Extending LP-Decoding for Permutation Codes from Euclidean to Kendall tau Metric

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Abstract

Invented in the 1960's, permutation codes have reemerged in recent years as a topic of great interest because of properties making them attractive for certain modern technological applications. In 2011 a decoding method called LP (linear programming) decoding was introduced for a class of permutation codes with a Euclidean distance induced metric. In this paper we comparatively analyze the Euclidean and Kendall tau metrics, ultimately providing conditions and examples for which LP-decoding methods can be extended to permutation codes with the Kendall tau metric. This is significant since contemporary research in permutation codes and their promising applications has incorporated the Kendall tau metric.

Contents

1. Introduction

In 1965, D. Slepian published a paper entitled "Permutation Modulation" [1], where he constructed a code book by beginning with an initial sequence and then taking all distinctive sequences formed by permuting the order of the numbers in this sequence. The class of codes he constructed were suitable for the transmission of digital information in the presence of white Gaussian noise. Since that time, other codes such as BCH codes and LDPC codes have become more prominent in terms of ubiquitous application. However, in recent years permutation codes have reemerged as a subject of modern interest.

Within the past 15 years, researchers have investigated various types and properties of permutation codes, claiming the existence of good permutation codes and expositing on some of their benefits [2, 3, 7, 8]. In the year 2000, Vinck discussed the application of permutation codes in power line communication [4]. Perhaps most notably, in 2008, Jiang *et. al.* proposed the implementation of rank modulation, a type of permutation code, in flash memory devices [6]. The scheme described would improve the stability and efficiency in programming flash memory cells. A metric known as the Kendall tau distance was central to the proposal of incorporating rank modulation in flash memory.

While the Hamming distance is likely the most well-studied metric in coding theory, many metrics are possible, including Euclidean and Kendall tau distance. Unfortunately, because of their disparate properties, coding or decoding methods for a code utilizing one type of metric will not necessarily translate to codes utilizing another type. Indeed, on the surface, the Euclidean distance and Kendall tau distance are nearly entirely unrelated. The Euclidean distance is perhaps the most standard and intuitive of all distances defined over $\mathbb{R}^n$. In two dimensions it is simply the distance between two points in the Cartesian plane. Alternatively, the Kendall tau distance measures the number of pairwise order disagreements between two permutations.

Nevertheless, one of the goals of this paper is to extend a decoding process devised for permutation codes endowed with the Euclidean distance as a metric to codes endowed with the Kendall tau distance metric. In July 2011, a novel class of permutation codes called LP (linear programming) decodable permutation codes were introduced [13]. In this formulation, Euclidean distance was incorporated in an effective algorithm, reducing the decoding process to an optimization problem solvable via linear programming methods. A main contribution of this paper is to provide conditions and examples for which this LP-decoding method can be extended to permutation codes with the Kendall tau metric.

Extending LP-decoding in this manner is significant, since the Kendall tau distance has been of recent interest in the context of permutation codes [5, 6, 8], including the aforementioned coding for flash memory. Moreover, it was proven by Buchheim *et. al.* that finding an element of a permutation group with minimal Kendall tau distance from a given permutation of S_n is an NP-complete problem [9]. This implies that it is unlikely that a polynomial time algorithm exists to solve the minimum distance decoding problem in general. However, linear programming problems are guaranteed to be solvable in polynomial time, implying that the LP-decoding process for minimal distance decoding has a polynomial time computational cost [10]. Extending LP-decoding is also interesting because LP-decoding was initially only applicable for continuous objects with a metric defined for vectors, but we apply LP-decoding to permutations.

Toward the goal of extending LP-decoding methods, we comparatively analyze the Euclidean and Kendall tau metrics. We simplify the comparison by considering the weights induced by both metrics. For certain subgroups of the symmetric group S_n, we provide complete characterizations of the Euclidean weight in terms of the Kendall tau weight. As corollaries, we also determine the minimal Kendall tau and Euclidean weights for the cases of Cyclic groups and Dihedral groups of any order.

2. Preliminaries

In this section we introduce notation and definitions necessary to read this paper. We also prove some elementary facts concerning the notation and definitions stated. Throughout this paper we denote the symmetric group of permutations on the set $\{1, 2, \ldots, n\}$ by S_n. For any permutation σ in S_n, we use the notation $\sigma := [\sigma_1, \sigma_2, \ldots, \sigma_n]$ as shorthand for the mapping which sends i to σ_i, where σ_i is in the set $\{1, 2, \ldots, n\}$ and whenever $i \neq j$, then $\sigma_i \neq \sigma_j$.

Permutations are multiplied in the typical manner, from right to left. For example, if $\sigma := [3, 1, 2, 5, 4]$ and $\tau := [2, 1, 5, 4, 3]$, then we have $\sigma\tau = [3, 1, 2, 5, 4][2, 1, 5, 4, 3] = [1, 3, 4, 5, 2]$. The identity permutation is denoted by $e := [1, 2, \ldots, n] \in S_n$, while the inverse of a permutation $\sigma \in S_n$ is denoted by σ^{-1}. Furthermore, if $\sigma := [\sigma_1, \sigma_2, \ldots, \sigma_n]$, then we write out the the inverse of σ as $\sigma^{-1} = [(\sigma^{-1})_1, (\sigma^{-1})_2, \ldots, (\sigma^{-1})_n]$.

Definition 2.1 (Group Action)**.** Let G be a group and let X be a set. A (left) group action is an operator $\circ : G \times X \to X$ such that $\circ$ satisfies the following:

(1) Associativity: If g_1 and g_2 are elements of G, and x is an element of X, then $(g_1 \cdot g_2) \circ x = g_1 \circ (g_2 \circ x)$.

(2) Identity: There exists an identity element e in G such that for any x in X, we have $e \circ x = x$.

We may define an action of S_n on $\mathbb{R}^n$ by allowing $\sigma := [\sigma_1, \sigma_2, \ldots, \sigma_n]$ to be the permutation sending the ith position of a vector $\vec{\mu} \in \mathbb{R}^n$ upon which σ acts to the σ_ith position. We denote this action by $\sigma \circ \vec{\mu}$.

We may easily verify that $\circ$ as defined previously satisfies the definition of a group action. Let $\vec{\mu} := (\mu_1, \mu_2, \ldots, \mu_n)$ be a vector of $\mathbb{R}^n$ and let $\sigma := [\sigma_1, \sigma_2, \ldots, \sigma_n]$ and $\tau := [\tau_1, \tau_2, \ldots, \tau_n]$ be members of S_n. Since we know that the identity, e, is an element of S_n, to prove that $\circ$ defines a group action, it remains only to show that associativity holds for $\circ : S_n \times \mathbb{R}^n \to \mathbb{R}^n$, *i.e.,* it remains to show that

$$\sigma \circ (\tau \circ \vec{\mu}) = (\sigma\tau) \circ \vec{\mu}$$

In the case of $(\sigma\tau) \circ \vec{\mu}$, from the multiplication of permutations we immediately obtain the equality

$$(\sigma\tau) \circ \vec{\mu} = ([\sigma_1, \sigma_2, \ldots, \sigma_n][\tau_1, \tau_2, \ldots, \tau_n]) \circ \mu = [\sigma_{\tau_1}, \sigma_{\tau_2}, \ldots, \sigma_{\tau_n}] \circ \vec{\mu}.$$

In the case of $\sigma \circ (\tau \circ \vec{\mu})$, we first have τ acting on $\vec{\mu}$ so that μ_i is sent to the τ_ith position. Subsequently σ acts upon $(\tau \circ \vec{\mu})$, so that the τ_ith position of $(\tau \circ \vec{\mu})$ is sent to the σ_{τ_i}th position. This is equivalent to sending μ_i to the σ_{τ_i}th position. Therefore

$$\sigma \circ (\tau \circ \vec{\mu}) = [\sigma_{\tau_1}, \sigma_{\tau_2}, \ldots, \sigma_{\tau_n}] \circ \vec{\mu}.$$

Continuing our discussion on notation, in this paper s_i in S_n denotes the adjacent transposition $(i, i+1)$, switching the ith and $(i+1)$th positions. That is, $s_i = [1, 2, \ldots, i-1, i+1, i, i+2, \ldots n]$. We embed S_n into $\mathbb{R}^n$ in the following manner. For any permutation $\sigma = [\sigma_1, \sigma_2, \ldots, \sigma_n]$ in S_n, we associate the vector $\vec{\sigma} := ((\sigma^{-1})_1, (\sigma^{-1})_2, \ldots, (\sigma^{-1})_n)$ in $\mathbb{R}^n$. It is worth noting that $\sigma \circ \vec{\mu} = \overrightarrow{\sigma\mu}$ and in particular $\sigma \circ \vec{e} = \vec{\sigma}$.

To see that why this is true, suppose that $\sigma := [\sigma_1, \sigma_2, \ldots, \sigma_n]$ and $\mu := [\mu_1, \mu_2, \ldots, \mu_n]$. We have $\mu \circ \vec{e} = ((\mu^{-1})_1, (\mu^{-1})_2, \ldots, (\mu^{-1})_n) = \vec{\mu} = \overrightarrow{\mu e}$. Therefore we have $\overrightarrow{\sigma\mu} = \overrightarrow{\sigma\mu e} = (\sigma\mu) \circ \vec{e} = \sigma \circ (\mu \circ \vec{e}) = \sigma \circ \vec{\mu}$. As an example, suppose $\sigma := [4, 5, 1, 3, 2]$ and $\mu := [3, 1, 5, 2, 4]$. Then $\vec{\mu} = (2, 4, 1, 5, 3)$. Thus $\sigma \circ \vec{\mu} = [4, 5, 1, 3, 2] \circ (2, 4, 1, 5, 3) = (1, 3, 5, 2, 4)$. On the other hand, $\overrightarrow{\sigma\mu} = (\overrightarrow{[4, 5, 1, 3, 2][3, 1, 5, 2, 4]}) = \overrightarrow{[1, 4, 2, 5, 3]} = (1, 3, 5, 2, 4)$. If instead $\mu := e$, we would have $\vec{\mu} = \vec{e}$, in which case $\sigma \circ \vec{e} = [4, 5, 1, 3, 2] \circ (1, 2, 3, 4, 5) = (3, 5, 4, 1, 2)$. Alternatively, $\overrightarrow{\sigma e} = (\overrightarrow{[4, 5, 1, 3, 2][1, 2, 3, 4, 5]}) = \overrightarrow{[4, 5, 1, 3, 2]} = (3, 5, 4, 1, 2)$.

Note also that in any group $G \subseteq S_n$, both the permutation $\sigma := [\sigma_1, \sigma_2 \ldots \sigma_n]$ and the permutation $\sigma^{-1} := [(\sigma^{-1})_1, (\sigma^{-1})_2, \ldots, (\sigma^{-1})_n]$ are in G, so that their associated vectors $((\sigma^{-1})_1, (\sigma^{-1})_2, \ldots, (\sigma^{-1})_n)$ and $(\sigma_1, \sigma_2, \ldots, \sigma_n)$ and are in the set of associated vectors for permutations of G.

Definition 2.2 (Permutation Code)**.** Let G be a subgroup of S_n, and $\vec{\mu} \in \mathbb{R}^n$. A permutation code $G\vec{\mu}$ is defined as the orbit of G acting on $\vec{\mu}$. In set-builder notation, $G\vec{\mu} := \{g \circ \vec{\mu} \mid g \in G\}$.

In general a permutation code can be comprised of any collection of permutations on the initial vector $\vec{\mu}$, but we consider only $G\vec{\mu}$ for some subgroup $G \subseteq S_n$ so that we may take advantage of algebraic structure.

We now define two previously studied distances [11], beginning with the Kendall tau distance. The Kendall tau distance between two permutations σ and τ measures the minimum number of pairwise adjacent transpositions necessary to transform σ into τ, or equivalently τ into σ.

Definition 2.3 (Kendall tau Distance)**.** Given $\sigma, \tau \in S_n$, the Kendall tau distance $\mathrm{d}_K(\sigma, \tau)$ between σ and τ is defined as

$$\mathrm{d}_K(\sigma, \tau) := \#\{(i, j) \mid 1 \leq i < j \leq n, (\sigma^{-1}\tau)_i > (\sigma^{-1}\tau)_j\},$$

where $\#A$ denotes the cardinality of the set A.

Proposition 2.4. *Kendall tau distance is left invariant, i.e., given any* $\sigma, \tau, \lambda \in S_n$, *we have* $\mathrm{d}_K(\sigma, \tau) = \mathrm{d}_K(\lambda\sigma, \lambda, \tau)$.

