq y$ and $x \neq y$, while $x > y$ is defined dually.

If a and b are elements in an ordered set P with $a < b$ such that there exists no $c \in P$ such that $a < c < b$, then we say that a is *covered by* b , and write $a \prec b$. In this situation we also say that b *covers* a . In addition we say that b is an *upper cover* of a and that a is a *lower cover* of b . We also define an equivalence relation on P by saying c is equivalent to d if there is a finite sequence $c = c_0, c_1, \dots, c_n = d$ such that c_i either covers or is covered by c_{i+1} . We say that a subset of P which is a block of this relation is called a *connected set*. If $a \leq b$, then we let b/a denote the interval $\{x : a \leq x \leq b\}$.

A relation which is reflexive and transitive, but not necessarily anti-symmetric, is called a *quasi-order*. If $\sqsubseteq$ is a quasi-order on a set X , then setting $a \equiv b$ if and only if $a \sqsubseteq b$ and $b \sqsubseteq a$ defines an equivalence relation on X . The quasi-order $\sqsubseteq$ then induces a natural partial order on $X/\equiv$.

A *chain* C in an ordered set P is a subset of X such that any two elements of C are comparable, i.e., if $x \in C$ and $y \in C$ then either $x \leq y$ or $y \leq x$. An *antichain* is a subset A of P such that no two elements of A are comparable.

Let S be a subset of P . When the set S has a least upper bound in P we refer to it as the *join* of S and denote it $\bigvee S$. Dually, when S has a greatest lower bound we call it the *meet* of S and denote it $\bigwedge S$. When $S = \{a, b\}$ we denote the join and meet by $a \vee b$ and $a \wedge b$ respectively.

A *semilattice* is an algebra $\mathcal{S} = \langle S, * \rangle$ with a single binary operation that is idempotent, commutative, and associative. One can define an order on a semilattice by $x \leq y$ if and only if $x = x * y$. With this ordering we say $\mathcal{S}$ is a *meet semilattice*, whereas the dual ordering forms a *join semilattice*.

A *lattice* is an algebra $\mathcal{L} = \langle L, \vee, \wedge \rangle$ with two binary operations that are both idempotent, commutative, and associative, and satisfy the following *absorptive laws*:

$$\begin{aligned} x \vee (y \wedge x) &= x \\ x \wedge (y \vee x) &= x \end{aligned}$$

for all $x, y \in L$. If $\mathcal{L} = \langle L, \vee, \wedge \rangle$ is a lattice, we can define a partial order on L by $x \leq y$ if and only if $x \wedge y = x$ or equivalently $x \vee y = y$. Under this partial order, $x \wedge y$ is the greatest lower bound of x and y . Conversely, an ordered set $\langle L, \leq \rangle$ such that each pair of elements of L has both a greatest lower bound and a least upper bound defines a lattice.

A lattice $\mathcal{L}$ is called *distributive* if the following identities hold for every $x, y, z \in L$:

$$\begin{aligned} x \wedge (y \vee z) &= (x \wedge y) \vee (x \wedge z) \\ x \vee (y \wedge z) &= (x \vee y) \wedge (x \vee z). \end{aligned}$$

A lattice $\mathcal{L}$ is called *modular*, if for every $x, y, z \in L$, if $x \leq z$ then

$$x \vee (y \wedge z) = (x \vee y) \wedge z.$$

It is an elementary exercise to show that every distributive lattice is modular, but that the converse is false.

If a lattice $\mathcal{L}$ has a least element, it is denoted $0_{\mathcal{L}}$, and if it has a greatest element, it is denoted by $1_{\mathcal{L}}$. We wish for any finite set to have a join and meet in a $\mathcal{L}$. As $\emptyset$ is finite, we define $0_{\mathcal{L}} = \bigvee \emptyset$, and $1_{\mathcal{L}} = \bigwedge \emptyset$. In an infinite lattice that has no greatest (respectively, least) element the meet (respectively, join) of $\emptyset$ is undefined.

An element p of a lattice $\mathcal{L}$ is *join irreducible* if $p = a \vee b$ implies that $a = p$ or $b = p$. An element p is *join prime* if $p \leq a \vee b$ implies that $p \leq a$ or $p \leq b$. The terms *meet irreducible* and *meet prime* are defined dually. If a lattice is finite every non-zero join irreducible element p has a unique lower cover, denoted p_* . For a non-maximum meet irreducible element m in a finite lattice, its unique upper cover is denoted m^* . Let $JI(\mathcal{L})$ denote the join irreducible elements of $\mathcal{L}$, and let $MI(\mathcal{L})$ denote the meet irreducible elements of $\mathcal{L}$.

An *order ideal* in an ordered set P is a subset S of P such that whenever $a \leq b$ and $b \in S$, then $a \in S$. An *order filter* is defined dually. The order ideal $\{x \in P : x \leq a\}$ is denoted $\downarrow a$ and called a *principal ideal*. A *principal filter* is defined dually and denoted $\uparrow a$. These definitions remain valid for $P = \langle X, \sqsubseteq \rangle$ where $\sqsubseteq$ is a quasi-order, the only difference being that if a is an element of a given order ideal (or order filter) then so is every $b \in a / \equiv$.

A subset I of a lattice $\mathcal{L}$ is called an *ideal* if it is an order ideal and it is closed under finite joins. A *filter* is defined dually. If $S \subseteq L$, then the ideal generated by S , the smallest ideal containing S , consists of all elements $a \in L$ such that $a \leq s_1 \vee \dots \vee s_k$ for some $s_1, \dots, s_k \in S$. The filter generated by a set $S \subseteq L$ is constructed dually. If a is an element of a lattice it follows that $\downarrow a$ and $\uparrow a$ are a lattice ideal and filter, respectively.

1.2 Day's Doubling: Intervals

The earliest use of the inflation construction was by Alan Day, who used it as a new proof of the correctness of Whitman's solution to the word problem in free lattices in [D 70]. Again, we follow the presentation in Freese, Ježek, and Nation [FJN 95]. The history here is actually rather convoluted. An earlier, more complicated, proof of this result had been presented by Ph. M. Whitman in [W 41] and [W 42] just before World War II. However, a different solution had in fact been found independently by Thoralf Skolem in 1920 (reprinted in [S 70]), but had been forgotten during the 1930s.

Because Whitman's paper became the standard reference on the subject, the key component of the result is known as Whitman's Condition, and it is here that Day found the first use for inflation.

Let $\mathcal{L}$ be a lattice. A subset C of L is *convex* if whenever a and c are in C and $a \leq b \leq c$, then $b \in C$. Of course an interval of a lattice is a convex set. Other examples of convex sets are *lower pseudo-intervals*, which are finite unions of intervals that share the same least element. An *upper pseudo-interval* is the dual concept, i.e., finite unions of intervals that share the same greatest element.

Let C be a convex subset of a lattice $\mathcal{L}$ and let $L[C]$ denote the disjoint union $(L \setminus C) \cup (C \times 2)$. Order $L[C]$ by $x \leq y$ if one of the following conditions holds:

- i $x, y \in L \setminus C$ and $x \leq_{\mathcal{L}} y$,

- ii $(x, i), (y, j) \in C \times 2$, and $x \leq_{\mathcal{L} \times 2} y$,
- iii $x \in L \setminus C, (y, j) \in C \times 2$, and $x \leq_{\mathcal{L}} y$,
- iv $(x, i) \in C \times 2, y \in L \setminus C$, and $x \leq_{\mathcal{L}} y$.

There is a natural map $h: \mathcal{L}[C] \rightarrow \mathcal{L}$ defined by

- $h(x) = x$ if $x \in L \setminus C$,
- $h(x, i) = x$ if $x \in C$.

In the next chapter we will prove a generalized result that demonstrates that under this order $\mathcal{L}[C]$ is a lattice, denoted $\mathcal{L}[C]$, and h is a lattice epimorphism.

A *free lattice* $\mathbf{FL}(X)$ is a lattice generated by the set X that satisfies the following universal mapping property: given a lattice $\mathcal{L}$, any map from $\phi: X \rightarrow \mathcal{L}$ can be extended to a homomorphism $h: \mathbf{FL}(X) \rightarrow \mathcal{L}$.

Theorem 1.1. *The free lattice $\mathbf{FL}(X)$ satisfies the following condition, denoted by (W) : if*

$$v = v_1 \wedge \dots \wedge v_r \leq u_1 \vee \dots \vee u_s = u,$$

then either $v_i \leq u$ for some i , or $v \leq u_j$ for some j .

Proof. Suppose $v = v_1 \wedge \dots \wedge v_r \leq u_1 \vee \dots \vee u_s = u$ but that $v_i \not\leq u$ and $v \not\leq u_j$ for all i and j . If $v \leq x \leq u$ for some $x \in X$, then since x is a generating element of the free lattice, $v_i \leq x \leq u$ for some i , contrary to supposition. Let I be the interval u/v and let $\mathbf{FL}(X)[I]$ be the lattice obtained by doubling I . From above, we know that $I \cap X = \emptyset$. It follows that X is a subset of $\mathbf{FL}(X)[I]$, whence the identity map on X extends to a homomorphism $f: \mathbf{FL}(X) \rightarrow \mathbf{FL}(X)[I]$. Since $x \notin I$, it follows that $h(x) = x$, where h is the canonical homomorphism. Hence $h(f(w)) = w$ for every $w \in \mathbf{FL}(X)$ and this implies $f(w) = w$ if $w \notin I$. It follows from the definition of the partial order on $\mathbf{FL}(X)[I]$ we have that

$$\begin{aligned} f(v) &= f(v_1) \wedge \dots \wedge f(v_r) \\ &= v_1 \wedge \dots \wedge v_r = (v, 1) \\ &\not\leq (u, 0) = u_1 \vee \dots \vee u_s \\ &= f(u_1) \vee \dots \vee f(u_s) = f(u). \end{aligned}$$

This contradicts the assumption that $v \leq u$ in $\mathbf{FL}(X)$. □

1.3 Upper and Lower Bounded Lattices

For a short time Day's contemporaries considered this doubling construction to be nothing more than a shortcut to proving that free lattices satisfy Whitman's Condition. However, in his paper [D 79] he showed there was more to the construction. Day was able to classify the so called *lower bounded* lattices introduced by McKenzie in [M 72]. To examine the first such result, we need to introduce some new concepts. What follows below is a presentation of the results in Chapter II of Freese, Ježek, and Nation [FJN 95] that are necessary for us to understand the more general results in the next chapter.

Let $\mathcal{K}$ and $\mathcal{L}$ be lattices. A homomorphism $h: \mathcal{K} \rightarrow \mathcal{L}$ is called *lower bounded* if for every $a \in \mathcal{L}$, the set $h^{-1}(\uparrow a)$ is either empty or has a least element. The least element of a nonempty $h^{-1}(\uparrow a)$ will be denoted $\beta_h(a)$, or if the context is clear $\beta(a)$. If h is lower bounded, $\beta_h: \mathcal{L} \rightarrow \mathcal{K}$ is a partial mapping whose domain is an ideal of $\mathcal{L}$. Dually, h is called *upper bounded* if $h^{-1}(\downarrow a)$ is empty or has a greatest element, which is denoted $\alpha_h(a)$ or $\alpha(a)$, whenever it is nonempty. For an upper bounded

homomorphism, the domain of α_h is a filter of $\mathcal{L}$. A homomorphism which is both upper and lower bounded is called *bounded*.

When h is an epimorphism, h is lower bounded if and only if $h^{-1}(a)$ has a least element for every $a \in \mathcal{L}$. Likewise, when $\mathcal{L}$ is finite, $h: \mathcal{K} \rightarrow \mathcal{L}$ is lower bounded if and only if $h^{-1}(a)$ has a least element whenever it is nonempty. On the other hand, every homomorphism with domain $\mathcal{K}$ where $\mathcal{K}$ is finite is bounded.

Note that β is monotonic and a left adjoint for h , that is $a \leq h(x)$ if and only if $\beta(a) \leq x$. Thus, $\beta(x \vee y) \geq \beta(x) \vee \beta(y)$. Further, as

$$h(\beta(x) \vee \beta(y)) = h(\beta(x)) \vee h(\beta(y)) \geq x \vee y,$$

we have that $\beta(x) \vee \beta(y) \geq \beta(x \vee y)$. It follows that β is join preserving on its domain, i.e., if $h^{-1}(\uparrow a) \neq \emptyset$, and $h^{-1}(\uparrow b) \neq \emptyset$ then $\beta(a \vee b) = \beta(a) \vee \beta(b)$. Similarly, α is a right adjoint of h , i.e., $h(y) \leq a$ if and only if $y \leq \alpha(a)$, and α is a meet preserving map on its domain. In particular, if h is an epimorphism, then α and β are meet and join homomorphisms respectively.

For an element a of a lattice $\mathcal{L}$, we say that a finite set $S \subseteq \mathcal{L}$ is a *join cover* of a if $a \leq \bigvee S$. If $a \not\leq s$ for all $s \in S$, we say S is a *nontrivial join cover*. Otherwise S is called *trivial*. For $A, B \subseteq \mathcal{L}$ we say A *refines* B , denoted $A \ll B$, if for every $a \in A$ there exists $b \in B$ such that $a \leq b$. Note that $A \ll B$ implies $\bigvee A \leq \bigvee B$.

We define a *minimal non-trivial join cover* of $a \in \mathcal{L}$ to be a non-trivial join cover S with the property that if $a \leq \bigvee T$ and $T \ll S$, then $S \subseteq T$. This can be shown to be equivalent to the following, more intuitive, definition: a non-trivial join cover S is minimal if and only if S is an antichain of join irreducible elements of $\mathcal{L}$, and whenever any element of S is deleted or replaced by a (finite) set of strictly smaller elements then the resulting set is no longer a join cover of a .

Following Day, we define the *join dependency relation* D on $JI(\mathcal{L})$ by $p D q$ if $p \neq q$ and there exists $x \in \mathcal{L}$ with $p \leq q \vee x$, and $p \not\leq a \vee x$ for every $a < q$. When $\mathcal{L}$ is finite we can replace the final condition with $p \not\leq q_* \vee x$. Note that the join prime elements of $\mathcal{L}$ are those such that p depends on no other join irreducibles in $\mathcal{L}$.

The D relation is of the utmost importance in analyzing the structure of finite lattices, as it the D relation that determines the congruences of a finite lattice. To show this, we need to take a brief detour.

Lemma 1.2. *If $\mathcal{L}$ is a finite lattice and $p, q \in JI(\mathcal{L})$, then $p D q$ if and only if q is in some minimal nontrivial join cover of p .*

Proof. Assume that $p D q$. Then there exists $x \in \mathcal{L}$ such that $p \leq x \vee q$ and $p \not\leq x \vee q_*$. Let $A = \downarrow x \cap JI(\mathcal{L})$. As $\mathcal{L}$ is finite, there exists B such that $B \ll A$, and $B \cup \{q\}$ is a minimal non-trivial join cover of x . We then conclude that $B \cup \{q\}$ is a minimal non-trivial join cover of p .

Conversely, assume that $q \in A$ where $A \subseteq \mathcal{L}$ is a minimal non-trivial join cover of p . Let $x = \bigvee (A \setminus \{q\})$. It follows that x witnesses $p D q$. $\square$

When discussing congruences of a lattice $\mathcal{L}$, we use the notation $Cg(a, b)$ to be the smallest congruence θ of $\mathcal{L}$ such that $a \theta b$. For a finite lattice $\mathcal{L}$ and a join irreducible $p \in JI(\mathcal{L})$, we use the notation $\Phi_p := Cg(p, p_*)$. Similarly for $m \in MI(\mathcal{L})$ we define $\Phi^m := Cg(m, m^*)$. We know from Funayama and Nakayama [FN 42] that the congruence lattice of a lattice is distributive. For finite lattices, every congruence is the unique join of join prime congruences, and the join prime congruences are precisely those that can be expressed in the form Φ_p or dually Φ^m [N2 95, Theorem 5.12]. For $p \in JI(\mathcal{L})$, or $m \in MI(\mathcal{L})$, there exists a unique congruence $\Psi_p = \bigvee \{\Phi_q: \Phi_p \not\leq \Phi_q\}$, respectively $\Psi^m = \bigvee \{\Phi^m: \Phi^n \not\leq \Phi^m\}$, which is maximal with respect to the condition that $p \theta p_*$, or respectively $m \theta m^*$, fails.

We say that $p D^n q$ if there exists a finite sequence of elements $r_i \in \mathcal{L}$ such that

$$p = r_0 D r_1 D \dots D r_n = q.$$

Lemma 1.3. *For a finite lattice $\mathcal{L}$ and $p, q \in JI(\mathcal{L})$, we have that $\Phi_p \leq \Phi_q$ if and only if $p = q$ or $p D^n q$ for some n .*

Proof. Assume $p D q$. Let $x \in \mathcal{L}$ be such that $p \leq q \vee x$, $p \not\leq q_* \vee x$. From these properties of x we have

$$((q \vee x) \wedge p) \vee p_* = p$$

$$((q_* \vee x) \wedge p) \vee p_* = p_*.$$

This implies that $p \Phi_q p_*$, whence $\Phi_p \leq \Phi_q$. It follows that if $p D^n q$ implies

$$\Phi_p = \Phi_{r_0} \leq \Phi_{r_1} \leq \dots \leq \Phi_{r_n} = \Phi_q.$$

To prove the converse, consider

$$S := \{r \in JI(\mathcal{L}) : (r, r_*) \in \Phi_q\}.$$

Clearly $q \in S$, and furthermore if $p D^n q$ then $p \in S$. Given any set $T \subseteq JI(\mathcal{L})$ such that if $p \in T$ and $p D q$, then $q \in T$ (we say such subsets of $JI(\mathcal{L})$ are *D-closed sets*), we can define a new lattice $T^\vee$ consisting of all joins of finite subsets of T . Clearly $T^\vee$ is a join subsemilattice of $\mathcal{L}$. Define the meet relation in $T^\vee$ by $x \wedge y$ is equal to the join of all elements in T below both x and y . Given a *D*-closed set $T \subseteq \mathcal{L}$ there is a canonical epimorphism $h: \mathcal{L} \rightarrow T^\vee$ defined by $h(x) = \bigvee \{t \in T : t \leq x\}$.

Clearly h is onto. To show h is a homomorphism let $x, y \in \mathcal{L}$. By Lemma 1.2 we can conclude that for every $t \in T$ any join cover of t can be refined to a join cover of t consisting of elements in T . As such, if $b \in \mathcal{L}$ is such that $b \leq t_1 \vee t_2 \vee \dots \vee t_k$ with $t_i \in T$ for very i such that $1 \leq i \leq k$, then every minimal nontrivial join cover of b is contained in T . Indeed, let U be a minimal nontrivial join cover of b . Fix $u \in U$. Since u is join irreducible, there exists some i such that $t_i \not\leq c \vee \bigvee (U \setminus \{u\})$ whenever $c < u$. It follows that $t_i D u$, whence $u \in T$.