Proof. By definition,

$$\mathrm{d}_K(\lambda\sigma, \lambda\tau) = \#\{(i, j) \mid 1 \leq i < j \leq n, ((\lambda\sigma)^{-1}\lambda\tau)_i > ((\lambda\sigma)^{-1}\lambda\tau)_j\}.$$

However, $(\lambda\sigma)^{-1} = \sigma^{-1}\lambda^{-1}$, so that

$$\begin{aligned}\mathrm{d}_K(\lambda\sigma, \lambda\tau) &= \#\{(i, j) \mid 1 \leq i < j \leq n, (\sigma^{-1}\lambda^{-1}\lambda\tau)_i > (\sigma^{-1}\lambda^{-1}\lambda\tau)_j\} \\ &= \#\{(i, j) \mid 1 \leq i < j \leq n, (\sigma^{-1}\tau)_i > (\sigma^{-1}\tau)_j\} = \mathrm{d}_K(\sigma, \tau).\end{aligned}$$

□

To see that the definition of Kendall tau distance $\mathrm{d}_K(\sigma, \tau)$ between two permutations σ and τ is equivalent to counting the minimum number of adjacent transpositions necessary to transform σ into τ, note first that by the above proposition $\mathrm{d}_K(\sigma, \tau) = \mathrm{d}_K(e, \sigma^{-1}\tau)$. Hence it suffices to show that $\#\{(i, j) \mid 1 \leq i < j \leq n, (\sigma^{-1}\tau)_i > (\sigma^{-1}\tau)_j\}$ is equal to the number of adjacent transpositions necessary to transform e into $\sigma^{-1}\tau$. The permutation $\sigma^{-1}\tau$ can be transformed into the identity element e by a series of adjacent transpositions, each decreasing the Kendall tau distance by 1 through an algorithm known as the insertion sort [21].

The insertion sort works in $n-1$ stages. In the ith stage of the algorithm, a series of adjacent transpositions are applied until the $(i+1)$th entry is in the correct position relative to the previous i entries. In this algorithm, every pair (i, j) such that $1 \leq i < j \leq n$ and $(\sigma^{-1}\tau)_i > (\sigma^{-1}\tau)_j$ corresponds exactly with the application of a single adjacent transposition decreasing the number of remaining adjacent transpositions necessary to transform $\sigma^{-1}\tau$ into e. Of course, applying the same adjacent transpositions in reverse order would transform e back into $\sigma^{-1}\tau$. Hence the Kendall tau distance between σ and τ as defined is equivalent to the minimum number of adjacent transpositions necessary to transform σ into τ.

As an expository example, we shall demonstrate the insertion sort applied to the permutation $\sigma := [2, 1, 4, 3]$. In the first stage, the second entry, 1, is compared to the first entry, 2, and since the first entry is larger than the second, that is $\sigma_1 > \sigma_2$, the adjacent transposition s_1 is applied, resulting in the permutation $[1, 2, 4, 3]$. In the second stage, we compare 4 to 2, and since they are in the proper order, that is $\sigma_2 < \sigma_3$, no transpositions are applied. In the final stage, 3 is compared to 4, and since $\sigma_3 > \sigma_4$, the adjacent transposition s_3 is applied, resulting in $[1, 2, 3, 4]$.

Remark 2.5. We can verify that the Kendall tau distance d_K satisfies the distance axioms over S_n.

Proof. Let $\sigma, \tau \in S_n$. Having shown that the Kendall tau distance between permutations σ and τ of S_n is equivalent to the minimum number of adjacent transpositions necessary to transform σ into τ, it is trivial that $\mathrm{d}_K(\sigma, \tau) = \mathrm{d}_K(\tau, \sigma)$. It is also trivial by the equivalence of definitions that $\mathrm{d}_K(\sigma, \tau) \geq 0$ and $\mathrm{d}_K(\sigma, \tau) = 0$ if and only if $\sigma = \tau$. To prove that $\mathrm{d}_K(\sigma, \tau) \leq \mathrm{d}_K(\sigma, \lambda) + \mathrm{d}_K(\lambda, \tau)$, we may simply observe that $\mathrm{d}_K(\sigma, \lambda) + \mathrm{d}_K(\lambda, \tau)$ is the minimum number of adjacent transpositions necessary to first transform σ into λ, and then subsequently λ into τ, so that the composition of both series of adjacent transpositions transforms σ into τ. However, by definition $\mathrm{d}_K(\sigma, \tau)$ is the minimum number of adjacent transpositions necessary to take σ to τ, so it follows that $\mathrm{d}_K(\sigma, \tau) \leq \mathrm{d}_K(\sigma, \lambda) + \mathrm{d}_K(\lambda, \tau)$ as desired. Thus d_K is symmetric, positive definite, and satisfies the triangle inequality. $\square$

Definition 2.6 (Euclidean Distance)**.** Given $\sigma, \tau \in S_n$ with associated vectors $\vec{\sigma}, \vec{\tau} \in \mathbb{R}^n$, the Euclidean distance $\mathrm{d}_E(\vec{\sigma}, \vec{\tau})$ between $\vec{\sigma}$ and $\vec{\tau}$ is defined as

$$\mathrm{d}_E(\vec{\sigma}, \vec{\tau}) := \sqrt{\langle \vec{\sigma} - \vec{\tau}, \vec{\sigma} - \vec{\tau} \rangle} = \sqrt{\sum_{i=1}^{n} ((\sigma^{-1})_i - (\tau^{-1})_i)^2}.$$

Here $\langle x, y \rangle$ denotes the standard dot product between $x, y \in \mathbb{R}^n$.

The following table compares the Kendall tau and Euclidean distances between permutations and their associated vectors respectively.

σ	τ	$\mathrm{d}_K(\sigma, \tau)$	$\mathrm{d}_E(\vec{\sigma}, \vec{\tau})$
$[2, 1, 4, 3]$	$[2, 1, 4, 3]$	0	0
$[2, 1, 4, 3]$	$[2, 1, 3, 4]$	1	$\sqrt{2} \approx 1.1421$
$[1, 2, 3, 4]$	$[2, 1, 4, 3]$	2	2
$[1, 2, 3, 4]$	$[1, 4, 2, 3]$	2	$\sqrt{6} \approx 2.44949$
$[1, 2, 3, 4]$	$[2, 4, 3, 1]$	4	$\sqrt{14} \approx 3.74166$
$[1, 2, 3, 4]$	$[3, 4, 1, 2]$	4	4
$[2, 1, 4, 3]$	$[3, 4, 1, 2]$	6	$2\sqrt{5} \approx 4.4721$

Notice that there is no apparent direct relationship, as for example we have $\mathrm{d}_K([1, 2, 3, 4], [1, 4, 2, 3]) < \mathrm{d}_E(\overrightarrow{[1, 2, 3, 4]}, \overrightarrow{[1, 4, 2, 3]})$, but on the other hand $\mathrm{d}_K([1, 2, 3, 4], [2, 4, 3, 1]) > \mathrm{d}_E(\overrightarrow{[1, 2, 3, 4]}, \overrightarrow{[2, 4, 3, 1]})$. We can also observe instances in which the Kendall tau and Euclidean distances are equal.

3. Analysis of Euclidean and Kendall tau weights

In this section we discuss a surprising connection between the Euclidean and Kendall tau distances by analyzing their induced weight functions. As mentioned in the preliminaries, Kendall tau distance is left invariant so that $\mathrm{d}_K(\sigma,\tau) = \mathrm{d}_K(e,\sigma^{-1}\tau)$. The distance $\mathrm{d}_K(e,\sigma)$ between the identity element e and $\sigma \in S_n$ is a previously studied value called the length of σ [14]. In coding theory the weight of a codeword is sometimes defined as the number of positions in which that codeword differs from the zero element. This definition, however, is for the Hamming weight, which is defined with respect to the Hamming distance. The Kendall tau distance between the identity e and an element has also been referred to by Barg as the weight of that element. [8].

Definition 3.1 (Kendall tau Weight)**.** Given $\sigma \in S_n$, the Kendall tau weight $\mathrm{wt}_K(\sigma)$ of σ is defined as

$$\mathrm{wt}_K(\sigma) := \mathrm{d}_K(e,\sigma) = \#\{(i,j) \mid 1 \le i < j \le n, \sigma_i > \sigma_j\}.$$

Euclidean distance is also left invariant, *i.e.* given $\lambda \in S_n$ and $\vec{\sigma},\vec{\tau}$ in $\mathbb{R}^n$, we have $\mathrm{d}_E(\vec{\sigma},\vec{\tau}) = \mathrm{d}_E(\lambda\vec{\sigma},\lambda\vec{\tau})$. Hence since $\mathrm{d}_E(\lambda\vec{\sigma},\lambda\vec{\tau}) = \mathrm{d}_E(\overrightarrow{\lambda\sigma},\overrightarrow{\lambda\tau})$, then $\mathrm{d}_E(\vec{\sigma},\vec{\tau}) = \mathrm{d}_E(\vec{e},\overrightarrow{\sigma^{-1}\tau})$. Thus it is natural to consider the distance between an element of $\mathbb{R}^n$ and the identity element $\vec{e}$. We introduce as a new innovation the analogue to the Kendall tau weight for the Euclidean distance metric.

Definition 3.2 (Euclidean Weight)**.** Given $\sigma \in S_n$, the Euclidean weight $\mathrm{wt}_E(\vec{\sigma})$ of $\vec{\sigma}$ is defined as

$$\mathrm{wt}_E(\vec{\sigma}) := \frac{1}{2}\mathrm{d}_E(\vec{e},\vec{\sigma})^2.$$

In our formulation of the Euclidean weight the distance is squared since we are concerned only with the relative weights of elements. We also include the factor of $\frac{1}{2}$ since all squared distances between $\vec{e}$ and another vector $\vec{\sigma}$ are even. Indeed, given a vector $\vec{\sigma} \in \mathbb{R}^n$, we have

$$\mathrm{d}_E(\vec{e},\vec{\sigma})^2 = \left(\sqrt{\langle \vec{e}-\vec{\sigma}, \vec{e}-\vec{\sigma}\rangle}\right)^2$$

$$= \sum_{i=1}^{n} \left(e_i - (\sigma^{-1})_i\right)^2 = \sum_{i=1}^{n} (e_i)^2 - 2e_i(\sigma^{-1})_i + \left((\sigma^{-1})_i\right)^2.$$

Note here that $\sum_{i=1}^{n}\left((\sigma^{-1})_i\right)^2 = \sum_{i=1}^{n}(e_i)^2$ since each $(\sigma^{-1})_i$ is in the set $\{1,2,\ldots,n\}$ and $i \neq j$ implies $(\sigma^{-1})_i \neq (\sigma^{-1})_j$. Therefore, as $\langle e,e\rangle = \sum_{i=1}^{n}(e_i)^2$ and $\langle e,\sigma\rangle = \sum_{i=1}^{n} e_i(\sigma^{-1})_i$, the following equality holds.

$$\mathrm{d}_E(\vec{e},\vec{\sigma})^2 = 2\langle\vec{e},\vec{e}\rangle - 2\langle\vec{e},\vec{\sigma}\rangle,$$

where $\langle\vec{e},\vec{e}\rangle$ and $\langle\vec{e},\vec{\sigma}\rangle$ are integers.

The following table compares Kendall tau and Euclidean weights for some permutations and their associated vectors.

σ	$\mathrm{wt}_K(\sigma)$	$\mathrm{wt}_E(\vec{\sigma})$
$[1,2,3,4,6,5]$	1	1
$[2,1,3,4,6,5]$	2	2
$[1,2,4,6,5,3]$	4	7
$[2,3,4,5,6,1]$	5	15
$[3,2,1,6,5,4]$	6	8

Notice that there are instances for which $\mathrm{wt}_K(\sigma) = \mathrm{wt}_E(\vec{\sigma})$, as well as instances for which $\mathrm{wt}_K(\sigma) < \mathrm{wt}_E(\vec{\sigma})$. Notice also that in the table $\mathrm{wt}_K([2,1,3,4,6,5]) < \mathrm{wt}_K([1,2,4,6,5,3]$ while for their associated vectors, $\mathrm{wt}_E(\overrightarrow{[2,1,3,4,6,5]} < \mathrm{wt}_E(\overrightarrow{[1,2,4,6,5,3]})$. However, in the case of $[2,3,4,5,6,1]$ and $[3,2,1,6,5,4]$, we have $\mathrm{wt}_K([2,3,4,5,6,1]) < \mathrm{wt}_K([3,2,1,6,5,4])$ while for their associated vectors, $\mathrm{wt}_E(\overrightarrow{[2,3,4,5,6,1]}) > \mathrm{wt}_E(\overrightarrow{[3,2,1,6,5,4]})$. These relations suggest the difficulty of comparing the two weights.

We begin our discussion of weight functions by characterizing both the Kendall tau and the Euclidean weights of elements in $C_n \subseteq S_n$, the cyclic subgroup of order n generated by $\sigma^{[n]} := [2,3,4\ldots,n,1]$.

Proposition 3.3. *Let $\sigma^{[i]} := (\sigma^{[n]})^{n-i+1}$, the unique permutation of C_n with 1 in the ith position. Then $\mathrm{wt}_K(\sigma^{[i]}) = (i-1)(n-i+1)$.*

Proof. For any $\sigma^{[i]} \in C_n$, we have

$$\sigma^{[i]} = [\underbrace{n-i, n-i+1, \ldots, n,}_{i-1}\underbrace{1, 2, \ldots, n-i+1}_{n-i+1}].$$

Note that $\sigma^{[i]}$ splits into two sub-sequences of length $i-1$ and $n-i+1$ respectively. For any element j of the left sub-seqence, $j > k$ for all k in the right sub-sequence. Thus the Kendall tau weight of $\sigma^{[i]}$ is determined by the product of the length of each sub-sequence, *i.e.*, $\mathrm{wt}_K(\sigma^{[i]}) = (i-1)(n-i+1)$. □

Example 3.4. The elements of C_5 and their respective Kendall tau weights are as follows:

$\sigma^{[1]} = [1,2,3,4,5]$, $\mathrm{wt}_K(\sigma^{[1]}) = 0$.
$\sigma^{[2]} = [5,1,2,3,4]$, $\mathrm{wt}_K(\sigma^{[2]}) = 1 \cdot 4 = 4$.
$\sigma^{[3]} = [4,5,1,2,3]$, $\mathrm{wt}_K(\sigma^{[3]}) = 2 \cdot 3 = 6$.
$\sigma^{[4]} = [3,4,5,1,2]$, $\mathrm{wt}_K(\sigma^{[4]}) = 3 \cdot 2 = 6$.
$\sigma^{[5]} = [2,3,4,5,1]$, $\mathrm{wt}_K(\sigma^{[5]}) = 4 \cdot 1 = 4$.

Although wt_K and wt_E have different domains and appear dissimilar superficially, amazingly there is a linear relationship between $\mathrm{wt}_K(\sigma)$ and $\mathrm{wt}_E(\vec{\sigma})$ for all $\sigma \in C_n$.

Proposition 3.5. *Let $\sigma^{[i]}$ denote the element of C_n with 1 in the ith position. Then $\mathrm{wt}_E(\overrightarrow{\sigma^{[i]}}) = \dfrac{n}{2}(i-1)(n-i+1)$ i.e., $\mathrm{wt}_E(\overrightarrow{\sigma^{[i]}}) = \dfrac{n}{2}\mathrm{wt}_K(\sigma^{[i]})$.*

Proof.

$$\overrightarrow{\sigma^{[i]}} = (\underbrace{i, i+1, \ldots, n}_{n-i+1}, \underbrace{1, 2, \ldots, i-1}_{i-1}).$$

Thus we have

$$\begin{aligned}
\mathrm{wt}_E(\overrightarrow{\sigma^{[i]}}) &= \frac{1}{2}\big(\underbrace{((i-1)^2 + \cdots + (n-(n-i+1))^2}_{n-i+1}\big) \\
&\quad + \frac{1}{2}\big(\underbrace{(((n-i+2)-1)^2 + \cdots + (n-(i-1))^2}_{i-1}\big) \\
&= \frac{1}{2}\big((n-i+1)(i-1)^2\big) + \frac{1}{2}\big((i-1)(n-i+1)^2)\big) \\
&= \frac{n}{2}(i-1)(n-i+1).
\end{aligned}$$

□

Example 3.6. The elements of C_5 and their respective Euclidean weights are as follows:

$\sigma^{[1]} = [1,2,3,4,5]$, $\mathrm{wt}_E(\overrightarrow{\sigma^{[1]}}) = 0$.
$\sigma^{[2]} = [5,1,2,3,4]$, $\mathrm{wt}_E(\overrightarrow{\sigma^{[2]}}) = \frac{1}{2}((4)1^2 + (1)4^2) = 10$.
$\sigma^{[3]} = [4,5,1,2,3]$, $\mathrm{wt}_E(\overrightarrow{\sigma^{[3]}}) = \frac{1}{2}((3)2^2 + (2)3^2) = 15$.
$\sigma^{[4]} = [3,4,5,1,2]$, $\mathrm{wt}_E(\overrightarrow{\sigma^{[4]}}) = \frac{1}{2}((2)3^2 + (3)2^2) = 15$.
$\sigma^{[5]} = [2,3,4,5,1]$, $\mathrm{wt}_E(\overrightarrow{\sigma^{[5]}}) = \frac{1}{2}((1)4^2 + (4)1^2) = 10$.

We now define the notions of minimum Kendall tau distance and minimum Euclidean distance.

Definition 3.7 (Minimum Kendall tau Distance)**.** Let G be a subgroup of S_n. The **minimum Kendall tau distance** for G is defined as

$$\min_{\sigma,\tau \in G, \sigma \neq \tau} \mathrm{d}_K(\sigma, \tau).$$

Definition 3.8 (Minimum Euclidean Distance)**.** Let G be a subgroup of S_n. The **minimum Euclidean distance** for G is defined as

$$\min_{\sigma,\tau \in G, \sigma \neq \tau} \mathrm{d}_K(\vec{\sigma}, \vec{\tau}).$$

We also define the notions of minimal Kendall tau weight element and minimal Euclidean weight element.

Definition 3.9 (Minimal Kendall tau Weight Element)**.** Let G be a subgroup of S_n. An element $m \neq e$ in G is of **minimal Kendall tau weight** if for all $\sigma \in G$, we have

$$\mathrm{wt}_K(m) \leq \mathrm{wt}_K(\sigma).$$

Definition 3.10 (Minimal Euclidean Weight Element)**.** Let G be a subgroup of S_n. An associated vector $\vec{m} \neq \vec{e}$ for some $m \in G$ is of **minimal Euclidean weight** if for all $\sigma \in G$, we have

$$\mathrm{wt}_E(\vec{m}) \leq \mathrm{wt}_K(\vec{\sigma}).$$

Having characterized the Kendall tau weights of all elements of C_n, as well as the Euclidean weights of associated vectors for elements of C_n, we are equipped to calculate the minimal weight elements. Determining minimal weights is often an important question in coding theory, since this provides insight into the minimum distance between codewords, an important concept in evaluating the error-correcting capabilities of a code. In the case of the Kendall tau and Euclidean distances, the minimum distance is equivalent to the minimum weight since in both cases we can rewrite the distance between any two elements as the weight of some element. We begin by determining the elements of minimal Kendall tau weight in C_n and explicitly calculating their values.

Corollary 3.11. *The elements of minimal Kendall tau weight in C_n are $\sigma^{[2]}$ and $\sigma^{[n]}$, with $\mathrm{wt}_K(\sigma^{[2]}) = \mathrm{wt}_K(\sigma^{[n]}) = n - 1$.*

Proof. By the previous proposition, to find the element $\sigma^{[i]} \in C_n \backslash \{e\}$ of minimal weight, it suffices to minimize $f(i) := (i-1)(n+i-1)$. Setting $f'(i) = -2i+n+2$ equal to 0, we determine that $\frac{n}{2}+1$ is a critical point. Two other critical points occur when $i = 2$ or $i = n$. Note that we exclude the possibility of $i = 1$ since $\sigma^{[1]} = e$. $f''(i) = -2$, so $f(i)$ is maximal at $i = \frac{n}{2}+1$. Checking our remaining critical points, we find that $f(2) = f(n) = n - 1$. Therefore $\sigma^{[2]}$ and $\sigma^{[n]}$ are the elements of $C_n \backslash \{e\}$ with minimal Kendall tau weight of $n - 1$. □

Since there is a linear relationship between the Kendall tau weight and the Euclidean weight for elements of C_n and their respective associated vectors, it is a simple matter to determine the elements of minimal Euclidean weight and their corresponding values in the C_n case.

Corollary 3.12. *The elements of minimal Euclidean weight for all associated vectors of permutations of C_n are $\overrightarrow{\sigma^{[2]}}$ and $\overrightarrow{\sigma^{[n]}}$, with $\mathrm{wt}_E(\overrightarrow{\sigma^{[2]}}) = \mathrm{wt}_E(\overrightarrow{\sigma^{[n]}}) = \frac{n}{2}(n-1)$.*

Proof. The desired result follows immediately from Proposition 3.5 and Proposition 3.11. □

We now shift our focus from minimal weight elements of subgroups of S_n to the maximal weight element over all of S_n. It is well-known that

$\omega_0 := [n, n-1, \ldots, 1]$ is the unique element of S_n having maximal Kendall tau weight, $\mathrm{wt}_K(\omega_0) = \frac{n(n-1)}{2}$; it is called the longest element. It is also a known property that for all σ in S_n, $\mathrm{wt}_K(\omega_0\sigma) = \mathrm{wt}_K(\omega_0) - \mathrm{wt}_K(\sigma)$ [14]. We proceed to show that a similar property also holds for the associated vector of ω_0 in terms of the wt_E, making $\vec{\omega_0}$ the element of maximal Euclidean weight.

Theorem 3.13. *Let ω_0 be the longest element and let $\sigma \in S_n$. Then* $\mathrm{wt}_E(\overrightarrow{\omega_0\sigma}) = \mathrm{wt}_E(\vec{\omega_0}) - \mathrm{wt}_E(\vec{\sigma})$.

Proof. Note first that

$2\mathrm{wt}_E(\overrightarrow{\omega_0\sigma}) = \langle \vec{e}, \vec{e} \rangle - 2\langle \vec{e}, \overrightarrow{\omega_0\sigma} \rangle + \langle \overrightarrow{\omega_0\sigma}, \overrightarrow{\omega_0\sigma} \rangle$

$2\mathrm{wt}_E(\vec{\omega_0}) = \langle \vec{e}, \vec{e} \rangle - 2\langle \vec{e}, \vec{\omega_0} \rangle + \langle \vec{\omega_0}, \vec{\omega_0} \rangle$ and

$2\mathrm{wt}_E(\vec{\sigma}) = \langle \vec{e}, \vec{e} \rangle - 2\langle \vec{e}, \vec{\sigma} \rangle + \langle \vec{\sigma}, \vec{\sigma} \rangle$.

Therefore

$$\begin{aligned}
& \mathrm{wt}_E(\overrightarrow{\omega_0\sigma}) = \mathrm{wt}_E(\vec{\omega_0}) - \mathrm{wt}_E(\vec{\sigma}) \\
\iff \quad & 2\langle \vec{e}, \vec{e} \rangle - 2\langle \vec{e}, \overrightarrow{\omega_0\sigma} \rangle \\
& = 2\langle \vec{e}, \vec{e} \rangle - 2\langle \vec{e}, \vec{\omega_0} \rangle - 2\langle \vec{e}, \vec{e} \rangle + 2\langle \vec{e}, \vec{\sigma} \rangle \\
\iff \quad & \langle \vec{e}, \vec{e} \rangle + \langle \vec{e}, \vec{\omega_0} \rangle = \langle \vec{e}, \vec{\sigma}, \rangle + \langle \vec{e}, \overrightarrow{\omega_0\sigma} \rangle \\
\iff \quad & \langle \vec{e}, \vec{e} \rangle + \langle \vec{e}, \vec{\omega_0} \rangle = \langle \vec{\sigma}, \vec{e} \rangle + \langle \vec{\sigma}, \vec{\omega_0} \rangle \\
\iff \quad & \langle \vec{e}, (1, 2, \ldots, n) \rangle + \langle \vec{e}, (n, n-1, \ldots, 1) \rangle \\
& = \langle \vec{\sigma}, (1, 2, \ldots, n) \rangle + \langle \vec{\sigma}, (n, n-1, \ldots, 1) \rangle \\
\iff \quad & \langle \vec{e}, (n+1, n+1, \ldots, n+1) \rangle \\
& = \langle \vec{\sigma}, (n+1, n+1, \ldots, n+1) \rangle.
\end{aligned}$$

Of course the last equality holds since $\vec{\sigma}$ is simply a permutation of $\vec{e}$. $\square$

Corollary 3.14. *The longest element $\omega_0 \in S_n$ is the unique element such that $\vec{\omega_0}$ is of maximal Euclidean weight.*

Proof. By Theorem 3.13, for any $\sigma \in S_n$, we have $\mathrm{wt}_E(\vec{\sigma}) = \mathrm{wt}_E(\vec{\omega_0}) - \mathrm{wt}_E(\overrightarrow{\omega_0\sigma}) \leq \mathrm{wt}_E(\vec{\omega_0})$, since $\mathrm{wt}_E(\overrightarrow{\omega_0\sigma}) \geq 0$. Equality holds only if $\omega_0\sigma = e$, which is true only when $\sigma = \omega_0$. Thus the statement holds. $\square$

Proposition 3.15. $\mathrm{wt}_E(\vec{\omega_0}) = \dfrac{1}{6}(n^3 - n) = \dfrac{n+1}{3}\mathrm{wt}_K(\omega_0)$.