We deduce that if T contains one minimal nontrivial join cover of a given element, it contains them all. This justifies

$$h(x \vee y) = \bigvee \{t \in T : t \leq x \vee y\} = \bigvee \{t \in T : t \leq x\} \vee \bigvee \{t \in T : t \leq y\} = h(x) \vee h(y),$$

$$h(x \wedge y) = \bigvee \{t \in T : t \leq x \wedge y\} = \bigvee \{t \in T : t \leq x, \text{ and } t \leq y\} = h(x) \wedge h(y).$$

Under such an epimorphism, for any $p \in JI(\mathcal{L})$ we have that $h(p) \neq h(p_*)$ if and only if $p \in T$.

By the definition of S we have $T := JI(\mathcal{L}) \setminus S$ is a *D*-closed set. Let $h: \mathcal{L} \rightarrow T^\vee$ be the canonical epimorphism. We see that the kernel of h is Φ_q . It follows that for $p \in JI(\mathcal{L})$ we have that $\Phi_p \leq \Phi_q$ if and only if $p \in S$. It can be seen that T is the smallest *D* closed set containing q , whence $p D^n q$ implies $p \in S$. We conclude $\Phi_p \leq \Phi_q$, as desired. $\square$

We can now order the join irreducibles of a finite lattice $\mathcal{L}$ based on the *D* relation. We say $p \leq_D q$ if $p D^n q$ or $p = q$. Under this quasi-order $p \equiv q$ if and only if $p D^m q D^n p$. Given the previous lemma, the following well known theorem then follows from the fact that in any finite lattice $x = \bigvee \{\downarrow x \cap JI(\mathcal{L})\}$.

Theorem 1.4. *If $\mathcal{L}$ is a finite lattice then*

$$\langle JI(\mathcal{L}) / \equiv, \leq_D \rangle \cong \langle JI(Con(\mathcal{L})), \leq \rangle,$$

and thus $Con(\mathcal{L})$ is isomorphic to lattice of order ideals of $\langle JI(\mathcal{L}), \leq_D \rangle$.

We note at this point that the relation *D* has a dual concept on the meet irreducibles of a finite lattice $\mathcal{L}$ denoted D^δ . It follows by duality then that $Con(\mathcal{L})$ is isomorphic to the lattice of order ideals of $\langle MI(\mathcal{L}), \leq_{D^\delta} \rangle$.

Recall the definition of a lower bounded lattice. Given the relationship between the D relation and the congruences of a finite lattice $\mathcal{L}$ it is perhaps not surprising that there is a characterization of finite lower bounded lattices, first proven by Jónsson and Nation in [JN 77].

Theorem 1.5. *A finite lattice $\mathcal{L}$ is lower (respectively upper) bounded if and only if it contains no D -cycle (respectively D^δ -cycle).*

Another way to formulate this result is that a finite lattice is lower (respectively upper) bounded if and only if the equivalence relation generated by the quasi-order $\leq_D$ (respectively $\leq_{D^\delta}$) is equality.

We are at last ready to show the connection established by Day in [D 79] between doubling and the characterization of finite lower bounded lattices. We present Day's proof below, as many of the techniques used will be of great importance in the sequel.

Theorem 1.6. *A finite lattice $\mathcal{L}$ is lower bounded if and only if $\mathcal{L}$ can be constructed from the one-element lattice by doubling a sequence of lower pseudo-intervals.*

Proof. We first show that if $\mathcal{L}$ is lower bounded and C is a lower pseudo-interval in $\mathcal{L}$ with unique least element b , then $\mathcal{L}[C]$ is lower bounded. It then follows from induction that any lattice constructed from $\mathbf{1}$ by doubling a sequence of lower pseudo-intervals is lower bounded.

We can classify the elements of $JI(\mathcal{L}[C])$ by noting that

$$JI(\mathcal{L}[C]) = J_{-1} \cup J_0 \cup J_1$$

where

$$\begin{aligned} J_{-1} &:= JI(\mathcal{L}) \setminus C, \\ J_0 &:= \{(p, 0) : p \in JI(\mathcal{L}) \cap C\}, \\ J_1 &:= \{(b, 1) : b \text{ is the unique minimal element of } C\}. \end{aligned}$$

Let $h: \mathcal{L}[C] \rightarrow \mathcal{L}$ be the canonical epimorphism, which is a bounded homomorphism. For any $p \in JI(\mathcal{L})$, $\beta_h(p) \in J_{-1} \cup J_0$. If $p \in JI(\mathcal{L}[C])$ and $p \leq \bigvee B$ for some $B \subseteq \mathcal{L}$ the set $A := \{\beta_h(b) : b \in B\}$ is such that $A \ll B$ and $p \leq \bigvee A$. This demonstrates that if $p \in J_{-1} \cup J_0$, p cannot be an element of a D -cycle in $JI(\mathcal{L}[C])$, as this would imply the existence of such a cycle in $\mathcal{L}$. The only other element we need concern ourselves with is $(b, 1)$. By definition $(b, 1) D p$ implies that $p \neq (b, 1)$, whence $p \in J_0 \cup J_{-1}$ and therefore by our previous argument that $(b, 1) D^n p$ implies $p \neq (b, 1)$.

Assume that $\mathcal{L}$ is lower bounded. We proceed by induction on $|\mathcal{L}|$. Let θ be an atom in $Con(\mathcal{L})$. As $\mathcal{L}$ is finite it is evident that any homomorphism $h: \mathcal{L} \rightarrow \mathcal{L}/\theta$ is lower bounded. As the composition of lower bounded homomorphisms is lower bounded and $\mathcal{L}$ is lower bounded, so is $\mathcal{L}/\theta$.

There exists $p \in JI(\mathcal{L})$ such that $\theta = \Phi_p$. Suppose there exists $q \in JI(\mathcal{L})$ such that $q D p$. By Lemma 1.3 this implies that $\Phi_q \leq \Phi_p$, whence $\Phi_q = \Phi_p$ as Φ_p is a minimal non-trivial congruence of $\mathcal{L}$. Combining the results of Theorem 1.4 and Theorem 1.5 we see lower bounded lattices have the property that there exists a one-to-one correspondence between join irreducible elements $r \in JI(\mathcal{L})$ and join irreducible congruences $\Phi_r \in JI(Con(\mathcal{L}))$, whence $q = p$. This contradicts that $q D p$, and we conclude that $q D p$ is false for every $q \in JI(\mathcal{L})$.

For a congruence ϕ of $\mathcal{L}$, define

$$\begin{aligned} R_\phi &= \{p \in JI(\mathcal{L}) : p \phi p_*\} = \{p \in JI(\mathcal{L}) : \Phi_p \leq \phi\}, \\ S_\phi &= JI(\mathcal{L}) \setminus R_\phi = \{p \in JI(\mathcal{L}) : (p, p_*) \notin \phi\}. \end{aligned}$$

By our argument above we have that $R_\theta = \{c \in JI(\mathcal{L}) : c \theta c_*\} = \{p\}$, whence $S_\theta = JI(\mathcal{L}) - \{p\}$. As S_θ is D -closed set, we have that

$$u \theta v \text{ iff } \downarrow u \cap (JI(\mathcal{L}) \setminus \{p\}) = \downarrow v \cap (JI(\mathcal{L}) \setminus \{p\}).$$

Thus, if u and v are distinct and $u \theta v$, then $\downarrow u$ and $\downarrow v$ consist of the same set of elements except that one of them, say u , contains p and the other does not. This demonstrates that $p_* \leq u < v = p \vee u$. Hence every congruence class a/θ contains either one or at most two elements for any $a \in \mathcal{L}$. Thus $p_* \leq u < u \vee p$ implies $u \prec u \vee p$ because congruence classes in lattices are convex sets and $p_* \leq u$ implies $u = u \vee p_* \theta p \vee u$.

As $\mathcal{L}$ is finite, the natural map $\mathcal{L} \rightarrow \mathcal{L}/\theta$ is a meet isomorphism onto the meet subsemilattice $\mathcal{S} = \alpha(\mathcal{L})$ consisting of the maximum element in each θ class. Given our work above, we have that

$$\mathcal{S} = \{x \in \mathcal{L} : p_* \not\leq x \text{ or } p \leq x\},$$

with the order, whence the meet operation inherited from $\mathcal{L}$. One can show that $\mathcal{S}$ is a lattice by defining the join operation on $\mathcal{S}$ by

$$x \vee y = \bigwedge \{s \in \mathcal{S} : s \geq x \text{ and } s \geq y\},$$

though $\mathcal{S}$ need not be a sublattice of $\mathcal{L}$. We can also identify the set of elements of $\mathcal{S}$ which come from two-element θ classes, i.e., the greatest element of each two element class:

$$C = \{x \in \mathcal{L} : \text{there exists } y \in \mathcal{L} \text{ with } p_* \leq y < x = y \vee p\}.$$

The element y in this definition is uniquely determined by the $x \theta y$, so we denote it as $\zeta(x) := y$. The proof will be complete if we demonstrate that C is a lower pseudo-interval of $\mathcal{S}$ and that $\mathcal{L} \cong \mathcal{S}[C]$.

Clearly p is the least element of C . We claim that if $p \leq t \leq x$ where $x \in C$, then $t \in C$. Clearly $p_* \leq t \wedge \zeta(x)$ and $t = t \wedge x \theta t \wedge \zeta(x)$. It follows that $t \wedge \zeta(x) \prec t$, and because $p \not\leq \zeta(x)$ we have that $t = p \vee (t \wedge \zeta(x))$. Therefore $t \in C$ with $\zeta(t) = t \wedge \zeta(x)$. Thus C is convex and has a unique least element, and therefore C is a lower pseudo-interval.

Define the mapping $h: \mathcal{L} \rightarrow \mathcal{S}[C]$ by

$$\begin{aligned} h(x) &= x \text{ if } x \notin C \cup \zeta(C), \\ h(x) &= (x, 1) \text{ if } x \in C, \\ h(\zeta(x)) &= (x, 0) \text{ if } x \in C. \end{aligned}$$

It is straightforward, though somewhat tedious, to show that h is well-defined, one-to-one, and onto. It is less trivial to show that the order on $\mathcal{L}$ corresponds to the order of $\mathcal{S}[C]$ given by the doubling construction, i.e., $x \leq y$ if and only if $h(x) \leq h(y)$. Assume that $x \leq y$. First note that $x \not\leq \zeta(y)$ for $x, y \in C$, because $p \leq x$ and $p \not\leq \zeta(y)$. If $x \in C$ and $y \in \mathcal{S}$, then $\zeta(x) \leq y$ implies $x = \zeta(x) \vee p \leq y$, and similarly $\zeta(x) \leq \zeta(y)$ implies $x \leq y$. These facts allow us to resolve all possible cases. Conversely, assume $h(x) \leq h(y)$. Recall that for $t, x \in C$ with $t \leq x$ we have that $\zeta(t) = t \wedge \zeta(x)$. It follows that $(t, 0) \leq (x, 0)$ in $\mathcal{S}[C]$ implies $\zeta(t) \leq \zeta(x)$, and the remaining cases are trivial. We conclude that h is a lattice isomorphism, as desired. $\square$

Figure 1.1: Doublings of lower pseudo-intervals

There is an alternative way to characterize finite lower bounded lattices which will be particularly important in the sequel. Following Jónsson in [JN 77], for a finite lattice $\mathcal{L}$ define the sets

$$D_0 = \{p \in JI(\mathcal{L}) : p \text{ is join prime}\}$$

$$D_{k+1} = \{p \in JI(\mathcal{L}) : p D q \text{ implies } q \in D_k\}.$$

In those finite lattices where the concept is well defined, we say that the D -rank of $p \in JI(\mathcal{L})$ is the least integer k such that $p \in D_k$ and $p \notin D_{k-1}$, and denote it as $rk_D(p)$ or simply $rk(p)$ when the context is clear. It is important to note however that it is easy to find examples of finite lattices such that the D -rank is undefined, e.g., $\mathcal{M}_3$ in Figure 1.2. It is not difficult to prove that $D_k(\mathcal{L}) = JI(\mathcal{L})$ for some $k \in \omega$ if and only if $\mathcal{L}$ admits no D -cycles, whence a finite lattice is lower bounded if and only if every join irreducible element has a well defined D -rank.

Figure 1.2: The lattice $\mathcal{M}_3$.

Some remarks are in order. The dual of Theorem 1.6 shows that a finite lattice is upper bounded if and only if it can be constructed from the one-element lattice by a sequence of doubling upper pseudo-intervals, with D^δ taking the place of D . Combining these two results gives us the theorem that a finite lattice is bounded if and only if it can be constructed from the one-element lattice by a sequence of doubling intervals.

Define the *depth* of an element x in a finite ordered set as the length of a maximal chain in $\uparrow x$. Given Lemma 1.3 and the proof of Theorem 1.6, we have Day's result that the depth of Φ_p in the poset $JI(Con(\mathcal{L}))$ is equal to the D -rank of p . As was shown in the proof of Theorem 1.6, if b is the unique minimum element of a lower pseudo-interval C , then $q D (b, 1)$ holds for no $q \in JI(\mathcal{L}[C])$. It follows $\Phi_{(b,1)}$ is a minimal element in the ordered set $JI(Con(\mathcal{L}[C]))$.

Recall that every join prime congruence of a finite lattice $\mathcal{L}$ is of the form Φ_p for some $p \in JI(\mathcal{L})$. Thus, there exists a surjection from the set of join irreducible elements of $\mathcal{L}$ and the join irreducible congruences (all of which are join prime since $Con(\mathcal{L})$ is distributive) of $\mathcal{L}$. In general, this correspondence need not be one-to-one, as in $\mathcal{M}_3$ shown in Figure 1.1. Combining Theorem 1.4 and 1.5 we deduce the well known result first shown by P. Pudlák and J. Tuma, in [PT 74], which classifies those finite lattices such that the correspondence is one-to-one.

Theorem 1.7. *The following are equivalent for a finite lattice $\mathcal{L}$:*

- $\mathcal{L}$ is lower bounded.
- $JI(\mathcal{L}) = D_k(\mathcal{L})$ for some $k \in \omega$.
- $\Phi_p = \Phi_q$ implies $p = q$ for all $p, q \in JI(\mathcal{L})$.

Several important classes of lattices are examples of lower bounded lattices. The class of splitting lattices introduced in [M 72], which are vital in studying the equational theory of lattices, are (both lower and upper) bounded. Projective lattices, one of the few known classes of lattices known to be embeddable into free lattices, are also bounded. The lattice of all subsemilattices of a finite semilattice is also lower bounded, see [A 91].

When doubling a lower pseudo-interval, the only change in the congruences of the original lattice is the addition of a single minimal join irreducible congruence. Indeed, the same is true if we allow arbitrary convex sets to be doubled, but the analysis requires some additional tools.

1.4 Congruence Normal Lattices

Recall from the previous section that if $\mathcal{L}$ is a lattice and $C \subseteq \mathcal{L}$ is a convex subset, then $\mathcal{L}[C]$ is a lattice. So far we have seen that if we restrict C to intervals or pseudo-intervals then there is a characterization of those lattices that are constructible by doubling such classes of subsets. The question arises: is there such a characterization for lattices constructible from arbitrary convex sets? The answer to this question was first discovered by Winfried Geyer in [G 94], who used an approach known as Formal Concept Analysis. Alan Day quickly followed this with a solution using more traditional lattice theoretical language in [D 94].

To understand this class of lattices, we first need to introduce some new concepts. Let $a \leq b$ and $u \leq v$ in a lattice $\mathcal{L}$. We say that the intervals b/a and v/u are *associates* if $b \vee u = v$ and $b \wedge u = a$, or if $v \vee a = b$ and $v \wedge a = b$. We say that b/a and v/u are *projective* and write $b/a \approx v/u$ if there exist sequences $a = x_0, x_1, x_2, \dots, x_{n-1}, x_n = u$ and $b = y_0, y_1, y_2, \dots, y_{n-1}, y_n = v$ such that for every $1 \leq i \leq n$ we have that y_{i-1}/x_{i-1} and y_i/x_i are associates. Note that $b/a \approx v/u$ implies $Cg(b, a) = Cg(v, u)$.

Of particular interest is the case when $\mathcal{L}$ is finite and we consider intervals that consist of covering pairs of the form p/p_* for $p \in JI(\mathcal{L})$ and m^*/m for $m \in MI(\mathcal{L})$. For $p, q \in JI(\mathcal{L})$ we say p is projective to q , and write $p \approx q$, to mean $p/p_* \approx q/q_*$. Similarly for $m, n \in MI(\mathcal{L})$ we say m is projective to n , and write $m \approx n$, to mean $m^*/m \approx n^*/n$ and say p is projective to m , written $p \approx m$ to mean $p/p_* \approx m^*/m$. When interpreted in this way, projectivity is an equivalence relation on $JI(\mathcal{L}) \cup MI(\mathcal{L})$. Note that for $p, q \in JI(\mathcal{L})$ we have that $p \approx q$ implies $\Phi_p = \Phi_q$, but not conversely, see the lattice in Figure 1.4.

Figure 1.3: $\mathcal{L}[C]$ is neither lower nor upper bounded.

We seek for formulate a classification of finite lattices constructible by doubling arbitrary convex subsets that is analogous to our classification of those constructible by doubling pseudo-intervals. Consider the example in Figure 1.3. The first lattice is clearly bounded, indeed, it is distributive. However in the second lattice the elements p, q witness a D -cycle, whilst m, n witness a D^δ -cycle. Notice also that both these cycles are contained within the same projectivity class, and consist of “new” join and meet irreducibles that were produced by the doubling.

Up until now we have been working with subclasses of convex sets that are connected by definition. It is important to note that if C_1, C_2 are connected convex sets such that $C_1 \cup C_2$ is not connected, then $\mathcal{L}[C_1][C_2] \cong \mathcal{L}[C_2][C_1]$. From this point forward we restrict our attention to doubling connected convex sets.

Recall that in the definition of the order relation on $\mathcal{L}[C]$ it is not possible for $(x, 1)$ to be below $(y, 0)$ for any $x, y \in C$. This fact, along with observations of the previous example, motivate the following definition: a lattice $\mathcal{L}$ is called *congruence normal* if for all $p \in JI(\mathcal{L})$ and $m \in MI(\mathcal{L})$,

if $\Phi_p = \Phi^m$ then $p \not\leq m$. While congruence normality is hardly as elegant a condition as lower boundedness, it does answer the question at hand.

Theorem 1.8. *A finite lattice $\mathcal{L}$ is constructible from the one element lattice by a sequence of doubling convex sets if and only if $\mathcal{L}$ is congruence normal.*

The proof of this is presented as a series of somewhat technical lemmas by Day. The specific techniques used will not be directly applicable in the sequel, and so we refer the reader to Day's original paper [D 94]. There are however, concepts in the appendix to that article [N 94] that are directly relevant to later chapters, which we discuss presently.