Proof. For any even positive integer n,

$$\begin{aligned}
\mathrm{wt}_E(\vec{\omega_0}) &= \sum_{i=0}^{\frac{n}{2}-1} (1+2i)^2 = \sum_{i=0}^{\frac{n}{2}-1} 1 + 4\sum_{i=0}^{\frac{n}{2}-1} i + 4\sum_{i=0}^{\frac{n}{2}-1} i^2 \\
&= \frac{n}{2} + 4(\frac{1}{2})(\frac{n}{2} - 1)(\frac{n}{2}) + 4(\frac{1}{6})(\frac{n}{2} - 1)(\frac{n}{2})(n-1) \\
&= \frac{1}{6}(n^3 - n).
\end{aligned}$$

For any odd positive integer n,

$$\begin{aligned}
\mathrm{wt}_E(\vec{\omega_0}) &= \sum_{i=0}^{\frac{n-1}{2}} (2i)^2 = \sum_{i=0} \frac{n-1}{2} 4i^2 \\
&= 4(\frac{1}{6})(\frac{n-1}{2})(\frac{n-1}{2}+1)(n-1+1) \\
&= \frac{1}{6}(n^3-n).
\end{aligned}$$

□

Continuing our exposition on the relationship between Kendall tau and Euclidean weights, we consider now their respective weight enumerator polynomials [18], also known as the generator functions [20]. The weight enumerator polynomial is defined as follows.

Definition 3.16 (Weight Enumerator Polynomials)**.** Let $G \subseteq S_n$. The weight enumerator polynomial W_K (resp. W_E) of G for wt_K (resp. wt_E) is:

$$W_K(G;t) := \sum_{\sigma \in G} t^{\mathrm{wt}_K(\sigma)} \text{ (resp. } W_E(G;t) := \sum_{\sigma \in G} t^{\mathrm{wt}_E(\vec{\sigma})} \text{).}$$

We shall begin our discussion of weight enumerator polynomials with the cyclic subgroup case.

Example 3.17. The Kendall tau weight enumerator polynomials of C_n for $n = 1, \ldots 7$ are as follows.

$$\begin{array}{llllll}
W_K(C_1;t) & = 1. & & & \\
W_K(C_2;t) & = 1 & +t. & & \\
W_K(C_3;t) & = 1 & +2t^2. & & \\
W_K(C_4;t) & = 1 & +2t^3 & +t^4. & \\
W_K(C_5;t) & = 1 & +2t^4 & +2t^6. & \\
W_K(C_6;t) & = 1 & +2t^5 & +2t^8 & +t^9. \\
W_K(C_7;t) & = 1 & +2t^6 & +2t^{10} & +2t^{12}.
\end{array}$$

Proposition 3.18. *The Kendall tau weight enumerator polynomial for the cyclic group of order n is characterized by*

$$W_K(C_n;t) = \sum_{i=1}^{n} t^{(i-1)(n-i+1)}.$$

Proof. The desired result follows immediately from Proposition 3.3. □

Example 3.19. The Euclidean weight enumerator polynomials of C_n for $n = 1, \ldots 7$ are as follows.

$$
\begin{array}{llllll}
W_E(C_1;t) & = 1. & & & \\
W_E(C_2;t) & = 1 & +t. & & \\
W_E(C_3;t) & = 1 & +2t^3. & & \\
W_E(C_4;t) & = 1 & +2t^6 & +t^8. & \\
W_E(C_5;t) & = 1 & +2t^{10} & +2t^{15}. & \\
W_E(C_6;t) & = 1 & +2t^{15} & +2t^{24} & +t^{27}. \\
W_E(C_7;t) & = 1 & +2t^{21} & +2t^{35} & +2t^{42}.
\end{array}
$$

Proposition 3.20. *The Euclidean weight enumerator polynomials for the cyclic group of order n is characterized by*

$$W_E(C_n;t) = \sum_{i=1}^{n} t^{\frac{n}{2}(i-1)(n-i+1)}.$$

Proof. The desired result follows immediately from Proposition 3.5. □

Remark 3.21. The above two propositions yield the following relationship between the Kendall tau and Euclidean weight enumerator polynomials in the cyclic group case.

$$W_E(C_n;t) = W_K(C_n;t^{\frac{n}{2}}).$$

With the preceding discussion of C_n and the maximal weight element ω_0 of S_n, we are now equipped to characterize the weights of all permutations in $D_{2n} := C_n \cup \omega_0 C_n$, a dihedral group of order $2n$.

Example 3.22. The Kendall tau weight enumerator polynomials of D_{2n} for $n = 3, \ldots, 7$ (There are no Dihedral groups for $n \leq 2$) are as follows.

$$
\begin{array}{rcllllll}
W_K(D_6;t) & = & 1 & +2t & +2t^2 & +t^3. & & \\
W_K(D_8;t) & = & 1 & +t^2 & +4t^3 & +t^4 & +t^6. & \\
W_K(D_{10};t) & = & 1 & +4t^4 & +4t^6 & +t^{10}. & & \\
W_K(D_{12};t) & = & 1 & +2t^5 & +t^6 & +2t^7 & +2t^8 & +t^9 \\
 & & & +2t^{10} & +t^{15}. & & & \\
W_K(D_{14};t) & = & 1 & +2t^6 & +2t^9 & +2t^{10} & +2t^{11} & +2t^{12} \\
 & & & +2t^{15} & +t^{21}. & & &
\end{array}
$$

Proposition 3.23.

$$
\begin{aligned}
W_K(D_{2n};t) &= W_K(C_n;t) + t^{\frac{1}{2}(n^2-n)} W_K(C_n;t^{-1}) \\
&= \sum_{i=1}^{n} t^{(i-1)(n-i+1)} + t^{\frac{1}{2}(n^2-n)} \sum_{i=1}^{n} t^{-(i-1)(n-i+1)}.
\end{aligned}
$$

Proof. Recall that D_{2n} can be split into its cyclic subgroup C_n of order n and the coset $\omega_0 C_n$. Hence the desired result follows immediately from Proposition 3.3, the property that for all $\sigma \in S_n$, that $\mathrm{wt}_K(\omega_0\sigma) = \mathrm{wt}_K(\omega) - \mathrm{wt}_K(\sigma)$, and the fact that $\mathrm{wt}_K(\omega_0) = \frac{1}{2}(n^2 - n)$. □

Example 3.24. The Euclidean weight enumerator polynomials of D_{2n} for $n = 3, \ldots, 7$ are as follows.

$$\begin{array}{rcllllll}
W_E(D_6;t) & = & 1 & +2t & +2t^3 & +t^4. & & \\
W_E(D_8;t) & = & 1 & +t^2 & +2t^4 & +2t^6 & +t^8 & +t^{10}. \\
W_E(D_{10};t) & = & 1 & +2t^5 & +4t^{10} & +2t^{15} & +t^{20}. & \\
W_E(D_{12};t) & = & 1 & +t^8 & +2t^{11} & +2t^{15} & +2t^{20} & +2t^{24} \\
 & & & +t^{27} & +t^{35}. & & & \\
W_E(D_{14};t) & = & 1 & +2t^{14} & +4t^{21} & +4t^{35} & +2t^{42} & +t^{56}.
\end{array}$$

Proposition 3.25.

$$\begin{aligned} W_E(D_{2n};t) &= W_E(C_n;t) + t^{\frac{1}{6}(n^3-n)} W_E(C_n;t^{-1}) \\ &= \sum_{i=1}^{n} t^{\frac{n}{2}(i-1)(n-i+1)} + t^{\frac{1}{6}(n^3-n)} \sum_{i=1}^{n} t^{-\frac{n}{2}(i-1)(n-i+1)}. \end{aligned}$$

Proof. As in the proof of Proposition 3.23, we may split D_{2n} into its cyclic subgroup C_n of order n and the coset $\omega_0 C_n$. The desired result follows immediately from Proposition 3.5, the property that for all $\sigma \in S_n$, that $\mathrm{wt}_E(\overrightarrow{\omega_0 \sigma} = \mathrm{wt}_E(\vec{\omega_0}) - \mathrm{wt}_E(\vec{\sigma})$, and the fact that $\mathrm{wt}_E(\vec{\omega_0}) = \frac{1}{6}(n^3-n)$. $\square$

Remark 3.26. The relationship between the Kendall tau and Euclidean weight enumerator polynomials is not as clear in the D_{2n} case as it was in the C_n case. However, based on the previous two propositions, we may write the Euclidean weight enumerator polynomial for D_{2n} in terms of Kendall tau weight enumerator polynomials as follows.

$$W_E(D_{2n};t) = W_K(C_n;t^{\frac{n}{2}}) + t^{\frac{1}{6}(n^3-n)} W_K(C_n;t^{-\frac{n}{2}}).$$

At this point, having characterized the weight enumerator polynomials for D_{2n}, we are now equipped to determine the elements of minimal weight in D_{2n} and their corresponding values. We begin by finding the element of D_{2n} having minimal Kendall tau weight.

Corollary 3.27. *Let $\sigma \in D_{2n}$. For $n \geq 5$,* $\min_{\substack{\sigma \in D_{2n}, \\ \sigma \neq e}} \mathrm{wt}_K(\sigma) = n-1.$

Proof. Consider the cyclic subgroup $C_n \subset D_{2n}$ of order n. By Proposition 3.11, the minimum Kendall tau weight among elements of C_n is $n-1$. Note that ω_0, the element of maximal length in S_n is contained in the coset $\omega_0 C_n \subseteq D_{2n}$. We know that $\mathrm{wt}_K(\omega_0) = \frac{1}{2}(n-1)(n)$ (a simple combinatorial proof verifies this fact). Therefore By Proposition 3.3 and the fact that $\mathrm{wt}_K(\omega_0 \circ \sigma) = \mathrm{wt}_K(\omega_0) - \mathrm{wt}_E$, the weights of all elements in the coset $\omega_0 C_n$ can be characterized by $\frac{1}{2}(n-1)(n) - (i-1)(n-i+1) = i^2 - (n+1)i + \frac{n^2+n+2}{2}$ where $i = 1, \ldots, n$. Minimizing $f(i) := i^2 - (n+1)i + \frac{n^2+n+2}{2}$ by elementary calculus methods as in the proof of Proposition 3.11, we determine that $f(i)$ is minimized when $i = \frac{n}{2}+1$ with minimum value of $\frac{n^2}{4} - \frac{n}{2}$. Comparing this value to $n-1$, we see that $n-1$ remains minimal for all $n \geq 6$. Explicitly calculating weights of all elements in D_{10}, we verify that 4 is the minimum value among elements in $D_{10}\backslash e$. Thus we have shown that $\min_{\substack{\sigma\in D_{2n},\\ \sigma\neq e}} \mathrm{wt}_K(\sigma) = n-1$ for all $n \geq 5$. $\square$

Corollary 3.28. *Let $\sigma \in D_{2n}$. For $n \geq 11$, $\min_{\substack{\sigma\in D_{2n}\\ \sigma\neq e}} \mathrm{wt}_E(\vec{\sigma}) = \frac{n}{2}(n-1)$.*

Proof. We begin as in the previous proposition, by dividing D_{2n} into its cyclic subgroup C_n of order n, and the coset $\omega_0 C_n \subseteq D_{2n}$. By Proposition 3.12, we have $\min_{\substack{\sigma\in C_n\\ \sigma\neq e}} \mathrm{wt}_E(\vec{\sigma}) = \frac{n}{2}(n-1)$. By Theorem 3.13 and Proposition 3.15, the weight of all associated vectors of elements in the coset $\omega_0 C_n$ can be characterized by $\frac{1}{6}(n^3-n) - i(n-i)\frac{n}{2} = \frac{1}{6}\left(3ni^2 - 3n^2 i + n^3 - n\right)$ where $i = 1, \ldots, n$.