Recall that a finite lattice is lower bounded, i.e., constructible from the one element lattice by doubling lower pseudo-intervals, if and only if $D_k(\mathcal{L}) = JI(\mathcal{L})$ for some $k \in \omega$. We wish to have an analogous result for congruence normal lattices. We define a relationship that is based on that used by Nation is [N 94], but which we have extended to be considerably more general.

$$F_0 = \{p \in JI(\mathcal{L}) : p D q \text{ implies } p \approx q\}$$

$$F_{k+1} = \{p \in JI(\mathcal{L}) : p D q \text{ implies } q \in F_k \text{ or } p \approx q\}.$$

Figure 1.4: For no $k \in \omega$ is $F_k(\mathcal{L}) = JI(\mathcal{L})$.

We define the *F-rank* of a join irreducible element p in a finite lattice $\mathcal{L}$ as we did the *D-rank*, and use the notation $rk_F(p)$, or just $rk(p)$ when there is no risk of ambiguity. As with *D-rank*, this may not be defined for every join irreducible in a every finite lattice, e.g., the “rice paddy” pictured in Figure 1.4.

Lemma 1.9. *The following are equivalent for a finite lattice $\mathcal{L}$.*

- $F_k(\mathcal{L}) = JI(\mathcal{L})$ for some $k \in \omega$.
- $\Phi_p = \Phi_q$ implies $p \approx q$ for all $p, q \in JI(\mathcal{L})$.

Proof. Assume that $F_k(\mathcal{L}) = JI(\mathcal{L})$ for some $k \in \omega$. Let $\Phi_p = \Phi_q$ for some $p, q \in JI(\mathcal{L})$. By Lemma 1.3 we have that $p = r_0 D r_1 D r_2 D \dots D r_n = q$ where $n \in \omega$ and $r_i \in JI(\mathcal{L})$ for every i such that $0 \leq i \leq n$. By assumption this implies that $r_{i-1} \approx r_i$ or $rk_F(r_{i-1}) > rk_F(r_i)$. In the latter case $r_i D^m r_{i-1}$ fails for all $m \in \omega$, whence $\Phi_{r_i} \not\leq \Phi_{r_{i-1}}$, contrary to our choice of p and q . We conclude that $r_{i-1} \approx r_i$ for all i , whence $p \approx q$.

Assume $\Phi_p = \Phi_q$ implies $p \approx q$ for all $p, q \in JI(\mathcal{L})$. Let $p \in JI(\mathcal{L})$. We proceed by induction on the depth of Φ_p . If Φ_p is maximal in the ordered set of join prime congruences of $\mathcal{L}$, then $p D q$

implies that $\Phi_p = \Phi_q$, whence by assumption $p \approx q$. It follows that $p \in F_0$. Let $rk_F(p) = k \geq 1$. If $p D q$, then $\Phi_p \leq \Phi_q$. If $\Phi_p = \Phi_q$ then $p \approx q$. Otherwise $\Phi_p < \Phi_q$ whence, by the inductive assumption, $q \in F_{k-1}(\mathcal{L})$. We conclude that $p \in F_k(\mathcal{L})$, as desired. $\square$

The second half of the proof of Lemma 1.9 give us the following fact that will be frequently referred to, so we state it as a corollary.

Corollary 1.10. *Let $\mathcal{L}$ be a finite lattice. If $F_k(\mathcal{L}) = JI(\mathcal{L})$ for some $k \in \omega$, then $rk_F(p)$ is equal to the depth of Φ_p in $JI(Con(\mathcal{L}))$ for every $p \in JI(\mathcal{L})$.*

Lemma 1.9 is a partial generalization of Pudlák and Tuma's result in Theorem 1.7. However, we are crucially missing a characterization of finite lattices that satisfy these conditions that is analogous to lower boundedness. Clearly lower bounded lattices satisfy these conditions, but so do modular lattices such as $\mathcal{M}_3$, geometric lattices, and lattices of equivalence relations on a finite set. Every class of finite lattices classified in the remaining chapters will be subclasses of class of finite lattices that satisfy these conditions as well. Clearly $\mathcal{M}_3$ cannot be obtained from the one element lattice by doubling convex sets, so we will need additional assumptions to characterize such lattices. First, we prove that lattices which satisfy the conditions of Lemma 1.9 are similar to lower bounded lattices in another way.

We say that a finite lattice $\mathcal{L}$ is **2-simple** if the only simple lattice that is a homomorphic image of $\mathcal{L}$ is the two-element lattice. All simple homomorphic images of a lattice $\mathcal{L}$ are those of the form $\mathcal{L}/\theta$, where θ is a maximal nontrivial congruence of $\mathcal{L}$. Recall that the meet irreducible congruences of $\mathcal{L}$ are those of the form Ψ_p for some $p \in JI(\mathcal{L})$, where Ψ_p is the maximal congruence that does not contain the pair (p, p_*) . Thus, every simple homomorphic image of $\mathcal{L}$ is of the form $\mathcal{L}/\Psi_p$ for some $p \in JI(\mathcal{L})$. Thus, a finite lattice $\mathcal{L}$ is **2-simple** if and only if Ψ_p being a maximal meet irreducible congruence implies p is join prime.

This definition is motivated by the observation that if one begins with the one-element lattice and creates new lattices, the first step will always be to construct the two element lattice. Given that there exists a canonical homomorphism from $\mathcal{L}[C] \rightarrow \mathcal{L}$ for every step of this sequence, the two element lattice is the only simple lattice that will appear in this sequence.

The conditions given in 1.9 do not guarantee that a finite lattice $\mathcal{L}$ is **2-simple**. If a finite lattice $\mathcal{L}$ satisfies the conditions of 1.9 it is not difficult to see that any simple homomorphic image $\mathcal{K}$ of $\mathcal{L}$ will satisfy $F_0(\mathcal{K}) = JI(\mathcal{K})$. However, there exist simple lattices other than **2**, such as $\mathcal{M}_3$, for which this holds. In [N 94], J.B. Nation introduced the following two equivalent conditions for a finite lattice $\mathcal{L}$ to exclude such possibilities:

T : for all $p, q \in JI(\mathcal{L})$, if $p \approx q$ then $q \leq q_* \vee p$,

T' : for all $p, q \in JI(\mathcal{L})$ and $m \in MI(\mathcal{L})$ if $p \approx q$ and q is an associate of m then $p \not\leq m$.

Nation did not address the issue of **2-simplicity**. The following lemma shows that the old result still holds.

Lemma 1.11. *Let $\mathcal{L}$ be a finite lattice such that $F_k(\mathcal{L}) = JI(\mathcal{L})$ and $\mathcal{L}$ satisfies **T**. Then $\mathcal{L}$ is **2-simple**.*

Proof. The simple homomorphic images of $\mathcal{L}$ are those isomorphic to $\mathcal{L}/\theta$ where θ is a maximal nontrivial congruence of $\mathcal{L}$. As $Con(\mathcal{L})$ is distributive it follows that every such θ is of the form Ψ_p for some $p \in JI(\mathcal{L})$. If p is join prime then the quotient map $f: \mathcal{L} \rightarrow \mathcal{L}/\Psi_p$ maps $\uparrow p$ to $1_{\mathcal{L}/\Psi_p}$, and all other elements to $0_{\mathcal{L}/\Psi_p}$. Thus it suffices to prove that for $p \in JI(\mathcal{L})$, if $rk(p) > 0$ then Φ_p is not maximal in $JI(Con(\mathcal{L}))$.

Assume $rk(p) > 0$. As p is not join prime, there exists $q \in JI(\mathcal{L})$ such that $p D q$, whence $\Phi_p \leq \Phi_q$. By hypothesis this implies $rk(q) < rk(p)$ or $p \approx q$. If $rk(q) < rk(p)$ then $rk(p) \not\leq rk(q)$, whence $p D^n q$ fails to hold for all n . This implies that $\Phi_q \not\leq \Phi_p$, which combined with the fact

that $\Phi_p \leq \Phi_q$ leads us to conclude $\Phi_p < \Phi_q$. If $p \approx q$, we have by **T** that $p \leq p_* \vee q$. As such there exists $r \in JI(\mathcal{L})$ such that $r \leq p_*$ and $p D r$. Suppose $p \approx r$. By **T** it follows that

$$p \leq p_* \vee r = p_* < p,$$

which is a contradiction. Thus $rk(r) < rk(p)$ and we can apply the previous argument to conclude $\Phi_p < \Phi_r$. $\square$

Finally we are ready to prove Nation's result in [N 94] about the classification of finite congruence normal lattices.

Theorem 1.12. *A finite lattice $\mathcal{L}$ is congruence normal if and only if $F_k(\mathcal{L}) = JI(\mathcal{L})$ for some $k \in \omega$ and $\mathcal{L}$ satisfies **T**.*

Proof. Assume that $F_k(\mathcal{L}) = JI(\mathcal{L})$ for some $k \in \omega$ and $\mathcal{L}$ satisfies **T**. For $p, q \in JI(\mathcal{L})$ we have in general that $\Phi_p \leq \Phi_q$ if and only if $p D^n q$ for some n . As $F_k(\mathcal{L}) = JI(\mathcal{L})$ it follows that $p \approx q$, as otherwise $rk_F(p) < rk_F(q)$, whence by Corollary 1.10 we have $\Phi_p < \Phi_q$. We conclude that $\Phi_p = \Phi_q$ implies $p \approx q$. If there exists $m \in MI(\mathcal{L})$ such that $\Phi_p = \Phi^m$, choose $q \in JI(\mathcal{L})$ to be such that m and q are associates. It follows that $\Phi_p = \Phi_q$, whence $p \approx q$. We conclude by **T** that $p \not\leq m$, which proves that $\mathcal{L}$ is congruence normal.

Assume that $\mathcal{L}$ is congruence normal. We claim every congruence normal lattice satisfies **T**. Indeed, if **T'** fails in $\mathcal{L}$ it follows that there exists $p, q \in JI(\mathcal{L})$ and $m \in MI(\mathcal{L})$ such that $\Phi_p = \Phi_q = \Phi^m$ and $p \leq m$, whence $\mathcal{L}$ is not congruence normal. Any distributive lattice satisfies $F_0(\mathcal{L}) = JI(\mathcal{L})$. By Day's characterization theorem, we know every congruence normal lattice is constructible from the one element lattice by doubling convex sets. It suffices to show that if $F_k(\mathcal{L}) = JI(\mathcal{L})$ then for any connected and convex $C \subseteq \mathcal{L}$ we have $F_{k+1}(\mathcal{L}[C]) = JI(\mathcal{L}[C])$.

We can classify the elements of $JI(\mathcal{L}[C])$ by noting that

$$JI(\mathcal{L}[C]) = J_{-1} \cup J_0 \cup J_1$$

where

$$\begin{aligned} J_{-1} &:= JI(\mathcal{L}) \setminus C, \\ J_0 &:= \{(p, 0) : p \in JI(\mathcal{L}) \cap C\}, \\ J_1 &:= \{(x, 1) : x \text{ is a minimal element of } C\}. \end{aligned}$$

Moreover we claim that if $p, q \in J_1$ then $p \approx q$. Indeed, let $p = (x, 1)$ and $q = (y, 1)$ for some $x, y \in C$. As C is connected, there exist elements u_i, v_i in C for $0 \leq i \leq n$ such that

$$x = u_0 \leq v_1 \geq u_1 \leq \dots \leq v_n \geq u_n = y.$$

As C is convex we may assume without loss of generality that u_i is minimal in C and v_i is maximal in C for every $0 \leq i \leq n$. As such $(u_i, 1) \in JI(\mathcal{L}[C])$ and $(v_i, 0) \in MI(\mathcal{L}[C])$ are such that $(u_i, 1)$ and $(v_i, 1)$ are associates for all i . Similarly $(u_{i-1}, 1)$ and $(v_i, 0)$ are associates for all i . We conclude that $p \approx q$.

Let $h: \mathcal{L}[C] \rightarrow \mathcal{L}$ be the canonical epimorphism, which is a bounded homomorphism. For any $p \in JI(\mathcal{L})$, $\beta_h(p) \in J_{-1} \cup J_0$. If $p \in JI(\mathcal{L}[C])$ and $p \leq \bigvee B$ for some $B \subseteq \mathcal{L}$ the set $A := \{\beta_h(b) : b \in B\}$ is such that $A \ll B$ and $p \leq \bigvee A$. This demonstrates that $J_{-1} \cup J_0 \subseteq F_k(\mathcal{L}[C])$. The argument above shows that $J_1 = \{q \in JI(\mathcal{L}[C]) : q \approx p\}$ for any $p \in J_1$. Therefore $JI(\mathcal{L}[C]) \subseteq F_k(\mathcal{L}[C]) \cup \{q \in JI(\mathcal{L}[C]) : q \approx p\}$ for any $p \in J_1$ and we conclude $JI(\mathcal{L}[C]) \subseteq F_{k+1}(\mathcal{L}[C])$. $\square$

Some remarks are in order. The first is that the proof of Theorem 1.12 shows that for any $p \in J_{-1} \cup J_0$, the only elements in any minimal nontrivial join cover of p are also in $J_{-1} \cup J_0$. It follows that $p D q$ fails for any $q \in J_1$. As J_1 consists of a single projectivity class it follows that Φ_p

is the only new join irreducible congruence in $Con(JI(\mathcal{L}))$, and that Φ_p is minimal in $JI(Con(\mathcal{L}))$. Also note that for a finite lower bounded lattice $\mathcal{L}$ and $p \in JI(\mathcal{L})$ we have $rk_D(p) = rk_F(p)$.

The question occurs: is this as far as we can go? There exist finite lattices $\mathcal{L}$ satisfying $F_k(\mathcal{L}) = JI(\mathcal{L})$ and $\mathcal{L}$ is **2**-simple, but that fail to satisfy **T**. We will see an example later in Figure 3.6. Can such lattices be constructed by doubling? If so, what kind of subsets are doubled to produce these lattices?

After Alan Day's death in 1990 his results on doubling convex sets were published posthumously in 1994. J.B. Nation observed in [N1 95] that doubling need not be limited to convex sets. In 2002 McNulty, Nation and Freese used a generalization of doubling as a critical tool in their work in [FMN 02]. The departure here was that instead of replacing old lattice elements with a two element interval, other lattices such as $\mathcal{M}_3$ were used. Most recently in his 2011 dissertation Heiko Reppe classified those subsets that can be used in the doubling construction. In the next chapter, we will combine these ideas to introduce a new way of looking at doubling and its generalizations.

CHAPTER 2

CONSTRUCTING FINITE LATTICES BY INFLATION

2.1 Motivating Examples

In the last chapter we restricted our attention to doubling convex subsets of finite lattices. While it is straightforward to see that doubling convex sets results in a new lattice, one cannot double arbitrary subsets in a finite lattice and produce a lattice. This is seen in the example given in Figure 2.1. Clearly the elements a and b have no least upper bound. Note however that while the doubling is not a lattice, it is still a well defined partially ordered set.

Figure 2.1: A doubling that fails to produce a lattice.

The question then is what kinds of non-convex sets can be doubled and still produce a lattice? A canonical example is due to Heiko Reppe and is reproduced below.

Figure 2.2: Reppe's lattice.

It is not hard to check that the doubling produces a lattice despite the fact that u witnesses the set that is doubled fails to be convex. A critical role in preserving meets and joins after the doubling is played by properties of u , as we will show in the next section. That said, the name u is chosen in this case to stand for “unnecessary.” The doubling could be performed on a three atom Boolean algebra and still produce a lattice as illustrated in the figure below.

This example is constructed differently from the way we have considered doubling so far. Even though the elements x and y are in the set of elements that is doubled, we ignore the covering edge $p \prec m$ during the construction. Imagining the edges of the Hasse diagram as threads in a game of cat's cradle, we think of the threads linking p and m as having gotten “twisted” and not doubling when stretched taut.

Figure 2.3: A, so far, unexplained doubling.

In both Reppe's example and the example of "twisting," we are confronted with a situation where in the constructed lattice we encounter $(a, 1) \leq (b, 0)$ for $a, b \in \mathcal{L}$. This never occurred when considering convex sets. Indeed, the whole point of using convex sets in the beginning was to avoid this situation so as to fix failures of Whitman's condition! Our first task is to define the new partial order created when doubling non-convex sets. In the interests of full generality, we will also introduce the concept of inflating by replacing elements of the original lattice with partially ordered sets other than $\mathbf{2}$.

2.2 Inflation of Partially Ordered Sets

Let $\mathcal{L}$ and $\mathcal{K}$ be partially ordered sets. Let E be a subset of the partial order $\leq_{\mathcal{L}}$ that is transitive, i.e., if $(x, y) \in E$ and $(y, z) \in E$ then $(x, z) \in E$. We will normally construct such sets by specifying a set of covering pairs and then taking their transitive closure. One can define a new partially ordered set $\mathcal{L} \star_E \mathcal{K}$ by "inflating" E as follows. Let E' denote the set of elements of $\mathcal{L}$ that are included in the relations of E . The universe of the ordered set $\mathcal{L} \star_E \mathcal{K}$ is $(L \setminus E') \cup (E' \times K)$. Let $x, y \in L$ and $a, b \in K$. We define a binary relation $\sqsubseteq$ on $(L \setminus E') \cup (E' \times K)$ by

- i $x \sqsubseteq y$ if $x \leq_{\mathcal{L}} y$,
- ii $(x, a) \sqsubseteq y$ if $x \leq_{\mathcal{L}} y$,
- iii $x \sqsubseteq (y, b)$ if $x \leq_{\mathcal{L}} y$,
- iv $(x, a) \sqsubseteq (y, b)$ if $x \leq_{\mathcal{L}} y$ and $a \leq_{\mathcal{K}} b$,
- v $(x, a) \sqsubseteq (y, b)$ if $x \leq_{\mathcal{L}} y$, and there exists $(u, v) \notin E$ such that $x \leq u < v \leq y$.

While our primary interest is the case when $\mathcal{L}$ and $\mathcal{K}$ are finite lattices, this is not required for the construction to produce an ordered set. Note how this new relation differs from the partial order constructed by doubling convex sets in a crucial way. In the previous chapter, it was assumed that if a and b were points which were to be doubled and $a \leq b$ then $(a, 1) \not\leq (b, 0)$. Because we allow covering relations to become "twisted" we are confronted with situations where, when doubling non-convex sets, we can have $(a, 1) \leq (b, 0)$. This necessitates condition (v) to ensure $\mathcal{L} \star_E \mathcal{K} = \langle (L \setminus E') \cup (E' \times K), \sqsubseteq \rangle$ is an ordered set.

Lemma 2.1. *The binary relation $\sqsubseteq$ is a partial order on $(L \setminus E') \cup (E' \times K)$.*

Proof. It is evident from conditions (i) and (iv) of the definition that the relation is reflexive. To show the relation is antisymmetric we consider a number of possible cases.

If $x \sqsubseteq y \sqsubseteq x$, then by i we have that $x \leq_{\mathcal{L}} y \leq_{\mathcal{L}} x$, whence $x = y$.

If either $(x, a) \sqsubseteq (y, b) \sqsubseteq (x, a)$ or $x \sqsubseteq (y, b) \sqsubseteq x$, then by **ii** and **iii** we may again conclude that $x = y$. However, this implies $x \in E'$ and $x \notin E'$, a contradiction. Therefore, neither of these cases occur.

If $(x, a) \sqsubseteq (y, b) \sqsubseteq (x, a)$ then, as before, we conclude that $x = y$. This indicates neither $(x, a) \sqsubseteq (y, b)$ nor $(y, b) \sqsubseteq (x, a)$ is due to **v**. We then have that $a \leq_{\mathcal{K}} b \leq_{\mathcal{K}} a$, whence $a = b$.

To prove transitivity we again must look at a number of different cases. In every case the original order is preserved on the first components of all the elements considered, thus we may assume that $x \leq z$ in each case.

If $x \sqsubseteq y \sqsubseteq z$ then transitivity follows directly from the definition.

If $x \sqsubseteq y \sqsubseteq (z, c)$ then by **iii** it follows $x \sqsubseteq (z, c)$.

If $x \sqsubseteq (y, b) \sqsubseteq z$ then by **i** it follows $x \sqsubseteq z$.

If $(x, a) \sqsubseteq y \sqsubseteq z$ then by **ii** it follows $(x, a) \sqsubseteq z$.

If $(x, a) \sqsubseteq (y, b) \sqsubseteq z$ by **ii** it follows $(x, a) \sqsubseteq z$.

If $x \sqsubseteq (y, b) \sqsubseteq (z, c)$ then by **iii** it follows $x \sqsubseteq (z, c)$.

If $(x, a) \sqsubseteq y \sqsubseteq (z, c)$ then $x \neq y$ and $y \neq z$ it follows $x < z$, whence by **v** we conclude $(x, a) \sqsubseteq (z, c)$.

If $(x, a) \sqsubseteq (y, b) \sqsubseteq (z, c)$ we must contend with several sub-cases. If $a \leq c$ we conclude by **iv** that $(x, a) \sqsubseteq (z, c)$. If $a \not\leq c$ then either $a \not\leq b$, whence there exists $(u, v) \notin E$ such that $x \leq u \leq v \leq y$, or $b \not\leq c$, whence there exists $(u, v) \notin E$ such that $y \leq u \leq v \leq z$. Given either possibility we can conclude by **v** that $(x, a) \sqsubseteq (z, c)$. $\square$

It is convenient to note that if U and V are connected subsets of $\leq_{\mathcal{L}}$ such that $U \cup V$ is not connected, then

$$\mathcal{L} \star_{U \cup V} \mathcal{K} \cong (\mathcal{L} \star_U \mathcal{K}) \star_V \mathcal{K} \cong (\mathcal{L} \star_V \mathcal{K}) \star_U \mathcal{K}.$$

This is somewhat different than in the previous chapter as we are considering a set of edges rather than a set of points. The proof follows from a direct application of the definition of the partial order in the construction. When inflating, we therefore restrict our attention to connected sets.

There exists a canonical map $h : \mathcal{L} \star_E \mathcal{K} \rightarrow \mathcal{L}$ that behaves as projection onto the first coordinate for $E' \times K$, and acts as the identity on $L \setminus E'$. Note that h is order preserving.

We take this opportunity to consider several examples which illustrate the subtle, but extremely important, ways that this definition of inflation differs from the doubling constructions in the previous chapter. Let $\mathcal{L}$ be the partially ordered set illustrated in Figure 2.4. Note that $\mathcal{L}$ is obviously not a lattice.

Figure 2.4: The edge sets E (left), and F (right).

Let

$$\begin{aligned} E &= \{(a, y), (b, x)\} \cup \{(m, m) : m \in \mathcal{L}\}, \\ F &= \{(a, x), (a, y), (b, x), (b, y)\} \cup \{(m, m) : m \in \mathcal{L}\}. \end{aligned}$$

The ordered set $\mathcal{L} \star_E \mathbf{2}$ and $\mathcal{L} \star_F \mathbf{2}$ differ in that in the former, the edges (a, x) and (b, y) are “twisted” and we have $(a, 1) \leq (x, 0)$ and $(b, 1) \leq (y, 0)$ whereas for the latter this does not occur. Thus though we have inflated, in this case doubled, the same points as $E' = F'$ we have produced different ordered sets. The difference in our construction here and the doubling in the previous chapter is that there if $a, x \in E'$ it was assumed that the edge (a, x) would be included in the doubling construction, whereas with our new approach we may “twist” the edge (a, x) and produce a different partial order, placing $(a, 1)$ below $(x, 0)$.

Figure 2.5: A “twisted” doubling (left), and doubling (right).

Note also that we need not restrict ourselves to doubling. Indeed, we not not even restrict ourselves to inflating by lattice ordered sets. In the sequel we will be most interested in inflating finite lattices by finite lattices, so we take as an example $\mathcal{L} \star_E \mathcal{M}_3$, where $\mathcal{L}$ is the ordered set illustrated on the left side of Figure 2.4.

Figure 2.6: Inflation by the ordered set $\mathcal{M}_3$.

Our primary interest in investigating this generalized inflation construction is in those instances where $\mathcal{L}$ and $\mathcal{K}$ are finite lattices. Our next step is to specify those sets $E \subseteq \leq_{\mathcal{L}}$ such that $\mathcal{L} \star_E \mathcal{K}$ is a lattice.

A subset $E \subseteq \leq_{\mathcal{L}}$ is called *all-or-nothing* if E' is connected and the existence of a chain

$$\{(x_0, x_1), (x_1, x_2), (x_3, x_4), \dots, (x_{n-1}, x_n)\} \subseteq E$$

implies that if u, v are such that $x_0 \leq u < v \leq x_n$, then $(u, v) \in E$. This is a stronger condition than transitivity. Indeed, let $x < t < v < y$ and $x < u < y$ in $\mathcal{L}$. Then the set $F := \{(x, t), (t, v), (v, y)\}$ is transitive, but is not all-or-nothing as $(x, t), (t, y) \notin F$.

Given a convex subset C of a finite lattice $\mathcal{L}$, we can define an all or nothing set $E_C := \{(a, b) \in \leq_{\mathcal{L}} : a, b \in C\}$. In this case we have that $\mathcal{L}[C] \cong \mathcal{L} \star_E \mathbf{2}$. We think of convex sets as those all-or-nothing sets where we do not exercise our option of “or nothing,” rather we simply insist on “all.”

Theorem 2.2. *Given a lattice $\mathcal{L}$, a subset $E \subseteq \leq_{\mathcal{L}}$, and a lattice $\mathcal{K}$ with minimum and maximum, the ordered set $\mathcal{L} \star_E \mathcal{K}$ is a lattice if and only if E is all-or-nothing.*

Proof. We closely follow Reppe’s argument [R 11, Theorem 3.31, pg. 97], modifying as necessary for our new perspective of doubling. Assume that E is not an all-or-nothing subset, and let x/y be such that there exist $x \leq u < v \leq y$ such that $(u, v) \notin E$. We proceed by induction on the size of the longest chain in $x/y \cap E'$, call it n .

Let $n = 2$, and let $(x, a), (a, y) \in E$. Consider now the upper bounds of $(a, 0_{\mathcal{K}}), (x, 1_{\mathcal{K}})$ in $\mathcal{L} \star_E \mathcal{K}$. By the definition of the ordered set $\mathcal{L} \star_E \mathcal{K}$ and our assumptions regarding x, a , and y , we have that $(a, 1_{\mathcal{K}})$ and $(y, 0_{\mathcal{K}})$ are both lower bounds of the set $\{(a, 0_{\mathcal{K}}), (x, 1_{\mathcal{K}})\}$. However, by the definition of the ordered set $\mathcal{L} \star_E \mathcal{K}$ these elements are incomparable, whence $(a, 0_{\mathcal{K}}) \vee (x, 1_{\mathcal{K}})$ is undefined.

Assume for $n \leq k$ that the desired conclusion holds. Let $n = k + 1$. Let $w \in x/y$ be maximal with respect to the condition that there exist $x \leq s < t \leq w$ such that $(s, t) \notin E$. Let $a \in x/y$ be maximal with respect to $x \leq s < t \leq y$ implies $(s, t) \in E$, and $a \not\leq x$. Such elements must exist, or else we have contradicted our assumption that the interval x/y witnesses the failure of E to be all-or-nothing.

By our choice of a and x , we have that $a \leq s < t \leq a \vee x$ implies $(s, t) \in E$. If $a \wedge x \leq s < t \leq a$ implies $(s, t) \in E$, then as in the base case the set $\{(a, 0_K), (a \wedge, 1_K)\}$ has maximal lower bounds $(a, 1_K)$ and $(a \vee x, 0_K)$ that are incomparable in $\mathcal{L} \star_E \mathcal{K}$. On the other hand, if there exist $a \wedge x \leq s < t \leq a$ such that $(s, t) \notin E$, then the interval $a/a \wedge x$ also witnesses the failure of E to be all-or-nothing, whence the proof follows by the inductive assumption.

Assume that E is all-or-nothing and let $x, y \in \mathcal{L}$, with $a, b \in \mathcal{K}$. We must show that the join and meet operations are defined in the ordered set $\mathcal{L} \star_E \mathcal{K}$. We prove that join is well defined, and meet follows by duality. Note that for all possible cases, if $x \vee y \notin E'$ then the join of the elements in question is $x \vee y$ in $\mathcal{L} \star_E \mathcal{K}$, so we assume below without loss of generality that $x \vee y \in E'$.

- $x \vee y = (x \vee y, 0_K)$.
- $x \vee (y, b) = (x \vee y, 0_K)$ if $\exists x \leq s < t \leq x \vee y$ such that $(s, t) \notin E$.
- $x \vee (y, b) = (x \vee y, b)$ if $x \leq s < t \leq x \vee y \Rightarrow (s, t) \in E$.
- $(x, a) \vee (y, b) = (x \vee y, 0_K)$ if $x \leq s < t \leq x \vee y \Rightarrow (s, t) \in E$ and $x \leq u < v \leq y \Rightarrow (u, v) \in E$.
- $(x, a) \vee (y, b) = (x \vee y, a)$ if $x \leq s < t \leq x \vee y \Rightarrow (s, t) \in E$ and $\exists y \leq u < v \leq x \vee y$ such that $(u, v) \notin E$.
- $(x, a) \vee (y, b) = (x \vee y, b)$ if $y \leq u < v \leq x \vee y \Rightarrow (u, v) \in E$ and $\exists x \leq s < t \leq x \vee y$ such that $(s, t) \notin E$.
- $(x, a) \vee (y, b) = (x \vee y, a \vee b)$ $\exists x \leq s < t \leq x \vee y$ such that $(s, t) \notin E$ and $\exists y \leq u < v \leq x \vee y$ such that $(u, v) \notin E$.

This completes the proof. $\square$

When $\mathcal{L} \star_E \mathcal{K}$ is a lattice, the canonical map $h: \mathcal{L} \star_E \mathcal{K} \rightarrow \mathcal{L}$ is a lattice homomorphism. Before proceeding we take some time to compare this new inflation construction to the method used by Heiko Reppe in [R 11], using sets of points as in the previous chapter.

Definition 2.3. Let $\mathcal{L}$ be a lattice. A connected subset $M \subseteq \mathcal{L}$ is called *municipal* if and only if the following condition is satisfied for all x in $\mathcal{L}$: if there exists $a \in M$ such that $a \wedge x \in M$ and $a \vee x \in M$, then $x \in M$.

Given a connected subset S in a lattice $\mathcal{L}$, let E_S be the transitive closure of the set of edges $\{(x, y) \in \leq_{\mathcal{L}} : x, y \in S \text{ and } x \prec y\}$. When S is municipal we refer to such E_S a *municipal edge set*. It is not hard to see that E_S is all-or-nothing. It follows that if $\mathcal{K}$ is a lattice with maximum and minimum elements and $S \subseteq \mathcal{L}$ is a municipal set, then $\mathcal{L} \star_{E_S} \mathcal{K}$ is a lattice. Moreover, the converse holds, i.e., if $\mathcal{L} \star_{E_S} \mathcal{K}$ is a lattice then S is municipal by Reppe's theorem [R 11, Theorem 3.31, pg. 97]. It is also clear that not every all-or-nothing set corresponds to E_M for a municipal subset M . Indeed, this occurs in the all-or-nothing set that is inflated in Figure 2.3.

Theorem 2.4. Let $\mathcal{L}$ be a lattice and let $\mathcal{K}$ be a lattice with maximum and minimum elements. If $E \subseteq \leq_{\mathcal{L}}$ is an all-or-nothing set, then there exists a lattice $\hat{\mathcal{L}}$ and a municipal set $M \subseteq \hat{\mathcal{L}}$ such that $\mathcal{L} \star_E \mathcal{K} \leq \hat{\mathcal{L}} \star_{E_M} \mathcal{K}$.

Proof. Define $U := \{(t, v) \in \leq_{\mathcal{L}} : t \prec v, t, v \in E', (t, v) \notin E\}$. We construct the lattice $\hat{\mathcal{L}}$ as follows. The universe of $\hat{\mathcal{L}}$ is the set of elements $L \cup \{u_{(t,v)} : (t, v) \in U\}$. Each $u_{(t,v)}$ is both a join and meet irreducible element with unique lower cover t and unique upper cover v . The rest of $\leq_{\hat{\mathcal{L}}}$ is inherited from $\leq_{\mathcal{L}}$. Define the set $M \subseteq \hat{\mathcal{L}}$ by

$$M := \{m \in \hat{\mathcal{L}} : m \in E'\}.$$

Clearly $\mathcal{L} \star_E \mathcal{K}$ can be embedded in $\hat{\mathcal{L}} \star_{E_M} \mathcal{K}$ as an ordered set. We claim $\hat{\mathcal{L}} \star_{E_M} \mathcal{K}$ is a lattice. For elements in $\hat{\mathcal{L}} \star_{E_M} \mathcal{K}$ other than $u_{t,v}$ for some $(t, v) \in U$, meets and joins are defined as in $\mathcal{L} \star_E \mathcal{K}$. Let $u = u_{t,v}$ for some $(t, v) \in U$ and $x \in \hat{\mathcal{L}} \star_{E_M} \mathcal{K}$. If $x \geq u$ or $x \leq u$, then $x \wedge u$ and $x \vee u$ are defined trivially. Otherwise, we have that

$$u \wedge x = (v, 0_{\mathcal{K}}) \wedge x,$$

$$u \vee x = (t, 1_{\mathcal{K}}) \vee x.$$

Thus, it follows that $\hat{\mathcal{L}} \star_{E_M} \mathcal{K}$ is a lattice. We conclude by [R 11, Theorem 3.31, pg. 97] that M is municipal, as desired. $\square$

The importance of Theorem 2.4 lies in that it shows that our new approach to inflation extends previously known results. Indeed, any finite lattice that can be constructed by inflating municipal sets, such as convex sets as in the previous chapter, can be produced by inflating all or nothing sets. The link between our new construction and Reppe's goes further. We could forgo dealing with edges in a lattice by adding new doubly irreducible elements, inflating a municipal set, and then taking a sublattice by removing the elements we just added. We adopt the approach of using edges and all-or-nothing sets as it is far easier to classify lattices constructible in this way than it is to directly classify sublattices of finite lattices that can be constructed by inflating municipal sets.

For finite lattices $\mathcal{L}$ and $\mathcal{K}$, and an all-or-nothing set E , we have that $JI(\mathcal{L} \star_E \mathcal{K}) = J_{-1} \cup J_0 \cup J_1$ where

$$J_{-1} := \{x \in JI(\mathcal{L}) : x \notin E'\},$$

$$J_0 := \{(p, 0) : p \in JI(\mathcal{L}) \cap E'\},$$

$$J_1 := \{(x, a) : a \in JI(\mathcal{K}), w \prec x \Rightarrow (w, x) \notin E\}.$$

The join covers in $\mathcal{L} \star_E \mathcal{K}$ are more complicated than those that arise from the doublings considered in the previous chapter. However, when the lattices in question are finite, the canonical epimorphism $h : \mathcal{L} \star_E \mathcal{K} \rightarrow \mathcal{L}$ is bounded. This allows us to classify the join covers of $\mathcal{L} \star_E \mathcal{K}$ as follows.

Lemma 2.5. *Let $q \in J_{-1} \cup J_0$ and $S \subseteq \mathcal{L} \star_E \mathcal{K}$. Then $q \leq \bigvee S$ if and only if $h(q) = q \leq \bigvee_{s \in S} h(s)$ in $\mathcal{L}$. Moreover if S is a minimal non-trivial join cover of $q \in J_{-1} \cup J_0$, then $S \subseteq J_{-1} \cup J_0$.*

Proof. Let $h : \mathcal{L} \star_E \mathcal{K} \rightarrow \mathcal{L}$ be the canonical epimorphism and $\beta = \beta_h$. That $q \leq \bigvee S$ implies $q = h(q) \leq \bigvee_{s \in S} h(s)$ is trivial. Conversely, if $q \leq \bigvee_{s \in S} h(s)$ we have that

$$q \leq h(\bigvee S) \Rightarrow q = \beta(s) \leq \bigvee S.$$

For any $p \in JI(\mathcal{L} \star_E \mathcal{K})$ we have that $\beta(p) \in J_0 \cup J_{-1}$. It follows that if $q \leq \bigvee S$, then the set $R := \{\beta(s) : s \in S\}$ is such that $R \ll S$ and $q \leq \bigvee R$. $\square$

Lemma 2.6. *Let $(x, a) \in J_1$ and $S \subseteq \mathcal{L} \star_E \mathcal{K}$ with $(x, a) \leq \bigvee S$. If S is a minimal non-trivial join cover of (x, a) , then one of the following conditions holds:*

- $\bigvee S = z$, where there exist $x \leq u < v \leq z$, then $S \subseteq J_{-1} \cup J_0$.
- $\bigvee S = (z, 0)$, where there exist $x \leq u < v \leq z$, then $S \subseteq J_{-1} \cup J_0$.

- $\bigvee S = (z, c)$, where $v \neq 0_K$, $S = U \cup \{(y_i, b_i)\}_{i=1}^n$, $U = S \cap (J_{-1} \cup J_0)$, $x \leq_{\mathcal{L}} z = \bigvee U \vee \bigvee_{i=1}^n y_i$, and $c \leq \bigvee_{i=1}^n b_i$ for all $1 \leq i \leq n$.