Let $f(i) = \frac{1}{6}\left(3ni^2 - 3n^2 i + n^3 - n\right)$. Then $f'(i) = \frac{1}{6}(6ni - 3n^2)$. Setting $f'(i) = 0$, we determine that $i = \frac{n}{2}$. Since $f''(i) > 0$, it follows that $i = \frac{n}{2}$ determines a minimum value of $f(i)$. Thus the minimum value of $f(i)$ is equal to $f(\frac{n}{2}) = \frac{1}{6}\left(3n(\frac{n}{2})^2 - 3n^2(\frac{n}{2}) + n^3 - n\right) = \frac{1}{6}(\frac{1}{4}n^3 - n)$. Hence to prove our proposition it suffices to show that for all $n \geq 12$, the inequality $\frac{1}{6}(\frac{1}{4}n^3 - n) \geq (\frac{n}{2})(n-1)$ is true.

$$\begin{aligned} \frac{1}{6}(\frac{1}{4}n^3 - n) &\geq \frac{n}{2}(n-1) \\ \iff n^2 - 4n &\geq 12n - 12 \\ \iff n^2 - 16n + 12 &\geq 0 \\ \iff n \leq 4 - 2\sqrt{13} &\text{ or } n \geq 4 + 2\sqrt{13} \end{aligned}$$

Of course $4 - 2\sqrt{13} < 0$, so the only viable option for n is $n \geq 4 + 2\sqrt{13}$. Since $12 > 4 + 2\sqrt{13}$, for all $n \geq 12$ the desired inequality holds. After explicit calculations we can also see that the minimum Euclidean weight of D_{22} is 55, completing the proof. $\square$

The following table shows the minimum Kendall tau and Euclidean weights of D_{2n} for $n = 3, \ldots, 11$ (there are no Dihedral groups for $n \leq 2$).

n	$\min_{\sigma\in D_{2n},\sigma\neq e}(\mathrm{wt}_K(\sigma))$	$\min_{\sigma\in D_{2n},\sigma\neq e}(\mathrm{wt}_E(\sigma))$
3	1	1
4	2	2
5	4	5
6	5	8
7	6	14
8	7	20
9	8	30
10	9	40
11	10	55

In the case of S_n, calculating minimal weight elements is a trivial matter, since for any natural number n, there exists a permutation of S_n with a Kendall tau or Euclidean weight of 1. Specifically, any adjacent permutation s_i in S_n will have both Kendall tau and Euclidean weight of 1. However, it remains an open question to characterize $W_E(S_n;t)$ in terms of $W_K(S_n;t)$ as in the cases of C_n and D_{2n}. Using a simple computer program, we have calculated the weight enumerator polynomials of S_n similarly to the previous examples.

Example 3.29. The Kendall tau weight enumerator polynomials of S_n for $n = 1, \ldots, 7$ are as follows.

$$
\begin{aligned}
W_K(S_1;t) &= 1 \\
W_K(S_2;t) &= 1 \quad +t \\
W_K(S_3;t) &= 1 \quad +2t \quad +2t^2 \quad +t^3 \\
W_K(S_4;t) &= 1 \quad +3t \quad +5t^2 \quad +6t^3 \quad +5t^4 \quad +3t^5 \quad +t^6 \\
W_K(S_5;t) &= 1 \quad +4t \quad +9t^2 \quad +15t^3 \quad +20t^4 \quad +22t^5 \quad +20t^6 \quad +15t^7 \\
&\qquad +9t^8 \quad +4t^9 \quad +t^{10} \\
W_K(S_6;t) &= 1 \quad +5t \quad +14t^2 \quad +29t^3 \quad +49t^4 \quad +71t^5 \quad +90t^6 \quad +101t^7 \\
&\qquad +101t^8 \quad +90t^9 \quad +71t^{10} \quad +49t^{11} \quad +29t^{12} \quad +14t^{13} \quad +5t^{14} \\
&\qquad +t^{15} \\
W_K(S_7;t) &= 1 \quad +6t \quad +20t^2 \quad +49t^3 \quad +98t^4 \quad +169t^5 \quad +259t^6 \quad +359t^7 \\
&\qquad +455t^8 \quad +531t^9 \quad +573t^{10} \quad +573t^{11} \quad +531q^{12} \quad +455t^{13} \quad +359t^{14} \\
&\qquad +259t^{15} \quad +169t^{16} \quad +98t^{17} \quad +49t^{18} \quad +20t^{19} \quad +6t^{20} \quad +t^{21}
\end{aligned}
$$

The following general formula of $W_K(S_n;t)$ for $n \geq 2$ is well-known [16].

$$W_K(S_n;t) = (1+t)(1+t+t^2)\cdots(1+t+\cdots+t^{n-1}).$$

This formula is a special case of Weyl's character formula for Lie thoery [17]. It is a relatively simple matter to see why it is true. From the above example the formula is easily verifiable for $n = 2$. Notice that any permutation of S_3 can be obtained from a permutation of S_2 by simply inserting the number 3 into some position. For example, the permutation $[1,2,3]$ is simply the permutation $[1,2]$, with 3 inserted into the third position. Similarly $[1,3,2]$ is simply the permutation $[1,2]$ with 3 inserted in the second position and $[3,1,2]$ is $[1,2]$ with 3 inserted into the first position. In general, inserting n into the $(n-i)$th position for $i = 0, \ldots, n-1$ in each of the permutation σ of S_{n-1} corresponds to a new permutation of S_n with weight $\mathrm{wt}_K(\sigma) + i$. Thus an inductive argument yields the desired formula.

Now that we are convinced the above formula is veracious, it is clear that the powers of $W_K(S_n;t)$ are consecutive, running from 1 through $\frac{1}{2}(n^2-n)$.

That is, for all values k between 0 and $\frac{1}{2}(n^2 - n)$, there exists a $\sigma \in S_n$ such that $\mathrm{wt}_E(\sigma) = k$. It remains an open question to find a concise formula for $W_E(S_n; t)$ similar to the formula above. It is also an open question to relate $W_E(S_n; t)$ and $W_K(S_n; t)$ as in the case of C_n and D_{2n}.

The size of the weight enumerator polynomials for large values of n suggests the difficulty of characterizing $W_E(S_n; t)$ in terms of $W_K(S_n; t)$. Moreover, there is a large disparity between the total number of weight values in $W_E(S_n; t)$ and the total number of weight values in $W_K(S_n; t)$. In fact, as we will later observe, there are exactly $\frac{1}{6}(n^3 - n) + 1$ total Euclidean weight values for S_n whenever $n \neq 3$, and from the fact that the powers of $W_K(S_n; t)$ are consecutive, there are exactly $\frac{1}{2}(n^2 - n) + 1$ weight values in $W_K(S_n; t)$. The following example shows the corresponding Euclidean weight enumerator polynomials to the example above.

Example 3.30. The Euclidean weight enumerator polynomials of S_n for $n = 1, \ldots, 7$ are as follows.

$$
\begin{array}{llllllllll}
W_E(S_1;t) & = 1 \\
\\
W_E(S_2;t) & = 1 & +t \\
\\
W_E(S_3;t) & = 1 & +2t & +2t^3 & +t^4 \\
\\
W_E(S_4;t) & = 1 & +3t & +t^2 & +4t^3 & +2t^4 & +2t^5 & +2t^6 & +4t^7 \\
 & & +t^8 & +3t^9 & +t^{10} \\
\\
W_E(S_5;t) & = 1 & +4t & +3t^2 & +6t^3 & +7t^4 & +6t^5 & +4t^6 & +10t^7 \\
 & & +6t^8 & +10t^9 & +6t^{10} & +10t^{11} & +6t^{12} & +10t^{14} & +4t^{14} \\
 & & +6t^{15} & +7t^{16} & +6t^{17} & +3t^{18} & +4t^{19} & +t^{20} \\
\\
W_E(S_6;t) & = 1 & +5t & +6t^2 & +9t^3 & +16t^4 & +12t^5 & +14t^6 & +24t^7 \\
 & & +20t^8 & +21t^9 & +23t^{10} & +28t^{11} & +24t^{12} & +34t^{13} & +20t^{14} \\
 & & +32t^{15} & +42t^{16} & +29t^{17} & +29t^{18} & +42t^{19} & +32t^{20} & +20t^{21} \\
 & & +34t^{22} & +24t^{23} & +28t^{24} & +23t^{25} & +21t^{26} & +20t^{27} & +24t^{28} \\
 & & +14t^{29} & +12t^{30} & +16t^{31} & +9t^{32} & +6t^{33} & +5t^{34} & +t^{35} \\
\\
W_E(S_7;t) & = 1 & +6t & +10t^2 & +14t^3 & +29t^4 & +26t^5 & +35t^6 & +46t^7 \\
 & & +55t^8 & +54t^9 & +74t^10 & +70t^{11} & +84t^{12} & +90t^{13} & +78t^{14} \\
 & & +90t^{15} & +129t^{16} & +106t^{17} & +123t^{18} & +134t^{19} & +147t^{20} & +98t^{21} \\
 & & +168t^{22} & +130t^{23} & +175t^{24} & +144t^{25} & +168t^{26} & +144t^{27} & +184t^{28} \\
 & & +144t^{29} & +168t^{30} & +144t^{31} & +175t^{32} & +130t^{33} & +168t^{34} & +98t^{35} \\
 & & +147t^{36} & +134t^{37} & +123t^{38} & +106t^{39} & +129t^{40} & +90t^{41} & +78t^{42} \\
 & & +90t^{43} & +84t^{44} & +70t^{45} & +74t^{46} & +54t^{47} & +55t^{48} & +46t^{49} \\
 & & +35t^{50} & +26t^{51} & +29t^{52} & +14t^{53} & +10t^{54} & +6t^{55} & +t^{56}
\end{array}
$$

From example above it can be observed that all the Euclidean weight values appear to be consecutive for $n \neq 3$. In other words, for all values k between 0 and $\frac{1}{6}(n^3 - n)$, there exists a $\sigma \in S_n$ such that $\mathrm{wt}_E(\vec{\sigma}) = k$. We now proceed to prove the validity of this observation.

Proposition 3.31. *For $n \geq 1$ and $n \neq 3$, the powers of t in $W_E(S_n; t)$ are consecutive.*

Proof. From the above example we can observe that the powers of t in $W_E(S_n; t)$ are consecutive for $n = 1$ and $n = 2$. We proceed to justify our claim by induction on n. For the base case of $n = 4$, note that $W_E(S_4; t) = 1 + 3t + t^2 + 4t^3 + 2t^4 + 2t^5 + 2t^6 + 4t^7 + t^8 + 3t^9 + t^{10}$, where we can easily observe that the powers of t are consecutive. Suppose now that the powers of t in $W_E(S_n; t)$ are consecutive. We will show that this implies that the powers of t are consecutive for $W_E(S_{n+1}; t)$.

Let us begin by showing that the first $\frac{1}{6}(n^3 - n) + 1$ powers of t are consecutive. Since $S_n \subseteq S_{n+1}$, we conclude that each of the powers 0 through $\max\{\mathrm{wt}_E(\vec{\sigma}) | \sigma \in S_n\} = \frac{1}{6}(n^3 - n)$ are contained in $W_E(S_{n+1}; t)$. Hence the first $\frac{1}{6}(n^3 - n) + 1$ powers are consecutive. Next, let us show that the last $\frac{1}{6}(n^3 - n) + 1$ powers of t are consecutive.

Let $j = 0, \ldots, \frac{1}{6}(n^3 - n)$ Then for each j there exists a $\sigma_j \in S_{n+1}$ such that $\mathrm{wt}_E(\overrightarrow{\sigma^{(j)}}) = j$. By Theorem 3.13, for each $\sigma_j \in S_{n+1}$ there exists a $\sigma_j^* \in S_{n+1}$ such that $\mathrm{wt}_E(\overrightarrow{\sigma^{(j^*)}}) = \mathrm{wt}_E(\vec{\omega_0}) - \mathrm{wt}_E(\vec{\sigma}) = \frac{1}{6}((n+1)^3 - (n+1)) - j$. Since $j = 0, \ldots, \frac{1}{6}(n^3 - n)$, the last $\frac{1}{6}(n^3 - n) + 1$ powers of t are in fact consecutive. Thus to show that the powers of t in $W_E(S_{n+1}; t)$ are consecutive, it suffices to show that $2 \cdot \frac{1}{6}(n^3 - n) \geq \frac{1}{6}((n+1)^3 - (n+1))$.

$$2 \cdot \frac{1}{6}(n^3 - n) \geq \frac{1}{6}((n+1)^3 - (n+1))$$
$$\iff n^3 - 3n^2 - 4n \geq 0 \iff n(n-4)(n+1) \geq 0$$

The above inequality is satisfied for all $n \geq 4$. Ergo by induction, the powers of $W_E(S_n; t)$ are consecutive for $n \geq 4$. $\square$

4. Extending LP-Decoding Methods

In this section we explain a necessary and sufficient condition for LP decoding methods to be utilized in permutation codes with the Kendall tau distance metric and provide examples of codes satisfying this condition. We begin with a brief explanation of LP-Decodable permutation codes. For a more detailed exposition on the topic, the reader is referred to [13] and [19].

We first recall the embedding of S_n into the set $M_n(\mathbb{R})$ of $n \times n$ matrices as described in [19]. With any permutation σ in S_n, we associate with σ the matrix X^σ with entries $X^\sigma_{i,j} := \delta_{i=\sigma_j}$, where $X_{i,j}$ is the (i, j)th entry of a matrix X, and δ is the Kronecker delta. For example,

$$[1,4,2,3] \text{ is associated with the matrix } \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Such matrices are known as permutation matrices. Note that if X^σ is the permutation matrix associated with the permutation σ and $\vec{\mu} \in \mathbb{R}^n$, then the action $\sigma \circ \vec{\mu}$ is equivalent to the natural action $X^\sigma \vec{\mu}^T$, where $\vec{\mu}^T$ denotes the transpose of $\vec{\mu}$.

Definition 4.1 (Linear Constraint)**.** A **linear constraint** for a matrix X is a linear equation or linear inequality on the entries $X_{i,j}$ of X.

Permutation matrices can be described in terms of linear constraints. Specifically, a permutation matrix is an $n \times n$ binary matrix X such that for all i and j in the set $\{1, 2, \ldots, n\}$, the following constraints are satisfied: $\sum_{i=1}^{n} X_{i,j} = 1$ and $\sum_{j=1}^{n} X_{i,j} = 1$. If we exclude the binary constraint on permutation matrices and specify non-negativity of matrix entries, we obtain the set of doubly stochastic matrices.

Definition 4.2 (Doubly Stochastic Matrix)**.** An $n \times n$ matrix is called a **doubly stochastic matrix** if it satisfies the following linear constraints:

(1) For $j = 1, \ldots, n$, $\sum\limits_{i=1}^{n} X_{i,j} = 1$.

(2) For $i = 1, \ldots, n$, $\sum\limits_{j=1}^{n} X_{i,j} = 1$.

(3) For all $1 \le i \le n$ and $1 \le j \le n$, the matrix entry $X_{i,j} \ge 0$.

The following theorem relating doubly stochastic matrices and permutation matrices is known as the Birkhoff-von Neumann theorem. The statement of the theorem is exactly as it appears in [13]

Theorem 4.3 (Birkhoff-von Neumann Theorem[23, 24])**.** *Every doubly stochastic matrix is a convex combination of permutation matrices.*

The set of $n \times n$ doubly stochastic matrices, denoted by B_n, is a convex polytope known as a Birkhoff polytope [23]. Convex polytopes are generalizations of convex polygons for arbitrarily large finite dimension. They are defined by Grünbaum as a compact convex set of $\mathbb{R}^n$ having a finite number of vertexes [22, p. 31]. Vertexes are defined formally subsequently.

Definition 4.4 (Doubly Stochastic Constraint)**.** A **doubly stochastic constraint** $\mathcal{L}$ for an $n \times n$ matrix X is a set of linear constraints such that if X satisfies the constraints of $\mathcal{L}$, then X is a doubly stochastic matrix.

Definition 4.5 (Doubly Stochastic Polytope)**.** A subset D of $M_n(\mathbb{R})$, the set of $n \times n$ matrices over $\mathbb{R}$ is a **doubly stochastic polytope** if there exists a doubly stochastic constraint $\mathcal{L}$ such that D is the set of all $n \times n$ matrices satisfying $\mathcal{L}$. This doubly stochastic polytope resulting from matrices satisfying $\mathcal{L}$ is denoted by $D(\mathcal{L})$.

Definition 4.6 (Vertex)**.** Let D be a doubly stochastic polytope. An element $X \in D$ is called a **vertex** if there do not exist elements X_1 and X_2 in D with $X_1 \neq X_2$ such that $X = c_1X_1 + c_2X_2$, where c_1, c_2 are strictly positive real numbers. The set of vertices for D is denoted by $\mathrm{Ver}(D)$.

By the Birkhoff-von Newmann theorem, it is clear that the extreme points, or vertexes, of B_n are exactly the $n \times n$ permutation matrices. Additional constraints may also be added to form new polytopes. Let G be a subset of S_n.

It was proven by Wadayama and Hagiwara that in the case of a doubly stochastic polytope D, having a vertex set $\mathrm{Ver}(D)$ such that $G = \mathrm{Ver}(D) \cap S_n$, then finding $\vec{x} = \arg \min\limits_{X\vec{\mu} \in G\vec{\mu}} \left(\mathrm{d}_E(\vec{\lambda}, X\vec{\mu})\right)$ is equivalent to solving a linear programming problem. In this setup $\vec{\lambda}$ is a fixed received vector and $\vec{\mu}$ is a fixed initial vector. The equivalent linear programming problem is to maximize $\vec{\lambda}^T X\vec{\mu}$, where X is a permutation matrix from G. The matrix X_0 maximizing $\vec{\lambda}^T X\vec{\mu}$ minimizes $\mathrm{d}_E(\vec{\lambda}, X\vec{\mu})$ so that $\vec{x}$ is equal to $X_0\vec{\mu}$ [13, 19].

Definition 4.7 (LP-Decodable Permutation Code)**.** A permutation code $G\vec{\mu}$ is **LP-Decodable** if there exists a doubly stochastic constraint $\mathcal{L}$ such that $G = \mathrm{Ver}(D(\mathcal{L})) \cap S_n$.

Suppose $G\vec{\mu}$ is an LP-decodable permutation code where $\mathcal{L}$ is a doubly stochastic constraint such that $G = \mathrm{Vec}(D(\mathcal{L})) \cap S_n$. The decoding process is summarized in the following algorithm.

LP-Decoding Algorithm

Let $\vec{\lambda}$ be a received word,

1) Solve the following linear programming problem:

$$\text{maximize } \vec{\lambda}^T X\vec{\mu} \text{ over all } X \text{ satisfying } \mathcal{L}.$$

2) For a solution X_0, set $\vec{\mu_0} = X_0\vec{\mu}$.

3) If X_0 is a permutation matrix, output $\vec{\mu_0}$.

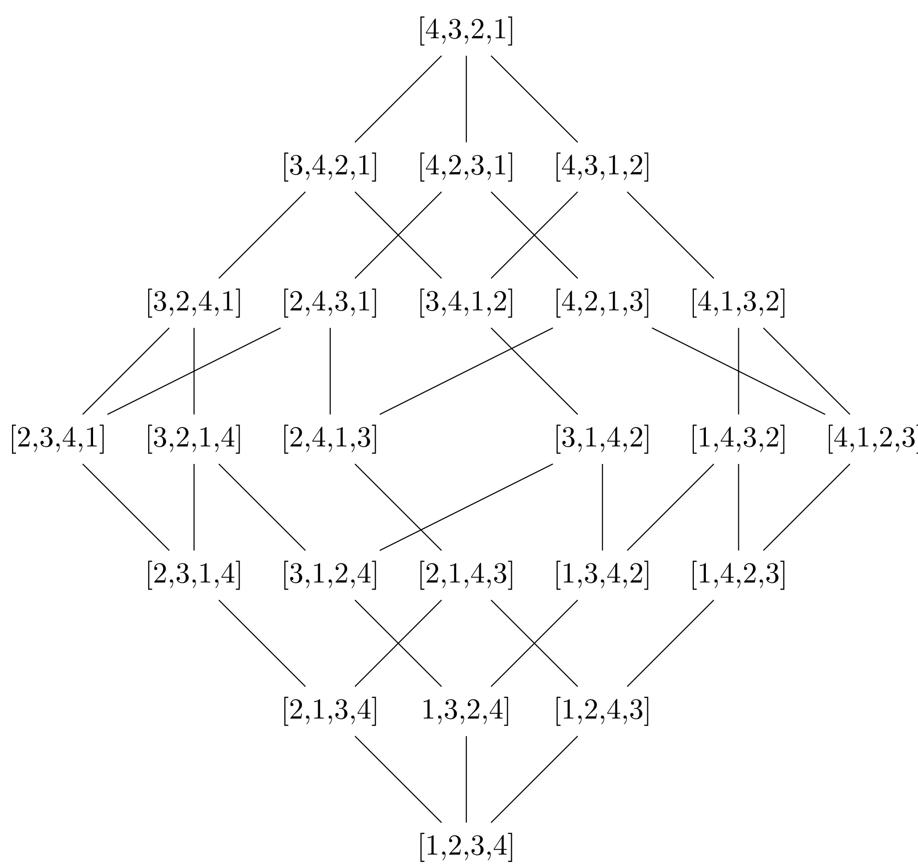
Otherwise, declare decoding failure.

Ideally, in an LP-Decodable Permutation Code $G\vec{\mu}$, there will be a doubly stochastic constraint $\mathcal{L}$ such that $G = \mathrm{Ver}(D(\mathcal{L}))$. This will ensure that solutions to the above maximization problem in the LP-decdoing algorithm will be permutation matrices.

With the goal of extending LP decoding methods in mind, we next prove a relation between $\mathrm{wt}_K(\sigma)$ and $\mathrm{wt}_E(\vec{\sigma})$ that holds for any general σ of S_n. To prove this relation we recall a partial ordering known as the weak Bruhat ordering.

Definition 4.8. Let $\sigma, \tau \in S_n$. Define $\sigma^{(0)} := \sigma$ and $\sigma^{(\mathrm{wt}_K(\tau)-\mathrm{wt}_K(\sigma))} := \tau$. The weak (right) Bruhat ordering on S_n is a partial ordering $\leq$ where $\sigma < \tau$ if and only if $\mathrm{wt}_K(\sigma) < \mathrm{wt}_K(\tau)$, and for all $1 \leq r \leq \mathrm{wt}_K(\tau) - \mathrm{wt}_K(\sigma)$, there exists $\sigma^{(r)} \in S_n$, and $1 \leq i_r < n$ such that $(\sigma^{(r-1)})^{-1}(\sigma^{(r)}) = (i_r, i_r + 1)$ and $\mathrm{wt}_K(\sigma^{(r)}) = \mathrm{wt}_K(\sigma) + r$. We say $\sigma \leq \tau$ if either $\sigma < \tau$ or $\sigma = \tau$.

Intuitively, the above definition states that a permutation σ is strictly less than a permutation τ if σ has a smaller Kendall tau weight and τ can be obtained by applying a series of adjacent transpositions to σ with the Kendall tau weight increasing by 1 with each adjacent transposition. The following figure illustrates the weak (right) Bruhat ordering for S_4.



In the diagram above, two permutations are comparable under the weak Bruhat ordering if there is a strictly ascending or strictly descending connected path between the two permutations. For example, $[1,2,3,4] < [2,1,3,4] < [2,3,1,4] < [2,3,4,1] < [3,2,4,1] < [3,4,2,1] < [4,3,2,1]$ forms an ascending chain under the weak Bruhat ordering. However, as an example, neither of the following statements is true: $[2,1,3,4] \leq [1,4,2,3]$ or $[1,4,2,3] \leq [2,1,3,4]$. Notice that both $\omega_0 = [4,3,2,1]$ and $e = [1,2,3,4]$ are comparable to all other permutations of S_n.

Theorem 4.9. *If $\sigma < \tau$ in the weak Bruhat ordering, then* $\mathrm{wt}_E(\vec{\sigma}) < \mathrm{wt}_E(\vec{\tau})$.