Proof. Let $h : \mathcal{L} \star_E \mathcal{K} \rightarrow \mathcal{L}$ be the canonical epimorphism and $\beta = \beta_h$. If $\bigvee S = z$ or $\bigvee S = (z, 0)$ holds the argument proceeds as in Lemma 2.5.

Assume that $\bigvee S = (z, c)$ for $c \neq 0_K$. We have that $(z, 0) \leq \bigvee \{\beta(s) : s \in S\}$. Let D be a minimal non-trivial join cover of c . Let t be a minimal element of the set $S \cap E'$. Define $R := \bigcup \{\beta(s) : s \in S\} \cup \bigcup \{(t, d) : d \in D\}$. We conclude that $(z, c) \leq \bigvee R$ and $R \ll S$. $\square$

Having defined our new way of constructing finite lattices, we will seek in the next chapter to answer the question: what finite lattices can be constructed by inflation?

CHAPTER 3

INFLATION AND CONGRUENCES

3.1 Introduction

Having defined a new construction involving inflation by finite lattices, our goal is to classify those finite lattices that can be constructed in this manner. In the case of Day's doubling construction, such lattices can be classified by the properties of their congruences. This in turn allows such lattices to be classified in terms of a ranking system on their join irreducible elements that measures the depth of Φ_p in the ordered set of join irreducible congruences of such a lattice.

The new inflation construction allows us to inflate by any finite lattice $\mathcal{L}$. Clearly $\mathcal{L} \cong \mathbf{1} \star_{\{(1,1)\}} \mathcal{L}$, where $\mathbf{1}$ is the one element lattice and 1 denotes its single element, so in a trivial way we can say any finite lattice can be constructed by inflation. As we have seen in Chapter 1, some finite lattices can be constructed by a sequence of inflations using smaller finite lattices. We say that a finite lattice $\mathcal{L}$ is *inflatable* if

- $\mathcal{L}$ is simple, i.e., $|\mathcal{L}| > 1$ and the only congruence relations on $\mathcal{L}$ are trivial, or
- $\mathcal{L}$ can be constructed by a sequence of inflations by lattices $\mathcal{K}_1, \mathcal{K}_2, \dots, \mathcal{K}_n$ such that $|\mathcal{K}_i| < |\mathcal{L}|$ for all $1 \leq i \leq n$.

Not every finite lattice can be constructed in such a manner. Indeed, consider the lattice shown in Figure 3.1. It will be proven later that this lattice is not inflatable. This can also be seen by trial and error.

Figure 3.1: A lattice that is not inflatable.

In this chapter we will classify those finite lattices $\mathcal{L}$ that are inflatable. The proof of this result shows how to identify those finite lattices $\mathcal{K}_i$ that are used in the construction of $\mathcal{L}$. This will be vital in the next chapter when we apply the classification theorem to resolve a specific case of Problem 1 of [SL 13].

3.2 Classification of Inflatable Lattices

As with those lattices constructible by doubling convex sets, we will classify inflatable lattices by the properties of their congruences. We begin by examining what kind of congruences can arise from inflation.

Observe that if $\mathcal{L}$ and $\mathcal{K}$ are finite lattices and $E \subseteq \leq_{\mathcal{L}}$ is an all-or-nothing set and $h: \mathcal{L} \star_E \mathcal{K} \rightarrow \mathcal{L}$ is the canonical epimorphism, then

$$\alpha_h(x)/\beta_h(x) \cong \mathcal{K}$$

for every $x \in E'$. A *weaving polynomial* on a lattice $\mathcal{L}$ is a member of the set W of unary functions defined recursively by

- $w = id_{\mathcal{L}} \in W$,
- if $w \in W$ and $a \in \mathcal{L}$, then the functions u, v such that $u(x) = w(x) \wedge a$ and $v(x) = w(x) \vee a$ are in W ,
- only these functions are in W .

Thus every weaving polynomial looks something like

$$w(x) = (\dots (x \vee s_1) \wedge s_2) \vee \dots \vee s_{n-1}) \wedge s_n),$$

or dually, or with some successive joins and meets.

We say that intervals y/x and t/s in a lattice $\mathcal{L}$ are *projectively isomorphic* if $y/x \approx t/s$ and there exists a weaving polynomial w such that when restricted y/x we have that w is an isomorphism of the ordered subsets y/x and t/s .

Lemma 3.1. *Let $\mathcal{L}$ and $\mathcal{K}$ be finite lattices and let $E \subseteq \leq_{\mathcal{L}}$ be an all-or-nothing set. If $x, y \in E'$, then $\alpha_h(x)/\beta_h(x)$ is projectively isomorphic to $\alpha_h(y)/\beta_h(y)$ in $\mathcal{L} \star_E \mathcal{K}$.*

Proof. The canonical epimorphism $h: \mathcal{L} \star_E \mathcal{K} \rightarrow \mathcal{L}$ is bounded as $\mathcal{L}$ and $\mathcal{K}$ are finite. Let $\alpha := \alpha_h$ and $\beta := \beta_h$. As E' is connected there exists a sequence of elements of E' such that

$$\beta(x) = u_0 \leq v_1 \geq u_1 \leq \dots \leq v_n \geq u_n = \beta(y).$$

From the definition of the join operation in $\mathcal{L} \star_E \mathcal{K}$ we have that for every $1 \leq i \leq n$ and $a \in \mathcal{K}$

$$\begin{aligned} (u_{i-1}, a) \vee \beta(v_i) &= (v_i, a), \\ (v_i, a) \wedge \alpha(u_i) &= (u_i, a). \end{aligned}$$

We conclude that the weaving polynomial

$$w(x) = (\dots (x \vee \beta(v_1)) \wedge (\alpha(u_1)) \vee \dots \vee \beta(v_n)) \wedge \alpha(u_n)).$$

witnesses that $\alpha(x)/\beta(x)$ is projectively isomorphic to $\alpha(y)/\beta(y)$. □

For a finite lattice $\mathcal{L}$, let $Con(\mathcal{L})$ be the lattice of congruence relations of $\mathcal{L}$ ordered by inclusion.

Definition 3.2. *Let*

$$\theta_0 \leq \theta_1 \leq \theta_2 \leq \dots \leq \theta_n$$

be a chain in $Con(\mathcal{L})$ such that θ_0 is the identity relation and θ_n is the total relation. Given such a chain define

$$h_i: \mathcal{L}/\theta_{i-1} \rightarrow \mathcal{L}/\theta_i$$

to be the canonical epimorphism. For ease of notation define $\alpha_i := \alpha_{h_i}$ and $\beta_i := \beta_{h_i}$. We call such an ascending chain from $0_{Con(\mathcal{L})}$ to $1_{Con(\mathcal{L})}$ an inflation sequence if for all i such that $1 \leq i \leq n$ and $x, y \in \mathcal{L}/\theta_i$ such that $\beta_i(x) \neq \alpha_i(x)$, and $\beta_i(y) \neq \alpha_i(y)$:

- i** *there is an isomorphism $f_x: \mathcal{K} \rightarrow \alpha_i(x)/\beta_i(x)$ for a fixed finite lattice $\mathcal{K}$,*
- ii** *there exists a weaving polynomial*

$$w_{x,y}(a) = (\dots (a \vee s_1) \wedge s_2) \vee \dots \vee s_{n-1}) \wedge s_n),$$

such that $\alpha_i(x)/\beta_i(x)$ is projectively isomorphic to $\alpha_i(y)/\beta_i(y)$ via $w_{x,y}$,

- iii** *for any such $w_{x,y}$ we have that $f_y = w_{x,y} \circ f_x$.*

Conditions **i** and **ii** do not imply condition **iii**, as can be seen in the example $\mathcal{L}$ shown below in Figure 3.2. It looks at first like a three atom boolean algebra with all elements, except $0_{\mathcal{L}}$ and $1_{\mathcal{L}}$, inflated by 2×2 . However, the edges shown in bold demonstrate that the ordering of the elements is not the same. This lattice is subdirectly irreducible, i.e., has a unique minimal nontrivial congruence θ . The canonical homomorphism $h: \mathcal{L} \rightarrow \mathcal{L}/\theta$ satisfies conditions **i** and **ii**, but not **iii**.

Figure 3.2: A lattice $\mathcal{L}$ that fails condition **iii**.

Theorem 3.3. *A finite lattice $\mathcal{L}$ is inflatable if and only if there exists an inflation sequence in $\text{Con}(\mathcal{L})$.*

Proof. Assume that $\mathcal{L}$ is inflatable and $\{\theta_i: 0 \leq i \leq n\} \subseteq \text{Con}(\mathcal{L})$ is an inflation sequence. Let $\mathcal{K}$ be a finite lattice and $E \subseteq \leq_{\mathcal{L}}$ an all-or-nothing set. One direction of the theorem will be proven if we can show there is an inflation sequence in $\text{Con}(\mathcal{L} \star_E \mathcal{K})$.

Let $h: \mathcal{L} \star_E \mathcal{K} \rightarrow \mathcal{L}$ be the canonical epimorphism. Define ψ_0 to be the identity relation on $\mathcal{L} \star_E \mathcal{K}$. Given $S \subseteq \mathcal{L}$ let $Cg(S)$ denote the smallest congruence θ of $\mathcal{L}$ such that $a \theta b$ for all $a, b \in S$. For $1 \leq i \leq n+1$ define

$$\psi_i := Cg(\{(a, b) \in \mathcal{L}^2: (h(a), h(b)) \in \theta_{i-1}\}).$$

Observe that ψ_1 is the kernel of h , ψ_{n+1} is the universal relation on $\mathcal{L} \star_E \mathcal{K}$, and

$$\psi_0 \leq \psi_1 \leq \psi_2 \leq \dots \leq \psi_{n+1}.$$

We claim that $\{\psi_i: 0 \leq i \leq n+1\}$ is an inflation sequence. For $1 \leq i \leq n+1$ let

$$h_i: (\mathcal{L} \star_E \mathcal{K})/\psi_{i-1} \rightarrow (\mathcal{L} \star_E \mathcal{K})/\psi_i$$

be the canonical epimorphism. In the case of $i = 1$, conditions **i** and **ii** of the definition for an inflation sequence are satisfied as a result of Lemma 3.1. Condition **iii** is satisfied as a direct result of the definition of the join and meet operations on $\mathcal{L} \star_E \mathcal{K}$. The desired conclusion then follows from the assumption that $\{\theta_i: 0 \leq i \leq n\} \subseteq \text{Con}(\mathcal{L})$ is an inflation sequence of $\mathcal{L}$ by observing that

$$(\mathcal{L} \star_E \mathcal{K})/\psi_i \cong \mathcal{L}/\theta_{i-1}$$

for every $1 \leq i \leq n+1$.

Assume that $\mathcal{L}$ is a finite lattice and that $\{\theta_i: 0 \leq i \leq n\} \subseteq \text{Con}(\mathcal{L})$ is an inflation sequence. Proceed by induction on $|\mathcal{L}|$. By the inductive assumption $\mathcal{L}/\theta_1$ is inflatable, as we have that $\theta_1 \leq \theta_2 \leq \dots \leq \theta_n$ is an inflation sequence for $\mathcal{L}/\theta_1$. Let $a \in \mathcal{L}$ be such that $\beta_1(a) \neq \alpha_1(a)$. Let $\mathcal{K}$ be a finite lattice such that $\mathcal{K} \cong \alpha_1(x)/\beta_1(x)$. Define

$$E_x := \{(x, y) \in \leq_{\mathcal{L}/\theta_1}: \alpha_1(x)/\beta_1(x) \text{ and } \alpha_1(y)/\beta_1(y) \text{ are associates}\},$$

and

$$E := \bigcup_{\beta_1(x) \neq \alpha_1(x)} E_x.$$

Note that by condition **ii** in the definition of an inflation sequence, E is a connected set. We claim that E is an all-or-nothing set. Let

$$x = y_0 < y_1 < y_2 < y_3 < \dots < y_{n-1} < y_n = z$$

be a chain in $\mathcal{L}/\theta_1$, where $(y_{i-1}, y_i) \in E$ for all i such that $1 \leq i \leq n$. By assumption we have that $\alpha_1(x)/\beta_1(x)$ is projectively isomorphic to $\alpha_1(z)/\beta_1(z)$ via the weaving polynomial $p(t) = t \vee \beta_1(z)$. By our assumptions about x, y, z , and condition **iii** we have that the weaving polynomial

$$q(t) = ((t \vee \beta_1(y)) \vee \beta_1(z)) = t \vee \beta_1(z)$$

also witnesses that $\alpha_1(x)/\beta_1(x)$ is projectively isomorphic to $\alpha_1(z)/\beta_1(z)$. Therefore, $\alpha_1(x)/\beta_1(x)$ is projectively isomorphic to $\alpha_1(y)/\beta_1(y)$ as well. This suffices to demonstrate that E is an all-or-nothing set.

Figure 3.3: An illustration of the proof that E is an all-or-nothing set.

We wish to show that $\mathcal{L} \cong \mathcal{L}/\theta_1 \star_E \mathcal{K}$. Fix $u \in \mathcal{L}$ such that $\alpha_1(u) \neq \beta_1(u)$, let $f_u: \mathcal{K} \rightarrow \alpha_1(u)/\beta_1(u)$ be a lattice isomorphism. Note that as E is all-or-nothing for any $x \in E'$ there exists a weaving polynomial $w_{u,x}$ such that for all $a \in \mathcal{K}$ we have that $w_{u,x}(u, a) = (x, a)$ in $\mathcal{L}/\theta_1 \star_E \mathcal{K}$. Define $\phi: \mathcal{L}/\theta_1 \star_E \mathcal{K} \rightarrow \mathcal{L}$ as follows.

- $\phi(x) = x$ if $x \in (\mathcal{L}/\theta_1) \setminus E'$.
- $\phi((x, a)) = w_{u,x} \circ f_u(a)$ if $x \in E' \times K$.

It is straightforward to see that ϕ is well defined, one to one, and onto. Note however that ϕ could fail to be well defined in the absence of condition **iii** in the definition of an inflation sequence. It is less trivial to show that the order induced by the inflation construction is the same as the order on $\mathcal{L}$, i.e., that $u \leq v$ in $\mathcal{L}/\theta_1 \star_E \mathcal{K}$ if and only if $\phi(u) \leq \phi(v)$ in $\mathcal{L}$. We proceed on a case by case basis. Let $x, y \in \mathcal{L}/\theta_1$ and $a, b \in \mathcal{K}$.

If $x, y \in \mathcal{L}/\theta_1 \setminus E'$, then $x \leq y$ implies $\phi(x) \leq \phi(y)$ and vice versa.

Let $x \in \mathcal{L}/\theta_1 \setminus E'$ and $(y, b) \in E' \times K$. If $x \leq (y, b)$ then $x \leq y$ in $\mathcal{L}/\theta_1$, whence $x \leq \beta_1(y) \leq \phi(y, b)$. Conversely, if $x \leq \beta_1(y) \vee f_y(b)$ then as $h_1(x) = x$ we have that $x \leq y$ in $\mathcal{L}/\theta_1$, whence $x \leq (y, b)$.

Let $(x, a) \in E' \times K$ and $y \in \mathcal{L}/\theta_1 \setminus E'$. If $(x, a) \leq y$ then $x \leq y$ in $\mathcal{L}/\theta_1$, whence $\phi(x, a) \leq \alpha_1(x) \leq y$. Conversely, if $\beta_1(x) \vee f_x(a) \leq y$ then as $h_1(y) = y$ we have that $x \leq y$ in $\mathcal{L}/\theta_1$, whence $(x, a) \leq y$.

Let $(x, a), (y, b) \in E' \times K$. Assume for all $s, t \in \mathcal{L}/\theta_1$ that if $x \leq s < t \leq y$ then $(s, t) \in E$. If $(x, a) \leq (y, b)$ then $x \leq y$ and $a \leq b$ by assumption and the definition of the ordering in $\mathcal{L} \star_E \mathcal{K}$. By assumption and the definition of E , we have that $\alpha_1(x)/\beta_1(x)$ is an associate of $\alpha_1(y)/\beta_1(y)$ and

the mapping

$$p: \alpha_1(x)/\beta_1(x) \rightarrow \alpha_1(y)/\beta_1(y)$$

defined by $p(w) = w \vee \beta_1(y)$ witnesses that $\alpha_1(x)/\beta_1(x)$ is projectively isomorphic to $\alpha_1(y)/\beta_1(y)$. It follows that

$$f_x(a) \leq f_x(a) \vee \beta_1(y) = f_y(a) \leq f_y(b).$$

Conversely, if $f_x(a) \leq f_y(b)$ then $h_1(f_x(a)) = x$ and $h_1(f_y(b)) = y$, whence $x \leq y$. As before we have by assumption that $\alpha_1(x)/\beta_1(x)$ is an associate of $\alpha_1(y)/\beta_1(y)$. The mapping p defined as above witnesses that $\alpha_1(x)/\beta_1(x)$ is projectively isomorphic to $\alpha_1(y)/\beta_1(y)$. It follows that

$$f_x(a) \leq f_y(b) \Leftrightarrow f_x(a) \vee \beta_1(y) \leq f_y(b) \vee \beta_1(y) \Leftrightarrow f_y(a) \leq f_y(b).$$

We conclude that $a \leq b$ in $\mathcal{K}$.

Let $(x, a), (y, b) \in E' \times K$ and $s, t \in \mathcal{L}/\theta_1$ be such that $x \leq s < t \leq y$ and $(s, t) \notin E$. In this case the desired conclusion follows immediately as $(x, a) < (y, b)$ by condition **v** of the definition of the partial order of Lemma 2.1 for all $a, b \in \mathcal{K}$ and $\alpha_1(x) < \beta_1(y)$. $\square$

3.3 Consequences of the Classification Theorem

The classification theorem for inflatable lattices looks, at first glance, quite different from the classification theorem for congruence normal lattices. Clearly, such lattices are inflatable. Indeed, given a finite lattice $\mathcal{L}$ and a connected convex subset $C \subseteq \mathcal{L}$, we may define an all-or-nothing set

$$E := \{(x, y) \in \leq_{\mathcal{L}} : x, y \in C\}.$$

This is an easy all-or-nothing set to construct as we simply take all edges between any two elements we wish to double. We do not exercise our option to included “nothing” between two elements we double, we insist on “all” such edges every time. It is easy to see that $\mathcal{L}[C] \cong \mathcal{L} \star_E \mathbf{2}$. It follows that every congruence normal lattice has an inflation sequence. Because we inflated only by convex sets to construct congruence normal lattices, we can describe their inflation sequences in more detail than in the general case.