Proof. We proceed by induction. Suppose $\sigma < \tau$. To prove the base case let $\sigma^{-1}\tau = (i, i+1)$ and $\mathrm{wt}_K(\tau) = \mathrm{wt}_K(\sigma) + 1$. Then $\tau = \sigma s_i$ where $s_i \in S_n$ is the transposition $(i, i+1)$. Hence $\mathrm{wt}_K(\sigma s_i) = \mathrm{wt}_K(\sigma) + 1$. It follows that $\sigma_{i+1} > \sigma_i$. Therefore $((i+1)-\sigma_i)^2 + (i-\sigma_{i+1})^2 > (i-\sigma_{i+1})^2 + ((i+1)-\sigma_i)^2$. Notice that $\vec{\sigma}$ and $\vec{\tau}$ differ only in the σ_i and σ_{i+1}th position, with $\vec{\sigma}_{\sigma_i} = i, \vec{\sigma}_{\sigma_{i+1}} = i+1, \vec{\tau}_{\sigma_i} = i+1$, and $\vec{\tau}_{\sigma_{i+1}} = i$. Hence $\mathrm{wt}_E(\vec{\sigma}) < \mathrm{wt}_E(\vec{\tau})$.

For our induction hypothesis, we shall suppose that $\tau = \sigma s_{i_1} \cdots s_{i_m}$ where $\mathrm{wt}_K(\sigma s_{i_1} \cdots s_{i_r}) = \mathrm{wt}_K(\sigma s_{i_1} \cdots s_{i_{r-1}}) + 1$ for all $1 \leq r \leq m$, implies that $\mathrm{wt}_E(\vec{\sigma}) < \mathrm{wt}_E(\vec{\tau})$. Consider now $\tau = \sigma s_{i_1} \cdots s_{i_{m+1}}$ where $\mathrm{wt}_K(\sigma s_{i_1} \cdots s_{i_r}) = \mathrm{wt}_K(\sigma s_{i_1} \cdots s_{i_{r-1}}) + 1$ for all $1 \leq r \leq m+1$. Then there exists $\sigma^{(m)}$ such that $\sigma^{(m)} = \sigma s_{i_1} \cdots s_{i_m}$ with $\mathrm{wt}_K(\sigma s_{i_1} \cdots s_{i_r}) = \mathrm{wt}_K(\sigma s_{i_1} \cdots s_{i_{r-1}}) + 1$ for all $1 \leq r \leq m$. We also have $\tau = \sigma^{(m)} s_{i_{m+1}}$ and $\mathrm{wt}_K(\tau) = \mathrm{wt}_K(\sigma^{(m)}) + 1$. Ergo by the base case and the induction hypothesis, $\mathrm{wt}_E(\vec{\sigma}) < \mathrm{wt}_E(\overrightarrow{\sigma^{(m)}}) < \mathrm{wt}_E(\vec{\tau})$. ☐

Corollary 4.10. $\mathrm{wt}_K(\sigma) \leq \mathrm{wt}_E(\vec{\sigma})$ *for all* $\sigma \in S_n$.

Proof. The identity element e has a weight of 0 under both the Kendall tau weight and the Euclidean weight. Any other permutation of S_n can be written as a composition of elements from the set $\{s_1, s_2, \ldots, s_{n-1}\}$. Moreover, for any $e \neq \sigma \in S_n$, we may write σ as $\sigma = s_{i_1} s_{i_2} \ldots s_{i_k}$ such that $e < \sigma$, where $i_j \in [1, n-1]$ for all j. For $\sigma = s_{i_1} s_{i_2} \ldots s_{i_k}$, we have $\mathrm{wt}_K(\sigma) = k$, but by Theorem 4.9, $\mathrm{wt}_E(\vec{\sigma}) \geq k$. Thus for any $\sigma \in S_n$, $\mathrm{wt}_K(\sigma) \leq \mathrm{wt}_E(\vec{\sigma})$. ☐

We will now explain the conditions necessary to extend LP decoding methods to permutation codes with the Kendall tau metric.

Definition 4.11 (Kendall tau LP-Decodable)**.** Let $\lambda, \mu \in S_n$, and G be a subgroup of S_n. Let $g_0 \in G$. We say $G\vec{\mu}$ is **Kendall tau LP-decodable** if the permutation code $G\vec{\mu}$ is LP-decodable *i.e.*, there exists a doubly stochastic constraint whose convex polytope has G as a vertex set, and the following statement called the **LP-decoding extension condition** is satisfied.

$$\begin{aligned} &\mathrm{d}_E(\vec{\lambda}, \overrightarrow{g_0\mu}) \leq \mathrm{d}_E(\vec{\lambda}, \overrightarrow{g\mu}) \quad \text{for all } g \in G \\ \Longrightarrow\ &\mathrm{d}_K(\lambda, g_0\mu) \leq \mathrm{d}_K(\lambda, g\mu) \quad \text{for all } g \in G. \end{aligned} \tag{4.1}$$

In this scheme, suppose a potentially corrupted transmitted vector $\vec{\lambda}$ is received. The decoder will attempt to find the closest codeword $g_0 \circ \vec{\mu} = \overrightarrow{g_0\mu}$ from $\vec{\lambda}$ in terms of d_E via linear programming methods [13]. If the LP-decoding extension condition holds, then $g_0\mu$ will be the closest permutation to λ in terms of d_K.

Lemma 4.12. *Let $\lambda, \mu \in S_n$, and G be a subgroup of S_n. Let $g_0 \in G$.*

$$\begin{aligned} &\mathrm{d}_E(\vec{\lambda}, \vec{\mu}) \leq \mathrm{d}_E(\vec{\lambda}, \overrightarrow{g\mu}) \text{ for all } g \in G \\ \Longrightarrow\ &\mathrm{d}_K(\lambda, \mu) \leq \mathrm{d}_K(\lambda, g\mu) \text{ for all } g \in G \end{aligned} \tag{4.2}$$

implies

$$\begin{aligned} &\mathrm{d}_E(\vec{\lambda}, \overrightarrow{g_0\mu}) \leq \mathrm{d}_E(\vec{\lambda}, \overrightarrow{g\mu}) \text{ for all } g \in G \\ \Longrightarrow\ &\mathrm{d}_K(\lambda, g_0\mu) \leq \mathrm{d}_K(\lambda, g\mu) \text{ for all } g \in G. \end{aligned}$$

Proof. To justify this lemma, note that both d_E and d_K are left invariant metrics. Thus $\mathrm{d}_E(\vec{\lambda}, \overrightarrow{g_0\mu}) = \mathrm{d}_E(\overrightarrow{g_0^{-1}\lambda}, \vec{\mu})$ and likewise $\mathrm{d}_K(\lambda, g_0\mu) = \mathrm{d}_K(g_0^{-1}\lambda, \mu)$. Since λ is taken over all of S_n, the desired result follows. □

For the case when $\mu = e$, we may simplify the LP-decoding extension condition even further.

Lemma 4.13. *Let $\lambda, \in S_n$, $\mu = e$, and G be a subgroup of S_n. Let $g_0 \in G$.*

$$\begin{aligned} &\mathrm{wt}_E(\overrightarrow{\lambda^{-1}}) \leq \mathrm{wt}_E(\overrightarrow{\lambda^{-1}g}) \quad \text{for all } g \in G \\ \Longrightarrow\ &\mathrm{wt}_K(\lambda^{-1}) \leq \mathrm{wt}_K(\lambda^{-1}g) \quad \text{for all } g \in G \end{aligned} \tag{4.3}$$

implies

$$\begin{aligned} &\mathrm{d}_E(\vec{\lambda}, \overrightarrow{g_0\mu}) \leq \mathrm{d}_E(\vec{\lambda}, \overrightarrow{g\mu}) \text{ for all } g \in G \\ \Longrightarrow\ &\mathrm{d}_K(\lambda, g_0\mu) \leq \mathrm{d}_K(\lambda, g\mu) \text{ for all } g \in G. \end{aligned}$$

Proof. The lemma follows immediately from the definitions of $\mathrm{d}_K, \mathrm{wt}_K, \mathrm{d}_E$, and wt_E. □

For the following examples we assume $\mu = e$. Thus by the previous two lemmas, to show that G satisfies the LP-decoding extension condition, it suffices to show that G satisfies statement (4.3).

Example 4.14 (Trivial Examples). $G = \{e\}$ and $G = S_n$ satisfy statement (4.3).

Remark 4.15. We can reformulate statement (4.3) in terms of cosets. A group $G \subseteq S_n$ will satisfy statement (4.3) if and only if the following implication holds: If $\lambda^{-1}g$ is an element of the coset $\lambda^{-1}G$ such that $\overrightarrow{\lambda^{-1}g}$ is of minimal Euclidean weight among all associated vectors $\vec{\sigma}$ for $\sigma \in \lambda^{-1}G$, then $\lambda^{-1}g$ is of minimal Kendall tau weight in $\lambda^{-1}G$. Here $\lambda^{-1} \in S_n$ and $g \in G$.

Example 4.16 (C_4). $C_4 \subset S_4$ satisfies the LP-decoding extension condition for $\mu = e$. Note first that $|S_4| = 24$ and $|C_4| = 4$, so we have 6 cosets of C_4 to consider. Of course C_4 itself satisfies statement (4.3) since $e \in C_4$ and $\mathrm{wt}_E(\vec{e}) = \mathrm{wt}_K(e) = 0$, which is always minimal. The cosets $[2,1,3,4]C_4$, $[1,3,2,4]C_4$, and $[1,2,4,3]C_4$ are disjoint and $\mathrm{wt}_E(\overrightarrow{[2,1,3,4]}) = \mathrm{wt}_E(\overrightarrow{[1,3,2,4]}) = \mathrm{wt}_E(\overrightarrow{[1,2,4,3]}) = 1$. These elements are the only elements of minimal weight in their respective cosets. By Corollary 4.10, it follows that these elements are also minimal in their cosets in terms of the Kendall tau weight. Thus it remains only to check the last two cosets. After explicit calculations, we determine that $\overrightarrow{[2,3,1,4]}, \overrightarrow{[1,4,2,3]}$, and $\overrightarrow{[2,1,4,3]}$ are the only elements in the remaining cosets for which wt_E is minimal among coset elements. It is easily verified that wt_K is also minimal among coset elements for $[2,3,1,4], [1,4,2,3]$, and $[2,1,4,3]$. Hence statement (4.3) is satisfied.

Note that each of $\{e\}$, S_n, and C_n can each be realized as the vertex set of the polytope for some doubly stochastic constraint [19]. Thus by the above examples $\{e\}\vec{\mu}$, $S_n\vec{\mu}$, and $C_4\vec{\mu}$ are Kendall tau LP-decodable when $\mu = e$. The following is an example of a subgroup that does not satisfy the LP-decoding extension condition.

Example 4.17 (D_{12}). The dihedral group D_{12} does not satisfy statement (4.3). Consider $\lambda^{-1} := [2,1,5,4,3,6]$. $\lambda^{-1}D_{12} = \{\lambda^{-1}g | g \in D_{12}\}$. It is easily verified that $\min\limits_{\lambda^{-1}g \,\in\, \lambda^{-1}D_{12}} \mathrm{wt}_E(\overrightarrow{\lambda^{-1}g}) = 5 = \mathrm{wt}_E(\overrightarrow{\lambda^{-1}})$. However, $g_0 := [2,1,6,5,4,3] \in D_{12}$ which implies that $\lambda^{-1}g_0 = [1,2,6,3,4,5] \in \lambda^{-1}D_{12}$ and $\mathrm{wt}_K(\lambda^{-1}g_0) = 3 < 4 = \mathrm{wt}_K(\lambda^{-1})$. Therefore $\mathrm{wt}_E(\overrightarrow{\lambda^{-1}}) \leq \mathrm{wt}_E(\overrightarrow{\lambda^{-1}g})$ for all $g \in G$, but there exists $g \in G$ such that $\mathrm{wt}_K(\lambda^{-1}) > \mathrm{wt}_K(\lambda^{-1}g)$.

Thus far we have only considered small nontrivial subgroups of S_n whose orbits are LP-decodable permutation codes. We would now like to consider parabolic subgroups $P \subseteq S_n$, which can potentially be quite large. Toward defining parabolic subgroups, we first recall the notion of a reflection group, as defined by Kane [15], beginning with the definition of a reflection.

Definition 4.18 (Reflection)**.** A **reflection** r_α is a linear operator acting upon a Euclidean space sending some nonzero vector α to its negative while fixing the hyperplane orthogonal to α.

Definition 4.19 (Reflection Group)**.** A **reflection group** is a group generated by a set of reflections.

The following exposition can be found in greater detail in [14], pages 5-6.

Remark 4.20. The symmetric group S_n is a reflection group.