Let $\mathcal{L}$ be congruence normal and let P be the ordered set of join irreducible congruences of $\mathcal{L}$. Recall that $P = \{\Phi_p : p \in JI(\mathcal{L})\}$. Extend the ordering of P to be a linear order by choosing representative join irreducible elements p_k for each $\Phi_p \in P$ such that if $i < j$ then $\Phi_i \not\leq \Phi_j$. We may do this by taking all join irreducible congruences of maximal depth, i.e., those generated by join irreducible elements of maximal F -rank k , followed by all join irreducible congruences generated by join irreducible elements of F -rank $k - 1$, and so on. We may order the congruences generated by join irreducible elements of a given F -rank k arbitrarily. Let $\{\Phi_{p_k} : 1 \leq k \leq n\}$ be such a linear ordering on $JI(Con(\mathcal{L}))$. We may define an ascending chain of congruences from $0_{Con(\mathcal{L})}$ to $1_{Con(\mathcal{L})}$ by defining $\theta_0 = 0_{Con(\mathcal{L})}$ and

$$\theta_i := \bigvee \{\Phi_{p_k} : 1 \leq k \leq i\}$$

for i such that $1 \leq i \leq n$.

We claim that $\{\theta_i : 0 \leq i \leq n\}$ is an inflation sequence. We know from the proof of Theorem 1.12 that if $x \in \mathcal{L}/\theta_{i-1}$ and $\alpha_i(x) \neq \beta_i(x)$ then $\alpha_i(x)/\beta_i(x)$ is projectively isomorphic to the interval $h_{i-1}(p_i)/h_{i-1}(p_i)_*$. We conclude that $\{\theta_i : 0 \leq i \leq n\}$ is an inflation sequence.

Given an inflation sequence $\{\theta_i : 0 \leq i \leq n\} \subseteq Con(\mathcal{L})$, consider the interval θ_i/θ_{i-1} in $Con(\mathcal{L})$. If $\theta_{i-1} \leq \theta \leq \theta_i$ then $(x, y) \in \theta/\theta_{i-1}$ implies (x, y) is in the kernel of h_1 . As such, the covering pairs that generate θ are all in $\alpha_i(x)/\beta_i(x)$ for some $x \in \mathcal{L}/\theta_i$. When such a pair is collapsed, it will collapse exactly those pairs that it is projectively isomorphic to as a result of the inflation sequence. It follows that the interval θ_i/θ_{i-1} in $Con(\mathcal{L})$ is isomorphic to $Con(\mathcal{K})$, where $\mathcal{K} \cong \alpha_i(x)/\beta_i(x)$ for $x \in \mathcal{L}/\theta_{i-1}$ such that $\alpha_i(x) \neq \beta_i(x)$.

Recall that the class of lattices constructed by Day's doubling form a *prevariety*: this class is closed under homomorphic images, sublattices, and finite direct products. It is natural to ask if the class of inflatable lattices is also a prevariety. The class of inflatable lattices is closed under finite direct products as a result of the fact that if $\mathcal{L}_1$ and $\mathcal{L}_2$ are lattices then $Con(\mathcal{L}_1 \times \mathcal{L}_2) \cong Con(\mathcal{L}_1) \times Con(\mathcal{L}_2)$ [N2 95, Exercise 8, pg. 60].

To see that the class of inflatable lattices is closed under sublattices requires more care. Assume inductively that $\mathcal{L}$ is an inflatable lattice such that every sublattice $\mathcal{S}_{\mathcal{L}} \leq \mathcal{L}$ is inflatable. Let $E \subseteq \leq_{\mathcal{L}}$ be an all or nothing set and $\mathcal{K}$ a finite lattice. Given a sublattice $\mathcal{S} \leq \mathcal{L} \star_E \mathcal{K}$ we can construct $\mathcal{S}$ by beginning with some $\mathcal{S}_{\mathcal{L}} \leq \mathcal{L}$ and inflating an all-or-nothing subset of E by a sublattice $\mathcal{S}_{\mathcal{K}} \leq \mathcal{K}$. Note however that for an arbitrary sublattice $\mathcal{S}$ we may need to perform multiple inflations, as there is no guarantee that $\mathcal{S} \cap (E' \times \mathcal{K})$ is connected.

The class of inflatable lattices is, in general, not preserved under homomorphisms. It is not difficult to see that if $\{\theta_i : 0 \leq i \leq n\} \subseteq Con(\mathcal{L})$ is an inflation sequence then $\mathcal{L}/\theta_i$ is inflatable for every $0 \leq i \leq n$. However, if $\theta \in Con(\mathcal{L})$ is not part of an inflation sequence, then $\mathcal{L}/\theta$ may not be inflatable.

An example given by Sun and Li in [SL 13] illustrates this. We will refer to this example again in the next chapter. The lattice pictured below in Figure 3.4 is congruence normal. We then inflate the all-or-nothing set E illustrated by bold edges by $\mathbf{2}$ to produce the lattice shown on the left hand side of Figure 3.5. Note that $\mathcal{L} \star_E \mathbf{2}$ fails to be congruence normal.

Figure 3.4: The congruence normal lattice $\mathcal{L}$.

The element labeled p in Figure 3.4 is doubled and produces two new join irreducible elements in $\mathcal{L} \star_E \mathbf{2}$. In Figure 3.5, we label these as $q := (p, 0)$ and $r := (p, 1)$.

Figure 3.5: The lattice $\mathcal{L} \star_E \mathbf{2}$ (left) and $Con(\mathcal{L} \star_E \mathbf{2})$ (right).

Because it is the congruence produced by the last inflation in the sequence that produced $\mathcal{L} \star_E \mathbf{2}$ the congruence Φ_r is the least element in the inflation sequence that corresponds to how the lattice was originally constructed. Although Φ_q has the same depth as Φ_r , we cannot use Φ_q in an inflation sequence for $\mathcal{L} \star_E \mathbf{2}$. This is because $(\mathcal{L} \star_E \mathbf{2})/\Phi_q$ is isomorphic to the lattice pictured in Figure 3.6. This lattice has a unique minimal nontrivial congruence, Φ_s , which has a block that contains a three element chain. It follows that there is no inflation sequence for $(\mathcal{L} \star_E \mathbf{2})/\Phi_q$.

Figure 3.6: An image of $\mathcal{L} \star_E \mathbf{2}$ that is not inflatable.

Thus the lattice $\mathcal{L} \star_E \mathbf{2}$ in Figure 3.5 is inflatable because we can form an inflation sequence by collapsing the join prime congruences Φ_r , Φ_q , Φ_t , Φ_u , and Φ_s in that order. However, if we collapse Φ_q first, we have no choice but to collapse Φ_r next, which, as can be seen in Figure 3.6, results in a sequence of congruences that is not an inflation sequence.

We say that if y/x and v/u are intervals in an ordered set and $y < u$ then the intervals are *comparable*. In congruence normal lattices, intervals that form blocks in the kernels of the maps generated by an inflation sequence are never projectively isomorphic and comparable. However, for inflatable lattices this may occur, and collapsing the interval u/y may produce a homomorphic image of an inflatable lattice that is not inflatable. In particular, if for an inflatable lattice $\mathcal{L}$ let y/x and v/u be projective intervals that are comparable. A homomorphism that collapses the interval u/y but does not collapse the interval y/x , as in the previous examples, may result in a lattice that is not inflatable.

While inflatable lattices do not form a prevariety, when we restrict our choices of what lattices $\mathcal{K}$ we use to inflate all-or-nothing sets there are still notable structural properties that are preserved by inflation. We say a lattice $\mathcal{L}$ is *projectively simple* if $\mathcal{L}$ is simple and $p \approx q$ for all $p, q \in JI(\mathcal{L})$. When we restrict ourselves to inflating by finite projectively simple lattices, we obtain a subclass of those lattices such that $F_k(\mathcal{L}) = JI(\mathcal{L})$ for some $k \in \omega$.

Theorem 3.4. *Let $\mathcal{L}$ be a finite lattice such that $F_k(\mathcal{L}) = JI(\mathcal{L})$ for some $k \in \omega$, $E \subseteq \leq_{\mathcal{L}}$ be an all-or-nothing set, and $\mathcal{K}$ be a finite projectively simple lattice. Then $F_{k+1}(\mathcal{L} \star_E \mathcal{K}) = JI(\mathcal{L} \star_E \mathcal{K})$.*

Proof. Recall that $JI(\mathcal{L} \star_E \mathcal{K}) = J_{-1} \cup J_0 \cup J_1$ where

$$\begin{aligned} J_{-1} &:= \{x \in JI(\mathcal{L}) : x \notin E'\}, \\ J_0 &:= \{(p, 0) : p \in JI(\mathcal{L}) \cap E'\}, \\ J_1 &:= \{(x, a) : a \in JI(\mathcal{K}), w \prec x \Rightarrow (w, x) \notin E\}. \end{aligned}$$

As $\mathcal{K}$ is simple, we have that if $p \in J_1$ then Φ_p is the kernel of the canonical homomorphism

$h: \mathcal{L} \star_E \mathcal{K} \rightarrow \mathcal{L}$. Let $p, q \in J_1$ with $p \neq q$. We have that $p \in \alpha_h(x)/\beta_h(x)$ and $q \in \alpha_h(y)/\beta_h(y)$ for some $x, y \in \mathcal{L}$. As $\mathcal{K}$ is projectively simple and $\alpha_h(x)/\beta_h(x)$ is projectively isomorphic to $\alpha_h(y)/\beta_h(y)$, we conclude that $p \approx q$. As $F_k(\mathcal{L}) = JI(\mathcal{L})$, it follows by Lemma 1.9 that $F_n(\mathcal{L} \star_E \mathcal{K}) = JI(\mathcal{L} \star_E \mathcal{K})$ for some $n \in \omega$. Moreover, by Corollary 1.10 we have that $rk_F(p)$ is equal to the depth of Φ_p for every $p \in JI(\mathcal{L} \star_E \mathcal{K})$. In particular, if $p \in J_1$, then $rk_F(p) \leq k + 1$. This completes the proof. $\square$

We can see from the lattice shown in Figure 3.7 that not every lattice for which $F_k(\mathcal{L}) = JI(\mathcal{L})$ for some $k \in \omega$ can be constructed from the one element lattice by inflating all-or-nothing sets by projectively simple lattices. The two minimal nontrivial congruences of $\mathcal{L}$ are Φ_p and Φ_q , where p and q are as shown in Figure 3.7. Neither Φ_p nor Φ_q satisfies the conditions necessary to be part of an inflation sequence, whence $\mathcal{L}$ is not inflatable.

Figure 3.7: A lattice $\mathcal{L}$ disproving the converse of Theorem 3.4.

CHAPTER 4

BINARY CUT-THROUGH CODABLE LATTICES

4.1 Introduction

The original question that began the investigation that led to the current results was: *can we classify those lattices obtainable by doubling Reppe’s municipal sets in a way similar to the way Day classified those lattices obtainable by doubling convex sets?* This investigation led us to formulate inflation by all-or-nothing sets in order to close this class under sublattices. A classification of lattices constructed in this way turns out to be closely related to the work of Sun and Li in [SL 13]. At first Reppe’s work seemed entirely independent of that of Sun and Li, but in fact “binary cut-through coding” (which we will describe presently) gives a characterization of those lattices which can be obtainable by inflating by the two element lattice.

In this chapter we summarize those findings in [SL 13] that are necessary to understand how their work relates to the inflation construction. We then show how the classification theorem for inflatable lattices can be used to address questions regarding binary cut-through codable lattices. Other facets of their work are of interest in their own right, as is the connection between their coding scheme and electrical engineering. We refer the reader to [SL 13] for those topics.

4.2 Cut-Through Codability

An n -ary *cut-through code* of a finite lattice $\mathcal{L}$ is a scheme to encode all elements of $\mathcal{L}$ by distinct n -ary strings of letters of uniform length k such that for $j \leq k$, the initial j letters in the strings that encode any two given elements $x, y \in \mathcal{L}$ determine the initial j letters in the strings that encode $x \wedge y$ and $x \vee y$. We will use an equivalent formulation first shown in [SL 13, Proposition 4, pg. 273]. Recall that for a lattice congruence θ we use $\|\theta\|$ to denote the cardinality of the largest block of θ .

Definition 4.1. *A finite lattice $\mathcal{L}$ is called n -ary cut-through codable if and only if there exists a chain*

$$0_{Con(\mathcal{L})} = \theta_0 \leq \theta_1 \leq \dots \leq \theta_k = 1_{Con(\mathcal{L})}$$

such that $\|\theta_j/\theta_{j-1}\| \leq n$ for all $j \leq k$.

Define the *cut-through index* of a finite lattice $\mathcal{L}$ to be the smallest integer n such that there exists an n -ary cut-through code on $\mathcal{L}$. We denote the cut-through index of $\mathcal{L}$ by $Ct(\mathcal{L})$.

A few remarks are in order. The first is that n -ary cut-through codability is preserved under the formation of sublattices and finite direct products. However, for every n there exists a lattice that is n -ary cut-through codable, but has a homomorphic image that fails to be n -ary cut-through codable. We refer the reader to [SL 13, pg. 275] for illustrations of lattices that prove this.

Of particular interest to us, however, is Sun and Li’s example showing that the class of binary cut-through codable lattices is not closed under homomorphic images. We have seen it before in the previous chapter when we showed that the class of inflatable lattices is not closed under homomorphic images. We construct this lattice by a sequence of inflations shown in Figure 4.1, culminating in the familiar example. We have added the binary cut-through coding of the elements at each stage. Note that in the second lattice the only edge that is inflated is the trivial edge $(01, 01)$, which we illustrate by enlarging the point 01 in the Hasse diagram. The last inflation will result in the lattice shown in Figure 4.2.

Constructing the code as we inflate, we see that if a particular element is on an edge that is inflated, we append a 0 to the existing string for the lower of the two resulting elements and a 1 to the existing string for the greater. If an element is not an edge that is doubled, we append a 1 to

Figure 4.1: The construction of the binary cut-through code.

Figure 4.2: A binary cut-through code (left), and inflation sequence (right).

its string if it is above an element that was on an edge that is doubled, otherwise we append a 0. Thus, the coding is a function mapping every $x \in \mathcal{L}$ to a sequence $\delta_1(x)\delta_2(x) \dots \delta_5(x)$. The value of $\delta_i(x)$ is either 0 or 1 as determined by the procedure above.

Referring to the elements of $\mathcal{L}$ by their binary cut-through code, we note that the inflation sequence corresponds to ideals of join irreducible congruences in the following way:

$$\begin{aligned}\theta_1 &= \Phi_{01001} \\ \theta_2 &= \theta_1 \vee \Phi_{00010} \\ \theta_3 &= \theta_2 \vee \Phi_{01110} \\ \theta_4 &= \theta_3 \vee \Phi_{01000} \\ \theta_5 &= \theta_4 \vee \Phi_{10000}.\end{aligned}$$

We can see from this example how to construct a binary cut-through code given a sequence of congruences that witnesses the lattice is binary cut-through codable. Taking the homomorphic images $\mathcal{L}/\theta_i$ for each θ_i in the inflation sequence, we may reconstruct how $\mathcal{L}$ was inflated. At this point, we may determine the coding as we did before.

Such binary cut through codes give a representation of a lattice as subsets of the power set of a k -element finite set. However, as can be seen from the example above such a representation does not respect joins and meets. The construction does, however, give the minimal k for which such a representation that respects $\leq_{\mathcal{L}}$ can be created.

Theorem 4.2. *A finite lattice $\mathcal{L}$ is binary cut-through codable if and only if it is inflatable and there exists a sequence of inflations to construct $\mathcal{L}$ such that every inflation $\mathcal{L}$ is by $\mathbf{2}$.*

Proof. Assume that $\mathcal{L}$ is inflatable and every inflation used to construct $\mathcal{L}$ is by $\mathbf{2}$. By Theorem 3.3 we know $\mathcal{L}$ has an inflation sequence $\{\theta_i: 0 \leq i \leq n\}$. By assumption for every $h_i: \mathcal{L}/\theta_{i-1} \rightarrow \mathcal{L}/\theta_i$ we have that $|h_i^{-1}(x)| \leq 2$ for all $x \in \mathcal{L}/\theta_{i-1}$. It follows that the inflation sequence witnesses that $\mathcal{L}$ is binary cut-through codable.

Conversely, assume that $\mathcal{L}$ is binary cut-through codable. Let $\{\theta_i: 0 \leq i \leq n\} \subseteq \text{Con}(\mathcal{L})$ be a chain that witnesses this. We claim that $\{\theta_i: 0 \leq i \leq n\}$ is an inflation sequence. If we prove that conditions **i**, **ii**, and **iii** of Definition 3.2 hold for $h_1: \mathcal{L} \rightarrow \mathcal{L}/\theta_1$, the desired conclusion holds by induction on n .

We first reduce to the case that $\theta_{i-1} \prec \theta_i$ for every i such that $1 \leq i \leq n$. Assume there exists some i such that there is a congruence θ that satisfies $\theta_{i-1} < \theta < \theta_i$. By the assumption that $\mathcal{L}$ is binary cut-through codable we have that $\|\theta_i/\theta\| = 2$ and $\|\theta/\theta_{i-1}\| = 2$. Thus we can include the congruence θ in a new sequence of congruences that would also witness that $\mathcal{L}$ is binary cut-through codable. As such from here on we will assume, without loss of generality, that $\theta_{i-1} \prec \theta_i$ for every i such that $1 \leq i \leq n$.

By assumption we know **i** holds for the given chain. To prove **ii**, note that we have $\theta_1 = \Phi_p$ for some $p \in \text{JI}(\mathcal{L})$, whence by assumption $\|\Phi_p\| = 2$. Moreover, $0_{\text{Con}(\mathcal{L})} \prec \Phi_p$, whence the set of elements $\{x \in \mathcal{L}: |x/\Phi_p| = 2\}$ is connected. Let $(a, b) \in \Phi_p$. If a and b were incomparable this would imply that $\|\Phi_p\| > 2$, so we say $a < b$. We claim that p/p_* is projectively isomorphic to b/a .

If $(a, b) \in \Phi_p$ with $a < b$ then there exists a sequence of weaving polynomials $w_1, w_2, \dots, w_k$ such that

$$a = w_1(p_*) < w_1(p) = w_2(p_*) < w_2(p) = \dots = w_k(p_*) < w_k(p) = b.$$

As $\|\Phi_p\| = 2$, we have that $k = 1$. For any weaving polynomial w we have that $w(p_*) \prec w(p)$, once again by the fact that $\|\Phi_p\| = 2$. It follows that p/p_* is projectively isomorphic to b/a by the weaving polynomial w_1 , whence $\mathcal{L}$ satisfies **ii**. We conclude that $\mathcal{L}$ satisfies **iii** from the fact that the only endomorphism of $\mathbf{2}$ is the identity map. $\square$

In the case of binary cut-through codable lattices, because every inflation is by $\mathbf{2}$, we know that any inflation sequence consists of covering relations. If $\mathcal{L}$ is binary cut-through codable and $\{\theta_i: 0 \leq i \leq n\} \subseteq \text{Con}(\mathcal{L})$ is a chain that witnesses this, we know from the proof of Theorem 4.2 that $\{\theta_i: 0 \leq i \leq n\}$ is an inflation sequence. As such, we may assume $\theta_{i-1} \prec \theta_i$ for all $1 \leq i \leq n$. As such, $\theta_i = \theta_{i-1} \vee \Phi_p$ for some $p \in \text{JI}(\mathcal{L})$. This resembles the kind of inflation sequence that we saw produced congruence normal lattices. We wish to know what other properties of congruence normal lattices are preserved in binary cut-through codable lattices.

Lemma 4.3. *If $\mathcal{L}$ is binary cut-through codable, then $F_k(\mathcal{L}) = \text{JI}(\mathcal{L})$ for some $k \in \omega$ and $\mathcal{L}$ is $\mathbf{2}$ -simple.*

Proof. Let $\mathcal{L}$ be binary cut-through codable. It suffices to show that if $F_k(\mathcal{L}) = \text{JI}(\mathcal{L})$ and $\mathcal{L}$ is $\mathbf{2}$ -simple, then for any all-or-nothing set $E \subseteq \leq_{\mathcal{L}}$ we have $F_{k+1}(\mathcal{L} \star_E \mathbf{2}) = \text{JI}(\mathcal{L} \star_E \mathbf{2})$ and $\mathcal{L} \star_E \mathbf{2}$ is $\mathbf{2}$ -simple.

We can classify the elements of $\text{JI}(\mathcal{L} \star_E \mathbf{2})$ using Lemma 2.5 and Lemma 2.6.

$$\text{JI}(\mathcal{L} \star_E \mathbf{2}) = J_{-1} \cup J_0 \cup J_1$$

where

$$\begin{aligned} J_{-1} &:= JI(\mathcal{L}) \setminus E', \\ J_0 &:= \{(p, 0) : p \in JI(\mathcal{L}) \cap E'\}, \\ J_1 &:= \{(x, 1) : x \text{ has no lower cover in } E'\}. \end{aligned}$$

Moreover, if $p, q \in J_1$ then $\Phi_p = \Phi_q$, as each of the congruences is the kernel of the canonical epimorphism $h : \mathcal{L} \star_E \mathbf{2} \rightarrow \mathcal{L}$. By Lemma 1.9 we conclude that $p \approx q$.

Let $p = (x, 1)$ for some $x \in E'$ be such that $J_1 = p / \approx$. If Φ_p is not a maximal join prime congruence of $\mathcal{L} \star_E \mathbf{2}$, then the simple homomorphic images of $\mathcal{L} \star_E \mathbf{2}$ are the same as the simple homomorphic images of $\mathcal{L}$ and the proof is complete. If Φ_p is a maximal join prime congruence, it follows that $p \not D q$ fails for any $q \in JI(\mathcal{L} \star_E \mathbf{2}) \setminus J_1$. It follows that J_1 is a D closed set and $(\mathcal{L} \star_E \mathbf{2}) / \Psi_p \cong J_1^\vee \cong \mathbf{2}$. $\square$

Figure 4.3: The lattice $\mathcal{L}$ (left), and its image $\mathcal{L}/\Phi_p$ (right).

While the conditions in Lemma 4.3 are necessary for a finite lattice $\mathcal{L}$ to be binary cut-through codable, they are not sufficient. Consider the two lattices in Figure 4.3. While $\mathcal{L}$ is binary cut-through codable, $\mathcal{L}/\Phi_p$ is not, despite the fact that $F_1(\mathcal{L}/\Phi_p) = JI(\mathcal{L})$ and $\mathcal{L}/\Phi_p$ is $\mathbf{2}$ -simple. Unlike in the case of congruence normal lattices, we cannot collapse the join prime congruences of $\mathcal{L}$ in any order that respects depth. Indeed, Φ_p is of maximal depth in $JI(Con(\mathcal{L}))$.

Given a finite lattice $\mathcal{L}$ such that $F_k(\mathcal{L}) = JI(\mathcal{L})$ for some $k \in \omega$ and $\mathcal{L}$ is $\mathbf{2}$ -simple, how can we ensure that there exists an inflation sequence in $Con(\mathcal{L})$ that witnesses that $\mathcal{L}$ is binary cut-through codable? If $\mathcal{L}$ is congruence normal, any sequence of congruences such that

$$0_{Con(\mathcal{L})} = \theta_0 \leq \theta_1 \leq \dots \leq \theta_k = 1_{Con(\mathcal{L})},$$

will witness that $\mathcal{L}$ is binary cut-through codable. Should there exist $p \in JI(\mathcal{L})$ and $m \in MI(\mathcal{L})$ such that $p \leq m$ and $p \approx m$, more care is needed, as shown by the example in Figure 4.3.

Specifically, let a be maximal with respect to the property that $a \leq m$ there exists $a_\dagger$ such that $a_\dagger < a$ and $(a_\dagger, a) \in \Phi_p$. Similarly, let b be minimal with respect to the property that $b \geq p$ there exists $b^\dagger$ such that $b^\dagger > b$ and $(b^\dagger, b) \in \Phi_p$. We have that there exists $Q \subseteq JI(\mathcal{L})$ such that $Cg(a, b) = \bigvee \{\Phi_q : q \in Q\}$. We need that there is an order in which we can collapse the join prime congruences of $\mathcal{L}$ such that Φ_p is collapsed before Φ_q for some $q \in Q$.

Theorem 4.4. *If $\mathcal{L}$ is binary cut-through codable, then $\mathcal{L}$ satisfies the following:*

- i $F_k(\mathcal{L}) = JI(\mathcal{L})$ for some $k \in \omega$.
- ii $\mathcal{L}$ is $\mathbf{2}$ -simple.
- iii If $p \in JI(\mathcal{L})$, $m \in MI(\mathcal{L})$, $p \approx m$ and $p \leq m$ then there exists $q \in JI(\mathcal{L})$ such that $q \leq m$, $q \not\leq p$, and $\Phi_q \not\subseteq \Phi_p$.

Proof. We have by Lemma 4.3 that $\mathcal{L}$ satisfies **i** and **ii**. Let $\{\theta_i : 1 \leq i \leq n\}$ be an inflation sequence for $\mathcal{L}$ that witnesses the binary cut-through codability of $\mathcal{L}$. Let $p \in JI(\mathcal{L})$ and $m \in MI(\mathcal{L})$ be such that $p \approx m$ and $p \leq m$. Choose i such that $\Phi_p \not\leq \theta_{i-1}$ and $\Phi_p \leq \theta_i$. Let $h : \mathcal{L} \rightarrow \mathcal{L}/\theta_{i-1}$ be the canonical homomorphism. We know $h(p) \neq h(m)$, for otherwise $\|\theta_i/\theta_{i-1}\| \geq 3$. As $\Phi_p \not\leq \theta_{i-1}$ we have $h(p_*) < h(p) < h(m) < h(m^*)$. This implies that there exists $q \in JI(\mathcal{L})$ such that $q \leq m$, $q \not\leq p$, $\Phi_q \not\leq \theta_{i-1}$, and $\Phi_q \not\leq \Phi_p$. $\square$

The question immediately occurs: are the necessary conditions for binary cut-through codability in Theorem 4.4 sufficient for binary cut-through codability? Let $\mathcal{L}$ be a finite lattice satisfying these conditions. Using condition **ii** shows that if the height of $Con(\mathcal{L})$, that is the cardinality of the largest chain from $0_{Con(\mathcal{L})}$ to $1_{Con(\mathcal{L})}$, is 2 then clearly $\mathcal{L}$ is binary cut-through codable. Proceeding by induction on the height of $Con(\mathcal{L})$, let $q \in JI(\mathcal{L})$ be of maximal rank k . By **i** we have that $\|\Phi_q\| = 2$, thus it suffices to show that $\mathcal{L}/\Phi_q$ satisfies **i**, **ii**, and **iii**.

Clearly $\mathcal{L}/\Phi_q$ satisfies **i** and **ii** for any $q \in JI(\mathcal{L})$. As long as we assume there exists some $q \in JI(\mathcal{L})$ of maximal rank k such that for no $p \in JI(\mathcal{L})$ such that $p \neq q$ does there exist $m \in MI(\mathcal{L})$ such that $p \approx m$, $p \leq m$, $q \leq m$, and $q \not\leq p$ then we may conclude $\mathcal{L}/\Phi_q$ satisfies **iii**. However, at present we are unable to rule out the possibility that for some finite lattice $\mathcal{L}$ satisfying **i**, **ii**, and **iii** no such $q \in JI(\mathcal{L})$ exists.

4.3 Lattice Varieties and Cut-through Codes

Let $\mathbf{K}$ be a class of lattices. The lattice *variety* generated by $\mathbf{K}$, denoted $Var(\mathbf{K})$, is the class of all homomorphic images of sublattices of direct products of lattices in $\mathbf{K}$. One of the most important consequences of lattices being congruence distributive is that varieties of lattices form a distributive lattice when ordered by inclusion, see [N2 95, Theorem 7.3, pg. 78]. We denote this lattice by Λ .

Definition 4.5. A lattice variety $\mathbf{V}$ is called *n-ary cut-through codable* if and only if for every finite $\mathcal{L} \in \mathbf{V}$ if $\mathcal{L}$ is binary cut-through codable.

As we have seen, *n*-ary codability is preserved by finite direct products and sublattices. It follows that $Var(\mathbf{K})$ is *n*-ary cut-through codable if and only if every finite lattice $\mathcal{L}$ that is a homomorphic image of a lattice in $\mathbf{K}$ satisfied $Ct(\mathcal{L}) \leq n$. The following was first shown by Sun and Li as [SL 13, Proposition 8, pg. 277].

Lemma 4.6. The class of *n*-ary cut-through codable varieties forms a lattice ideal in Λ , the lattice of all lattice varieties ordered by inclusion.

Proof. Let $\mathbf{V}_1$ and $\mathbf{V}_2$ be *n*-ary cut-through codable lattice varieties. Clearly $\mathbf{V}_1 \wedge \mathbf{V}_2$ is *n*-ary cut-through codable. Let $\mathcal{L} \in \mathbf{V}_1 \vee \mathbf{V}_2$ be a finite lattice. We know by Birkhoff's Subdirect Representation Theorem, see [B 35], that $\mathcal{L}$ can be represented as a sublattice of subdirectly irreducible lattices in the variety $Var(\{\mathcal{L}\})$, which is the smallest variety containing $\mathcal{L}$. As a consequence of Jónnson's Lemma there are finitely many subdirectly irreducible lattices in $Var(\{\mathcal{L}\})$, see [J 67], and each of these subdirectly irreducible lattices is finite. It follows that $\mathcal{L}$ is a sublattice of a direct product of finite lattices $\mathcal{L}_1, \mathcal{L}_2, \dots, \mathcal{L}_k$ in $Var(\{\mathcal{L}\}) \subseteq \mathbf{V}_1 \vee \mathbf{V}_2$. As another consequence of Jónnson's Lemma $\mathcal{L}_i \in \mathbf{V}_1$ or $\mathcal{L}_i \in \mathbf{V}_2$ for every i such that $1 \leq i \leq k$. Each such $\mathcal{L}_i$ is *n*-ary cut-through codable, whence the proof is complete. $\square$

It is known that for any $n \geq 2$ there exists an *n*-ary cut-through codable lattice $\mathcal{L}$ and a congruence $\theta \in Con(\mathcal{L})$ such that $Ct(\mathcal{L}/\theta) > n$, see [SL 13, Proposition 6, pg. 274]. It follows that there exists a strictly ascending chain $\mathbf{I}_1 \subset \mathbf{I}_2 \subset \dots \subset \mathbf{I}_n \subset \dots$ of ideals in Λ where $\mathbf{I}_n$ consists of all *n*-ary cut-through codable lattice varieties.

Clearly any finite lattice $\mathcal{L}$ such that $Var(\mathcal{L}) \in \mathbf{I}_2$ is binary cut-through codable, whence $\mathcal{L}$ satisfies conditions **i**, **ii**, and **iii** from Theorem 4.4. However, not every lattice $\mathcal{L}$ that is binary

cut-through codable satisfies $\text{Var}(\mathcal{L}) \in \mathbf{I}_2$, e.g., the lattice $\mathcal{L}$ illustrated in Figure 4.3. We say that a finite lattice $\mathcal{L}$ is *strongly n -ary cut-through codable* if and only if $\text{Var}(\mathcal{L}) \in \mathbf{I}_n$, i.e., every $0 - 1$ spanning sequence in $\text{Con}(\mathcal{L})$ is an inflation sequence witnessing that $\mathcal{L}$ is n -ary cut-through codable.

Recall that for a finite lattice $\mathcal{L}$ and $p \in \text{JI}(\mathcal{L})$, we let Ψ_p denote the greatest congruence such that does not contain (p, p_*) . Every meet irreducible congruence of $\mathcal{L}$ is of the form Ψ_p for some $p \in \text{JI}(\mathcal{L})$.

Corollary 4.7. *A finite lattice $\mathcal{L}$ is strongly n -ary cut through codable if and only if $\mathcal{L}/\Psi_p$ is strongly n -ary cut-through codable for every $p \in \text{JI}(\mathcal{L})$.*

Proof. Every finite lattice $\mathcal{L}$ is isomorphic to a sublattice of the direct product of subdirectly irreducible lattices $\prod_{p \in \text{JI}(\mathcal{L})} \mathcal{L}/\Psi_p$, see [N2 95, Theorem 5.5, pg. 54]. The proof then follows immediately as strongly n -ary cut-through codable lattices are closed under homomorphic images of sublattices of direct products. $\square$

This reduces the problem of determining whether a given finite lattice is n -ary cut-through codable to determining whether, given a finite subdirectly irreducible lattice $\mathcal{L}$ with unique minimal nontrivial congruence μ , if $\|u\| \leq n$ and $\mathcal{L}/\mu$ is n -ary cut-through codable. Note that while Theorem 4.7 gives us a way to check whether a given finite lattice is strongly n -ary cut through codable, we do not have a similar result using the weaker assumption of n -ary cut-through codability. Indeed, referring back to Figure 3.5 and Figure 3.6, note that $\Phi_q = \Psi_r$. Thus $\mathcal{L}$ from Figure 3.5 is binary cut-through codable, but its subdirectly irreducible homomorphic image $\mathcal{L}/\Psi_r$, pictured in Figure 3.6, is not binary cut through codable.

Theorem 4.8. *A finite lattice $\mathcal{L}$ is strongly binary cut-through codable if and only if*

- i** $F_k(\mathcal{L}) = \text{JI}(\mathcal{L})$ for some $k \in \omega$,
- ii** $\mathcal{L}$ is 2-simple,
- iii** If $p \in \text{JI}(\mathcal{L})$, $m \in \text{MI}(\mathcal{L})$, $p \approx m$, and $p \leq m$, then there exists $q \in \text{JI}(\mathcal{L})$ such that $q \leq m$, $q \not\leq p$, and $\Phi_p < \Phi_q$.

Proof. Assume that $\mathcal{L}$ is strongly binary cut-through codable. By Theorem 4.4 we know $\mathcal{L}$ satisfies **i** and **ii**. Let $p \in \text{JI}(\mathcal{L})$ and $m \in \text{MI}(\mathcal{L})$ be such that $p \approx m$, and $p \leq m$. Let $m' \in \mathcal{L}$ be minimal such that $p \leq m'$ and there exists $x \in \mathcal{L}$ such that $(m', x) \in \Phi_p$. We claim $(p, m') \notin \Psi_p$. Indeed, if $(p, m') \in \Psi_p$ then $\mathcal{L}/\Psi_p$ is a subdirectly irreducible lattice whose minimal non-trivial congruence has a block containing a three element chain, contrary to assumption. It follows that there exists $q \in \text{JI}(\mathcal{L})$ such that $q \leq m' \leq m$, $q \not\leq p$, and $\Phi_q \not\leq \Psi_p$, whence $\Phi_q \geq \Phi_p$. We know $\Phi_q \neq \Phi_p$, as otherwise there would exist $n \in \mathcal{L}$ such that $p \leq n < m'$ such that $(n, x) \in \Phi_p$, contrary to choice of m' . We conclude $\Phi_q > \Phi_p$.

Assume the $\mathcal{L}$ satisfies **i**, **ii**, and **iii**. We proceed by induction on $|\mathcal{L}|$. If $\mathcal{L}$ is a simple lattice the desired conclusion follows immediately from **ii**. Assume the height of $\text{Con}(\mathcal{L}) > 2$. Let Φ_r be a join prime of maximum depth in $\text{Con}(\mathcal{L})$. We wish to show that $\mathcal{L}/\Phi_r$ satisfies **i**, **ii**, and **iii**. As Φ_r was an arbitrary minimal nontrivial congruence of $\mathcal{L}$, the proof will then be complete by the Second Isomorphism Theorem. As **i** and **ii** are preserved by arbitrary homomorphisms, we need only concern ourselves with **iii**.

Let $p \in \text{JI}(\mathcal{L})$, $m \in \text{MI}(\mathcal{L})$ be such that $p \approx m$ and $p \leq m$. It follows from **iii** that there exists $q \in \text{JI}(\mathcal{L})$ such that $q \leq m$, $q \not\leq p$, we have $\Phi_p < \Phi_q$. Let $S := \{s \in \text{JI}(\mathcal{L}) : (s, s_*) \in \Phi_r\}$. Recall from the proof of Lemma 1.3 that $T := \text{JI}(\mathcal{L}) \setminus S$ is D -closed, if $a \in T$ and $a D^n b$ then $b \in T$. Recall that we define the lattice $T^\vee$ to be the lattice of joins of finite subsets of T . There exists a canonical epimorphism $h: \mathcal{L} \rightarrow T^\vee$ defined by $h(x) = \bigvee \{t \in T : t \leq x\}$. By the definition of S and T we have that $\Phi_s \leq \Phi_r$ if and only if $s \in S$. It follows that Φ_r is the kernel of h , whence $T^\vee \cong \mathcal{L}/\Phi_r$.

If $\Phi_p \leq \Phi_r$, then $p \notin T$ and there is nothing to prove. Otherwise $\Phi_p \not\leq \Phi_r$, whence $\Phi_q \not\leq \Phi_r$ as $\Phi_q > \Phi_p$. It follows that $q \in T$, whence in $T^\vee$ we have $q \leq m$ and $q \not\leq p$. As $\Phi_q > \Phi_p$ in $\mathcal{L}$ we have by Lemma 1.3 that $p D^n q$ and not vice versa. As homomorphisms create no new join irreducible elements, we have that $q D^n p$ does not hold in $T^\vee$. Every $t \in JI(\mathcal{L})$ used to witness that $p D^n q$ in $\mathcal{L}$ is also an element of t . Indeed, if $t \notin T$, then $t \in S$ which would imply $\Phi_p \leq \Phi_r$, contrary to assumption. We conclude that $p D^n q$ in $T^\vee$, whence $\Phi_q > \Phi_p$ in $Con(T^\vee)$, as desired. $\square$

Corollary 4.9. *Congruence normal lattices are strongly binary cut-through codable.*

We have already seen an example of a strongly binary cut-through codable lattice that is not congruence normal in Figure 2.3. We can construct infinitely many examples in a similar way.