Proof. We claim that $\{s_1, \ldots, s_{n-1}\}$ is a set of reflections generating S_n. To see that s_i is a reflection, note first that S_n can be viewed as a subgroup of the group of orthogonal matrices by allowing any element of S_n to act on the standard basis $\{e_1, \ldots, e_n\}$ for $\mathbb{R}^n$. Then s_i sends $e_i - e_{i+1}$ to its negative, $-e_i + e_{i+1}$. The set of vectors orthogonal to $e_i - e_{i+1}$ consists of all vectors whose ith and $(i+1)$th components are equal. Therefore s_i fixes the hyperplane orthogonal to $e_i - e_{i+1}$. It is well-known that $s_1, \ldots, s_{n-1}$ generate S_n [12]. Indeed, any permutation can be seen to be a composition of general transpositions (i, j), and any transposition is obtained by a composition of adjacent transpositions from the set $\{s_1, \ldots, s_{n-1}\}$. □

We introduce the terms root system, fundamental system, fundamental reflections, and parabolic subgroup as defined by Kane [15] in pages 25, 35, 45, and 57 respectively.

Definition 4.21 (Root System)**.** A **root system** for a reflection group W is a set of nonzero vectors $\Delta \subset \mathbb{R}^n$ satisfying the following statements.

1) $W = \{r_\alpha \mid \alpha \in \Delta\}$.
2) If $\alpha \in \Delta$, then $c\alpha \in \Delta$ if and only if $c = \pm 1$.
3) If $\alpha, \beta \in \Delta$, then $r_\alpha \circ \beta \in \Delta$
   (here $r_\alpha \circ \beta$ denotes the action of r_α on β).

Suppose i, j, i', and j' are in the set $\{1, 2, \ldots, n\}$. Notice that the set of vectors reflected by transpositions of the form $(i, j) \in S_n$ are exactly vectors of the form $e_i - e_j$. It is clear that $ce_i - ce_{i+1}$ is of the form $e_{i'} - e_{j'}$ if and only if $c = \pm 1$. It is also clear that $s_i \circ (e_j - e_{j+1})$ is of the form $e_{i'} - e_{j'}$ for some i' and j'. Since the set of transpositions of the form (i, j) generate all of S_n, we see that the set $\Delta_{S_n} := \{e_i - e_j \mid 1 \leq i < j \leq n\}$ is a root system for S_n.

Definition 4.22 (Fundamental System)**.** Given a root system $\Delta \subseteq \mathbb{R}^n$, then $\Sigma \subseteq \Delta$ is a **fundamental system** of Δ if

1) Σ is linearly independent
2) every element of Δ is a linear combination of elements of Σ, where the coefficients are all nonnegative or all nonpositive.

Note here that $\sigma_{S_n} := \{e_i - e_{i+1} \mid 1 \leq i \leq n-1\}$ is a fundamental system of Δ_{S_n} as defined above. Indeed, if the sum $\sum_{i=1}^{n-1} c_i(e_i - e_{i+1}) = 0$ where $c_i \in \mathbb{R}$, not all zero, then there must exist some $c_i(e_i - e_{i+1}) \neq 0$. This implies that $c_{i-1} = -c_{i-1}$, which in turn implies that $c_{i-2} = c_i$. Continuing in this manner, $c_1 = \pm c_i$, which implies that $\sum_{i=1}^{n-1} c_i(e_i - e_{i+1}) \neq 0$, a contradiction. Therefore Σ_{S_n} is linearly independent. It is also true that any element of Δ_{S_n} is a linear combination of Σ_{S_n} where the coefficients are all nonnegative. To see this fact, consider any $e_i - e_j$ where $1 \leq i < j \leq n$. We have $e_i - e_j = (e_i - e_{i+1}) + (e_{i+1} - e_{i+2}) + \cdots + (e_{j-1} - e_j)$.

Definition 4.23 (Fundamental Reflections)**.** Let $\Delta \subset \mathbb{R}^n$ be a root system of a reflection group W. The reflections $\{r_\alpha \mid \alpha \in \Sigma\}$ corresponding to a fundamental system Σ of Δ are called a set of **fundamental reflections** for W.

It is immediate from the previous discussion and the previous definition that the set $\{s_1, s_2, \ldots, s_{n-1}\}$ is a set of fundamental reflections for S_n. We next define the parabolic subgroup, which is a type of subgroup of a reflection group.

Definition 4.24 (Parabolic Subgroup)**.** A parabolic subgroup P of a reflection group W is a group generated by a subset of a set of fundamental reflections generating W.

In the case of S_n, a parabolic subgroup $P \subseteq S_n$ may be taken to be a subgroup of S_n generated by a subset of the set $\{s_1, \ldots, s_{n-1}\}$. In this paper we only consider such parabolic subgroups, corresponding to the aforementioned fundamental system Σ_{S_n}. For a discussion of conjugate parabolic subgroups corresponding to different fundamental systems, the reader is referred to [15], pages 57-63.

We would like to show that a parabolic subgroup $P \subseteq S_n$ is Kendall tau LP-decodable. We first show that P is LP-decodable. Notice that for any σ in P, the associated permutation matrix $X_{i,j}^\sigma$ will have blocks of permutation matrices along the main diagonal. For example, consider the parabolic subgroup $P \subseteq S_6$ such that P is generated by $s_1, s_4,$ and s_5. Then any permutation of P will have an associated matrix of the following form:

$$\begin{pmatrix} X_{1,1} & X_{1,2} & 0 & 0 & 0 & 0 \\ X_{2,1} & X_{2,2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & X_{4,4} & X_{4,5} & X_{4,6} \\ 0 & 0 & 0 & X_{5,4} & X_{5,5} & X_{5,6} \\ 0 & 0 & 0 & X_{6,4} & X_{6,5} & X_{6,6} \end{pmatrix}.$$

Here the matrices $\begin{pmatrix} X_{1,1} & X_{1,2} \\ X_{2,1} & X_{2,2} \end{pmatrix}$ and $\begin{pmatrix} X_{4,4} & X_{4,5} & X_{4,6} \\ X_{5,4} & X_{5,5} & X_{5,6} \\ X_{6,4} & X_{6,5} & X_{6,6} \end{pmatrix}$

have the form of permutation matrices for S_2 and S_3 respectively. In other words, for any parabolic subgroup $P \subseteq S_n$, the set of all permutations in P is simply the set of all permutations of S_n with the added constraint that $X_{i,j} = X_{j,i} = 0$ for all (i, j) in a subset of $\{1, \ldots, n\} \times \{1, \ldots, n\}$. We can explicitly construct this subset by first characterizing the entries of matrices in P that potentially have non-zero entries.

Definition 4.25 (Block Index Set)**.** Given a permutation subgroup P of S_n, the **block index set** $C_P := \{(i, \sigma_i) \mid 1 \leq i \leq n, \sigma \in P\}$ is the set of all ordered pairs (i,j) such that there exists a permutation matrix X of P with $X_{i,j} \neq 0$.

As an example, let us calculate C_P corresponding to the parabolic subgroup P of S_6 generated by s_1, s_4, and s_5. For any permutation $\sigma \in P$, either $\sigma_1 = 1$ or $\sigma_1 = 2$ and similarly $\sigma_2 = 1$ or $\sigma_2 = 2$. Hence the ordered pairs $(1,1), (1,2), (2,1)$, and $(2,2)$ are in C_P, but no other ordered paris beginning with 1 or 2 are in C_P. For all $\sigma \in P$, we have $\sigma_3 = 3$, so that $(3,3)$ is the only ordered pair C_P beginning with 3. Continuing in this manner, the only other entries that are included in C_P are $(4,4), (4,5), (4,6), (5,4), (5,5), (5,6), (6,4), (6,5)$, and $(6,6)$. For any $(i,j) \notin C_P$, we have $X_{i,j} = 0$.

Theorem 4.26. *Let P be a parabolic subgroup of S_n generated by a subset $\{s_{k_1}, \ldots s_{k_m}\}$ of $\{s_1, \ldots, s_{n-1}\}$. Then P is LP-decodable, i.e., there exists a doubly stochastic constraint $\mathcal{L}$ such that $P = Ver(D(\mathcal{L})) \cap S_n$, where L consists of the following linear constraints:*

(1) For $j = 1, \ldots, n$, $\sum_{i=1}^{n} X_{i,j} = 1$.

(2) For $i = 1, \ldots, n$, $\sum_{j=1}^{n} X_{i,j} = 1$.

(3) For all $1 \leq i,j \leq n$, $X_{i,j} \geq 0$.

(4) For all $(i,j) \in \{1, \ldots, n\} \times \{1, \ldots, n\}$ such that $(i,j) \notin C_P$,

$$X_{i,j} = X_{j,i} = 0.$$

Proof. Notice that the first three constraints of $\mathcal{L}$ are exactly those in the definition of a doubly stochastic matrix. By the Birkhoff von Neumann theorem, we know that the Birkhoff polytope B_n comprised of stochastic matrices is the convex polytope satisfying the first three constraints, and $\mathrm{Ver}(B_n) = S_n$. Since constraint (4) is comprised strictly of linear equations, including constraint (4) will remove permutations matrices such that $X_{i,j} = 1$ or $X_{j,i} = 1$ from the vertex set $\mathrm{Ver}(D(\mathcal{L}))$ but will have no effect on other vertexes. Thus $\mathrm{Ver}(D(\mathcal{L})) \cap S_n$ consists of all permutation matrices such that $X_{i,j} \neq 1$ and $X_{j,i} \neq 1$, for all $(i,j) \in \{1, \ldots, n\} \times \{1, \ldots, n\}$ but $(i,j) \notin C_P$. From the previous discussion, the linear constraint (4) is exactly the constraint that retains permutation matrices of P while excluding permutation matrices outside of P. This implies that $\mathrm{Ver}(D(\mathcal{L})) \cap S_n = P$. $\square$

A stronger result actually holds for parabolic subgroups P of S_n, namely that there exists a doubly stochastic constraint $\mathcal{L}$ such that $P = \mathrm{Ver}(D(\mathcal{L}))$. This result is a consequence of a type of constraint called a "consolidation" of linear constraints and a theorem proven in [19]. This is significant, since in the decoding algorithm described above, if the vertex set of the linear constraint $\mathcal{L}$ is a subset of S_n, then the solution to the linear programming problem will be a permutation matrix. The decoding performance of such a code will obviously be improved. In any case, we have verified that $P\vec{\mu}$ is an LP-decodable permutation code. Therefore to prove that P is Kendall tau LP-decodable, it remains only to show that P satisfies the LP-decoding extension condition.

Theorem 4.27. *If $P \subseteq S_n$ is a parabolic subgroup generated by a subset of $\{s_1, \ldots, s_{n-1}\}$ and $\mu = e$, then P will satisfy the LP-decoding extension condition.*

Proof. By Theorem 10 and Theorem 12 of [25], for any reflection subgroup G of S_n, there exists a unique element g_0 of minimal Kendall tau weight in the coset g_0G. Therefore, since P is a reflection subgroup of S_n, for each coset of P in S_n there exists a unique element $\lambda^{-1} \in S_n$ such that $\mathrm{wt}_K(\lambda^{-1}) < \mathrm{wt}_K(\lambda^{-1}\sigma)$ for any $\sigma \in P$. We saw previously that for any element $e \neq \sigma \in S_n$, $e < \sigma$ in the weak (right) Bruhat ordering. Thus for all $\sigma \in P$, it follows that $\lambda^{-1} < \lambda^{-1}\sigma$. By Theorem 4.9, $\overrightarrow{\lambda^{-1}}$ is the unique element of minimal Euclidean weight among all associated vectors for permutations in the coset $\lambda^{-1}P$, which implies that the LP-decoding extension condition is satisfied. □

5. Conclusion

In this paper we introduced the Euclidean weight and compared this function to the previously studied Kendall tau weight. We discussed the weight enumerator polynomials for certain subgroups and proved relations holding for general permutations of the symmetric group S_n. We also stated conditions for LP-decoding methods originally invented for use in permutation codes with the Euclidean distance as a metric to be extended to permutation codes with the Kendall tau metric. Finally we provided examples of groups satisfying these conditions, including parabolic subgroups.

Concerning future research directions, it remains an open question to completely characterize $W_E(S_n; t)$ in terms of $W_K(S_n; t)$. Similarly, it remains an open question to characterize $W_E(P; t)$ in terms of $W_K(P; t)$. It is also natural to carefully analyze every aspect of the performance of different Kendall tau LP-decodable codes, including the capacity, error-correcting capabilities, etc. One might also consider the effect of choosing an initial vector other than the identity.

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