Theorem 4.10. *Let $\mathcal{B}(n)$ be the boolean algebra on n elements, i.e., the lattice of subsets of an n element set ordered by inclusion. If $n \geq 3$ then there exists $E \subseteq \leq_{\mathcal{B}(n)}$ such that $\mathcal{B}(n) \star_E \mathbf{2}$ is strongly binary cut through codable, and $\mathcal{B}(n) \star_E \mathbf{2}$ is not congruence normal.*

Proof. Given $\mathcal{B}(n)$ with $n \geq 3$, choose $a, b \in \mathcal{B}(n)$ such that $a \neq 0$, $b \neq 1$ and $a \prec b$. We construct a sequence $a = c_0, c_1, c_2, \dots, c_k = b$ such that for all i odd with $1 \leq i \leq k-1$ we have $c_{i-1} \prec c_i$ and $c_{i+1} \prec c_i$. The existence of such a sequence is a consequence of the assumption that $n \geq 3$. We define

$$E := \{(c_0, c_1), (c_1, c_2), (c_2, c_3), \dots, (c_{k-1}, c_k)\} \setminus \{(a, b)\}.$$

Thus, E consists of edges between elements of $\mathcal{B}(n)$ whose heights differ by at most one, with at least one edge between them excluded.

By construction E is an all-or-nothing set. Consider $\mathcal{B}(n) \star_E \mathbf{2}$. This new lattice is not congruence normal, as by construction $(a, 1) \approx (b, 0)$ and $(a, 1) \leq (b, 0)$. The join irreducible elements of $\mathcal{B}(n) \star_E \mathbf{2}$ are exactly those elements p such that $p \approx (a, 1)$ or $0 \prec p$. There exists some p such that $0 \prec p$ and $p \not\leq (a, 0)$. It follows that $(a, 1) < (a, 0) \vee p$. Thus if $\Phi_{(a,1)} \neq \Phi_p$ then $\Phi_{(a,1)} < \Phi_p$. We conclude that $\mathcal{B}(n) \star_E \mathbf{2}$ is a subdirectly irreducible lattice and its unique minimal nontrivial congruence $\Phi_{(a,1)}$ is such that $\|\Phi_{(a,1)}\| = 2$. The desired result follows immediately from the Second Isomorphism Theorem. $\square$

The proof of 4.10 not only gives us infinitely many examples of strongly binary cut-through codable lattices that are not congruence normal, but each of these examples is subdirectly irreducible. One of the consequences of Jónsson's Lemma is that given subdirectly irreducible lattices $\mathcal{L}$ and $\mathcal{K}$ we have $Var(\mathcal{L}) \leq Var(\mathcal{K})$ if and only if $\mathcal{L}$ is a homomorphic image of a sublattice of $\mathcal{K}$. As a lattice variety is defined by its subdirectly irreducible members, we obtain the following corollary.

Corollary 4.11. *There exist infinitely many are binary cut-through codable lattice varieties that contain finite lattices that are not congruence normal.*

4.4 A Summary of Known Classes of Inflatable Lattices

In general, the only method we have to decide if a given finite lattice $\mathcal{L}$ is inflatable is to use Theorem 3.3. This requires us to determine if $Con(\mathcal{L})$ contains an inflation sequence. At present there is no way to do this other than test every sequence from $0_{Con(\mathcal{L})}$ to $1_{Con(\mathcal{L})}$ to see if it satisfies Definition 3.2.

When we restrict our attention to inflatable lattices such that every inflation used to construct $\mathcal{L}$ is inflation by $\mathbf{2}$ the situation becomes less daunting. In this case, it is clear such a lattice is $\mathbf{2}$ -simple. Furthermore, as each inflation creates a single projectivity class in the new lattice we have that $F_k(\mathcal{L}) = JI(\mathcal{L})$ for some $k \in \omega$. The goal becomes to find conditions under which lattices $\mathcal{L}$ such that $\mathcal{L}$ is $\mathbf{2}$ -simple and $F_k(\mathcal{L}) = JI(\mathcal{L})$ for some $k \in \omega$ have an inflation sequence.

We say a sequence in $Con(\mathcal{L})$ from $0_{Con(\mathcal{L})}$ to $1_{Con(\mathcal{L})}$ is a *0-1 spanning sequence* in $Con(\mathcal{L})$. How can a 0-1 spanning sequence in $Con(\mathcal{L})$ in a finite lattice $\mathcal{L}$ that satisfies conditions **i** and **ii** of

Figure 4.4: The lattice $\mathcal{L}$ (left), and $Con(\mathcal{L})$ (right).

4.4 fail to be an inflation sequence? The only way this can occur is if there exist non-trivial intervals b/a and y/x such that $b \leq x$ and $(x, y) \in Cg(a, b)$. If $Cg(b, y) \not\leq Cg(a, b)$, then one could find a sequence in $Con(\mathcal{L})$ from $0_{Con(\mathcal{L})}$ to $1_{Con(\mathcal{L})}$ such that for some $Cg(b, y) \leq \theta_i$ and $Cg(a, b) \not\leq \theta_i$ for some i . In this case, clearly $\{\theta_i : 0 \leq i \leq n\}$ is not an inflation sequence.

Nation's condition **T** from Theorem 1.12 ensures that intervals such as b/a and y/x described above do not occur. A non-trivial example of such a congruence normal lattice, along with its congruence lattice, is shown in Figure 4.4. We can, however, permit such intervals in $\mathcal{L}$. If $\mathcal{L}$ satisfies **iii** from Theorem 4.8, then despite the existence of such intervals we still have that every $0 - 1$ spanning sequence in $Con(\mathcal{L})$ is an inflation sequence. Indeed, we have encountered examples of such strongly binary cut-through codable lattices before, see Figure 2.3, which we reproduce here in Figure 4.5 along with its congruence lattice. We have seen in Theorem 4.10 we can construct infinitely many examples of such strongly binary cut-through codable, but not congruence normal, lattices in a similar manner.

Figure 4.5: The lattice $\mathcal{L}$ (left), and $Con(\mathcal{L})$ (right).

Such strong conditions are not necessary to ensure that a given finite lattice $\mathcal{L}$ is binary cut-through codable however. We do not need that every $0 - 1$ spanning sequence in $Con(\mathcal{L})$ is an inflation sequence, only that at least one inflation sequence exists. We have seen an example of such a lattice $\mathcal{L}$ in Figure 4.2. The homomorphism illustrated in Figure 4.3 witnesses that not every $0 - 1$ spanning sequence in $Con(\mathcal{L})$ is an inflation sequence. Condition **iii** in Theorem 4.4 is still necessary for a finite lattice $\mathcal{L}$ to be binary cut-through codable. However, it may not be strong enough to be sufficient for binary cut-through codability.

CHAPTER 5

OPEN PROBLEMS

We have a partial analogue of Pudlák and Tuma's result in Lemma 1.9. We do not yet know what class of finite lattices would play the role of lower bounded finite lattices in Theorem 1.7. Such a class would have to be very broad, as it will include not only binary-cut through codable lattices, but finite modular lattices, geometric lattices, and lattices of equivalence relations on a finite set as well.

Problem 5.1. *Can Lemma 1.9 be extended to an analogue of Theorem 1.7?*

We have been able to classify binary cut-through codable lattices as those that can be constructed by inflation using only the two element lattice. However, we only have a partial analogue of Theorems 1.6 and 1.12 in Theorem 4.4.

Problem 5.2. *Are conditions i, ii, and iii of Theorem 4.4 sufficient to prove a finite lattice $\mathcal{L}$ is binary cut-through codable? If not, can they be suitably modified to form a set of necessary and sufficient conditions for binary cut-through codability?*

We have seen that the property of a finite lattice being strongly binary cut-through codable is not preserved by inflating arbitrary all-or-nothing sets by **2**. In some cases, e.g. the construction shown in Figure 2.3, such a construction results in a new strongly binary cut-through codable lattice. Other times, e.g. the construction shown in Figure 3.4 and Figure 3.5, the construction results in a lattice is binary cut-through codable but not strongly binary cut-through codable. If $\mathcal{L}$ is a strongly binary cut-through codable lattice and $E \subseteq \leq_{\mathcal{L}}$ is such that $\mathcal{L} \star_E \mathbf{2}$ is strongly binary cut-through codable then we say E is *stable*. We have yet to find an explicit description of these sets.

Problem 5.3. *Explicitly classify stable all-or-nothing sets of strongly binary cut-through codable lattices.*

Our understanding of strongly binary cut-through codable lattices is lacking in another way compared to some of the prior classes considered in Chapter 1. In the case of finite lower bounded lattices, and thus by duality and composition finite upper bounded and finite bounded lattices, Nation showed that a lattice is lower bounded if and only if it satisfies a set of identities given in [N 90, Theorem 5.3, pg. 541]. At present we do not have an analogous result for congruence normal lattices, let alone strongly binary cut-through codable lattices.

Problem 5.4. *Does there exist a set of identities that classify congruence normal lattices?*

Problem 5.5. *Does there exist a set of identities that classify strongly binary cut-through codable lattices?*

A lattice $\mathcal{L}$ is called *join semidistributive* if and only if for all $a, b, c \in \mathcal{L}$ we have that $a \vee b = a \vee c$ implies $a \vee b = a \vee (b \wedge c)$. A *meet semidistributive* lattice is defined dually, and a lattice is called *semidistributive* if and only if it is both join and meet semidistributive. Lower bounded lattices are a subclass of join semidistributive lattices. A lattice variety $\mathbf{V}$ is called *semidistributive* if and only if every lattice $\mathcal{L} \in \mathbf{V}$ is semidistributive. Jónsson and Rival showed a variety $\mathbf{V}$ is semidistributive if and only if $\mathcal{M}_3, \mathcal{L}_2, \mathcal{L}_3, \mathcal{L}_4, \mathcal{L}_5 \notin \mathbf{V}$ [JR 79]. Like semidistributive varieties, $\mathbf{I}_2$ and the class of congruence normal varieties, i.e., those lattice varieties whose finite members are congruence normal, form ideals in Λ . It is currently unknown whether there are similar exclusion results for these classes.

Problem 5.6. *Do there exist finitely many minimal lattice varieties that are not congruence normal? Is there an exclusion result for congruence normal varieties?*

Problem 5.7. *Do there exist finitely many minimal lattice varieties that are not in $\mathbf{I}_2$? Is there an exclusion result for $\mathbf{I}_2$?*

A finite lattice is called *representable* if and only if it is isomorphic to the lattice of congruence relations of a finite algebra. One of the most important open problems in lattice theory is to determine whether or not every finite lattice is representable. Recent progress was made on this problem by William Demeo in [WD 13]. Demeo's new technique for expanding finite algebras demonstrates that if $\mathcal{L}$ is a representable lattice, $E = \{(a, a)\}$ for some $a \in \mathcal{L}$, and $Eq(n)$ is defined to be the lattice of equivalence relations on a set of cardinality $n \in \omega$, then $\mathcal{L} \star_E Eq(n)$ is representable. There is reason to hope Demeo's construction can be extended to hold for more substantial all-or-nothing sets.

Problem 5.8. *Given representable lattices $\mathcal{L}$ and $\mathcal{K}$, and an all-or-nothing set $E \subseteq \leq_{\mathcal{L}}$, under what conditions is $\mathcal{L} \star_E \mathcal{K}$ representable?*

APPENDIX

The Universal Algebra Calculator created by Ralph Freese, which can be found at <http://www.uacalc.org/>, was used extensively for computations on the examples of finite lattices seen throughout this text. To aid data entry, several macros were made in the Python programming language. These macros allow one to enter the covering relations of a finite lattice, which is then used to generate the lattice in a format usable by the calculator. Permission is granted by the author for the free reproduction and distribution of this code to aid fellow investigators. Please note that an installation of Python must be able to access the “numpy” package to utilize these macros.

```
import numpy

def covers(A):
    cov = A
    for i in range(0,A.shape[1]):
        cov[i,i] = 1
    ans = "n"
    while not ans == "y":
        x = int(input("Row? "))
        y = int(input("Column? "))
        cov[x,y] = 1
        ans = input("Done?: ")
    return cov

def latgen(m):
    x = m[1].size
    m = m ** x
    for i in range(0,x):
        for j in range(0,x):
            if not m[i,j] == 0:
                m[i,j] = 1
    lat = m
    return lat

def meetop(A):
    meettable = numpy.matrix(numpy.ones(
    (A.shape)), int)
    x = A.shape[1]
    for i in range(0,x):
        meettable[i,i] = i
        for j in range(i,x):
            meet = A[0]
            for k in range(0,x):
                if all(A[k,y] >= A[i,y] and
                A[k,y] >= A[j,y] for
                y in range(0,x)) and
                all(A[k,y] <= meet[0,y] for
                y in range(0,x)):
                    meet = A[k]
            else:
                pass
```

```

        for k in range(0,x):
            if all(A[k,y] == meet[0,y] for
                y in range(0,x)):
                meettable[i,j] = k
                meettable[j,i] = k
            else:
                pass
    return meettable

def joinop(A):
    jointable = numpy.matrix(numpy.ones(
        (A.shape)), int)
    x = A.shape[1]
    for i in range(0,x):
        jointable[i,i] = i
        for j in range(i,x):
            join = A[x-1]
            for k in range(0,x):
                if all(A[k,y] <= A[i,y] and
                    A[k,y] <= A[j,y] for
                    y in range(0,x)) and
                    all(A[k,y] >= join[0,y] for
                    y in range(0,x)):
                    join = A[k]
            else:
                pass
        for k in range(0,x):
            if all(A[k,y] == join[0,y] for
                y in range(0,x)):
                jointable[i,j] = k
                jointable[j,i] = k
            else:
                pass
    return jointable

def uacalc(name, M, J):
    alg = '<?xml version="1.0"?><algebra>
    <basicAlgebra><algName>'
    + name + '</algName><cardinality>'
    + str(M.shape[1]) +
    '</cardinality><operations>'
    alg = alg + '<op><opSymbol>
    <opName>meet</opName><arity>
    2</arity></opSymbol>
    <opTable><intArray>'
    for i in range(0,M.shape[1]):
        alg = alg + '<row r="' + str(i) + '">'
        for j in range(0,M.shape[1]):
            alg = alg + str(M[i,j])
            if j < M.shape[1]:
                alg = alg + ', '

```

```

        alg = alg + '</row>'
    alg = alg + '</intArray></opTable></op>'
    alg = alg + '<op><opSymbol><opName>join</opName>'
    <arity>2</arity></opSymbol></opTable></intArray>'
    for i in range(0,J.shape[1]):
        alg = alg + '<row r="' + str(i) + '">'
        for j in range(0,J.shape[1]):
            alg = alg + str(J[i,j])
            if j < J.shape[1]:
                alg = alg + ', '
        alg = alg + '</row>'
    alg = alg + '</intArray></opTable></op>'
    alg = alg + '</operations>'
    </basicAlgebra></algebra>'
    return alg

```

BIBLIOGRAPHY

- [A 91] Kira V. Adaricheva, *A characterization of finite lattices of subsemilattices*, Algebra i Logika **30**, 1991, 385–404 (Russian)
- [B 35] Garrett Birkhoff, *On the structure of abstract algebras*, Proc. Cambridge Phil. Soc. **31**, 1935, 433–454.
- [D 70] Alan Day, *A simple solution to the word problem for lattices*, Can. Math. Bull. **13**, 1970, 253–254
- [D 79] Alan Day, *Characterizations of finite lattices that are bounded-homomorphic images or sublattices of free lattices*, Can. J. Math. **31**, 1979, 69–78
- [D 94] Alan Day, *Congruence Normality: The characterization of the doubling class of convex sets*, Algebra Universalis **31** 397–406, Birkhäuser Verlag, 1994
- [WD 13] William Demeo, *Expansions of finite algebras and their congruence lattices*, Algebra Universalis **69**, 2013, 257–278
- [FJN 95] Ralph Freese, Jaroslav Ježek, and J.B. Nation, *Free Lattices*, Mathematical Surveys and Monographs **42**, American Mathematical Society, 1995
- [FMN 02] Ralph Freese, George McNulty, and J.B. Nation, *Inherently nonfinitely based lattices*, Annals of Pure and Applied Logic **115**, 2002, 175–193
- [FN 42] N. Funayama and T. Nakayama, *On the distributivity of a lattice of lattice-congruences*, Proc. Imp. Aca. Tokyou **18**, 1942, 553–554
- [G 94] W. Geyer, *The generalized doubling construction and formal concept analysis*, Algebra Universalis **32**, 1994, 341–367
- [J 67] Bjarni Jónsson, *Algebras whose congruence lattices are distributive*, Math. Scand **21**, 1967, 110–121
- [JN 77] Bjarni Jónsson and J.B. Nation, *A report on sublattices of a free lattice*, Contributions to universal algebra, Coll. Math. Soc. János Bolyai, **17**, 233–257, North-Holland Publishing Co., Amsterdam, 1977
- [JR 79] Bjarni Jónsson and Ivan Rival, *Lattice varieties covering the smallest non-modular variety*, Pacific J. Math., **82**, 1979, 463–478
- [M 72] Ralph McKenzie, *Equational bases and non-modular lattice varieties*, Trans. Amer. Math. Soc., **174**, 1–43, 1972
- [N 94] J.B. Nation, *Addendum to ‘Congruence Normality’ of Alan Day*, Algebra Universalis **31**, 307–310, Birkhäuser Verlag, 1994
- [N 90] J.B. Nation, *An approach to lattice varieties of finite height*, Algebra Universalis **27**, 1990, 521–543
- [N1 95] J.B. Nation, *Alan Day’s Doubling Construction*, Algebra Universalis **34**, 1995, 24–34
- [N2 95] J.B. Nation, *Notes on Lattice Theory*, unpublished, <http://math.hawaii.edu/~jb/books.html>

- [PT 74] Pavel Pudlák and Jiří Tůma, *Yeast graphs and fermentation of algebraic lattices*, Coll. Math. Soc. János Bolyai, **14**, 1974, 301-341
- [R 11] Heiko Reppe, *Three Generalizations of Lattice Distributivity: An FCA Perspective*, Shaker Verlag, 2011
- [S 70] T. Skolem, *Select Works in Logic*, Scandinavian University Books, Oslo, 1970
- [SL 13] Qifu Tyler Sun and Shuo-Yen Robert Li, *Lattice classification by Cut-Through coding*, Order, **31**, 2014, 271-278
- [W 41] Ph. M. Whitman, *Free Lattices*, Ann. of Math. (2) **42**, 1941, 325–330
- [W 42] Ph. M. Whitman, *Free Lattices II*, Ann. of Math. (2) **43**, 1942, 104–115